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ON THE REGULARITY OF THE GREEN CURRENT FOR SEMI-EXTREMAL ENDOMORPHISMS OF $\mathbb{P}^2$

Christophe Dupont and Axel Rogue

Université de Rennes, CNRS, IRMAR - UMR 6625
F-35000 Rennes, France

Abstract. We study the regularity of the Green current for semi-extremal endomorphisms of $\mathbb{P}^2$. Under suitable assumptions, we show that the pointwise lower Radon-Nikodym derivative of stable slices with respect to the one dimensional Lebesgue measure is bounded at almost every point for the equilibrium measure. This provides a weak amount of metric regularity for the Green current along holomorphic discs.

1. **Introduction.** Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let T be the Green current and let $\mu := T \wedge T$ be the equilibrium measure of f, see [13, 26]. The measure μ is invariant and ergodic, we denote $\mathcal{J}$ its support. The critical set of f satisfies $\mu(\mathcal{C}) = 0$. The Lyapunov exponents $\lambda_2 \leq \lambda_1$ of μ will play a central role in this article. They are bounded below by $\frac{1}{2}\log d$, see [8].

1.1. **Extremal and semi-extremal endomorphisms.** We say that f is *extremal* if $\lambda_1 = \lambda_2 = \frac{1}{2}\log d$. Those endomorphisms are characterized by the four equivalent properties: $\mu \ll Leb_{\mathbb{P}^2}$, $\dim_H \mu = 4$, T is a positive smooth $(1,1)$-form on some open subset of $\mathbb{P}^2$, and f is a Lattès map, see [6, 4, 10]. We recall that a Lattès map on $\mathbb{P}^k$ is the projection of an affine dilation on a complex k-torus by means of a finite galoisian covering σ, see [15] for examples. Similar characterizations were proved for rational maps on $\mathbb{P}^1$, in which case $\lambda = \frac{1}{2}\log d$ and $T = \mu$, see [23, 24, 27].

We say that f is *semi-extremal* if $\lambda_1 > \lambda_2 = \frac{1}{2}\log d$. Examples are provided by endomorphisms f of $\mathbb{P}^2$ having an invariant pencil of lines on which f induces a one dimensional Lattès map, see [18]. A natural question is to find characterizations of semi-extremal endomorphisms, as we have for extremal ones. We already know that they satisfy $\dim_H \mu = 2 + \frac{\log d}{\lambda_1}$, see [7, 10, 17], and that f is semi-extremal once $\mu \ll T \wedge \omega$, see [14]. Here ω stands for the Fubini-Study $(1,1)$-form on $\mathbb{P}^2$, so that $T \wedge \omega$ is the trace measure of the Green current. Dujardin asked in [14, Section 3] the following questions.

Question 1.1. *Does every semi-extremal endomorphism satisfy $\mu \ll T \wedge \omega$? If $\mu \ll T \wedge \omega$, does f contain (in some sense) a one-dimensional Lattès map?*

A strategy for the first question would to show that semi-extremal endomorphisms have an invariant pencil of lines. Indeed, in this case the radial Lyapunov exponent of f is $\geq \log d$ (by [18, Theorem 1.7]), hence the Lyapunov exponent of the action of f on the pencil of lines (which belongs to the spectrum of μ by [18,

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Theorem 1.5]) must be equal to $\frac{1}{2}\log d$ (semi-extremality is assumed). This implies that f induces a Lattès map on the pencil of lines, which ensures that $\mu \ll T \wedge \omega$ by [18, Corollary 1.3].

We can make a similar observation for the second question. If $\mu \ll T \wedge \omega$ then $\frac{1}{2}\log d$ is one of the Lyapunov exponents of μ by [14]. Using the same arguments as before (relying on [18]), the Lyapunov exponent of the action of f on the pencil of lines must be $\frac{1}{2}\log d$, ensuring that f induces a Lattès map on this pencil.

In the general case (without assuming the existence of an invariant pencil of lines), the condition $\mu \ll T \wedge \omega$, equivalent to $T \wedge T \ll T \wedge \omega$, formally says that $T \ll \omega$ with respect to T. A first step to exhibit such regular properties for T consists in establishing regular properties for slices. Theorems 1.3 and 1.4 below provide results in that direction: they bound the pointwise lower Radon-Nikodym derivative of stable slices of T at μ-almost every point. In Section 1.4 we shall examine the guideline given by polynomial lifts of Lattès mappings on $\mathbb{P}^1$: such regular slices of course exist, moreover there exist regular slices which do not intersect the support of μ (see Remark 1.6).

1.2. Statements of the results.

Definition 1.2. Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Assume that the Lyapunov exponents of the equilibrium measure of f are different, hence $\lambda_2 < \lambda_1$. Let $v_s(x) \in \mathbb{P}(T_x\mathbb{P}^2)$ denote the stable Oseledec direction, which is defined for every x in an invariant μ-generic Borel subset A. Let R be the set of points $x \in A$ such that there exists a holomorphic disc $\xi_x : \mathbb{D} \to \mathbb{P}^2$ satisfying $\xi_x(0) = x$, $[\xi_x'(0)] = v_s(x)$ and

$$\liminf_{r\to 0} \frac{\xi_x^* T(\mathbb{D}(r))}{\mathrm{Leb}(\mathbb{D}(r))} < \infty. \tag{1.1}$$

In that definition $[v] \in \mathbb{P}(T_x\mathbb{P}^2)$ stands for the line of $T_x\mathbb{P}^2$ directed by v. Since $\mu(\mathcal{C}) = 0$, one can assume that A does not intersect $\cup_{n\in\mathbb{Z}} f^n(\mathcal{C})$. The relation $f^*T = dT$ then implies that R is totally f-invariant, in particular, $\mu(R) \in \{0, 1\}$ by ergodicity. We prove that $\mu(R) > 0$ under two different sets of assumptions related to semi-extremality.

Theorem 1.3. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Assume that $\mu \ll T \wedge \omega$ and $\lambda_2 < \lambda_1 < 2\lambda_2$. Then $\mu(R) = 1$.*

Theorem 1.4. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Assume that f is semi-extremal and that $\lambda_2 < \lambda_1 < 2\lambda_2$. Assume also that $\mathcal{J} \cap \mathcal{C} = \emptyset$ and that v_s is Hölder continuous on A. Then $\mu(R) = 1$.*

For the second result, we shall see in Section 2.2 that the Hölder continuity of v_s on A is satisfied when f is partially hyperbolic on $\mathcal{J}$.

1.3. Outline of the proofs.

To prove Theorems 1.3 and 1.4, a first step will consist in showing

$$d^{n_j} e^{-2M_1} \leq \|D_x f^{n_j}(\vec{v}_s(x))\|^2 \leq d^{n_j} e^{2M_1} \tag{1.2}$$

for every x in a subset of positive μ-measure, where $(n_j)_j$ is an increasing sequence of integers depending on x and $\vec{v}_s(x) \in T_x\mathbb{P}^2$ is a unitary vector in $v_s(x) \in \mathbb{P}(T_x\mathbb{P}^2)$ defined in Definition 1.2. It is crucial that no exponential error term $e^{\pm n\epsilon}$ occur in (1.2). Such error terms are actually inherent to (our context of) non uniform hyperbolicity, a difficulty is thus to get rid of them. A second step will be to integrate

(1.2), namely to construct a holomorphic disc on which f^{n_j} is a multiplication by $d^{n_j/2}e^{\pm M_1}$. We will use for that purpose a normal form Theorem for the inverse branches of f. Theorem 3.1 (see Section 3) summarizes both steps, it provides workable sufficient conditions ensuring that a point belongs to R.

