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ABOUT THE UNIFICATION TYPE OF FUSIONS OF MODAL LOGICS

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Abstract

In a modal logic \mathbf{L} , a unifier of a formula φ is a substitution σ such that $\sigma(\varphi)$ is in \mathbf{L} . When unifiable formulas have no minimal complete sets of unifiers, they are nullary. Otherwise, they are either infinitary, or finitary, or unitary depending on the cardinality of their minimal complete sets of unifiers. The fusion $\mathbf{L}_1 \otimes \mathbf{L}_2$ of modal logics \mathbf{L}_1 and \mathbf{L}_2 respectively based on the modal connectives \Box_1 and \Box_2 is the least modal logic based on these modal connectives and containing both \mathbf{L}_1 and \mathbf{L}_2 . In this paper, we prove that if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is unitary then \mathbf{L}_1 and \mathbf{L}_2 are unitary and if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is finitary then \mathbf{L}_1 and \mathbf{L}_2 are either unitary, or finitary. We also prove that the fusion of arbitrary consistent extensions of $\mathbf{S5}$ is nullary when these extensions are different from \mathbf{Triv} ¹.

Keywords: Fusions of modal logics. Unification type.

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¹Dzik conjectured that the fusion $\mathbf{S5} \otimes \mathbf{S5}$ of $\mathbf{S5}$ with itself is either nullary, or infinitary [11, Chapter 6].

1 Introduction

The unification problem in a modal logic \mathbf{L} is to determine, given a formula φ , whether there exists a substitution σ such that $\sigma(\varphi)$ is in \mathbf{L} [1]². In that case, σ is a unifier of φ . We shall say that a set of unifiers of a unifiable formula φ is complete if for all unifiers σ of φ , there exists a unifier τ of φ in that set such that τ is more general than σ . When unifiable formulas have no minimal complete sets of unifiers, they are nullary. Otherwise, they are either infinitary, or finitary, or unitary depending on the cardinality of their minimal complete sets of unifiers [11]. To be nullary is considered to be the worst situation for a unifiable formula whereas to be unitary is considered to be better than to be finitary which is itself considered to be better than to be infinitary. The unification type of a modal logic is the worst unification type of its unifiable formulas³.

The fusion $\mathbf{L}_1 \otimes \mathbf{L}_2$ of modal logics \mathbf{L}_1 and \mathbf{L}_2 respectively based on the modal connectives \Box_1 and \Box_2 is the least modal logic based on these modal connectives and containing both \mathbf{L}_1 and \mathbf{L}_2 . A first immediate result is that $\mathbf{L}_1 \otimes \mathbf{L}_2$ is a conservative extension of the modal logics \mathbf{L}_1 and \mathbf{L}_2 when \mathbf{L}_1 and \mathbf{L}_2 are consistent. A number of other results — transfer results — have been obtained as well. They concern properties preserved under the operation of forming fusions: the fusion of decidable modal logics is decidable, the fusion of modal logics having uniform interpolation property has uniform interpolation property, etc. See [15, Chapter 4] and [23, 24, 31]. To the best of our knowledge, the preservation of properties related to the unification problem has not been studied yet.

Owing to its strong connections with the admissibility problem [29], the unification problem is an important problem in Applied Non-Classical Logics [1], a domain of investigations where fusions of modal logics are omnipresent [24]. It is therefore natural to ask how the unification types of modal logics are related to the unification type of their fusion. In this paper, we prove that if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is unitary then \mathbf{L}_1 and \mathbf{L}_2 are unitary and if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is finitary then \mathbf{L}_1 and \mathbf{L}_2 are either unitary, or finitary. In other respects, Dzik conjectured that the fusion $\mathbf{S5} \otimes \mathbf{S5}$ of $\mathbf{S5}$ with itself is either nullary, or infinitary [11, Chapter 6]. Cla-

²We assume that the reader is at home with tools and techniques in modal logics. In particular, we follow the standard conventions for talking about modal logics: $\mathbf{S5}$ is the least modal logic containing the formulas usually denoted (\mathbf{T}) , (4) and (\mathbf{B}) , \mathbf{KT} is the least modal logic containing the formula usually denoted (\mathbf{T}) , etc. For more on this, see [7, 8, 22]. As a result, in the main body of the paper, we will present neither the algebraic semantics of modal logics, nor the relational semantics of modal logics, preferring to introduce semantic tools and techniques when they are needed.

³About the unification type of modal logics, it is known that \mathbf{KB} , \mathbf{KD} , \mathbf{KDB} , \mathbf{KT} and \mathbf{KTB} are nullary [3, 4, 5], $\mathbf{S5}$ and $\mathbf{S4.3}$ are unitary [10, 12], some transitive modal logics like $\mathbf{K4}$ and $\mathbf{S4}$ are finitary [18, 19], \mathbf{K} is nullary [20] and $\mathbf{K4D1}$ is unitary [21], the nullariness of \mathbf{KB} , \mathbf{KD} , \mathbf{KDB} , \mathbf{KT} and \mathbf{KTB} having only been obtained within the context of unification with parameters. No modal logic is known to be infinitary.

rifying Dzik’s conjecture, we prove that the fusion of arbitrary consistent extensions of **S5** is nullary when these extensions are different from **Triv**. An Appendix includes the proofs of some of our results.

Jeřábek has proved that **K** is nullary by showing that the **K**-unifiable formula $x \rightarrow \Box x$ has no minimal complete sets of unifiers [20]. In Jeřábek’s line of reasoning, the fact that for all $d \geq 0$, $\Box^{d+1} \perp \rightarrow \Box^d \perp \notin \mathbf{K}$ plays an important role. Unfortunately, for all $d \geq 0$, $\Box^d \perp$ is either equivalent to \perp , or equivalent to $\Box \perp$ in **KB**, **KD**, **KDB**, **KT** and **KTB**. It follows that Jeřábek’s line of reasoning has to be seriously adapted if one wants to apply it to **KB**, **KD**, **KDB**, **KT** and **KTB**. This has been done in [3, 4, 5] by using parameters and by considering much more complicated formulas than $x \rightarrow \Box x$. For the fusion of arbitrary consistent extensions of **S5** different from **Triv**, a new adaptation of Jeřábek’s line of reasoning is described in the course of Lemmas 14–29 and Propositions 7, 8 and 9.

2 Syntax

2.1 Formulas and substitutions

Let *VAR* be a countably infinite set of *propositional variables* (with typical members denoted x, y , etc). Let *PAR* be a countably infinite set of *propositional parameters* (with typical members denoted p, q , etc). *Atoms* (denoted α, β , etc) are either variables, or parameters. Let I be a non-empty subset of $\{1, 2\}$. The set *FOR_I* of *I-formulas* (with typical members denoted φ, ψ , etc) is inductively defined as follows:

- $\varphi, \psi ::= \alpha \mid \perp \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box_i \varphi$

where i ranges over I . We adopt the standard rules for omission of the parentheses. For all I -formulas φ , we write “ φ^0 ” to mean “ $\neg\varphi$ ” and we write “ φ^1 ” to mean “ φ ”. For all I -formulas φ , let $\text{var}(\varphi)$ be the set of all variables occurring in φ . For all I -formulas φ , the *degree* of φ (denoted $\text{deg}(\varphi)$) is defined as usual. An *I-substitution* is a function σ associating to each variable x an I -formula $\sigma(x)$ ⁴. We shall say that an I -substitution σ *moves* a variable x if $\sigma(x) \neq x$. Following the standard assumption considered in the literature [1], we will always assume that I -substitutions move at most finitely many variables. For all $\{1, 2\}$ -formulas $\varphi(x_1, \dots, x_m)$, let $\sigma(\varphi(x_1, \dots, x_m))$ be the $\{1, 2\}$ -formula $\varphi(\sigma(x_1), \dots, \sigma(x_m))$. The *composition* $\sigma \circ \tau$ of the I -substitutions σ and τ is the I -substitution associating to each variable x the I -formula $\tau(\sigma(x))$. Obviously, for all $\{1, 2\}$ -formulas $\varphi(x_1, \dots, x_m)$, $(\sigma \circ \tau)(\varphi(x_1, \dots, x_m))$ is the $\{1, 2\}$ -formula $\varphi(\tau(\sigma(x_1)), \dots, \tau(\sigma(x_m)))$.

⁴Occasionally, we will slightly abuse notation by considering that $\{1\}$ -substitutions and $\{2\}$ -substitutions are also $\{1, 2\}$ -substitutions.

2.2 Abbreviations and translation functions

The Boolean connectives \top , \wedge , \rightarrow and \leftrightarrow are defined by the usual abbreviations. For all finite sets X of variables, we will use \top_X as a shorthand for $\bigwedge\{x \vee \top : x \in X\}$. As it is traditionally done, in the extreme case when the finite set X of variables is empty, \top_X will be a shorthand for \top . The role of the finite set X of variables in the definition of \top_X will become clear in Propositions 3 and 6. Nevertheless, we can already mention that this role is connected to the fact that for all finite sets X of variables, \top_X is a tautology such that $\text{var}(\top_X) = X$. The modal connectives \diamond_1 and \diamond_2 are defined as follows:

- $\diamond_1\varphi ::= \neg\Box_1\neg\varphi$,
- $\diamond_2\varphi ::= \neg\Box_2\neg\varphi$.

From now on in this paper,

let p, q, r be fixed distinct parameters.

Now, let us define modal connectives that will be useful in Section 6 for proving Proposition 9 saying that the fusion of arbitrary consistent extensions of **S5** is nullary when these extensions are different from **Triv**. The modal connectives \boxplus and \boxminus are defined as follows:

- $\boxplus\varphi ::= p^1 \wedge q^0 \wedge r^1 \rightarrow \Box_1(p^0 \wedge q^0 \wedge r^0 \rightarrow \Box_2(p^0 \wedge q^0 \wedge r^1 \rightarrow \Box_1(p^0 \wedge q^1 \wedge r^0 \rightarrow \Box_2(p^0 \wedge q^1 \wedge r^1 \rightarrow \Box_1(p^1 \wedge q^0 \wedge r^0 \rightarrow \Box_2(p^1 \wedge q^0 \wedge r^1 \rightarrow \varphi))))))$,
- $\boxminus\varphi ::= p^1 \wedge q^0 \wedge r^1 \rightarrow \Box_2(p^1 \wedge q^0 \wedge r^0 \rightarrow \Box_1(p^0 \wedge q^1 \wedge r^1 \rightarrow \Box_2(p^0 \wedge q^1 \wedge r^0 \rightarrow \Box_1(p^0 \wedge q^0 \wedge r^1 \rightarrow \Box_2(p^0 \wedge q^0 \wedge r^0 \rightarrow \Box_1(p^1 \wedge q^0 \wedge r^1 \rightarrow \varphi))))))$.

For all $k \geq 0$, the modal connectives \boxplus^k and \boxminus^k are inductively defined as follows:

- $\boxplus^0\varphi ::= \varphi$,
- $\boxminus^0\varphi ::= \varphi$,
- $\boxplus^{k+1}\varphi ::= \boxplus\boxplus^k\varphi$,
- $\boxminus^{k+1}\varphi ::= \boxminus\boxminus^k\varphi$.

For all $k \geq 0$, the modal connectives $\boxplus^{<k}$ and $\boxminus^{<k}$ are inductively defined as follows:

- $\boxplus^{<0}\varphi ::= \top$,
- $\boxminus^{<0}\varphi ::= \top$,
- $\boxplus^{<k+1}\varphi ::= \boxplus^{<k}\varphi \wedge \boxplus^k\varphi$,

- $\Box^{<k+1}\varphi ::= \Box^{<k}\varphi \wedge \Box^k\varphi.$

Now, let us define translation functions that will be useful in Section 5 for proving Propositions 4 and 5 saying that if the fusion of arbitrary consistent modal logics is unitary then both of them are unitary and if the fusion of arbitrary consistent modal logics is finitary then both of them are either unitary, or finitary. We inductively define for all finite sets X of variables and for all $i \in \{1, 2\}$, the translation functions $\text{tr}_i^T : FOR_{\{1,2\}} \longrightarrow FOR_{\{i\}}$ and $\text{tr}_{X,i}^V : FOR_{\{1,2\}} \longrightarrow FOR_{\{i\}}$ as follows:

- $\text{tr}_i^T(\alpha) = \alpha,$
- $\text{tr}_{X,i}^V(\alpha) = \alpha,$
- $\text{tr}_i^T(\perp) = \perp,$
- $\text{tr}_{X,i}^V(\perp) = \perp,$
- $\text{tr}_i^T(\neg\varphi) = \neg\text{tr}_i^T(\varphi),$
- $\text{tr}_{X,i}^V(\neg\varphi) = \neg\text{tr}_{X,i}^V(\varphi),$
- $\text{tr}_i^T(\varphi \vee \psi) = \text{tr}_i^T(\varphi) \vee \text{tr}_i^T(\psi),$
- $\text{tr}_{X,i}^V(\varphi \vee \psi) = \text{tr}_{X,i}^V(\varphi) \vee \text{tr}_{X,i}^V(\psi),$
- $\text{tr}_i^T(\Box_j\varphi) = \Box_j\text{tr}_i^T(\varphi)$ when $i = j,$
- $\text{tr}_{X,i}^V(\Box_j\varphi) = \Box_j\text{tr}_{X,i}^V(\varphi)$ when $i = j,$
- $\text{tr}_i^T(\Box_j\varphi) = \text{tr}_i^T(\varphi)$ when $i \neq j,$
- $\text{tr}_{X,i}^V(\Box_j\varphi) = \top_X$ when $i \neq j.$

As the reader can see from the above definition, the translation function tr_i^T does not depend on X . The reader is invited to appreciate the use of the abbreviation \top_X in the definition of the translation function $\text{tr}_{X,i}^V$.

Lemma 1. *Let X be a finite set of variables and $i \in \{1, 2\}$. For all $\{i\}$ -formulas $\varphi,$*

- $\text{tr}_i^T(\varphi) = \varphi,$
- $\text{tr}_{X,i}^V(\varphi) = \varphi.$

Lemma 2. *Let $i \in \{1, 2\}$. For all $\{1, 2\}$ -formulas $\varphi,$*

- $\text{var}(\text{tr}_i^T(\varphi)) = \text{var}(\varphi)$,
- $\text{var}(\text{tr}_{\text{var}(\varphi),i}^V(\varphi)) = \text{var}(\varphi)$.

For all finite sets X of variables, for all $i \in \{1, 2\}$ and for all $\{1, 2\}$ -substitutions σ , let σ_i^T and $\sigma_{X,i}^V$ be the $\{i\}$ -substitutions defined as follows:

- for all variables x , $\sigma_i^T(x) = \text{tr}_i^T(\sigma(x))$,
- for all variables x , $\sigma_{X,i}^V(x) = \text{tr}_{X,i}^V(\sigma(x))$.

