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Contact Logic is finitary for unification with constants

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Abstract. Contact Logic is a formalism for reasoning about the relation of contact between regions. We introduce a new inference problem for Contact Logic, the unification problem, which extends the validity problem by allowing one to replace variables by terms before testing for validity. Our main result is the proof that unification with constants in Contact Logic is finitary.

Keywords: Contact Logic, unification with constants, unification type.

1 Introduction

Contact Logic is a formalism for reasoning about the relation of contact between regions represented by Boolean terms. Its language is obtained from the language of Boolean algebras by the addition of a binary predicate representing this relation. Its role in artificial intelligence and computer science stems from the importance of spatial information in systems for natural language understanding, robotic navigation, etc [8, 22]. Different variants of Contact Logic have been proposed [19–21, 24, 26]. Some of them are based on the predicate of boundedness or the predicate of connectedness. Their semantics can be given by interpreting terms and formulas either in topological spaces [9–11], or in frames [7, 25].

The above-mentioned applications of Contact Logic require the resolution of unification problems. There is a wide variety of situations where unification problems arise. Suppose the formula $\varphi(p_1, \dots, p_m)$ describes a given geographic configuration of constant regions p_1, \dots, p_m and the formula $\psi(x_1, \dots, x_n)$ represents a desirable geographic property of variable regions x_1, \dots, x_n . It may happen that $\varphi(p_1, \dots, p_m) \rightarrow \psi(x_1, \dots, x_n)$ is not valid in the considered geographic environment. Hence, one may ask whether there are n -tuples (a_1, \dots, a_n)

of terms such that $\varphi(p_1, \dots, p_m) \rightarrow \psi(a_1, \dots, a_n)$ is valid in this environment. Moreover, one may be interested to obtain, if possible, the most general n -tuples (a_1, \dots, a_n) of terms such that $\varphi(p_1, \dots, p_m) \rightarrow \psi(a_1, \dots, a_n)$ is valid.

Little is known about the unification problem in Contact Logic. For example, without the unary predicate of connectedness, elementary unification is NP -complete whereas unification with constants is decidable, its exact complexity being still unknown. See [5] for details about the computability of the unification problem in Contact Logic without the unary predicate of connectedness. A logic is said to be unitary when the solutions of a unification problem can always be represented by a most general one whereas it is said to be finitary when the solutions of a unification problem can always be represented by finitely many maximal ones. In this paper, we prove that unification with constants in Contact Logic either with, or without the unary predicate of connectedness is finitary.

2 Syntax

We adopt the standard rules for omission of the parentheses.

Atoms Let **CON** be a countable set of *propositional constants* (with typical members denoted p, q , etc) and **VAR** be a countable set of *propositional variables* (with typical members denoted x, y , etc). Let (p_1, p_2, \dots) be an enumeration of **CON** without repetitions and (x_1, x_2, \dots) be an enumeration of **VAR** without repetitions. A *propositional atom* is either a constant, or a variable. Let **ATO** be the set of all atoms (with typical members denoted α, β , etc).

Terms The *Boolean terms* (denoted a, b , etc) are defined as follows:

$$- a ::= \alpha \mid 0 \mid a^* \mid (a \cup b).$$

The other Boolean constructs for terms (for instance, 1 and \cap) are defined as usual. We use the following notations for terms: a^0 for a^* and a^1 for a . Reading terms as regions, the constructs 0 , $*$ and \cup should be regarded as the empty region, the complement operation and the union operation. As a result, the constructs 1 and \cap should be regarded as the full region and the intersection operation. For all nonnegative integers m, n , let **TER** $_{m,n}$ be the set of all terms whose constants form a subset of $\{p_1, \dots, p_m\}$ and whose variables form a subset of $\{x_1, \dots, x_n\}$. Let **TER** be the set of all terms.

Formulas The *formulas* (denoted φ, ψ , etc) are defined as follows:

$$- \varphi ::= C(a, b) \mid a \equiv b \mid \text{con}(a) \mid \perp \mid \neg\varphi \mid (\varphi \vee \psi).$$

Here, a and b are terms whereas C is the binary predicate of contact, \equiv is the binary predicate of equality and con is the unary predicate of connectedness. For C , \equiv and con , we propose the following readings:

- $C(a, b)$: “the regions denoted a and b are in contact”,
- $a \equiv b$: “the regions denoted a and b are equal”,
- $con(a)$: “the region denoted a is connected”.

The other connectives for formulas (for instance, \top and \wedge) are defined as usual. We use the following notation for formulas: $a \leq b$ for $a \cup b \equiv b$. A formula φ is *equational* if \equiv is the only predicate possibly occurring in φ . For all nonnegative integers m, n , let $\mathbf{FOR}_{m,n}$ be the set of all formulas whose constants form a subset of $\{p_1, \dots, p_m\}$ and whose variables form a subset of $\{x_1, \dots, x_n\}$. Let \mathbf{FOR} be the set of all formulas. Note that \mathbf{FOR} is denoted $\mathcal{C}c$ in [19–21] and L^c in [24].

Example 1. Within the context of the unification problem, we will interest with the following formulas:

- $\varphi_{01} = x \equiv 0 \vee x \equiv 1$,
- $\varphi_{pq} = C(p, q) \rightarrow x \neq 0 \wedge x \leq p \cup q$.

The first formula is equational. It says that the region denoted x is either the empty region, or the full region. As for the second formula, it says that if the regions denoted p and q are in contact then the region denoted x is a nonempty region included in the region denoted $p \cup q$.

Substitutions A *substitution* is a function $\sigma : \mathbf{VAR} \rightarrow \mathbf{TER}$ which moves at most finitely many variables. Given a substitution σ , let $\bar{\sigma} : \mathbf{TER} \cup \mathbf{FOR} \rightarrow \mathbf{TER} \cup \mathbf{FOR}$ be the endomorphism such that for all variables x , $\bar{\sigma}(x) = \sigma(x)$. The *composition* of the substitutions σ and τ is the substitution $\sigma \circ \tau$ such that for all $x \in \mathbf{VAR}$, $(\sigma \circ \tau)(x) = \bar{\tau}(\sigma(x))$.

Example 2. Within the context of the unification problem, we will interest with the following substitutions:

- the substitution σ_0 such that $\sigma_0(x) = 0$ and for all variables y , if $x \neq y$ then $\sigma_0(y) = y$,
- the substitution σ_1 such that $\sigma_1(x) = 1$ and for all variables y , if $x \neq y$ then $\sigma_1(y) = y$,
- the substitution σ_p such that $\sigma_p(x) = p \cup (q \cap x)$ and for all variables y , if $x \neq y$ then $\sigma_p(y) = y$,
- the substitution σ_q such that $\sigma_q(x) = q \cup (p \cap x)$ and for all variables y , if $x \neq y$ then $\sigma_q(y) = y$.

3 Semantics

The semantics of Contact Logic can be given by interpreting terms and formulas either in topological spaces [7, 26], or in frames [7, 25].

Topological semantics The best way to understand the meaning of the predicates of contact and connectedness is by interpreting them in *topological spaces*, i.e. structures of the form (X, τ) where X is a nonempty set (with typical members denoted A, B , etc) and τ is a set of subsets of X such that

- $\emptyset \in \tau$,
- $X \in \tau$,
- if $\{A_i : i \in I\}$ is a finite subset of τ then $\bigcap\{A_i : i \in I\} \in \tau$,
- if $\{A_i : i \in I\}$ is a subset of τ then $\bigcup\{A_i : i \in I\} \in \tau$.

The subsets of X in τ are called *open sets* whereas their complements are called *closed sets*. The most important topological spaces considered within the context of Contact Logic are for all $n \geq 1$, the topological space $(\mathbb{R}^n, \tau_{\mathbb{R}^n})$, i.e. the real space \mathbb{R}^n together with its ordinary topology $\tau_{\mathbb{R}^n}$. See [19–21, 26] for details. Let (X, τ) be a topological space. Let Int_τ and Cl_τ denote the *interior operator* and the *closure operator* in (X, τ) . A subset A of X is *regular closed* if $Cl_\tau(Int_\tau(A)) = A$. Regular closed subsets of X will also be called *regions*. It is well-known that the set $RC(X, \tau)$ of all regular closed subsets of X forms a Boolean algebra $(RC(X, \tau), 0_X, \star_X, \cup_X)$ where for all $A, B \in RC(X, \tau)$:

- $0_X = \emptyset$,
- $A^{\star_X} = Cl_\tau(X \setminus A)$,
- $A \cup_X B = A \cup B$.

As a result, for all $A, B \in RC(X, \tau)$, $1_X = X$ and $A \cap_X B = Cl_\tau(Int_\tau(A \cap B))$. Regions being regular closed subsets of X , two regions are in contact if they have a nonempty intersection whereas a region is connected if it cannot be represented as the union of two disjoint nonempty regions. For this reason, we associate to the predicates C and con the relations $C^{(X, \tau)}$ and $con^{(X, \tau)}$ on the set of all regular closed subsets of X as follows:

- $C^{(X, \tau)}(A, B)$ if $A \cap B \neq \emptyset$,
- $con^{(X, \tau)}(A)$ if for all nonempty regular closed subsets B, B' of X , if $B \cup B' = A$ then $B \cap B' \neq \emptyset$.

The following conditions hold for all regular closed subsets A, B of X :

- if $A \neq \emptyset$ then $C^{(X, \tau)}(A, A)$,
- if $C^{(X, \tau)}(A, B)$ then $C^{(X, \tau)}(B, A)$.

If τ only lets \emptyset and X being open then the following condition holds for all regular closed subsets A, B of X :

- if $A \neq \emptyset$ and $B \neq \emptyset$ then $C^{(X, \tau)}(A, B)$.

If X cannot be represented as the union of two disjoint nonempty open sets then the following conditions hold for all regular closed subsets A of X :

- if $A \neq \emptyset$ and $A \neq X$ then $C^{(X, \tau)}(A, Cl_\tau(X \setminus A))$,
- $con^{(X, \tau)}(X)$.

