



HAL
open science

Kähler spaces with zero first Chern class: Bochner principle, fundamental groups, and the Kodaira problem

Benoît Claudon, Patrick Graf, Henri Guenancia, Philipp Naumann

► To cite this version:

Benoît Claudon, Patrick Graf, Henri Guenancia, Philipp Naumann. Kähler spaces with zero first Chern class: Bochner principle, fundamental groups, and the Kodaira problem. *Journal für die reine und angewandte Mathematik*, 2022, 10.1515/crelle-2022-0001 . hal-02936352v1

HAL Id: hal-02936352

<https://hal.science/hal-02936352v1>

Submitted on 11 Sep 2020 (v1), last revised 21 Jul 2022 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

KÄHLER SPACES WITH ZERO FIRST CHERN CLASS: BOCHNER PRINCIPLE, FUNDAMENTAL GROUPS, AND THE KODAIRA PROBLEM

BENOÎT CLAUDON, PATRICK GRAF, HENRI GUENANCIA, AND PHILIPP NAUMANN

ABSTRACT. Let X be a compact Kähler space with klt singularities and vanishing first Chern class. We prove the Bochner principle for holomorphic tensors on the smooth locus of X : any such tensor is parallel with respect to the singular Ricci-flat metrics. As a consequence, after a finite quasi-étale cover X splits off a complex torus of the maximum possible dimension. We then proceed to decompose the tangent sheaf of X according to its holonomy representation. In particular, we classify those X which have strongly stable tangent sheaf: up to quasi-étale covers, these are either Calabi–Yau or irreducible holomorphic symplectic.

Several applications of these results are given, the strongest ones of which apply in dimension four. In this case, we prove Campana’s Abelianity Conjecture for X . If in addition X has only algebraic singularities then a cover of X admits small projective deformations and the Beauville–Bogomolov Decomposition Theorem holds for X .

CONTENTS

1.	Introduction	1
2.	Preliminaries	5
3.	Bochner principle for reflexive tensors	10
4.	Structure of the Albanese map	14
5.	Spaces with flat tangent sheaf on the smooth locus	17
6.	Holonomy of singular Ricci-flat metrics	21
7.	Fundamental groups	26
8.	Kodaira problem and decomposition theorem	29
	References	32

1. INTRODUCTION

Let X be a compact Kähler manifold such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$. The celebrated Beauville–Bogomolov Decomposition Theorem states that X admits an étale cover that splits as a product of a complex torus, Calabi–Yau manifolds and irreducible holomorphic symplectic manifolds, cf. [Bea83]. The proof relies in a crucial way on Yau’s solution of the Calabi conjecture [Yau78] that enables one to equip X with a Kähler, Ricci-flat metric ω . From there, one can use the powerful tools of differential geometry (de Rham’s splitting theorem, Cheeger–Gromoll

Date: September 11, 2020.

2010 Mathematics Subject Classification. 32J27, 14E30, 14J32.

Key words and phrases. Kähler spaces, klt singularities, vanishing first Chern class, holonomy, Albanese map, fundamental groups, Kodaira problem, decomposition theorem.

H.G. was partially supported by the ANR project GRACK. B.C. was partially supported by the ANR projects Foliage ANR–16–CE40–0008 and Hodgefun ANR–16–CE40–0011.

core theorem, Berger–Simons holonomy classification and Bieberbach theorem) to exhibit the sought cover.

Over the last few decades, the tremendous advances of the (algebraic) Minimal Model Program in birational geometry have highlighted the importance to understand and classify varieties with mild singularities. By variety, we mean here either a complex projective variety or a compact Kähler space.

It is in this context that a lot of attention has been drawn in recent years towards generalizing the Beauville–Bogomolov Decomposition Theorem to compact Kähler spaces X with klt singularities and trivial first Chern class. A first important step in that direction was made by [EGZ09] who generalized Yau’s theorem and constructed singular Kähler–Einstein metrics ω on such varieties X . Unfortunately, these metrics viewed on X_{reg} are geodesically incomplete, preventing most of the classical results in differential geometry from applying. However, breakthroughs relying on the algebraic MMP (e.g. [GKKP11, Xu14, GKP16a]) and the theory of algebraic foliations [Dru18] allowed for a better understanding of projective varieties with klt singularities and trivial first Chern class. Down the line, this enabled [GGK19] to compute the holonomy of the singular Ricci-flat metrics ω and shortly after, Höring and Peternell gave a proof of the decomposition theorem in the projective setting, based on an algebraic integrability result for vector bundles [HP19].