3 Fusions of modal logics

From now on in this paper,

we write “ $\bar{1}$ ” to mean “ 2 ” and we write “ $\bar{2}$ ” to mean “ 1 ”.

3.1 Modal logics

Let I be a non-empty subset of $\{1, 2\}$. An I -logic is a set \mathbf{L} of I -formulas such that

- \mathbf{L} contains all I -tautologies
- for all $i \in I$, \mathbf{L} contains all I -formulas of the form $\Box_i(\varphi \rightarrow \psi) \rightarrow (\Box_i\varphi \rightarrow \Box_i\psi)$,
- \mathbf{L} is closed under modus ponens (if $\varphi \in \mathbf{L}$ and $\varphi \rightarrow \psi \in \mathbf{L}$ then $\psi \in \mathbf{L}$),
- \mathbf{L} is closed under I -generalization (if $\varphi \in \mathbf{L}$ then for all $i \in I$, $\Box_i\varphi \in \mathbf{L}$).

As is well-known, the intersection of I -logics is an I -logic. Hence, for every set of I -formulas, there exists a least I -logic containing it. We shall say that an I -logic \mathbf{L} is *consistent* if $\mathbf{L} \neq \text{FOR}_I$. In this paper, it will be useful to remember that for all $i \in \{1, 2\}$,

- if an $\{i\}$ -logic \mathbf{L} is consistent then either \mathbf{L} is contained in the least $\{i\}$ -logic \mathbf{Triv}_i containing all $\{i\}$ -formulas of the form $\Box_i\varphi \leftrightarrow \varphi$, or \mathbf{L} is contained in the least $\{i\}$ -logic \mathbf{Verum}_i containing all $\{i\}$ -formulas of the form $\Box_i\varphi$.

See [25]. For all $i \in \{1, 2\}$, we shall say that an $\{i\}$ -logic \mathbf{L} is a *non-trivial extension of S5* if \mathbf{L} contains the least $\{i\}$ -logic $\mathbf{S5}_i$ containing all $\{i\}$ -formulas of the form $\Box_i\chi \rightarrow \chi$, $\Box_i\chi \rightarrow \Box_i\Box_i\chi$ and $\chi \rightarrow \Box_i\Diamond_i\chi$ and \mathbf{L} is strictly contained in \mathbf{Triv}_i . In this paper, it will be useful to remember that for all $i \in \{1, 2\}$, if an $\{i\}$ -logic \mathbf{L} is a non-trivial extension of $\mathbf{S5}$ then one of the following conditions holds:

- there exists $\mathbf{kk} \geq 2$ such that \mathbf{L} is equal to the least extension $\mathbf{S5}_i^{\mathbf{kk}}$ of $\mathbf{S5}_i$ containing all $\{i\}$ -formulas of the form $\bigwedge\{\diamond_i\varphi_m : 0 \leq m \leq \mathbf{kk}\} \rightarrow \bigvee\{\diamond_i(\varphi_m \wedge \varphi_n) : 0 \leq m < n \leq \mathbf{kk}\}$,
- $\mathbf{L} = \mathbf{S5}_i$.

See [26, 27]. It will also be useful to remember that for all $i \in \{1, 2\}$,

- for all $\mathbf{kk} \geq 2$, $\mathbf{S5}_i^{\mathbf{kk}}$ is a Kripke complete modal logic characterized by the class of all Kripke frames (W, R_i) where R_i is an equivalence relation on W for which each equivalence class is a finite set of exactly \mathbf{kk} possible worlds,
- for all $\mathbf{kk} \geq 2$, $\mathbf{S5}_i^{\mathbf{kk}}$ is a Kripke complete modal logic characterized by the class of all Kripke frames (W, R_i) where R_i is an equivalence relation on W for which each equivalence class is a finite set of at most \mathbf{kk} possible worlds,
- $\mathbf{S5}_i$ is a Kripke complete modal logic characterized by the class of all Kripke frames (W, R_i) where R_i is an equivalence relation on W for which each equivalence class is a countably infinite set of possible worlds.

3.2 Fusions

Let \mathbf{L}_1 be a $\{1\}$ -logic and \mathbf{L}_2 be a $\{2\}$ -logic. The *fusion* of \mathbf{L}_1 and \mathbf{L}_2 is the least $\{1, 2\}$ -logic (denoted $\mathbf{L}_1 \otimes \mathbf{L}_2$) containing \mathbf{L}_1 and \mathbf{L}_2 ⁵. As is well-known, if \mathbf{L}_1 is consistent and \mathbf{L}_2 is consistent then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is a conservative extension of \mathbf{L}_1 and \mathbf{L}_2 [14, 30]⁶. A number of other results — transfer results — have been obtained as well. They concern properties preserved under the operation of forming fusions [15, Chapter 4]: the fusion of decidable modal logics is decidable, the fusion of modal logics having uniform interpolation property has uniform interpolation property, etc. We shall say that $\mathbf{L}_1 \otimes \mathbf{L}_2$ is *tensed* if $\mathbf{L}_1 \otimes \mathbf{L}_2$ contains all $\{1, 2\}$ -formulas of the form $\varphi \rightarrow \Box_1 \Diamond_1 \varphi$ and $\varphi \rightarrow \Box_2 \Diamond_2 \varphi$. We shall say that $\mathbf{L}_1 \otimes \mathbf{L}_2$ is *smooth* if for all $k, l \geq 0$, if $k > l$ then $\Box^k \perp \rightarrow \Box^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\Box^k \perp \rightarrow \Box^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$.

⁵Generalized to logics formulated in languages with an arbitrary number of modal connectives, the operation of forming fusions is associative. Therefore it makes sense to define the fusion of an arbitrary number of logics $\mathbf{L}_1, \dots, \mathbf{L}_n$ respectively formulated in languages with the modal connectives \Box_1, \dots, \Box_n as being the least logic formulated in the language with the modal connectives \Box_1, \dots, \Box_n and containing $\mathbf{L}_1, \dots, \mathbf{L}_n$. For instance, the multimodal logics considered in [9, 13] are fusions of finitely many logics of knowledge. In this paper, we will only consider the operation of forming fusions of two unimodal logics.

⁶That is to say, when \mathbf{L}_1 and \mathbf{L}_2 are consistent, for all $i \in \{1, 2\}$ and for all $\{i\}$ -formulas φ , if $\varphi \in \mathbf{L}_1 \otimes \mathbf{L}_2$ then $\varphi \in \mathbf{L}_i$. Obviously, if either \mathbf{L}_1 is inconsistent, or \mathbf{L}_2 is inconsistent then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is inconsistent. In actual fact, as noticed by Kracht and Wolter [23], $\mathbf{L}_1 \otimes \mathbf{L}_2$ is a conservative extension of \mathbf{L}_1 and \mathbf{L}_2 if and only if either \mathbf{L}_1 is consistent and \mathbf{L}_2 is consistent, or \mathbf{L}_1 is inconsistent and \mathbf{L}_2 is inconsistent.

Lemma 3. *If \mathbf{L}_1 and \mathbf{L}_2 are non-trivial extensions of $\mathbf{S5}$ then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed and smooth.*

Within the context of this paper, it is relevant to investigate the properties of the translation functions $\text{tr}_i^T : \text{FOR}_{\{1,2\}} \longrightarrow \text{FOR}_{\{i\}}$ and $\text{tr}_{X,i}^V : \text{FOR}_{\{1,2\}} \longrightarrow \text{FOR}_{\{i\}}$ in $\mathbf{L}_1 \otimes \mathbf{L}_2$ for each finite set X of variables and for each $i \in \{1, 2\}$. The following results will be used in Section 5.

Lemma 4. *Let X, Y be finite sets of variables and $i \in \{1, 2\}$. For all $\{1, 2\}$ -formulas φ , $\text{tr}_{X,i}^V(\varphi) \leftrightarrow \text{tr}_{Y,i}^V(\varphi) \in \mathbf{L}_i$.*

Lemma 5. *Let X be a finite set of variables, $i \in \{1, 2\}$ and σ be a $\{1, 2\}$ -substitution. For all $\{1, 2\}$ -formulas φ ,*

- $\sigma_i^T(\text{tr}_i^T(\varphi)) \leftrightarrow \text{tr}_i^T(\sigma(\varphi)) \in \mathbf{L}_i$,
- $\sigma_{X,i}^V(\text{tr}_{X,i}^V(\varphi)) \leftrightarrow \text{tr}_{X,i}^V(\sigma(\varphi)) \in \mathbf{L}_i$.

Lemma 6. *Let X be a finite set of variables, $i \in \{1, 2\}$ and σ be an $\{i\}$ -substitution. For all $\{1, 2\}$ -formulas φ ,*

- $\sigma(\text{tr}_i^T(\varphi)) \leftrightarrow \text{tr}_i^T(\sigma(\varphi)) \in \mathbf{L}_i$,
- $\sigma(\text{tr}_{X,i}^V(\varphi)) \leftrightarrow \text{tr}_{X,i}^V(\sigma(\varphi)) \in \mathbf{L}_i$.

Lemma 7. *Let X be a finite set of variables. For all $\{1, 2\}$ -formulas φ ,*

- $\text{tr}_1^T(\varphi) \leftrightarrow \varphi \in \mathbf{L}_1 \otimes \mathbf{Triv}_2$,
- $\text{tr}_{X,1}^V(\varphi) \leftrightarrow \varphi \in \mathbf{L}_1 \otimes \mathbf{Verum}_2$,
- $\text{tr}_2^T(\varphi) \leftrightarrow \varphi \in \mathbf{Triv}_1 \otimes \mathbf{L}_2$,
- $\text{tr}_{X,2}^V(\varphi) \leftrightarrow \varphi \in \mathbf{Verum}_1 \otimes \mathbf{L}_2$.

Lemma 8. *Let X be a finite set of variables, $i \in \{1, 2\}$ and φ be a $\{1, 2\}$ -formula. If $\varphi \in \mathbf{L}_1 \otimes \mathbf{L}_2$ then*

1. *if $\mathbf{L}_{\bar{i}} \subseteq \mathbf{Triv}_{\bar{i}}$ then $\text{tr}_i^T(\varphi) \in \mathbf{L}_i$,*
2. *if $\mathbf{L}_{\bar{i}} \subseteq \mathbf{Verum}_{\bar{i}}$ then $\text{tr}_{X,i}^V(\varphi) \in \mathbf{L}_i$.*

Within the context of this paper, it is relevant to investigate the properties of the modal connectives \boxplus and \boxminus in $\mathbf{L}_1 \otimes \mathbf{L}_2$. The following results will be used in Section 6.

Lemma 9. *1. $\mathbf{L}_1 \otimes \mathbf{L}_2$ contains all $\{1, 2\}$ -formulas of the form $\boxplus(\varphi \rightarrow \psi) \rightarrow (\boxplus\varphi \rightarrow \boxplus\psi)$,*

2. $\mathbf{L}_1 \otimes \mathbf{L}_2$ contains all $\{1, 2\}$ -formulas of the form $\boxplus(\varphi \rightarrow \psi) \rightarrow (\boxplus\varphi \rightarrow \boxplus\psi)$,
3. $\mathbf{L}_1 \otimes \mathbf{L}_2$ is closed under the rule $\frac{\varphi}{\boxplus\varphi}$,
4. $\mathbf{L}_1 \otimes \mathbf{L}_2$ is closed under the rule $\frac{\varphi}{\boxminus\varphi}$,
5. if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is closed under the rule $\frac{\neg\varphi \rightarrow \boxplus\psi}{\neg\psi \rightarrow \boxplus\varphi}$,
6. if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is closed under the rule $\frac{\neg\varphi \rightarrow \boxminus\psi}{\neg\psi \rightarrow \boxminus\varphi}$.

Lemma 10. For all $k \geq 0$,

1. $\boxplus^k \top \in \mathbf{L}_1 \otimes \mathbf{L}_2$,
2. $\boxminus^k \top \in \mathbf{L}_1 \otimes \mathbf{L}_2$,
3. $\boxplus^{<k} \top \in \mathbf{L}_1 \otimes \mathbf{L}_2$,
4. $\boxminus^{<k} \top \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Lemma 11. Let $k \geq 0$. For all $\{1, 2\}$ -formulas φ ,

1. $\boxplus^{<k+1} \varphi \leftrightarrow \varphi \wedge \boxplus \boxplus^{<k} \varphi \in \mathbf{L}_1 \otimes \mathbf{L}_2$,
2. $\boxminus^{<k+1} \varphi \leftrightarrow \varphi \wedge \boxminus \boxminus^{<k} \varphi \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Lemma 12. Let $k \geq 0$. If $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth then

1. $\boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$,
2. $\boxminus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$.

Lemma 13. Let $k \geq 0$. If $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed and smooth then for all $l \geq 0$, $\boxplus^k \perp \vee \boxminus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$.

In anticipation of our results about the unification type of fusions in Sections 5 and 6, we complete this section by defining the families $(\sigma_k)_{k \geq 0}$, $(\tau_k)_{k \geq 0}$, $(\lambda_k)_{k \geq 0}$ and $(\mu_k)_{k \geq 0}$ of $\{1, 2\}$ -substitutions and by proving some of their properties. From now on in this paper,

let x be a fixed variable.

For all $k \geq 0$, let σ_k and τ_k be the $\{1, 2\}$ -substitutions inductively defined as follows:

- $\sigma_0(x) = \perp$,
- for all variables y distinct from x , $\sigma_0(y) = y$,

- $\tau_0(x) = \top$,
- for all variables y distinct from x , $\tau_0(y) = y$,
- $\sigma_{k+1}(x) = x \wedge \boxplus \sigma_k(x)$,
- for all variables y distinct from x , $\sigma_{k+1}(y) = y$,
- $\tau_{k+1}(x) = \neg(\neg x \wedge \boxminus \neg \tau_k(x))$,
- for all variables y distinct from x , $\tau_{k+1}(y) = y$.

For all $k \geq 0$, let λ_k and μ_k be the $\{1, 2\}$ -substitutions defined as follows:

- $\lambda_k(x) = x \wedge \boxplus^k \perp$,
- for all variables y distinct from x , $\lambda_k(y) = y$,
- $\mu_k(x) = \neg(\neg x \wedge \boxminus^k \perp)$,
- for all variables y distinct from x , $\mu_k(y) = y$.