A *valuation* on X is a map \mathcal{V} associating with every atom α a regular closed subset $\mathcal{V}(\alpha)$ of X . Given a valuation \mathcal{V} on X , we define

- $\bar{\mathcal{V}}(\alpha) = \mathcal{V}(\alpha)$,
- $\bar{\mathcal{V}}(0) = \emptyset$,
- $\bar{\mathcal{V}}(a^*) = Cl_\tau(X \setminus \bar{\mathcal{V}}(a))$,
- $\bar{\mathcal{V}}(a \cup b) = \bar{\mathcal{V}}(a) \cup \bar{\mathcal{V}}(b)$.

Thus, every term is interpreted as a regular closed subset of X . A *model* on (X, τ) is a structure $\mathcal{M} = (X, \tau, \mathcal{V})$ where \mathcal{V} is a valuation on X . The connectives \perp , \neg and \vee being classically interpreted, the *satisfiability* of a formula φ in \mathcal{M} (in symbols $\mathcal{M} \models \varphi$) is defined as follows:

- $\mathcal{M} \models C(a, b)$ if $C^{(X, \tau)}(\bar{\mathcal{V}}(a), \bar{\mathcal{V}}(b))$,
- $\mathcal{M} \models a \equiv b$ if $\bar{\mathcal{V}}(a) = \bar{\mathcal{V}}(b)$,
- $\mathcal{M} \models con(a)$ if $con^{(X, \tau)}(\bar{\mathcal{V}}(a))$.

A formula φ is *valid* in (X, τ) if for all valuations \mathcal{V} on X , $(X, \tau, \mathcal{V}) \models \varphi$. A formula φ is *satisfiable* in (X, τ) if there exists a valuation \mathcal{V} on X such that $(X, \tau, \mathcal{V}) \models \varphi$.

Galton's perspective A relational perspective is suggested by Galton [13] who introduces the notion of adjacency space. Galton's spaces are frames (W, R) where W is a nonempty set of cells and R is an adjacency relation between cells. Galton defines regions to be sets of cells. He also defines two regions A and B to be in contact if some point in A is adjacent to some point in B . This definition relates Galton's adjacency spaces to the relational semantics of modal logic which makes it possible to use methods from modal logic for studying region-based theories of space. The truth is that the above-mentioned topological semantics and the relational perspective suggested by Galton are equivalent [24, 25]. In this paper, we adopt Galton's perspective.

Relational semantics A *frame* is a structure of the form (W, R) where W is a nonempty set (with typical members denoted s, t , etc) and R is a binary relation on W . Let (W, R) be a frame. We will denote by R^+ the transitive closure of R , i.e. the least transitive relation on W containing R . Subsets of W will also be called *regions*. Naturally, two regions are in contact if they have R -related elements whereas a region is connected if there is an R -path between any pair of elements in that region. For this reason, we associate to the predicates C and con the relations $C^{(W, R)}$ and $con^{(W, R)}$ on the set of all subsets of W as follows:

- $C^{(W, R)}(A, B)$ if $R \cap (A \times B) \neq \emptyset$,
- $con^{(W, R)}(A)$ if $A \times A \subseteq R^+$.

If R is reflexive and symmetric then the following conditions hold for all subsets A, B of W :

- if $A \neq \emptyset$ then $C^{(W, R)}(A, A)$,

- if $C^{(W,R)}(A, B)$ then $C^{(W,R)}(B, A)$.

If $R = W \times W$ then the following condition holds for all subsets A, B of W :

- if $A \neq \emptyset$ and $B \neq \emptyset$ then $C^{(W,R)}(A, B)$.

If $R^+ = W \times W$ then the following conditions hold for all subsets A of W :

- if $A \neq \emptyset$ and $A \neq W$ then $C^{(W,R)}(A, W \setminus A)$,
- $con^{(W,R)}(W)$.

A *valuation* on W is a map \mathcal{V} associating with every atom α a subset $\mathcal{V}(\alpha)$ of W . Given a valuation \mathcal{V} on W , we define

- $\bar{\mathcal{V}}(\alpha) = \mathcal{V}(\alpha)$,
- $\bar{\mathcal{V}}(0) = \emptyset$,
- $\bar{\mathcal{V}}(a^*) = W \setminus \bar{\mathcal{V}}(a)$,
- $\bar{\mathcal{V}}(a \cup b) = \bar{\mathcal{V}}(a) \cup \bar{\mathcal{V}}(b)$.

Thus, every term is interpreted as a subset of W . A *model* on (W, R) is a structure $\mathcal{M} = (W, R, \mathcal{V})$ where \mathcal{V} is a valuation on W . The connectives \perp , \neg and \vee being classically interpreted, the *satisfiability* of a formula φ in \mathcal{M} (in symbols $\mathcal{M} \models \varphi$) is defined as follows:

- $\mathcal{M} \models C(a, b)$ if $C^{(W,R)}(\bar{\mathcal{V}}(a), \bar{\mathcal{V}}(b))$,
- $\mathcal{M} \models a \equiv b$ if $\bar{\mathcal{V}}(a) = \bar{\mathcal{V}}(b)$,
- $\mathcal{M} \models con(a)$ if $con^{(W,R)}(\bar{\mathcal{V}}(a))$.

A formula φ is *valid* in (W, R) if for all valuations \mathcal{V} on W , $(W, R, \mathcal{V}) \models \varphi$. A formula φ is *satisfiable* in (W, R) if there exists a valuation \mathcal{V} on W such that $(W, R, \mathcal{V}) \models \varphi$.

Validity and satisfiability Let \mathcal{C} be a class of frames. We say that a formula φ is \mathcal{C} -*valid* if for all frames (W, R) in \mathcal{C} , φ is valid in (W, R) . We say that a formula φ is \mathcal{C} -*satisfiable* if there exists a frame (W, R) in \mathcal{C} such that φ is satisfiable in (W, R) . The \mathcal{C} -*satisfiability problem* consists in determining whether a given formula is \mathcal{C} -satisfiable. A frame (W, R) is *indiscrete* if $R = W \times W$. A frame (W, R) is *connected* if $R^+ = W \times W$. Let \mathcal{C}_{all} denote the class of all reflexive and symmetric frames, \mathcal{C}_{ind} denote the class of all reflexive, symmetric and indiscrete frames and \mathcal{C}_{con} denote the class of all reflexive, symmetric and connected frames. The topological counterparts of these classes of frames are the class of all topological spaces, the class of all indiscrete spaces (those spaces that only let \emptyset and the full set being open) and the class of all connected spaces (those spaces that cannot be represented as the union of two disjoint nonempty open sets). For *con*-free formulas, the satisfiability problem is known to be *NP*-complete in \mathcal{C}_{all} and \mathcal{C}_{ind} and *PSPACE*-complete in \mathcal{C}_{con} , the situation being generally more complex for arbitrary formulas. See [7, 19–21, 26] for details. The following formulas are \mathcal{C}_{all} -valid:

- $x \neq 0 \rightarrow C(x, x)$,

– $C(x, y) \rightarrow C(y, x)$.

The following formula is \mathcal{C}_{ind} -valid:

– $x \neq 0 \wedge y \neq 0 \rightarrow C(x, y)$.

The following formulas are \mathcal{C}_{con} -valid:

– $x \neq 0 \wedge x \neq 1 \rightarrow C(x, x^*)$,
– $con(1)$.

4 About classes of frames

Let \mathcal{C} be a class of frames. We say that \mathcal{C} *agrees with unions* if for all disjoint frames $(W, R), (W', R')$ in \mathcal{C} , there exists a frame (W'', R'') in \mathcal{C} such that $W \cup W' = W''$ and $R'' \not\subseteq Id_{W''}$. We should stress here that many natural classes of frames agree with unions. In particular,

Proposition 1. $\mathcal{C}_{all}, \mathcal{C}_{ind}$ and \mathcal{C}_{con} agree with unions.

Proof. By the definition of what it means for classes of frames to agree with unions.

We say that \mathcal{C} is *determined* by a formula φ if \mathcal{C} is the class of all frames validating φ .

Proposition 2. \mathcal{C}_{all} is determined by $x \neq 0 \rightarrow C(x, x)$ and $C(x, y) \rightarrow C(y, x)$, \mathcal{C}_{ind} is determined by $x \neq 0 \wedge y \neq 0 \rightarrow C(x, y)$ and \mathcal{C}_{con} is both determined by $x \neq 0 \wedge x \neq 1 \rightarrow C(x, x^*)$ and $con(1)$.

Proof. Left to the reader.

We say that \mathcal{C} is *large* if there exists a frame (W, R) in \mathcal{C} such that $\mathbf{Card}(W) \geq 3$ and $R \not\subseteq Id_W$. We should stress here that many natural classes of frames are large. In particular, obviously, if a nonempty class of frames agrees with unions then it is large. A valuation \mathcal{V} on a nonempty set W is *balanced* if for all terms a , either $\bar{\mathcal{V}}(a) = \emptyset$, or $\bar{\mathcal{V}}(a) = W$, or $\bar{\mathcal{V}}(a)$ is infinite and coinfinite. We say that \mathcal{C} is *balanced* if for all formulas φ , if φ is \mathcal{C} -satisfiable then there exists a countable frame (W, R) in \mathcal{C} and there exists a balanced valuation \mathcal{V} on W such that $(W, R, \mathcal{V}) \models \varphi$. We should stress here that many natural classes of frames are balanced. In particular,

Proposition 3. $\mathcal{C}_{all}, \mathcal{C}_{ind}$ and \mathcal{C}_{con} are balanced.

