



**HAL**  
open science

# Courant-sharp property for eigenfunctions of the Klein bottle

Pierre Bérard, Bernard Helffer, Rola Kiwan

► **To cite this version:**

Pierre Bérard, Bernard Helffer, Rola Kiwan. Courant-sharp property for eigenfunctions of the Klein bottle. 2020. hal-02937338v1

**HAL Id: hal-02937338**

**<https://hal.science/hal-02937338v1>**

Preprint submitted on 13 Sep 2020 (v1), last revised 11 Apr 2021 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# COURANT-SHARP PROPERTY FOR EIGENFUNCTIONS OF THE KLEIN BOTTLE

PIERRE BÉRARD, BERNARD HELFFER, AND ROLA KIWAN

ABSTRACT. The question of determining for which eigenvalues there exists an eigenfunction which has the same number of nodal domains as the label of the associated eigenvalue (Courant-sharp property) was motivated by the analysis of minimal spectral partitions. In previous works, many examples have been analyzed corresponding to Möbius strips, squares, rectangles, disks, triangles, tori, . . . . A natural toy model for further investigations is the Klein bottle, a non-orientable surface with Euler characteristic 0, and particularly the Klein bottle associated with the square torus, whose eigenvalues have higher multiplicities. In this note, we prove that the only Courant-sharp eigenvalues of the Klein bottle associated with the square torus (resp. of the Klein bottle with square fundamental domain) are the first and second eigenvalues.

## 1. INTRODUCTION

Given a closed Riemannian surface  $(M, g)$ , we write the eigenvalues of the Laplace-Beltrami operator  $-\Delta_g$ ,

$$(1.1) \quad 0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots,$$

in nondecreasing order, with multiplicities accounted for, and starting from the label 1.

Courant's nodal domain theorem (1923) states that any eigenfunction associated with the eigenvalue  $\lambda_k$  has at most  $k$  nodal domains (connected components of the complement of the zero set of  $u$ ). The eigenvalue  $\lambda_k$  is called *Courant-sharp* if there exists an associated eigenfunction with precisely  $k$  nodal domains. It follows from Courant's theorem that the eigenvalues  $\lambda_1$  and  $\lambda_2$  are Courant-sharp, and that  $\lambda_{k-1} < \lambda_k$  whenever  $\lambda_k$  is Courant-sharp. Courant-sharp eigenvalues appear naturally in the context of partitions, [13].

Back in 1956, Pleijel<sup>1</sup> proved that there are only finitely many Courant-sharp eigenvalues. The purpose of this note is to prove the following theorem.

**Theorem 1.1.** *Let  $\mathbb{K}$  denote the flat Klein bottle associated with the square torus  $\mathbb{T}$ . Then the only Courant-sharp eigenvalues of  $\mathbb{K}$  are  $\lambda_1$  and  $\lambda_2$ .*

Courant-sharp eigenvalues have previously been determined for several compact surfaces. We refer to the following papers and their bibliographies.

- ◇ Closed surfaces: round 2-sphere and projective plane, [21]; flat tori, [12, 3, 19].

---

*Date:* September 13, 2020 (BHK-klein-200717.tex).

*2010 Mathematics Subject Classification.* 58C40, 49Q10.

*Key words and phrases.* Spectral theory, Courant theorem, Laplacian, Nodal sets, Klein bottle.

<sup>1</sup>Pleijel's original proof applied to Dirichlet eigenvalues of bounded domains in  $\mathbb{R}^2$ , and was later extended to more general domains, [22], and to closed Riemannian manifolds, [6].

- ◇ Compact surfaces with boundary (different boundary conditions might be considered): square, [23, 2, 15, 9, 10]; equilateral triangle, [3]; 2-rep-tiles, [1]; cylinders, [11]; Möbius strip, [5].

There are also a few results in higher dimensions, see for example [16, 20, 14].

Most of the papers mentioned above adapt the method introduced by Pleijel in [23] to the example at hand.

The note is organized as follows. In Section 2, we recall some basic facts concerning Klein bottles. In Section 3, we adapt Pleijel's method [23] to the flat Klein bottle  $\mathbb{K}$  associated with the square torus. More precisely, we establish a lower bound on the counting function of the eigenvalues of  $\mathbb{K}$ , and we prove a Faber-Krahn inequality for the first Dirichlet eigenvalue of domains  $\omega \subset \mathbb{K}$  whose volume is small enough. As a consequence, we obtain that the only possible Courant-sharp eigenvalues of  $\mathbb{K}$  are  $\lambda_1, \lambda_2, \lambda_3, \lambda_5$ . In Section 4, we show that the eigenvalues  $\lambda_3$  and  $\lambda_5$  of  $\mathbb{K}$  are not Courant-sharp, by analyzing the nodal sets of associated eigenfunctions. In Appendix A, we consider the case of the flat Klein bottle whose fundamental domain is a square (Theorem A.1).

## 2. PRELIMINARIES

**2.1. Klein bottles.** In this note, we are interested in the flat Klein bottles<sup>2</sup>. More precisely, given  $a, b > 0$ , we consider the isometries of  $\mathbb{R}^2$  given by

$$(2.1) \quad \begin{cases} \tau_1 : (x, y) \mapsto (x, y + b), \\ \tau_2 : (x, y) \mapsto (x + a, y), \\ \tau : (x, y) \mapsto (x + \frac{a}{2}, b - y). \end{cases}$$

We denote by  $G_2$  (resp.  $G$ ) the group generated by  $\tau_1$  and  $\tau_2$  (resp. by  $\tau_1$  and  $\tau$ ). These groups act properly and freely by isometries on  $\mathbb{R}^2$  equipped with the usual scalar product. Since  $\tau^2 = \tau_2$ , the group  $G_2$  is a subgroup of index 2 of the group  $G$ . We denote by  $\mathbb{T}_{a,b}$  (resp.  $\mathbb{K}_{a,b}$ ) the torus  $\mathbb{R}^2/G_2$  (resp. the Klein bottle  $\mathbb{R}^2/G$ ). We equip  $\mathbb{T}_{a,b}$  and  $\mathbb{K}_{a,b}$  with the induced flat Riemannian metrics.