Lemma 14. *Let $k \geq 0$. We have $\boxplus^{<k} x \wedge \boxplus^k \perp \rightarrow \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^{<k} \neg x \wedge \boxminus^k \perp \rightarrow \neg \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.*

Lemma 15. *Let $k \geq 0$. We have $\sigma_k(x) \rightarrow x \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_k(x) \rightarrow \neg x \in \mathbf{L}_1 \otimes \mathbf{L}_2$.*

Lemma 16. *Let $k \geq 0$. We have $\sigma_k(x) \rightarrow \boxplus \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_k(x) \rightarrow \boxminus \neg \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.*

Lemma 17. *Let $k \geq 0$. For all $l \geq 0$, if $k \leq l$ then $\sigma_k(x) \rightarrow \boxplus^l \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_k(x) \rightarrow \boxminus^l \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$.*

Lemma 18. *Let $k \geq 0$. For all $l \geq 0$, if $k \leq l$ then $\boxplus^k \perp \wedge \sigma_l(x) \leftrightarrow \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \wedge \neg \tau_l(x) \leftrightarrow \neg \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.*

Lemma 19. *Let $k \geq 0$. For all $l \geq 0$, if $k \leq l$ then $\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\mu_l(\tau_k(x)) \leftrightarrow \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.*

Lemma 20. *Let $k \geq 0$. For all $l \geq 0$, if $k \geq l$ then $\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\mu_l(\tau_k(x)) \leftrightarrow \tau_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.*

Lemma 21. *Let $k \geq 0$. If $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth then for all $l \geq 0$, if $k > l$ then $\sigma_k(x) \rightarrow \boxplus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_k(x) \rightarrow \boxminus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$.*

Lemma 22. *Let $k \geq 0$. If $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth then for all $l \geq 0$, $\boxplus^k \perp \vee \neg \tau_l(x) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \vee \sigma_l(x) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$.*

4 Unification

4.1 Unifiable formulas and unification types

Let I be a non-empty subset of $\{1, 2\}$. Let \mathbf{L} be an I -logic. We shall say that an I -substitution σ is *equivalent* in \mathbf{L} to an I -substitution τ with respect to a set \mathcal{X} of variables (in symbols $\sigma \simeq_{\mathbf{L}}^{\mathcal{X}} \tau$) if for all variables $y \in \mathcal{X}$, $\sigma(y) \leftrightarrow \tau(y) \in \mathbf{L}$. We shall say that an I -substitution σ is more *general* in \mathbf{L} than an I -substitution τ with respect to a set \mathcal{X} of variables (in symbols $\sigma \preceq_{\mathbf{L}}^{\mathcal{X}} \tau$) if there exists an I -substitution ν such that $\sigma \circ \nu \simeq_{\mathbf{L}}^{\mathcal{X}} \tau$. Obviously, for all sets \mathcal{X} of variables and for all I -substitutions σ, τ , if $\sigma \simeq_{\mathbf{L}}^{\mathcal{X}} \tau$ then $\sigma \preceq_{\mathbf{L}}^{\mathcal{X}} \tau$. Moreover, for all sets \mathcal{X} of variables, on the set of all I -substitutions, the binary relation $\simeq_{\mathbf{L}}^{\mathcal{X}}$ is reflexive, symmetric and transitive and the binary relation $\preceq_{\mathbf{L}}^{\mathcal{X}}$ is reflexive and transitive. We shall say that an I -formula φ is **L-unifiable** if there exists an I -substitution σ such that $\sigma(\varphi) \in \mathbf{L}$. In that case, σ is an **L-unifier** of φ . We shall say that a set Σ of **L-unifiers** of an **L-unifiable** I -formula φ is **L-complete** if for all **L-unifiers** σ of φ , there exists $\tau \in \Sigma$ such that $\tau \preceq_{\mathbf{L}}^{\text{var}(\varphi)} \sigma$. As is well-known, if an **L-unifiable** I -formula has minimal **L-complete** sets of **L-unifiers** then these sets have the same cardinality⁷. About the type of **L-unifiable** I -formulas, we shall say that an **L-unifiable** I -formula

- φ is **L-nullary** (or of type 0) if there exists no minimal **L-complete** set of **L-unifiers** of φ ,
- φ is **L-infinitary** (or of type ∞) if there exists a minimal **L-complete** set of **L-unifiers** of φ but there exists no finite one,
- φ is **L-finitary** (or of type ω) if there exists a finite minimal **L-complete** set of **L-unifiers** of φ but there exists no with cardinality 1,
- φ is **L-unitary** (or of type 1) if there exists a minimal **L-complete** set of **L-unifiers** of φ with cardinality 1.

Obviously, the types “**L-nullary**”, “**L-infinitary**”, “**L-finitary**” and “**L-unitary**” constitute a set of jointly exhaustive and pairwise distinct situations. To be of type 0 is considered to be the worst situation whereas to be of type 1 is considered to be better than to be of type ω which is itself considered to be better than to be of type ∞ . As for the type of \mathbf{L} , we traditionally distinguish between *elementary unification* and *unification with parameters*:

⁷Suppose Σ and Δ are minimal **L-complete** sets of **L-unifiers** of the same **L-unifiable** I -formula φ . By the **L-completeness** of Σ and Δ , one can readily define functions $f : \Sigma \rightarrow \Delta$ and $g : \Delta \rightarrow \Sigma$ such that $f(\sigma) \preceq_{\mathbf{L}}^{\text{var}(\varphi)} \sigma$ for each $\sigma \in \Sigma$ and $g(\delta) \preceq_{\mathbf{L}}^{\text{var}(\varphi)} \delta$ for each $\delta \in \Delta$. By the minimality of Σ and Δ , it follows that f and g are injective. Hence, Σ and Δ have the same cardinality.

- elementary unification in \mathbf{L} is the problem of asking whether a given parameter-free I -formula is \mathbf{L} -unifiable,
- unification with parameters in \mathbf{L} is the problem of asking whether a given I -formula is \mathbf{L} -unifiable.

We shall say that

- \mathbf{L} is *nullary (or of type 0) for elementary unification* if there exists an \mathbf{L} -nullary \mathbf{L} -unifiable parameter-free I -formula,
- \mathbf{L} is *infinitary (or of type ∞) for elementary unification* if every \mathbf{L} -unifiable parameter-free I -formula is either \mathbf{L} -unitary, or \mathbf{L} -finitary, or \mathbf{L} -infinitary and there exists an \mathbf{L} -infinitary \mathbf{L} -unifiable parameter-free I -formula,
- \mathbf{L} is *finitary (or of type ω) for elementary unification* if every \mathbf{L} -unifiable parameter-free I -formula is either \mathbf{L} -unitary, or \mathbf{L} -finitary and there exists an \mathbf{L} -finitary \mathbf{L} -unifiable parameter-free I -formula,
- \mathbf{L} is *unitary (or of type 1) for elementary unification* if every \mathbf{L} -unifiable parameter-free I -formula is \mathbf{L} -unitary.

We shall say that

- \mathbf{L} is *nullary (or of type 0) for unification with parameters* if there exists an \mathbf{L} -nullary \mathbf{L} -unifiable I -formula,
- \mathbf{L} is *infinitary (or of type ∞) for unification with parameters* if every \mathbf{L} -unifiable I -formula is either \mathbf{L} -unitary, or \mathbf{L} -finitary, or \mathbf{L} -infinitary and there exists an \mathbf{L} -infinitary \mathbf{L} -unifiable I -formula,
- \mathbf{L} is *finitary (or of type ω) for unification with parameters* if every \mathbf{L} -unifiable I -formula is either \mathbf{L} -unitary, or \mathbf{L} -finitary and there exists an \mathbf{L} -finitary \mathbf{L} -unifiable I -formula,
- \mathbf{L} is *unitary (or of type 1) for unification with parameters* if every \mathbf{L} -unifiable I -formula is \mathbf{L} -unitary.

Obviously, both for elementary unification and for unification with parameters, the types “nullary”, “infinitary”, “finitary” and “unitary” constitute a set of jointly exhaustive and pairwise distinct situations. In other respects, the unification type of \mathbf{L} for elementary unification is at least better than the unification type of \mathbf{L} for unification with parameters and there is a priori no guarantee that the unification type for elementary unification and the

unification type for unification with parameters are equal. For instance, the implication fragment of Boolean logic is unitary for elementary unification and finitary for unification with parameters [6]⁸. To the extent that in cases such as **KB**, **KD**, **KDB**, **KT** and **KTB**, the unification type for unification with parameters is known whereas the unification type for elementary unification is still a mystery⁹. Of course, seeing that the unification type of an equational theory depends not only on the equational theory itself but also on the set of symbols that can occur in the considered unification problems, this phenomenon is already well-known from the theory of unification [2]. Finally, as already noticed by several authors within the context of unimodal logics, there is no I -logic \mathbf{L} that is known to be infinitary either for elementary unification, or for unification with parameters. See [11].

4.2 Playing with formulas and substitutions

Let \mathbf{L}_1 be a consistent $\{1\}$ -logic and \mathbf{L}_2 be a consistent $\{2\}$ -logic.

Lemma 23. *Let $k \geq 0$. For all $l \geq 0$, if $k \leq l$ then $\sigma_l \circ \lambda_k \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_k$ and $\tau_l \circ \mu_k \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \tau_k$.*

Lemma 24. *Let $k \geq 0$. For all $l \geq 0$, if $k \leq l$ then $\sigma_l \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_k$ and $\tau_l \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \tau_k$.*

Lemma 25. *Let $k \geq 0$. If $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth then for all $l \geq 0$, if $k < l$ then $\sigma_k \not\preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_l$ and $\tau_k \not\preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \tau_l$.*

Lemma 26. *Let $k \geq 0$. If $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth then for all $l \geq 0$, $\sigma_k \not\preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \tau_l$ and $\tau_k \not\preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_l$.*

From now on in this paper,

let φ be the $\{1, 2\}$ -formula $x \rightarrow \boxplus x$ and ψ be the $\{1, 2\}$ -formula $\neg x \rightarrow \boxminus \neg x$.

The $\{1, 2\}$ -formulas φ and ψ will be the keys in Section 6 to the determination of the unification type of the fusion of arbitrary consistent extensions of **S5**. In the meantime, by Lemma 9, if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed then φ and ψ have the same unifiers in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, in that case, as long as we only consider φ and ψ through their unifiers in $\mathbf{L}_1 \otimes \mathbf{L}_2$, it does not matter if we are talking about either φ , or ψ .

Lemma 27. *Let $k \geq 0$. For all unifiers σ of φ in $\mathbf{L}_1 \otimes \mathbf{L}_2$, $\sigma(x) \rightarrow \boxplus^{<k} \sigma(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and for all unifiers τ of ψ in $\mathbf{L}_1 \otimes \mathbf{L}_2$, $\neg \tau(x) \rightarrow \boxminus^{<k} \neg \tau(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.*

⁸For all parameter-free formulas φ with \rightarrow as its sole connective, φ is unifiable in Boolean logic and the so-called Löwenheim substitution ϵ defined by $\epsilon(y) = \varphi \rightarrow y$ for each $y \in \text{var}(\varphi)$ constitutes a minimal complete set of unifiers of it.

⁹**KB**, **KD**, **KDB**, **KT** and **KTB** are nullary for unification with parameters [3, 4, 5].

Lemma 28. For all $k \geq 0$, σ_k is a unifier of φ in $\mathbf{L}_1 \otimes \mathbf{L}_2$ and τ_k is a unifier of ψ in $\mathbf{L}_1 \otimes \mathbf{L}_2$.

Lemma 29. Let v be a $\{1, 2\}$ -substitution. If v is a unifier of φ in $\mathbf{L}_1 \otimes \mathbf{L}_2$ then for all $k \geq 0$, the following conditions are equivalent:

(a) $\sigma_k \circ v \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$,

(b) $\sigma_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$,

(c) $v(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$

and if v is a unifier of ψ in $\mathbf{L}_1 \otimes \mathbf{L}_2$ then for all $k \geq 0$, the following conditions are equivalent:

(d) $\tau_k \circ v \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$,

(e) $\tau_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$,

(f) $\neg v(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

5 General results about the unification type of fusions

Let \mathbf{L}_1 be a consistent $\{1\}$ -logic and \mathbf{L}_2 be a consistent $\{2\}$ -logic.

Proposition 1. Let $i \in \{1, 2\}$ and χ be an $\{i\}$ -formula. If χ is unifiable in $\mathbf{L}_1 \otimes \mathbf{L}_2$ then χ is \mathbf{L}_i -unifiable.

Proof. Suppose χ is unifiable in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, there exists a $\{1, 2\}$ -substitution σ such that $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Without loss of generality, suppose $i = 1$. Since \mathbf{L}_2 is consistent, either $\mathbf{L}_2 \subseteq \mathbf{Triv}_2$, or $\mathbf{L}_2 \subseteq \mathbf{Verum}_2$. In the former case, $\mathbf{L}_1 \otimes \mathbf{L}_2 \subseteq \mathbf{L}_1 \otimes \mathbf{Triv}_2$ ¹⁰. Since $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{Triv}_2$. Thus, by Lemma 7, $\text{tr}_1^T(\sigma(\chi)) \in \mathbf{L}_1 \otimes \mathbf{Triv}_2$. Since $\mathbf{L}_1 \otimes \mathbf{Triv}_2$ is a conservative extension of \mathbf{L}_1 , it follows that $\text{tr}_1^T(\sigma(\chi)) \in \mathbf{L}_1$. Consequently, by Lemma 5, $\sigma_1^T(\text{tr}_1^T(\chi)) \in \mathbf{L}_1$. Hence, by Lemma 1, $\sigma_1^T(\chi) \in \mathbf{L}_1$. Thus, χ is \mathbf{L}_1 -unifiable. \square

Proposition 2. Let $i \in \{1, 2\}$ and χ be an $\{i\}$ -formula. For all complete sets Σ of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$,

1. if $L_{\bar{i}} \subseteq \mathbf{Triv}_{\bar{i}}$ then $\{\sigma_i^T : \sigma \in \Sigma\}$ is an \mathbf{L}_i -complete set of \mathbf{L}_i -unifiers of χ ,
2. if $L_{\bar{i}} \subseteq \mathbf{Verum}_{\bar{i}}$ then $\{\sigma_{\emptyset, i}^V : \sigma \in \Sigma\}$ is an \mathbf{L}_i -complete set of \mathbf{L}_i -unifiers of χ .

¹⁰In the latter case, the proof can be similarly done.

Proof. Let Σ be a complete set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. The proof of Item (1) can be done as follows¹¹.

Suppose $i = 1$ and $L_2 \subseteq \mathbf{Triv}_2$.

Claim $\{\sigma_1^T : \sigma \in \Sigma\}$ is a set of \mathbf{L}_1 -unifiers of χ .

