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AN HYPERBOLIC-PARABOLIC PREDATOR-PREY MODEL INVOLVING A VOLE POPULATION STRUCTURED IN AGE

G. M. COCLITE, C. DONADELLO, AND T. N. T. NGUYEN

ABSTRACT. We prove existence and stability of entropy solutions for a predator-prey system consisting of an hyperbolic equation for predators and a parabolic-hyperbolic equation for preys. The preys' equation, which represents the evolution of a population of voles as in [2], depends on time, t , age, a , and on a 2-dimensional space variable x , and it is supplemented by a nonlocal boundary condition at $a = 0$. The drift term in the predators' equation depends nonlocally on the density of preys and the two equations are also coupled via classical source terms of Lotka-Volterra type, as in [4]. We establish existence of solutions by applying the vanishing viscosity method, and we prove stability by a doubling of variables type argument.

1. INTRODUCTION

1.1. The model and the assumptions. Our goal in this paper is to investigate the wellposedness of a predator-prey model extending the model for a vole population structured in age we introduced in [2]. To this end we couple the latter equation to the hyperbolic equation for predators proposed in [4] in which the drift depends nonlocally on the density of preys, so that the predators tend to move toward the regions in which preys are more abundant. The system we consider writes as follows

$$(1.1) \quad \begin{cases} \partial_t u + \operatorname{div}_x (uv(\phi)) = (\mathbf{b}(\phi) - \beta)u, & (t, x) \in (0, T) \times \mathbb{R}^2, \\ \partial_t \rho + \partial_a \rho + \operatorname{div}_x (\rho \chi_1(a) \mathbf{v}(x) Y_\theta(\phi - R)) = \mu \Delta_x \rho - \mathfrak{d}(t, a, x) \rho - \mathbf{p}(a, u) \rho, & (t, a, x) \in (0, T)^2 \times \mathbb{R}^2, \\ \rho(t, 0, x) = \mathcal{A}(\phi) \left(\int_0^\infty \rho(t, a, x) \chi_3(a) da \right) \omega(t, x), & (t, x) \in (0, T) \times \mathbb{R}^2, \\ \rho(0, a, x) = \rho_0(a, x), & (a, x) \in (0, T) \times \mathbb{R}^2, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^2, \end{cases}$$

where $u = u(t, x)$ and $\phi = \phi(t, x)$ represent the respective density of predators and preys at (t, x) . Since the prey population is also structured on age its dynamics is better described by $\rho = \rho(t, a, x)$, which is the density of preys of age a at (t, x) . More precisely, the relation between ϕ and ρ is given by

$$(1.2) \quad \phi(t, x) = \int_0^\infty \rho(t, a, x) \chi_2(a) da,$$

where $\chi_2(a)$ is an approximation of the indicator function of the interval (σ, T) , where T is the target time of our observation and $0 < \sigma \ll 1$. The parameter σ does not play a role in the modeling, but allows to avoid technical difficulties in our analysis. In the first equation, the function $\mathbf{b}(\phi)$ represents the reproduction rate of predators depending on preys' availability, while $\beta > 0$, the predators' mortality rate, is assumed to be constant. As in [4] the flux of u is driven to the direction of higher preys'

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concentration by a nonlinear, nonlocal velocity ν of the form

$$\nu(\phi) = \kappa \frac{\nabla(\phi * \eta)}{\sqrt{1 + \|\nabla(\phi * \eta)\|^2}},$$

where $\kappa > 0$ is the maximal speed of predators and η is a positive smooth mollifier with $\int_{\mathbb{R}^2} \eta dx = 1$ so that the convolution $(\phi(t) * \eta)(x)$ represents an average of the density of preys in a neighborhood of x at time t .

The equation for the preys, introduced in [2], is related to classical models for the dynamics of a population structured in age, see [3, 9], but the choice of the coefficients and boundary conditions at $a = 0$ takes into account the data collections and ecological considerations in [1, 5, 6, 7, 11]. We recall here the essential assumptions on the form of the coefficients.

We introduce constants $0 < A_1 < A_2$ so that a vole is young (baby) if its age a is in $(0, A_1)$, juvenile if its age is in (A_1, A_2) and adult otherwise. The three age classes differ as babies do not reproduce, adults' mortality rate is lower and juveniles exhibit a significant spatial dynamic during dispersals. Dispersal is a characteristic phenomenon of vole populations, correlated to overcrowding. Whenever the density of voles ϕ rises above a threshold value $R > 0$, representing a fraction of the capacity of the environment, the juvenile individuals leave their original colony and disperse over relatively large distances (0.5 to 5 km) with velocity $\mathbf{v}(x)$. We fix $\theta > 0$ and we consider an approximation of the Heaviside function, Y_θ , defined as

$$Y(\xi) = \begin{cases} 1, & \text{if } \xi \geq 0, \\ 0, & \text{if } \xi \leq -1, \end{cases} \quad Y'(\xi) \geq 0, \quad Y_\theta(\xi) = Y\left(\frac{\xi}{\theta}\right).$$

From Y_θ we construct the approximations of the indicator functions of the intervals (σ, T) , (A_1, A_2) , and (A_1, T)

$$\chi_1(a) = Y_\theta(a - A_1)Y_\theta(A_2 - a), \quad \chi_2(a) = Y_\theta(a - \sigma)Y_\theta(T - a), \quad \chi_3(a) = Y_\theta(a - A_1)Y_\theta(T - a).$$

The mortality rate of voles splits into two terms: $\mathbf{p} = \mathbf{p}(a, u)$ represents the mortality due to the presence of the specific predator whose density is u , while $\mathfrak{d} = \mathfrak{d}(t, a, x)$ stands for all other casualties (sickness, starvation, generic predation, etc).

The second-order term $\mu \Delta_x \rho$ represents short range spatial dynamics related to foraging activities. Everywhere in the following θ and $\mu > 0$ are fixed.

In the boundary condition at $a = 0$, the function $\omega = \omega(t, x)$ is the reproduction rate of voles depending on time and position. The function $\mathcal{A}(\phi)$ describes the influence of the total density of voles on natality.

The fourth and fifth equations are the respective initial conditions at $t = 0$ for voles and predators.

1.2. Assumptions. The coefficients $\mathbf{b}, \mathbf{v}, \mathfrak{d}, \mathbf{p}, \mathcal{A}, \omega$ and the initial data ρ_0, u_0 of system (1.1) satisfy the following conditions:

$$(1.3) \quad \mathbf{b} \in C^\infty(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}), \quad \mathbf{b}(\cdot) \geq 0,$$

$$(1.4) \quad \mathbf{v} \in C^\infty(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), \quad \operatorname{div}_x(\mathbf{v}) \in L^1(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2), \quad \mathbf{v} > 0,$$

$$(1.5) \quad \mathfrak{d} \in C^\infty([0, \infty) \times [0, \infty) \times \mathbb{R}^2) \cap W^{1,\infty}((0, \infty) \times (0, \infty) \times \mathbb{R}^2), \quad 0 < d_* \leq \mathfrak{d}(\cdot, \cdot, \cdot) \leq d^*,$$

$$(1.6) \quad \mathbf{p} \in C^\infty([0, \infty) \times \mathbb{R}) \cap W^{2,\infty}((0, \infty) \times \mathbb{R}), \quad 0 < p_* \leq \mathbf{p}(\cdot, \cdot) \leq p^*,$$

$$(1.7) \quad \mathcal{A} \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \mathcal{A}(\cdot) \geq 0, \quad \mathcal{A}(0) = 0, \quad |\mathcal{A}'(\xi)\xi|, |\mathcal{A}''(\xi)\xi| \leq C_0,$$

$$(1.8) \quad \omega \in C^\infty([0, \infty) \times \mathbb{R}^2) \cap W^{1,\infty}((0, \infty) \times \mathbb{R}^2), \quad \omega(\cdot, \cdot) \geq 0,$$

$$(1.9) \quad \rho_0 \in L^1((0, \infty) \times \mathbb{R}^2) \cap L^\infty((0, \infty) \times \mathbb{R}^2), \quad \rho_0 \geq 0,$$

$$(1.10) \quad \sup_{x \in \mathbb{R}^2} \|\rho_0(\cdot, x)\|_{L^1(0, \infty)}, \sup_{a > 0} \|\rho_0(a, \cdot)\|_{L^1(\mathbb{R}^2)}, \int_{\mathbb{R}^2} TV(\rho_0(\cdot, x)) dx \leq C_0,$$

$$(1.11) \quad u_0 \in L^1(\mathbb{R}^2) \cap BV(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), \quad u_0 \geq 0.$$

for some positive constants d_* , d^* , p_* , p^* , C_0 .

For what the velocity of predators, ν , is concerned, we fix the mollifier η so to ensure the estimates in the following Lemma, whose proof is postponed to Section 4 (see also [4, Lemma 4.1])

Lemma 1.1. *Let η be such that*

$$(1.12) \quad \nabla_x \eta \in (C^2 \cap W^{2,2} \cap W^{1,\infty})(\mathbb{R}^2, \mathbb{R}^2).$$

Then the map $\nu : L^1(\mathbb{R}^2; \mathbb{R}) \mapsto C^\infty(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ satisfies

$$(1.13) \quad \|\nu(\phi)\|_{L^\infty(\mathbb{R}^2; \mathbb{R}^2)} \leq K \|\phi\|_{L^1(\mathbb{R}^2; \mathbb{R})},$$

$$(1.14) \quad \|\operatorname{div}_x(\nu(\phi))\|_{L^2(\mathbb{R}^2; \mathbb{R})} \leq K \|\phi\|_{L^1(\mathbb{R}^2; \mathbb{R})},$$

$$(1.15) \quad \|\nabla_x \nu(\phi)\|_{L^\infty(\mathbb{R}^2, \mathbb{R}^{2 \times 2})} \leq K \|\phi\|_{L^1(\mathbb{R}^2; \mathbb{R})},$$

$$(1.16) \quad \|\nu(\phi_1) - \nu(\phi_2)\|_{L^\infty(\mathbb{R}^2, \mathbb{R}^2)} \leq K \|\phi_1 - \phi_2\|_{L^1(\mathbb{R}^2; \mathbb{R})},$$

$$(1.17) \quad \|\nabla_x(\operatorname{div}_x(\nu(\phi)))\|_{L^2(\mathbb{R}^2, \mathbb{R}^2)} \leq K \left(1 + K \|\phi\|_{L^1(\mathbb{R}^2; \mathbb{R})}\right) \|\phi\|_{L^1(\mathbb{R}^2; \mathbb{R})},$$

$$(1.18) \quad \|\operatorname{div}_x(\nu(\phi_1) - \nu(\phi_2))\|_{L^2(\mathbb{R}^2, \mathbb{R})} \leq K \left(1 + K \|\phi_2\|_{L^1(\mathbb{R}^2; \mathbb{R})}\right) \|\phi_1 - \phi_2\|_{L^1(\mathbb{R}^2; \mathbb{R})},$$

where K is a positive constant.

1.3. Main result. Our main result is the wellposedness of entropy weak solutions for system (1.1), stated in Theorem 1.1. We adopt the following definitions of weak solution and entropy solution

Definition 1.1. *We say that the pair (u, ρ) is a weak solution of (1.1) if the following holds for every $T > 0$*

$$(D.1) \quad \rho \geq 0, \rho \in L^\infty(0, T; L^1((0, \infty) \times \mathbb{R}^2)) \cap L^\infty((0, T) \times (0, \infty) \times \mathbb{R}^2) \cap L^2((0, T) \times (0, \infty); H^2(\mathbb{R}^2)).$$

$$(D.2) \quad u \in L^1((0, T) \times \mathbb{R}^2) \cap BV((0, T) \times \mathbb{R}^2).$$

(D.3) *For almost every $(t, x) \in (0, T) \times \mathbb{R}^2$, $\rho(t, \cdot, x) \in BV(0, \infty)$ and*

$$\rho(t, 0^+, x) = \mathcal{A}(\phi) \left(\int_0^\infty \rho(t, a, x) \chi_3(a) da \right) \omega(t, x),$$

where $\rho(t, 0^+, x)$ is the trace of $\rho(t, \cdot, x)$ at $a = 0$.

(D.4) *For every test function $\xi \in C_c^\infty(\mathbb{R}^4)$*

$$\begin{aligned} & \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} (\rho \partial_t \xi + \rho \partial_a \xi + \rho \chi_1(a) \mathbf{v} \cdot \nabla_x \xi Y_\theta(\phi - R) + \mu \rho \Delta_x \xi - \mathfrak{d} \rho \xi - \mathfrak{p} \rho \xi) dx da dt \\ & + \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} \mathcal{A}(\phi) \rho(t, a, x) \chi_3(a) \omega(t, x) \xi(t, 0, x) dx da dt \\ & + \int_0^\infty \int_{\mathbb{R}^2} \rho_0(a, x) \xi(0, a, x) dx da = 0, \end{aligned}$$

$$\int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} (u \partial_t \xi + u \nu(\phi) \cdot \nabla_x \xi + (\mathbf{b}(\phi) - \beta) u \xi) dx da dt + \int_0^\infty \int_{\mathbb{R}^2} u_0(x) \xi(0, a, x) dx da = 0.$$

Definition 1.2. We say that a weak solution (u, ρ) is an entropy solution of (1.1) if for any non-negative test function $\xi \in C^\infty(\mathbb{R}^4)$ with compact support and for any constant $\mathbf{c} \in \mathbb{R}$ there hold

$$\begin{aligned}
(1.19) \quad & \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} (|\rho - \mathbf{c}| (\partial_t \xi + \partial_a \xi) - \operatorname{div}_x (|\rho - \mathbf{c}| \chi_1 \mathbf{v} Y_\theta (\phi - R))) \xi \\
& \quad + \mu \Delta_x |\rho - \mathbf{c}| \xi - \operatorname{sign}(\rho - \mathbf{c}) (\mathfrak{d} + \mathfrak{p}) \rho \xi \, dx \, da \, dt \\
& + \int_0^\infty \int_{\mathbb{R}^2} |\rho(t, 0^+, x) - \mathbf{c}| \xi(t, 0, x) \, dx \, dt \\
& + \int_0^\infty \int_{\mathbb{R}^2} |\rho_0(a, x) - \mathbf{c}| \xi(0, a, x) \, dx \, da \\
& \geq \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} \mathbf{c} \operatorname{sign}(\rho - \mathbf{c}) \chi_1(a) \operatorname{div}_x (\mathbf{v}(x) Y_\theta (\phi - R)) \xi \, dx \, da \, dt
\end{aligned}$$

and

$$\begin{aligned}
(1.20) \quad & \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} (|u - \mathbf{c}| \partial_t \xi + |u - \mathbf{c}| \nu(\phi) \cdot \nabla_x \xi + \operatorname{sign}(u - \mathbf{c}) (\mathbf{b}(\phi) - \beta) u \xi) \, dx \, da \, dt \\
& + \int_0^\infty \int_{\mathbb{R}^2} |u_0(x) - \mathbf{c}| \xi(0, a, x) \, dx \, da \geq \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} \mathbf{c} \operatorname{sign}(u - \mathbf{c}) \operatorname{div}_x (\nu(\phi)) \xi \, dx \, da \, dt.
\end{aligned}$$