Proof. Let \mathcal{C} be one of the above-mentioned classes of frames. In order to demonstrate that \mathcal{C} is balanced, let φ be a \mathcal{C} -satisfiable formula. By [7, Theorem 4.2], φ is satisfiable in a finite frame of \mathcal{C} . Let (W, R) be a finite frame of \mathcal{C} and \mathcal{V} be a valuation on it such that $(W, R, \mathcal{V}) \models \varphi$. Let $W' = W \times \mathbb{N}$ and R' be the binary relation on W' defined by $(x, i)R'(y, j)$ if xRy . Since the frame (W, R) is finite, therefore the frame (W', R') is countable. Let \mathcal{V}' be the valuation on

(W', R') defined by $\mathcal{V}'(\alpha) = \mathcal{V}(\alpha) \times \mathbb{N}$ for each atom α . As the reader may easily verify by induction, $\mathcal{V}'(a) = \mathcal{V}(a) \times \mathbb{N}$ for each term a . It follows that the model (W', R', \mathcal{V}') is balanced. As the reader may easily verify by induction, $(W, R, \mathcal{V}) \models \psi$ iff $(W', R', \mathcal{V}') \models \psi$ for each formula ψ . Since $(W, R, \mathcal{V}) \models \varphi$, therefore $(W', R', \mathcal{V}') \models \varphi$. Finally, it suffices now to remark that if (W, R) is in \mathcal{C}_{all} (respectively, in \mathcal{C}_{ind} , in \mathcal{C}_{con}) then (W', R') is in \mathcal{C}_{all} (respectively, in \mathcal{C}_{ind} , in \mathcal{C}_{con}) too.

5 Unifiability

Now, following the standard notations and definitions about the unifiability problem in modal and description logics [2–4, 14–17, 23], we introduce the unifiability problem in Contact Logic. Let \mathcal{C} be a class of frames.

Unifiable formulas We say that a substitution σ is \mathcal{C} -equivalent to a substitution τ (in symbols $\sigma \simeq_{\mathcal{C}} \tau$) if for all variables x , $\sigma(x) \equiv \tau(x)$ is \mathcal{C} -valid.

Example 3. As the reader can easily show, the substitutions σ_0 and σ_1 introduced in Example 2 are respectively \mathcal{C} -equivalent to the substitutions σ'_0 and σ'_1 such that $\sigma'_0(x) = x \cap x^*$, $\sigma'_1(x) = x \cup x^*$ and for all variables y , if $x \neq y$ then $\sigma'_0(y) = y$ and $\sigma'_1(y) = y$.

We say that a substitution σ is *more \mathcal{C} -general* than a substitution τ (in symbols $\sigma \preceq_{\mathcal{C}} \tau$) if there exists a substitution v such that $\sigma \circ v \simeq_{\mathcal{C}} \tau$. In that case, we also say that τ is *less \mathcal{C} -general* than σ .

Example 4. As the reader can easily show, the substitutions σ_p and σ_q introduced in Example 2 are respectively less \mathcal{C} -general than the substitutions σ'_p and σ'_q such that $\sigma'_p(x) = x \cup p$, $\sigma'_q(x) = x \cup q$ and for all variables y , if $x \neq y$ then $\sigma'_p(y) = y$ and $\sigma'_q(y) = y$.

We say that a formula φ is \mathcal{C} -unifiable if there exists a substitution σ such that $\bar{\sigma}(\varphi)$ is \mathcal{C} -valid. In that case, we say that σ is a \mathcal{C} -unifier of φ .

Example 5. As the reader can easily show, the substitutions σ_0 and σ_1 introduced in Example 2 are \mathcal{C} -unifiers of the formula φ_{01} introduced in Example 1. In other respect, the substitutions σ_p and σ_q introduced in Example 2 are \mathcal{C} -unifiers of the formula φ_{pq} introduced in Example 1.

The \mathcal{C} -unifiability problem with constants consists in determining whether a given formula is \mathcal{C} -unifiable. We say that a set of \mathcal{C} -unifiers of a formula φ is *complete* if for all \mathcal{C} -unifiers σ of φ , there exists a \mathcal{C} -unifier τ of φ in that set such that $\tau \preceq_{\mathcal{C}} \sigma$. An important question is: when a formula is \mathcal{C} -unifiable, has it a minimal complete set of \mathcal{C} -unifiers? When the answer is “yes”, how large is this set?

Unification types We say that a \mathcal{C} -unifiable formula φ is \mathcal{C} -finitary if there exists a finite minimal complete set of \mathcal{C} -unifiers of φ but there exists no with cardinality 1. We say that a \mathcal{C} -unifiable formula φ is \mathcal{C} -unitary if there exists a minimal complete set of \mathcal{C} -unifiers of φ with cardinality 1. We say that unification in \mathcal{C} is *finitary* if every \mathcal{C} -unifiable formula is either \mathcal{C} -finitary, or \mathcal{C} -unitary and there exists a \mathcal{C} -finitary formula. We say that unification in \mathcal{C} is *unitary* if every \mathcal{C} -unifiable formula is \mathcal{C} -unitary. See [6, 12, 18] for an introduction to the unification types in logics.

6 Examples

From now on, when we write “**CPL**”, we mean “Classical Propositional Logic”. Let \mathcal{C} be a class of frames.

As remarked above, the substitutions σ_0 and σ_1 introduced in Example 2 are \mathcal{C} -unifiers of the formula φ_{01} introduced in Example 1. Moreover,

- $\sigma_0 \not\preceq_{\mathcal{C}} \sigma_1$ and $\sigma_1 \not\preceq_{\mathcal{C}} \sigma_0$.

To see why, suppose $\sigma_0 \preceq_{\mathcal{C}} \sigma_1$, or $\sigma_1 \preceq_{\mathcal{C}} \sigma_0$. Without loss of generality, suppose $\sigma_0 \preceq_{\mathcal{C}} \sigma_1$. Let τ be a substitution such that $\sigma_0 \circ \tau \simeq_{\mathcal{C}} \sigma_1$. Hence, $\bar{\tau}(\sigma_0(x)) \equiv \sigma_1(x)$ is \mathcal{C} -valid. Since $\sigma_0(x) = 0$ and $\sigma_1(x) = 1$, therefore $0 \equiv 1$ is \mathcal{C} -valid: a contradiction. In other respect,

- if \mathcal{C} agrees with unions then σ_0 and σ_1 form a complete set of \mathcal{C} -unifiers of φ_{01} .

To see why, suppose \mathcal{C} agrees with unions and σ_0 and σ_1 do not form a complete set of \mathcal{C} -unifiers of φ_{01} . Let τ be a \mathcal{C} -unifier of φ_{01} such that neither $\sigma_0 \preceq_{\mathcal{C}} \tau$, nor $\sigma_1 \preceq_{\mathcal{C}} \tau$. Thus, neither $0 \equiv \tau(x)$ is \mathcal{C} -valid, nor $1 \equiv \tau(x)$ is \mathcal{C} -valid. Let (W, R) and (W', R') be disjoint frames in \mathcal{C} , \mathcal{V} be a valuation on W and \mathcal{V}' be a valuation on W' such that neither $\bar{\mathcal{V}}(\tau(x)) = \emptyset$, nor $\bar{\mathcal{V}}'(\tau(x)) = W'$. Since \mathcal{C} agrees with unions, therefore let (W'', R'') be a frame in \mathcal{C} such that $W \cup W' = W''$. Let \mathcal{V}'' be a valuation on W'' such that for all atoms α , $\mathcal{V}''(\alpha) = \mathcal{V}(\alpha) \cup \mathcal{V}'(\alpha)$. Obviously, for all terms a , $\bar{\mathcal{V}}''(a) = \bar{\mathcal{V}}(a) \cup \bar{\mathcal{V}}'(a)$. Since neither $\bar{\mathcal{V}}(\tau(x)) = \emptyset$, nor $\bar{\mathcal{V}}'(\tau(x)) = W'$, therefore neither $\bar{\mathcal{V}}''(\tau(x)) = \emptyset$, nor $\bar{\mathcal{V}}''(\tau(x)) = W''$. Consequently, τ is not a \mathcal{C} -unifier of φ_{01} : a contradiction. Remark that the above argument shows that if \mathcal{C} agrees with unions then φ_{01} is either \mathcal{C} -unitary, or \mathcal{C} -finitary. Finally,

- if \mathcal{C} agrees with unions then φ_{01} is \mathcal{C} -finitary.

To see why, suppose \mathcal{C} agrees with unions and φ_{01} is not \mathcal{C} -finitary. Since if \mathcal{C} agrees with unions then φ_{01} is either \mathcal{C} -unitary, or \mathcal{C} -finitary, therefore φ_{01} is \mathcal{C} -unitary. Let τ be a \mathcal{C} -unifier of φ_{01} such that $\tau \not\preceq_{\mathcal{C}} \sigma_0$ and $\tau \not\preceq_{\mathcal{C}} \sigma_1$. Since σ_0 and σ_1 form a complete set of \mathcal{C} -unifiers of φ_{01} , therefore $\sigma_0 \preceq_{\mathcal{C}} \tau$, or $\sigma_1 \preceq_{\mathcal{C}} \tau$. Without loss of generality, suppose $\sigma_0 \preceq_{\mathcal{C}} \tau$. Since $\tau \not\preceq_{\mathcal{C}} \sigma_1$, therefore $\sigma_0 \preceq_{\mathcal{C}} \sigma_1$: a contradiction.

As remarked above, the substitutions σ_p and σ_q introduced in Example 2 are \mathcal{C} -unifiers of the formula φ_{pq} introduced in Example 1. Moreover,

- if $p \neq q$ then $\sigma_p \not\leq_{\mathcal{C}} \sigma_q$ and $\sigma_q \not\leq_{\mathcal{C}} \sigma_p$.

