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About the temporal logic of the lexicographic products of unbounded dense linear orders: a new study of its computability

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Abstract. This article considers the temporal logic of the lexicographic products of unbounded dense linear orders and provides via mosaics a new proof of the membership in NP of the satisfiability problem it gives rise to.

Keywords: Linear temporal logic, lexicographic product, satisfiability problem, decidability, complexity, mosaic method, decision procedure.

1 Introduction

The mosaic method has been applied for proving completeness and decidability of temporal logics over multifarious linear flows of time [9, 19, 20, 22]. The operation of lexicographic product of Kripke frames has been introduced as a variant of the more classical operation of Cartesian product [12, 14, 17, 23]. See [5] for details. It has been used for defining the semantical basis of different languages designed for time representation and temporal reasoning from the perspective of non-standard analysis [1, 2]. In [3, 4], the temporal logic of the lexicographic products of unbounded dense linear orders has been considered, its complete axiomatization has been given and its computability has been studied.

The purpose of this paper is give a new proof of the membership in NP of the satisfiability problem of the lexicographic products of linear temporal logics. Its section-by-section breakdown is as follows. Section 2 studies the elementary properties of the lexicographic products of unbounded dense linear orders. In section 3, we present the syntax and the semantics of the temporal logic we will be working with. Sections 4 and 5 define mosaics and maps. In sections 6 and 7, we prove that the satisfiability problem in our temporal logic is in NP .

2 Products of unbounded dense linear orders

Let $\mathcal{F}_1 = (T_1, <_1)$ and $\mathcal{F}_2 = (T_2, <_2)$ be linear orders. Their *lexicographic product* is the structure $\mathcal{F} = (T, \prec_1, \prec_2)$ where

- $T = T_1 \times T_2$,
- \prec_1 and \prec_2 are the binary relations on T defined by putting $(s_1, s_2) \prec_1 (t_1, t_2)$ iff $s_1 <_1 t_1$ and $(s_1, s_2) \prec_2 (t_1, t_2)$ iff $s_1 = t_1$ and $s_2 <_2 t_2$.

We define the binary relation \prec on T by putting $(s_1, s_2) \prec (t_1, t_2)$ iff $s_1 <_1 t_1$, or $s_1 = t_1$ and $s_2 <_2 t_2$. For all $(s_1, s_2), (t_1, t_2) \in T$, if $(s_1, s_2) \prec (t_1, t_2)$ then let

- $] (s_1, s_2), (t_1, t_2) [$ be the set of all $(u_1, u_2) \in T$ such that $(s_1, s_2) \prec (u_1, u_2)$ and $(u_1, u_2) \prec (t_1, t_2)$,
- $] (s_1, s_2), +\infty [$ be the set of all $(u_1, u_2) \in T$ such that $(s_1, s_2) \prec (u_1, u_2)$,
- $] -\infty, (t_1, t_2) [$ be the set of all $(u_1, u_2) \in T$ such that $(u_1, u_2) \prec (t_1, t_2)$.

The effect of the operation of lexicographic product may be described informally as follows: given 2 linear orders, their lexicographic product is the structure obtained by replacing each point of the first one by a copy of the second one. The global intuitions underlying such an operation is based upon the fact that, depending on the accuracy required or the available knowledge, one can describe a temporal situation at different levels of abstraction. See [6, section I.2.2], or [10] for details. In *Fig. 1* below, we have $s_1 <_1 t_1$ and $s_2 <_2 t_2$. As a result, we have $(s_1, s_2) \prec_2 (s_1, t_2)$, $(s_1, s_2) \prec_1 (t_1, s_2)$, $(s_1, s_2) \prec_1 (t_1, t_2)$, $(s_1, t_2) \prec_1 (t_1, s_2)$, $(s_1, t_2) \prec_1 (t_1, t_2)$ and $(t_1, s_2) \prec_2 (t_1, t_2)$.

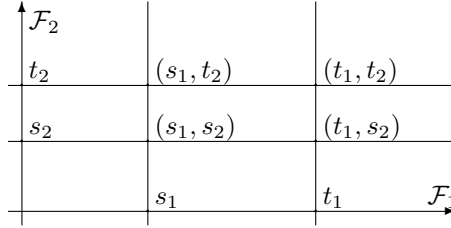


Fig. 1.

In order to characterize the elementary properties of the operation of lexicographic product, we introduce a first-order language. Let Var denote a countable set of individual variables (with typical members denoted x, y , etc). The set \mathcal{L}_{fo} of all *formulas* (with typical members denoted φ, ψ , etc) of the first-order language is given by the rule

$$- \varphi ::= x <_1 y \mid x <_2 y \mid \perp \mid \neg\varphi \mid (\varphi \vee \psi) \mid \forall x \varphi \mid x = y,$$

the formulas $x <_1 y$ and $x <_2 y$ being read “ x precedes but is not infinitely close to y ” and “ x precedes and is infinitely close to y ”. We adopt the standard definitions for the remaining Boolean connectives. As usual, we define for all individual variables x ,

$$- \exists x \varphi ::= \neg \forall x \neg \varphi.$$

The notion of a *subformula* is standard. We adopt the standard rules for omission of the parentheses. The *size* of a formula φ , in symbols $|\varphi|$, is its length over the alphabet of \mathcal{L}_{fo} . Formulas in which every individual variable in an atomic subformula is in the scope of a corresponding quantifier are called *sentences*. *Models* for the first-order language are flows $\mathcal{F} = (T, \prec_1, \prec_2)$ where T is a nonempty set and \prec_1 and \prec_2 are binary relations on T . An *assignment* on \mathcal{F} is a function $f: Var \rightarrow T$. *Satisfaction* is a 3-place relation \models between a flow $\mathcal{F} = (T, \prec_1, \prec_2)$, an assignment f on \mathcal{F} and a formula φ . It is inductively defined as usual. In particular,

- $\mathcal{F} \models_f x <_1 y$ iff $f(x) \prec_1 f(y)$,
- $\mathcal{F} \models_f x <_2 y$ iff $f(x) \prec_2 f(y)$.

Obviously, if 2 linear orders are unbounded and dense then their lexicographic product satisfies the following sentences:

- IRRE* - $\forall x x \not<_1 x$,
- $\forall x x \not<_2 x$,
- DISJ* - $\forall x \forall y (x \not<_1 y \vee x \not<_2 y)$,
- TRAN* - $\forall x \forall y (\exists z (x <_1 z \wedge z <_1 y) \rightarrow x <_1 y)$,
- $\forall x \forall y (\exists z (x <_1 z \wedge z <_2 y) \rightarrow x <_1 y)$,
- $\forall x \forall y (\exists z (x <_2 z \wedge z <_1 y) \rightarrow x <_1 y)$,
- $\forall x \forall y (\exists z (x <_2 z \wedge z <_2 y) \rightarrow x <_2 y)$,
- DENS* - $\forall x \forall y (x <_1 y \rightarrow \exists z (x <_1 z \wedge z <_1 y))$,
- $\forall x \forall y (x <_1 y \rightarrow \exists z (x <_1 z \wedge z <_2 y))$,
- $\forall x \forall y (x <_1 y \rightarrow \exists z (x <_2 z \wedge z <_1 y))$,
- $\forall x \forall y (x <_2 y \rightarrow \exists z (x <_2 z \wedge z <_2 y))$,
- SERI* - $\forall x \exists y x <_1 y$,
- $\forall x \exists y x <_2 y$,
- $\forall x \exists y y <_1 x$,
- $\forall x \exists y y <_2 x$,
- UNIV* - $\forall x \forall y (x = y \vee x <_1 y \vee x <_2 y \vee y <_1 x \vee y <_2 x)$,

IRRE, *DISJ*, *TRAN*, *DENS*, *SERI* and *UNIV* standing for “irreflexive”, “disjoint”, “transitive”, “dense”, “serial” and “universal”. Obviously, the sentences as above have not the finite model property. By Löwenheim-Skolem theorem, they have models in each infinite power. A flow $\mathcal{F} = (T, \prec_1, \prec_2)$ is said to be *standard* iff it satisfies the sentences as above. The *first-order theory SF* of standard flows has the following list of proper axioms: *IRRE*, *DISJ*, *TRAN*, *DENS*, *SERI* and *UNIV*. The membership problem in the first-order theory *SF* is this:

- determine whether a given sentence is in the first-order theory *SF*.