Theorem 1.1. Assume (1.3)-(1.12), then the initial boundary value problem (1.1) admits a unique entropy solution (u, ρ) in the sense of Definition 1.2. Moreover, if (u_1, ρ_1) and (u_2, ρ_2) are the two entropy solutions of (1.1) having initial data $(u_{1,0}, \rho_{1,0})$ and $(u_{2,0}, \rho_{2,0})$, then there exists a positive constant C such that the following estimate holds for almost every $t \geq 0$

$$\begin{aligned}
(1.21) \quad & \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\mathbb{R}^2)} + \|\rho_1(t, \cdot, \cdot) - \rho_2(t, \cdot, \cdot)\|_{L^1((0, \infty) \times \mathbb{R}^2)} \\
& \leq C e^{C e^{Ct}} \|\rho_{1,0} - \rho_{2,0}\|_{L^1((0, \infty) \times \mathbb{R}^2)} + C e^{C e^{Ct}} \|u_{1,0} - u_{2,0}\|_{L^1(\mathbb{R}^2)}.
\end{aligned}$$

The fixed point argument used in [4] to prove existence and stability for a predator-prey system does not apply to our system in a straightforward way because it requires extremely fine information on the coefficients appearing in the a priori estimates for both predators' and preys' equations. This is not easy to obtain in our case, as the equation we use for preys comes from a specific population model and its analytical study is rather technical.

In Section 2 we introduce a sequence of parabolic approximations of problem (1.1), for which we prove suitable a priori estimates. Then we apply the compensated compactness lemma by Panov, see [10], to show the strong compactness of the sequence for voles. Lemma 2.11 establishes the existence of an entropy solution in the sense of Definition 1.2. The uniqueness and stability of such solutions are proved in Section 3 using a doubling of variables type argument.

2. EXISTENCE

The existence argument is based on the compactness analysis of a sequence of solutions to approximating problems defined as follows. For any given $\varepsilon > 0$, we let $(\rho_\varepsilon = \rho_\varepsilon(t, a, x), u_\varepsilon = u_\varepsilon(t, x))$ be a

solution of the problem

$$(2.1) \quad \begin{cases} \partial_t u_\varepsilon + \operatorname{div}_x (u_\varepsilon \nu(\phi_\varepsilon)) = (\mathbf{b}(\phi_\varepsilon) - \beta)u_\varepsilon + \varepsilon \Delta_x u_\varepsilon, & (t, x) \in (0, T) \times \mathbb{R}^2, \\ \partial_t \rho_\varepsilon + \partial_a \rho_\varepsilon + \operatorname{div}_x (\rho_\varepsilon \chi_1(a) \mathbf{v}(x) Y_\theta(\phi_\varepsilon - R)) \\ \quad = \mu \Delta_x \rho_\varepsilon - \mathfrak{d}(t, a, x) \rho_\varepsilon - \mathfrak{p}(a, u_\varepsilon) \rho_\varepsilon + \varepsilon \partial_{aa}^2 \rho_\varepsilon, & (t, a, x) \in (0, T)^2 \times \mathbb{R}^2, \\ \rho_\varepsilon(t, 0, x) = \mathcal{A}(|\phi_\varepsilon|) \left(\int_0^\infty |\rho_\varepsilon(t, a, x)| \chi_3(a) da \right) \omega(t, x), & (t, x) \in (0, T) \times \mathbb{R}^2, \\ \rho_\varepsilon(0, a, x) = \rho_{0,\varepsilon}(a, x), & (a, x) \in (0, T)^2 \times \mathbb{R}^2, \\ u_\varepsilon(0, x) = u_{0,\varepsilon}(x), & x \in \mathbb{R}^2, \end{cases}$$

where

$$\phi_\varepsilon(t, x) = \int_0^\infty \rho_\varepsilon(t, a, x) \chi_2(a) da,$$

and $\{(\rho_{0,\varepsilon}, u_{0,\varepsilon})\}_\varepsilon$ is a family of approximations of the initial condition (ρ_0, u_0) such that

$$(2.2) \quad \begin{aligned} \rho_{0,\varepsilon} &\in C^\infty((0, \infty) \times \mathbb{R}^2), \quad u_{0,\varepsilon} \in C^\infty(\mathbb{R}^2), & \varepsilon > 0, \\ \rho_{0,\varepsilon} &\rightarrow \rho_0, \quad \text{a.e. and in } L^p((0, \infty) \times \mathbb{R}^2), \quad 1 \leq p < \infty, \text{ as } \varepsilon \rightarrow 0, \\ u_{0,\varepsilon} &\rightarrow u_0, \quad \text{a.e. and in } L^p(\mathbb{R}^2), \quad 1 \leq p < \infty, \text{ as } \varepsilon \rightarrow 0, \\ \rho_{0,\varepsilon} &\geq 0, \quad \|\rho_{0,\varepsilon}\|_{L^1((0,\infty) \times \mathbb{R}^2)} \leq C, & \varepsilon \geq 0, \\ \sup_{x \in \mathbb{R}^2} \|\rho_{0,\varepsilon}(\cdot, x)\|_{L^1(\mathbb{R})}, \sup_{a \geq 0} \|\rho_{0,\varepsilon}(a, \cdot)\|_{L^1(\mathbb{R}^2)}, \|\partial_a \rho_{0,\varepsilon}\|_{L^1((0,\infty) \times \mathbb{R}^2)} &\leq C, & \varepsilon \geq 0, \\ \|u_{0,\varepsilon}\|_{L^1(\mathbb{R}^2)}, \|\nabla_x u_{0,\varepsilon}\|_{L^1(\mathbb{R}^2)}, \varepsilon \|\Delta_x u_{0,\varepsilon}\|_{L^1(\mathbb{R}^2)} &\leq C, \quad u_{0,\varepsilon} \geq 0, & \varepsilon > 0. \end{aligned}$$

2.1. A priori estimates. In this section we establish the a priori estimates on $(u_\varepsilon, \rho_\varepsilon)$ which are necessary to pass to the limit as $\varepsilon \rightarrow 0$ in (2.1).

From now on we use the notation C for all the positive constants independent on ε appearing in the text or in the statements of our results, while in proofs we write c to indicate any positive constant non depending on ε , and c_T for quantities of the form ce^{ct} , $t \in (0, T)$.

Lemma 2.1 (Nonnegativity of $\rho_\varepsilon, \phi_\varepsilon, u_\varepsilon$). *We have that*

$$(2.3) \quad \rho_\varepsilon \geq 0, \quad \phi_\varepsilon \geq 0, \quad u_\varepsilon \geq 0.$$

Proof. Consider the function

$$x \mapsto \eta(x) = -\rho \mathbb{1}_{(-\infty, 0)}(x).$$

and observe that

$$\eta'(x) = -\mathbb{1}_{(-\infty, 0)}(x), \quad \eta(x) = x\eta'(x).$$

From (2.1) we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \int_{\mathbb{R}^2} \eta(\rho_\varepsilon) dx da &= \int_0^\infty \int_{\mathbb{R}^2} \eta'(\rho_\varepsilon) \partial_t \rho_\varepsilon dx da \\ &= - \int_0^\infty \int_{\mathbb{R}^2} \eta'(\rho_\varepsilon) \partial_a \rho_\varepsilon dx da - \int_0^\infty \int_{\mathbb{R}^2} \operatorname{div}_x (\rho_\varepsilon \chi_1 \mathbf{v} Y_\theta) \eta'(\rho_\varepsilon) dx da \\ &\quad + \mu \int_0^\infty \int_{\mathbb{R}^2} \eta'(\rho_\varepsilon) \Delta_x \rho_\varepsilon dx da - \int_0^\infty \int_{\mathbb{R}^2} \eta'(\rho_\varepsilon) (\mathfrak{d} + \mathfrak{p}) \rho_\varepsilon dx da + \varepsilon \int_0^\infty \int_{\mathbb{R}^2} \eta'(\rho_\varepsilon) \partial_{aa}^2 \rho_\varepsilon dx da \\ &= \int_{\mathbb{R}^2} \underbrace{\eta(\rho_\varepsilon(t, 0, x))}_{=0} dx + \underbrace{\int_0^\infty \int_{\mathbb{R}^2} \rho_\varepsilon \chi_1 (\mathbf{v} \cdot \nabla_x \rho_\varepsilon) Y_\theta \eta''(\rho_\varepsilon) dx da}_{=0} \end{aligned}$$

$$\begin{aligned}
& \underbrace{-\mu \int_0^\infty \int_{\mathbb{R}^2} \eta''(\rho_\varepsilon) (\nabla_x \rho_\varepsilon)^2 dx da}_{\leq 0} - \underbrace{\int_0^\infty \int_{\mathbb{R}^2} (\mathfrak{d} + \mathfrak{p}) \eta(\rho_\varepsilon) dx da}_{\leq 0} \\
& + \varepsilon \int_0^\infty \int_{\mathbb{R}^2} \partial_a (\eta'(\rho_\varepsilon) \partial_a \rho_\varepsilon) dx da - \underbrace{\varepsilon \int_0^\infty \int_{\mathbb{R}^2} \eta''(\rho_\varepsilon) (\partial_a \rho_\varepsilon)^2 dx da}_{\leq 0} \\
& \leq -\varepsilon \int_{\mathbb{R}^2} \underbrace{\eta'(\rho_\varepsilon(t, 0, x))}_{=0} \partial_a \rho_\varepsilon(t, 0, x) dx = 0.
\end{aligned}$$

Thus, integrating on $(0, t)$ we obtain $\eta(\rho_\varepsilon(t, a, x)) = 0$ which implies that $\rho_\varepsilon \geq 0$, and then $\phi_\varepsilon \geq 0$.

From (2.1) we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^2} \eta(u_\varepsilon) dx &= \int_{\mathbb{R}^2} \eta'(u_\varepsilon) \partial_t u_\varepsilon dx \\
&= - \underbrace{\int_{\mathbb{R}^2} \eta'(u_\varepsilon) \operatorname{div}_x (u_\varepsilon \nu(\phi_\varepsilon)) dx}_{=0} + \int_{\mathbb{R}^2} \eta'(u_\varepsilon) \underbrace{(\mathfrak{b}(\phi) - \beta)}_{\leq c} u_\varepsilon dx + \varepsilon \underbrace{\int_{\mathbb{R}^2} \eta'(u_\varepsilon) \Delta_x u_\varepsilon dx}_{\leq 0} \\
&\leq c \int_{\mathbb{R}^2} \eta(u_\varepsilon) dx.
\end{aligned}$$

Integrating on $(0, t)$ and applying Gronwall Lemma we obtain $\eta(u_\varepsilon(t, x)) = 0$, so that $u_\varepsilon \geq 0$. \square

Remark 2.1. *Without loss of information, we can remove the absolute value in the boundary condition for ρ_ε in (2.1).*

We consider the class of functions

$$\psi_\varepsilon(t, x) = \int_0^\infty \rho_\varepsilon(t, a, x) \xi(a) da,$$

for $\xi \in C^\infty([0, \infty])$ such that

$$(2.4) \quad \operatorname{supp}(\xi) \subset (0, T).$$

In particular any of the functions χ_i , $i = 1, 2, 3$, can play the role of ξ , so that the estimates obtained for ψ_ε apply to ϕ_ε .

Lemma 2.2 (L^1 estimate on ψ_ε). *For all $t \geq 0$, we have that*

$$(2.5) \quad \|\psi_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R}^2)} \leq e^{Ct} C.$$

Proof. We multiply by $\xi(a)$ the equation for ρ_ε in system (2.1) then, integrating with respect to a , we get

$$(2.6) \quad \partial_t \psi_\varepsilon - \mu \Delta_x \psi_\varepsilon + \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) = \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi - \mathfrak{p}(a, u_\varepsilon) \xi) da.$$

Due to the nonnegativity of ρ_ε , ψ_ε and the boundedness of ξ'/ξ , ξ''/ξ , \mathfrak{d} , \mathfrak{p} we have

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^2} |\psi_\varepsilon| dx &= \frac{d}{dt} \int_{\mathbb{R}^2} \psi_\varepsilon dx = \mu \underbrace{\int_{\mathbb{R}^2} \Delta_x \psi_\varepsilon dx}_{=0} - \underbrace{\int_{\mathbb{R}^2} \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) dx}_{=0} \\
&+ \int_{\mathbb{R}^2} \int_0^\infty \rho_\varepsilon \underbrace{(\varepsilon \xi'' + \xi' - \mathfrak{d} \xi - \mathfrak{p} \xi)}_{\leq c \xi} da dx \leq c \int_{\mathbb{R}^2} \psi_\varepsilon dx.
\end{aligned}$$

The result comes from the Gronwall Lemma and the assumptions in (2.2). \square

Lemma 2.3 (L^2 estimate on ψ_ε). *For any $t \geq 0$, we have*

$$(2.7) \quad \|\psi_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)}, \|\nabla_x \psi_\varepsilon\|_{L^2((0,t) \times \mathbb{R}^2)} \leq e^{Ct} C.$$

Proof. We multiply (2.6) by ψ_ε and have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} \frac{\psi_\varepsilon^2}{2} dx &= \int_{\mathbb{R}^2} \psi_\varepsilon \partial_t \psi_\varepsilon dx \\ &= \mu \int_{\mathbb{R}^2} \psi_\varepsilon \Delta_x \psi_\varepsilon dx - \int_{\mathbb{R}^2} \psi_\varepsilon \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) dx \\ &\quad + \int_{\mathbb{R}^2} \int_0^\infty \rho_\varepsilon \underbrace{(\varepsilon \xi'' + \xi' - \mathfrak{d} \xi - \mathfrak{p} \xi)}_{\leq c \xi} \psi_\varepsilon da dx \\ &\leq -\mu \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 dx + \int_{\mathbb{R}^2} \nabla_x \psi_\varepsilon \cdot \mathbf{v} \left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) Y_\theta dx + c \int_{\mathbb{R}^2} \psi_\varepsilon^2 dx \\ &\leq -\frac{\mu}{2} \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 dx + \frac{1}{2\mu} \int_{\mathbb{R}^2} |\mathbf{v}|^2 \underbrace{\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right)^2}_{\leq c \psi_\varepsilon^2} Y_\theta dx + c \int_{\mathbb{R}^2} \psi_\varepsilon^2 dx \\ &\leq -\frac{\mu}{2} \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \psi_\varepsilon^2 dx. \end{aligned}$$

Integrating over $(0, t)$ and using the Gronwall Lemma we get (2.7). \square

Lemma 2.4 (L^∞ estimate on u_ε). *For every $t \geq 0$, we have*

$$(2.8) \quad \|u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq C e^{Ct}.$$

Proof. Let \mathcal{C} be a positive constant that will be fixed later. We define

$$\bar{u}_\varepsilon(t, x) = e^{-\mathcal{C}t} u_\varepsilon(t, x),$$

and we consider the associated problem

$$\begin{cases} \partial_t \bar{u}_\varepsilon + \operatorname{div}_x (\bar{u}_\varepsilon \nu(\phi_\varepsilon)) = (\mathfrak{b}(\phi_\varepsilon) - \beta - \mathcal{C}) \bar{u}_\varepsilon + \varepsilon \Delta_x \bar{u}_\varepsilon, \\ \bar{u}_\varepsilon(0, x) = u_{0,\varepsilon}(x). \end{cases}$$

We claim that for any given $T > 0$ there exist a sufficiently large constant $k > 0$ and a suitable \mathcal{C} such that $u_\varepsilon(t, x) \leq k$ for any $t \leq T$ and $x \in \mathbb{R}^2$, provided $u_{0,\varepsilon}(x) \leq k$ for all $x \in \mathbb{R}^2$.