Proof: It suffices to prove that for all $\sigma \in \Sigma$, $\sigma_1^T(\chi) \in \mathbf{L}_1$. Let $\sigma \in \Sigma$. The proof that $\sigma_1^T(\chi) \in \mathbf{L}_1$ is essentially the one described in the body of the proof of Proposition 1. We include it here for the sake of the completeness. Since Σ is a set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$, we obtain $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $L_2 \subseteq \mathbf{Triv}_2$, $\mathbf{L}_1 \otimes \mathbf{L}_2 \subseteq \mathbf{L}_1 \otimes \mathbf{Triv}_2$. Since $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{Triv}_2$. Hence, by Lemma 7, $\text{tr}_1^T(\sigma(\chi)) \in \mathbf{L}_1 \otimes \mathbf{Triv}_2$. Since $\mathbf{L}_1 \otimes \mathbf{Triv}_2$ is a conservative extension of \mathbf{L}_1 , $\text{tr}_1^T(\sigma(\chi)) \in \mathbf{L}_1$. Thus, by Lemma 5, $\sigma_1^T(\text{tr}_1^T(\chi)) \in \mathbf{L}_1$. Hence, by Lemma 1, $\sigma_1^T(\chi) \in \mathbf{L}_1$.

Claim $\{\sigma_1^T : \sigma \in \Sigma\}$ is an \mathbf{L}_1 -complete set of \mathbf{L}_1 -unifiers of χ .

Proof: By the previous Claim, it suffices to prove that for all $\{1\}$ -substitutions σ , if $\sigma(\chi) \in \mathbf{L}_1$ then there exists $\tau \in \Sigma$ such that $\tau_1^T \preceq_{\mathbf{L}_1}^{\text{var}(\chi)} \sigma$. Let σ be a $\{1\}$ -substitution. Suppose $\sigma(\chi) \in \mathbf{L}_1$. Hence, $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since Σ is a complete set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$, there exists $\tau \in \Sigma$ such that $\tau \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\text{var}(\chi)} \sigma$. Thus, there exists a $\{1, 2\}$ -substitution v such that $\tau \circ v \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\text{var}(\chi)} \sigma$. Hence, for all variables $y \in \text{var}(\chi)$, $v(\tau(y)) \leftrightarrow \sigma(y) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $L_2 \subseteq \mathbf{Triv}_2$, $\mathbf{L}_1 \otimes \mathbf{L}_2 \subseteq \mathbf{L}_1 \otimes \mathbf{Triv}_2$. Since for all variables $y \in \text{var}(\chi)$, $v(\tau(y)) \leftrightarrow \sigma(y) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, for all variables $y \in \text{var}(\chi)$, $v(\tau(y)) \leftrightarrow \sigma(y) \in \mathbf{L}_1 \otimes \mathbf{Triv}_2$. Thus, by Lemma 7, for all variables $y \in \text{var}(\chi)$, $\text{tr}_1^T(v(\tau(y))) \leftrightarrow \sigma(y) \in \mathbf{L}_1 \otimes \mathbf{Triv}_2$. Since $\mathbf{L}_1 \otimes \mathbf{Triv}_2$ is a conservative extension of \mathbf{L}_1 , for all variables $y \in \text{var}(\chi)$, $\text{tr}_1^T(v(\tau(y))) \leftrightarrow \sigma(y) \in \mathbf{L}_1$. Thus, for all variables $y \in \text{var}(\chi)$, $\text{tr}_1^T(v(\tau(y))) \leftrightarrow \text{tr}_1^T(\sigma(y)) \in \mathbf{L}_1$. Consequently, by Lemma 5, for all variables $y \in \text{var}(\chi)$, $v_1^T(\tau_1^T(y)) \leftrightarrow \text{tr}_1^T(\sigma(y)) \in \mathbf{L}_1$. Hence, by Lemma 1, for all variables $y \in \text{var}(\chi)$, $v_1^T(\tau_1^T(y)) \leftrightarrow \sigma(y) \in \mathbf{L}_1$. Thus, $\tau_1^T \circ v_1^T \simeq_{\mathbf{L}_1}^{\text{var}(\chi)} \sigma$. Consequently, $\tau_1^T \preceq_{\mathbf{L}_1}^{\text{var}(\chi)} \sigma$.

This ends the proof of Proposition 2. □

Proposition 3. Let $i \in \{1, 2\}$ and χ be a $\{1, 2\}$ -formula.

1. For all minimal \mathbf{L}_i -complete sets Σ of \mathbf{L}_i -unifiers of $\text{tr}_i^T(\chi)$, if $L_i = \mathbf{Triv}_i$ then Σ is a minimal complete set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$,

¹¹The proof of Items (2)–(4) can be similarly done.

2. for all minimal \mathbf{L}_i -complete sets Σ of \mathbf{L}_i -unifiers of $\text{tr}_{\text{var}(\chi),i}^V(\chi)$, if $L_i = \text{Verum}_i$ then Σ is a minimal complete set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof. The proof of Item (1) can be done as follows¹².

Let Σ be a minimal \mathbf{L}_i -complete set of \mathbf{L}_i -unifiers of $\text{tr}_i^T(\chi)$. Suppose $i = 1$ and $L_2 = \text{Triv}_2$.

Claim Σ is a set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof: It suffices to prove that for all $\sigma \in \Sigma$, $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Let $\sigma \in \Sigma$. Since Σ is a set of \mathbf{L}_i -unifiers of $\text{tr}_i^T(\chi)$, $\sigma(\text{tr}_i^T(\chi)) \in \mathbf{L}_i$. Hence, by Lemma 6, $\text{tr}_i^T(\sigma(\chi)) \in \mathbf{L}_i$. Since $i = 1$ and $L_2 = \text{Triv}_2$, then by Lemma 7, $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Claim Σ is a complete set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof: By the previous Claim, it suffices to prove that for all $\{1, 2\}$ -substitutions σ , if $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ then there exists $\tau \in \Sigma$ such that $\tau \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\text{var}(\chi)} \sigma$. Let σ be a $\{1, 2\}$ -substitution. Suppose $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $i = 1$ and $L_2 = \text{Triv}_2$, then by Lemma 7, $\text{tr}_i^T(\sigma(\chi)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is a conservative extension of \mathbf{L}_i , $\text{tr}_i^T(\sigma(\chi)) \in \mathbf{L}_i$. Hence, by Lemma 5, $\sigma_i^T(\text{tr}_i^T(\chi)) \in \mathbf{L}_i$. Thus, σ_i^T is an \mathbf{L}_i -unifier of $\text{tr}_i^T(\chi)$. Since Σ is an \mathbf{L}_i -complete set of \mathbf{L}_i -unifiers of $\text{tr}_i^T(\chi)$, there exists $\tau \in \Sigma$ such that $\tau \preceq_{\mathbf{L}_i}^{\text{var}(\text{tr}_i^T(\chi))} \sigma_i^T$.

Consequently, there exists an $\{i\}$ -substitution v such that $\tau \circ v \simeq_{\mathbf{L}_i}^{\text{var}(\text{tr}_i^T(\chi))} \sigma_i^T$. Hence, for all variables $y \in \text{var}(\text{tr}_i^T(\chi))$, $v(\tau(y)) \leftrightarrow \sigma_i^T(y) \in \mathbf{L}_i$. Thus, by Lemma 1, for all variables $y \in \text{var}(\text{tr}_i^T(\chi))$, $\text{tr}_i^T(v(\tau(y))) \leftrightarrow \text{tr}_i^T(\sigma(y)) \in \mathbf{L}_i$. Consequently, for all variables $y \in \text{var}(\text{tr}_i^T(\chi))$, $\text{tr}_i^T(v(\tau(y))) \leftrightarrow \text{tr}_i^T(\sigma(y)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $i = 1$ and $L_2 = \text{Triv}_2$, then by Lemma 7, for all variables $y \in \text{var}(\text{tr}_i^T(\chi))$, $v(\tau(y)) \leftrightarrow \sigma(y) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, by Lemma 2, for all variables $y \in \text{var}(\chi)$, $v(\tau(y)) \leftrightarrow \sigma(y) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $\tau \circ v \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\text{var}(\chi)} \sigma$. Consequently, $\tau \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\text{var}(\chi)} \sigma$.

Claim Σ is a minimal complete set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof: By the previous Claim, it suffices to prove that for all $\sigma, \tau \in \Sigma$, if $\sigma \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\text{var}(\chi)} \tau$ then $\sigma = \tau$. Let $\sigma, \tau \in \Sigma$. Suppose $\sigma \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\text{var}(\chi)} \tau$. Hence, there exists a $\{1, 2\}$ -substitution v such that $\sigma \circ v \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\text{var}(\chi)} \tau$. Thus, for all variables $y \in \text{var}(\chi)$, $v(\sigma(y)) \leftrightarrow \tau(y) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, by Lemma 2, for all variables $y \in \text{var}(\text{tr}_i^T(\chi))$, $v(\sigma(y)) \leftrightarrow \tau(y) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

¹²The proof of Item (2) can be similarly done.

Since $i = 1$ and $L_2 = \mathbf{Triv}_2$, then by Lemma 7, for all variables $y \in \text{var}(\text{tr}_i^T(\chi))$, $\text{tr}_i^T(v(\sigma(y))) \leftrightarrow \text{tr}_i^T(\tau(y)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is a conservative extension of \mathbf{L}_i , for all variables $y \in \text{var}(\text{tr}_i^T(\chi))$, $\text{tr}_i^T(v(\sigma(y))) \leftrightarrow \text{tr}_i^T(\tau(y)) \in \mathbf{L}_i$. Consequently, by Lemmas 1 and 5, for all variables $y \in \text{var}(\text{tr}_i^T(\chi))$, $v_i^T(\sigma(y)) \leftrightarrow \tau(y) \in \mathbf{L}_i$. Thus, $\sigma \preceq_{\mathbf{L}_i}^{\text{var}(\text{tr}_i^T(\chi))} \tau$. Since Σ is a minimal \mathbf{L}_i -complete set of \mathbf{L}_i -unifiers of $\text{tr}_i^T(\chi)$, $\sigma = \tau$.

This ends the proof of Proposition 3. \square

In the above proof, the reader is invited to appreciate the uses of Lemma 2.

Proposition 4. *Both for elementary unification and for unification with parameters, if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type 1 then for all $i \in \{1, 2\}$, \mathbf{L}_i is of type 1.*

Proof. Suppose $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type 1. Suppose $i = 1$.

It suffices to prove that for all \mathbf{L}_1 -unifiable $\{1\}$ -formulas χ , χ is \mathbf{L}_1 -unitary. Let χ be an \mathbf{L}_1 -unifiable $\{1\}$ -formula. Hence, there exists a $\{1\}$ -substitution σ such that $\sigma(\chi) \in \mathbf{L}_1$. Thus, $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, χ is unifiable in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type 1, there exists a minimal complete set Σ of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$ with cardinality 1. Since \mathbf{L}_2 is consistent, either $\mathbf{L}_2 \subseteq \mathbf{Triv}_2$, or $\mathbf{L}_2 \subseteq \mathbf{Verum}_2$. In the former case, by Proposition 2, $\{\sigma_1^T : \sigma \in \Sigma\}$ is an \mathbf{L}_1 -complete set of \mathbf{L}_1 -unifiers of χ ¹³. Since the cardinality of Σ is 1, the cardinality of $\{\sigma_1^T : \sigma \in \Sigma\}$ is 1. Consequently, χ is \mathbf{L}_1 -unitary. \square

Notice that the converse of the statement established in Proposition 4 is not always true. For instance, as proved in Section 6, $\mathbf{S5} \otimes \mathbf{S5}$ is of type 0.

Proposition 5. *Both for elementary unification and for unification with parameters, if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type ω then for all $i \in \{1, 2\}$, \mathbf{L}_i is either of type 1, or of type ω .*

Proof. Suppose $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type ω . Suppose $i = 1$.

It suffices to prove that for all \mathbf{L}_1 -unifiable $\{1\}$ -formulas χ , χ is either \mathbf{L}_1 -unitary, or \mathbf{L}_1 -finitary. Let χ be an \mathbf{L}_1 -unifiable $\{1\}$ -formula. The proof that χ is either \mathbf{L}_1 -unitary, or \mathbf{L}_1 -finitary is essentially the one described in the body of the proof of Proposition 4. We include it here for the sake of the completeness. Since χ is an \mathbf{L}_1 -unifiable $\{1\}$ -formula, there exists a $\{1\}$ -substitution σ such that $\sigma(\chi) \in \mathbf{L}_1$. Thus, $\sigma(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, χ is unifiable in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type ω , there exists a finite minimal complete set Σ of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Since \mathbf{L}_2 is consistent, either $\mathbf{L}_2 \subseteq \mathbf{Triv}_2$, or $\mathbf{L}_2 \subseteq \mathbf{Verum}_2$. In the former case, by Proposition 2, $\{\sigma_1^T : \sigma \in \Sigma\}$ is an \mathbf{L}_1 -complete set of \mathbf{L}_1 -unifiers

¹³In the latter case, the proof can be similarly done.

of χ^{14} . Since Σ is finite, $\{\sigma_1^T : \sigma \in \Sigma\}$ is finite. Consequently, χ is either \mathbf{L}_1 -unitary, or \mathbf{L}_1 -finitary. \square

Notice that the converse of the statement established in Proposition 5 is not always true. For instance, as proved in [28, Chapter 6], $\mathbf{K4} \otimes \mathbf{K4}$ and $\mathbf{S4} \otimes \mathbf{S4}$ are of type 0. After Propositions 4 and 5, it is natural to ask whether both for elementary unification and for unification with parameters, if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type ∞ then for all $i \in \{1, 2\}$, \mathbf{L}_i is either of type 1, or of type ω , or of type ∞ . Unfortunately, we have not been able to answer this question, seeing that in Proposition 2, it is not clear that if the set Σ considered there is an infinite minimal complete set of unifiers then, when either $L_{\bar{i}} \subseteq \mathbf{Triv}_{\bar{i}}$, or $L_{\bar{i}} \subseteq \mathbf{Verum}_{\bar{i}}$, the corresponding set among $\{\sigma_i^T : \sigma \in \Sigma\}$ and $\{\sigma_{\emptyset,i}^V : \sigma \in \Sigma\}$ is minimal complete too.

Proposition 6. *Both for elementary unification and for unification with parameters, for all $i \in \{1, 2\}$, if either $\mathbf{L}_i = \mathbf{Triv}_i$, or $\mathbf{L}_i = \mathbf{Verum}_i$ then the type of $\mathbf{L}_1 \otimes \mathbf{L}_2$ and the type of $\mathbf{L}_{\bar{i}}$ are equal.*

Proof. Suppose $i = 1$.

Suppose either $\mathbf{L}_1 = \mathbf{Triv}_1$, or $\mathbf{L}_1 = \mathbf{Verum}_1$. In the former case, for the sake of the contradiction, suppose that the type of $\mathbf{L}_1 \otimes \mathbf{L}_2$ and the type of \mathbf{L}_2 are not equal¹⁵. We consider the following cases.