To see why, suppose $p \neq q$ and $\sigma_p \leq_{\mathcal{C}} \sigma_q$, or $\sigma_q \leq_{\mathcal{C}} \sigma_p$. Without loss of generality, suppose $\sigma_p \leq_{\mathcal{C}} \sigma_q$. Let τ be a substitution such that $\sigma_p \circ \tau \simeq_{\mathcal{C}} \sigma_q$. Hence, $\bar{\tau}(\sigma_p(x)) \equiv \sigma_q(x)$ is \mathcal{C} -valid. Since $\sigma_p(x) = p \cup (q \cap x)$ and $\sigma_q(x) = q \cup (p \cap x)$, therefore $p \cup (q \cap \tau(x)) \equiv q \cup (p \cap x)$ is \mathcal{C} -valid. Thus, after replacing in $p \cup (q \cap \tau(x)) \equiv q \cup (p \cap x)$ each occurrence of q and x by 0 , $p \equiv 0$ is \mathcal{C} -valid: a contradiction. In other respect,

- if \mathcal{C} agrees with unions and $p \neq q$ then σ_p and σ_q form a complete set of \mathcal{C} -unifiers of φ_{pq} .

To see why, suppose \mathcal{C} agrees with unions, $p \neq q$ and σ_p and σ_q do not form a complete set of \mathcal{C} -unifiers of φ_{pq} . As a result, remark that \mathcal{C} is large. Let τ be a \mathcal{C} -unifier of φ_{pq} such that neither $\sigma_p \leq_{\mathcal{C}} \tau$, nor $\sigma_q \leq_{\mathcal{C}} \tau$. Thus, $C(p, q) \rightarrow \tau(x) \neq 0 \wedge \tau(x) \leq p \cup q$ is \mathcal{C} -valid. Without loss of generality, we can assume that $\tau(x) = (p^0 \cap q^0 \cap a_{00}) \cup (p^0 \cap q^1 \cap a_{01}) \cup (p^1 \cap q^0 \cap a_{10}) \cup (p^1 \cap q^1 \cap a_{11})$ for some $\{p, q\}$ -free terms $a_{00}, a_{01}, a_{10}, a_{11}$. Since \mathcal{C} is large, therefore let (W, R) be a frame in \mathcal{C} such that $\mathbf{Card}(W) \geq 3$ and $R \not\subseteq Id_W$. As a result, remark that $R \neq \emptyset$. Suppose we succeed in proving the following:

- (1) $a_{00} \equiv 0$ is \mathcal{C} -valid,
- (2) $a_{11} \equiv 1$ is \mathcal{C} -valid,
- (3) $a_{01} \cup a_{10} \equiv 1$ is \mathcal{C} -valid,
- (4) $a_{01} \leq a_{10}$ is \mathcal{C} -valid, or $a_{10} \leq a_{01}$ is \mathcal{C} -valid.

Consequently, by (1) and (2), considered as formulas in **CPL**, $\tau(x)$ and $(p^0 \cap q^1 \cap a_{01}) \cup (p^1 \cap q^0 \cap a_{10}) \cup (p^1 \cap q^1)$ are equivalent. Moreover, by (3) and (4), $a_{10} \equiv 1$ is \mathcal{C} -valid, or $a_{01} \equiv 1$ is \mathcal{C} -valid. In the former case, considered as formulas in **CPL**, $\tau(x)$ and $p \cup (q \cap a_{01})$ are equivalent. Hence, $\sigma_p \leq_{\mathcal{C}} \tau$: a contradiction. In the latter case, considered as formulas in **CPL**, $\tau(x)$ and $q \cup (p \cap a_{10})$ are equivalent. Thus, $\sigma_q \leq_{\mathcal{C}} \tau$: a contradiction. Consequently, it remains to prove (1), (2), (3) and (4).

(1) Suppose $a_{00} \equiv 0$ is not \mathcal{C} -valid. Let \mathcal{V} be a valuation on (W, R) such that $\bar{\mathcal{V}}(a_{00}) = W$. Since a_{00} is $\{p, q\}$ -free, $\mathbf{Card}(W) \geq 3$ and $R \neq \emptyset$, therefore we can assume that $\mathcal{V}(p) \cup \mathcal{V}(q) \neq W$ and $R \cap (\mathcal{V}(p) \times \mathcal{V}(q)) \neq \emptyset$. Hence, $(W, R, \mathcal{V}) \models C(p, q)$ and $\bar{\mathcal{V}}(\tau(x)) \not\subseteq \bar{\mathcal{V}}(p \cup q)$: a contradiction with the \mathcal{C} -validity of $C(p, q) \rightarrow \tau(x) \neq 0 \wedge \tau(x) \leq p \cup q$. Thus, $a_{00} \equiv 0$ is \mathcal{C} -valid and (1) is proved. Moreover, considered as formulas in **CPL**, $\tau(x)$ and $(p^0 \cap q^1 \cap a_{01}) \cup (p^1 \cap q^0 \cap a_{10}) \cup (p^1 \cap q^1 \cap a_{11})$ are equivalent.

(2) Suppose $a_{11} \equiv 1$ is not \mathcal{C} -valid. Let \mathcal{V} be a valuation on (W, R) such that $\bar{\mathcal{V}}(a_{11}) = \emptyset$. Since a_{11} is $\{p, q\}$ -free and $R \neq \emptyset$, therefore we can assume that

$\mathcal{V}(p) = \mathcal{V}(q)$ and $R \cap (\mathcal{V}(p) \times \mathcal{V}(q)) \neq \emptyset$. Consequently, $(W, R, \mathcal{V}) \models C(p, q)$ and $\bar{\mathcal{V}}(\tau(x)) = \emptyset$: a contradiction with the \mathcal{C} -validity of $C(p, q) \rightarrow \tau(x) \neq 0 \wedge \tau(x) \leq p \cup q$. Hence, $a_{11} \equiv 1$ is \mathcal{C} -valid and (2) is proved. Moreover, considered as formulas in **CPL**, $\tau(x)$ and $(p^0 \cap q^1 \cap a_{01}) \cup (p^1 \cap q^0 \cap a_{10}) \cup (p^1 \cap q^1)$ are equivalent.

(3) Suppose $a_{01} \cup a_{10} \equiv 1$ is not \mathcal{C} -valid. Let \mathcal{V} be a valuation on (W, R) such that $\bar{\mathcal{V}}(a_{01}) = \emptyset$ and $\bar{\mathcal{V}}(a_{10}) = \emptyset$. Since $p \neq q$, a_{01} and a_{10} are $\{p, q\}$ -free and $R \not\subseteq Id_W$, therefore we can assume that $\mathcal{V}(p) \cap \mathcal{V}(q) = \emptyset$ and $R \cap (\mathcal{V}(p) \times \mathcal{V}(q)) \neq \emptyset$. Thus, $(W, R, \mathcal{V}) \models C(p, q)$ and $\bar{\mathcal{V}}(\tau(x)) = \emptyset$: a contradiction with the \mathcal{C} -validity of $C(p, q) \rightarrow \tau(x) \neq 0 \wedge \tau(x) \leq p \cup q$. Consequently, $a_{01} \cup a_{10} \equiv 1$ is \mathcal{C} -valid and (3) is proved.

(4) Suppose neither $a_{01} \leq a_{10}$ is \mathcal{C} -valid, nor $a_{10} \leq a_{01}$ is \mathcal{C} -valid. By using an argument similar to the one showing above that σ_0 and σ_1 form a complete set of unifiers of φ_{01} , we obtain a frame (W'', R'') in \mathcal{C} such that $R'' \not\subseteq Id_W$ and a valuation \mathcal{V}'' on W'' such that $\bar{\mathcal{V}}''(a_{01}) \not\subseteq \bar{\mathcal{V}}''(a_{10})$ and $\bar{\mathcal{V}}''(a_{10}) \not\subseteq \bar{\mathcal{V}}''(a_{01})$. Since $p \neq q$ and a_{01} and a_{10} are $\{p, q\}$ -free, therefore we can assume that $\mathcal{V}''(p)$ and $\mathcal{V}''(q)$ are such that $\mathcal{V}''(p) \subseteq \bar{\mathcal{V}}''(a_{01})$, $\mathcal{V}''(p) \cap \bar{\mathcal{V}}''(a_{10}) = \emptyset$, $\mathcal{V}''(q) \cap \bar{\mathcal{V}}''(a_{01}) = \emptyset$, $\mathcal{V}''(q) \subseteq \bar{\mathcal{V}}''(a_{10})$ and $R \cap (\mathcal{V}''(p) \times \mathcal{V}''(q)) \neq \emptyset$. Thus, $\bar{\mathcal{V}}''(\tau(x)) = \emptyset$ and $(W'', R'', \mathcal{V}'') \models C(p, q)$: a contradiction with the \mathcal{C} -validity of $C(p, q) \rightarrow \tau(x) \neq 0 \wedge \tau(x) \leq p \cup q$. Thus, $a_{01} \leq a_{10}$ is \mathcal{C} -valid, or $a_{10} \leq a_{01}$ is \mathcal{C} -valid and (4) is proved.

Remark that the above argument shows that if \mathcal{C} agrees with unions and $p \neq q$ then φ_{pq} is either \mathcal{C} -unitary, or \mathcal{C} -finitary. Finally,

- if \mathcal{C} agrees with unions and $p \neq q$ then φ_{pq} is \mathcal{C} -finitary.

To see why, suppose \mathcal{C} agrees with unions, $p \neq q$ and φ_{pq} is not \mathcal{C} -finitary. Since if \mathcal{C} agrees with unions and $p \neq q$ then φ_{pq} is either \mathcal{C} -unitary, or \mathcal{C} -finitary, therefore φ_{pq} is either \mathcal{C} -unitary, or \mathcal{C} -finitary. Since φ_{pq} is not \mathcal{C} -finitary, therefore φ_{pq} is \mathcal{C} -unitary. Let τ be a \mathcal{C} -unifier of φ_{pq} such that $\tau \preceq_{\mathcal{C}} \sigma_p$ and $\tau \preceq_{\mathcal{C}} \sigma_q$. Since σ_p and σ_q form a complete set of \mathcal{C} -unifiers of φ_{pq} , therefore $\sigma_p \preceq_{\mathcal{C}} \tau$, or $\sigma_q \preceq_{\mathcal{C}} \tau$. Without loss of generality, suppose $\sigma_p \preceq_{\mathcal{C}} \tau$. Since $\tau \preceq_{\mathcal{C}} \sigma_q$, therefore $\sigma_p \preceq_{\mathcal{C}} \sigma_q$: a contradiction.