Consider the function

$$x \mapsto \eta(x) = (x - k) \mathbb{1}_{(k, +\infty)}(x),$$

and observe that

$$\eta'(x) = \mathbb{1}_{(k, +\infty)}(x), \quad x \eta'(x) = \eta(x) + k \eta'(x).$$

We have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} \eta(\bar{u}_\varepsilon) dx &= \int_{\mathbb{R}^2} \eta'(\bar{u}_\varepsilon) \partial_t \bar{u}_\varepsilon dx \\ &= - \int_{\mathbb{R}^2} \eta'(\bar{u}_\varepsilon) \operatorname{div}_x (\bar{u}_\varepsilon \nu(\phi_\varepsilon)) dx + \int_{\mathbb{R}^2} \eta'(\bar{u}_\varepsilon) (\mathfrak{b}(\phi_\varepsilon) - \beta - \mathcal{C}) \bar{u}_\varepsilon dx + \varepsilon \underbrace{\int_{\mathbb{R}^2} \eta'(\bar{u}_\varepsilon) \Delta_x \bar{u}_\varepsilon dx}_{\leq 0} \\ &\leq - \underbrace{\int_{\mathbb{R}^2} \eta'(\bar{u}_\varepsilon) \operatorname{div}_x ((\bar{u}_\varepsilon - k) \nu(\phi_\varepsilon)) dx}_{=0} - k \int_{\mathbb{R}^2} \eta'(\bar{u}_\varepsilon) \operatorname{div}_x (\nu(\phi_\varepsilon)) dx \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^2} (\eta(\bar{u}_\varepsilon) + k\eta'(\bar{u}_\varepsilon))(\mathbf{b}(\phi_\varepsilon) - \beta - \mathcal{C}) \, dx \\
& \leq \int_{\mathbb{R}^2} \eta(\bar{u}_\varepsilon)(\mathbf{b}(\phi_\varepsilon) - \beta - \mathcal{C}) \, dx - k \int_{\mathbb{R}^2} \eta'(\bar{u}_\varepsilon)(\mathcal{C} + \beta + \operatorname{div}_x(\nu(\phi_\varepsilon)) - \mathbf{b}(\phi_\varepsilon)) \, dx.
\end{aligned}$$

From the inequality $\|\operatorname{div}_x(\nu(\phi_\varepsilon))\|_{L^\infty} \leq 2\|\nabla\nu(\phi_\varepsilon)\|_{L^\infty}$ and the estimates in (1.15) and (2.5), it follows that for \mathcal{C} large enough

$$\mathcal{C} + \beta - \mathbf{b}(\phi_\varepsilon) \geq 0, \quad \mathcal{C} + \beta + \operatorname{div}_x(\nu(\phi_\varepsilon)) - \mathbf{b}(\phi_\varepsilon) \geq 0,$$

thus,

$$\frac{d}{dt} \int_{\mathbb{R}^2} \eta(\bar{u}_\varepsilon) \, dx \leq 0.$$

Integrating over $(0, t)$ we obtain

$$0 \leq \int_{\mathbb{R}^2} \eta(\bar{u}_\varepsilon(t, x)) \, dx \leq \int_{\mathbb{R}^2} \eta(u_{0,\varepsilon}(x)) \, dx = 0,$$

which means $\bar{u}_\varepsilon \leq k$. The inequality (2.8) follows. \square

Lemma 2.5. *For all $t \geq 0$, the following estimates on u_ε hold*

$$(2.9) \quad \|u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R}^2)} \leq e^{Ct} C,$$

$$(2.10) \quad \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)}, \|\nabla_x u_\varepsilon\|_{L^2((0,t) \times \mathbb{R}^2)} \leq e^{Ct} C,$$

$$(2.11) \quad \|\nabla_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)}, \|\Delta_x u_\varepsilon\|_{L^2((0,t) \times \mathbb{R}^2)} \leq e^{Ct} C.$$

Proof. We recall the equation of u_ε

$$(2.12) \quad \partial_t u_\varepsilon + \operatorname{div}_x(u_\varepsilon \nu(\phi_\varepsilon)) = (\mathbf{b}(\phi_\varepsilon) - \beta)u_\varepsilon + \varepsilon \Delta_x u_\varepsilon.$$

(2.9). Using the nonnegativity of u_ε we have

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^2} |u_\varepsilon| \, dx & = \frac{d}{dt} \int_{\mathbb{R}^2} u_\varepsilon \, dx = \varepsilon \underbrace{\int_{\mathbb{R}^2} \Delta_x u_\varepsilon \, dx - \int_{\mathbb{R}^2} \operatorname{div}_x(u_\varepsilon \nu(\phi_\varepsilon)) \, dx}_{=0} \\
& \quad + \int_{\mathbb{R}^2} \underbrace{(\mathbf{b}(\phi_\varepsilon) - \beta)}_{\leq c} u_\varepsilon \, dx \leq c \int_{\mathbb{R}^2} u_\varepsilon \, dx.
\end{aligned}$$

Then, applying the Gronwall Lemma we gain

$$\|u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R}^2)} \leq \|u_{0,\varepsilon}(\cdot)\|_{L^1(\mathbb{R}^2)} e^{ct}.$$

(2.10). We multiply (2.12) by u_ε to obtain

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^2} \frac{u_\varepsilon^2}{2} \, dx & = \int_{\mathbb{R}^2} u_\varepsilon \partial_t u_\varepsilon \, dx \\
& = - \int_{\mathbb{R}^2} \operatorname{div}_x(u_\varepsilon \nu(\phi_\varepsilon)) u_\varepsilon \, dx + \int_{\mathbb{R}^2} (\mathbf{b}(\phi_\varepsilon) - \beta) u_\varepsilon^2 \, dx + \varepsilon \int_{\mathbb{R}^2} \Delta_x u_\varepsilon u_\varepsilon \, dx \\
& \leq \int_{\mathbb{R}^2} u_\varepsilon \nu(\phi_\varepsilon) \cdot \nabla_x u_\varepsilon \, dx + c \int_{\mathbb{R}^2} u_\varepsilon^2 \, dx - \varepsilon \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon|^2 \, dx \\
& \leq \int_{\mathbb{R}^2} \nabla_x \left(\frac{u_\varepsilon^2}{2} \right) \cdot \nu(\phi_\varepsilon) \, dx + c \int_{\mathbb{R}^2} u_\varepsilon^2 \, dx - \varepsilon \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon|^2 \, dx \\
& \leq \int_{\mathbb{R}^2} \frac{u_\varepsilon^2}{2} \operatorname{div}_x(\nu(\phi_\varepsilon)) \, dx + c \int_{\mathbb{R}^2} u_\varepsilon^2 \, dx - \varepsilon \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon|^2 \, dx \\
& \leq c_T \int_{\mathbb{R}^2} |u_\varepsilon| \, dx + c \int_{\mathbb{R}^2} u_\varepsilon^2 \, dx - \varepsilon \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon|^2 \, dx.
\end{aligned}$$

Integrating over $(0, t)$ and then using the Gronwall Lemma we get

$$\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + 2\varepsilon e^{ct} \int_0^t e^{-cs} \|\nabla_x u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R}^2)}^2 ds \leq e^{ct} \|u_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + ce^{ct}.$$

(2.11). We multiply (2.12) by $-\Delta_x u_\varepsilon$ to obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} \frac{|\nabla_x u_\varepsilon|^2}{2} dx = \int_{\mathbb{R}^2} \nabla_x u_\varepsilon \cdot \partial_t \nabla_x u_\varepsilon dx = - \int_{\mathbb{R}^2} \Delta_x u_\varepsilon \partial_t u_\varepsilon dx \\ & = \int_{\mathbb{R}^2} \Delta_x u_\varepsilon \operatorname{div}_x (u_\varepsilon \nu(\phi_\varepsilon)) dx - \int_{\mathbb{R}^2} \Delta_x u_\varepsilon (\mathbf{b}(\phi_\varepsilon) - \beta) u_\varepsilon dx - \varepsilon \int_{\mathbb{R}^2} |\Delta_x u_\varepsilon|^2 dx \\ & = - \int_{\mathbb{R}^2} \nabla_x u_\varepsilon \nabla_x (\nabla_x u_\varepsilon \cdot \nu(\phi) + u_\varepsilon \operatorname{div}_x (\nu(\phi))) dx + \int_{\mathbb{R}^2} (\mathbf{b}(\phi) - \beta) |\nabla_x u_\varepsilon|^2 dx \\ & \quad + \int_{\mathbb{R}^2} \mathbf{b}'(\phi) \nabla_x \phi \cdot \nabla_x u_\varepsilon u_\varepsilon dx - \varepsilon \int_{\mathbb{R}^2} |\Delta_x u_\varepsilon|^2 dx \\ & \leq -\varepsilon \int_{\mathbb{R}^2} |\Delta_x u_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 dx \\ & \quad + \int_{\mathbb{R}^2} \nabla_x \left(\frac{|\nabla_x u_\varepsilon|^2}{2} \right) \cdot \nu(\phi) dx + \int_{\mathbb{R}^2} |\nabla_x u|^2 \underbrace{|\nabla_x \nu(\phi)|}_{\leq c_T} dx + \int_{\mathbb{R}^2} |\nabla_x u|^2 \underbrace{|\operatorname{div}_x (\nu(\phi))|}_{\leq c_T} dx \\ & \quad + \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon| u_\varepsilon |\nabla_x \operatorname{div}_x (\nu(\phi))| dx \\ & \leq -\varepsilon \int_{\mathbb{R}^2} |\Delta_x u_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x \operatorname{div}_x (\nu(\phi))|^2 dx. \end{aligned}$$

Integrating over $(0, t)$ and using the estimates in (1.17) and (2.7) we gain

$$\|\nabla_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)} + \varepsilon \|\Delta_x u_\varepsilon(\cdot, \cdot)\|_{L^2((0,t) \times \mathbb{R}^2)} \leq ce^{ct}.$$

□

Lemma 2.6. *For every $t \geq 0$, the following estimates on ψ_ε hold*

$$(2.13) \quad \|\nabla_x \psi_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)}, \|D_x^2 \psi_\varepsilon\|_{L^2((0,t) \times \mathbb{R}^2)} \leq e^{Ct} C,$$

$$(2.14) \quad \|D_x^2 \psi_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)}, \|D_x^3 \psi_\varepsilon\|_{L^2((0,t) \times \mathbb{R}^2)} \leq e^{Ct} C,$$

$$(2.15) \quad \|\psi_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq e^{Ct} C,$$

$$(2.16) \quad \|D_x^3 \psi_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)}, \|D_x^4 \psi_\varepsilon\|_{L^2((0,t) \times \mathbb{R}^2)} \leq e^{Ct} C,$$

$$(2.17) \quad \|\partial_t \psi_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R}^2)}, \|\partial_t \psi_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)}, \|\nabla_x \psi_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq e^{Ct} C.$$

Proof. (2.13). The proof is identical to the proof of [2, Lemma 2.2, Eq. (2.8)], due to the boundedness of the mortality rate related to predation $\mathbf{p}(\cdot, u)$.