Case \mathbf{L}_2 is of type 0: Hence, there exists an \mathbf{L}_2 -unifiable $\{2\}$ -formula χ of type 0. Thus, there exists an \mathbf{L}_2 -unifier v of χ . Hence, v is a unifier of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is not of type 0, there exists a minimal complete set Σ of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 = \mathbf{Triv}_1$, then by Proposition 2, $\{\sigma_2^T : \sigma \in \Sigma\}$ is an \mathbf{L}_2 -complete set of \mathbf{L}_2 -unifiers of χ . Since χ is of type 0, $\{\sigma_2^T : \sigma \in \Sigma\}$ is not a minimal \mathbf{L}_2 -complete set of \mathbf{L}_2 -unifiers of χ . Consequently, there exists $\sigma, \tau \in \Sigma$ such that $\sigma_2^T \preceq_{\mathbf{L}_2}^{\text{var}(\chi)} \tau_2^T$ and $\sigma_2^T \neq \tau_2^T$. Thus, $\sigma \neq \tau$. Since $\sigma_2^T \preceq_{\mathbf{L}_2}^{\text{var}(\chi)} \tau_2^T$, it follows that there exists a $\{2\}$ -substitution λ such that for all variables $y \in \text{var}(\chi)$, $\lambda(\sigma_2^T(y)) \leftrightarrow \tau_2^T(y) \in \mathbf{L}_2$. Consequently, for all variables $y \in \text{var}(\chi)$, $\lambda(\text{tr}_2^T(\sigma(y))) \leftrightarrow \text{tr}_2^T(\tau(y)) \in \mathbf{L}_2$. Thus, by Lemma 6, for all variables $y \in \text{var}(\chi)$, $\text{tr}_2^T(\lambda(\sigma(y))) \leftrightarrow \text{tr}_2^T(\tau(y)) \in \mathbf{L}_2$. Hence, for all variables $y \in \text{var}(\chi)$, $\text{tr}_2^T(\lambda(\sigma(y))) \leftrightarrow \text{tr}_2^T(\tau(y)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 = \mathbf{Triv}_1$, then by Lemma 7, for all variables $y \in \text{var}(\chi)$, $\lambda(\sigma(y)) \leftrightarrow \tau(y) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, $\sigma \circ \lambda \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\text{var}(\chi)} \tau$. Thus, $\sigma \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\text{var}(\chi)} \tau$. Since Σ is a minimal complete set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$, $\sigma = \tau$: a contradiction.

¹⁴In the latter case, the proof can be similarly done.

¹⁵In the latter case, the proof can be similarly done.

Case \mathbf{L}_2 is of type ∞ : Hence, by Propositions 4 and 5, neither $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type 1, nor $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type ω . Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is not of type ∞ , $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type 0. Thus, there exists a unifiable $\{1, 2\}$ -formula χ of type 0 in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, there exists a unifier v of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, $v(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 = \mathbf{Triv}_1$, then by Lemma 7, $\text{tr}_2^T(v(\chi)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, by Lemma 5, $v_2^T(\text{tr}_2^T(\chi)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is a conservative extension of \mathbf{L}_2 , it follows that $v_2^T(\text{tr}_2^T(\chi)) \in \mathbf{L}_2$. Consequently, v_2^T is an \mathbf{L}_2 -unifier of $\text{tr}_2^T(\chi)$. Since \mathbf{L}_2 is of type ∞ , there exists a minimal \mathbf{L}_2 -complete set Σ of \mathbf{L}_2 -unifiers of $\text{tr}_2^T(\chi)$. Since $L_1 = \mathbf{Triv}_1$, then by Proposition 3, Σ is a minimal complete set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, χ is not of type 0 in $\mathbf{L}_1 \otimes \mathbf{L}_2$: a contradiction.

Case \mathbf{L}_2 is of type ω : Thus, by Proposition 4, $\mathbf{L}_1 \otimes \mathbf{L}_2$ is not of type 1. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is not of type ω , either $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type 0, or $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type ∞ . Consequently, there exists a unifiable $\{1, 2\}$ -formula χ either of type 0, or of type ∞ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, there exists a unifier v of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, $v(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 = \mathbf{Triv}_1$, then by Lemma 7, $\text{tr}_2^T(v(\chi)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, by Lemma 5, $v_2^T(\text{tr}_2^T(\chi)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is a conservative extension of \mathbf{L}_2 , we obtain $v_2^T(\text{tr}_2^T(\chi)) \in \mathbf{L}_2$. Hence, v_2^T is an \mathbf{L}_2 -unifier of $\text{tr}_2^T(\chi)$. Since \mathbf{L}_2 is of type ω , there exists a finite minimal \mathbf{L}_2 -complete set Σ of \mathbf{L}_2 -unifiers of $\text{tr}_2^T(\chi)$. Since $L_1 = \mathbf{Triv}_1$, then by Proposition 3, Σ is a minimal complete set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, neither χ is of type 0 in $\mathbf{L}_1 \otimes \mathbf{L}_2$, nor χ is of type ∞ in $\mathbf{L}_1 \otimes \mathbf{L}_2$: a contradiction.

Case \mathbf{L}_2 is of type 1: Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is not of type 1, there exists a unifiable $\{1, 2\}$ -formula χ either of type 0, or of type ∞ , or of type ω in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, there exists a unifier v of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $v(\chi) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 = \mathbf{Triv}_1$, then by Lemma 7, $\text{tr}_2^T(v(\chi)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, by Lemma 5, $v_2^T(\text{tr}_2^T(\chi)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is a conservative extension of \mathbf{L}_2 , $v_2^T(\text{tr}_2^T(\chi)) \in \mathbf{L}_2$. Thus, v_2^T is an \mathbf{L}_2 -unifier of $\text{tr}_2^T(\chi)$. Since \mathbf{L}_2 is of type 1, there exists a finite minimal \mathbf{L}_2 -complete set Σ of \mathbf{L}_2 -unifiers of $\text{tr}_2^T(\chi)$ with cardinality 1. Since $L_1 = \mathbf{Triv}_1$, then by Proposition 3, Σ is a minimal complete set of unifiers of χ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, neither χ is of type 0 in $\mathbf{L}_1 \otimes \mathbf{L}_2$, nor χ is of type ∞ in $\mathbf{L}_1 \otimes \mathbf{L}_2$, nor χ is of type ω in $\mathbf{L}_1 \otimes \mathbf{L}_2$: a contradiction. \square

In the above proof, the reader is invited to appreciate the uses of Proposition 3.

6 Specific results about the unification type of fusions

Let \mathbf{L}_1 be a consistent $\{1\}$ -logic and \mathbf{L}_2 be a consistent $\{2\}$ -logic. In Propositions 7 and 8, for all $m \geq 0$, σ_m and τ_m are the $\{1, 2\}$ -substitutions defined in Section 3 and φ and ψ are the $\{1, 2\}$ -formulas defined in Section 4.

Proposition 7. *If \mathbf{L}_1 and \mathbf{L}_2 are non-trivial extensions of $\mathbf{S5}$ then for all unifiers v of $\varphi \wedge \psi$ in $\mathbf{L}_1 \otimes \mathbf{L}_2$, there exists $m \geq 0$ such that either $\sigma_m \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$, or $\tau_m \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$.*

Proof. Suppose \mathbf{L}_1 and \mathbf{L}_2 are non-trivial extensions of $\mathbf{S5}$. Let v be a unifier of $\varphi \wedge \psi$ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, v is a unifier of φ in $\mathbf{L}_1 \otimes \mathbf{L}_2$ and v is a unifier of ψ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Let $m \geq 0$ be such that $\deg(v(x)) \leq 6m$. Suppose $\sigma_m \not\preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$ and $\tau_m \not\preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$. Since v is a unifier of φ in $\mathbf{L}_1 \otimes \mathbf{L}_2$ and v is a unifier of ψ in $\mathbf{L}_1 \otimes \mathbf{L}_2$, then by Lemma 29, $v(x) \rightarrow \boxplus^m \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg v(x) \rightarrow \boxplus^m \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Since \mathbf{L}_1 and \mathbf{L}_2 are non-trivial extensions of $\mathbf{S5}$, either there exists $\mathbf{kk} \geq 2$ such that $\mathbf{L}_1 = \mathbf{S5}_1^{\mathbf{kk}}$, or $\mathbf{L}_1 = \mathbf{S5}_1$ and either there exists $\mathbf{ll} \geq 2$ such that $\mathbf{L}_2 = \mathbf{S5}_2^{\mathbf{ll}}$, or $\mathbf{L}_2 = \mathbf{S5}_2$. Consequently, we have to consider the following cases:

1. there exists $\mathbf{kk} \geq 2$ such that $\mathbf{L}_1 = \mathbf{S5}_1^{\mathbf{kk}}$ and there exists $\mathbf{ll} \geq 2$ such that $\mathbf{L}_2 = \mathbf{S5}_2^{\mathbf{ll}}$,
2. there exists $\mathbf{kk} \geq 2$ such that $\mathbf{L}_1 = \mathbf{S5}_1^{\mathbf{kk}}$ and $\mathbf{L}_2 = \mathbf{S5}_2$,
3. $\mathbf{L}_1 = \mathbf{S5}_1$ and there exists $\mathbf{ll} \geq 2$ such that $\mathbf{L}_2 = \mathbf{S5}_2^{\mathbf{ll}}$,
4. $\mathbf{L}_1 = \mathbf{S5}_1$ and $\mathbf{L}_2 = \mathbf{S5}_2$.

The proof in Case (1) can be done as follows¹⁶.

Remind that

- $\mathbf{S5}_1^{\mathbf{kk}}$ is a Kripke complete $\{1\}$ -logic characterized by the class of all Kripke frames (W, R_1) where R_1 is an equivalence relation on W for which each equivalence class is a finite set of exactly \mathbf{kk} possible worlds,
- $\mathbf{S5}_2^{\mathbf{ll}}$ is a Kripke complete $\{2\}$ -logic characterized by the class of all Kripke frames (W, R_2) where R_2 is an equivalence relation on W for which each equivalence class is a finite set of exactly \mathbf{ll} possible worlds,
- $\mathbf{S5}_1^{\mathbf{kk}}$ is characterized by the class of all Kripke frames (W, R_1) where R_1 is an equivalence relation on W for which each equivalence class is a finite set of at most \mathbf{kk} possible worlds,
- $\mathbf{S5}_2^{\mathbf{ll}}$ is also characterized by the class of all Kripke frames (W, R_2) where R_2 is an equivalence relation on W for which each equivalence class is a finite set of at most \mathbf{ll} possible worlds.

Since these classes of Kripke frames are closed under the formation of disjoint unions and isomorphic copies, then by [15, Theorem 4.1],

¹⁶The proof of Cases (2)–(4) can be similarly done.

- $\mathbf{L}_1 \otimes \mathbf{L}_2$ is a Kripke complete $\{1, 2\}$ -logic characterized both by the class $\mathcal{C}^=$ of all Kripke frames (W, R_1, R_2) where R_1 is an equivalence relation on W for which each equivalence class is a finite set of exactly **kk** possible worlds and R_2 is an equivalence relation on W for which each equivalence class is a finite set of exactly **ll** possible worlds,
- $\mathbf{L}_1 \otimes \mathbf{L}_2$ is a Kripke complete $\{1, 2\}$ -logic characterized both by the class \mathcal{C}^{\leq} of all Kripke frames (W, R_1, R_2) where R_1 is an equivalence relation on W for which each equivalence class is a finite set of at most **kk** possible worlds and R_2 is an equivalence relation on W for which each equivalence class is a finite set of at most **ll** possible worlds.

Since $v(x) \rightarrow \boxplus^m \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg v(x) \rightarrow \boxminus^m \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$,

- there exists a Kripke frame $\mathcal{F} = (W, R_1, R_2)$ in $\mathcal{C}^=$, there exists a model $\mathcal{M} = (W, R_1, R_2, V)$ based on \mathcal{F} and there exists $t_0 \in W$ such that $\mathcal{M}, t_0 \models v(x) \wedge \neg \boxplus^m \perp$,
- there exists a Kripke frame $\mathcal{F}' = (W', R'_1, R'_2)$ in $\mathcal{C}^=$, there exists a model $\mathcal{M}' = (W', R'_1, R'_2, V')$ based on \mathcal{F}' and there exists $t'_0 \in W'$ such that $\mathcal{M}', t'_0 \models \neg v(x) \wedge \neg \boxminus^m \perp$.

Now, let us transform \mathcal{M} and \mathcal{M}' into kinds of tree-like models without affecting satisfiability. An adaptation of the transformation called unravelling [7, Definition 4.51] will enable us to do this. We describe the transformation of \mathcal{M} as follows¹⁷. A t_0 -tip in \mathcal{M} is a tuple of the form $(u_0, a_1, u_1, \dots, a_k, u_k)$ where $u_0 = t_0$, $k \geq 0$, $a_1, \dots, a_k \in \{1, 2\}$ and $u_1, \dots, u_k \in W$ are such that

- for all $i \in \{1, \dots, k\}$, $u_{i-1} R_{a_i} u_i$,
- for all $i \in \{1, \dots, k\}$, $u_{i-1} \neq u_i$,
- for all $i \in \{2, \dots, k\}$, $a_{i-1} \neq a_i$.

Let W'' be the set of all t_0 -tips in \mathcal{M} . Notice that $(t_0) \in W''$. For all $i \in \{1, 2\}$, let R''_i be the equivalence relation on W'' such that for all $(u_0, a_1, u_1, \dots, a_k, u_k), (v_0, b_1, v_1, \dots, b_l, v_l) \in W''$, $(u_0, a_1, u_1, \dots, a_k, u_k) R''_i (v_0, b_1, v_1, \dots, b_l, v_l)$ iff one of the following conditions holds:

- $(u_0, a_1, u_1, \dots, a_k, u_k) = (v_0, b_1, v_1, \dots, b_l, v_l)$,
- $k \geq 1$, $(u_0, a_1, u_1, \dots, a_{k-1}, u_{k-1}) = (v_0, b_1, v_1, \dots, b_l, v_l)$ and $a_k = i$,

¹⁷The description of the transformation of \mathcal{M}' can be similarly done.

- $l \geq 1$, $(u_0, a_1, u_1, \dots, a_k, u_k) = (v_0, b_1, v_1, \dots, b_{l-1}, v_{l-1})$ and $b_l = i$,
- $k \geq 1, l \geq 1$, $(u_0, a_1, u_1, \dots, a_{k-1}, u_{k-1}) = (v_0, b_1, v_1, \dots, b_{l-1}, v_{l-1})$, $a_k = i$, and $b_l = i$.