There are several results about the first-order theory *SF*:

Proposition 1 ([1]). (i) *SF* is countably categorical; (ii) *SF* is not categorical in any uncountable power; (iii) *SF* is maximal consistent; (iv) *SF* is complete with respect to the lexicographic product of any unbounded dense linear orders.

The results about the membership problem in the first-order theory SF are summarized in the following proposition:

Proposition 2. (i) SF is decidable; (ii) SF is PSPACE-complete.

See [1] for the proof of propositions 1 and 2.

3 Syntax and semantics

It is now time to meet the temporal language we will be working with. Let At be a countable set of atomic formulas (with typical members denoted p, q , etc). We define the set \mathcal{L}_t of *formulas* of our temporal language (with typical members denoted φ, ψ , etc) as follows:

$$- \varphi ::= p \mid \perp \mid \neg\varphi \mid (\varphi \vee \psi) \mid G_1\varphi \mid G_2\varphi \mid H_1\varphi \mid H_2\varphi,$$

the formulas $G_1\varphi, G_2\varphi, H_1\varphi$ and $H_2\varphi$ being read “ φ will be true at each point within the future of but not infinitely close to the present point”, “ φ will be true at each instant within the future of and infinitely close to the present instant”, “ φ has been true at each point within the past of but not infinitely close to the present point” and “ φ has been true at each point within the past of and infinitely close to the present point”. We adopt the standard definitions for the remaining Boolean connectives. As usual, we define for all $i \in \{1, 2\}$,

$$\begin{aligned} - F_i\varphi &::= \neg G_i\neg\varphi \text{ and} \\ - P_i\varphi &::= \neg H_i\neg\varphi. \end{aligned}$$

The notion of a *subformula* is standard. It is usual to omit parentheses if this does not lead to any ambiguity. The *size* of a formula φ , in symbols $|\varphi|$, is the number of symbols of φ . A set Γ of formulas is said to be *full* iff there exists $i \in \{1, 2\}$ such that $G_i\perp \in \Gamma$, or $H_i\perp \in \Gamma$ and the following unique condition is satisfied:

$$- \text{if there exists } i \in \{1, 2\} \text{ such that } G_i\varphi \in \Gamma, \text{ or } H_i\varphi \in \Gamma \text{ then for all } i \in \{1, 2\}, G_i\varphi \in \Gamma \text{ and } H_i\varphi \in \Gamma.$$

Remark that if Γ is a subformula closed full set of formulas then $\perp \in \Gamma$. Moreover, for all $i \in \{1, 2\}$, $G_i\perp \in \Gamma$ and $H_i\perp \in \Gamma$. In other respect,

Lemma 1. *Let φ be a formula. Let Γ be the least subformula closed full set of formulas containing φ . Then $\text{card}(\Gamma) \leq 4 \times |\varphi| + 2$.*

Proof. By induction on φ .

A *model* is a structure $\mathcal{M} = (\mathcal{F}_1, \mathcal{F}_2, V)$ where $\mathcal{F}_1 = (T_1, <_1)$ and $\mathcal{F}_2 = (T_2, <_2)$ are linear orders and $V: At \rightarrow 2^{T_1 \times T_2}$ is a function. *Satisfaction* is a 3-place relation \models between a model $\mathcal{M} = (\mathcal{F}_1, \mathcal{F}_2, V)$, a pair $(s_1, s_2) \in T_1 \times T_2$ and a formula φ . It is defined by induction on φ as usual. In particular, for all $i \in \{1, 2\}$,

- $\mathcal{M}, (s_1, s_2) \models G_i\varphi$ iff $\mathcal{M}, (t_1, t_2) \models \varphi$ for each pair $(t_1, t_2) \in T_1 \times T_2$ such that $(s_1, s_2) \prec_i (t_1, t_2)$,
- $\mathcal{M}, (s_1, s_2) \models H_i\varphi$ iff $\mathcal{M}, (t_1, t_2) \models \varphi$ for each pair $(t_1, t_2) \in T_1 \times T_2$ such that $(t_1, t_2) \prec_i (s_1, s_2)$.

As a result, for all $i \in \{1, 2\}$,

- $\mathcal{M}, (s_1, s_2) \models F_i\varphi$ iff $\mathcal{M}, (t_1, t_2) \models \varphi$ for some pair $(t_1, t_2) \in T_1 \times T_2$ such that $(s_1, s_2) \prec_i (t_1, t_2)$,
- $\mathcal{M}, (s_1, s_2) \models P_i\varphi$ iff $\mathcal{M}, (t_1, t_2) \models \varphi$ for some pair $(t_1, t_2) \in T_1 \times T_2$ such that $(t_1, t_2) \prec_i (s_1, s_2)$.

\mathcal{M} is said to be a *model* for φ iff there exists $(s_1, s_2) \in T_1 \times T_2$ such that $\mathcal{M}, (s_1, s_2) \models \varphi$. Let \mathcal{C}_1 and \mathcal{C}_2 be classes of linear orders. We shall say that a formula φ is *satisfiable* with respect to $(\mathcal{C}_1, \mathcal{C}_2)$ iff there exists a linear order $\mathcal{F}_1 = (T_1, <_1)$ in \mathcal{C}_1 , there exists a linear order $\mathcal{F}_2 = (T_2, <_2)$ in \mathcal{C}_2 and there exists a function $V: At \rightarrow 2^{T_1 \times T_2}$ such that $(\mathcal{F}_1, \mathcal{F}_2, V)$ is a model for φ . The *temporal logic* of $(\mathcal{C}_1, \mathcal{C}_2)$ is the set of all formulas φ such that $\neg\varphi$ is not satisfiable with respect to $(\mathcal{C}_1, \mathcal{C}_2)$. The class of all unbounded dense linear orders will be denoted \mathcal{C}_{ud} . [3] considers the temporal logic of $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$ and gives its complete axiomatization based on the following 32 axioms:

- $\phi \rightarrow G_1P_1\phi$,
- $\phi \rightarrow H_1F_1\phi$,
- $\phi \rightarrow G_2P_2\phi$,
- $\phi \rightarrow H_2F_2\phi$,
- $F_1F_1\phi \rightarrow F_1\phi$,
- $P_1P_1\phi \rightarrow P_1\phi$,
- $F_1F_2\phi \rightarrow F_1\phi$,
- $P_1P_2\phi \rightarrow P_1\phi$,
- $F_2F_1\phi \rightarrow F_1\phi$,
- $P_2P_1\phi \rightarrow P_1\phi$,
- $F_2F_2\phi \rightarrow F_2\phi$,
- $P_2P_2\phi \rightarrow P_2\phi$,
- $F_1\phi \rightarrow F_1F_1\phi$,
- $P_1\phi \rightarrow P_1P_1\phi$,
- $F_1\phi \rightarrow F_1F_2\phi$,
- $P_1\phi \rightarrow P_1P_2\phi$,
- $F_1\phi \rightarrow F_2F_1\phi$,
- $P_1\phi \rightarrow P_2P_1\phi$,
- $F_2\phi \rightarrow F_2F_2\phi$,
- $P_2\phi \rightarrow P_2P_2\phi$,
- $F_1\top$,
- $P_1\top$,
- $F_2\top$,
- $P_2\top$,
- $F_1\phi \wedge F_1\psi \rightarrow F_1(\phi \wedge \psi) \vee F_1(\phi \wedge F_1\psi) \vee F_1(\phi \wedge F_2\psi) \vee F_1(\psi \wedge F_1\phi) \vee F_1(\psi \wedge F_2\phi)$,

- $P_1\phi \wedge P_1\psi \rightarrow P_1(\phi \wedge \psi) \vee P_1(\phi \wedge P_1\psi) \vee P_1(\phi \wedge P_2\psi) \vee P_1(\psi \wedge P_1\phi) \vee P_1(\psi \wedge P_2\phi)$,
- $F_1\phi \wedge F_2\psi \rightarrow F_2(\psi \wedge F_1\phi)$,
- $P_1\phi \wedge P_2\psi \rightarrow P_2(\psi \wedge P_1\phi)$,
- $F_2\phi \wedge F_1\psi \rightarrow F_2(\phi \wedge F_1\psi)$,
- $P_2\phi \wedge P_1\psi \rightarrow P_2(\phi \wedge P_1\psi)$,
- $F_2\phi \wedge F_2\psi \rightarrow F_2(\phi \wedge \psi) \vee F_2(\phi \wedge F_2\psi) \vee F_2(\psi \wedge F_2\phi)$,
- $P_2\phi \wedge P_2\psi \rightarrow P_2(\phi \wedge \psi) \vee P_2(\phi \wedge P_2\psi) \vee P_2(\psi \wedge P_2\phi)$.