7 Monomials and polynomials

The purpose of this section is to introduce definitions and properties about terms. These definitions and properties are purely Boolean. Let k, m, n be nonnegative integers. An m -vector is a map \mathbf{s} associating with every positive integer $i \leq m$ an element $\mathbf{s}(i)$ of $\{0, 1\}$. A (k, m, n) -correspondence is a map f associating with every m -vector \mathbf{s} a function $f_{\mathbf{s}} : \{0, 1\}^k \rightarrow \{0, 1\}^n$. Let f be a (k, m, n) -correspondence. An n -monomial is a term of the form

$$- x_1^{\beta_1} \cap \dots \cap x_n^{\beta_n}$$

where $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$. For all m -vectors \mathbf{s} , considering a term a in $\mathbf{TER}_{m,n}$ as a formula in Classical Propositional Logic (**CPL**), we define:

$$- \text{mon}_{\mathbf{s}}(n, a) = \{x_1^{\beta_1} \cap \dots \cap x_n^{\beta_n} : a \text{ is a tautological consequence of } p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap x_1^{\beta_1} \cap \dots \cap x_n^{\beta_n}\}.$$

Hence, for all m -vectors \mathbf{s} , considering a as a formula in **CPL**, $\text{mon}_{\mathbf{s}}(n, a)$ is nothing but the set of all conjunctions of the form $x_1^{\beta_1} \cap \dots \cap x_n^{\beta_n}$ that appear in the Disjunctive Normal Form of a in conjunction with the conjunction $p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)}$. An n -polynomial is a term of the form

$$- (x_1^{\beta_{11}} \cap \dots \cap x_n^{\beta_{1n}}) \cup \dots \cup (x_1^{\beta_{j1}} \cap \dots \cap x_n^{\beta_{jn}})$$

where j is a nonnegative integer and $(\beta_{11}, \dots, \beta_{1n}), \dots, (\beta_{j1}, \dots, \beta_{jn}) \in \{0, 1\}^n$. Remark that for all m -vectors \mathbf{s} and for all terms a in $\mathbf{TER}_{m,n}$, $\bigcup \text{mon}_{\mathbf{s}}(n, a)$ is an n -polynomial.

Proposition 4. *Let $a \in \mathbf{TER}_{m,n}$. Considered as formulas in **CPL**, the following terms are equivalent:*

$$- a,$$

$$- \bigcup \{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap \bigcup \text{mon}_{\mathbf{s}}(n, a) : \mathbf{s} \text{ is an } m\text{-vector}\}.$$

Proof. By the definition of $\text{mon}_{\mathbf{s}}$, \mathbf{s} being an arbitrary m -vector.

For all positive integers $i \leq n$, let $\pi_i : \{0, 1\}^n \rightarrow \{0, 1\}$ be the function such that for all $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$,

$$- \pi_i(\beta_1, \dots, \beta_n) = \beta_i.$$

For all m -vectors \mathbf{s} and for all $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$, we define:

$$- f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n) = \{(\alpha_1, \dots, \alpha_k) \in \{0, 1\}^k : f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = (\beta_1, \dots, \beta_n)\}.$$

Obviously, for all m -vectors \mathbf{s} and for all $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$, $f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n) \subseteq \{0, 1\}^k$. For all m -vectors \mathbf{s} and for all positive integers $i \leq n$, we define:

$$- \Delta_{\mathbf{s},i} = \{(\alpha_1, \dots, \alpha_k) \in \{0, 1\}^k : \pi_i(f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k)) = 1\},$$

$$- c_{\mathbf{s},i} = \bigcup \{x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} : (\alpha_1, \dots, \alpha_k) \in \Delta_{\mathbf{s},i}\}.$$

Obviously, for all positive integers $i \leq n$, $\Delta_{\mathbf{s},i} \subseteq \{0, 1\}^k$ and $c_{\mathbf{s},i}$ is a k -polynomial. Remark that $\Delta_{\mathbf{s},i}$ and $c_{\mathbf{s},i}$ depend on f — more precisely, on $f_{\mathbf{s}}$ — too.

Proposition 5. *For all m -vectors \mathbf{s} and for all $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$, considered as formulas in **CPL**, the following terms are equivalent:*

$$- \bigcup \{x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} : (\alpha_1, \dots, \alpha_k) \in f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n)\},$$

$$- c_{\mathbf{s},1}^{\beta_1} \cap \dots \cap c_{\mathbf{s},n}^{\beta_n}.$$

Proof. Let \mathbf{s} be an m -vector and $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$. It suffices to show that considered as formulas in **CPL**, for all $\theta_1, \dots, \theta_k \in \{0, 1\}$, if x_1, \dots, x_k are respectively interpreted by $\theta_1, \dots, \theta_k$ then $\bigcup\{x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} : (\alpha_1, \dots, \alpha_k) \in f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n)\}$ is equivalent to 1 iff $c_{\mathbf{s},1}^{\beta_1} \cap \dots \cap c_{\mathbf{s},n}^{\beta_n}$ is equivalent to 1. Let $\theta_1, \dots, \theta_k \in \{0, 1\}$. Let x_1, \dots, x_k be respectively interpreted by $\theta_1, \dots, \theta_k$.

Suppose $\bigcup\{x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} : (\alpha_1, \dots, \alpha_k) \in f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n)\}$ is equivalent to 1. Hence, $(\theta_1, \dots, \theta_k) \in f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n)$. Thus, $f_{\mathbf{s}}(\theta_1, \dots, \theta_k) = (\beta_1, \dots, \beta_n)$. For the sake of the contradiction, suppose $c_{\mathbf{s},1}^{\beta_1} \cap \dots \cap c_{\mathbf{s},n}^{\beta_n}$ is equivalent to 0. Let $i \leq n$ be a positive integer such that $c_{\mathbf{s},i}^{\beta_i}$ is equivalent to 0. Since either $\beta_i = 0$, or $\beta_i = 1$, therefore we have to consider two cases. In the former case, $\beta_i = 0$ and therefore $\bigcup\{x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} : (\alpha_1, \dots, \alpha_k) \in \Delta_{\mathbf{s},i}\}$ is equivalent to 1. Consequently, $(\theta_1, \dots, \theta_k) \in \Delta_{\mathbf{s},i}$. Hence, $\pi_i(f_{\mathbf{s}}(\theta_1, \dots, \theta_k)) = 1$. Since $f_{\mathbf{s}}(\theta_1, \dots, \theta_k) = (\beta_1, \dots, \beta_n)$, therefore $\beta_i = 1$: a contradiction. In the latter case, $\beta_i = 1$ and therefore $\bigcup\{x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} : (\alpha_1, \dots, \alpha_k) \in \Delta_{\mathbf{s},i}\}$ is equivalent to 0. Thus, $(\theta_1, \dots, \theta_k) \notin \Delta_{\mathbf{s},i}$. Hence, $\pi_i(f_{\mathbf{s}}(\theta_1, \dots, \theta_k)) = 0$. Since $f_{\mathbf{s}}(\theta_1, \dots, \theta_k) = (\beta_1, \dots, \beta_n)$, therefore $\beta_i = 0$: a contradiction.

Suppose $c_{\mathbf{s},1}^{\beta_1} \cap \dots \cap c_{\mathbf{s},n}^{\beta_n}$ is equivalent to 1. Let $i \leq n$ be an arbitrary positive integer. Since $c_{\mathbf{s},1}^{\beta_1} \cap \dots \cap c_{\mathbf{s},n}^{\beta_n}$ is equivalent to 1, therefore $c_{\mathbf{s},i}^{\beta_i}$ is equivalent to 1. Since either $\beta_i = 0$, or $\beta_i = 1$, therefore we have to consider two cases. In the former case, $\beta_i = 0$ and therefore $c_{\mathbf{s},i}$ is equivalent to 0. Hence, $(\theta_1, \dots, \theta_k) \notin \Delta_{\mathbf{s},i}$. Thus, $\pi_i(f_{\mathbf{s}}(\theta_1, \dots, \theta_k)) = 0$. Since $\beta_i = 0$, therefore $\pi_i(f_{\mathbf{s}}(\theta_1, \dots, \theta_k)) = \beta_i$. In the latter case, $\beta_i = 1$ and therefore $c_{\mathbf{s},i}$ is equivalent to 1. Consequently, $(\theta_1, \dots, \theta_k) \in \Delta_{\mathbf{s},i}$. Hence, $\pi_i(f_{\mathbf{s}}(\theta_1, \dots, \theta_k)) = 1$. Since $\beta_i = 1$, therefore $\pi_i(f_{\mathbf{s}}(\theta_1, \dots, \theta_k)) = \beta_i$. In both cases, $\pi_i(f_{\mathbf{s}}(\theta_1, \dots, \theta_k)) = \beta_i$. Since i was arbitrary, therefore $f_{\mathbf{s}}(\theta_1, \dots, \theta_k) = (\beta_1, \dots, \beta_n)$. Thus, $(\theta_1, \dots, \theta_k) \in f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n)$. Consequently, $\bigcup\{x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} : (\alpha_1, \dots, \alpha_k) \in f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n)\}$ is equivalent to 1.

The property about m -vector \mathbf{s} and $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$ described in Proposition 5 is purely Boolean. It will be used in the proof of Proposition 6 which will itself be used in the proof of Proposition 12, our main result.