(2.14). At first, we consider the following estimate

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} \frac{(\psi_\varepsilon u_\varepsilon)^2}{2} dx = \int_{\mathbb{R}^2} (u_\varepsilon^2 \psi_\varepsilon \partial_t \psi_\varepsilon + \psi_\varepsilon^2 u_\varepsilon \partial_t u_\varepsilon) dx \\ & = \mu \int_{\mathbb{R}^2} u_\varepsilon^2 \psi_\varepsilon \Delta_x \psi_\varepsilon dx - \int_{\mathbb{R}^2} u_\varepsilon^2 \psi_\varepsilon \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) dx \\ & \quad + \int_{\mathbb{R}^2} u_\varepsilon^2 \int_0^\infty \rho_\varepsilon \underbrace{(\varepsilon \xi'' + \xi' - \mathfrak{d} \xi - \mathbf{p} \xi)}_{\leq c, \text{ see (2.4)}} \psi_\varepsilon da dx - \int_{\mathbb{R}^2} u_\varepsilon \psi_\varepsilon^2 \operatorname{div}_x (u_\varepsilon \nu(\phi_\varepsilon)) dx \\ & \quad + \int_{\mathbb{R}^2} \psi_\varepsilon^2 (\mathbf{b}(\phi_\varepsilon) - \beta) u_\varepsilon^2 dx + \varepsilon \int_{\mathbb{R}^2} \psi_\varepsilon^2 \Delta_x u_\varepsilon u_\varepsilon dx \end{aligned}$$

$$\begin{aligned}
&\leq -\mu \int_{\mathbb{R}^2} \nabla_x(u_\varepsilon^2 \psi_\varepsilon) \nabla_x \psi_\varepsilon dx - \varepsilon \int_{\mathbb{R}^2} \nabla_x(\psi_\varepsilon^2 u_\varepsilon) \nabla_x u_\varepsilon dx + c \int_{\mathbb{R}^2} (\psi_\varepsilon u_\varepsilon)^2 dx \\
&\quad + c_T \int_{\mathbb{R}^2} \psi_\varepsilon^2 dx + c_T \int_{\mathbb{R}^2} \left(\operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) \right)^2 dx \\
&\quad + \int_{\mathbb{R}^2} \nabla_x(\psi_\varepsilon^2 u_\varepsilon) \cdot \nu(\phi_\varepsilon) u_\varepsilon dx \\
&\leq -\mu \int_{\mathbb{R}^2} u_\varepsilon^2 |\nabla_x \psi_\varepsilon|^2 dx - \varepsilon \int_{\mathbb{R}^2} \psi_\varepsilon^2 |\nabla_x u_\varepsilon|^2 dx - 2(\mu + \varepsilon) \int_{\mathbb{R}^2} \nabla_x \left(\frac{u_\varepsilon^2}{2} \right) \nabla_x \left(\frac{\psi_\varepsilon^2}{2} \right) dx \\
&\quad + c \int_{\mathbb{R}^2} (\psi_\varepsilon u_\varepsilon)^2 dx + c_T \int_{\mathbb{R}^2} \psi_\varepsilon^2 dx + c_T \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|\mathbf{v}|^2 Y_\theta^2}_{\leq c} dx \\
&\quad + c_T \int_{\mathbb{R}^2} \left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right)^2 \underbrace{|\operatorname{div}_x(\mathbf{v})|^2 Y_\theta^2}_{\leq c} dx + c_T \int_{\mathbb{R}^2} \underbrace{\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right)^2 (Y_\theta')^2 |\mathbf{v}|^2}_{\leq c, \text{ see (2.4)}} |\nabla_x \phi_\varepsilon|^2 dx \\
&\quad + 2 \int_{\mathbb{R}^2} \nabla_x \psi_\varepsilon \psi_\varepsilon \underbrace{u_\varepsilon^2 \nu(\phi_\varepsilon)}_{\leq c_T} dx + \int_{\mathbb{R}^2} \psi_\varepsilon^2 \nabla_x \left(\frac{u_\varepsilon^2}{2} \right) \cdot \nu(\phi_\varepsilon) dx \\
&\leq -\mu \int_{\mathbb{R}^2} u_\varepsilon^2 |\nabla_x \psi_\varepsilon|^2 dx - \varepsilon \int_{\mathbb{R}^2} \psi_\varepsilon^2 |\nabla_x u_\varepsilon|^2 dx + (\mu + \varepsilon) \int_{\mathbb{R}^2} \underbrace{u_\varepsilon^2}_{\leq c_T} (|\nabla_x \psi_\varepsilon|^2 + \psi_\varepsilon \Delta_x \psi_\varepsilon) dx \\
&\quad + c \int_{\mathbb{R}^2} (\psi_\varepsilon u_\varepsilon)^2 dx + c_T \int_{\mathbb{R}^2} \psi_\varepsilon^2 dx + c_T \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx \\
&\quad + c_T \int_{\mathbb{R}^2} \left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right)^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x \phi_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} \nabla_x \psi_\varepsilon \psi_\varepsilon dx + \int_{\mathbb{R}^2} \underbrace{\frac{u_\varepsilon^2}{2} \operatorname{div}_x(\nu(\phi_\varepsilon)) \psi_\varepsilon^2}_{\leq c_T} dx \\
&\leq -\mu \int_{\mathbb{R}^2} u_\varepsilon^2 |\nabla_x \psi_\varepsilon|^2 dx - \varepsilon \int_{\mathbb{R}^2} \psi_\varepsilon^2 |\nabla_x u_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} |\Delta_x \psi_\varepsilon|^2 dx \\
&\quad + c \int_{\mathbb{R}^2} (\psi_\varepsilon u_\varepsilon)^2 dx + c_T \int_{\mathbb{R}^2} \psi_\varepsilon^2 dx + c_T \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx \\
&\quad + c_T \int_{\mathbb{R}^2} \left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right)^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x \phi_\varepsilon|^2 dx.
\end{aligned}$$

Using the Gronwall Lemma and the estimates in (2.7), (2.13) we gain

$$\begin{aligned}
(2.18) \quad &\|\psi_\varepsilon(t, \cdot) u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + 2\mu e^{ct} \int_0^t e^{-cs} \int_{\mathbb{R}^2} u_\varepsilon^2(s, x) |\nabla_x \psi_\varepsilon(s, x)|^2 dx ds \\
&\quad + 2\varepsilon e^{ct} \int_0^t e^{-cs} \int_{\mathbb{R}^2} \psi_\varepsilon^2(s, x) |\nabla_x u_\varepsilon(s, x)|^2 dx ds \leq c e^{ct}.
\end{aligned}$$

Then, we multiply (2.6) by $D_x^4 \psi_\varepsilon$ to have

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^2} \frac{|\Delta_x \psi_\varepsilon|^2}{2} dx = \int_{\mathbb{R}^2} \Delta_x \psi_\varepsilon \cdot \partial_t \Delta_x \psi_\varepsilon dx = \int_{\mathbb{R}^2} D_x^4 \psi_\varepsilon \partial_t \psi_\varepsilon dx \\
&= \mu \int_{\mathbb{R}^2} \Delta_x \psi_\varepsilon D_x^4 \psi_\varepsilon dx - \int_{\mathbb{R}^2} D_x^4 \psi_\varepsilon \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) dx \\
&\quad + \int_{\mathbb{R}^2} \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi - \mathfrak{p}(a, u_\varepsilon) \xi) D_x^4 \psi_\varepsilon da dx \\
&= -\mu \int_{\mathbb{R}^2} |D_x^3 \psi_\varepsilon|^2 dx + \int_{\mathbb{R}^2} D_x^3 \psi_\varepsilon \cdot \nabla_x \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) dx \\
&\quad - \int_{\mathbb{R}^2} \nabla_x \left(\int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi - \mathfrak{p}(a, u_\varepsilon) \xi) da \right) \cdot D_x^3 \psi_\varepsilon dx
\end{aligned}$$

$$\begin{aligned}
 &\leq -\frac{\mu}{2} \int_{\mathbb{R}^2} |D_x^3 \psi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \left(\nabla_x \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) \right)^2 dx \\
 &\quad + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi - \mathbf{p}(a, u_\varepsilon) \xi) da \right|^2 dx \\
 &\leq -\frac{\mu}{2} \int_{\mathbb{R}^2} |D_x^3 \psi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|\mathbf{v}|^2 Y_\theta^2}_{\leq c} dx + c \int_{\mathbb{R}^2} \left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|D_x^2 \mathbf{v}|^2 Y_\theta^2}_{\leq c} dx \\
 &\quad + c \int_{\mathbb{R}^2} \underbrace{\left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2}_{\leq c, \text{ see (2.4)}} |\mathbf{v}|^2 (Y_\theta'')^2 |\nabla_x \phi_\varepsilon|^4 dx + c \int_{\mathbb{R}^2} \underbrace{\left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2}_{\leq c, \text{ see (2.4)}} |\mathbf{v}|^2 (Y_\theta')^2 |D_x^2 \phi_\varepsilon|^2 dx \\
 &\quad + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|\nabla_x \mathbf{v}|^2 Y_\theta^2}_{\leq c} dx + c \int_{\mathbb{R}^2} \underbrace{\left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2}_{\leq c, \text{ see (2.4)}} |\nabla_x \mathbf{v}|^2 (Y_\theta')^2 |\nabla_x \phi_\varepsilon|^2 dx \\
 &\quad + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|\mathbf{v}|^2 (Y_\theta')^2}_{\leq c} |\nabla_x \phi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi) da \right|^2 dx \\
 &\quad + c \int_{\mathbb{R}^2} \left| \int_0^\infty \nabla_x \rho_\varepsilon \mathbf{p}(a, u_\varepsilon) \xi da \right|^2 dx + c \int_{\mathbb{R}^2} \left| \int_0^\infty \rho_\varepsilon \underbrace{\partial_u \mathbf{p}(a, u_\varepsilon)}_{\leq c} \nabla_x u_\varepsilon \xi da \right|^2 dx \\
 &\leq -\frac{\mu}{2} \int_{\mathbb{R}^2} |D_x^3 \psi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx + c \int_{\mathbb{R}^2} \left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx \\
 &\quad + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx + c \int_{\mathbb{R}^2} |\nabla_x \phi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} |\nabla_x \phi_\varepsilon|^4 dx \\
 &\quad + c \int_{\mathbb{R}^2} |D_x^2 \phi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^4 dx \\
 &\quad + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi)_+ da \right|^2 dx + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi)_- da \right|^2 dx \\
 &\quad + c \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \psi_\varepsilon^2 |\nabla_x u_\varepsilon|^2 dx,
 \end{aligned}$$

where

$$\begin{aligned}
 (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi)_+ &= \max\{(\varepsilon \xi'' + \xi' - \mathfrak{d} \xi), 0\}, \\
 (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi)_- &= \max\{-(\varepsilon \xi'' + \xi' - \mathfrak{d} \xi), 0\}.
 \end{aligned}$$

Integrating over $(0, t)$ we get

$$\begin{aligned}
 &\|D_x^2 \psi_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + \mu \|D_x^3 \psi_\varepsilon\|_{L^2((0, t) \times \mathbb{R}^2)}^2 \\
 &\leq \|D_x^2 \psi_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \left\| \left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) (s) \right\|_{H^2(\mathbb{R}^2)}^2 ds + \int_0^t \|\phi_\varepsilon(s, \cdot)\|_{H^2(\mathbb{R}^2)}^2 ds \\
 &\quad + \int_0^t \left\| \left(\nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) (s) \right\|_{L^4(\mathbb{R}^2)}^4 ds + \int_0^t \|\nabla_x \phi_\varepsilon(s, \cdot)\|_{L^4(\mathbb{R}^2)}^4 ds \\
 &\quad + \int_0^t \left\| \left(\nabla_x \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi)_+ da \right) (s) \right\|_{L^2(\mathbb{R}^2)}^2 ds \\
 &\quad + \int_0^t \left\| \left(\nabla_x \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi)_- da \right) (s) \right\|_{L^2(\mathbb{R}^2)}^2 ds
 \end{aligned}$$

$$+ \int_0^t \|\nabla_x \psi_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R}^2)}^2 ds + \int_0^t \int_{\mathbb{R}^2} \psi_\varepsilon^2(s, x) |\nabla_x u_\varepsilon(s, x)|^2 dx ds.$$

We remark that a classical regularization argument applied to the continuous cut-off functions in

$$I_\pm = \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d}\xi)_\pm da,$$

shows that estimates (2.7), (2.13), apply to I_\pm .

Then (2.7), (2.13), (2.18) and the embedding $H^1(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$ give us (2.14).

(2.15). Directly follows from (2.7), (2.13), (2.14) and the embedding $H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$.

(2.16). We prove first the following estimate using the L^∞ bounds on u_ε and ψ_ε

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} \frac{|\nabla_x \psi_\varepsilon|^2 u_\varepsilon^2}{2} dx &= \int_{\mathbb{R}^2} u_\varepsilon^2 \nabla_x \psi_\varepsilon \partial_t \nabla_x \psi_\varepsilon dx + \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 u_\varepsilon \partial_t u_\varepsilon dx \\ &= - \int_{\mathbb{R}^2} u_\varepsilon^2 \Delta_x \psi \partial_t \psi_\varepsilon dx - \int_{\mathbb{R}^2} \nabla_x (u_\varepsilon^2) \cdot \nabla_x \psi_\varepsilon \partial_t \psi_\varepsilon dx + \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 u_\varepsilon \partial_t u_\varepsilon dx \\ &= -\mu \int_{\mathbb{R}^2} u_\varepsilon^2 |D_x^2 \psi_\varepsilon|^2 dx + \int_{\mathbb{R}^2} u_\varepsilon^2 \Delta_x \psi_\varepsilon \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) dx \\ &\quad - \int_{\mathbb{R}^2} u_\varepsilon^2 \int_0^\infty \underbrace{\rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d}\xi - \mathfrak{p}\xi)}_{\leq c\xi} \Delta_x \psi_\varepsilon da dx \\ &\quad - \mu \int_{\mathbb{R}^2} \nabla_x (u_\varepsilon^2) \cdot \nabla_x \psi_\varepsilon \Delta_x \psi_\varepsilon dx + \int_{\mathbb{R}^2} \nabla_x (u_\varepsilon^2) \cdot \nabla_x \psi_\varepsilon \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) dx \\ &\quad - \int_{\mathbb{R}^2} \nabla_x (u_\varepsilon^2) \cdot \nabla_x \psi_\varepsilon \int_0^\infty \underbrace{\rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d}\xi - \mathfrak{p}\xi)}_{\leq c\xi} \Delta_x \psi_\varepsilon da dx \\ &\quad - \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 u_\varepsilon \operatorname{div}_x (u_\varepsilon \nu(\phi_\varepsilon)) dx + \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 (\mathbf{b}(\phi_\varepsilon) - \beta) u_\varepsilon^2 dx + \varepsilon \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 \Delta_x u_\varepsilon u_\varepsilon dx \\ &\leq -\frac{\mu}{2} \int_{\mathbb{R}^2} u_\varepsilon^2 |D_x^2 \psi_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} \left(\operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) \right)^2 dx + c_T \int_{\mathbb{R}^2} \psi_\varepsilon^2 u_\varepsilon^2 dx \\ &\quad + c_T \int_{\mathbb{R}^2} |D_x^2 \psi_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} |D_x^3 \psi_\varepsilon|^2 dx \\ &\quad + c_T \int_{\mathbb{R}^2} \left(\nabla_x \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) \right)^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^4 dx \\ &\quad + 3 \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon| |D_x^2 \psi_\varepsilon| \underbrace{u_\varepsilon^2 \nu(\phi_\varepsilon)}_{\leq c_T} dx + \int_{\mathbb{R}^2} \underbrace{\frac{u_\varepsilon^2}{2} |\operatorname{div}_x (\nu(\phi_\varepsilon))|}_{\leq c_T} |\nabla_x \psi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 u_\varepsilon^2 dx \\ &\quad - 2\varepsilon \int_{\mathbb{R}^2} \nabla_x \psi_\varepsilon D_x^2 \psi_\varepsilon \nabla_x \left(\frac{u_\varepsilon^2}{2} \right) dx - \varepsilon \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 |\nabla_x u_\varepsilon|^2 dx \\ &\leq -\frac{\mu}{2} \int_{\mathbb{R}^2} u_\varepsilon^2 |D_x^2 \psi_\varepsilon|^2 dx - \varepsilon \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 |\nabla_x u_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx \\ &\quad + c_T \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx + c_T \int_{\mathbb{R}^2} \left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x \phi_\varepsilon|^2 dx \\ &\quad + c_T \int_{\mathbb{R}^2} |\nabla_x \phi_\varepsilon|^4 dx + c \int_{\mathbb{R}^2} |D_x^2 \phi_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} \psi_\varepsilon^2 u_\varepsilon^2 dx + c_T \int_{\mathbb{R}^2} |D_x^2 \psi_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 dx \\ &\quad + c_T \int_{\mathbb{R}^2} |D_x^3 \psi_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^4 dx + c \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 u_\varepsilon^2 dx. \end{aligned}$$