Let us specify how the estimates (1.2) will be obtained. The lower bound is actually true for every endomorphism of $\mathbb{P}^2$ satisfying $\lambda_2 \leq \lambda_1 < 2\lambda_2$, it does not need any semi-extremality assumption, see Theorem 2.7. The upper bound in (1.2) needs semi-extremality: for Theorem 1.3 we use arguments developed by Dujardin [14] to construct Fatou directions (see Section 4), and for Theorem 1.4 we use the Central Limit Theorem for the logarithm of the tangent map on the stable direction $v_s(x)$ (see Section 5).

Let us notice that the Central Limit Theorem was used in [27] to prove regularity properties for the equilibrium measure of rational maps on $\mathbb{P}^1$ and in [16] to provide a new proof of the fact that μ is absolutely continuous when f is extremal on $\mathbb{P}^k$.

1.4. **Study of the lifts of one dimensional Lattès mappings.** The polynomial lifts of Lattès mappings on $\mathbb{P}^1$ are the most simple examples of semi-extremal endomorphisms of $\mathbb{P}^2$. In this section we verify for them $\mu(R) = 1$ and $\mu \ll T \wedge \omega$ by using normal forms for their Green function.

Proposition 1.5. *Let $f[z : w : t] := [P(z, w) : Q(z, w) : t^d]$, where $[P : Q]$ is a Lattès mapping of $\mathbb{P}^1$ of degree $d \geq 2$. Then $\mu(R) = 1$.*

Proof. The mapping f belongs to the class of regular polynomial endomorphisms of $\mathbb{C}^2$ studied by Bedford-Jonsson [2]. Let $L := (P, Q)$ and let $L^n = (P^n, Q^n)$ denote the n-th iterate of L. The Green function of the homogeneous polynomial mapping L is

$$G(z, w) := \lim_n \frac{1}{d^n} \log ||(P^n(z, w), Q^n(z, w))||.$$

Let us denote by $\Omega := \{G < 0\}$ the basin of attraction of the origin in $\mathbb{C}^2$, and by $\partial\Omega = \{G = 0\}$ the boundary of Ω. In the affine chart $\{t = 1\}$, the Green current T of f is equal to dd^cG^+, where $G^+ := \max\{G, 0\}$. Since $[P : Q]$ is a Lattès mapping, the support of the equilibrium measure $\mu = T \wedge T$ of f coincides with $\partial\Omega$.

The results [3, Theorem 1.2 and Lemma 4.2] due to Berteloot-Loeb provide normal forms for the function G when $[P : Q]$ is a Lattès map of $\mathbb{P}^1$ of degree d. Precisely, if x does not belong to the union S of a finite number of complex lines in $\mathbb{C}^2$ passing through the origin (which has zero μ-measure), then there exist a biholomorphism $p : (\mathbb{C}^2, (0, 0)) \to (\mathbb{C}^2, x)$ and $\delta \in \mathbb{R}$ such that

$$G \circ p(Z, W) = |Z|^2 + \Re(W) \quad \text{and} \quad p^{-1} \circ L \circ p(Z, W) = (e^{i\delta}\sqrt{d}\, Z, d\, W).$$

In particular, up to a holomorphic change of coordinates near x, $\partial\Omega$ is a piece of the 3-euclidian sphere. Now observe that the function $G \circ p$ is strictly subharmonic on the Z-axis (the unique complex line in the tangent space at $(0, 0)$ of the real hypersurface $\{G \circ p = 0\}$), which is included in $\{G \circ p \geq 0\}$. Hence G^+ is strictly subharmonic along the holomorphic disc

$$\xi_x(u) := p(u, 0).$$

The limit of $\frac{\xi_x^* T(\mathbb{D}(r))}{\mathrm{Leb}(\mathbb{D}(r))}$ when r tends to 0 then exists and is equal to $dd^c(G^+ \circ \xi_x)(0) \in \mathbb{R}_+^*$. The fact that $[\xi_x'(0)] = v_s(x)$ comes from [3, Lemma 4.2], which lifts the Lattès commuting diagram to line bundles, the local coordinate in the fibers being W. We thus have verified that μ-almost every x belongs to R, as desired. $\square$

Remark 1.6. In the preceding proof, for every $W_0 \in (\mathbb{C}, 0)$ such that $\Re(W_0) > 0$, the function G^+ is strictly subharmonic along the holomorphic disc $\xi(u, W_0) := p(u, W_0)$. Such a disc is included in $\{G \circ p > 0\}$, which does not intersect the support of μ.

Proposition 1.7. *Let $f[z : w : t] := [P(z, w) : Q(z, w) : t^d]$, where $[P : Q]$ is a Lattès mapping of $\mathbb{P}^1$ of degree $d \geq 2$. Then $\mu \ll T \wedge \omega$.*

Proof. We take the notations of the proof of Proposition 1.5. It suffices to verify $\mu \ll T \wedge \omega$ at every point $x \in \partial\Omega \setminus S$, since $\mu(S) = 0$. We work with the local coordinates provided by p. Let $G_0(Z, W) := |Z|^2 + \Re(W)$, $T_0 := dd^c G_0^+$ and $\omega_0 := \frac{i}{2} dZ \wedge d\overline{Z} + \frac{i}{2} dW \wedge d\overline{W}$. For every test function φ on $(\mathbb{C}^2, (0, 0))$, we have by using and adapting [22, Proposition 6.5.5]:

$$\int_{(\mathbb{C}^2,(0,0))} \varphi\, T_0 \wedge \omega_0 = \int_{\{G_0=0\}} \varphi\, d^c G_0 \wedge \omega_0 + \int_{\{G_0 \geq 0\}} \varphi\, dd^c G_0 \wedge \omega_0, \tag{1.3}$$

$$\int_{(\mathbb{C}^2,(0,0))} \varphi\, T_0 \wedge T_0 = \int_{\{G_0=0\}} \varphi\, d^c G_0 \wedge dd^c G_0, \tag{1.4}$$

where the orientation on $\{G_0 = 0\}$ is induced by the one of $\{G_0 \leq 0\}$. A straigthforward computation shows that

$$d^c G_0 \wedge \omega_0 = d^c G_0 \wedge dd^c G_0, \tag{1.5}$$

on the spherical 3-manifold $\{G_0 = 0\}$. Finally, one obtains $T_0 \wedge T_0 \ll T_0 \wedge \omega_0$ by using (1.5) and comparing (1.3) and (1.4). □

Remark 1.8. The measure given by (1.5) is equal to $d\Im W \wedge \frac{i}{2} dZ \wedge d\overline{Z}$, which belongs to the Lebesgue measure class of the 3-manifold $\{G_0 = 0\}$.

2. Oseledec theorem, stable directions, normal forms.

2.1. Oseledec theorem.

Let us state the Oseledec Theorem in our context.

Theorem 2.1. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let $\lambda_2 \leq \lambda_1$ be the Lyapunov exponents of the equilibrium measure μ of f.*

1. *If $\lambda_1 = \lambda_2 = \lambda$, then there exists an invariant Borel subset A of full μ-measure and disjoint from $\cup_{n\in\mathbb{Z}} f^n(\mathcal{C})$ such that for every $x \in A$:*

$$\forall \vec{v} \in T_x\mathbb{P}^2 \setminus \{0\}\ ,\ \lim_{n\to+\infty} \frac{1}{n} \log \|D_x f^n(\vec{v})\| = \lambda.$$

2. *If $\lambda_1 > \lambda_2$, then there exists an invariant Borel subset A of full μ-measure and disjoint from $\cup_{n\in\mathbb{Z}} f^n(\mathcal{C})$ such that for every $x \in A$, there exists $v_s(x) \in \mathbb{P}(T_x\mathbb{P}^2)$ satisfying:*

$$\forall \vec{v} \in T_x\mathbb{P}^2 \setminus v_s(x)\ ,\ \lim_{n\to+\infty} \frac{1}{n} \log \|D_x f^n(\vec{v})\| = \lambda_1.$$

$$\forall \vec{v} \in v_s(x) \setminus \{0\}\ ,\ \lim_{n\to+\infty} \frac{1}{n} \log \|D_x f^n(\vec{v})\| = \lambda_2.$$

Moreover v_s is measurable and satisfies $[D_x f](v_s(x)) = v_s(f(x))$ for every $x \in A$, where $[D_x f]$ is the projectivization of the tangent map $D_x f$.