Notice that for all $(u_0, a_1, u_1, \dots, a_k, u_k) \in W''$, the equivalence class of $(u_0, a_1, u_1, \dots, a_k, u_k)$ modulo R_1'' contains exactly $\mathbf{k}\mathbf{k}$ elements and the equivalence class of $(u_0, a_1, u_1, \dots, a_k, u_k)$ modulo R_2'' contains exactly \mathbf{II} elements. Moreover, the intersection of these equivalence classes is the singleton $\{(u_0, a_1, u_1, \dots, a_k, u_k)\}$. Let V'' be the valuation on W'' such that for all atoms α , $V''(\alpha) = \{(u_0, a_1, u_1, \dots, a_k, u_k) \in W'' : u_k \in V(\alpha)\}$. Let the *unravelling of \mathcal{M} around t_0* be the structure

- $\mathcal{M}'' = (W'', R_1'', R_2'', V'')$.

Similarly, let the *unravelling of \mathcal{M}' around t'_0* be the structure

- $\mathcal{M}''' = (W''', R_1''', R_2''', V''')$.

Let f'' be the function defined from W'' to W and associating to each t_0 -tip $(u_0, a_1, u_1, \dots, a_k, u_k)$ in W'' the possible world u_k in W . Similarly, let f''' be the function defined from W''' to W' and associating to each t'_0 -tip $(u'_0, a'_1, u'_1, \dots, a'_k, u'_k)$ in W''' the possible world u'_k in W' . Obviously, f'' is a bounded morphism from \mathcal{M}'' to \mathcal{M} such that $f''((t_0)) = t_0$ ¹⁸. Similarly, obviously, f''' is a bounded morphism from \mathcal{M}''' to \mathcal{M}' such that $f'''((t'_0)) = t'_0$. Since $\mathcal{M}, t_0 \models v(x) \wedge \neg \boxplus^m \perp$, then by [7, Proposition 2.14], $\mathcal{M}'', (t_0) \models v(x)$ and $\mathcal{M}'', (t_0) \not\models \boxplus^m \perp$. Similarly, since $\mathcal{M}', t'_0 \models \neg v(x) \wedge \neg \boxminus^m \perp$, then by [7, Proposition 2.14], $\mathcal{M}''', (t'_0) \not\models v(x)$ and $\mathcal{M}''', (t'_0) \not\models \boxminus^m \perp$. Since $\mathcal{M}'', (t_0) \not\models \boxplus^m \perp$, there exists $t_{1,1}, t_{1,2}, t_{1,3}, t_{1,4}, t_{1,5}, t_{1,6}, \dots, t_{m,1}, t_{m,2}, t_{m,3}, t_{m,4}, t_{m,5}, t_{m,6} \in W$ such that

- $t_0 R_1 t_{1,1} R_2 t_{1,2} R_1 t_{1,3} R_2 t_{1,4} R_1 t_{1,5} R_2 t_{1,6} \dots R_1 t_{m,1} R_2 t_{m,2} R_1 t_{m,3} R_2 t_{m,4} R_1 t_{m,5} R_2 t_{m,6}$.

¹⁸To see this, notice that

- by the definition of the valuation V'' on W'' , for all t_0 -tips $(u_0, a_1, u_1, \dots, a_k, u_k)$ in W'' , $(u_0, a_1, u_1, \dots, a_k, u_k)$ and u_k satisfy the same atoms,
- for all $i \in \{1, 2\}$, by the definition of the equivalence relation R_i'' on W'' , knowing that R_i is an equivalence relation on W , for all t_0 -tips $(u_0, a_1, u_1, \dots, a_k, u_k), (v_0, b_1, v_1, \dots, b_l, v_l)$ in W'' , if $(u_0, a_1, u_1, \dots, a_k, u_k) R_i'' (v_0, b_1, v_1, \dots, b_l, v_l)$ then $u_k R_i v_l$,
- for all $i \in \{1, 2\}$, by the definition of the equivalence relation R_i'' on W'' , knowing that R_i is an equivalence relation on W , for all t_0 -tips $(u_0, a_1, u_1, \dots, a_k, u_k)$ in W'' and for all v in W , if $u_k R_i v$ then there exists a t_0 -tip $(v_0, b_1, v_1, \dots, b_l, v_l)$ in W'' such that $(u_0, a_1, u_1, \dots, a_k, u_k) R_i'' (v_0, b_1, v_1, \dots, b_l, v_l)$ and $v_l = v$.

- $\mathcal{M}, t_0 \models p^1 \wedge q^0 \wedge r^1$, $\mathcal{M}, t_{1,1} \models p^0 \wedge q^0 \wedge r^0$, $\mathcal{M}, t_{1,2} \models p^0 \wedge q^0 \wedge r^1$, $\mathcal{M}, t_{1,3} \models p^0 \wedge q^1 \wedge r^0$, $\mathcal{M}, t_{1,4} \models p^0 \wedge q^1 \wedge r^1$, $\mathcal{M}, t_{1,5} \models p^1 \wedge q^0 \wedge r^0$, $\mathcal{M}, t_{1,6} \models p^1 \wedge q^0 \wedge r^1$, \dots , $\mathcal{M}, t_{m,1} \models p^0 \wedge q^0 \wedge r^0$, $\mathcal{M}, t_{m,2} \models p^0 \wedge q^0 \wedge r^1$, $\mathcal{M}, t_{m,3} \models p^0 \wedge q^1 \wedge r^0$, $\mathcal{M}, t_{m,4} \models p^0 \wedge q^1 \wedge r^1$, $\mathcal{M}, t_{m,5} \models p^1 \wedge q^0 \wedge r^0$, $\mathcal{M}, t_{m,6} \models p^1 \wedge q^0 \wedge r^1$.

This implies that W'' contains the t_0 -tip $\mathbf{tt} = (t_0, 1, t_{1,1}, 2, t_{1,2}, 1, t_{1,3}, 2, t_{1,4}, 1, t_{1,5}, 2, t_{1,6}, \dots, 1, t_{m,1}, 2, t_{m,2}, 1, t_{m,3}, 2, t_{m,4}, 1, t_{m,5}, 2, t_{m,6})$. Similarly, since \mathcal{M}''' , $(t'_0) \not\models \boxplus^m \perp$, there exists $t'_{1,1}, t'_{1,2}, t'_{1,3}, t'_{1,4}, t'_{1,5}, t'_{1,6}, \dots, t'_{m,1}, t'_{m,2}, t'_{m,3}, t'_{m,4}, t'_{m,5}, t'_{m,6} \in W'$ such that

- $t'_0 R_2 t'_{1,1} R_1 t'_{1,2} R_2 t'_{1,3} R_1 t'_{1,4} R_2 t'_{1,5} R_1 t'_{1,6} \dots R_2 t'_{m,1} R_1 t'_{m,2} R_2 t'_{m,3} R_1 t'_{m,4} R_2 t'_{m,5} R_1 t'_{m,6}$,
- $\mathcal{M}', t'_0 \models p^1 \wedge q^0 \wedge r^1$, $\mathcal{M}', t'_{1,1} \models p^1 \wedge q^0 \wedge r^0$, $\mathcal{M}', t'_{1,2} \models p^0 \wedge q^1 \wedge r^1$, $\mathcal{M}', t'_{1,3} \models p^0 \wedge q^1 \wedge r^0$, $\mathcal{M}', t'_{1,4} \models p^0 \wedge q^0 \wedge r^1$, $\mathcal{M}', t'_{1,5} \models p^0 \wedge q^0 \wedge r^0$, $\mathcal{M}', t'_{1,6} \models p^1 \wedge q^0 \wedge r^1$, \dots , $\mathcal{M}', t'_{m,1} \models p^1 \wedge q^0 \wedge r^0$, $\mathcal{M}', t'_{m,2} \models p^0 \wedge q^1 \wedge r^1$, $\mathcal{M}', t'_{m,3} \models p^0 \wedge q^1 \wedge r^0$, $\mathcal{M}', t'_{m,4} \models p^0 \wedge q^0 \wedge r^1$, $\mathcal{M}', t'_{m,5} \models p^0 \wedge q^0 \wedge r^0$, $\mathcal{M}', t'_{m,6} \models p^1 \wedge q^0 \wedge r^1$.

Similarly, this implies that W''' contains the t'_0 -tip $\mathbf{tt}' = (t'_0, 2, t'_{1,1}, 1, t'_{1,2}, 2, t'_{1,3}, 1, t'_{1,4}, 2, t'_{1,5}, 1, t'_{1,6}, \dots, 2, t'_{m,1}, 1, t'_{m,2}, 2, t'_{m,3}, 1, t'_{m,4}, 2, t'_{m,5}, 1, t'_{m,6})$. Let $\mathcal{M}^\cup = (W^\cup, R_1^\cup, R_2^\cup, V^\cup)$ be the model obtained from the disjoint union of \mathcal{M}'' and \mathcal{M}''' by deleting all possible worlds in $R_1''(\mathbf{tt})$ but \mathbf{tt} and by deleting all possible worlds in $R_2'''(\mathbf{tt}')$ but \mathbf{tt}' . Notice that consequently, $R_1^\cup(\mathbf{tt}) = \{\mathbf{tt}\}$ and $R_2^\cup(\mathbf{tt}') = \{\mathbf{tt}'\}$. Obviously, the Kripke frame $(W^\cup, R_1^\cup, R_2^\cup)$ is in \mathcal{C}^\leq . Moreover, notice that in this frame, the length of the shortest path from (t_0) to \mathbf{tt} is equal to $6m$ and the length of the shortest path from (t'_0) to \mathbf{tt}' is equal to $6m$. Since $\deg(v(x)) \leq 6m$, \mathcal{M}'' , $(t_0) \models v(x)$ and \mathcal{M}''' , $(t'_0) \not\models v(x)$, \mathcal{M}^\cup , $(t_0) \models v(x)$ and \mathcal{M}^\cup , $(t'_0) \not\models v(x)$. Let $\mathcal{M}^\boxplus = (W^\boxplus, R_1^\boxplus, R_2^\boxplus, V^\boxplus)$ be the least model model obtained from $\mathcal{M}^\cup = (W^\cup, R_1^\cup, R_2^\cup, V^\cup)$ by adding new states u_1, u_2, u_3, u_4 and u_5 such that

$$(\star) \quad \mathbf{tt} R_1^\boxplus u_1 R_2^\boxplus u_2 R_1^\boxplus u_3 R_2^\boxplus u_4 R_1^\boxplus u_5 R_2^\boxplus \mathbf{tt}',$$

$$(\star\star) \quad R_1^\boxplus \text{ and } R_2^\boxplus \text{ are reflexive and symmetric}^{19},$$

$$(\star\star\star) \quad \mathcal{M}^\boxplus, u_1 \models p^0 \wedge q^0 \wedge r^0, \mathcal{M}^\boxplus, u_2 \models p^0 \wedge q^0 \wedge r^1, \mathcal{M}^\boxplus, u_3 \models p^0 \wedge q^1 \wedge r^0, \\ \mathcal{M}^\boxplus, u_4 \models p^0 \wedge q^1 \wedge r^1 \text{ and } \mathcal{M}^\boxplus, u_5 \models p^1 \wedge q^0 \wedge r^0.$$

Notice that consequently, $R_1^\boxplus(\mathbf{tt}) = \{\mathbf{tt}, u_1\}$ and $R_2^\boxplus(\mathbf{tt}') = \{\mathbf{tt}', u_5\}$. Obviously, the Kripke frame $(W^\boxplus, R_1^\boxplus, R_2^\boxplus)$ is in \mathcal{C}^\leq . Moreover, notice that in this frame, the length of the shortest path from (t_0) to \mathbf{tt} is still equal to $6m$ and the length of the shortest path from (t'_0) to \mathbf{tt}' is still equal to $6m$. Since $\deg(v(x)) \leq 6m$, \mathcal{M}^\cup , $(t_0) \models v(x)$ and \mathcal{M}^\cup , $(t'_0) \not\models v(x)$, we obtain \mathcal{M}^\boxplus , $(t_0) \models v(x)$ and \mathcal{M}^\boxplus , $(t'_0) \not\models v(x)$. Since v is a unifier

¹⁹The transitivity of R_1^\boxplus and R_2^\boxplus is a consequence of the definition of \mathcal{M}^\boxplus .

of φ and v is a unifier of ψ , $v(x) \rightarrow \boxplus v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg v(x) \rightarrow \boxminus \neg \sigma(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since the Kripke frame $(W^\boxplus, R_1^\boxplus, R_2^\boxplus)$ is in \mathcal{C}^{\leq} , $\mathcal{M}^\boxplus, (t_0) \models v(x)$ and $\mathcal{M}^\boxplus, (t'_0) \not\models v(x)$, it follows that $\mathcal{M}^\boxplus, \mathbf{tt} \models v(x)$ and $\mathcal{M}^\boxplus, \mathbf{tt}' \not\models v(x)$. Since $v(x) \rightarrow \boxplus v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $\neg v(x) \rightarrow \boxminus \neg \sigma(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and the Kripke frame $(W^\boxplus, R_1^\boxplus, R_2^\boxplus)$ is in \mathcal{C}^{\leq} , then by (\star) , $(\star\star)$ and $(\star\star\star)$, $\mathcal{M}^\boxplus, \mathbf{tt} \not\models v(x)$ and $\mathcal{M}^\boxplus, \mathbf{tt}' \models v(x)$: a contradiction.

This ends the proof of Proposition 7. \square

Proposition 8. *If \mathbf{L}_1 and \mathbf{L}_2 are non-trivial extensions of $\mathbf{S5}$ then $\varphi \wedge \psi$ is of type 0 in $\mathbf{L}_1 \otimes \mathbf{L}_2$.*

Proof. Suppose \mathbf{L}_1 and \mathbf{L}_2 are non-trivial extensions of $\mathbf{S5}$. Suppose $\varphi \wedge \psi$ is not of type 0 in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, there exists a minimal complete set Σ of unifiers of $\varphi \wedge \psi$ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 28, σ_0 is a unifier of $\varphi \wedge \psi$ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Since Σ is a minimal complete set of unifiers of $\varphi \wedge \psi$ in $\mathbf{L}_1 \otimes \mathbf{L}_2$, let $v \in \Sigma$ be such that $v \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_0$. Thus, by Proposition 7, let $k \geq 0$ be such that either $\sigma_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$, or $\tau_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$. In the former case, by Lemma 28, σ_{k+1} is a unifier of $\varphi \wedge \psi$ in $\mathbf{L}_1 \otimes \mathbf{L}_2$. Since Σ is a minimal complete set of unifiers of $\varphi \wedge \psi$ in $\mathbf{L}_1 \otimes \mathbf{L}_2$, let $v' \in \Sigma$ be such that $v' \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_{k+1}$. Since $\sigma_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$, then by Lemma 24, $v' \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$. Since Σ is a minimal complete set of unifiers of $\varphi \wedge \psi$ in $\mathbf{L}_1 \otimes \mathbf{L}_2$, $v' = v$. Since $\sigma_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$ and $v' \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_{k+1}$, $\sigma_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_{k+1}$: a contradiction with Lemmas 3 and 25. In the latter case, since $v \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_0$, $\tau_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_0$: a contradiction with Lemmas 3 and 26. \square

Proposition 9. *If \mathbf{L}_1 and \mathbf{L}_2 are non-trivial extensions of $\mathbf{S5}$ then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type 0 for unification with parameters.*

Proof. By Proposition 8. \square

7 Conclusion

After Propositions 4 and 5, it is natural to ask whether if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type ∞ then for all $i \in \{1, 2\}$, \mathbf{L}_i is either of type 1, or of type ω , or of type ∞ . Unfortunately, we have not been able to answer this question, seeing that in Proposition 2, it is not clear that if the set Σ considered there is an infinite minimal complete set of unifiers then, when $L_{\bar{i}} \subseteq \mathbf{Triv}_{\bar{i}}$, or $L_{\bar{i}} \subseteq \mathbf{Verum}_{\bar{i}}$, the corresponding set among $\{\sigma_i^T : \sigma \in \Sigma\}$ and $\{\sigma_{\emptyset, i}^V : \sigma \in \Sigma\}$ is minimal complete too²⁰.