A fundamental domain for the action of  $G_2$  (resp.  $G$ ) on  $\mathbb{R}^2$  is the rectangle  $\mathcal{T}_{a,b} = (0, a) \times (0, b)$  (resp. the rectangle  $\mathcal{K}_{a,b} = (0, \frac{a}{2}) \times (0, b)$ , see Figure 2.1 (A)). The horizontal sides of  $\mathcal{K}_{a,b}$  are identified with the same orientation, the vertical sides are identified with the opposite orientations.

The geodesics of the Klein bottle are the images of the lines in  $\mathbb{R}^2$  under the Riemannian covering map  $\mathbb{R}^2 \rightarrow \mathbb{K}_{a,b}$ . They can be looked at in the fundamental domain  $\mathcal{K}_{a,b}$ , taking into account the identifications  $(x, 0) \sim (x, b)$  and  $(0, y) \sim (\frac{a}{2}, b - y)$ . Among them, we have some special geodesics, see Figure 2.1 (B)-(C),

- ◇  $t \mapsto (t, 0)$  and  $t \mapsto (t, \frac{b}{2})$ , for  $0 \leq t \leq \frac{a}{2}$ , which are periodic geodesics of length  $\frac{a}{2}$ ;
- ◇ for  $0 < y_0 < \frac{b}{2}$ ,  $\gamma_{y_0} : t \mapsto \begin{cases} (t, y_0), & 0 \leq t \leq \frac{a}{2}, \\ (t - \frac{a}{2}, b - y_0), & \frac{a}{2} \leq t \leq a, \end{cases}$  which is a periodic geodesic of length  $a$ ; the two lines in blue in Figure 2.1 (C) yield a periodic geodesic of the Klein bottle;

---

<sup>2</sup>For the classification of complete, flat surfaces, we refer to [18, p. 222-223], or [24, Chap. 2.5]. A summary is given in Appendix B.

◇ for  $0 \leq x_0 \leq \frac{a}{2}$ ,  $t \mapsto (x_0, t)$ , with  $0 \leq t \leq b$ , is a periodic geodesic of length  $b$ .

**Remark 2.1.** *The description of geodesics of the Klein bottles as projected lines implies that the shortest, nontrivial, periodic geodesic of  $\mathbb{K}_{a,b}$  has length  $\min\{\frac{a}{2}, b\}$ .*

**Remark 2.2.** *Scissoring the Klein bottle along the blue geodesic  $t \mapsto \gamma_{y_0}(t)$ , with  $0 < y_0 < \frac{b}{2}$  and  $0 \leq t \leq a$ , divides the surface into two Möbius strips whose souls are the geodesics  $t \mapsto (t, 0)$  and  $t \mapsto (t, \frac{b}{2})$ , see Figure 2.1 (D).*

The isometry  $\tau$  of  $\mathbb{R}^2$  induces an isometry on the torus  $\mathbb{T}_{a,b}$  so that we can identify  $\mathbb{K}_{a,b}$  with the quotient  $\mathbb{T}_{a,b}/\{Id, \tau\}$ . It follows that the eigenfunctions of the Klein bottle  $\mathbb{K}_{a,b}$  are precisely the eigenfunctions of the torus  $\mathbb{T}_{a,b}$  which are invariant under the map  $\tau$ . Because  $\tau$  is orientation reversing, the surface  $\mathbb{K}_{a,b}$  is non-orientable with orientation double cover  $\mathbb{T}_{a,b}$ .