Let us assume that $\lambda_2 < \lambda_1$. For every $x \in A$, we denote

$$\psi(x) := \log \|D_x f(\vec{v}_s(x))\| ,$$

where $\vec{v}_s(x) \in \{\vec{v} \in v_s(x) \subset T_x\mathbb{P}^2 \ , \ \|\vec{v}\| = 1\}$. We have $\psi \in L^1(\mu)$, see [26, Section 3.7].

Lemma 2.2. *For every $x \in A$ and every $n \geq 1$, we have*

$$\frac{1}{n}\sum_{i=0}^{n-1} \psi(f^i(x)) = \frac{1}{n} \log \|D_x f^n(\vec{v}_s(x))\| .$$

This implies $\int \psi \, d\mu = \lambda_2$.

Proof. The first formula follows from the definition of ψ and the relation $[D_x f](v_s(x)) = v_s(f(x))$. We obtain the second formula by taking limits μ-almost everywhere: the left hand side converges to $\int \psi \, d\mu$ by Birkhoff ergodic Theorem, and the right hand side converges to λ_2 by Theorem 2.1. □

2.2. **Hölder continuity of the stable direction.** The following result, due to Brin, is [25, Proposition 3.9] adapted to our setting, see also [1, Section 5.3]. We use the notation dist for the distances on $\mathbb{P}^2$ and $\mathbb{P}(T\mathbb{P}^2)$.

Theorem 2.3. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Assume that f is partially hyperbolic on $\mathcal{J}$: that means that there exist $0 < \lambda_- < \lambda_+$ and $c \geq 1$ such that for every $x \in \mathcal{J}$, there exist unitary vectors $\vec{w}_s(x), \vec{w}_u(x) \in T_x\mathbb{P}^2$ whose angle is uniformly bounded from below on $\mathcal{J}$ and which satisfy*

$$\forall n \geq 1 \ , \ \|D_x f^n(\vec{w}_s(x))\| \leq c\lambda_-^n \ , \ \|D_x f^n(\vec{w}_u(x))\| \geq c^{-1}\lambda_+^n. \tag{2.1}$$

Then the direction $w_s = [\vec{w}_s] : \mathcal{J} \to \mathbb{P}(T\mathbb{P}^2)$ is Hölder continuous. More precisely, if $b := max\{1, \|f\|_{C^1}\}$ then for every $a > b^2$ there exists $D_a > 1$ such that:

$$\mathsf{dist}(x,y) < D_a^{-1} \ \Rightarrow \ \mathsf{dist}(w_s(x), w_s(y)) \leq 3c^2 \frac{\lambda_+}{\lambda_-} \left(D_a \, \mathsf{dist}(x,y)\right)^{\frac{\log(\lambda_+/\lambda_-)}{\log(a/\lambda_-)}} .$$

Remark 2.4. If f is partially hyperbolic on $\mathcal{J}$ and if $\lambda_2 < \lambda_1$, then one verifies that the stable Oseledec direction $v_s(x)$ provided by Theorem 2.1 coincides with $w_s(x)$ for every $x \in A$, hence v_s is Hölder continuous on A by Theorem 2.3. In particular, every partially hyperbolic endomorphism satisfying $\lambda_2 = \frac{1}{2}\log d$, $\lambda_2 < \lambda_1 < 2\lambda_2$ and $\mathcal{J} \cap \mathcal{C} = \emptyset$ fulfills the assumptions of Theorem 1.4.

Actually it is not easy to find examples of semi-extremal endomorphisms partially hyperbolic on $\mathcal{J}$. For instance the lift $f[z : w : t] = [z^2 - 2w^2 : z^2 : t^2]$ of the Lattès mapping $L[z : w] = [z^2 - 2w^2 : z^2]$ is not partially hyperbolic on $\mathcal{J}$. To see this, observe that the Lyapunov exponents of (f, μ) are $\frac{1}{2}\log 2$ (coming from the Lattès mapping L) and $\log 2$ (coming from t^2). In particular, the modulus of the multiplier of every n-repulsive cycle of L not intersecting the critical values of σ is equal to $(\sqrt{2})^n$. On the other hand, that modulus is equal to 4 for the fixed point $[1 : 1]$ (this is a critical value of σ, see [3, Section 3]). Hence f has n-periodic points in $\mathcal{J}$ with multipliers $\{(\sqrt{2})^n, 2^n\}$ and $\{4^n, 2^n\}$ respectively. It follows that no $\lambda_- < \lambda_+$ satisfy (2.1) over $\mathcal{J}$.

Examples of partially hyperbolic endomorphisms should be found among uniformly expanding endomorphisms. Jonsson [20] obtained examples in the family of polynomial skew products. For instance, $f(z, w) = (z^2, w^2 + zw + cz)$ is uniformly expanding on $\mathcal{J}$ if c is large enough (see Theorems 4.8, 8.2 and Examples 9.2 of

this article). Corollary 8.3 of [20] moreover characterizes the Axiom A property in terms of the dynamics of the critical set: $f(z,w) = (z^2-6, w^2+3-z)$ is uniformly expanding on $\mathcal{J}$. We refer to [19] for results on a family of Axiom A endomorphisms of $\mathbb{P}^2$ (called *s-hyperbolic*): basic sets, attractors and invariant currents are investigated there in Sections 3, 4 and 5.

To deduce partial hyperbolicity from uniform expansivity is also not straightforward. This should involve the Lyapunov exponents of μ: the stable Oseledec direction $v_s(x)$ is a natural candidate for $w_s(x)$. Nonetheless, that direction is only defined μ-a.e. on $\mathcal{J}$, and an other difficulty comes from the fact that the Lyapunov exponents only give an asymptotic growth rate. Hence, even if the Lyapunov exponents are distinct, the existence of $c \geq 1$ and $\lambda_- < \lambda_+$ satisfying (2.1) for every $n \geq 1$ is not obvious.

Before stating the next Proposition, let us recall Remark 2.4: if f is partially hyperbolic on $\mathcal{J}$ and if $\lambda_2 < \lambda_1$, then the stable Oseledec direction v_s provided by Theorem 2.1 is Hölder continuous on A.

Proposition 2.5. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Assume that $\lambda_2 < \lambda_1$ and that v_s is Hölder continuous on A.*

1. *Then v_s extends to a Hölder continuous map on $\mathcal{J}$, still denoted v_s.*
2. *If $\mathcal{J} \cap \mathcal{C} = \emptyset$, then $\psi(x) = \log \|D_x f(\vec{v}_s(x))\|$ is Hölder continuous on $\mathcal{J}$.*

Proof. The first item is classical by using Cauchy sequences and the fact that A is dense in $\mathcal{J}$. Let us write $\mathsf{dist}(v_s(x), v_s(y)) \leq c' d(x,y)^\alpha$ on $\mathcal{J}$ and prove the second item. By working in charts of $T\mathbb{P}^2$, let us fix a section of the unit tangent bundle $x \mapsto \vec{v}_s(x)$ such that $\vec{v}_s(x) \in v_s(x)$ and $\|\vec{v}_s(x) - \vec{v}_s(y)\| \leq c' \,\mathsf{dist}(x,y)^\alpha$. We get

$$\begin{aligned}
&\|D_x f(\vec{v}_s(x)) - D_y f(\vec{v}_s(y))\| \\
&\leq \|D_x f(\vec{v}_s(x)) - D_x f(\vec{v}_s(y))\| + \|D_x f(\vec{v}_s(y)) - D_y f(\vec{v}_s(y))\| \\
&\leq \|D_x f\| \, \|\vec{v}_s(x) - \vec{v}_s(y)\| + \|D_x f - D_y f\| \, \|\vec{v}_s(y)\| \\
&\leq \|f\|_{C^2} \, (1 + c') d(x,y)^\alpha.
\end{aligned}$$