The *satisfiability problem* this temporal logic gives rise to is this:

- determine whether a given formula is satisfiable with respect to $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$.

In order to provide a complete decision procedure in nondeterministic polynomial time for it, we use mosaics and maps.

4 Mosaics

Until the end of this paper, φ will denote a formula and Γ will denote the least subformula closed full set of formulas containing φ . A function $\sigma: \Gamma \rightarrow \{0, 1\}$ is said to be *adequate* iff $\sigma(\perp) = 0$, for all $i \in \{1, 2\}$, $\sigma(G_i\perp) = 0$ and $\sigma(H_i\perp) = 0$ and the 2 following conditions are satisfied:

1. if $\neg\psi \in \Gamma$ then $\sigma(\neg\psi) = 1 - \sigma(\psi)$,
2. if $\psi \vee \chi \in \Gamma$ then $\sigma(\psi \vee \chi) = \max\{\sigma(\psi), \sigma(\chi)\}$.

A *mosaic* is a structure (σ, τ) where $\sigma: \Gamma \rightarrow \{0, 1\}$ and $\tau: \Gamma \rightarrow \{0, 1\}$ are adequate functions. We shall say that a mosaic (σ, τ) is *1-temporal* iff the 2 following conditions are satisfied:

1. if $G_1\psi \in \Gamma$ then $\sigma(G_1\psi) \leq \min\{\tau(\psi), \tau(G_1\psi), \tau(G_2\psi), \tau(H_2\psi)\}$,
2. if $H_1\psi \in \Gamma$ then $\tau(H_1\psi) \leq \min\{\sigma(\psi), \sigma(G_2\psi), \sigma(H_1\psi), \sigma(H_2\psi)\}$.

A mosaic (σ, τ) is said to be *2-temporal* iff the 6 following conditions are satisfied:

1. if $G_1\psi \in \Gamma$ then $\sigma(G_1\psi) \leq \tau(G_1\psi)$,
2. if $G_2\psi \in \Gamma$ then $\sigma(G_2\psi) \leq \min\{\tau(\psi), \tau(G_2\psi)\}$,
3. if $H_1\psi \in \Gamma$ then $\sigma(H_1\psi) \leq \tau(H_1\psi)$,
4. if $G_1\psi \in \Gamma$ then $\tau(G_1\psi) \leq \sigma(G_1\psi)$,
5. if $H_1\psi \in \Gamma$ then $\tau(H_1\psi) \leq \sigma(H_1\psi)$,
6. if $H_2\psi \in \Gamma$ then $\tau(H_2\psi) \leq \min\{\sigma(\psi), \sigma(H_2\psi)\}$.

A *premodel* is a structure (σ_0, M_1, M_2) where $\sigma_0: \Gamma \rightarrow \{0, 1\}$ is an adequate function, M_1 is a set of 1-temporal mosaics and M_2 is a set of 2-temporal mosaics. The *cardinality* of a premodel (σ_0, M_1, M_2) , in symbols $\|(\sigma_0, M_1, M_2)\|$, is the number of mosaics in M_1 plus the number of mosaics in M_2 . We shall say that a premodel (σ_0, M_1, M_2) is *0-saturated* iff the 4 following conditions are satisfied:

1. if $G_1\psi \in \Gamma$ then $\sigma_0(G_1\psi) = 1$, or there exists an adequate function $\tau: \Gamma \rightarrow \{0, 1\}$ such that $(\sigma_0, \tau) \in M_1$ and $\tau(\psi) = 0$,

2. if $G_2\psi \in \Gamma$ then $\sigma_0(G_2\psi) = 1$, or there exists an adequate function $\tau: \Gamma \rightarrow \{0, 1\}$ such that $(\sigma_0, \tau) \in M_2$ and $\tau(\psi) = 0$,
3. if $H_1\psi \in \Gamma$ then $\sigma_0(H_1\psi) = 1$, or there exists an adequate function $\tau: \Gamma \rightarrow \{0, 1\}$ such that $(\tau, \sigma_0) \in M_1$ and $\tau(\psi) = 0$,
4. if $H_2\psi \in \Gamma$ then $\sigma_0(H_2\psi) = 1$, or there exists an adequate function $\tau: \Gamma \rightarrow \{0, 1\}$ such that $(\tau, \sigma_0) \in M_2$ and $\tau(\psi) = 0$.

A premodel (σ_0, M_1, M_2) is said to be *1-saturated* iff for all mosaics $(\sigma, \tau) \in M_1$, the 8 following conditions are satisfied:

1. if $G_1\psi \in \Gamma$ then $\sigma(G_1\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\sigma, \mu) \in M_1$, $(\mu, \tau) \in M_1 \cup M_2$ and $\mu(\psi) = 0$, or $\tau(\psi) = 0$, or $\tau(G_1\psi) = 0$, or $\tau(G_2\psi) = 0$,
2. if $G_2\psi \in \Gamma$ then $\sigma(G_2\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\sigma, \mu) \in M_2$, $(\mu, \tau) \in M_1$ and $\mu(\psi) = 0$,
3. if $H_1\psi \in \Gamma$ then $\sigma(H_1\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\mu, \sigma) \in M_1$ and $\mu(\psi) = 0$,
4. if $H_2\psi \in \Gamma$ then $\sigma(H_2\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\mu, \sigma) \in M_2$ and $\mu(\psi) = 0$,
5. if $G_1\psi \in \Gamma$ then $\tau(G_1\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\tau, \mu) \in M_1$ and $\mu(\psi) = 0$,
6. if $G_2\psi \in \Gamma$ then $\tau(G_2\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\tau, \mu) \in M_2$ and $\mu(\psi) = 0$,
7. if $H_1\psi \in \Gamma$ then $\tau(H_1\psi) = 1$, or $\sigma(H_1\psi) = 0$, or $\sigma(H_2\psi) = 0$, or $\sigma(\psi) = 0$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\sigma, \mu) \in M_1 \cup M_2$, $(\mu, \tau) \in M_1$ and $\mu(\psi) = 0$,
8. if $H_2\psi \in \Gamma$ then $\tau(H_2\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\sigma, \mu) \in M_1$, $(\mu, \tau) \in M_2$ and $\mu(\psi) = 0$.

We shall say that a premodel (σ_0, M_1, M_2) is *2-saturated* iff for all mosaics $(\sigma, \tau) \in M_2$, the 8 following conditions are satisfied:

1. if $G_1\psi \in \Gamma$ then $\sigma(G_1\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\tau, \mu) \in M_1$ and $\mu(\psi) = 0$,
2. if $G_2\psi \in \Gamma$ then $\sigma(G_2\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\sigma, \mu) \in M_2$, $(\mu, \tau) \in M_2$ and $\mu(\psi) = 0$, or $\tau(\psi) = 0$, or $\tau(G_2\psi) = 0$,
3. if $H_1\psi \in \Gamma$ then $\sigma(H_1\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\mu, \sigma) \in M_1$ and $\mu(\psi) = 0$,
4. if $H_2\psi \in \Gamma$ then $\sigma(H_2\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\mu, \sigma) \in M_2$ and $\mu(\psi) = 0$,
5. if $G_1\psi \in \Gamma$ then $\tau(G_1\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\tau, \mu) \in M_1$ and $\mu(\psi) = 0$,
6. if $G_2\psi \in \Gamma$ then $\tau(G_2\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\tau, \mu) \in M_2$ and $\mu(\psi) = 0$,
7. if $H_1\psi \in \Gamma$ then $\tau(H_1\psi) = 1$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\mu, \sigma) \in M_1$ and $\mu(\psi) = 0$,

8. if $H_2\psi \in \Gamma$ then $\tau(H_2\psi) = 1$, or $\sigma(H_2\psi) = 0$, or $\sigma(\psi) = 0$, or there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\sigma, \mu) \in M_2$, $(\mu, \tau) \in M_2$ and $\mu(\psi) = 0$.