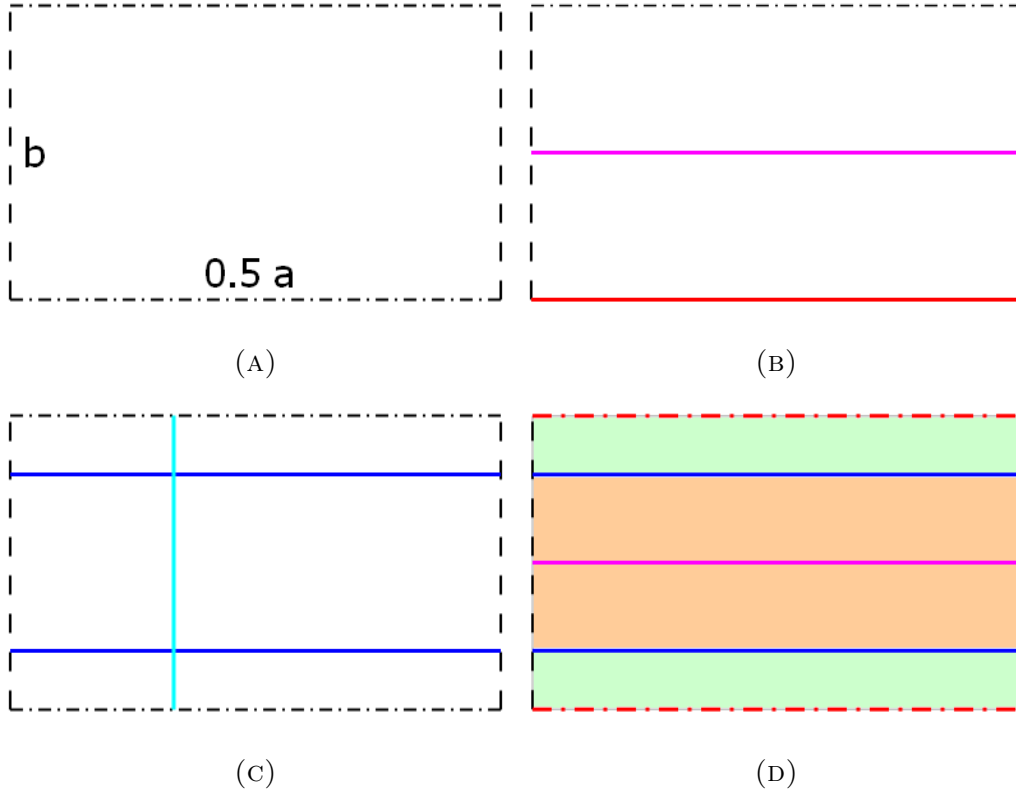


FIGURE 2.1. Fundamental domain, geodesics, partition into Möbius strips

**2.2. The spectrum of Klein bottles.** A complete family of (complex) eigenfunctions of the flat torus  $\mathbb{T}_{a,b}$  is

$$(2.2) \quad f_{m,n}(x, y) = \exp\left(i\frac{2\pi mx}{a}\right) \exp\left(i\frac{2\pi ny}{b}\right), \quad m, n \in \mathbb{Z},$$

with associated eigenvalues  $\hat{\lambda}(m, n) = 4\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$ . Given some eigenvalue  $\lambda$  of  $\mathbb{T}_{a,b}$ , we introduce the set,

$$(2.3) \quad \mathcal{L}_\lambda = \left\{ (m, n) \in \mathbb{Z}^2 \mid \hat{\lambda}(m, n) = \lambda \right\}$$

A general (complex) eigenfunction of  $\mathbb{T}_{a,b}$ , with eigenvalue  $\lambda$ , is of the form

$$(2.4) \quad \phi = \sum_{(m,n) \in \mathcal{L}_\lambda} \alpha_{m,n} f_{m,n},$$

with  $\alpha_{m,n} \in \mathbb{C}$ .

The function  $\phi$  is invariant under  $\tau$ ,  $\phi = \phi \circ \tau$ , if and only if

$$\sum_{(m,n) \in \mathcal{L}_\lambda} \alpha_{m,n} f_{m,n}(x,y) = \sum_{(m,n) \in \mathcal{L}_\lambda} \alpha_{m,n} (-1)^m f_{m,-n}(x,y),$$

or, equivalently, if and only if

$$(2.5) \quad \alpha_{m,-n} = (-1)^m \alpha_{m,n}, \quad \forall (m,n) \in \mathcal{L}_\lambda.$$

We can rewrite a  $\tau$ -invariant eigenfunction  $\phi$ ,

$$(2.6) \quad \phi = \sum_{(m,0) \in \mathcal{L}_\lambda, m \text{ even}} \alpha_{m,0} f_{m,0} + \sum_{(m,n) \in \mathcal{L}_\lambda, n > 0} \alpha_{m,n} (f_{m,n} + (-1)^m f_{m,-n}).$$

The following lemma follows readily.

**Lemma 2.3.** *A complete family of real eigenfunctions of the flat Klein bottle  $\mathbb{K}_{a,b}$  is given by the following functions.*

$$(2.7) \quad \begin{cases} \text{For } m = 0, n \in \mathbb{N} : & \cos\left(\frac{2\pi ny}{b}\right); \\ \text{for } m \in \mathbb{N}^\bullet \text{ even}, n \in \mathbb{N} : & \cos\left(\frac{2\pi mx}{a}\right) \cos\left(\frac{2\pi ny}{b}\right); \sin\left(\frac{2\pi mx}{a}\right) \cos\left(\frac{2\pi ny}{b}\right); \\ \text{for } m \in \mathbb{N}^\bullet \text{ odd}, n \in \mathbb{N}^\bullet : & \cos\left(\frac{2\pi mx}{a}\right) \sin\left(\frac{2\pi ny}{b}\right); \sin\left(\frac{2\pi mx}{a}\right) \sin\left(\frac{2\pi ny}{b}\right). \end{cases}$$

Here,  $\mathbb{N}$  denotes the set of non-negative integers, and  $\mathbb{N}^\bullet$  the set of positive integers.

**Remark 2.4.** *If  $\mathcal{L}_\lambda \cap (\{0\} \times \mathbb{Z}) \neq \emptyset$ , the multiplicity of  $\lambda$  is odd; if  $\mathcal{L}_\lambda \cap (\{0\} \times \mathbb{Z}) = \emptyset$ , the multiplicity of  $\lambda$  is even.*

**2.3. Choices for  $a$  and  $b$ .** In this paper, we mainly restrict our attention to the case,  $a = b = 2\pi$ , i.e. to the flat Klein bottle associated with the square flat torus (Theorem 1.1, Sections 3 and 4). In Appendix A, we consider the flat Klein bottle  $\mathbb{K}_{2\pi,\pi}$  whose fundamental domain is the square  $\mathcal{K}_{2\pi,\pi} = (0, \pi) \times (0, \pi)$ . As in [12] for flat tori, we could consider other values of the pair  $(a, b)$ .

We denote the associated square flat torus and flat Klein bottle by  $\mathbb{T}$  and  $\mathbb{K}$  respectively, with corresponding fundamental domains  $\mathcal{T} = (0, 2\pi) \times (0, 2\pi)$  and  $\mathcal{K} = (0, \pi) \times (0, 2\pi)$ .