This implies that $x \mapsto \|D_x f(\vec{v}_s(x))\|$ is Hölder continuous. Since $\mathcal{J} \cap \mathcal{C} = \emptyset$, that function is bounded below by a constant $\rho > 0$ on $\mathcal{J}$. By using that log is ρ^{-1}-Lipschitz on $[\rho, +\infty[$, we get that $\psi(x) = \log \|D_x f(\vec{v}_s(x))\|$ is Hölder continuous on $\mathcal{J}$. □

2.3. **Normal forms.** Recall that the subset A defined in Definition 1.2 is invariant, does not intersect $\cup_{n\in\mathbb{Z}} f^n(\mathcal{C})$ and satisfies $\mu(A) = 1$. Let

$$\hat{A} := \left\{ \hat{x} = (x_n)_{n\in\mathbb{Z}} \in A^{\mathbb{Z}} \,,\, x_{n+1} = f(x_n) \right\}$$

and let $\hat{f} : \hat{A} \to \hat{A}$ be the left shift. We denote $\hat{x}_n := \hat{f}^n(\hat{x})$ for every $n \in \mathbb{Z}$. A function $\varphi_\epsilon : \hat{A} \to]0, +\infty[$ is *ϵ-tempered* if $e^{-\epsilon}\varphi(\hat{x}) \leq \varphi(\hat{f}(\hat{x})) \leq e^{\epsilon}\varphi(\hat{x})$. Let $f^{-n}_{\hat{x}_n}$ be the inverse branch of f^n sending a neighbourhood of x_n to a neighbourhood of x_0. The following result provides normal forms for those mappings, see [5, 21]. The measure $\hat{\mu}$ stands for the unique $\hat{f}$-invariant measure on $\hat{A}$ such that $(\pi_0)_*\hat{\mu} = \mu$, where $\pi_0(\hat{x}) := x_0$. Let $B_x(r)$ denote the ball centered at x of radius r in $\mathbb{P}^2$.

Theorem 2.6. [5, Proposition 4.3] *For every $\epsilon > 0$ there exist ϵ-tempered functions $\eta_\epsilon, \rho_\epsilon : \hat{A} \to]0,1]$, $\beta_\epsilon, L_\epsilon, M_\epsilon : \hat{A} \to [1, +\infty[$ and a function $N : \hat{A} \to \mathbb{N}$ satisfying the following properties. There exist injective holomorphic mappings $\xi_{\hat{x}} : B_{x_0}(\eta_\epsilon(\hat{x})) \to \mathbb{D}^2(\rho_\epsilon(\hat{x}))$ satisfying for every $\hat{x} \in \hat{A}$:*

1. *$\xi_{\hat{x}}(x_0) = 0$ and $D_{x_0}\xi_{\hat{x}}(v_s(x_0))$ is the vertical axis in $\mathbb{C}^2$,*
2. *$\forall p, q \in B_{x_0}(\eta_\epsilon(\hat{x}))$, $\frac{1}{2}\,\mathsf{dist}(p,q) \leq \|\xi_{\hat{x}}(p) - \xi_{\hat{x}}(q)\| \leq \beta_\epsilon(\hat{x})\,\mathsf{dist}(p,q)$,*

and such that the following diagram commutes for every $n \geq N(\hat{x})$:

$$\begin{array}{ccc} B_{x_0}(\eta_\epsilon(\hat{x})) & \xleftarrow{\;f_{\hat{x}_n}^{-n}\;} & B_{x_n}(\eta_\epsilon(\hat{x}_n)) \\ \downarrow{\scriptstyle \xi_{\hat{x}}} & & {\scriptstyle \xi_{\hat{x}_n}}\downarrow \\ \mathbb{D}^2(\rho_\epsilon(\hat{x})) & \xleftarrow[\;R_{n,\hat{x}_n}\;]{} & \mathbb{D}^2(\rho_\epsilon(\hat{x}_n)) \end{array}$$

The mappings $R_{n,\hat{x}_n}$ have the following form depending on (λ_1, λ_2):

1. *If $\lambda_1 = \lambda_2 = \lambda$, then $R_{n,\hat{x}_n}$ is a linear mapping satisfying*
$$e^{-n(\lambda+\epsilon)}\,\|(z,w)\| \leq \|R_{n,\hat{x}_n}(z,w)\| \leq e^{-n(\lambda-\epsilon)}\,\|(z,w)\|\,.$$
2. *If $\lambda_1 = k\lambda_2$ for some $k \geq 2$ then $R_{n,\hat{x}_n}(z,w) = (\alpha_{n,\hat{x}_n} z, \beta_{n,\hat{x}_n} w) + (\gamma_{n,\hat{x}_n} w^k, 0)$.*
3. *If $\lambda_1 \notin \{k\lambda_2, k \geq 1\}$, then $R_{n,\hat{x}_n}(z,w) = (\alpha_{n,\hat{x}_n} z, \beta_{n,\hat{x}_n} w)$.*

Moreover, in the cases 2 and 3, we have
$$e^{-n(\lambda_1+\epsilon)} \leq |\alpha_{n,\hat{x}_n}| \leq e^{-n(\lambda_1-\epsilon)} \quad , \quad |\gamma_{n,\hat{x}_n}| \leq M_\epsilon(\hat{x}) e^{-n(\lambda_1-2\epsilon)}$$
and
$$e^{-n(\lambda_2+\epsilon)} \leq |\beta_{n,\hat{x}_n}| \leq e^{-n(\lambda_2-\epsilon)}. \tag{2.2}$$
In particular, if $\lambda_2 < \lambda_1$, the second coordinate of $R_{n,\hat{x}_n}$ has the form $w \mapsto \beta_{n,\hat{x}_n} w$.

2.4. **Estimates from pluripotential theory.** The following result holds for every endomorphism of $\mathbb{P}^2$ satisfying $\lambda_1 < 2\lambda_2$. The proof relies on Briend-Duval's strategy [8] by taking into account $\lambda_2 \geq \frac{1}{2}\log d$: the arguments use the facts that T has continuous potentials, satisfies $f^*T = dT$ and defines μ by the Monge-Ampère equation $\mu = T \wedge T$. Let us set

$$B_n(\rho) := \left\{ x \in \mathbb{P}^2 \ , \ u \mapsto f^n \circ (x + (D_x f^n)^{-1}(u)) : \mathbb{D}^2(\rho) \to \mathbb{P}^2 \text{ is injective } \right\},$$

$$R_n(\tau) := \left\{ x \in \mathbb{P}^2 \ , \ \left\|(D_x f^n)^{-1}\right\|^{-1} \geq \tau^{-1} d^{n/2} \right\}.$$

For the definition of $B_n(\rho)$, we used charts of $T\mathbb{P}^2$ so that the mapping $x + (D_x f^n)^{-1}$ is defined on $\mathbb{D}^2(\rho)$ and takes its values in a neighbourhood of x in $\mathbb{P}^2$.