A premodel is said to be *saturated* iff it is 0-, 1- and 2-saturated. We shall say that a premodel (σ_0, M_1, M_2) is *1-dense* iff for all mosaics $(\sigma, \tau) \in M_1$, the following unique condition is satisfied:

- there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\sigma, \mu) \in M_1$ and $(\mu, \tau) \in M_1$.

A premodel (σ_0, M_1, M_2) is said to be *2-dense* iff for all mosaics $(\sigma, \tau) \in M_2$, the following unique condition is satisfied:

- there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\sigma, \mu) \in M_2$ and $(\mu, \tau) \in M_2$.

We shall say that a premodel is *dense* iff it is 1- and 2-dense. A premodel (σ_0, M_1, M_2) is said to be *for* φ iff $\sigma_0(\varphi) = 1$.

5 Maps

Until the end of this section, (σ_0, M_1, M_2) will denote a saturated dense premodel. A *map* is a partial function $l: \mathbb{Q} \times \mathbb{Q} \rightarrow (\Gamma \rightarrow \{0, 1\})$ with finite domain and such that for all pairs $(s_1, s_2) \in \text{dom}(l)$, $l(s_1, s_2)$ is an adequate function. A map l is said to be *respectful* iff $(0, 0) \in \text{dom}(l)$, $l(0, 0) = \sigma_0$ and for all pairs $(s_1, s_2), (t_1, t_2) \in \text{dom}(l)$, the 2 following conditions are satisfied:

1. if $(s_1, s_2) \prec_1 (t_1, t_2)$ and $](s_1, s_2), (t_1, t_2)[\cap \text{dom}(l) = \emptyset$ then $(l(s_1, s_2), l(t_1, t_2)) \in M_1$,
2. if $(s_1, s_2) \prec_2 (t_1, t_2)$ and $](s_1, s_2), (t_1, t_2)[\cap \text{dom}(l) = \emptyset$ then $(l(s_1, s_2), l(t_1, t_2)) \in M_2$.

We first prove a simple lemma.

Lemma 2. *Let l_0 be the map defined by $\text{dom}(l_0) = \{(0, 0)\}$ and $l_0(0, 0) = \sigma_0$. Then l_0 is respectful.*

Proof. By definition of l_0 .

The map l_0 defined by lemma 2 is called *init map* with respect to (σ_0, M_1, M_2) . We shall say that a map l' *extends* a map l iff $\text{dom}(l') \supseteq \text{dom}(l)$ and $l'|_{\text{dom}(l)} = l$. Much more difficult than lemma 2 are the following lemmas.

Lemma 3. *Let $i \in \{1, 2\}$ and ψ be a formula such that $G_i\psi \in \Gamma$. Let l be a respectful map and $(s_1, s_2) \in \text{dom}(l)$ such that $l(s_1, s_2)(G_i\psi) = 0$. There exists a respectful map l' and there exists $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ such that l' extends l , $(t_1, t_2) \in \text{dom}(l')$, $(s_1, s_2) \prec_i (t_1, t_2)$ and $l'(t_1, t_2)(\psi) = 0$.*

Proof. Since $\text{dom}(l)$ is finite, then there exists a nonnegative integer k and there exists $(u_1^1, u_2^1), \dots, (u_1^k, u_2^k) \in \mathbb{Q} \times \mathbb{Q}$ such that $](s_1, s_2), +\infty[\cap \text{dom}(l) = \{(u_1^1, u_2^1), \dots, (u_1^k, u_2^k)\}$. Without loss of generality, we may assume that $(s_1, s_2) \prec (u_1^1, u_2^1) \dots \prec (u_1^k, u_2^k)$. Moreover, $(s_1, s_2) \prec_{j_1} (u_1^1, u_2^1) \dots \prec_{j_k} (u_1^k, u_2^k)$ for exactly one k -tuple $(j_1, \dots, j_k) \in \{1, 2\}^k$. Now, we proceed by induction on k .

Basis. Suppose $k = 0$. Now, consider the 2 following cases.

1. Suppose $]-\infty, (s_1, s_2)[\cap \text{dom}(l) = \emptyset$. Since l is respectful and $l(s_1, s_2)(G_i\psi) = 0$, then $(s_1, s_2) = (0, 0)$ and there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(\sigma_0, \mu) \in M_i$ and $\mu(\psi) = 0$. Let $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ be such that $(0, 0) \prec_i (t_1, t_2)$ and l' be the least extension of l such that $(t_1, t_2) \in \text{dom}(l')$ and $l'(t_1, t_2) = \mu$. Obviously, l' is respectful.
2. Suppose $]-\infty, (s_1, s_2)[\cap \text{dom}(l) \neq \emptyset$. Since $\text{dom}(l)$ is finite, then there exists $(v_1, v_2) \in \text{dom}(l)$ such that $(v_1, v_2) \prec (s_1, s_2)$ and $](v_1, v_2), (s_1, s_2)[\cap \text{dom}(l) = \emptyset$. Moreover, $(v_1, v_2) \prec_j (s_1, s_2)$ for exactly one $j \in \{1, 2\}$. Since l is respectful and $l(s_1, s_2)(G_i\psi) = 0$, then $(l(v_1, v_2), l(s_1, s_2)) \in M_j$ and there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(l(s_1, s_2), \mu) \in M_i$ and $\mu(\psi) = 0$. Let $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ be such that $(s_1, s_2) \prec_i (t_1, t_2)$ and l' be the least extension of l such that $(t_1, t_2) \in \text{dom}(l')$ and $l'(t_1, t_2) = \mu$. Obviously, l' is respectful.

Step. Suppose $k \geq 1$. Now, consider the 4 following cases.

1. Suppose $i = 1$ and $j_1 = 1$. Since l is respectful and $l(s_1, s_2)(G_1\psi) = 0$, then $(l(s_1, s_2), l(u_1^1, u_2^1)) \in M_1$ and **(i)** there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(l(s_1, s_2), \mu) \in M_1$, $(\mu, l(u_1^1, u_2^1)) \in M_1$ and $\mu(\psi) = 0$, or **(ii)** there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(l(s_1, s_2), \mu) \in M_1$, $(\mu, l(u_1^1, u_2^1)) \in M_2$ and $\mu(\psi) = 0$, or **(iii)** $l(u_1^1, u_2^1)(\psi) = 0$, or **(iv)** $l(u_1^1, u_2^1)(G_1\psi) = 0$, or **(v)** $l(u_1^1, u_2^1)(G_2\psi) = 0$. In case **(i)**, let $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ be such that $(s_1, s_2) \prec_1 (t_1, t_2)$ and $(t_1, t_2) \prec_1 (u_1^1, u_2^1)$ and l' be the least extension of l such that $(t_1, t_2) \in \text{dom}(l')$ and $l'(t_1, t_2) = \mu$. In case **(ii)**, let $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ be such that $(s_1, s_2) \prec_1 (t_1, t_2)$ and $(t_1, t_2) \prec_2 (u_1^1, u_2^1)$ and l' be the least extension of l such that $(t_1, t_2) \in \text{dom}(l')$ and $l'(t_1, t_2) = \mu$. In case **(iii)**, let $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ be (u_1^1, u_2^1) and l' be l . In case **(iv)**, by induction hypothesis, there exists a respectful map l' and there exists $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ such that l' extends l , $(t_1, t_2) \in \text{dom}(l')$, $(u_1^1, u_2^1) \prec_1 (t_1, t_2)$ and $l'(t_1, t_2)(\psi) = 0$. In case **(v)**, by induction hypothesis, there exists a respectful map l' and there exists $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ such that l' extends l , $(t_1, t_2) \in \text{dom}(l')$, $(u_1^1, u_2^1) \prec_2 (t_1, t_2)$ and $l'(t_1, t_2)(\psi) = 0$. Obviously, in all cases, l' is respectful.
2. Suppose $i = 1$ and $j_1 = 2$. Since l is respectful and $l(s_1, s_2)(G_1\psi) = 0$, then $(l(s_1, s_2), l(u_1^1, u_2^1)) \in M_2$ and $l(u_1^1, u_2^1)(G_1\psi) = 0$. By induction hypothesis, there exists a respectful map l' and there exists $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ such that l' extends l , $(t_1, t_2) \in \text{dom}(l')$, $(u_1^1, u_2^1) \prec_1 (t_1, t_2)$ and $l'(t_1, t_2)(\psi) = 0$. Obviously, l' is respectful.
3. Suppose $i = 2$ and $j_1 = 1$. Since l is respectful and $l(s_1, s_2)(G_2\psi) = 0$, then $(l(s_1, s_2), l(u_1^1, u_2^1)) \in M_1$ and there exists an adequate function $\mu: \Gamma \rightarrow$