As points of the spectrum, the eigenvalues of the flat Klein bottle  $\mathbb{K}$  are the numbers of the form  $\hat{\lambda}(p, q) = p^2 + q^2$ , with  $p, q \in \mathbb{N}$ , and the extra condition that  $p$  is even when  $q = 0$ . As usual, the eigenvalues of  $\mathbb{K}$  are listed in nondecreasing order, multiplicities accounted for, starting from the label 1,

$$(2.8) \quad 0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

For  $\lambda \geq 0$ , we introduce the *Weyl counting function*,

$$(2.9) \quad W(\lambda) = \#\{j \mid \lambda_j < \lambda\}.$$

Weyl's asymptotic law tells us that

$$(2.10) \quad W(\lambda) = \frac{|\mathbb{K}|}{4\pi} \lambda + O(\sqrt{\lambda}),$$

where  $|\mathbb{K}|$  denotes the area of the Klein bottle, namely  $|\mathbb{K}| = 2\pi^2$ .

For later purposes, we also introduce the *lattice counting function*,

$$(2.11) \quad L(\lambda) = \# \left\{ (m, n) \in \mathbb{N}^2 \mid m^2 + n^2 < \lambda \right\} .$$

### 3. A PRIORI INEQUALITIES ON THE COURANT-SHARP EIGENVALUES

The purpose of this section, following an original idea of Pleijel [23], is to establish a priori inequalities satisfied by the Courant-sharp eigenvalues of the Klein bottle  $\mathbb{K} = \mathbb{K}_{2\pi, 2\pi}$ . To do so we will need the following lower bound on the Weyl counting function.

**Lemma 3.1.** *The Weyl counting function  $W$  of the Klein bottle  $\mathbb{K}$  (see Section 2), satisfies,*

$$(3.1) \quad W(\lambda) \geq \frac{\pi}{2}\lambda - 2\sqrt{\lambda} - 3, \quad \text{for all } \lambda \geq 0 .$$

*Proof.* From the description of the spectrum of  $\mathbb{K}$ , a pair  $(m, n) \in \mathbb{N}^2$ , with  $m^2 + n^2 < \lambda$ , contributes to  $W(\lambda)$  by

- a) 2 if  $m, n \geq 1$ ;
- b) 1 if  $m = 0$ , and  $n \geq 0$ ;
- c) 2 if  $m \geq 2$  is even, and  $n = 0$ ;
- d) 0 otherwise.

Multiplicities also arise from the number of solutions of the equation  $m^2 + n^2 = m_0^2 + n_0^2$ . It follows that,

$$W(\lambda) = 2 \left( L(\lambda) - 2\lfloor\sqrt{\lambda}\rfloor - 1 \right) + \left( \lfloor\sqrt{\lambda}\rfloor + 1 \right) + 2\lfloor\frac{\sqrt{\lambda}}{2}\rfloor ,$$

where  $\lfloor x \rfloor$  denotes the integer part of  $x \geq 0$ . On the other hand, we have  $L(\lambda) \geq \frac{\pi}{4}\lambda$ . Indeed, to each  $(m, n)$  we can attach the square  $[m, m+1] \times [n, n+1]$ , and we have

$$\left\{ (x, y) \mid 0 \leq x, y, x^2 + y^2 < \lambda \right\} \subset \bigcup_{0 \leq m, n, m^2 + n^2 < \lambda} [m, m+1] \times [n, n+1] .$$

The lemma follows. □

**Lemma 3.2.** *Let  $\lambda_k$  be a Courant-sharp eigenvalue. Then,  $W(\lambda_k) = k - 1$ .*

*Proof.* Indeed, if  $\lambda_k$  is a Courant-sharp eigenvalue, we have  $\lambda_{k-1} < \lambda_k$  by applying Courant's theorem. □

**Lemma 3.3.** *Let  $\lambda_k$  be a Courant-sharp eigenvalue, and assume that  $k \geq 7$ . Then,*

$$(3.2) \quad \lambda_k \geq \frac{j_{0,1}^2}{2\pi} k > 0.92042 k ,$$

where  $j_{0,1} \approx 2.404825$  is the first positive zero of the Bessel function  $J_0$ .

*Proof.* As pointed out in Remark 2.1, the shortest nontrivial closed geodesic of  $\mathbb{K}$  has length  $\pi$ . It follows from [17, § 7], that any domain  $\omega \subset \mathbb{K}$ , with area  $|\omega|$  less than  $\pi$  satisfies the Euclidean isoperimetric inequality  $|\partial\omega|^2 \geq 4\pi |\omega|$  (here  $|\partial\omega|$  denotes the length of the boundary of  $\omega$ ). The usual symmetrization procedure implies that the least Dirichlet eigenvalue  $\delta_1(\omega)$  of the domain  $\omega$  satisfies the Faber-Krahn inequality,

$$\delta_1(\omega) |\omega| \geq \pi j_{0,1}^2 .$$

Let  $u$  be an eigenfunction associated with the Courant-sharp eigenvalue  $\lambda_k$ ,  $k \geq 7$ , with precisely  $k$  nodal domains. Then there exists at least one nodal domain, call it  $\omega$ , whose area satisfies

$$|\omega| \leq \frac{|\mathbb{K}|}{k} = \frac{2\pi^2}{k} \leq \frac{2\pi^2}{7} < \pi.$$

We can then apply the above Faber-Krahn inequality to the nodal domain  $\omega$ , and conclude that,

$$\lambda_k = \delta_1(\omega) \geq \frac{\pi j_{0,1}^2}{|\omega|} \geq \frac{j_{0,1}^2}{2\pi} k.$$