Unfortunately, these spectacular results leave the Kähler case of the decomposition theorem largely open, which may sound odd in light of the fact that the proof in the smooth case relies entirely on transcendental methods! In this paper, we prove the Kähler version of several building blocks that underly the proof of the decomposition theorem in the projective case.

Bochner principle and Albanese map. As a first step towards the above goal, we prove the so-called *Bochner principle*.

Theorem A (Bochner principle, Theorem 3.4). *Let (X, ω_X) be a normal compact Kähler space with klt singularities such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$ and let $\omega \in \{\omega_X\}$ be the singular Ricci-flat metric. Let $p, q \geq 0$ be non-negative integers. Then any holomorphic tensor*

$$\tau \in H^0(X_{\text{reg}}, \mathcal{F}_X^{\otimes p} \otimes \Omega_X^{\otimes q})$$

is parallel with respect to ω on X_{reg} .

Quite generally, a sacrilegious bottleneck of the theory in its current state is that if X is klt, we have no way of comparing the fundamental group of the smooth locus $\pi_1(X_{\text{reg}})$ to the whole $\pi_1(X)$. This is important because differential-geometric methods only control the former group. The issue is that the algebraic results of [GKP16a] about *maximally quasi-étale covers* are not yet available for complex spaces. See Remark 6.10 for an in-depth discussion. Note also that the problem does not occur for *locally algebraic* complex spaces (Proposition 5.9) and hence Conjecture K below yields an alternative approach to the bottleneck in the special case $c_1(X) = 0$.

In fact, a large part of the present work is devoted to finding ways to bypass the above-mentioned lamentable limitation of the literature. This can already be seen in Theorem A: even though its statement is a straightforward generalization of [GGK19, Theorem A], the proof is quite different because it needs to avoid the use of maximally quasi-étale covers. We refer to Remark 3.5 for a more thorough comparison of the two results.

Our next theorem is about the Albanese map of Kähler spaces X as above: alb_X is surjective and after a finite étale base change, it becomes globally trivial (Theorem 4.1). This generalizes [Kaw85, Theorem 8.3] from the projective case,

but the proof relies on the Bochner principle. An important consequence is the existence of so-called *torus covers*.

Theorem B (Torus covers, Corollary 4.2). *Let X be a normal compact Kähler space with klt singularities such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$. Then there exist normal compact Kähler spaces T and Z with canonical singularities together with a quasi-étale cover $\gamma: T \times Z \rightarrow X$ such that:*

- T is a complex torus of dimension $\tilde{q}(X)$.
- The canonical sheaf of Z is trivial, $\omega_Z \cong \mathcal{O}_Z$.
- The augmented irregularity of Z vanishes, $\tilde{q}(Z) = 0$.

For the definition of the augmented irregularity $\tilde{q}(X)$, see Definition 2.1. Also, we remark that the torus cover is essentially unique by Proposition 4.4.

Direct applications of Theorem B include an alternative proof of the Abundance Conjecture in the setting of compact Kähler spaces with *canonical* singularities and trivial first Chern class (Corollary 4.5) and a characterization of torus quotients in terms of the augmented irregularity (Corollary 4.6). Concerning the further study of klt Kähler spaces with trivial first Chern class, Theorem B enables us to reduce most questions to the case of canonical singularities, trivial canonical bundle and, most importantly, vanishing augmented irregularity.

Holonomy and the flat factor. The natural next step is to understand the holonomy of the Ricci-flat metrics on X_{reg} . This leads to a decomposition of the tangent sheaf of X that already reflects the conjectural Beauville–Bogomolov decomposition on an infinitesimal level. The missing last steps are the algebraic integrability of the summands and a splitting theorem for such foliations.