Theorem 2.7. [6, Propositions 1 and 2] *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. If $\lambda_1 < 2\lambda_2$, then for every $\beta > 0$,*

1. *$\exists \rho > 0$, $\forall n \geq 1$, $\mu(B_n(\rho)) \geq 1 - \beta$,*
2. *$\exists \rho > 0$, $\forall n \geq 1$, $\forall \tau > 0$, $\mu(B_n(\rho) \cap R_n(\tau)) \geq 1 - \beta - (\rho\tau)^{-2}$.*
3. *In particular, $\forall a > 0$, $\exists \tau > 0$, $\forall n \geq 1$, $\mu(R_n(\tau)) \geq 1 - a$.*

The following Corollary will be used to prove Proposition 5.2.

Corollary 2.8. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$ satisfying $\lambda_2 < \lambda_1 < 2\lambda_2$. For every $\beta > 0$ let $\rho > 0$ given by Theorem 2.7. There exists a Borel set H of full μ-measure satisfying for every $x \in H$, there exists $n(x) \geq 1$ such that*
$$\forall n \geq n(x), \quad x \notin B_n(\rho) \text{ or } \|D_x f^n(\vec{v}_s(x))\| \geq \frac{d^{n/2}}{n}.$$

Proof. According to Theorem 2.7, we have $\sum_{n\geq 1}\mu(B_n(\rho)\cap R_n(n)^c)\leq \rho^{-2}\sum_{n\geq 1}\frac{1}{n^2}$. By Borel-Cantelli's Lemma, there exists H such that $\mu(H)=1$ and

$$\forall x\in H,\ \exists n(x)\geq 1,\ \forall n\geq n(x),\quad x\in B_n(\rho)^c\cup R_n(n).$$

Let $x\in H$. If $x\in B_n(\rho)^c$, the proof is complete. If not, $x\in R_n(n)$, and since $\vec{v}_s(x)$ is unitary we get $\|D_xf^n(\vec{v}_s(x))\|\geq \left\|(D_xf^n)^{-1}\right\|^{-1}\geq \frac{d^{n/2}}{n}$. □

3. **A criterion for bounded Radon-Nikodym derivatives.** The following theorem gives sufficient conditions ensuring that a point x belongs to the set R defined in Definition 1.2. The integer $N(\hat{x})$ is defined in Theorem 2.6.

Theorem 3.1. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d\geq 2$. Assume that $\lambda_1>\lambda_2$ and let $\hat{x}\in\hat{A}$. If there exist $\eta_0,M_1,M_2>0$ and an increasing sequence $(n_j)_{j\in\mathbb{N}}$ of integers such that $n_0=0$ and $n_1\geq N(\hat{x})$ satisfying:*

1. $\eta_\epsilon(\hat{x}_{n_j})\geq 4\eta_0$, $\beta_\epsilon(\hat{x}_{n_j})\leq e^{M_2}$,
2. $e^{-M_1}d^{n_j/2}\leq \|D_xf^{n_j}(\vec{v}_s(\pi_0(\hat{x})))\|\leq e^{M_1}d^{n_j/2}$,

then $x:=\pi_0(\hat{x})\in R$: there exists a holomorphic disc $\xi_x:\mathbb{D}\to\mathbb{P}^2$ satisfying $\xi_x(0)=x$, $[\xi_x'(0)]=v_s(x)$ and

$$\liminf_{r\to 0}\frac{\xi_x^*T(\mathbb{D}(r))}{\mathrm{Leb}(\mathbb{D}(r))}<\infty. \tag{3.1}$$

Proof. Let us apply Theorem 2.6 which gives normal forms for the inverse branches of f. Since $\eta_\epsilon(\hat{x}_{n_j})\geq 4\eta_0$, the image of $B_{x_{n_j}}(\eta_\epsilon(\hat{x}_{n_j}))$ by $\xi_{\hat{x}_{n_j}}$ contains $\mathbb{D}^2(2\eta_0)$. Similarly, the image of $B_{x_0}(\eta_\epsilon(\hat{x}))$ by $\xi_{\hat{x}}$ contains $\mathbb{D}^2(2\eta_0)$. Let V be the vertical disc

$$V:\begin{array}{ccc}\mathbb{D}(2\eta_0) & \to & \mathbb{D}(2\eta_0)\times\mathbb{D}(2\eta_0)\\ w & \mapsto & (0,w)\end{array},$$

and let us set

$$\tilde{\xi}_0:=(\xi_{\hat{x}})^{-1}\circ V\quad\text{and}\quad \tilde{\xi}_{n_j}:=(\xi_{\hat{x}_{n_j}})^{-1}\circ V.$$

By pulling back $(f^{n_j})^*T=d^{n_j}T$ by $\tilde{\xi}_0$, we get

$$(f^{n_j}\circ\tilde{\xi}_0)^*T=d^{n_j}\,\tilde{\xi}_0^*T\ \text{ on }\ \mathbb{D}(\eta_0). \tag{3.2}$$

Let us observe that

$$f^{n_j}\circ\tilde{\xi}_0=f^{n_j}\circ(\xi_{\hat{x}})^{-1}\circ V=(\xi_{\hat{x}_{n_j}})^{-1}\circ R^{-1}_{n_j,\hat{x}_{n_j}}\circ V,$$

where the first equality comes from the definition of $\tilde{\xi}_0$ and the second one from Theorem 2.6. If $\beta_{n_j}:\mathbb{C}\to\mathbb{C}$ denotes the multiplication by $\beta_{n_j,\hat{x}_{n_j}}$, we also have the relation

$$R^{-1}_{n_j,\hat{x}_{n_j}}\circ V=V\circ\beta_{n_j}^{-1},$$

since the second coordinate of $R_{n_j,\hat{x}_{n_j}}$ is linear. Hence we get:

$$f^{n_j}\circ\tilde{\xi}_0=\tilde{\xi}_{n_j}\circ\beta_{n_j}^{-1}\ \text{on}\ \beta_{n_j}(\mathbb{D}(\eta_0)).$$

Equation (3.2) restricted to $\beta_{n_j}(\mathbb{D}(\eta_0))\subset\mathbb{D}(\eta_0)$ then implies:

$$\tilde{\xi}_{n_j}^*T=d^{n_j}\,(\beta_{n_j}^{-1})_*\tilde{\xi}_0^*T\ \text{on}\ \mathbb{D}(\eta_0),$$

which gives

$$\tilde{\xi}_{n_j}^*T(\mathbb{D}(\eta_0))=d^{n_j}\,\tilde{\xi}_0^*T(\mathbb{D}(\eta_0\cdot|\beta_{n_j,\hat{x}_{n_j}}|)). \tag{3.3}$$

Now we bound from below the right hand side of Equation (3.3). This is where we use the second hypothesis of Theorem 3.1.

Lemma 3.2. *We have* $|\beta_{n_j,\hat{x}_{n_j}}| \in [d^{-n_j/2}e^{-M}, d^{-n_j/2}e^{M}]$, *where* $M := M_1 + M_2 + \ln 2$.