8 Equivalence relations between tuples of bits

Let k, m, n be nonnegative integers. Let $(a_1, \dots, a_n) \in \mathbf{TER}_{m,k}^n$. For all m -vectors \mathbf{s} , we define on $\{0, 1\}^k$ the equivalence relation $\sim_{(a_1, \dots, a_n)}^{k, \mathbf{s}}$ as follows:

- $(\alpha_1, \dots, \alpha_k) \sim_{(a_1, \dots, a_n)}^{k, \mathbf{s}} (\alpha'_1, \dots, \alpha'_k)$ if for all positive integers $i \leq n$, $x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} \in \text{mon}_{\mathbf{s}}(k, a_i)$ iff $x_1^{\alpha'_1} \cap \dots \cap x_k^{\alpha'_k} \in \text{mon}_{\mathbf{s}}(k, a_i)$.

Obviously, for all m -vectors \mathbf{s} , $\sim_{(a_1, \dots, a_n)}^{k, \mathbf{s}}$ has at most 2^n equivalence classes on $\{0, 1\}^k$. Let f be a (k, m, n) -correspondence such that for all m -vectors \mathbf{s} and

for all $(\alpha_1, \dots, \alpha_k), (\alpha'_1, \dots, \alpha'_k) \in \{0, 1\}^k$, if $f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = f_{\mathbf{s}}(\alpha'_1, \dots, \alpha'_k)$ then $(\alpha_1, \dots, \alpha_k) \sim_{(a_1, \dots, a_n)}^{k, \mathbf{s}} (\alpha'_1, \dots, \alpha'_k)$. For all m -vectors \mathbf{s} , by means of f — more precisely, of $f_{\mathbf{s}}$ —, we define the n -tuple $(b_{\mathbf{s},1}, \dots, b_{\mathbf{s},n})$ of n -polynomials as follows:

$$- b_{\mathbf{s},i} = \bigcup \{x_1^{\beta_1} \cap \dots \cap x_n^{\beta_n} : x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} \in \text{mon}_{\mathbf{s}}(k, a_i) \text{ and } f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = (\beta_1, \dots, \beta_n)\}.$$

An n -tuple $(b_1, \dots, b_n) \in \mathbf{TER}_{m,n}^n$ of terms is *properly obtained* from (a_1, \dots, a_n) if for all positive integers $i \leq n$, considered as formulas in **CPL**, the following terms are equivalent:

$$- b_i, \\ - \bigcup \{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap b_{\mathbf{s},i} : \mathbf{s} \text{ is an } m\text{-vector}\}.$$

For all m -vectors \mathbf{s} and for all $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$, let $f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n)$ be as in Section 7. For all m -vectors \mathbf{s} and for all positive integers $i \leq n$, let $\Delta_{\mathbf{s},i}$ and $c_{\mathbf{s},i}$ be as in Section 7. A substitution v is *properly obtained* from (a_1, \dots, a_n) if for all variables y , if $y \notin \{x_1, \dots, x_n\}$ then $v(y) = y$ and for all positive integers $i \leq n$, considered as formulas in **CPL**, the following terms are equivalent:

$$- v(x_i), \\ - \bigcup \{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap c_{\mathbf{s},i} : \mathbf{s} \text{ is an } m\text{-vector}\}.$$

The proof of the following result uses the purely Boolean property described in Proposition 5.

Proposition 6. *Let $(a_1, \dots, a_n) \in \mathbf{TER}_{m,k}^n$, $(b_1, \dots, b_n) \in \mathbf{TER}_{m,n}^n$ be an n -tuple of terms and v be a substitution. If (b_1, \dots, b_n) and v are properly obtained from (a_1, \dots, a_n) then for all positive integers $i \leq n$, considered as formulas in **CPL**, the following terms are equivalent:*

$$- a_i, \\ - \bar{v}(b_i).$$

Proof. Suppose (b_1, \dots, b_n) and v are properly obtained from (a_1, \dots, a_n) . Let $i \leq n$ be a positive integer. Considered as formulas in **CPL**, the following terms are equivalent:

1. $\bar{v}(b_i)$.
2. $\bigcup \{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap \bar{v}(b_{\mathbf{s},i}) : \mathbf{s} \text{ is an } m\text{-vector}\}.$
3. $\bigcup \{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap v(x_1)^{\beta_1} \cap \dots \cap v(x_n)^{\beta_n} : \mathbf{s} \text{ is an } m\text{-vector}, x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} \in \text{mon}_{\mathbf{s}}(k, a_i) \text{ and } f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = (\beta_1, \dots, \beta_n)\}.$
4. $\bigcup \{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap c_{\mathbf{s},1}^{\beta_1} \cap \dots \cap c_{\mathbf{s},n}^{\beta_n} : \mathbf{s} \text{ is an } m\text{-vector}, x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} \in \text{mon}_{\mathbf{s}}(k, a_i) \text{ and } f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = (\beta_1, \dots, \beta_n)\}.$
5. $\bigcup \{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap x_1^{\alpha'_1} \cap \dots \cap x_k^{\alpha'_k} : \mathbf{s} \text{ is an } m\text{-vector}, x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} \in \text{mon}_{\mathbf{s}}(k, a_i), f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = (\beta_1, \dots, \beta_n) \text{ and } (\alpha'_1, \dots, \alpha'_k) \in f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n)\}.$

6. $\bigcup\{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap \bigcup \text{mon}_{\mathbf{s}}(k, a_i) : \mathbf{s} \text{ is an } m\text{-vector}\}$.
7. a_i .

The equivalence between 1 and 2 is a consequence of the definition of (b_1, \dots, b_n) . The equivalence between 2 and 3 is a consequence of the definition of $(b_{\mathbf{s},1}, \dots, b_{\mathbf{s},n})$, \mathbf{s} being an arbitrary m -vector. The equivalence between 3 and 4 is a consequence of the definition of v . The equivalence between 4 and 5 is a consequence of Proposition 5. The equivalence between 5 and 6 is a consequence of the definitions of $\text{mon}_{\mathbf{s}}$ and $\sim_{(a_1, \dots, a_n)}^{k, \mathbf{s}}$ and the fact that for all $(\alpha_1, \dots, \alpha_k), (\alpha'_1, \dots, \alpha'_k) \in \{0, 1\}^k$, if $f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = f_{\mathbf{s}}(\alpha'_1, \dots, \alpha'_k)$ then $(\alpha_1, \dots, \alpha_k) \sim_{(a_1, \dots, a_n)}^{k, \mathbf{s}} (\alpha'_1, \dots, \alpha'_k)$, \mathbf{s} being an arbitrary m -vector. The equivalence between 6 and 7 is a consequence of Proposition 4.

Proposition 7. *Let $(a_1, \dots, a_n) \in \mathbf{TER}_{m,k}^n$ and $(b_1, \dots, b_n) \in \mathbf{TER}_{m,n}^n$ be an n -tuple of terms. Let W be a nonempty set. If (b_1, \dots, b_n) is properly obtained from (a_1, \dots, a_n) then:*

- for all valuations \mathcal{V} on W , there exists a valuation \mathcal{V}' on W such that for all positive integers $i \leq n$, $\bar{\mathcal{V}}(a_i) = \bar{\mathcal{V}}'(b_i)$,
- for all valuations \mathcal{V} on W , there exists a valuation \mathcal{V}' on W such that for all positive integers $i \leq n$, $\bar{\mathcal{V}}(b_i) = \bar{\mathcal{V}}'(a_i)$.

Proof. Let \mathcal{V} be a valuation on W . Let \mathcal{V}' be a valuation on W such that for all positive integers $i \leq m$, $\mathcal{V}'(p_i) = \mathcal{V}(p_i)$ and for all positive integers $i \leq n$, $\mathcal{V}'(x_i) = \bar{\mathcal{V}}(\bigcup\{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap c_{\mathbf{s},i} : \mathbf{s} \text{ is an } m\text{-vector}\})$. For all positive integers $i \leq n$, the following subsets of W are equal:

1. $\bar{\mathcal{V}}'(b_i)$.
2. $\bigcup\{\bar{\mathcal{V}}'(p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)}) \cap \bar{\mathcal{V}}'(b_{\mathbf{s},i}) : \mathbf{s} \text{ is an } m\text{-vector}\}$.
3. $\bigcup\{\bar{\mathcal{V}}'(p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)}) \cap \bar{\mathcal{V}}'(x_1^{\beta_1} \cap \dots \cap x_n^{\beta_n}) : \mathbf{s} \text{ is an } m\text{-vector}, x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} \in \text{mon}_{\mathbf{s}}(k, a_i) \text{ and } f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = (\beta_1, \dots, \beta_n)\}$.
4. $\bar{\mathcal{V}}(\bigcup\{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap c_{\mathbf{s},1}^{\beta_1} \cap \dots \cap c_{\mathbf{s},n}^{\beta_n} : \mathbf{s} \text{ is an } m\text{-vector}, x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} \in \text{mon}_{\mathbf{s}}(k, a_i) \text{ and } f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = (\beta_1, \dots, \beta_n)\})$.
5. $\bar{\mathcal{V}}(\bigcup\{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap x_1^{\alpha'_1} \cap \dots \cap x_k^{\alpha'_k} : \mathbf{s} \text{ is an } m\text{-vector}, x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} \in \text{mon}_{\mathbf{s}}(k, a_i), f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = (\beta_1, \dots, \beta_n) \text{ and } (\alpha'_1, \dots, \alpha'_k) \in f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n)\})$.
6. $\bar{\mathcal{V}}(\bigcup\{p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap \bigcup \text{mon}_{\mathbf{s}}(k, a_i) : \mathbf{s} \text{ is an } m\text{-vector}\})$.
7. $\bar{\mathcal{V}}(a_i)$.