Theorem C (Holonomy covers, p. 23). *Let (X, ω_X) be as in Theorem A. Then after replacing X by a finite quasi-étale cover, there exists a direct sum decomposition of the tangent sheaf of X ,*

$$\mathcal{T}_X = \mathcal{F} \oplus \bigoplus_{k \in K} \mathcal{E}_k,$$

where the reflexive sheaves \mathcal{F} and \mathcal{E}_k satisfy the following:

- The sheaves \mathcal{F} and \mathcal{E}_k are foliations with trivial determinant.
- The sheaf $\mathcal{F}|_{X_{\text{reg}}}$ is flat. More precisely, it is given by a special unitary representation of $\pi_1(X_{\text{reg}})$.
- Each factor $\mathcal{E}_k|_{X_{\text{reg}}}$ is parallel and has full holonomy group either $\text{SU}(n_k)$ or $\text{Sp}(n_k/2)$, with respect to the pullback of the singular Ricci-flat metric ω . Here, $n_k = \text{rk}(\mathcal{E}_k)$. Moreover, \mathcal{E}_k is strongly stable with respect to any Kähler class.
- After passing to a torus cover $\gamma: T \times Z \rightarrow X$ as in Theorem B, the tangent sheaf of T becomes a direct summand of $\gamma^{[*]} \mathcal{F}$. If X is locally algebraic, then we have equality $\text{pr}_T^* \mathcal{T}_T = \gamma^{[*]} \mathcal{F}$.

As far as the flat factor \mathcal{F} is concerned, Theorem C only provides partial information. We do however have a complete understanding of X in the case where the other summands \mathcal{E}_k vanish. In particular, in this case we know that $\text{pr}_T^* \mathcal{T}_T = \gamma^{[*]} \mathcal{F}$ without making any algebraicity assumptions:

Theorem D (Characterization of torus quotients, Theorem 5.2 and Proposition 6.8). *Let X be a normal complex space with klt singularities. If $\mathcal{T}_{X_{\text{reg}}}$ is flat, then X has only finite quotient singularities.*

Moreover, if X is compact and Kähler, then it is a quotient of a complex torus by a finite group acting freely in codimension one.

Theorem D would follow directly from Theorem C as soon as the maximally quasi-étale covers of [GKP16a] are available. In our non-algebraic setting, we cannot rely on that result, but capitalizing on the classical fact that *isolated* singularities are algebraic, we are able to prove a “generic” version of [GKP16a], cf. Proposition 5.8. Coupling this with the resolution of the (log canonical) Lipman–Zariski conjecture due to Kovács and the second author [GK14], this weaker statement turns out to be sufficient to obtain Theorem D.

Varieties with strongly stable tangent sheaf. Theorems A–D, together with some standard representation theory, imply Corollary E below. This result characterizes the conjectural building blocks of compact Kähler spaces with klt singularities and trivial first Chern class via stability properties of their tangent sheaf. We refer to Section 6.D for the relevant definitions.

Corollary E (Spaces with strongly stable tangent sheaf, Corollary 6.14). *Let X be as in Theorem B. If \mathcal{T}_X is strongly stable, then X admits a quasi-étale cover that is either a Calabi–Yau variety or an irreducible holomorphic symplectic variety.*

Corollary E is the Kähler version of [GGK19, Theorem E]. Its proof bypasses the use of Druel’s splitting result for flat summands [Dru18] as well as the existence of maximally quasi-étale covers.

Fundamental groups. According to Campana’s Abelianity Conjecture [Cam04b, Conjecture 7.3], the fundamental group of the regular locus X_{reg} of a compact Kähler space with klt singularities such that $c_1(X) = 0$ should be virtually abelian, and finite if the augmented irregularity of X vanishes. Unfortunately, this result seems out of reach for the moment, even in the projective case. It is yet crucial to control $\pi_1(X_{\text{reg}})$, for instance by relating it to $\pi_1(X)$, in order to trivialize flat subsheaves of the tangent sheaf or to clear the difference between restricted holonomy groups (which are classified by Berger–Simons) and holonomy groups (which govern the geometry of X via the Bochner principle).

The fundamental group of the whole space $\pi_1(X)$ is in general easier to understand but it might be much smaller than $\pi_1(X_{\text{reg}})$, as the standard Example 6.2 shows. Nevertheless, the following results about $\pi_1(X)$ are useful and assuming the existence of maximally quasi-étale covers, they also help in understanding $\pi_1(X_{\text{reg}})$.