The lemma is proved.  $\square$

**Lemma 3.4.** *Let  $\lambda_k$  be a Courant-sharp eigenvalue, and assume that  $k \geq 7$ . Then,  $\lambda_k < 25$ .*

*Proof.* From Lemma 3.2 and Lemma 3.3, we have

$$W(\lambda_k) + 1 = k \leq \frac{2\pi}{j_{0,1}^2} \lambda_k.$$

Using Lemma 3.1, we obtain the inequality,

$$\frac{\pi}{2} \lambda_k - 2\sqrt{\lambda_k} - 2 \leq \frac{2\pi}{j_{0,1}^2} \lambda_k,$$

so that  $\lambda_k$  satisfies the inequality

$$(3.3) \quad \frac{\pi}{2} \left(1 - \frac{4}{j_{0,1}^2}\right) \lambda_k - 2\sqrt{\lambda_k} - 2 \leq 0.$$

This inequality implies that  $\lambda_k$  is less than or equal to the square of the positive root of the corresponding equation of degree 2,

$$\lambda_k \leq \left[ \frac{2 + 2\sqrt{1 + \pi\left(1 - \frac{4}{j_{0,1}^2}\right)}}{\pi\left(1 - \frac{4}{j_{0,1}^2}\right)} \right]^2 < 25.$$

$\square$

The following table displays the eigenvalues of  $\mathbb{K}$  less than or equal to 25, the corresponding labeled eigenvalues, and the ratio  $\frac{\lambda_{k_{\min}}}{k_{\min}}$  which should be larger than or equal to  $\frac{j_{0,1}^2}{2\pi} \approx 0.92042$  if  $\lambda_{k_{\min}}$  is Courant-sharp.

Lemma 3.4 tells us that for  $\lambda_k \geq 25$ , the eigenvalue  $\lambda_k$  is not Courant-sharp. Multiplicities, see also Lemma 3.3, imply that the eigenvalues  $\lambda_{k_{\min}+1}, \dots, \lambda_{k_{\max}}$  are not Courant-sharp. Since the eigenvalues  $\lambda_1, \lambda_2$  are Courant-sharp, in order to prove Theorem 1.1, it remains to investigate  $\lambda_3$  and  $\lambda_5$ . This is done in the next section.

Value	$\lambda_{k_{\min}}$	$\lambda_{k_{\max}}$	$\frac{\lambda_{k_{\min}}}{k_{\min}}$
2	$\lambda_3$	$\lambda_4$	—
4	$\lambda_5$	$\lambda_7$	—
5	$\lambda_8$	$\lambda_{11}$	0.625
8	$\lambda_{12}$	$\lambda_{13}$	0.6666
9	$\lambda_{14}$	$\lambda_{14}$	0.6428
10	$\lambda_{15}$	$\lambda_{18}$	0.6666
13	$\lambda_{19}$	$\lambda_{22}$	0.6842
16	$\lambda_{23}$	$\lambda_{25}$	0.6956
17	$\lambda_{26}$	$\lambda_{29}$	0.6538
18	$\lambda_{30}$	$\lambda_{31}$	0.6
20	$\lambda_{32}$	$\lambda_{35}$	0.625
25	$\lambda_{36}$	$\lambda_{40}$	0.6944

TABLE 3.1. Eigenvalues of the Klein bottle  $\mathbb{K}$

#### 4. THE FIRST EIGENVALUES OF $\mathbb{K}$

Recall that we work with the Klein bottle  $\mathbb{K} = \mathbb{K}_{2\pi, 2\pi}$ . In order to prove Theorem 1.1, we investigate the eigenvalues  $\lambda_3 = \hat{\lambda}(1, 1)$  and  $\lambda_5 = \hat{\lambda}(2, 0) = \hat{\lambda}(0, 2)$ .

**4.1. The eigenvalue  $\lambda_3$  is not Courant-sharp.** A general eigenfunction associated with  $\lambda_3$  has the form  $(a \cos(x) + b \sin(x)) \sin(y)$ . It is sufficient to look at eigenfunctions of the form  $\sin(x - \alpha) \sin(y)$ . These eigenfunctions have exactly two nodal domains in  $\mathbb{K}$ . It follows that  $\lambda_3$  is not Courant-sharp, see Figure 4.1.

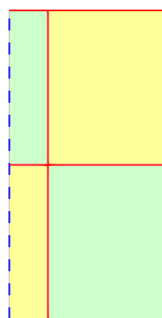


FIGURE 4.1. Nodal domains of  $\sin(x - \alpha) \sin(y)$



**4.2. The eigenvalue  $\lambda_5$  is not Courant-sharp.** A general eigenfunction associated with  $\lambda_5$  has the form  $a \cos(2x) + b \sin(2x) + c \cos(2y)$ . Up to multiplication by a scalar, it suffices to consider the family  $\cos \theta \cos(2x - \alpha) + \sin \theta \cos(2y)$ , with  $\theta \in [0, \pi)$  and  $\alpha \in [0, \pi)$ . Choosing the fundamental domain appropriately, we can assume that  $\alpha = 0$ . Changing  $y$  to  $y + \frac{\pi}{2}$ , we see that it suffices to consider  $\theta \in [0, \frac{\pi}{2}]$ . The nodal sets are known explicitly when  $\theta = 0$  or  $\frac{\pi}{2}$ . We now consider the family

$$(4.1) \quad \phi_\theta(x, y) = \cos \theta \cos(2x) + \sin \theta \cos(2y), \quad \theta \in (0, \frac{\pi}{2}).$$