Using the Gronwall Lemma and the estimates in (2.7), (2.13), (2.14), (2.18) we gain

$$(2.19) \quad \begin{aligned} & \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon(t, x)|^2 u_\varepsilon(t, x)^2 dx + \mu e^{ct} \int_0^t e^{-cs} \int_{\mathbb{R}^2} u_\varepsilon(s, x)^2 |D_x^2 \psi_\varepsilon(s, x)|^2 dx ds \\ & + 2\varepsilon e^{ct} \int_0^t e^{-cs} \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon(s, x)|^2 |\nabla_x u_\varepsilon(s, x)|^2 dx ds \leq c e^{ct}. \end{aligned}$$

We continue by proving (2.16). We multiply (2.6) by $-D_x^6 \psi_\varepsilon$

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} \frac{|D_x^3 \psi_\varepsilon|^2}{2} dx = \int_{\mathbb{R}^2} D_x^3 \psi_\varepsilon \cdot \partial_t D_x^3 \psi_\varepsilon dx = - \int_{\mathbb{R}^2} D_x^6 \psi_\varepsilon \partial_t \psi_\varepsilon dx \\ & = - \mu \int_{\mathbb{R}^2} \Delta_x \psi_\varepsilon D_x^6 \psi_\varepsilon dx + \int_{\mathbb{R}^2} D_x^6 \psi_\varepsilon \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) dx \\ & \quad - \int_{\mathbb{R}^2} \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi - \mathbf{p}(a, u_\varepsilon) \xi) D_x^6 \psi_\varepsilon da dx \\ & = - \mu \int_{\mathbb{R}^2} |D_x^4 \psi_\varepsilon|^2 dx + \int_{\mathbb{R}^2} D_x^4 \psi_\varepsilon \Delta_x \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) dx \\ & \quad - \int_{\mathbb{R}^2} \Delta_x \left(\int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi - \mathbf{p}(a, u_\varepsilon) \xi) da \right) D_x^4 \psi_\varepsilon dx \\ & \leq - \frac{\mu}{2} \int_{\mathbb{R}^2} |D_x^4 \psi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \left(\Delta_x \operatorname{div}_x \left(\left(\int_0^\infty \rho_\varepsilon \chi_1 \xi da \right) \mathbf{v} Y_\theta \right) \right)^2 dx \\ & \quad + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi)_+ da \right|^2 dx + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi)_- da \right|^2 dx \\ & \quad + c \int_{\mathbb{R}^2} \left| \Delta_x \left(\int_0^\infty \rho_\varepsilon \mathbf{p}(a, u_\varepsilon) \xi da \right) \right|^2 dx \\ & \leq - \frac{\mu}{2} \int_{\mathbb{R}^2} |D_x^4 \psi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \left| D_x^3 \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|\mathbf{v}|^2 Y_\theta^2}_{\leq c} dx \\ & \quad + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|\nabla_x \mathbf{v}|^2 Y_\theta^2}_{\leq c} dx + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|D_x^2 \mathbf{v}|^2 Y_\theta^2}_{\leq c} dx \\ & \quad + c \int_{\mathbb{R}^2} \left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|D_x^3 \mathbf{v}|^2 Y_\theta^2}_{\leq c} dx + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|\mathbf{v}|^2 (Y'_\theta)^2}_{\leq c} |\nabla_x \phi_\varepsilon|^2 dx \\ & \quad + c \int_{\mathbb{R}^2} \underbrace{\left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2}_{\leq c} |D_x^2 \mathbf{v}|^2 (Y'_\theta)^2 |\nabla_x \phi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|\nabla_x \mathbf{v}|^2 (Y'_\theta)^2}_{\leq c} |\nabla_x \phi_\varepsilon|^2 dx \\ & \quad + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|\mathbf{v}|^2 (Y''_\theta)^2}_{\leq c} |\nabla_x \phi_\varepsilon|^4 dx + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 \underbrace{|\mathbf{v}|^2 (Y''_\theta)^2}_{\leq c} |D_x^2 \phi_\varepsilon|^2 dx \\ & \quad + c \int_{\mathbb{R}^2} \underbrace{\left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2}_{\leq c} |\nabla_x \mathbf{v}|^2 (Y''_\theta)^2 |\nabla_x \phi_\varepsilon|^4 dx + c \int_{\mathbb{R}^2} \underbrace{\left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2}_{\leq c} |\nabla_x \mathbf{v}|^2 (Y'_\theta)^2 |D_x^2 \phi_\varepsilon|^2 dx \\ & \quad + c \int_{\mathbb{R}^2} \underbrace{\left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2}_{\leq c} |\mathbf{v}|^2 (Y'''_\theta)^2 |\nabla_x \phi_\varepsilon|^6 dx + c \int_{\mathbb{R}^2} \underbrace{\left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2}_{\leq c} |\mathbf{v}|^2 (Y''_\theta)^2 |D_x^2 \phi_\varepsilon|^2 |\nabla_x \phi_\varepsilon|^2 dx \\ & \quad + c \int_{\mathbb{R}^2} \underbrace{\left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2}_{\leq c} |\mathbf{v}|^2 (Y'_\theta)^2 |D_x^3 \phi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d} \xi)_+ da \right|^2 dx \end{aligned}$$

$$\begin{aligned}
& + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d}\xi)_- da \right|^2 dx + c \int_{\mathbb{R}^2} \left| \int_0^\infty \nabla_x \rho_\varepsilon \cdot \nabla_x u_\varepsilon \xi \underbrace{\partial_u \mathfrak{p}(a, u_\varepsilon)}_{\leq c} da \right|^2 dx \\
& + c \int_{\mathbb{R}^2} \left| \int_0^\infty \Delta_x \rho_\varepsilon \xi \mathfrak{p}(a, u_\varepsilon) da \right|^2 dx + c \int_{\mathbb{R}^2} \left| \int_0^\infty \rho_\varepsilon \xi \underbrace{\partial_{uu}^2 \mathfrak{p}(a, u_\varepsilon)}_{\leq c} |\nabla_x u_\varepsilon|^2 da \right|^2 dx \\
& + c \int_{\mathbb{R}^2} \left| \int_0^\infty \rho_\varepsilon \xi \underbrace{\partial_u \mathfrak{p}(a, u_\varepsilon)}_{\leq c} \Delta_x u_\varepsilon da \right|^2 dx \\
& \leq -\frac{\mu}{2} \int_{\mathbb{R}^2} |D_x^4 \psi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} \left| D_x^3 \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx \\
& + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx + c \int_{\mathbb{R}^2} \left| \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^2 dx + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^4 dx \\
& + c \int_{\mathbb{R}^2} \left| \nabla_x \int_0^\infty \rho_\varepsilon \chi_1 \xi da \right|^4 dx + c \int_{\mathbb{R}^2} |D_x^3 \phi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} |D_x^2 \phi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} |D_x^2 \phi_\varepsilon|^4 dx \\
& + c \int_{\mathbb{R}^2} |\nabla_x \phi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} |\nabla_x \phi_\varepsilon|^4 dx + c \int_{\mathbb{R}^2} |\nabla_x \phi_\varepsilon|^6 dx + c \int_{\mathbb{R}^2} |\nabla_x \phi_\varepsilon|^8 dx \\
& + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d}\xi)_+ da \right|^2 dx + c \int_{\mathbb{R}^2} \left| D_x^2 \int_0^\infty \rho_\varepsilon (\varepsilon \xi'' + \xi' - \mathfrak{d}\xi)_- da \right|^2 dx \\
& + c \int_{\mathbb{R}^2} |D_x^2 \psi_\varepsilon|^2 dx + c \int_{\mathbb{R}^2} |\nabla_x \psi_\varepsilon|^2 |\nabla_x u_\varepsilon|^2 dx + c_T \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon|^4 dx + c_T \int_{\mathbb{R}^2} |\Delta_x u_\varepsilon|^2 dx.
\end{aligned}$$

We integrate over $(0, t)$, and we remark once again that, up to a regularization argument, the estimates obtained for ψ_ε apply to I_\pm as well. Then (2.7), (2.13) and (2.14), (2.19), (2.11), together with the embedding $H^1(\mathbb{R}^2) \subset L^p(\mathbb{R}^2)$ for very $1 \leq p < \infty$ give us

$$\|D_x^3 \psi_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + \mu \|D_x^4 \psi_\varepsilon\|_{L^2((0,t) \times \mathbb{R}^2)}^2 \leq ce^{ct}.$$

(2.17). It follows from (2.7), (2.13), (2.14), (2.16), (2.6) and the embedding $H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$. \square

The following two lemmas provide keys estimates on ρ_ε , its derivatives and its trace at $a = 0$. We omit their proofs as they closely follow the argument used in the proofs of [2, Corollary 2.1, Lemma 2.3], using Lemmas 2.2, 2.3, 2.6, 2.8, and the boundedness of \mathfrak{p} .

Lemma 2.7 (Estimates on the trace of ρ_ε at $a = 0$). *Define $\rho_\varepsilon^0(t, x) = \rho_\varepsilon(t, 0, x)$.*

For every $t \geq 0$ the following estimates hold

$$\begin{aligned}
& \|\rho_\varepsilon^0(t, \cdot)\|_{L^1(\mathbb{R}^2)}, \|D_x^2 \rho_\varepsilon^0\|_{L^1((0,t) \times \mathbb{R}^2)}, \|\partial_t \rho_\varepsilon^0\|_{L^1((0,t) \times \mathbb{R}^2)} \leq e^{Ct} C, \\
& \|\rho_\varepsilon^0(t, \cdot)\|_{L^2(\mathbb{R}^2)}, \|\nabla_x \rho_\varepsilon^0(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq e^{Ct} C, \\
& \|\nabla_x \rho_\varepsilon^0\|_{L^2((0,t) \times \mathbb{R}^2)}, \|D_x^2 \rho_\varepsilon^0\|_{L^2((0,t) \times \mathbb{R}^2)}, \|\partial_t \rho_\varepsilon^0\|_{L^2((0,t) \times \mathbb{R}^2)} \leq e^{Ct} C, \\
& \|\rho_\varepsilon^0(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq e^{Ct} C.
\end{aligned}$$

Lemma 2.8 (Estimates on ρ_ε). *For every $t \geq 0$ the following estimates hold*

$$(2.20) \quad \|\rho_\varepsilon(t, \cdot, \cdot)\|_{L^\infty((0,\infty) \times \mathbb{R}^2)} \leq Ce^{Ct},$$

$$(2.21) \quad \|\rho_\varepsilon(t, \cdot, \cdot)\|_{L^1((0,\infty) \times \mathbb{R}^2)} \leq Ce^{Ct},$$

$$(2.22) \quad \|\rho_\varepsilon(t, \cdot, \cdot)\|_{L^2((0,\infty) \times \mathbb{R}^2)}, \|\nabla_x \rho_\varepsilon\|_{L^2((0,t) \times (0,\infty) \times \mathbb{R}^2)}, \sqrt{\varepsilon} \|\partial_a \rho_\varepsilon\|_{L^2((0,t) \times (0,\infty) \times \mathbb{R}^2)} \leq Ce^{Ct},$$

$$(2.23) \quad \|\nabla_x \rho_\varepsilon(t, \cdot, \cdot)\|_{L^2((0,\infty) \times \mathbb{R}^2)}, \|D_x^2 \rho_\varepsilon\|_{L^2((0,t) \times (0,\infty) \times \mathbb{R}^2)}, \sqrt{\varepsilon} \|\partial_a \nabla_x \rho_\varepsilon\|_{L^2((0,t) \times (0,\infty) \times \mathbb{R}^2)} \leq Ce^{Ct},$$

$$(2.24) \quad \|\partial_a \rho_\varepsilon(t, \cdot, \cdot)\|_{L^1((0,\infty) \times \mathbb{R}^2)} \leq Ce^{Ct}.$$

Lemma 2.9 (BV estimate w.r.t. x on u_ε). For every $t \geq 0$ the following estimate holds

$$(2.25) \quad \|\nabla_x u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R}^2)} \leq C e^{Ct}.$$

Proof. From the equation of u_ε in (2.12) we have the equation

$$\partial_t \nabla_x u_\varepsilon + \nabla_x (\nabla u_\varepsilon \cdot \nu(\phi_\varepsilon) + u_\varepsilon \operatorname{div}_x (\nu(\phi_\varepsilon))) = (\mathbf{b}(\phi_\varepsilon) - \beta) \nabla_x u_\varepsilon + \mathbf{b}'(\phi_\varepsilon) \nabla_x \phi_\varepsilon u_\varepsilon + \varepsilon \nabla_x \Delta_x u_\varepsilon.$$