$\{0, 1\}$ such that $(l(s_1, s_2), \mu) \in M_2$, $(\mu, l(u_1^1, u_2^1)) \in M_1$ and $\mu(\psi) = 0$. Let $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ be such that $(s_1, s_2) \prec_2 (t_1, t_2)$ and $(t_1, t_2) \prec_1 (u_1^1, u_2^1)$ and l' be the least extension of l such that $(t_1, t_2) \in \text{dom}(l')$ and $l'(t_1, t_2) = \mu$. Obviously, l' is respectful.

4. Suppose $i = 2$ and $j_1 = 2$. Since l is respectful and $l(s_1, s_2)(G_2\psi) = 0$, then $(l(s_1, s_2), l(u_1^1, u_2^1)) \in M_2$ and **(i)** there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(l(s_1, s_2), \mu) \in M_2$, $(\mu, l(u_1^1, u_2^1)) \in M_2$ and $\mu(\psi) = 0$, or **(ii)** $l(u_1^1, u_2^1)(\psi) = 0$, or **(iii)** $l(u_1^1, u_2^1)(G_2\psi) = 0$. In case **(i)**, let $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ be such that $(s_1, s_2) \prec_2 (t_1, t_2)$ and $(t_1, t_2) \prec_2 (u_1^1, u_2^1)$ and l' be the least extension of l such that $(t_1, t_2) \in \text{dom}(l')$ and $l'(t_1, t_2) = \mu$. In case **(ii)**, let $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ be (u_1^1, u_2^1) and l' be l . In case **(iii)**, by induction hypothesis, there exists a respectful map l' and there exists $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ such that l' extends l , $(t_1, t_2) \in \text{dom}(l')$, $(u_1^1, u_2^1) \prec_2 (t_1, t_2)$ and $l'(t_1, t_2)(\psi) = 0$. Obviously, in all cases, l' is respectful.

Lemma 4. *Let $i \in \{1, 2\}$ and ψ be a formula such that $H_i\psi \in \Gamma$. Let l be a respectful map and $(s_1, s_2) \in \text{dom}(l)$ such that $l(s_1, s_2)(H_i\psi) = 0$. There exists a respectful map l' and there exists $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ such that l' extends l , $(t_1, t_2) \in \text{dom}(l')$, $(t_1, t_2) \prec_i (s_1, s_2)$ and $l'(t_1, t_2)(\psi) = 0$.*

Proof. Similar to the proof of lemma 3.

Lemma 5. *Let $i \in \{1, 2\}$. Let l be a respectful map and $(s_1, s_2), (t_1, t_2) \in \text{dom}(l)$ such that $(s_1, s_2) \prec_i (t_1, t_2)$ and $](s_1, s_2), (t_1, t_2)[\cap \text{dom}(l) = \emptyset$. There exists a respectful map l' and there exists $(u_1, u_2) \in \mathbb{Q} \times \mathbb{Q}$ such that l' extends l , $(u_1, u_2) \in \text{dom}(l')$, $(s_1, s_2) \prec_i (u_1, u_2)$ and $(u_1, u_2) \prec_i (t_1, t_2)$.*

Proof. Since l is respectful and $(s_1, s_2) \prec_i (t_1, t_2)$, then $(l(s_1, s_2), l(t_1, t_2)) \in M_i$ and there exists an adequate function $\mu: \Gamma \rightarrow \{0, 1\}$ such that $(l(s_1, s_2), \mu) \in M_i$ and $(\mu, l(t_1, t_2)) \in M_i$. Let $(u_1, u_2) \in \mathbb{Q} \times \mathbb{Q}$ be such that $(s_1, s_2) \prec_i (u_1, u_2)$ and $(u_1, u_2) \prec_i (t_1, t_2)$ and l' be the least extension of l such that $(u_1, u_2) \in \text{dom}(l')$ and $l'(t_1, t_2) = \mu$. Obviously, l' is respectful.

The maps l' defined by lemmas 3–5 are respectively called *right completion* of l with respect to i , ψ and (s_1, s_2) , *left completion* of l with respect to i , ψ and (s_1, s_2) and *dense completion* of l with respect to i , (s_1, s_2) and (t_1, t_2) .

6 Correctness and completeness of the mosaic method

Now, we are ready to formulate our main propositions.

6.1 Correctness

First, the correctness of the mosaic method.

Proposition 3. *If there exists a saturated dense premodel for φ then φ is satisfiable with respect to $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$.*

Proof. Suppose there exists a saturated dense premodel (σ_0, M_1, M_2) for φ . Following the line of reasoning suggested in [19], we think of the construction of a model for φ as a process approaching a limit via a sequence l_0, l_1, \dots of respectful maps. Lemma 2 is used to initiate the construction whereas lemmas 3–5 are used to make improvements at each step of the construction. Consider an enumeration $(i^0, \psi^0, (s_1^0, s_2^0), (t_1^0, t_2^0)), (i^1, \psi^1, (s_1^1, s_2^1), (t_1^1, t_2^1)), \dots$ of $\{1, 2\} \times \mathcal{L}_t \times (\mathbb{Q} \times \mathbb{Q}) \times (\mathbb{Q} \times \mathbb{Q})$ where each item appears infinitely often. We inductively define a sequence l_0, l_1, \dots of respectful maps as follows:

Basis. Let l_0 be the init map with respect to (σ_0, M_1, M_2) .

Step. Let l_n^{right}, l_n^{left} and l_{n+1} be the respectful maps defined as follows:

- (i) if i^n, ψ^n, l_n and (s_1^n, s_2^n) satisfy the conditions of lemma 3 then let l_n^{right} be the right completion of l_n with respect to i^n, ψ^n and (s_1^n, s_2^n) else let l_n^{right} be l_n ,
- (ii) if i^n, ψ^n, l_n^{right} and (s_1^n, s_2^n) satisfy the conditions of lemma 4 then let l_n^{left} be the left completion of l_n^{right} with respect to i^n, ψ^n and (s_1^n, s_2^n) else let l_n^{left} be l_n^{right} ,
- (iii) if $i^n, l_n^{left}, (s_1^n, s_2^n)$ and (t_1^n, t_2^n) satisfy the conditions of lemma 5 then let l_{n+1} be the dense completion of l_n^{left} with respect to $i^n, (s_1^n, s_2^n)$ and (t_1^n, t_2^n) else let l_{n+1} be l_n^{left} .