²⁰By the way, no modal logic (either unimodal, or multimodal) is known to be infinitary and it is also an open problem to determine if such modal logic exists.

Following the same line of reasoning as the one used in Section 6, other fusions such as $\mathbf{K4} \otimes \mathbf{K4}$ and $\mathbf{S4} \otimes \mathbf{S4}$ can also be proved to be nullary. See [28, Chapter 6]. The results obtained there as well as the results obtained in Section 6 lead us to the conjecture that every non-trivial fusion is of type 0, that is to say: if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is not of type 0 then either $\mathbf{L}_1 = \mathbf{Triv}_1$, or $\mathbf{L}_1 = \mathbf{Verum}_1$, or $\mathbf{L}_2 = \mathbf{Triv}_2$, or $\mathbf{L}_2 = \mathbf{Verum}_2$. These results also lead us to the conjecture that if either \mathbf{L}_1 is of type 0, or \mathbf{L}_2 is of type 0 then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type 0²¹. By Propositions 4 and 5, this conjecture is equivalent to the one saying that if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type ∞ then for all $i \in \{1, 2\}$, \mathbf{L}_i is either of type 1, or of type ω , or of type ∞ .

Finally, Proposition 9 only constitutes a partial answer to Dzik's conjecture that the fusion $\mathbf{S5} \otimes \mathbf{S5}$ of $\mathbf{S5}$ with itself is either nullary, or infinitary [11, Chapter 6], seeing that it is still unknown when \mathbf{L}_1 and \mathbf{L}_2 are non-trivial extensions of $\mathbf{S5}$ whether $\mathbf{L}_1 \otimes \mathbf{L}_2$ is of type 0 for elementary unification. In the case of elementary unification, what will play the role of the parameters p , q and r used in the formulas φ and ψ ? What will play the role of the formulas φ and ψ ? Is $\mathbf{S5} \otimes \mathbf{S5}$ itself of type 0 for elementary unification?

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²¹In specific cases when among \mathbf{L}_1 and \mathbf{L}_2 one is either \mathbf{K} , or \mathbf{KB} , or \mathbf{KD} , or \mathbf{KDB} , or \mathbf{KT} , or \mathbf{KTB} and the other is characterized by a class of Kripke frames closed under the formation of disjoint unions and isomorphic copies, the line of reasoning developed in [20] for \mathbf{K} and adapted in [3, 4, 5] for \mathbf{KB} , \mathbf{KD} , \mathbf{KDB} , \mathbf{KT} and \mathbf{KTB} can be used to show that the fusion of \mathbf{L}_1 and \mathbf{L}_2 is of type 0. See [28, Chapter 6].

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Appendix

This Appendix includes the proofs of some of our results. Most of these proofs are relatively simple and we have included them here just for the sake of the completeness.

Proof of Lemma 1: The proof is done by induction on φ .

Proof of Lemma 2: The proof is done by induction on φ .

Proof of Lemma 3: Suppose \mathbf{L}_1 and \mathbf{L}_2 are non-trivial extensions of $\mathbf{S5}$.

Firstly, we prove that $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed. Since $\mathbf{S5}_1$ contains all $\{1\}$ -formulas of the form $\varphi \rightarrow \Box_1 \Diamond_1 \varphi$ and $\mathbf{S5}_2$ contains all $\{2\}$ -formulas of the form $\varphi \rightarrow \Box_2 \Diamond_2 \varphi$, \mathbf{L}_1 contains all $\{1\}$ -formulas of the form $\varphi \rightarrow \Box_1 \Diamond_1 \varphi$ and \mathbf{L}_2 contains all $\{2\}$ -formulas of the form $\varphi \rightarrow \Box_2 \Diamond_2 \varphi$. Hence, $\mathbf{L}_1 \otimes \mathbf{L}_2$ contains all $\{1, 2\}$ -formulas of the form $\varphi \rightarrow \Box_1 \Diamond_1 \varphi$ and $\varphi \rightarrow \Box_2 \Diamond_2 \varphi$. Thus, $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed.

Secondly, we prove that $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth. More precisely, we prove that for all $k, l \geq 0$, if $k > l$ then $\Box^k \perp \rightarrow \Box^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ ²². Let $k, l \geq 0$. Suppose $k > l$. Let $\mathcal{M} = (W, R_1, R_2, V)$ be a model such that

- $W = \{i \geq 0 : 0 \leq i \leq 6l\}$,
- for all $i, j \in W$, iR_1j iff $|j - i| \leq 1$ and either $i = j$, or $\max\{i, j\}$ is odd,
- for all $i, j \in W$, iR_2j iff $|j - i| \leq 1$ and either $i = j$, or $\max\{i, j\}$ is even,
- $V(p) = \{i \geq 0 : 0 \leq i \leq 6l \text{ and either } i \bmod 6 = 0, \text{ or } i \bmod 6 = 5\}$,
- $V(q) = \{i \geq 0 : 0 \leq i \leq 6l \text{ and either } i \bmod 6 = 3, \text{ or } i \bmod 6 = 4\}$,
- $V(r) = \{i \geq 0 : 0 \leq i \leq 6l \text{ and either } i \bmod 6 = 0, \text{ or } i \bmod 6 = 2, \text{ or } i \bmod 6 = 4\}$.

Obviously, $\mathcal{M}, 0 \models \Box^k \perp$ and $\mathcal{M}, 0 \not\models \Box^l \perp$. Consequently, the Kripke frame (W, R_1, R_2) does not validate $\Box^k \perp \rightarrow \Box^l \perp$. In other respects, each equivalence class modulo R_1 contains exactly two possible worlds and each equivalence class modulo R_2 contains exactly two possible worlds. Hence, the Kripke frame (W, R_1, R_2) validates $\mathbf{S5}_1^2 \otimes \mathbf{S5}_2^2$. Since the Kripke frame (W, R_1, R_2) does not validate $\Box^k \perp \rightarrow \Box^l \perp$, $\Box^k \perp \rightarrow \Box^l \perp \notin \mathbf{S5}_1^2 \otimes \mathbf{S5}_2^2$. Since \mathbf{L}_1 and \mathbf{L}_2 are non-trivial extensions of $\mathbf{S5}$, $\mathbf{L}_1 \otimes \mathbf{L}_2 \subseteq \mathbf{S5}_1^2 \otimes \mathbf{S5}_2^2$. Since $\Box^k \perp \rightarrow$

²²The proof that for all $k, l \geq 0$, if $k > l$ then $\Box^k \perp \rightarrow \Box^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ can be similarly done.

$\boxplus^l \perp \notin \mathbf{S}5_1^2 \otimes \mathbf{S}5_2^2, \boxplus^k \perp \rightarrow \boxplus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2.$

Proof of Lemma 4: The proof is done by induction on φ .

Proof of Lemma 5: The proof is done by induction on φ .

Proof of Lemma 6: The proof is done by induction on φ .

Proof of Lemma 7: The proof is done by induction on φ .

Proof of Lemma 8: The proof of Item (1) can be done as follows²³.

Suppose $i = 1$.

Suppose $\varphi \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\mathbf{L}_2 \subseteq \mathbf{Triv}_2$. Hence, $\mathbf{L}_1 \otimes \mathbf{L}_2 \subseteq \mathbf{L}_1 \otimes \mathbf{Triv}_2$. Since $\varphi \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $\varphi \in \mathbf{L}_1 \otimes \mathbf{Triv}_2$. Thus, by Lemma 7, $\text{tr}_1^T(\varphi) \in \mathbf{L}_1 \otimes \mathbf{Triv}_2$. Since $\mathbf{L}_1 \otimes \mathbf{Triv}_2$ is a conservative extension of \mathbf{L}_1 , $\text{tr}_1^T(\varphi) \in \mathbf{L}_1$.

Proof of Lemma 9: The proofs of Items (1)–(4) are left to the reader. The proofs of Items (5) and (6) are done by using the well-known fact that if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed then $\mathbf{L}_1 \otimes \mathbf{L}_2$ is closed under the rules $\frac{\neg\varphi \rightarrow \square_1\psi}{\neg\psi \rightarrow \square_1\varphi}$ and $\frac{\neg\varphi \rightarrow \square_2\psi}{\neg\psi \rightarrow \square_2\varphi}$.

Proof of Lemma 10: The proof is done by induction on k .

Proof of Lemma 11: The proof is done by induction on k .

Proof of Lemma 12: Suppose $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth. Hence, $\boxplus^{k+1} \perp \rightarrow \boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^{k+1} \perp \rightarrow \boxminus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $\boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof of Lemma 13: The proof is done by induction on k . Suppose for all $k' \geq 0$, if $k' < k$ then if $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed and smooth then for all $l \geq 0$, $\boxplus^{k'} \perp \vee \boxminus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$.

Case $k = 0$: Suppose $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed and smooth. Let $l \geq 0$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth, then by Lemma 12, $\boxminus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $\boxplus^k \perp \vee \boxminus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$.

Case $k \geq 1$: Suppose $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed and smooth. Let $l \geq 0$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed and smooth, then by induction hypothesis, $(\boxplus^{k-1} \perp \vee \boxminus^{l+1} \perp) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus,

²³The proof of Item (2) can be similarly done.

$(\neg \boxplus^{k-1} \perp \rightarrow \boxminus \boxplus^l \perp) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is tensed, then by Lemma 9, $(\neg \boxplus^l \perp \rightarrow \boxplus \boxplus^{k-1} \perp) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, $(\boxplus^k \perp \vee \boxplus^l \perp) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof of Lemma 14: The proof is done by induction on k . Suppose for all $k' \geq 0$, if $k' < k$ then $\boxplus^{<k'} x \wedge \boxplus^{k'} \perp \rightarrow \sigma_{k'}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxplus^{<k'} \neg x \wedge \boxplus^{k'} \perp \rightarrow \neg \tau_{k'}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Case $k = 0$: This case is left to the reader.

Case $k \geq 1$: By induction hypothesis, $\boxplus^{<k-1} x \wedge \boxplus^{k-1} \perp \rightarrow \sigma_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxplus^{<k-1} \neg x \wedge \boxplus^{k-1} \perp \rightarrow \neg \tau_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $\boxplus \boxplus^{<k-1} x \wedge \boxplus \boxplus^{k-1} \perp \rightarrow \boxplus \sigma_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxplus \boxplus^{<k-1} \neg x \wedge \boxplus \boxplus^{k-1} \perp \rightarrow \boxplus \neg \tau_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $x \wedge \boxplus \boxplus^{<k-1} x \wedge \boxplus \boxplus^{k-1} \perp \rightarrow x \wedge \boxplus \sigma_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg x \wedge \boxplus \boxplus^{<k-1} \neg x \wedge \boxplus \boxplus^{k-1} \perp \rightarrow \neg x \wedge \boxplus \neg \tau_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, by Lemma 11, $\boxplus^{<k} x \wedge \boxplus^k \perp \rightarrow \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxplus^{<k} \neg x \wedge \boxplus^k \perp \rightarrow \neg \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof of Lemma 15: The proof is left to the reader.

Proof of Lemma 16: The proof is done by induction on k . Suppose for all $k' \geq 0$, if $k' < k$ then $\sigma_{k'}(x) \rightarrow \boxplus \sigma_{k'}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_{k'}(x) \rightarrow \boxminus \neg \tau_{k'}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Case $k = 0$: This case is left to the reader.

Case $k \geq 1$: By induction hypothesis, $\sigma_{k-1}(x) \rightarrow \boxplus \sigma_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_{k-1}(x) \rightarrow \boxminus \neg \tau_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, by Lemma 15, $\sigma_{k-1}(x) \rightarrow x \wedge \boxplus \sigma_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_{k-1}(x) \rightarrow \neg x \wedge \boxminus \neg \tau_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $\sigma_{k-1}(x) \rightarrow \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_{k-1}(x) \rightarrow \neg \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, $\boxplus \sigma_{k-1}(x) \rightarrow \boxplus \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus \neg \tau_{k-1}(x) \rightarrow \boxminus \neg \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\sigma_k(x) \rightarrow \boxplus \sigma_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_k(x) \rightarrow \boxminus \neg \tau_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $\sigma_k(x) \rightarrow \boxplus \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_k(x) \rightarrow \boxminus \neg \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof of Lemma 17: The proof is done by induction on k . Suppose for all $k' \geq 0$, if $k' < k$ then for all $l \geq 0$, if $k' \leq l$ then $\sigma_{k'}(x) \rightarrow \boxplus^l \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_{k'}(x) \rightarrow \boxminus^l \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Let $l \geq 0$. Suppose $k \leq l$.

Case $k = 0$: This case is left to the reader.

Case $k \geq 1$: Since $k \leq l$, $k-1 \leq l-1$ and by induction hypothesis, $\sigma_{k-1}(x) \rightarrow \boxplus^{l-1} \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_{k-1}(x) \rightarrow \boxminus^{l-1} \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $\boxplus \sigma_{k-1}(x) \rightarrow \boxplus \boxplus^{l-1} \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus \neg \tau_{k-1}(x) \rightarrow \boxminus \boxminus^{l-1} \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $x \wedge \boxplus \sigma_{k-1}(x) \rightarrow \boxplus \boxplus^{l-1} \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$

and $\neg x \wedge \boxminus \neg \tau_{k-1}(x) \rightarrow \boxminus \boxplus^{l-1} \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, $\sigma_k(x) \rightarrow \boxplus^l \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_k(x) \rightarrow \boxplus^l \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof of Lemma 18: The proof is done by induction on k . Suppose for all $k' \geq 0$, if $k' < k$ then for all $l \geq 0$, if $k' \leq l$ then $\boxplus^{k'} \perp \wedge \sigma_l(x) \leftrightarrow \sigma_{k'}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^{k'} \perp \wedge \neg \tau_l(x) \leftrightarrow \neg \tau_{k'}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Let $l \geq 0$. Suppose $k \leq l$.