We define

$$\operatorname{sign}(\nabla_x u_\varepsilon) = (\operatorname{sign}(\partial_{x_1}(u_\varepsilon)), \operatorname{sign}(\partial_{x_2}(u_\varepsilon))),$$

where $x = (x_1, x_2)$. Then, using the L^∞ bounds on $\nabla_x \phi_\varepsilon$ and $\nu(\phi_\varepsilon)$ in (2.17), (1.13), we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon| dx &= \int_{\mathbb{R}^2} \operatorname{sign}(\nabla_x u_\varepsilon) \partial_t \nabla_x u_\varepsilon dx \\ &= - \underbrace{\int_{\mathbb{R}^2} \nabla_x (\nabla_x u_\varepsilon \cdot \nu(\phi_\varepsilon)) \operatorname{sign}(\nabla_x u_\varepsilon) dx}_{=0} - \int_{\mathbb{R}^2} \nabla_x (u_\varepsilon \operatorname{div}_x (\nu(\phi_\varepsilon))) \operatorname{sign}(\nabla_x u_\varepsilon) dx \\ &\quad + \underbrace{\varepsilon \int_{\mathbb{R}^2} \nabla_x \Delta_x u_\varepsilon \operatorname{sign}(\nabla_x u_\varepsilon) dx}_{\leq 0} + \int_{\mathbb{R}^2} \underbrace{\mathbf{b}'(\phi_\varepsilon) \nabla_x \phi_\varepsilon u_\varepsilon \operatorname{sign}(\nabla_x u_\varepsilon)}_{\leq c_T} dx + \int_{\mathbb{R}^2} \underbrace{(\mathbf{b}(\phi_\varepsilon) - \beta) |\nabla_x u_\varepsilon|}_{\leq c} dx \\ &\leq \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon| |\operatorname{div}_x (\nu(\phi_\varepsilon))| dx - \int_{\mathbb{R}^2} u_\varepsilon \nabla_x (\operatorname{div}_x (\nu(\phi_\varepsilon))) \operatorname{sign}(\nabla_x u_\varepsilon) dx \\ &\quad + c_T \int_{\mathbb{R}^2} u_\varepsilon dx + c \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon| dx \\ &\leq c \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon| dx + c_T \int_{\mathbb{R}^2} |u_\varepsilon| dx + c \int_{\mathbb{R}^2} u_\varepsilon^2 dx + c \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon|^2 dx \\ &\quad + \int_{\mathbb{R}^2} |\operatorname{div}_x (\nu(\phi_\varepsilon))|^2 dx + \int_{\mathbb{R}^2} |\nabla_x (\operatorname{div}_x (\nu(\phi_\varepsilon)))|^2 dx. \end{aligned}$$

We obtain the desired inequality (2.25) integrating over $(0, t)$, using the Gronwall Lemma and the estimates in Lemma 2.5, (1.14) and (1.17). \square

Lemma 2.10 (BV estimate w.r.t. t on u_ε). For every $t \geq 0$, the following estimate holds

$$(2.26) \quad \|\partial_t u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R}^2)} \leq C e^{Ct}.$$

Proof. From the definition of $\nu(\phi)$ in (1.2) we compute

$$\partial_t \nu(\phi_\varepsilon) = \kappa \frac{\partial_t \phi_\varepsilon * \nabla_x \eta}{\left(1 + \|\phi_\varepsilon * \nabla_x \eta\|^2\right)^{3/2}},$$

then from (2.17) we have

$$\|\partial_t \nu(\phi_\varepsilon)\|_{L^\infty(\mathbb{R}^2; \mathbb{R}^2)} \leq \kappa \|\nabla_x \eta\|_{L^\infty(\mathbb{R}^2; \mathbb{R}^2)} \|\partial_t \phi_\varepsilon\|_{L^1(\mathbb{R}^2, \mathbb{R})} \leq C e^{Ct}.$$

Similarly, explicit computations give us

$$(2.27) \quad \operatorname{div}_x (\nu(\phi_\varepsilon)) = \kappa \frac{\phi_\varepsilon * \Delta_x \eta}{\left(1 + \|\phi_\varepsilon * \nabla_x \eta\|^2\right)^{3/2}},$$

and

$$\partial_t \operatorname{div}_x (\nu(\phi_\varepsilon)) = \kappa \frac{\partial_t \phi_\varepsilon * \Delta_x \eta}{\left(1 + \|\phi_\varepsilon * \nabla_x \eta\|^2\right)^{3/2}} - 3\kappa (\phi_\varepsilon * \Delta_x \eta) \frac{\phi_\varepsilon * \nabla_x \eta}{\sqrt{1 + \|\phi_\varepsilon * \nabla_x \eta\|^2}} \frac{\partial_t \phi_\varepsilon * \nabla_x \eta}{\left(1 + \|\phi_\varepsilon * \nabla_x \eta\|^2\right)^2}.$$

Therefore, from (2.5), (2.17), we obtain

$$\begin{aligned}
& \|\partial_t \operatorname{div}_x (\nu(\phi_\varepsilon))\|_{L^2(\mathbb{R}^2; \mathbb{R})} \\
& \leq \kappa \|\partial_t \phi_\varepsilon * \Delta_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R})} + 3\kappa \|\phi_\varepsilon * \Delta_x \eta\|_{L^\infty(\mathbb{R}^2; \mathbb{R})} \|\partial_t \phi_\varepsilon * \nabla_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} \\
& \leq \kappa \|\partial_t \phi_\varepsilon\|_{L^1(\mathbb{R}^2; \mathbb{R})} \left(\|\Delta_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R})} + 3 \|\Delta_x \eta\|_{L^\infty(\mathbb{R}^2; \mathbb{R})} \|\nabla_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} \|\phi_\varepsilon\|_{L^1(\mathbb{R}^2; \mathbb{R})} \right) \\
& \leq C e^{Ct}.
\end{aligned}$$

The estimates above allow us to prove (2.26). From the equation (2.12) for u_ε we have

$$\partial_{tt}^2 u_\varepsilon + \operatorname{div}_x (\partial_t u_\varepsilon \nu(\phi_\varepsilon)) + \operatorname{div}_x (u_\varepsilon \partial_t \nu(\phi_\varepsilon)) = (\mathbf{b}(\phi_\varepsilon) - \beta) \partial_t u_\varepsilon + \mathbf{b}'(\phi_\varepsilon) \partial_t \phi_\varepsilon u_\varepsilon + \varepsilon \Delta_x \partial_t u_\varepsilon,$$

then we consider

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^2} |\partial_t u_\varepsilon| dx &= - \underbrace{\int_{\mathbb{R}^2} \operatorname{div}_x (\partial_t u_\varepsilon \nu(\phi_\varepsilon)) \operatorname{sign}(\partial_t u_\varepsilon) dx}_{=0} + \underbrace{\varepsilon \int_{\mathbb{R}^2} \Delta_x \partial_t u_\varepsilon \operatorname{sign}(\partial_t u_\varepsilon) dx}_{\leq 0} \\
&\quad - \int_{\mathbb{R}^2} (\nabla_x u_\varepsilon \underbrace{\partial_t \nu(\phi_\varepsilon)}_{\leq cT} + u_\varepsilon \partial_t \operatorname{div}_x (\nu(\phi_\varepsilon))) \operatorname{sign}(\partial_t u_\varepsilon) dx \\
&\quad + \int_{\mathbb{R}^2} \underbrace{(\mathbf{b}(\phi_\varepsilon) - \beta)}_{\leq c} |\partial_t u_\varepsilon| dx + \int_{\mathbb{R}^2} \mathbf{b}'(\phi_\varepsilon) \partial_t \phi_\varepsilon u_\varepsilon \operatorname{sign}(\partial_t u_\varepsilon) dx \\
&\leq cT \int_{\mathbb{R}^2} |\nabla_x u_\varepsilon| dx + c \int_{\mathbb{R}^2} u_\varepsilon^2 dx + \int_{\mathbb{R}^2} |\partial_t \operatorname{div}_x (\nu(\phi_\varepsilon))|^2 dx \\
&\quad + cT \int_{\mathbb{R}^2} |\partial_t \phi_\varepsilon| dx + c \int_{\mathbb{R}^2} |\partial_t u_\varepsilon| dx.
\end{aligned}$$

Integrating on $(0, t)$ we obtain (2.26) thanks to (2.10), (2.17) and (2.25) and the Gronwall Lemma. \square

We are now ready to prove the compactness of the families $\{\rho_\varepsilon\}_\varepsilon$ and $\{u_\varepsilon\}_\varepsilon$, and the first part of Theorem 1.1, establishing the existence of entropy solutions for the system (1.1).

Lemma 2.11 (Strong compactness of $\{\rho_\varepsilon\}_\varepsilon$ and $\{u_\varepsilon\}_\varepsilon$). *There exists a couple of functions (u, ρ) and a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}} \in (0, \infty)$, $\varepsilon_k \rightarrow 0$, such that, for every $T > 0$*

$$\begin{aligned}
(2.28) \quad & \rho_{\varepsilon_k} \rightarrow \rho, \quad \text{a.e. in } (0, T) \times (0, \infty) \times \mathbb{R}^2 \quad \text{and in } L_{loc}^p((0, \infty) \times (0, \infty) \times \mathbb{R}^2), \quad 1 \leq p < \infty, \\
& \rho(\cdot, \cdot, \cdot) \geq 0, \quad \rho(t, \cdot, x) \in BV(0, \infty), \quad \text{for a.e. } (t, x) \in (0, \infty) \times \mathbb{R}^2, \\
& \rho \in L^\infty(0, T; L^1((0, \infty) \times \mathbb{R}^2)) \cap L^\infty((0, T) \times (0, \infty) \times \mathbb{R}^2) \cap L^2((0, T) \times (0, \infty); H^2(\mathbb{R}^2)),
\end{aligned}$$

and

$$\begin{aligned}
(2.29) \quad & u_{\varepsilon_{k_h}} \rightarrow u \quad \text{a.e. in } (0, T) \times \mathbb{R}^2 \quad \text{and in } L_{loc}^p((0, \infty) \times \mathbb{R}^2), \quad 1 \leq p < \infty, \\
& u(\cdot, \cdot) \geq 0, \quad u \in L^\infty((0, T) \times \mathbb{R}^2) \cap BV((0, T) \times \mathbb{R}^2).
\end{aligned}$$

Proof. The proof of the strong convergence (up to a subsequence) for $\{\rho_\varepsilon\}_\varepsilon$ follows the argument used in [2, Lemma 2.9], which is based on the Murat Lemma, [8], and the compensated compactness result in [10, Th. 5]. Thanks to Lemmas 2.8, 2.10, 2.9, $\{u_\varepsilon\}_\varepsilon$ is bounded in $L^\infty((0, T) \times \mathbb{R}^2) \cap BV((0, T) \times \mathbb{R}^2)$ so that Helly's Theorem applies. \square

Lemma 2.12. *The couple of functions (u, ρ) introduced in Lemma 2.11 is an entropy solution of (1.1) in the sense of Definition 1.2.*

Proof. It is clear that the couple (u, ρ) is a weak solution of (1.1) in the sense of Definition 1.1 thanks to the strong convergence results in Lemma 2.11. In particular, the fact that ρ is a weak solution comes directly from [10, Th. 5].

We obtain (1.19) as in [2, Lemma 2.10], then we only have to verify that (1.20) holds. Let $\xi \in C^\infty(\mathbb{R}^4)$ be a nonnegative test function with compact support and $\mathbf{c} \in \mathbb{R}$ be a constant. Multiplying the equation of u_ε in (2.1) by $\text{sign}(u_\varepsilon - \mathbf{c})$ we have

$$\begin{aligned} \partial_t |u_\varepsilon - \mathbf{c}| + \text{div}_x (|u_\varepsilon - \mathbf{c}| \nu(\phi_\varepsilon)) + \mathbf{c} \text{sign}(u_\varepsilon - \mathbf{c}) \text{div}_x (\nu(\phi_\varepsilon)) \\ \leq \text{sign}(u_\varepsilon - \mathbf{c}) (\mathbf{b}(\phi_\varepsilon) - \beta) u_\varepsilon + \varepsilon \Delta_x |u_\varepsilon - \mathbf{c}|. \end{aligned}$$

Then,

$$\begin{aligned} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} (|u_\varepsilon - \mathbf{c}| \partial_t \xi + |u_\varepsilon - \mathbf{c}| \nu(\phi_\varepsilon) \cdot \nabla_x \xi + \text{sign}(u_\varepsilon - \mathbf{c}) (\mathbf{b}(\phi_\varepsilon) - \beta) u_\varepsilon \xi \\ + \varepsilon |u_\varepsilon - \mathbf{c}| \Delta_x \xi - \mathbf{c} \text{sign}(u_\varepsilon - \mathbf{c}) \text{div}_x (\nu(\phi_\varepsilon)) \xi) dx da dt \\ + \int_0^\infty \int_{\mathbb{R}^2} |u_{\varepsilon,0}(x) - \mathbf{c}| \xi(0, a, x) dx da \geq 0. \end{aligned}$$

By taking the limit for $\varepsilon \rightarrow 0$, we get (1.20). \square

3. UNIQUENESS AND STABILITY

In this section we establish the inequality (1.21), which concludes the proof of Theorem 1.1. To this end we introduce the following preliminary lemma.