The reader may easily verify that the sequence l_0, l_1, \dots of respectful maps is such that $dom(l_0) \subseteq dom(l_1) \subseteq \dots$ and for all nonnegative integers n , $l_{n+1}|_{dom(l_n)} = l_n$. Let $l: \mathbb{Q} \times \mathbb{Q} \rightarrow (\Gamma \rightarrow \{0, 1\})$ be the partial function defined by $dom(l) = \bigcup \{dom(l_n) : n \text{ is a nonnegative integer}\}$ and $l(s_1, s_2) = l_n(s_1, s_2)$ for each $(s_1, s_2) \in dom(l)$, n being a nonnegative integer such that $(s_1, s_2) \in dom(l_n)$. Obviously, $(0, 0) \in dom(l)$ and $l(0, 0) = \sigma_0$. Let $\mathcal{F} = (T, \triangleleft_1, \triangleleft_2)$ be the structure defined by

- $T = dom(l)$,
- \triangleleft_1 and \triangleleft_2 are the binary relations on T defined by putting $(s_1, s_2) \triangleleft_1 (t_1, t_2)$ iff $s_1 < t_1$ and $(s_1, s_2) \triangleleft_2 (t_1, t_2)$ iff $s_1 = t_1$ and $s_2 < t_2$.

We define the binary relation \triangleleft on T by putting $(s_1, s_2) \triangleleft (t_1, t_2)$ iff $s_1 < t_1$, or $s_1 = t_1$ and $s_2 < t_2$. By lemmas 3–5, \mathcal{F} satisfies *SERI* and *DENS*. Since \mathcal{F} is a substructure of $\mathbb{Q} \times \mathbb{Q}$, then it satisfies *IRRE*, *DISJ*, *TRAN* and *UNIV*. Hence, \mathcal{F} is standard. Thus, by item 1 of proposition 1, \mathcal{F} is isomorphic with $\mathbb{Q} \times \mathbb{Q}$. Without loss of generality, we may assume that \mathcal{F} is equal to $\mathbb{Q} \times \mathbb{Q}$. Let $V: At \rightarrow 2^T$ be the function defined by $(s_1, s_2) \in V(p)$ iff $p \in \Gamma$ and $l(s_1, s_2)(p) = 1$ — i.e. V makes p true at (s_1, s_2) iff l says so — and $\mathcal{M} = (T, \triangleleft_1, \triangleleft_2, V)$. By induction on ψ , let us demonstrate that if $\psi \in \Gamma$ then

- if $l(s_1, s_2)(\psi) = 1$ then $\mathcal{M}, (s_1, s_2) \models \psi$,
- if $l(s_1, s_2)(\psi) = 0$ then $\mathcal{M}, (s_1, s_2) \not\models \psi$.

Basis. Suppose $\psi = p$. Now, if $\psi \in \Gamma$ then consider the 2 following cases.

1. Suppose $l(s_1, s_2)(\psi) = 1$. Therefore, $(s_1, s_2) \in V(p)$. Consequently, $\mathcal{M}, (s_1, s_2) \models \psi$.

2. Suppose $l(s_1, s_2)(\psi) = 0$. Hence, $(s_1, s_2) \notin V(p)$. Thus, $\mathcal{M}, (s_1, s_2) \not\models \psi$.

Step. Suppose $\psi = \perp$, or $\psi = \neg\chi$, or $\psi = \chi \vee \rho$, or $\psi = G_1\chi$, or $\psi = G_2\chi$, or $\psi = H_1\chi$, or $\psi = H_2\chi$. Leaving the cases $\psi = \perp$, $\psi = \neg\chi$ and $\psi = \chi \vee \rho$ to the reader, we only consider the cases $\psi = G_1\chi$ and $\psi = G_2\chi$, the cases $\psi = H_1\chi$ and $\psi = H_2\chi$ being similar to them. Now, if $G_i\chi \in \Gamma$ then consider the 2 following cases.

1. Suppose $l(s_1, s_2)(G_i\chi) = 1$. Therefore, $l_m(s_1, s_2)(G_i\chi) = 1$, m being a nonnegative integer such that $(s_1, s_2) \in \text{dom}(l_m)$. Let $(t_1, t_2) \in T$ such that $(s_1, s_2) \leq_i (t_1, t_2)$, we demonstrate $\mathcal{M}, (t_1, t_2) \models \chi$. Since $(t_1, t_2) \in T$, then there exists a nonnegative integer n such that $(t_1, t_2) \in \text{dom}(l_n)$. Let $o = \max\{m, n\}$. Since $l_m(s_1, s_2)(G_i\chi) = 1$, then $l_o(s_1, s_2)(G_i\chi) = 1$. Moreover, since $\text{dom}(l_o)$ is finite, then there exists a nonnegative integer k and there exists $(u_1^1, u_2^1), \dots, (u_1^k, u_2^k) \in T$ such that $](s_1, s_2), (t_1, t_2)[\cap \text{dom}(l_o) = \{(u_1^1, u_2^1), \dots, (u_1^k, u_2^k)\}$. Without loss of generality, we may assume that $(s_1, s_2) \leq (u_1^1, u_2^1) \dots \leq (u_1^k, u_2^k)$ and $(u_1^k, u_2^k) \leq (t_1, t_2)$. Moreover, $(s_1, s_2) \leq_{j_1} (u_1^1, u_2^1) \dots \leq_{j_k} (u_1^k, u_2^k)$ for exactly one k -tuple $(j_1, \dots, j_k) \in \{1, 2\}^k$ and $(u_1^k, u_2^k) \leq_j (t_1, t_2)$ for exactly one $j \in \{1, 2\}$. Since l_o is respectful, then $(l_o(s_1, s_2), l_o(u_1^1, u_2^1)) \in M_{j_1}, \dots, (l_o(u_1^{k-1}, u_2^{k-1}), l_o(u_1^k, u_2^k)) \in M_{j_k}$ and $(l_o(u_1^k, u_2^k), l_o(t_1, t_2)) \in M_j$. Remark that $\min\{\min\{j_1, \dots, j_k\}, j\} = i$. Since $l_o(s_1, s_2)(G_i\chi) = 1$, mosaics in M_1 are 1-temporal and mosaics in M_2 are 2-temporal, then $l_o(t_1, t_2)(\chi) = 1$. Consequently, $l(t_1, t_2)(\chi) = 1$. By induction hypothesis, $\mathcal{M}, (t_1, t_2) \models \chi$.
2. Suppose $l(s_1, s_2)(G_i\chi) = 0$. Hence, $l_m(s_1, s_2)(G_i\chi) = 0$, m being a nonnegative integer such that $(s_1, s_2) \in \text{dom}(l_m)$. By definition of the sequence l_0, l_1, \dots of respectful maps, there exists a nonnegative integer n and there exists $(t_1, t_2) \in T$ such that $(t_1, t_2) \in \text{dom}(l_n)$, $(s_1, s_2) \leq_i (t_1, t_2)$ and $l_n(t_1, t_2)(\chi) = 0$. Consequently, $l(t_1, t_2)(\chi) = 0$. By induction hypothesis, $\mathcal{M}, (t_1, t_2) \not\models \chi$. Since $(s_1, s_2) \leq_i (t_1, t_2)$, then $\mathcal{M}, (s_1, s_2) \not\models G_i\chi$.

Since (σ_0, M_1, M_2) is for φ , then $\sigma_0(\varphi) = 1$. Since $(0, 0) \in \text{dom}(l)$ and $l(0, 0) = \sigma_0$, then $l(0, 0)(\varphi) = 1$ and $\mathcal{M}, (0, 0) \models \varphi$. Since \mathcal{F} is isomorphic with $\mathbb{Q} \times \mathbb{Q}$, then φ is satisfiable with respect to $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$.

6.2 Completeness

Second, the completeness of the mosaic method.