The critical zeros (points at which both the function and its differential vanish) satisfy the system,

$$(4.2) \quad \begin{cases} \cos \theta \cos(2x) + \sin \theta \cos(2y) = 0, \\ \sin(2x) = 0, \\ \sin(2y) = 0. \end{cases}$$

It follows that critical zeros only occur for  $\theta = \frac{\pi}{4}$ , so that the nodal set of  $\phi_\theta$  is a regular curve when  $\theta \neq \frac{\pi}{4}$ . The nodal set of the eigenfunction

$$(4.3) \quad \phi_\theta(x, y) = \cos(2x) + \cos(2y) = 2 \cos(x + y) \cos(x - y)$$

is explicit, see Figure 4.2 (B). An analysis à la Stern, see [5, Section 5] or [4], shows that the nodal sets of  $\phi_\theta$  for  $0 < \theta < \frac{\pi}{4}$  and  $\frac{\pi}{4} < \theta < \frac{\pi}{2}$  are given by Figures 4.2 (A) and (C) respectively. More precisely, we first note that the common zeros to  $\phi_0$  and  $\phi_{\frac{\pi}{2}}$  are common zeros to all  $\phi_\theta$ . Since  $\theta \in (0, \frac{\pi}{2})$ , and hence  $\sin \theta \cos \theta > 0$ , it follows that, except for the common zeros, the nodal set of  $\phi_\theta$  is contained in the set  $\cos(2x) \cos(2y) < 0$ . Finally, depending on the sign of  $\frac{\pi}{4} - \theta$ , we can use the nodal set of  $\cos(2x)$  or the nodal set of  $\cos(2y)$  as barriers to obtain the behaviour described in the figures.

It follows that an eigenfunction associated with  $\lambda_5$  has at most 4 nodal domains in  $\mathbb{K}$ , and hence that  $\lambda_5$  is not Courant-sharp.

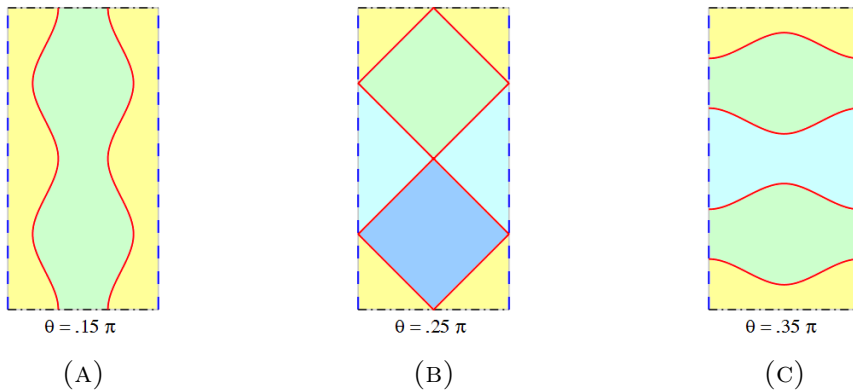


FIGURE 4.2. Nodal sets for  $\cos \theta \cos(2x) + \sin \theta \cos(2y)$

This completes the proof of Theorem 1.1.

APPENDIX A. THE KLEIN BOTTLE  $\mathbb{K}_{2\pi,\pi}$

In this appendix, we consider the case  $a = 2\pi$ ,  $b = \pi$ . We denote the associated rectangular flat torus by  $\mathbb{T}'$ , and the associated flat Klein bottle by  $\mathbb{K}'$ , with corresponding fundamental domains  $\mathcal{T}' = (0, 2\pi) \times (0, \pi)$  and  $\mathbb{K}' = (0, \pi) \times (0, \pi)$ . We prove,

**Theorem A.1.** *The only Courant-sharp eigenvalues of the flat Klein bottle  $\mathbb{K}'$ , with square fundamental domain, are the first and second eigenvalues.*

Since the proof is similar to the proof of Theorem 1.1, Sections 3 and 4, we only sketch it.

As points of the spectrum, the eigenvalues of the flat Klein bottle  $\mathbb{K}'$  are the numbers  $\hat{\lambda}'(p, q) = p^2 + 4q^2$ , with  $p, q \in \mathbb{N}$ , and the extra condition that  $p$  is even when  $q = 0$ . As usual, the eigenvalues of  $\mathbb{K}'$  are listed in nondecreasing order, multiplicities accounted for, starting with the label 1,

$$(A.1) \quad 0 = \lambda'_1 < \lambda'_2 \leq \lambda'_3 \leq \dots$$

For  $\lambda \geq 0$ , we introduce the Weyl counting function

$$(A.2) \quad W'(\lambda) = \#\{j \mid \lambda'_j < \lambda\}.$$

Weyl's asymptotic law tells us that

$$(A.3) \quad W'(\lambda) = \frac{|\mathbb{K}'|}{4\pi} \lambda + O(\sqrt{\lambda}) = \frac{\pi}{4} \lambda + O(\sqrt{\lambda}),$$

where  $|\mathbb{K}'|$  denotes the area of  $\mathbb{K}'$ , namely  $|\mathbb{K}'| = \pi^2$ .