Lemma 3.1. *Let (u_1, ρ_1) and (u_2, ρ_2) be two entropy solutions of (1.1) obtained from the initial data $(u_{1,0}, \rho_{1,0})$ and $(u_{2,0}, \rho_{2,0})$ respectively. For every nonnegative test function $\xi \in C_c^\infty(\mathbb{R}^4)$ the following inequalities hold*

$$\begin{aligned} (3.1) \quad & \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} \left(|\rho_1 - \rho_2| (\partial_t \xi + \partial_a \xi) - \text{div}_x (|\rho_1 - \rho_2| \chi_1 \mathbf{v} (Y_\theta(\phi_1 - R) + Y_\theta(\phi_2 - R))) \xi \right. \\ & \quad + \mu \Delta_x |\rho_1 - \rho_2| \xi - |\rho_1 - \rho_2| \partial \xi - |\rho_1 - \rho_2| \mathbf{p}(a, u_1) \xi \\ & \quad \left. - \text{sign}(\rho_1 - \rho_2) (\mathbf{p}(a, u_1) - \mathbf{p}(a, u_2)) \rho_2 \xi \right) dx da dt \\ & + \int_0^\infty \int_{\mathbb{R}^2} |\rho_1(t, 0^+, x) - \rho_2(t, 0^+, x)| \xi(t, 0, x) dx dt + \int_0^\infty \int_{\mathbb{R}^2} |\rho_{1,0}(a, x) - \rho_{2,0}(a, x)| \xi(0, a, x) dx da \\ & \geq \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} \text{sign}(\rho_1 - \rho_2) \chi_1 (\rho_2 \text{div}_x (\mathbf{v} Y_\theta(\phi_1 - R)) - \rho_1 \text{div}_x (\mathbf{v}(x) Y_\theta(\phi_2 - R))) \xi dx da dt, \end{aligned}$$

where

$$\phi_i(t, x) := \int_0^\infty \rho_i(t, a, x) \chi_2(a) da, \quad \text{for } i = 1, 2,$$

and

$$\begin{aligned} (3.2) \quad & \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} \left(|u_1 - u_2| \partial_t \xi - \text{div}_x (|u_1 - u_2| (\nu(\phi_1) + \nu(\phi_2))) \xi \right. \\ & \quad \left. + |u_1 - u_2| (\mathbf{b}(\phi_1) - \beta) \xi + \text{sign}(u_1 - u_2) (\mathbf{b}(\phi_1) - \mathbf{b}(\phi_2)) u_2 \xi \right) dx da dt \\ & + \int_0^\infty \int_{\mathbb{R}^2} |u_{1,0}(x) - u_{2,0}(x)| \xi(0, a, x) dx da \\ & \geq \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} \text{sign}(u_1 - u_2) (u_2 \text{div}_x (\nu(\phi_1)) - u_1 \text{div}_x (\nu(\phi_2))) \xi dx da dt. \end{aligned}$$

Proof. We double the variables and write

$$\rho_1 = \rho_1(t, a, x), \quad \rho_2 = \rho_2(s, b, y), \quad \phi_1 = \phi_1(t, x), \quad \phi_2 = \phi_2(s, y) \quad u_1 = u_1(t, x) \quad u_2 = u_2(s, y),$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Consider the test function

$$\Xi_n(t, s, a, b, x, y) = \xi \left(\frac{t+s}{2}, \frac{a+b}{2}, \frac{x+y}{2} \right) \lambda_n \left(\frac{s-t}{2} \right) \lambda_n \left(\frac{b-a}{2} \right) \lambda_n \left(\frac{y_1-x_1}{2} \right) \lambda_n \left(\frac{y_2-x_2}{2} \right),$$

where

$$\lambda_n(u) = n\lambda(nu), \quad \lambda \in C^\infty(\mathbb{R}), \quad \lambda \geq 0, \quad \|\lambda\|_{L^1(\mathbb{R})} = 1, \quad \text{supp}(\lambda) \subset [-1, 1].$$

To prove inequality (3.1) we follow the doubling of variables argument appearing in [2, Lemma 3.1], and use the regularity of \mathbf{p} and the L^∞ bounds on u_1, u_2 . Then we have only to verify (3.2). We write (1.20) for $u_1(t, x)$ using $u_2(s, y)$ as a constant and integrate over (s, y)

$$(3.3) \quad \begin{aligned} & \iiint \iiint \iiint (|u_1 - u_2| \partial_t \Xi_n - \text{div}_x (|u_1 - u_2| \nu(\phi_1)) \Xi_n + \text{sign}(u_1 - u_2) (\mathbf{b}(\phi_1) - \beta) u_1 \Xi_n) dx dy da db dt ds \\ & + \iiint \iiint |u_{1,0}(x) - u_2| \Xi_n(0, s, a, b, x, y) dx dy da db ds \\ & \geq \iiint \iiint \iiint \text{sign}(u_1 - u_2) u_2 \text{div}_x (\nu(\phi_1)) \Xi_n dx dy da db dt ds, \end{aligned}$$

and we write (1.20) for $u_2(s, y)$ using $u_1(t, x)$ as a constant and integrate over (t, x)

$$(3.4) \quad \begin{aligned} & \iiint \iiint \iiint (|u_1 - u_2| \partial_s \Xi_n - \text{div}_y (|u_1 - u_2| \nu(\phi_2)) \Xi_n - \text{sign}(u_1 - u_2) (\mathbf{b}(\phi_2) - \beta) u_2 \Xi_n) dx dy da db dt ds \\ & + \iiint \iiint |u_1 - u_{2,0}(y)| \Xi_n(t, 0, a, b, x, y) dx dy da db dt \\ & \geq - \iiint \iiint \iiint \text{sign}(u_1 - u_2) u_1 \text{div}_y (\nu(\phi_2)) \Xi_n dx dy da db dt ds. \end{aligned}$$

Summing (3.3) and (3.4) we have

$$\begin{aligned} & \iiint \iiint \iiint (|u_1 - u_2| (\partial_t \Xi_n + \partial_s \Xi_n) - \text{div}_x (|u_1 - u_2| \nu(\phi_1)) \Xi_n - \text{div}_y (|u_1 - u_2| \nu(\phi_2)) \Xi_n \\ & \quad + |u_1 - u_2| (\mathbf{b}(\phi_1) - \beta) \Xi_n + \text{sign}(u_1 - u_2) (\mathbf{b}(\phi_1) - \mathbf{b}(\phi_2)) u_2 \Xi_n) dx dy da db dt ds \\ & + \iiint \iiint |u_{1,0}(x) - u_2| \Xi_n(0, s, a, b, x, y) dx dy da db ds \\ & + \iiint \iiint |u_1 - u_{2,0}(y)| \Xi_n(t, 0, a, b, x, y) dx dy da db dt \\ & \geq \iiint \iiint \iiint \text{sign}(u_1 - u_2) u_2 \text{div}_x (\nu(\phi_1)) \Xi_n dx dy da db dt ds \\ & \quad - \iiint \iiint \iiint \text{sign}(u_1 - u_2) u_1 \text{div}_y (\nu(\phi_2)) \Xi_n dx dy da db dt ds. \end{aligned}$$

As $n \rightarrow \infty$ we get (3.2). □

Lemma 3.2. *For every $t \geq 0$, the following inequality holds*

(3.5)

$$\begin{aligned} \|\phi_1(t, \cdot) - \phi_2(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + \mu e^{Ct} \int_0^t \int_{\mathbb{R}^2} e^{-Cs} |\nabla_x(\phi_1 - \phi_2)|^2 dx ds &\leq C e^{Ct} \|\rho_{1,0} - \rho_{2,0}\|_{L^1((0,\infty) \times \mathbb{R}^2)}^2 \\ &+ C e^{Ct} \int_0^t \int_0^\infty \int_{\mathbb{R}^2} e^{-Cs} (\rho_1 - \rho_2)^2 dx da ds + C e^{Ct} \int_0^t \int_{\mathbb{R}^2} e^{-Cs} (u_1 - u_2)^2 dx ds. \end{aligned}$$

In particular, we have that

(3.6)

$$\begin{aligned} \left(\int_0^t \int_{\mathbb{R}^2} e^{-Cs} |\nabla_x(\phi_1 - \phi_2)|^2 dx ds \right)^{1/2} &\leq C e^{Ct} \|\rho_{1,0} - \rho_{2,0}\|_{L^1((0,\infty) \times \mathbb{R}^2)} \\ &+ C e^{Ct} \left(\int_0^t \|\rho_1(s, \cdot, \cdot) - \rho_2(s, \cdot, \cdot)\|_{L^1((0,\infty) \times \mathbb{R}^2)}^2 ds \right)^{1/2} + C e^{Ct} \left(\int_0^t \|u_1(s, \cdot) - u_2(s, \cdot)\|_{L^1(\mathbb{R}^2)}^2 ds \right)^{1/2}. \end{aligned}$$

Proof. Since ϕ_{ε_k} satisfies

$$\begin{aligned} \partial_t \phi_{\varepsilon_k} - \mu \Delta_x \phi_{\varepsilon_k} + \operatorname{div}_x \left(\left(\int_0^\infty \rho_{\varepsilon_k} \chi_1 da \right) \mathbf{v} Y_\theta(\phi_{\varepsilon_k} - R) \right) \\ = \int_0^\infty \rho_{\varepsilon_k} (\varepsilon_k \chi_2'' + \chi_2' - \mathfrak{d} \chi_2 - \mathfrak{p}(a, u_{\varepsilon_k}) \chi_2) da, \end{aligned}$$

as $k \rightarrow \infty$ we get the equation of ϕ . Then, subtracting the equation for ϕ_2 from the equation for ϕ_1 we obtain

$$\begin{aligned} \partial_t(\phi_1 - \phi_2) - \mu \Delta_x(\phi_1 - \phi_2) + \operatorname{div}_x \left(\left(\int_0^\infty (\rho_1 - \rho_2) \chi_1 da \right) \mathbf{v} Y_\theta(\phi_1 - R) \right) \\ + \operatorname{div}_x \left(\left(\int_0^\infty \rho_2 \chi_1 da \right) \mathbf{v} (Y_\theta(\phi_1 - R) - Y_\theta(\phi_2 - R)) \right) \\ = \int_0^\infty (\rho_1 - \rho_2) (\chi_2' - \mathfrak{d} \chi_2 - \mathfrak{p}(a, u_1) \chi_2) da - \int_0^\infty \rho_2 (\mathfrak{p}(a, u_1) - \mathfrak{p}(a, u_2)) \chi_2 da. \end{aligned} \quad (3.7)$$

Then

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} \frac{(\phi_1 - \phi_2)^2}{2} dx + \mu \int_{\mathbb{R}^2} |\nabla_x(\phi_1 - \phi_2)|^2 dx \\ = \int_{\mathbb{R}^2} (\phi_1 - \phi_2) \left(\int_0^\infty (\rho_1 - \rho_2) (\chi_2' - \mathfrak{d} \chi_2 - \mathfrak{p}(a, u_1) \chi_2) da \right) dx \\ - \int_{\mathbb{R}^2} (\phi_1 - \phi_2) \left(\int_0^\infty \underbrace{\rho_2 (\mathfrak{p}(a, u_1) - \mathfrak{p}(a, u_2)) \chi_2 da}_{\leq c|u_1 - u_2|} \right) dx \\ - \int_{\mathbb{R}^2} (\phi_1 - \phi_2) \operatorname{div}_x \left(\left(\int_0^\infty (\rho_1 - \rho_2) \chi_1 da \right) \mathbf{v} Y_\theta(\phi_1 - R) \right) dx \\ - \int_{\mathbb{R}^2} (\phi_1 - \phi_2) \operatorname{div}_x \left(\left(\int_0^\infty \rho_2 \chi_1 da \right) \mathbf{v} (Y_\theta(\phi_1 - R) - Y_\theta(\phi_2 - R)) \right) dx \\ \leq c \int_{\mathbb{R}^2} (\phi_1 - \phi_2)^2 dx + c e^{ct} \int_{\mathbb{R}^2} (u_1 - u_2)^2 dx + c e^{ct} \int_0^\infty \int_{\mathbb{R}^2} (\rho_1 - \rho_2)^2 dx da \\ + \frac{\mu}{2} \int_{\mathbb{R}^2} |\nabla_x(\phi_1 - \phi_2)|^2 dx. \end{aligned}$$

Using Gronwall Lemma we get

$$\begin{aligned} \|\phi_1(t, \cdot) - \phi_2(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + \mu e^{ct} \int_0^t \int_{\mathbb{R}^2} e^{-cs} |\nabla_x(\phi_1 - \phi_2)|^2 dx ds \\ \leq e^{ct} \|\phi_{1,0} - \phi_{2,0}\|_{L^2(\mathbb{R}^2)}^2 + c e^{ct} \int_0^t \int_0^\infty \int_{\mathbb{R}^2} e^{-cs} (\rho_1 - \rho_2)^2 dx da ds + c e^{ct} \int_0^t \int_{\mathbb{R}^2} e^{-cs} (u_1 - u_2)^2 dx ds. \end{aligned}$$

Finally, we obtain (3.5) and (3.6) from the definition of ϕ_1 and ϕ_2 . □

We are now ready to complete the proof of Theorem 1.1.

Proof. Our goal is to prove the inequality (1.21). We rewrite (3.1) and (3.2) as

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} (|\rho_1 - \rho_2| (\partial_t \xi_n + \partial_a \xi_n) - |\rho_1 - \rho_2| \chi_1 (Y_\theta(\phi_1 - R) + Y_\theta(\phi_2 - R)) \mathbf{v} \cdot \nabla_x \xi_n \\
& \quad + \mu |\rho_1 - \rho_2| \Delta_x \xi_n - |\rho_1 - \rho_2| \mathfrak{D} \xi_n - |\rho_1 - \rho_2| \mathfrak{p}(a, u_1) \xi_n \\
& \quad - \text{sign}(\rho_1 - \rho_2) (\mathfrak{p}(a, u_1) - \mathfrak{p}(a, u_2)) \rho_2 \xi_n) dx da dt \\
(3.8) \quad & + \int_0^\infty \int_{\mathbb{R}^2} |\rho_1(t, 0^+, x) - \rho_2(t, 0^+, x)| \xi_n(t, 0, x) dx dt \\
& + \int_0^\infty \int_{\mathbb{R}^2} |\rho_{1,0}(a, x) - \rho_{2,0}(a, x)| \xi_n(0, a, x) dx da \\
& \geq \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} \text{sign}(\rho_1 - \rho_2) \chi_1 (\rho_2 \text{div}_x (\mathbf{v} Y_\theta(\phi_1 - R)) - \rho_1 \text{div}_x (\mathbf{v}(x) Y_\theta(\phi_2 - R))) \xi_n dx da dt,
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} (|u_1 - u_2| \partial_t \xi_n + |u_1 - u_2| (\nu(\phi_1) + \nu(\phi_2)) \cdot \nabla_x \xi_n + |u_1 - u_2| (\mathfrak{b}(\phi_1) - \beta) \xi_n \\
& \quad + \text{sign}(u_1 - u_2) (\mathfrak{b}(\phi_1) - \mathfrak{b}(\phi_2)) u_2 \xi_n) dx da dt \\
(3.9) \quad & + \int_0^\infty \int_{\mathbb{R}^2} |u_{1,0}(x) - u_{2,0}(x)| \xi_n(0, a, x) dx da \\
& \geq \int_0^\infty \int_0^\infty \int_{\mathbb{R}^2} \text{sign}(u_1 - u_2) (u_2 \text{div}_x (\nu(\phi_1)) - u_1 \text{div}_x (\nu(\phi_2))) \xi_n dx da dt,
\end{aligned}$$

where $\{\xi_n\}_n$ is a sequence of nonnegative test functions approximating the characteristic function of the strip $(-\infty, t) \times \mathbb{R} \times \mathbb{R}^2$. Sending $n \rightarrow \infty$, we have that