The equivalence between 1 and 2 is a consequence of the definition of (b_1, \dots, b_n) . The equivalence between 2 and 3 is a consequence of the definition of $(b_{\mathbf{s},1}, \dots, b_{\mathbf{s},n})$, \mathbf{s} being an arbitrary m -vector. The equivalence between 3 and 4 is a consequence of the definition of \mathcal{V}' . The equivalence between 4 and 5 is a consequence of Proposition 5. The equivalence between 5 and 6 is a consequence of the definitions of $\text{mon}_{\mathbf{s}}$ and $\sim_{(a_1, \dots, a_n)}^{k, \mathbf{s}}$ and the fact that for all $(\alpha_1, \dots, \alpha_k), (\alpha'_1, \dots, \alpha'_k) \in \{0, 1\}^k$, if $f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = f_{\mathbf{s}}(\alpha'_1, \dots, \alpha'_k)$ then $(\alpha_1, \dots, \alpha_k) \sim_{(a_1, \dots, a_n)}^{k, \mathbf{s}} (\alpha'_1, \dots,$

α'_k), \mathbf{s} being an arbitrary m -vector. The equivalence between 6 and 7 is a consequence of Proposition 4.

Let \mathcal{V} be a valuation on W . Let \mathcal{V}' be a valuation on W such that for all positive integers $i \leq m$, $\mathcal{V}'(p_i) = \mathcal{V}(p_i)$ and for all m -vectors \mathbf{s} and for all $(\alpha_1, \dots, \alpha_k) \in \{0, 1\}^k$, $\bar{\mathcal{V}}'(p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k}) = \bar{\mathcal{V}}(p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap x_1^{\beta_1} \cap \dots \cap x_n^{\beta_n})$ where $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$ is such that $f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = (\beta_1, \dots, \beta_n)$. For all positive integers $i \leq n$, the following subsets of W are equal:

1. $\bar{\mathcal{V}}'(a_i)$.
2. $\bigcup \{ \bar{\mathcal{V}}'(p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)}) \cap \bar{\mathcal{V}}'(\bigcup \text{mon}_{\mathbf{s}}(k, a_i)) : \mathbf{s} \text{ is an } m\text{-vector} \}$.
3. $\bar{\mathcal{V}}(\bigcup \{ p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap x_1^{\beta_1} \cap \dots \cap x_n^{\beta_n} : \mathbf{s} \text{ is an } m\text{-vector, } x_1^{\alpha_1} \cap \dots \cap x_k^{\alpha_k} \in \text{mon}_{\mathbf{s}}(k, a_i) \text{ and } f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = (\beta_1, \dots, \beta_n) \})$.
4. $\bar{\mathcal{V}}(\bigcup \{ p_1^{\mathbf{s}(1)} \cap \dots \cap p_m^{\mathbf{s}(m)} \cap b_{\mathbf{s},i} : \mathbf{s} \text{ is an } m\text{-vector} \})$.
5. $\bar{\mathcal{V}}(b_i)$.

The equivalence between 1 and 2 is a consequence of Proposition 4. The equivalence between 2 and 3 is a consequence of the definitions of $\text{mon}_{\mathbf{s}}$ and \mathcal{V}' , \mathbf{s} being an arbitrary m -vector. The equivalence between 3 and 4 is a consequence of the definition of $(b_{\mathbf{s},1}, \dots, b_{\mathbf{s},n})$, \mathbf{s} being an arbitrary m -vector. The equivalence between 4 and 5 is a consequence of the definition of (b_1, \dots, b_n) .

9 Equivalence relations between formulas

Let k, m, n be nonnegative integers and \mathcal{C} be a class of frames. We define on $\mathbf{FOR}_{m,n}$ the equivalence relation $\equiv_{\mathcal{C}}^{m,n}$ as follows:

- $\varphi \equiv_{\mathcal{C}}^{m,n} \psi$ if $\varphi \leftrightarrow \psi$ is \mathcal{C} -valid.

Since each formula φ in $\mathbf{FOR}_{m,n}$ is a combination of formulas of the form $C(a, b)$, $a \equiv b$ and $\text{con}(a)$ where a and b are terms in $\mathbf{TER}_{m,n}$, therefore $\equiv_{\mathcal{C}}^{m,n}$ has finitely many equivalence classes on $\mathbf{FOR}_{m,n}$. Let A_n be the set of all n -tuples of terms. Note that n -tuples of terms in A_n may contain occurrences of constants outside $\{p_1, \dots, p_m\}$ and occurrences of variables outside $\{x_1, \dots, x_n\}$. Given a model (W, R, \mathcal{V}) on a frame in \mathcal{C} and $(a_1, \dots, a_n) \in A_n$, let $\Phi_{(a_1, \dots, a_n)}^{(W, R, \mathcal{V})}$ be the set of all equational formulas $\varphi(x_1, \dots, x_n)$ in $\mathbf{FOR}_{m,n}$ such that $(W, R, \mathcal{V}) \models \varphi(a_1, \dots, a_n)$. Consider a complete list of representatives for each equivalence class on $\Phi_{(a_1, \dots, a_n)}^{(W, R, \mathcal{V})}$ modulo $\equiv_{\mathcal{C}}^{m,n}$ and let $\varphi_{(a_1, \dots, a_n)}^{(W, R, \mathcal{V})}(x_1, \dots, x_n)$ be their conjunction. We define on A_n the equivalence relation $\cong_{\mathcal{C}}^{m,n}$ as follows:

- $(a_1, \dots, a_n) \cong_{\mathcal{C}}^{m,n} (b_1, \dots, b_n)$ if for all formulas $\varphi(x_1, \dots, x_n)$ in $\mathbf{FOR}_{m,n}$, $\varphi(a_1, \dots, a_n)$ is \mathcal{C} -valid iff $\varphi(b_1, \dots, b_n)$ is \mathcal{C} -valid.

Since $\equiv_{\mathcal{C}}^{m,n}$ has finitely many equivalence classes on $\mathbf{FOR}_{m,n}$, therefore $\cong_{\mathcal{C}}^{m,n}$ has finitely many equivalence classes on A_n . Now, we define on A_n the equivalence relation $\simeq_{\mathcal{C}}^{m,n}$ as follows:

- $(a_1, \dots, a_n) \simeq_{\mathcal{C}}^{m,n} (b_1, \dots, b_n)$ if for all equational formulas $\varphi(x_1, \dots, x_n)$ in $\mathbf{FOR}_{m,n}$, $\varphi(a_1, \dots, a_n)$ is \mathcal{C} -valid iff $\varphi(b_1, \dots, b_n)$ is \mathcal{C} -valid.

Obviously, $\simeq_{\mathcal{C}}^{m,n}$ is finer than $\simeq_{\mathcal{C}}^{m,n}$. Since $\simeq_{\mathcal{C}}^{m,n}$ has finitely many equivalence classes on A_n , therefore $\simeq_{\mathcal{C}}^{m,n}$ has finitely many equivalence classes on A_n .

Proposition 8. *Let $(a_1, \dots, a_n) \in \mathbf{TER}_{m,k}^n$ and $(b_1, \dots, b_n) \in \mathbf{TER}_{m,n}^n$ be an n -tuple of terms. If (b_1, \dots, b_n) is properly obtained from (a_1, \dots, a_n) then $(a_1, \dots, a_n) \simeq_{\mathcal{C}}^{m,n} (b_1, \dots, b_n)$.*

Proof. By Proposition 7.

Proposition 9. $\mathbf{TER}_{m,n}^n$ constitutes a complete set of representatives for each equivalence class on A_n modulo $\simeq_{\mathcal{C}}^{m,n}$.

Proof. By Proposition 8.

Proposition 10. *Let (W, R, \mathcal{V}) be a model on a frame in \mathcal{C} and $(a_1, \dots, a_n) \in A_n$. $(W, R, \mathcal{V}) \models \varphi_{(a_1, \dots, a_n)}^{(W, R, \mathcal{V})}(a_1, \dots, a_n)$.*

Proof. By the definition of $\varphi_{(a_1, \dots, a_n)}^{(W, R, \mathcal{V})}(x_1, \dots, x_n)$.

Proposition 11. *If \mathcal{C} is balanced then for all $(a_1, \dots, a_n), (b_1, \dots, b_n) \in A_n$, if $(a_1, \dots, a_n) \simeq_{\mathcal{C}}^{m,n} (b_1, \dots, b_n)$ then $(a_1, \dots, a_n) \cong_{\mathcal{C}}^{m,n} (b_1, \dots, b_n)$.*