Proof. By Theorem 2.6, we can write $f^{n_j} = (\xi_{\hat{x}_{n_j}})^{-1} \circ R^{-1}_{n_j,\hat{x}_{n_j}} \circ \xi_{\hat{x}}$. We study the differential of that composition. By using

$$\forall \vec{v} \in T_{f^{n_i}(x)}\mathbb{P}^2 \quad , \quad \frac{1}{2}\|\vec{v}\| \leq \left\|D_{f^{n_i}(x)}\xi_{\hat{x}_{n_i}}(\vec{v})\right\| \leq e^{M_2}\|\vec{v}\|$$

for $i = 0$ and $i = j$ (provided by Theorem 2.6) and the fact that the differential of $\xi_{\hat{x}}$ (resp. $\xi_{\hat{x}_{n_j}}$) at x (resp. $f^{n_j}(x)$) sends the line $v_s(x) \subset T_x\mathbb{P}^2$ (resp. $v_s(f^{n_j}(x)) \subset T_{f^{n_j}(x)}\mathbb{P}^2$) on the vertical axis, we obtain

$$\|D_x f^{n_j}(\vec{v}_s(x))\|\,(2e^{M_2})^{-1} \leq \left|\beta_{n_j,\hat{x}_{n_j}}\right|^{-1} \leq \|D_x f^{n_j}(\vec{v}_s(x))\|\,(2e^{M_2}).$$

Now we use the assumption $e^{-M_1}d^{n_j/2} \leq \|D_x f^{n_j}(\vec{v}_s(x))\| \leq e^{M_1}d^{n_j/2}$ to deduce

$$d^{-n_j/2}e^{-M_1}(2e^{M_2})^{-1} \leq \left|\beta_{n_j,\hat{x}_{n_j}}\right| \leq d^{-n_j/2}e^{M_1}(2e^{M_2}),$$

which concludes the proof. □

The right hand side term of Equation (3.3) hence satisfies:

$$d^{n_j}\,\tilde{\xi}_0^* T(\mathbb{D}(\eta_0 \cdot |\beta_{n_j,\hat{x}_{n_j}}|)) \geq \frac{\tilde{\xi}_0^* T(\mathbb{D}(\eta_0\, d^{-n_j/2}\, e^{-M}))}{d^{-n_j}} = c_0 \frac{\tilde{\xi}_0^* T(\mathbb{D}(\eta_0\, d^{-n_j/2}\, e^{-M}))}{\pi(\eta_0\, d^{-n_j/2}\, e^{-M})^2}, \tag{3.4}$$

where $c_0 = \pi(\eta_0 e^{-M})^2$. By setting $r_j = \eta_0\, d^{-n_j/2}\, e^{-M}$, we obtain:

$$d^{n_j}\,\tilde{\xi}_0^* T(\mathbb{D}(\eta_0 \cdot |\beta_{n_j,\hat{x}_{n_j}}|)) \geq c_0 \frac{\tilde{\xi}_0^* T(\mathbb{D}(r_j))}{\mathrm{Leb}(\mathbb{D}(r_j))}.$$

Now Lemma 3.3 below bounds the left hand side term of Equation (3.3), hence

$$\liminf_{r\to 0} \frac{\tilde{\xi}_0^* T(\mathbb{D}(r))}{\mathrm{Leb}(\mathbb{D}(r))} < \infty.$$

To conclude the proof of Theorem 3.1, we set $\xi_x : \mathbb{D} \to \mathbb{P}^2$ by $\xi_x(z) := \tilde{\xi}_0(\eta_0 z)$. □

Lemma 3.3. *There exists a constant* $M_3 > 0$ *such that for every* $j \geq 0$,

$$\tilde{\xi}_{n_j}^* T(\mathbb{D}(\eta_0)) \leq M_3.$$

Proof. Let $(B_i)_{i\in I}$ be a finite open covering of $\mathbb{P}^2$ by small balls, such that T admits for every $i \in I$ a bounded continuous plurisubharmonic potential G_i on the $4\eta_0$-tubular neighbourhood $\tilde{B}_i$ of B_i. Let $\|G\|_\infty := \max_{i\in I}\|G_i\|_\infty$ and let $\psi : \mathbb{D}(2\eta_0) \to \mathbb{R}^+$ be a function with compact support which is equal to 1 on $\mathbb{D}(\eta_0)$. Let us fix $j \geq 0$ and let $i \in I$ be such that $x_{n_j} \in B_i$, hence $B_{x_{n_j}}(4\eta_0) \subset \tilde{B}_i$. Since $\tilde{\xi}_{n_j} = (\xi_{\hat{x}_{n_j}})^{-1} \circ V$, we get

$$\tilde{\xi}_{n_j}(\mathbb{D}(2\eta_0)) \subset (\xi_{\hat{x}_{n_j}})^{-1}(\mathbb{D}^2(2\eta_0)) \subset B_{x_{n_j}}(4\eta_0).$$

We deduce

$$\tilde{\xi}_{n_j}^* T(\mathbb{D}(\eta_0)) \leq \int_{\mathbb{D}(2\eta_0)} \Delta(G_\alpha \circ \tilde{\xi}_{n_j})\psi = \int_{\mathbb{D}(2\eta_0)} G_\alpha \circ \tilde{\xi}_{n_j}\Delta\psi \leq \|G\|_\infty \max_{\mathbb{D}(2\eta_0)} \Delta\psi.$$

We complete the proof by setting $M_3 := \|G\|_\infty \max_{\mathbb{D}(2\eta_0)} \Delta\psi$. □

4. Proof of Theorem 1.3.

4.1. **Decomposable currents.** This subsection is borrowed from Dujardin's article [14]. Let ω be the Fubini-Study $(1,1)$-form on $\mathbb{P}^2$ and let S be a closed positive current of bidegree $(1,1)$ on $\mathbb{P}^2$. Let $\sigma_S = S \wedge \omega$ denote the trace measure of S, this is a positive measure on $\mathbb{P}^2$. We shall denote $S(\varphi) = \int_{\mathbb{P}^2} S \wedge \varphi$ for every $(1,1)$-test form φ on $\mathbb{P}^2$. Locally, one can write S as

$$S = \sum_{j,k \in \{1,2\}} S_{j,k}\, i dz_j \wedge d\overline{z_k}. \tag{4.1}$$

where $S_{1,1}$ and $S_{2,2}$ are positive measures and $S_{1,2} = \overline{S_{2,1}}$ are complex measures. Let (e_1, e_2) be a basis of $\mathbb{C}^2$. Let us recall that a $(1,1)$-positive vector of $\mathbb{C}^2$ is an element of $\Lambda^{1,1}\mathbb{C}^2$ of the form $i\sum_{j,k\in\{1,2\}} t_{j,k} e_j \wedge \overline{e_k}$ where $(t_{j,k})_{(j,k)}$ is a positive hermitian matrix. A positive vector of $\mathbb{C}^2$ is decomposable if it is equal to $iu \wedge \overline{u}$ for some $u \in \mathbb{C}^2$. In Equation (4.1), since S is a positive current, the measures $|S_{1,2}|$ ans $|S_{2,1}|$ are dominated by the trace measure $\sigma_S = S_{1,1} + S_{2,2}$. By the Radon-Nikodym Theorem, there exists a measurable field Λ_S of $(1,1)$-positive covectors of trace 1 such that

$$S = \Lambda_S \sigma_S.$$

By duality, there exists a measurable field t_S of $(1,1)$-positive vectors of trace 1 such that

$$\forall \eta\ (1,1) - \text{test form on } \mathbb{P}^2\ , \quad S(\eta) = \int_{\mathbb{P}^2} < t_S, \eta > \sigma_S. \tag{4.2}$$

By the Lebesgue density Theorem, there exists $E \subset \operatorname{Supp} S$ such that $S \wedge \omega(E) = 1$ and for every $x \in E$,

$$\lim_{r\to 0} \frac{1}{\sigma_S(B_x(r))} \int_{B_x(r)} t_S(y)\sigma_S(y) = t_S(x).$$

We say that S is decomposable at $x \in E$ if the $(1,1)$-positive vector $t_S(x)$ is decomposable. For instance, if $S = [L]$ where L is a smooth curve, then $t_S(x) = i\vec{u}_x \wedge \overline{\vec{u}_x}$ where $\vec{u}_x$ is a unitary tangent vector of L at x. The laminar currents are also examples of decomposable currents.

In [14], Dujardin established the following nice property: if f is not a Lattès map, then the Green current T is decomposable σ_T-almost everywhere, in particular it carries a directional information. This is the content of the following result.

Theorem 4.1. [14, Theorem 3.3] *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$ and let T be its Green current. If f is not a Lattès map, then there exists $E' \subset E$ of full σ_T-measure satisfying the following property. For every $x \in E'$, there exists $\vec{v}_x \in T_x\mathbb{P}^2$ satisfying $\|\vec{v}_x\| = 1$ and*

$$t_T(x) = i\vec{v}_x \wedge \overline{\vec{v}_x}.$$

Moreover the vectors $\vec{v}_x$ satisfy the following property.