$$\begin{aligned}
& \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\mathbb{R}^2)} \\
& \leq \|u_{1,0} - u_{2,0}\|_{L^1(\mathbb{R}^2)} - \int_0^t \int_{\mathbb{R}^2} \text{sign}(u_1 - u_2) (u_2 \text{div}_x (\nu(\phi_1)) - u_1 \text{div}_x (\nu(\phi_2))) dx ds \\
& \quad + \int_0^t \int_{\mathbb{R}^2} |u_1 - u_2| \underbrace{\mathfrak{b}(\phi_1)}_{\leq c} + \text{sign}(u_1 - u_2) \underbrace{(\mathfrak{b}(\phi_1) - \mathfrak{b}(\phi_2))}_{\leq c|\phi_1 - \phi_2|} u_2 dx ds \\
& \leq \|u_{1,0} - u_{2,0}\|_{L^1(\mathbb{R}^2)} + \int_0^t \int_{\mathbb{R}^2} |u_1 - u_2| \text{div}_x (\nu(\phi_1)) dx ds \\
& \quad - \int_0^t \int_{\mathbb{R}^2} \text{sign}(u_1 - u_2) u_1 \underbrace{\text{div}_x (\nu(\phi_1) - \nu(\phi_2))}_{\text{see (1.18)}} dx ds \\
& \quad + c \int_0^t \int_{\mathbb{R}^2} |u_1 - u_2| dx ds + ce^{ct} \int_0^t \int_{\mathbb{R}^2} |\phi_1 - \phi_2| dx ds \\
& \leq \|u_{1,0} - u_{2,0}\|_{L^1(\mathbb{R}^2)} + ce^{ct} \int_0^t \int_{\mathbb{R}^2} |u_1 - u_2| dx ds \\
& \quad + ce^{ct} \int_0^t \left(\int_{\mathbb{R}^2} |\text{div}_x (\nu(\phi_1) - \nu(\phi_2))|^2 dx \right)^{1/2} ds \\
& \quad + c \int_0^t \int_{\mathbb{R}^2} |u_1 - u_2| dx ds + ce^{ct} \int_0^t \int_{\mathbb{R}^2} |\phi_1 - \phi_2| dx ds \\
& \leq \|u_{1,0} - u_{2,0}\|_{L^1(\mathbb{R}^2)} + c(1 + e^{ct}) \int_0^t \|u_1(s, \cdot) - u_2(s, \cdot)\|_{L^1(\mathbb{R}^2)} ds
\end{aligned}$$

$$+ ce^{ct} \int_0^t \|\rho_1(s, \cdot, \cdot) - \rho_2(s, \cdot, \cdot)\|_{L^1((0, \infty) \times \mathbb{R}^2)} ds,$$

and

$$\begin{aligned} & \|\rho_1(t, \cdot, \cdot) - \rho_2(t, \cdot, \cdot)\|_{L^1((0, \infty) \times \mathbb{R}^2)} \\ & \leq \|\rho_{1,0} - \rho_{2,0}\|_{L^1((0, \infty) \times \mathbb{R}^2)} + \int_0^t \int_{\mathbb{R}^2} |\rho_1(s, 0^+, x) - \rho_2(s, 0^+, x)| dx ds \\ & \quad - \int_0^t \int_0^\infty \int_{\mathbb{R}^2} \text{sign}(\rho_1 - \rho_2) \chi_1(a) (\rho_2 \text{div}_x(\mathbf{v}Y_\theta(\phi_1 - R)) - \rho_1 \text{div}_x(\mathbf{v}Y_\theta(\phi_2 - R))) dx da ds \\ & \quad - \int_0^t \int_0^\infty \int_{\mathbb{R}^2} \text{sign}(\rho_1 - \rho_2) (\mathbf{p}(a, u_1) - \mathbf{p}(a, u_2)) \rho_2 dx da ds \\ & \leq \|\rho_{1,0} - \rho_{2,0}\|_{L^1((0, \infty) \times \mathbb{R}^2)} + \int_0^t \int_0^\infty \int_{\mathbb{R}^2} |\mathcal{A}(\phi_1)\rho_1 - \mathcal{A}(\phi_2)\rho_2| \chi_3 \omega dx da ds \\ & \quad - \int_0^t \int_0^\infty \int_{\mathbb{R}^2} \text{sign}(\rho_1 - \rho_2) \chi_1(a) (\rho_2 \text{div}_x(\mathbf{v}Y_\theta(\phi_1 - R)) - \rho_1 \text{div}_x(\mathbf{v}Y_\theta(\phi_2 - R))) dx da ds \\ & \quad - \int_0^t \int_0^\infty \int_{\mathbb{R}^2} \text{sign}(\rho_1 - \rho_2) (\mathbf{p}(a, u_1) - \mathbf{p}(a, u_2)) \rho_2 dx da ds \\ & \leq \|\rho_{1,0} - \rho_{2,0}\|_{L^1((0, \infty) \times \mathbb{R}^2)} + \int_0^t \int_0^\infty \int_{\mathbb{R}^2} \mathcal{A}(\phi_1) |\rho_1 - \rho_2| \chi_3 \omega dx da ds \\ & \quad + \int_0^t \int_0^\infty \int_{\mathbb{R}^2} |\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)| \rho_2 \chi_3 \omega dx da ds \\ & \quad + \int_0^t \int_0^\infty \int_{\mathbb{R}^2} |\rho_1 - \rho_2| \chi_1(a) \text{div}_x(\mathbf{v}Y_\theta(\phi_1 - R)) dx da ds \\ & \quad - \int_0^t \int_0^\infty \int_{\mathbb{R}^2} \text{sign}(\rho_1 - \rho_2) \chi_1(a) \rho_1 \text{div}_x(\mathbf{v})(Y_\theta(\phi_1 - R) - Y_\theta(\phi_2 - R)) dx da ds \\ & \quad - \int_0^t \int_0^\infty \int_{\mathbb{R}^2} \text{sign}(\rho_1 - \rho_2) \chi_1(a) \rho_1 \mathbf{v} \cdot \nabla_x (Y_\theta(\phi_1 - R) - Y_\theta(\phi_2 - R)) dx da ds \\ & \quad + ce^{ct} \int_0^t \|u_1(s, \cdot) - u_2(s, \cdot)\|_{L^1(\mathbb{R}^2)} ds \\ & \leq \|\rho_{1,0} - \rho_{2,0}\|_{L^1((0, \infty) \times \mathbb{R}^2)} + c \int_0^t \|\rho_1(s, \cdot, \cdot) - \rho_2(s, \cdot, \cdot)\|_{L^1((0, \infty) \times \mathbb{R}^2)} ds \\ & \quad + ce^{ct} \int_0^t \int_{\mathbb{R}^2} |\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)| dx ds \\ & \quad + ce^{ct} \int_0^t \int_{\mathbb{R}^2} |Y_\theta(\phi_1 - R) - Y_\theta(\phi_2 - R)| dx ds \\ & \quad + ce^{ct} \left(\int_0^t \int_{\mathbb{R}^2} |\nabla_x (Y_\theta(\phi_1 - R) - Y_\theta(\phi_2 - R))|^2 dx ds \right)^{1/2} \\ & \quad + ce^{ct} \int_0^t \|u_1(s, \cdot) - u_2(s, \cdot)\|_{L^1(\mathbb{R}^2)} ds \\ & \leq \|\rho_{1,0} - \rho_{2,0}\|_{L^1((0, \infty) \times \mathbb{R}^2)} + c(1 + e^{ct}) \int_0^t \|\rho_1(s, \cdot, \cdot) - \rho_2(s, \cdot, \cdot)\|_{L^1((0, \infty) \times \mathbb{R}^2)} ds \\ & \quad + ce^{ct} \left(\int_0^t \int_{\mathbb{R}^2} |\nabla_x(\phi_1 - \phi_2)|^2 dx ds \right)^{1/2} + ce^{ct} \int_0^t \|u_1(s, \cdot) - u_2(s, \cdot)\|_{L^1(\mathbb{R}^2)} ds. \end{aligned}$$

Using (3.6), we have

$$\|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\mathbb{R}^2)}^2 + \|\rho_1(t, \cdot, \cdot) - \rho_2(t, \cdot, \cdot)\|_{L^1((0, \infty) \times \mathbb{R}^2)}^2$$

$$\begin{aligned} &\leq \|u_{1,0} - u_{2,0}\|_{L^1(\mathbb{R}^2)}^2 + c(1 + e^{ct}) \int_0^t \|u_1(s, \cdot) - u_2(s, \cdot)\|_{L^1(\mathbb{R}^2)}^2 ds \\ &\quad + c(1 + e^{ct}) \|\rho_{1,0} - \rho_{2,0}\|_{L^1((0,\infty)\times\mathbb{R}^2)}^2 + c(1 + e^{ct}) \int_0^t \|\rho_1(s, \cdot, \cdot) - \rho_2(s, \cdot, \cdot)\|_{L^1((0,\infty)\times\mathbb{R}^2)}^2 ds. \end{aligned}$$

Finally, we use the Gronwall Lemma to obtain the result (1.21). \square

4. Proof of Lemma 1.1

Proof. This lemma is similar to [4, Lemma 4.1], in particular, (1.13) and (1.16) already appeared there. For completeness we sketch the proof of the other estimates.

(1.14). directly comes from the definition of ν in (1.2) and the expression of $\operatorname{div}_x(\nu(\phi))$ in (2.27).

(1.15). We compute $\nabla_x \nu(\phi)$ by using the fact that $\nabla_x(f\nu) = f\nabla_x \nu + \nu \otimes \nabla_x f$

$$\begin{aligned} \nabla_x \nu(\phi) &= \kappa \frac{1}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{1/2}} \nabla_x(\phi * \nabla_x \eta) + \kappa(\phi * \nabla_x \eta) \otimes \nabla_x \frac{1}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{1/2}} \\ &= \kappa \frac{\phi * \nabla_x^2 \eta}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{1/2}} - \kappa(\phi * \nabla_x \eta) \otimes \frac{(\phi * \nabla_x^2 \eta)(\phi * \nabla_x \eta)}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{3/2}}, \end{aligned}$$

then (notice that to shorten the notations we write L^∞ for $L^\infty(\mathbb{R}^2; \mathbb{R}^{2 \times 2})$)

$$\begin{aligned} &\|\nabla_x \nu(\phi)\|_{L^\infty} \\ &\leq \kappa \frac{\|\phi * \nabla_x^2 \eta\|_{L^\infty}}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{1/2}} + \kappa \left\| \frac{\phi * \nabla_x \eta}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{1/2}} \otimes \frac{\phi * \nabla_x^2 \eta}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{1/2}} \frac{\phi * \nabla_x \eta}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{1/2}} \right\|_{L^\infty} \\ &\leq \kappa \|\phi * \nabla_x^2 \eta\|_{L^\infty} + \kappa \left\| \frac{\phi * \nabla_x \eta}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{1/2}} \right\|_{L^\infty} \|\phi * \nabla_x^2 \eta\|_{L^\infty} \left\| \frac{\phi * \nabla_x \eta}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{1/2}} \right\|_{L^\infty} \\ &\leq 2\kappa \|\phi * \nabla_x^2 \eta\|_{L^\infty} \leq 2\kappa \|\nabla_x^2 \eta\|_{L^\infty} \|\phi\|_{L^1(\mathbb{R}^2; \mathbb{R})}. \end{aligned}$$

(1.17). We compute gradient of (2.27)

$$\nabla_x \operatorname{div}_x(\nu(\phi)) = \kappa \frac{\phi * \nabla \Delta_x \eta}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{3/2}} - 3\kappa(\phi * \Delta_x \eta) \frac{\phi * \nabla_x^2 \eta}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^2} \frac{\phi * \nabla_x \eta}{\left(1 + \|\phi * \nabla_x \eta\|^2\right)^{1/2}},$$

then

$$\begin{aligned} &\|\nabla_x \operatorname{div}_x(\nu(\phi))\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} \\ &\leq \kappa \|\phi * \nabla \Delta_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} + 3\kappa \|\phi * \Delta_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} \|\phi * \nabla_x^2 \eta\|_{L^\infty(\mathbb{R}^2; \mathbb{R}^{2 \times 2})} \\ &\leq \kappa \|\phi\|_{L^1(\mathbb{R}^2; \mathbb{R})} \left(\|\nabla \Delta_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} + 3 \|\Delta_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R})} \|\nabla_x^2 \eta\|_{L^\infty(\mathbb{R}^2; \mathbb{R}^{2 \times 2})} \|\phi\|_{L^1(\mathbb{R}^2; \mathbb{R})} \right). \end{aligned}$$

(1.18). We have that

$$\begin{aligned} &\operatorname{div}_x(\nu(\phi_1) - \nu(\phi_2)) \\ &= \kappa \frac{(\phi_1 - \phi_2) * \Delta_x \eta}{\left(1 + \|\phi_1 * \nabla_x \eta\|^2\right)^{3/2}} + \kappa(\phi_2 * \Delta_x \eta) \left(\frac{1}{\left(1 + \|\phi_1 * \nabla_x \eta\|^2\right)^{3/2}} - \frac{1}{\left(1 + \|\phi_2 * \nabla_x \eta\|^2\right)^{3/2}} \right). \end{aligned}$$

Using the inequality $|(1 + x^2)^{-3/2} - (1 + y^2)^{-3/2}| \leq \frac{48}{25\sqrt{5}}|x - y|$, we obtain

$$\|\operatorname{div}_x(\nu(\phi_1) - \nu(\phi_2))\|_{L^2(\mathbb{R}^2; \mathbb{R})}$$

$$\begin{aligned} &\leq \kappa \|(\phi_1 - \phi_2) * \Delta_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R})} + \frac{48}{25\sqrt{5}} \kappa \|\phi_2 * \Delta_x \eta\|_{L^\infty(\mathbb{R}^2; \mathbb{R})} \|(\phi_1 - \phi_2) * \nabla_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} \\ &\leq \kappa \|\phi_1 - \phi_2\|_{L^1(\mathbb{R}^2; \mathbb{R})} \left(\|\Delta_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R})} + \frac{48}{25\sqrt{5}} \|\nabla_x \eta\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} \|\Delta_x \eta\|_{L^\infty(\mathbb{R}^2; \mathbb{R})} \|\phi_2\|_{L^1(\mathbb{R}^2; \mathbb{R})} \right). \end{aligned}$$

□

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