For later purposes, we also introduce the lattice counting function,

$$(A.4) \quad L'(\lambda) = \#\{(m, n) \in \mathbb{N}^2 \mid m^2 + 4n^2 < \lambda\}.$$

Taking Lemma 2.3 into account, we can write the Weyl counting function as,

$$(A.5) \quad \begin{cases} W'(\lambda) = 2 \left[ L'(\lambda) - \lfloor \sqrt{\lambda} \rfloor - \lfloor \frac{\sqrt{\lambda}}{2} \rfloor - 1 \right] \\ \quad + \left[ \lfloor \frac{\sqrt{\lambda}}{2} \rfloor + 1 \right] \\ \quad + 2 \lfloor \frac{\sqrt{\lambda}}{2} \rfloor, \end{cases}$$

so that

$$(A.6) \quad W'(\lambda) = 2L'(\lambda) - 2\lfloor \sqrt{\lambda} \rfloor + \lfloor \frac{\sqrt{\lambda}}{2} \rfloor - 1.$$

As in the proof of Lemma 3.1, we see that  $L'(\lambda) \geq \frac{\pi}{4} \lambda$ , and it follows that

$$(A.7) \quad W'(\lambda) \geq \frac{\pi}{4} \lambda - \frac{3}{2} \sqrt{\lambda} - 2.$$

The smallest, nontrivial closed geodesic of  $\mathbb{K}'$  has length  $\pi$ . It follows that for any domain  $\omega \subset \mathbb{K}'$ , with  $|\omega| < \pi$ , we have

$$(A.8) \quad \delta_1(\omega) |\omega| \geq \pi j_{0,1}^2.$$

By the same argument as in Section 3, any Courant-sharp eigenvalue  $\lambda'_k$  of  $\mathbb{K}'$ , with  $k \geq 4$ , satisfies,

$$(A.9) \quad \lambda'_k \geq \frac{j_{0,1}^2}{\pi} k > 1.84 k$$

For any Courant-sharp eigenvalue  $\lambda'_k$  of  $\mathbb{K}'$ , we have  $W'(\lambda'_k) = k - 1$ . Using the preceding inequalities, we conclude that any Courant-sharp eigenvalue  $\lambda'_k$  of  $\mathbb{K}'$ , with  $k \geq 4$ , satisfies the inequality

$$(A.10) \quad \frac{\pi}{4} \left( 1 - \frac{4}{j_{0,1}^2} \right) \lambda'_k - \frac{3}{2} \sqrt{\lambda'_k} - 1 \leq 0,$$

which implies that  $\lambda'_k \leq 47$ . For the eigenvalues of  $\mathbb{K}'$ , we have the following table, similar to Table 3.1.

Value	$\lambda'_{k_{\min}}$	$\lambda'_{k_{\max}}$	$\frac{\lambda'_{k_{\min}}}{k_{\min}}$
4	$\lambda'_2$	$\lambda'_4$	—
5	$\lambda'_5$	$\lambda'_6$	1
8	$\lambda'_7$	$\lambda'_8$	1.1429
13	$\lambda'_9$	$\lambda'_{10}$	1.4444
16	$\lambda'_{11}$	$\lambda'_{13}$	1.4545
17	$\lambda'_{14}$	$\lambda'_{15}$	1.2143
20	$\lambda'_{16}$	$\lambda'_{19}$	1.2500
25	$\lambda'_{20}$	$\lambda'_{21}$	1.2500
29	$\lambda'_{22}$	$\lambda'_{23}$	1.3182
32	$\lambda'_{24}$	$\lambda'_{25}$	1.3333
36	$\lambda'_{26}$	$\lambda'_{28}$	1.3846
37	$\lambda'_{29}$	$\lambda'_{30}$	1.2759
40	$\lambda'_{31}$	$\lambda'_{34}$	1.2903
41	$\lambda'_{35}$	$\lambda'_{36}$	1.1714
45	$\lambda'_{37}$	$\lambda'_{38}$	1.2162
52	$\lambda'_{39}$	$\lambda'_{42}$	1.3333

TABLE A.1. Eigenvalues of the Klein bottle  $\mathbb{K}'$

The above table and the preceding inequalities show that the only Courant-sharp eigenvalues of  $\mathbb{K}'$  are  $\lambda'_1$  and  $\lambda'_2$ . This proves Theorem A.1.

#### APPENDIX B. CLASSIFICATION OF COMPLETE FLAT SURFACES

Classifying complete, flat, Riemannian 2-manifolds is equivalent to classifying discrete groups of isometries acting freely on  $\mathbb{R}^2$ . According to [18, p. 222-223], see also [24, Chap. 2.5], there are four types of such surfaces (up to a scaling factor). Each one is given by its fundamental group, and the way it acts on the universal cover  $\mathbb{R}^2$  in terms of Cartesian coordinates  $(x, y)$ .

- (1) The cylinder<sup>3</sup> (orientable surface),  
 (B.1)  $(x, y) \mapsto (x + n, y), \quad n \in \mathbb{Z} ;$
- (2) The torus (orientable surface),  
 (B.2)  $(x, y) \mapsto (x + ma + n, y + mb), \quad m, n \in \mathbb{Z}, \quad a, b \in \mathbb{R}, \quad b \neq 0 ;$
- (3) The Möbius band (non-orientable surface),  
 (B.3)  $(x, y) \mapsto (x + n, (-1)^n y), \quad n \in \mathbb{Z} ;$
- (4) The Klein bottle (non-orientable surface),  
 (B.4)  $(x, y) \mapsto (x + n, (-1)^n y + mb), \quad m, n \in \mathbb{Z}, \quad b \in \mathbb{R}, \quad b \neq 0 .$

For the flat tori and Klein bottles, we also refer to [8] (see 1.89, 2.25, 2.82, 2.83 and 4.46), and [7].