Proposition 4. *If φ is satisfiable with respect to $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$ then there exists a saturated dense premodel for φ of cardinality bounded by $2 \times (8 \times |\varphi| + 6)^8$.*

Proof. Suppose φ is satisfiable with respect to $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$. Hence, there exists a linear order $\mathcal{F}_1 = (T_1, \leq_1)$ in \mathcal{C}_{ud} , there exists a linear order $\mathcal{F}_2 = (T_2, \leq_2)$ in \mathcal{C}_{ud} and there exists a function $V: At \rightarrow 2^{T_1 \times T_2}$ such that $(\mathcal{F}_1, \mathcal{F}_2, V)$ is a model for φ . For all $(s_1, s_2) \in T_1 \times T_2$, let $\sigma(s_1, s_2): \Gamma \rightarrow \{0, 1\}$ be the function such that if $\psi \in \Gamma$ then $\sigma(s_1, s_2)(\psi) = 1$ if $(\mathcal{F}_1, \mathcal{F}_2, V), (s_1, s_2) \models \psi$ then 1 else 0. Obviously,

$\sigma(s_1, s_2)$ is an adequate function. Let $s_1 \in T_1$. Let \equiv_{s_1} be the binary relation on T_2 defined as follows: $t_2 \equiv_{s_1} u_2$ iff $\sigma(s_1, t_2) = \sigma(s_1, u_2)$. Obviously, \equiv_{s_1} is an equivalence relation on T_2 . Moreover, since Γ is finite, then there exists finitely many equivalence classes modulo \equiv_{s_1} . We define

- $\Delta_2^{s_1} = \{G_2\psi: G_2\psi \in \Gamma \text{ is such that } \sigma(s_1, t_2)(\psi) = 0 \text{ for some } t_2 \in T_2\}$,
- $\Lambda_2^{s_1} = \{H_2\psi: H_2\psi \in \Gamma \text{ is such that } \sigma(s_1, t_2)(\psi) = 0 \text{ for some } t_2 \in T_2\}$.

Obviously, $\Delta_2^{s_1} \subseteq \Gamma$ and $\Lambda_2^{s_1} \subseteq \Gamma$. Therefore, by lemma 1, $\text{card}(\Delta_2^{s_1}) \leq 4 \times |\varphi| + 2$ and $\text{card}(\Lambda_2^{s_1}) \leq 4 \times |\varphi| + 2$. Let

- for all $G_2\psi \in \Delta_2^{s_1}$, $f_2^{s_1}(G_2\psi) = \{t_2: t_2 \in T_2 \text{ is such that } \sigma(s_1, t_2)(\psi) = 0\}$,
- for all $H_2\psi \in \Lambda_2^{s_1}$, $g_2^{s_1}(H_2\psi) = \{t_2: t_2 \in T_2 \text{ is such that } \sigma(s_1, t_2)(\psi) = 0\}$.

We shall say that

- $t_2 \in f_2^{s_1}(G_2\psi)$ is maximal iff for all $u_2 \in f_2^{s_1}(G_2\psi)$, $t_2 \equiv_{s_1} u_2$, or there exists $v_2 \in f_2^{s_1}(G_2\psi)$ such that $u_2 <_2 v_2$ and $t_2 \equiv_{s_1} v_2$,
- $t_2 \in g_2^{s_1}(H_2\psi)$ is minimal iff for all $u_2 \in g_2^{s_1}(H_2\psi)$, $t_2 \equiv_{s_1} u_2$, or there exists $v_2 \in g_2^{s_1}(H_2\psi)$ such that $v_2 <_2 u_2$ and $t_2 \equiv_{s_1} v_2$.

Since there exists finitely many equivalence classes modulo \equiv_{s_1} , then for all $G_2\psi \in \Delta_2^{s_1}$, there exists $t_2(G_2\psi) \in f_2^{s_1}(G_2\psi)$ such that $t_2(G_2\psi)$ is maximal and for all $H_2\psi \in \Lambda_2^{s_1}$, there exists $t_2(H_2\psi) \in g_2^{s_1}(H_2\psi)$ such that $t_2(H_2\psi)$ is minimal. For all $s_2 \in T_2$, let

- $\Pi_2^{s_1}(s_2) = \{s_2\} \cup \{t_2(G_2\psi): G_2\psi \in \Delta_2^{s_1}\} \cup \{t_2(H_2\psi): H_2\psi \in \Lambda_2^{s_1}\}$.

Since $\text{card}(\Delta_2^{s_1}) \leq 4 \times |\varphi| + 2$ and $\text{card}(\Lambda_2^{s_1}) \leq 4 \times |\varphi| + 2$, then $\text{card}(\Pi_2^{s_1}(s_2)) \leq 8 \times |\varphi| + 5$. Let $t_2, u_2 \in \Pi_2^{s_1}(s_2)$ be such that $t_2 \equiv_{s_1} \circ <_2 \circ \equiv_{s_1} u_2$. A witness of 2-density for t_2 and u_2 is $v_2 \in T_2$ such that $t_2 \equiv_{s_1} \circ <_2 \circ \equiv_{s_1} v_2$, $v_2 \equiv_{s_1} \circ <_2 \circ \equiv_{s_1} u_2$ and $v_2 \equiv_{s_1} \circ <_2 \circ \equiv_{s_1} v_2$. Since there exists finitely many equivalence classes modulo \equiv_{s_1} and $t_2 \equiv_{s_1} \circ <_2 \circ \equiv_{s_1} u_2$, then such a witness exists. Now, let

- $\Pi_{2,witness}^{s_1}(s_2)$ be the least subset of T_2 containing $\Pi_2^{s_1}(s_2)$ and such that for all $t_2, u_2 \in \Pi_2^{s_1}(s_2)$, if $t_2 \equiv_{s_1} \circ <_2 \circ \equiv_{s_1} u_2$ then $\Pi_{2,witness}^{s_1}(s_2)$ contains a witness of 2-density for t_2 and u_2 .

Remark that the cardinality of $\Pi_{2,witness}^{s_1}(s_2)$ is bounded by $(8 \times |\varphi| + 6)^2$. Let \cong be the binary relation on $T_1 \times T_2$ defined as follows: $(s_1, s_2) \cong (t_1, t_2)$ iff $\{\sigma(s_1, u_2): u_2 \in \Pi_2^{s_1}(s_2)\} = \{\sigma(t_1, v_2): v_2 \in \Pi_2^{t_1}(t_2)\}$. Obviously, \cong is an equivalence relation on $T_1 \times T_2$. Moreover, since Γ is finite, then there exists finitely many equivalence classes modulo \cong . We define

- $\Delta_1 = \{G_1\psi: G_1\psi \in \Gamma \text{ is such that } \sigma(s_1, s_2)(\psi) = 0 \text{ for some } (s_1, s_2) \in T_1 \times T_2\}$,
- $\Lambda_1 = \{H_1\psi: H_1\psi \in \Gamma \text{ is such that } \sigma(s_1, s_2)(\psi) = 0 \text{ for some } (s_1, s_2) \in T_1 \times T_2\}$.

Obviously, $\Delta_1 \subseteq \Gamma$ and $\Lambda_1 \subseteq \Gamma$. Therefore, by lemma 1, $\text{card}(\Delta_1) \leq 4 \times |\varphi| + 2$ and $\text{card}(\Lambda_1) \leq 4 \times |\varphi| + 2$. Let

- for all $G_1\psi \in \Delta_1$, $f_1(G_1\psi) = \{(s_1, s_2): (s_1, s_2) \in T_1 \times T_2 \text{ is such that } \sigma(s_1, s_2)(\psi) = 0\}$,
- for all $H_1\psi \in \Lambda_1$, $g_1(H_1\psi) = \{(s_1, s_2): (s_1, s_2) \in T_1 \times T_2 \text{ is such that } \sigma(s_1, s_2)(\psi) = 0\}$.

We shall say that

- $(s_1, s_2) \in f_1(G_1\psi)$ is maximal iff for all $(t_1, t_2) \in f_1(G_1\psi)$, $(s_1, s_2) \cong (t_1, t_2)$, or there exists $(u_1, u_2) \in f_1(G_1\psi)$ such that $(t_1, t_2) \prec (u_1, u_2)$ and $(s_1, s_2) \cong (u_1, u_2)$,
- $(s_1, s_2) \in g_1(H_1\psi)$ is minimal iff for all $(t_1, t_2) \in g_1(H_1\psi)$, $(s_1, s_2) \cong (t_1, t_2)$, or there exists $(u_1, u_2) \in g_1(H_1\psi)$ such that $(u_1, u_2) \prec (t_1, t_2)$ and $(s_1, s_2) \cong (u_1, u_2)$.