Proposition 4.2. [14, Section 3.2] *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. We assume that f is not a Lattès map (hence $t_T(x) = i\vec{v}_x \wedge \overline{\vec{v}_x}$ σ_T-almost everywhere, with $\|\vec{v}_x\| = 1$). Then,*

$$\forall n \geq 1, \quad \int_{\mathbb{P}^2} \|D_x f^n(\vec{v}_x)\|^2 \ d\sigma_T(x) \leq d^n.$$

Markov's inequality then implies:

$$\forall n \geq 1,\ \forall a > 0, \quad (\sigma_T)\left\{x \in \mathbb{P}^2,\ \|D_x f^n(\vec{v}_x)\|^2 \geq a d^n\right\} \leq \frac{1}{a}.$$

Proof. For σ_T-almost every x, we have

$$\|D_x f^n(\vec{v}_x)\|^2 \leq \mathrm{Trace}\,(D_x f^n)_*(i\vec{v}_x \wedge \overline{\vec{v}_x}).$$

We use $t_T(x) = i\vec{v}_x \wedge \overline{\vec{v}_x}$ and the definition of the trace to get:

$$\begin{aligned}\|D_x f^n(\vec{v}_x)\|^2 &\leq \langle\, (D_x f^n)_*(t_T(x)), \omega(f^n(x))\,\rangle \\ &\leq \langle\, t_T(x), (D_x f^n)^*\omega(f^n(x))\,\rangle\end{aligned}$$

since $D_x f^n$ is invertible σ_T-almost everywhere. By integrating this inequality on $\mathbb{P}^2$:

$$\int_{\mathbb{P}^2} \|D_x f^n(\vec{v}_x)\|^2\, \sigma_T(x) \leq \int_{\mathbb{P}^2} \langle\, t_T(x), (D_x f^n)^*\omega(f^n(x))\,\rangle \sigma_T(x).$$

By using Equation (4.2), we obtain

$$\int_{\mathbb{P}^2} \|D_x f^n(\vec{v}_x)\|^2\, \sigma_T(x) \leq \int_{\mathbb{P}^2} T \wedge (f^n)^*\omega = d^n,$$

where the last equality uses the fact that $(f^n)^*\omega$ is cohomologous to $d^n\omega$, see for instance Sections 1.2 and A.3 of [13]. □

4.2. Application of Theorem 3.1 to prove Theorem 1.3.

Proposition 4.3. *Assume that f is not a Lattès map (hence $t_T(x) = i\vec{v}_x \wedge \overline{\vec{v}_x}$ σ_T-almost everywhere, with $\|\vec{v}_x\| = 1$). We also assume that $\mu \ll \sigma_T$. Then, there exist $q \geq 1$ and $\gamma > 0$ such that*

$$\forall n \geq 1, \quad \mu\left\{x \in \mathbb{P}^2,\ \|D_x f^n(\vec{v}_x)\|^2 \leq \frac{4}{\gamma} d^n\right\} \geq \frac{\gamma}{4q}.$$

Proof. Let us write $\mu = \varphi\,\sigma_T$ where $\varphi \in L^1(\sigma_T)$ and $\varphi \geq 0$. Let us set $\gamma := \sigma_T\,\{\varphi > 0\}$. It satisfies $\gamma > 0$ since $1 = \mu(\mathbb{P}^2) = \mu\,\{\varphi > 0\} = \int_{\{\varphi>0\}} \varphi\sigma_T$. Moreover $\sigma_T\left\{\frac{1}{q} \leq \varphi\right\}$ tends to $\sigma_T\,\{0 < \varphi\} = \gamma$ by Lebesgue convergence Theorem. Let us set $q \geq 1$ such that

$$\sigma_T\left\{\frac{1}{q} \leq \varphi\right\} \geq \gamma/2. \tag{4.3}$$

For every $n \geq 1$, we denote

$$A_n := \left\{x \in \mathbb{P}^2,\ \|D_x f^n(\vec{v}_x)\|^2 \leq \frac{4}{\gamma} d^n\right\}.$$

We want to show $\mu(A_n) \geq \frac{\gamma}{4q}$. Proposition 4.2 gives with $a = 4/\gamma$

$$\forall n \geq 1, \quad \sigma_T(A_n) \geq 1 - \gamma/4. \tag{4.4}$$

Equations (4.3) and (4.4) imply $\sigma_T(A_n \cap \left\{\frac{1}{q} \leq \varphi\right\}) \geq \frac{\gamma}{4}$. Coming back to μ,

$$\mu(A_n) \geq \mu(A_n \cap \left\{\frac{1}{q} \leq \varphi\right\}) = \int_{A_n \cap \left\{\frac{1}{q} \leq \varphi\right\}} \varphi\,\sigma_T.$$

We deduce

$$\mu(A_n) \geq \frac{1}{q}(\sigma_T)(A_n \cap \left\{\frac{1}{q} \leq \varphi\right\}) \geq \frac{1}{q}\frac{\gamma}{4},$$

which is the desired lower bound. □

Now let us prove Theorem 1.3. We assume that $\mu \ll \sigma_T$ and $\lambda_2 < \lambda_1 < 2\lambda_2$. Let

$$B_n := \left\{ x \in \mathbb{P}^2,\ \frac{1}{\tau_0^2} d^n \leq \|D_x f^n(\vec{v}_x)\|^2 \leq \frac{4}{\gamma} d^n \right\}.$$

By combining Proposition 4.3 and the third item of Theorem 2.7 (with $a = \frac{\gamma}{8q}$ and using $\|D_x f^n(\vec{v}_x)\| \geq \left\|(D_x f^n)^{-1}\right\|^{-1}$), we get $\mu(B_n) \geq \frac{\gamma}{8q}$. The set $\hat{B}_n := \pi_0^{-1}(B_n)$ then satisfies $\hat{\mu}(\hat{B}_n) \geq \frac{\gamma}{8q}$. Let M_2 be large enough and η_0 be small enough such that

$$\hat{E} := \left\{ \beta_\epsilon(\hat{x}) \leq e^{M_2}\ ,\ \eta_\epsilon(\hat{x}) \geq 4\eta_0 \right\}$$

satisfies $\hat{\mu}(\hat{E} \cap \hat{f}^{-n}(E)) \geq 1 - \frac{\gamma}{16q}$. Therefore the $\hat{\mu}$-measure of $\hat{C}_n := \hat{B}_n \cap \hat{E} \cap \hat{f}^{-n}(\hat{E})$ is larger than $\frac{\gamma}{16q}$, and the same property holds for $\hat{C} := \limsup \hat{C}_n$.

Lemma 4.4. *Every $\hat{x} \in \hat{C}$ satisfies the hypothesis of Theorem 3.1.*

Proof. Let $\hat{x} \in \hat{C}$ and $x = \pi_0(\hat{x})$. By definition of $\hat{C}$, there exists $(n_j)_j$ depending on $\hat{x}$ such that $n_0 = 0$, $n_1 \geq N(\hat{x})$ and $\hat{x} \in \hat{C}_{n_j}$ for every $j \geq 0$. In particular, $\eta_\epsilon(\hat{x}_{n_j}) \geq 4\eta_0$, $\beta_\epsilon(\hat{x}_{n_j}) \leq e^{M_2}$, and

$$\frac{1}{\tau_0^2} d^{n_j} \leq \|D_x f^{n_j}(\vec{v}_x)\|^2 \leq \frac{4}{\gamma} d^{n_j}.$$

Those estimates combined with the definition of the stable Oseledec direction given in Theorem 2.1 imply $v_s(x) = v_x$ (recall that $\lambda_2 < \lambda_1$). By taking M_1 large enough such that $e^{-2M_1} \leq \frac{1}{\tau_0^2} \leq \frac{4}{\gamma} \leq e^{2M_1}$, we get

$$e^{-M_1} d^{n_j/2} \leq \|D_x f^{n_j}(\vec{v}_s(x))\| \leq e^{M_1} d^{n_j/2},$$

which completes the proof of the Lemma. □

Lemma 4.4 ensures that $C := \pi_0(\hat{C}) \subset R$. But C has positive μ-measure, since $\mu(C) \geq \hat{\mu}(\hat{C}) \geq \frac{\gamma}{16q}$. Finally $\mu(R) > 0$, completing the proof of Theorem 1.3.