Proof. Suppose \mathcal{C} is balanced. Let $(a_1, \dots, a_n), (b_1, \dots, b_n) \in A_n$ be such that $(a_1, \dots, a_n) \simeq_{\mathcal{C}}^{m,n} (b_1, \dots, b_n)$ and $(a_1, \dots, a_n) \not\cong_{\mathcal{C}}^{m,n} (b_1, \dots, b_n)$. Let $\varphi(x_1, \dots, x_n)$ be a formula in $\mathbf{FOR}_{m,n}$ such that $\varphi(a_1, \dots, a_n)$ is \mathcal{C} -valid not-iff $\varphi(b_1, \dots, b_n)$ is \mathcal{C} -valid. Without loss of generality, let us assume that $\varphi(a_1, \dots, a_n)$ is \mathcal{C} -valid and $\varphi(b_1, \dots, b_n)$ is not \mathcal{C} -valid. Remind that \equiv may not be the only predicate occurring in $\varphi(x_1, \dots, x_n)$. Since \mathcal{C} is balanced, therefore let (W, R, \mathcal{V}) be a balanced model on a countable frame in \mathcal{C} such that $(W, R, \mathcal{V}) \not\models \varphi(b_1, \dots, b_n)$. By Proposition 10, $(W, R, \mathcal{V}) \models \varphi_{(b_1, \dots, b_n)}^{(W, R, \mathcal{V})}(b_1, \dots, b_n)$. Hence, $\neg \varphi_{(b_1, \dots, b_n)}^{(W, R, \mathcal{V})}(b_1, \dots, b_n)$ is not \mathcal{C} -valid. Remind that $\varphi_{(b_1, \dots, b_n)}^{(W, R, \mathcal{V})}(x_1, \dots, x_n)$ is equational. Since $(a_1, \dots, a_n) \simeq_{\mathcal{C}}^{m,n} (b_1, \dots, b_n)$, therefore $\neg \varphi_{(b_1, \dots, b_n)}^{(W, R, \mathcal{V})}(a_1, \dots, a_n)$ is not \mathcal{C} -valid. Since \mathcal{C} is balanced, therefore let (W', R', \mathcal{V}') be a balanced model on a countable frame in \mathcal{C} such that $(W', R', \mathcal{V}') \models \varphi_{(b_1, \dots, b_n)}^{(W, R, \mathcal{V})}(a_1, \dots, a_n)$. Now, consider $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$. If $\bar{\mathcal{V}}(b_1^{\beta_1} \cap \dots \cap b_n^{\beta_n}) = \emptyset$ then $(W, R, \mathcal{V}) \models b_1^{\beta_1} \cap \dots \cap b_n^{\beta_n} \equiv 0$. Thus, $\varphi_{(b_1, \dots, b_n)}^{(W, R, \mathcal{V})}(x_1, \dots, x_n) \rightarrow x_1^{\beta_1} \cap \dots \cap x_n^{\beta_n} \equiv 0$ is \mathcal{C} -valid. Since $(W', R', \mathcal{V}') \models \varphi_{(b_1, \dots, b_n)}^{(W, R, \mathcal{V})}(a_1, \dots, a_n)$, therefore $(W', R', \mathcal{V}') \models a_1^{\beta_1} \cap \dots \cap a_n^{\beta_n} \equiv 0$. Consequently, $\bar{\mathcal{V}}'(a_1^{\beta_1} \cap \dots \cap a_n^{\beta_n}) = \emptyset$. Similarly, the reader may easily verify that if $\bar{\mathcal{V}}(b_1^{\beta_1} \cap \dots \cap b_n^{\beta_n}) = W$ then $\bar{\mathcal{V}}'(a_1^{\beta_1} \cap \dots \cap a_n^{\beta_n}) = W'$ and if $\bar{\mathcal{V}}(b_1^{\beta_1} \cap \dots \cap b_n^{\beta_n})$ is infinite and coinfinite then $\bar{\mathcal{V}}'(a_1^{\beta_1} \cap \dots \cap a_n^{\beta_n})$ is infinite and coinfinite. In all cases, there exists a bijection $g_{(\beta_1, \dots, \beta_n)}$ from $\bar{\mathcal{V}}(b_1^{\beta_1} \cap \dots \cap b_n^{\beta_n})$ to $\bar{\mathcal{V}}'(a_1^{\beta_1} \cap \dots \cap a_n^{\beta_n})$. Let g be the union of all $g_{(\beta_1, \dots, \beta_n)}$ when $(\beta_1, \dots, \beta_n)$ describes $\{0, 1\}^n$. The reader

may easily verify that g is a bijection from W to W' such that for all $u \in W$ and for all $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$, $u \in \bar{\mathcal{V}}(b_1^{\beta_1} \cap \dots \cap b_n^{\beta_n})$ iff $g(u) \in \bar{\mathcal{V}}(a_1^{\beta_1} \cap \dots \cap a_n^{\beta_n})$. Let R'_g be the binary relation on W' defined by $u'R'_g v'$ if $g^{-1}(u')Rg^{-1}(v')$. Obviously, g is an isomorphism from (W, R) to (W', R'_g) . Since $\varphi(a_1, \dots, a_n)$ is \mathcal{C} -valid, therefore $(W', R'_g, \mathcal{V}') \models \varphi(a_1, \dots, a_n)$. Hence, $(W, R, \mathcal{V}) \models \varphi(b_1, \dots, b_n)$: a contradiction.

10 Unification type

Let \mathcal{C} be a class of frames.

Proposition 12. *If \mathcal{C} is balanced then unification in \mathcal{C} is either finitary, or unitary. Moreover, if \mathcal{C} agrees with unions then unification in \mathcal{C} is finitary.*

Proof. Suppose \mathcal{C} is balanced. Let φ be an arbitrary \mathcal{C} -unifiable formula. Let σ be an arbitrary substitution such that $\bar{\sigma}(\varphi)$ is \mathcal{C} -valid. Let $m, n \geq 0$ be such that φ 's constants form a subset of $\{p_1, \dots, p_m\}$ and φ 's variables form a subset of $\{x_1, \dots, x_n\}$. Let σ' be the substitution defined by $\sigma'(x_i) = \sigma(x_i)$ for all $i = 1 \dots n$ and $\sigma'(y) = y$ for all y not in $\{x_1, \dots, x_n\}$. Obviously, σ' is a unifier of φ too. Now, it may happen that for some $i \in \{1, \dots, n\}$, $\sigma'(x_i)$ contains extra constants outside $\{p_1, \dots, p_m\}$. If it is, then let q_1, \dots, q_l be the list of these extra constants. Take new variables z_1, \dots, z_l and define σ'' by uniformly replacing in $\sigma'(x_1), \dots, \sigma'(x_n)$ each occurrence of q_1, \dots, q_l by, respectively, z_1, \dots, z_l . Obviously, σ'' is a unifier of φ too. As a result, for all constants q , if $q \notin \{p_1, \dots, p_m\}$ then for all positive integers $i \leq n$, q does not occur in $\sigma''(x_i)$ and for all variables y , if $y \notin \{x_1, \dots, x_n\}$ then $\sigma''(y) = y$. Let k be a nonnegative integer and $(a_1, \dots, a_n) \in \mathbf{TER}_{m,k}^n$ be such that for all positive integers $i \leq n$, $\sigma''(x_i) = a_i$. Since $\sigma''(\varphi)$ is \mathcal{C} -valid, therefore $\varphi(a_1, \dots, a_n)$ is \mathcal{C} -valid. For all m -vectors \mathbf{s} , let $\sim_{(a_1, \dots, a_n)}^{k, \mathbf{s}}$ be as in Section 8. Let f be a (k, m, n) -correspondence such that for all m -vectors \mathbf{s} and for all $(\alpha_1, \dots, \alpha_k), (\alpha'_1, \dots, \alpha'_k) \in \{0, 1\}^k$, if $f_{\mathbf{s}}(\alpha_1, \dots, \alpha_k) = f_{\mathbf{s}}(\alpha'_1, \dots, \alpha'_k)$ then $(\alpha_1, \dots, \alpha_k) \sim_{(a_1, \dots, a_n)}^{k, \mathbf{s}} (\alpha'_1, \dots, \alpha'_k)$. For all m -vectors \mathbf{s} and for all $(\beta_1, \dots, \beta_n) \in \{0, 1\}^n$, let $f_{\mathbf{s}}^{-1}(\beta_1, \dots, \beta_n)$ be as in Section 7. For all m -vectors \mathbf{s} and for all positive integers $i \leq n$, let $\Delta_{\mathbf{s}, i}$ and $c_{\mathbf{s}, i}$ be as in Section 7. Let $(b_1, \dots, b_n) \in \mathbf{TER}_{m,n}^n$ be an n -tuple of terms properly obtained from (a_1, \dots, a_n) . By Proposition 8, $(a_1, \dots, a_n) \simeq_{\mathcal{C}}^{m,n} (b_1, \dots, b_n)$. Since \mathcal{C} is balanced, therefore by Proposition 11, $(a_1, \dots, a_n) \cong_{\mathcal{C}}^{m,n} (b_1, \dots, b_n)$. Let τ be the substitution such that for all variables y , if $y \notin \{x_1, \dots, x_n\}$ then $\tau(y) = y$ and for all positive integers $i \leq n$, $\tau(x_i) = b_i$. Remark that $(\tau(x_1), \dots, \tau(x_n)) \in \mathbf{TER}_{m,n}^n$. Moreover, since $\varphi(a_1, \dots, a_n)$ is \mathcal{C} -valid and $(a_1, \dots, a_n) \cong_{\mathcal{C}}^{m,n} (b_1, \dots, b_n)$, therefore $\varphi(b_1, \dots, b_n)$ is \mathcal{C} -valid. Hence, τ is a \mathcal{C} -unifier of φ . Let v be a substitution properly obtained from (a_1, \dots, a_n) . By Proposition 6, for all positive integers $i \leq n$, considered as formulas in **CPL**, the terms a_i and $\bar{v}(b_i)$ are equivalent. Thus, for all positive integers $i \leq n$, $\bar{v}(\tau(x_i)) \equiv \sigma''(x_i)$ is \mathcal{C} -valid. Consequently, $\tau \circ v \simeq_{\mathcal{C}} \sigma''$. Hence, $\tau \preceq_{\mathcal{C}} \sigma''$. By the construction of τ , one can deduce that $\tau \preceq_{\mathcal{C}} \sigma$. Since σ was arbitrary and

$(\tau(x_1), \dots, \tau(x_n)) \in \mathbf{TER}_{m,n}^n$, therefore φ is either \mathcal{C} -finitary, or \mathcal{C} -unitary. Since φ was arbitrary, therefore unification in \mathcal{C} is either finitary, or unitary. Now, suppose \mathcal{C} agrees with unions. By the examples considered in Section 6, unification in \mathcal{C} is not unitary. Since unification in \mathcal{C} is either finitary, or unitary, therefore unification in \mathcal{C} is finitary.

It follows from the above discussion that unification in \mathcal{C}_{all} , \mathcal{C}_{ind} and \mathcal{C}_{con} is finitary.

11 Conclusion

We anticipate a number of further investigations. For example, about the computability of the unification problem in Contact Logic. It is proved in [5] that with respect to most classes of frames, in Contact Logic, when one restricts the discussion to *con*-free formulas, elementary unification is *NP*-complete whereas unification with constants is decidable, its exact complexity being still unknown. By means of Propositions 9 and 11, one can prove that with respect to balanced classes of frames, elementary unification and unification with constants are decidable in Contact Logic when one does not restrict the discussion to *con*-free formulas, but its exact complexity is unknown. In this respect, we believe that arguments developed in [1] could be used.

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