#### REFERENCES

- [1] R. Band, M. Bersudsky, and D. Fajman. Courant-sharp eigenvalues of Neumann 2-rep-tiles. *Lett. Math. Phys.* 107:5 (2017) 821–859. [2](#)
- [2] P. Bérard and B. Helffer. Dirichlet eigenfunctions of the square membrane: Courant’s property, and A. Stern’s and Å. Pleijel’s analyses. *Springer Proceedings in Mathematics & Statistics* (2015), MIMS-GGTM conference in memory of M. S. Baouendi. A. Baklouti, A. El Kacimi, S. Kallel, and N. Mir Editors. [2](#)
- [3] P. Bérard and B. Helffer. Courant-sharp eigenvalues for the equilateral torus, and for the equilateral triangle. *Lett. Math. Phys.* 106 (2016) 1729–1789. [1](#), [2](#)
- [4] P. Bérard and B. Helffer. Nodal sets of eigenfunctions, Antonie Stern’s results revisited. *Actes du séminaire Théorie spectrale et géométrie* 32 (2014-2015) 1–37. [8](#)
- [5] P. Bérard, B. Helffer, and R. Kiwan. Courant-sharp property for Dirichlet eigenfunctions on the Möbius strip. *arXiv:2005.01175*. [2](#), [8](#)
- [6] P. Bérard and D. Meyer. Inégalités isopérimétriques et applications. *Annales scientifiques de l’É.N.S. série 4, tome 15, n 3 (18) (1982) p.513–541*. [1](#)
- [7] M. Berger, P. Gauduchon, E. Mazet. *Le spectre d’une variété riemannienne*. Lecture notes 192. Springer, 1971. [11](#)
- [8] S. Gallot, D. Hulin and J. Lafontaine. *Riemannian geometry*. Third edition. Springer, 2004. [11](#)
- [9] K. Gittins, B. Helffer. Courant-sharp Robin eigenvalues for the square and other planar domains. *Port. Math.* 76 (2019), 57-100. doi: 10.4171/PM/2027 [2](#)
- [10] K. Gittins, B. Helffer. Courant-sharp Robin eigenvalues for the square: the case with small Robin parameter. *Ann. Math. Québec* (2019). <https://doi.org/10.1007/s40316-019-00120-7> [2](#)
- [11] B. Helffer and T. Hoffmann-Ostenhof. On minimal partitions on thin strips on a cylinder or thin annulus like domains. *Proceedings de OTAMP 2010 in Operator Theory : Advances and Applications*, Vol. 227, p. 107-115 (2011). [2](#)
- [12] B. Helffer and T. Hoffmann-Ostenhof. Minimal partitions for anisotropic tori. *J. Spectral theory* 4 (2) (2014) 221-233. [1](#), [4](#)
- [13] B. Helffer, T. Hoffmann-Ostenhof, and S. Terracini. Nodal domains and spectral minimal partitions. *Ann. Inst. H. Poincaré AN* 26, 101–138, 2009. [1](#)

---

<sup>3</sup>We point out that both the cylinder and the Möbius band mentioned here are infinite complete surfaces, not surfaces with boundary.

- [14] B. Helffer and R. Kiwan. Dirichlet eigenfunctions in the cube, sharpening the Courant nodal inequality. In “Functional analysis and operator theory for quantum physics. The Pavel Exner anniversary volume. Dedicated to Pavel Exner on the occasion of his 70th birthday”. J. Dittrich (ed.). European Math. Soc. Series of Congress Reports (2017), pp. 353–371. [2](#)
- [15] B. Helffer and M. Persson Sundqvist. Nodal domains in the square. The Neumann case. *Mosc. Math. J.* 15 (2015) 455–495. [2](#)
- [16] B. Helffer and M. Persson Sundqvist. On nodal domains in Euclidean balls. *Proc. Amer. Math. Soc.* 144:11 (2016) 4777–4791. [2](#)
- [17] H. Howards, M. Hutchings, and F. Morgan. The isoperimetric problem on surfaces. *Amer. Math. Monthly* 106, 430–439 (1999). [5](#)
- [18] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. Vol. 1. Interscience publishers, 1963. [2](#), [10](#)
- [19] C. Léna. Courant-sharp eigenvalues of a two-dimensional torus. *C.R. Math. Acad. Sci. Paris* 353 (6) (2015) 535–539. [1](#)
- [20] C. Léna. Courant-sharp eigenvalues of the three-dimensional square torus. *Proc. Amer. Math. Soc.* 144:9 (2016) 3949–3958. [2](#)
- [21] J. Leydold. On the number of nodal domains of spherical harmonics. *Topology* 35:2 (1996) 301–321. [1](#)
- [22] J. Peetre. A generalization of Courant’s nodal domain theorem. *Math. Scand.* Vol 5, 1957, p 15–20. [1](#)
- [23] Å. Pleijel. Remarks on Courant’s nodal theorem. *Comm. Pure. Appl. Math.* **9** (1956), 543–550. [2](#), [5](#)
- [24] J. Wolf. *Spaces of Constant Curvature*. AMS chelsea Publishing, 2011. [2](#), [10](#)

PB: UNIVERSITÉ GRENOBLE ALPES AND CNRS, INSTITUT FOURIER, CS 40700, 38058 GRENOBLE CEDEX 9, FRANCE.

*E-mail address:* pierreherberard@gmail.com

BH: LABORATOIRE JEAN LERAY, UNIVERSITÉ DE NANTES AND CNRS, F44322 NANTES CEDEX, FRANCE AND LMO (UNIVERSITÉ PARIS-SUD).

*E-mail address:* Bernard.Helffer@univ-nantes.fr

RK: AMERICAN UNIVERSITY IN DUBAI, P.O.BOX 28282, DUBAI, UNITED ARAB EMIRATES.

*E-mail address:* rkiwan@aud.edu