5. Proof of Theorem 1.4.

5.1. Central Limit Theorem.

The dynamical system $(\mathbb{P}^2, f, \mu)$ satisfies a Central Limit Theorem, see [9, 11, 12, 16].

Theorem 5.1. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$ and let μ be its equilibrium measure. Let $\phi : \mathcal{J} \to \mathbb{R}$ be a Hölder continuous function such that $\int \phi \, d\mu = 0$ and let $S_n(\phi) := \sum_{i=0}^{n-1} \phi \circ f^i$. Then the asymptotic variance*

$$\sigma_\phi := \lim_n \frac{1}{\sqrt{n}} \|S_n(\phi)\|_2$$

exists and the following alternative holds:

1. *if $\sigma_\phi = 0$ there exists $u \in L^2(\mu)$ such that $\phi = u \circ f - u$.*
2. *if $\sigma_\phi > 0$ then $\frac{1}{\sqrt{n}} S_n(\phi)$ converges in law towards the Gaussian $\mathcal{N}(0, \sigma_\phi)$.*

5.2. **The asymptotic variance of ψ vanishes.** We assume that $\lambda_2 < \lambda_1$ and consider

$$\psi(x) = \log \|D_x f(\vec{v}_s(x))\|$$

where $v_s(x)$ is the stable direction at $x \in A$. By Proposition 2.5, if $\mathcal{J} \cap \mathcal{C} = \emptyset$ and if v_s is Hölder continuous on A, then ψ extends to a Hölder continuous function on $\mathcal{J}$. If moreover $\lambda_2 = \frac{1}{2}\log d$, then $\psi_0 := \psi - \frac{1}{2}\log d$ satisfies $\int \psi_0 \, d\mu = 0$ by Lemma 2.2.

Proposition 5.2. *Assume that $\lambda_2 = \frac{1}{2}\log d$ and that $\lambda_2 < \lambda_1 < 2\lambda_2$. Assume also that $\mathcal{J} \cap \mathcal{C} = \emptyset$ and that v_s is Hölder continuous on A. Then $\sigma_{\psi_0} = 0$. In particular, there exists $u \in L^2(\mu)$ such that for μ-almost every x:*

$$\log \|D_x f(\vec{v}_s(x))\| = \lambda_2 + u \circ f - u. \tag{5.1}$$

Proof. Assume to the contrary that $\sigma_{\psi_0} > 0$. By Lemma 2.2 one can write

$$\frac{1}{\sigma_{\psi_0}\sqrt{n}} S_n(\psi_0) = \frac{1}{\sigma_{\psi_0}\sqrt{n}} \left(\log \|D_x f^n(\vec{v}_s(x))\| - n\frac{\log d}{2} \right).$$

Theorem 5.1 ensures that $\frac{1}{\sigma_{\psi_0}\sqrt{n}} S_n(\psi_0)$ converges in law towards $\mathcal{N}(0,1)$. By setting

$$\beta_0 := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} e^{-u^2/2} \, \mathrm{d}u \quad \text{and} \quad G_n := \left\{ x \in \mathcal{J} \mid S_n(\psi_0)(x) \leq -\sigma_{\psi_0}\sqrt{n} \right\},$$

we get

$$\lim_{n\to\infty} \mu(G_n) = \beta_0.$$

Let $N \geq 1$ such that $\mu(G_n) \geq \frac{\beta_0}{2}$ for every $n \geq N$. Let $\rho > 0$ provided by Theorem 2.7 with $\beta = \frac{\beta_0}{4}$. We have $\mu(B_n(\rho)) \geq 1 - \frac{\beta_0}{4}$ for every $n \geq 1$. Let us define

$$F_n := B_n(\rho) \cap G_n, \quad F := \limsup F_n.$$

For every $n \geq N$, we have $\mu(F_n) \geq \frac{\beta_0}{4}$. We also have $\mu(F) \geq \frac{\beta_0}{4}$. Let H be given by Corollary 2.8. Since $\mu(H) = 1$ and $\mu(F) > 0$, there exists $x \in F \cap H$. Let $(n_i)_i$ be an increasing sequence of integers such that $x \in F_{n_i}$ for every $i \in \mathbb{N}$. Since $F_{n_i} \subset G_{n_i}$, we have:

$$S_{n_i}(\psi_0)(x) = \log \|D_x f^{n_i}(\vec{v}_s(x))\| - n_i \frac{\log d}{2} \leq -\sigma_{\psi_0}\sqrt{n_i},$$

which implies

$$d^{n_i/2} e^{-\sigma_{\psi_0}\sqrt{n_i}} \geq \|D_x f^{n_i}(\vec{v}_s(x))\|. \tag{5.2}$$

But x belongs to H and to $F_{n_i} \subset B_{n_i}(\rho)$, hence Corollary 2.8 ensures for every $n_i > n(x)$:

$$\|D_x f^{n_i}(\vec{v}_s(x))\| \geq \frac{d^{n_i/2}}{n_i}. \tag{5.3}$$

By combining (5.2) and (5.3), we obtain $e^{-\sigma_{\psi_0}\sqrt{n_i}} \geq \frac{1}{n_i}$ for every i large enough. This contradiction implies $\sigma_\psi = 0$ and completes the proof of Proposition 5.2. □

5.3. **Application of Theorem 3.1 to prove Theorem 1.4.** By summing Equation (5.1) over an orbit of length n and using Lemma 2.2, we obtain:

$$\forall x \ \mu - a.e.\,,\ \forall n \geq 1, \quad \log \|D_x f^n(\vec{v}_s(x))\| = n\lambda_2 + u \circ f^n - u.$$

The μ-measure of $\{|u| \leq R\}$ tends to 1 when R tends to infinity. The same property holds for $\{|u| \leq R\} \cap \{|u \circ f^n| \leq R\}$ since μ is invariant. Therefore, there exists $R > 0$ such that

$$B_n := \left\{ x \in \mathbb{P}^2\,,\ d^{n/2} e^{-2R} \leq \|D_x f^n(\vec{v}_s(x))\| \leq d^{n/2} e^{2R} \right\}$$

satisfies $\mu(B_n) \geq 1/2$ for every $n \geq 1$. Hence $\hat{B}_n := \pi_0^{-1}(B_n)$ satisfies $\hat{\mu}(\hat{B}_n) \geq 1/2$. Now, to complete the proof of Theorem 1.4, it suffices to repeat the end of Section 4.2, namely to introduce the subsets

$$\hat{E} := \left\{ \beta_\epsilon(\hat{x}) \leq e^{M_2}\ ,\ \eta_\epsilon(\hat{x}) \geq 4\eta_0 \right\},$$

$\hat{C}_n := \hat{B}_n \cap \hat{E} \cap \hat{f}^{-n}(\hat{E})$, $\hat{C} := \limsup \hat{C}_n$ and to apply similarly Theorem 3.1.

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E-mail address: `christophe.dupont@univ-rennes1.fr`
E-mail address: `axel.rogue@univ-rennes1.fr`