Case $k = 0$: This case is left to the reader.

Case $k \geq 1$: Since $k \leq l$, $k - 1 \leq l - 1$ and by induction hypothesis, $\boxplus^{k-1} \perp \wedge \sigma_{l-1}(x) \leftrightarrow \sigma_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^{k-1} \perp \wedge \neg \tau_{l-1}(x) \leftrightarrow \neg \tau_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $\boxplus \boxplus^{k-1} \perp \wedge x \wedge \boxplus \sigma_{l-1}(x) \leftrightarrow x \wedge \boxplus \sigma_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus \boxminus^{k-1} \perp \wedge \neg x \wedge \boxminus \neg \tau_{l-1}(x) \leftrightarrow \neg x \wedge \boxminus \neg \tau_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $\boxplus^k \perp \wedge \sigma_l(x) \leftrightarrow \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \wedge \neg \tau_l(x) \leftrightarrow \neg \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof of Lemma 19: The proof is done by induction on k . Suppose for all $k' \geq 0$, if $k' < k$ then for all $l \geq 0$, if $k' \leq l$ then $\lambda_l(\sigma_{k'}(x)) \leftrightarrow \sigma_{k'}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\mu_l(\tau_{k'}(x)) \leftrightarrow \tau_{k'}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Let $l \geq 0$. Suppose $k \leq l$.

Case $k = 0$: This case is left to the reader.

Case $k \geq 1$: Since $k \leq l$, $k - 1 \leq l$ and by induction hypothesis, $\lambda_l(\sigma_{k-1}(x)) \leftrightarrow \sigma_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\mu_l(\tau_{k-1}(x)) \leftrightarrow \tau_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $x \wedge \boxplus^l \perp \wedge \boxplus \lambda_l(\sigma_{k-1}(x)) \leftrightarrow \boxplus^l \perp \wedge x \wedge \boxplus \sigma_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg x \wedge \boxplus^l \perp \wedge \boxminus \neg \mu_l(\tau_{k-1}(x)) \leftrightarrow \boxplus^l \perp \wedge \neg x \wedge \boxminus \neg \tau_{k-1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $\lambda_l(x \wedge \boxplus \sigma_{k-1}(x)) \leftrightarrow \boxplus^l \perp \wedge \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\mu_l(\neg x \wedge \boxminus \neg \tau_{k-1}(x)) \leftrightarrow \boxplus^l \perp \wedge \neg \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $k \leq l$, by Lemma 17, $\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\mu_l(\tau_k(x)) \leftrightarrow \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof of Lemma 20: The proof is done by induction on k . Suppose for all $k' \geq 0$, if $k' < k$ then for all $l \geq 0$, if $k' \geq l$ then $\lambda_l(\sigma_{k'}(x)) \leftrightarrow \sigma_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\mu_l(\tau_{k'}(x)) \leftrightarrow \tau_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Let $l \geq 0$. Suppose $k \geq l$.

Case $k = l$: This case is left to the reader.

Case $k \geq l + 1$: Hence, $k - 1 \geq l$ and by induction hypothesis, $\lambda_l(\sigma_{k-1}(x)) \leftrightarrow \sigma_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\mu_l(\tau_{k-1}(x)) \leftrightarrow \tau_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $x \wedge \boxplus^l \perp \wedge \boxplus \lambda_l(\sigma_{k-1}(x)) \leftrightarrow \boxplus^l \perp \wedge x \wedge \boxplus \sigma_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg x \wedge \boxplus^l \perp \wedge \boxminus \neg \mu_l(\tau_{k-1}(x)) \leftrightarrow \boxplus^l \perp \wedge \neg x \wedge \boxminus \neg \tau_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, $\lambda_l(x \wedge \boxplus \sigma_{k-1}(x)) \leftrightarrow \boxplus^l \perp \wedge \sigma_{l+1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\mu_l(\neg x \wedge \boxminus \neg \tau_{k-1}(x)) \leftrightarrow \boxplus^l \perp \wedge \neg \tau_{l+1}(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, by Lemma 18, $\lambda_l(\sigma_k(x)) \leftrightarrow \sigma_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and

$\mu_l(\tau_k(x)) \leftrightarrow \tau_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof of Lemma 21: Suppose $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth. Let $l \geq 0$. Suppose $k > l$. Let v and θ be the $\{1, 2\}$ -substitutions defined as follows:

- $v(x) = \top$,
- for all variables y distinct from x , $v(y) = y$,
- $\theta(x) = \perp$,
- for all variables y distinct from x , $\theta(y) = y$.

By Lemma 14, $\boxplus^{<k} x \wedge \boxplus^k \perp \rightarrow \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^{<k} \neg x \wedge \boxminus^k \perp \rightarrow \neg \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $\boxplus^{<k} v(x) \wedge \boxplus^k \perp \rightarrow v(\sigma_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^{<k} \neg \theta(x) \wedge \boxminus^k \perp \rightarrow \neg \theta(\tau_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $v(x) = \top$ and $\theta(x) = \perp$, then by Lemma 10, $\boxplus^k \perp \rightarrow v(\sigma_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \rightarrow \neg \theta(\tau_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth and $k > l$, $\boxplus^k \perp \rightarrow \boxplus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \rightarrow \boxminus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\boxplus^k \perp \rightarrow v(\sigma_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \rightarrow \neg \theta(\tau_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $v(\sigma_k(x)) \rightarrow \boxplus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \theta(\tau_k(x)) \rightarrow \boxminus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $\sigma_k(x) \rightarrow \boxplus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau_k(x) \rightarrow \boxminus^l \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof of Lemma 22: Suppose $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth. Let $l \geq 0$. Let v and θ be the $\{1, 2\}$ -substitutions defined as follows:

- $v(x) = \top$,
- for all variables y distinct from x , $v(y) = y$,
- $\theta(x) = \perp$,
- for all variables y distinct from x , $\theta(y) = y$.

Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth, then by Lemma 12, $\boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $v(x) = \top$ and $\theta(x) = \perp$, $\boxplus^k \perp \vee \neg v(x) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \vee \theta(x) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $\boxplus^k \perp \vee \neg x \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \vee x \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 15, $\neg \tau_k(x) \rightarrow \neg x \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\sigma_l(x) \rightarrow x \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\boxplus^k \perp \vee \neg x \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \vee x \notin \mathbf{L}_1 \otimes \mathbf{L}_2$, $\boxplus^k \perp \vee \neg \tau_k(x) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\boxminus^k \perp \vee \sigma_l(x) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof of Lemma 23: By Lemma 20.

Proof of Lemma 24: By Lemma 23.

Proof of Lemma 25: Suppose $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth. Let $l \geq 0$. Suppose $k < l$. Suppose either $\sigma_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_l$, or $\tau_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \tau_l$. In the former case, let λ be a $\{1, 2\}$ -substitution such that $\sigma_k \circ \lambda \simeq^{\{x\}} \sigma_l$. Hence, $\lambda(\sigma_k(x)) \leftrightarrow \sigma_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth and $k < l$, then by Lemma 21, $\sigma_l(x) \rightarrow \boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 17, $\sigma_k(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $\lambda(\sigma_k(x)) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\sigma_l(x) \rightarrow \boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$, $\lambda(\sigma_k(x)) \leftrightarrow \sigma_l(x) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$: a contradiction. In the latter case, let μ be a $\{1, 2\}$ -substitution such that $\tau_k \circ \mu \simeq^{\{x\}} \tau_l$. Consequently, $\mu(\tau_k(x)) \leftrightarrow \tau_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth and $k < l$, by Lemma 21, $\neg \tau_l(x) \rightarrow \boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 17, $\neg \tau_k(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $\neg \mu(\tau_k(x)) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\neg \tau_l(x) \rightarrow \boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$, $\mu(\tau_k(x)) \leftrightarrow \tau_l(x) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$: a contradiction.

Proof of Lemma 26: Suppose $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth. Let $l \geq 0$. Suppose either $\sigma_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \tau_l$, or $\tau_k \preceq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} \sigma_l$. In the former case, let λ be a $\{1, 2\}$ -substitution such that $\sigma_k \circ \lambda \simeq^{\{x\}} \tau_l$. Hence, $\lambda(\sigma_k(x)) \leftrightarrow \tau_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 17, $\sigma_k(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth, then by Lemma 22, $\boxplus^k \perp \vee \neg \tau_l(x) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\lambda(\sigma_k(x)) \leftrightarrow \tau_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, then $\lambda(\sigma_k(x)) \rightarrow \boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $\sigma_k(x) \rightarrow \boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$: a contradiction. In the latter case, let μ be a $\{1, 2\}$ -substitution such that $\tau_k \circ \mu \simeq^{\{x\}} \sigma_l$. Consequently, $\mu(\tau_k(x)) \leftrightarrow \sigma_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 17, $\neg \tau_k(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mathbf{L}_1 \otimes \mathbf{L}_2$ is smooth, then by Lemma 22, $\boxplus^k \perp \vee \sigma_l(x) \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\mu(\tau_k(x)) \leftrightarrow \sigma_l(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $\neg \mu(\tau_k(x)) \rightarrow \boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $\neg \tau_k(x) \rightarrow \boxplus^k \perp \notin \mathbf{L}_1 \otimes \mathbf{L}_2$: a contradiction.

Proof of Lemma 27: The proof is done by induction on k . Suppose for all $k' \geq 0$, if $k' < k$ then for all $\mathbf{L}_1 \otimes \mathbf{L}_2$ -unifiers σ of φ , $\sigma(x) \rightarrow \boxplus^{<k'} \sigma(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and for all $\mathbf{L}_1 \otimes \mathbf{L}_2$ -unifiers τ of ψ , $\neg \tau(x) \rightarrow \boxplus^{<k'} \neg \tau(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Case $k = 0$: This case is left to the reader.

Case $k \geq 1$: Let σ be an $\mathbf{L}_1 \otimes \mathbf{L}_2$ -unifier of φ and τ be an $\mathbf{L}_1 \otimes \mathbf{L}_2$ -unifier of ψ . By induction hypothesis, $\sigma(x) \rightarrow \boxplus^{<k-1} \sigma(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau(x) \rightarrow \boxplus^{<k-1} \neg \tau(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since σ is an $\mathbf{L}_1 \otimes \mathbf{L}_2$ -unifier of φ and τ is an $\mathbf{L}_1 \otimes \mathbf{L}_2$ -unifier of ψ , $\sigma(x) \rightarrow \boxplus^{k-1} \sigma(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau(x) \rightarrow \boxplus^{k-1} \neg \tau(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\sigma(x) \rightarrow \boxplus^{<k-1} \sigma(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau(x) \rightarrow \boxplus^{<k-1} \neg \tau(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $\sigma(x) \rightarrow \boxplus^{<k-1} \sigma(x) \wedge \boxplus^{k-1} \sigma(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau(x) \rightarrow \boxplus^{<k-1} \neg \tau(x) \wedge \boxplus^{k-1} \neg \tau(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $\sigma(x) \rightarrow \boxplus^{<k} \sigma(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$ and $\neg \tau(x) \rightarrow \boxplus^{<k} \neg \tau(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

Proof of Lemma 28: By Lemma 16.

Proof of Lemma 29: Suppose v is an $\mathbf{L}_1 \otimes \mathbf{L}_2$ -unifier of φ . Let $k \geq 0$.

(a) \Rightarrow (b): Suppose $\sigma_k \circ v \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$. Hence, $\sigma_k \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$.

(b) \Rightarrow (c): Suppose $\sigma_k \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$. Let v' be a $\{1, 2\}$ -substitution such that $\sigma_k \circ v' \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$. Thus, $v'(\sigma_k(x)) \leftrightarrow v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 17, $\sigma_k(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, $v'(\sigma_k(x)) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $v'(\sigma_k(x)) \leftrightarrow v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $v(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

(c) \Rightarrow (a): Suppose $v(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since v is an $\mathbf{L}_1 \otimes \mathbf{L}_2$ -unifier of φ , by Lemma 27, $v(x) \rightarrow \boxplus^{<k} v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $v(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $v(x) \rightarrow \boxplus^{<k} v(x) \wedge \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 14, $\boxplus^{<k} x \wedge \boxplus^k \perp \rightarrow \sigma_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $\boxplus^{<k} v(x) \wedge \boxplus^k \perp \rightarrow v(\sigma_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $v(x) \rightarrow \boxplus^{<k} v(x) \wedge \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $v(x) \rightarrow v(\sigma_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 15, $\sigma_k(x) \rightarrow x \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $v(\sigma_k(x)) \rightarrow v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $v(x) \rightarrow v(\sigma_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $v(\sigma_k(x)) \leftrightarrow v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, $\sigma_k \circ v \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$.

Suppose v is an $\mathbf{L}_1 \otimes \mathbf{L}_2$ -unifier of ψ . Let $k \geq 0$.

(d) \Rightarrow (e): Suppose $\tau_k \circ v \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$. Hence, $\tau_k \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$.

(e) \Rightarrow (f): Suppose $\tau_k \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$. Let v' be a $\{1, 2\}$ -substitution such that $\tau_k \circ v' \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$. Thus, $v'(\tau_k(x)) \leftrightarrow v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 17, $\neg \tau_k(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, $v'(\neg \tau_k(x)) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $v'(\tau_k(x)) \leftrightarrow v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $\neg v(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$.

(f) \Rightarrow (d): Suppose $\neg v(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since v is an $\mathbf{L}_1 \otimes \mathbf{L}_2$ -unifier of ψ , by Lemma 27, $\neg v(x) \rightarrow \boxplus^{<k} \neg v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\neg v(x) \rightarrow \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $\neg v(x) \rightarrow \boxplus^{<k} \neg v(x) \wedge \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 14, $\boxplus^{<k} \neg x \wedge \boxplus^k \perp \rightarrow \neg \tau_k(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Hence, $\boxplus^{<k} \neg v(x) \wedge \boxplus^k \perp \rightarrow v(\neg \tau_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\neg v(x) \rightarrow \boxplus^{<k} \neg v(x) \wedge \boxplus^k \perp \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $\neg v(x) \rightarrow v(\neg \tau_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. By Lemma 15, $\neg \tau_k(x) \rightarrow \neg x \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Thus, $v(\neg \tau_k(x)) \rightarrow \neg v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Since $\neg v(x) \rightarrow v(\neg \tau_k(x)) \in \mathbf{L}_1 \otimes \mathbf{L}_2$, $v(\tau_k(x)) \leftrightarrow v(x) \in \mathbf{L}_1 \otimes \mathbf{L}_2$. Consequently, $\tau_k \circ v \simeq_{\mathbf{L}_1 \otimes \mathbf{L}_2}^{\{x\}} v$.