Since there exists finitely many equivalence classes modulo \cong , then for all $G_1\psi \in \Delta_1$, there exists $(s_1(G_1\psi), s_2(G_1\psi)) \in f_1(G_1\psi)$ such that $(s_1(G_1\psi), s_2(G_1\psi))$ is maximal and for all $H_1\psi \in \Lambda_1$, there exists $(s_1(H_1\psi), s_2(H_1\psi)) \in g_1(H_1\psi)$ such that $(s_1(H_1\psi), s_2(H_1\psi))$ is minimal. Since $(\mathcal{F}_1, \mathcal{F}_2, V)$ is a model for φ , there exists $(s_1^0, s_2^0) \in T_1 \times T_2$ such that $(\mathcal{F}_1, \mathcal{F}_2, V), (s_1^0, s_2^0) \models \varphi$. Let

- $\Pi_1 = \{(s_1^0, s_2^0)\} \cup \{(s_1(G_1\psi), s_2(G_1\psi)): G_1\psi \in \Delta_1\} \cup \{(s_1(H_1\psi), s_2(H_1\psi)): H_1\psi \in \Lambda_1\}$.

Since $\text{card}(\Delta_1) \leq 4 \times |\varphi| + 2$ and $\text{card}(\Lambda_1) \leq 4 \times |\varphi| + 2$, then $\text{card}(\Pi_1) \leq 8 \times |\varphi| + 5$. Let $(s_1, s_2), (t_1, t_2) \in \Pi_1$ be such that $(s_1, s_2) \cong \circ \prec_1 \circ \cong (t_1, t_2)$. A witness of 1-density for (s_1, s_2) and (t_1, t_2) is $(u_1, u_2) \in T_1 \times T_2$ such that $(s_1, s_2) \cong \circ \prec_1 \circ \cong (u_1, u_2)$, $(u_1, u_2) \cong \circ \prec_1 \circ \cong (t_1, t_2)$ and $(u_1, u_2) \cong \circ \prec_1 \circ \cong (u_1, u_2)$. Since there exists finitely many equivalence classes modulo \cong and $(s_1, s_2) \cong \circ \prec_1 \circ \cong (t_1, t_2)$, then such a witness exists. Now, let

- $\Pi_{1, \text{witness}}$ be the least subset of $T_1 \times T_2$ containing Π_1 and such that for all $(s_1, s_2), (t_1, t_2) \in \Pi_1$, if $(s_1, s_2) \cong \circ \prec_1 \circ \cong (t_1, t_2)$ then $\Pi_{1, \text{witness}}$ contains a witness of 1-density for (s_1, s_2) and (t_1, t_2) .

Remark that the cardinality of $\Pi_{1, \text{witness}}$ is bounded by $(8 \times |\varphi| + 6)^2$. Now, let (σ_0, M_1, M_2) be the structure defined by

- $\sigma_0 = \sigma(s_1^0, s_2^0)$,
- M_1 is the set of all mosaics of the form (σ, τ) such that for some $(s_1, s_2), (t_1, t_2) \in \Pi_{1, \text{witness}}$, there exists $u_2 \in \Pi_{2, \text{witness}}^{s_1}(s_2)$ and there exists $v_2 \in \Pi_{2, \text{witness}}^{t_1}(t_2)$ such that $\sigma = \sigma(s_1, u_2)$, $\tau = \sigma(t_1, v_2)$ and $(s_1, s_2) \cong \circ \prec_1 \circ \cong (t_1, t_2)$,
- M_2 is the set of all mosaics of the form (σ, τ) such that for some $(s_1, s_2) \in \Pi_{1, \text{witness}}$, there exists $t_2, u_2 \in \Pi_{2, \text{witness}}^{s_1}(s_2)$ such that $\sigma = \sigma(s_1, t_2)$, $\tau = \sigma(s_1, u_2)$ and $t_2 \equiv_{s_1} \circ \prec_2 \circ \equiv_{s_1} u_2$.

Remark that the cardinality of (σ_0, M_1, M_2) is bounded by $2 \times (8 \times |\varphi| + 6)^8$. Moreover, $\sigma_0(\varphi) = 1$. Hence, it suffices to prove the following properties:

- (σ_0, M_1, M_2) is a premodel,
- (σ_0, M_1, M_2) is saturated,
- (σ_0, M_1, M_2) is dense.

The proofs of these properties are left to the reader.

7 Decidability and complexity

Now, we are ready to provide a complete decision procedure in nondeterministic polynomial time for the satisfiability problem the temporal logic of the lexicographic products of unbounded dense linear orders gives rise to.

Theorem 1. *The satisfiability problem with respect to $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$ is in NP.*

Proof. Given a formula φ , let us proceed as follows:

1. compute the least subformula closed full set Γ of formulas containing φ ,
2. nondeterministically choose a premodel (σ_0, M_1, M_2) for φ of cardinality bounded by $2 \times (8 \times |\varphi| + 6)^8$,
3. check whether (σ_0, M_1, M_2) is saturated and dense.

By lemma 1, $card(\Gamma) \leq 4 \times |\varphi| + 2$. Hence, (σ_0, M_1, M_2) is polysize in $|\varphi|$. Thus, we can execute steps (i), (ii) and (iii) in time polynomial in $|\varphi|$. Obviously, the execution of steps (i), (ii) and (iii) leads to an “accept” iff there exists a saturated dense premodel for φ of cardinality bounded by $2 \times (8 \times |\varphi| + 6)^8$. Therefore, by propositions 3 and 4, the execution of steps (i), (ii) and (iii) leads to an “accept” iff φ is satisfiable with respect to $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$. Consequently, the satisfiability problem with respect to $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$ is in NP.

8 Conclusion

Temporal logics in which one can assign a proper meaning to the association of statements about different grained temporal domains have been considered. See [10, 15, 21] for details. Nevertheless, it seems that the results concerning the issues of axiomatization/completeness and decidability/complexity presented in [3, 4] and in this paper constitute the first steps towards a temporal logic based on different levels of abstraction. Much remains to be done.

Let us, for instance, consider the lexicographic products of special linear flows of time like \mathbb{Z} , \mathbb{Q} and \mathbb{R} . For each of these products, what is the corresponding first-order theory in terms of the binary predicates $<_1$ and $<_2$? Is this first-order theory decidable? What is the corresponding modal logic in terms of the temporal connectives G_1 , G_2 , H_1 and H_2 ? Is this modal logic decidable?

Concerning the problems of decidability and complexity, all normal extensions of $S4.3$, as proved in [8, 11], possess the finite model property and all finitely axiomatizable normal extensions of $K4.3$, as proved in [25], are decidable. Is it possible to demonstrate similar results in our lexicographic setting? Or could undecidability results similar to the ones obtained by Reynolds and Zakharyashev [23] within the context of the products of the modal logics determined by arbitrarily long linear orders be demonstrated in our lexicographic setting?

In other respect, we can associate with $<_1$ and $<_2$ the until-like connectives U_1 and U_2 and the since-like connectives S_1 and S_2 , the formulas $\varphi U_1 \psi$, $\varphi U_2 \psi$, $\varphi S_1 \psi$ and $\varphi S_2 \psi$ being read as one reads the formulas $\varphi U \psi$ and $\varphi S \psi$ in classical temporal logic, this time with $<_1$ and $<_2$. What can be done concerning the problems of axiomatization/completeness and decidability/complexity these new temporal connectives give rise to?

An important result in temporal logic is Kamp's Theorem. It concerns the functional completeness of the until operator [13]. Kamp's Theorem says that for all monadic formulas of the first-order theory of linear orders, there exists a temporal formula using only the until and since temporal operators such that both formulas — the monadic one and the temporal one — are equally interpreted in any temporal model. What can be done about Kamp's Theorem in terms of the binary predicates $<_1$ and $<_2$ and the temporal connectives U_1 , U_2 , S_1 and S_2 ?

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