Theorem F (Fundamental groups, Theorems 7.2 and 7.3). *Let X be as in Theorem B.*

- *If $\dim X$ is even and \mathcal{T}_X is strongly stable, then $\pi_1(X)$ is finite.*
- *If X is irreducible holomorphic symplectic or an even-dimensional Calabi–Yau variety, then X is simply connected.*
- *If $\tilde{q}(X) = 0$, then any flat vector bundle defined on the whole of X becomes trivial after a finite étale cover.*
- *If X is locally algebraic and $\tilde{q}(X) = 0$, then any flat vector bundle on X_{reg} becomes trivial after a finite quasi-étale cover.*

If we restrict our attention to low dimensions, we arrive at much stronger statements.

Theorem G (Fundamental groups in dimension four, Theorem 7.4). *Let X be as in Theorem B and of dimension ≤ 4 . Then:*

- *$\pi_1(X)$ is virtually abelian.*
- *If $\tilde{q}(X) = 0$, then $\pi_1(X)$ is finite.*

Assuming the MMP for Kähler fourfolds, this implies Campana’s Abelianity Conjecture in Kodaira dimension zero (Corollary 7.5). Note also that since every finitely generated abelian group (of even rank) is projective, the above result in particular sheds some new light on the long-standing open problem of whether every Kähler group is projective [ABC⁺96, (1.26)].

Kodaira problem and Beauville–Bogomolov decomposition. The Kodaira problem asked whether every compact Kähler manifold admits an algebraic approximation, i.e. whether it can be deformed (over a small germ) to a projective algebraic variety. As counterexamples were given by Voisin [Voi04], Peternell and independently Campana have proposed a refined “bimeromorphic” version of the original question: here one only tries to approximate *minimal* Kähler spaces [Gra18, Conjecture 1.2]. This has recently been settled affirmatively in dimension three [Gra18, CHL19], but the higher-dimensional case remains largely open.

Using our main results, we can draw some consequences about the above questions in Kodaira dimension zero. From Corollary E it follows that spaces with strongly stable tangent sheaf indeed admit an algebraic approximation, at least up to a quasi-étale cover. More generally, we show that the singular BB (= Beauville–Bogomolov) decomposition would imply approximability, again up to a cover (Theorem 8.3).

At least in dimension four, Kodaira problem and BB decomposition are actually equivalent and we can (almost) prove both statements. Unfortunately, here we are unable to completely circumvent the use of maximally quasi-étale covers and therefore we have to make the technical assumption that X is locally algebraic. We refer to Section 2.D for the precise notion of strong locally trivial algebraic approximation.

Theorem H (Kodaira problem in dimension four, Theorem 8.4). *Let X be as in Theorem B and of dimension ≤ 4 . If X is locally algebraic, then:*

- *Up to a quasi-étale Galois cover, X admits a strong locally trivial algebraic approximation.*
- *The variety X admits a Beauville–Bogomolov decomposition: some quasi-étale cover of X splits as a product of a complex torus, Calabi–Yau and irreducible holomorphic symplectic varieties.*

We stress that there is no deformation-theoretic reason for the algebraicity assumption in Theorem H. Quite the opposite: we expect the theorem to hold without that assumption and this would then imply *a posteriori* that the singularities of X are algebraic. More precisely, we make the following conjecture.

Conjecture K. *Let X be as in Theorem B. Then X admits a strong locally trivial algebraic approximation. In particular, X is locally algebraic.*

Note that by [GS20, Theorem 1.2], to prove Conjecture K it is sufficient to show smoothness of the semiuniversal locally trivial deformation space $\text{Def}^{\text{lt}}(X)$.

Acknowledgements. H.G. and B.C. would like to thank Stéphane Druel and Matei Toma for several enlightening discussions. P.G. and P.N. would like to thank Mihai Păun and Thomas Peternell for sharing their insight with them.

2. PRELIMINARIES

In this section we gather some basic material to be used in the rest of the article.

2.A. Global conventions. Unless otherwise stated, complex spaces are assumed to be countable at infinity, separated, reduced and connected. Algebraic varieties and schemes are always assumed to be defined over the complex numbers.

2.B. General definitions. As a courtesy to the reader, we recall the following standard definitions.

Definition 2.1 (Irregularity). The *irregularity* of a compact complex space X is $q(X) := h^1(Y, \mathcal{O}_Y)$, where $Y \rightarrow X$ is any resolution of singularities. The *augmented irregularity* of X is

$$\tilde{q}(X) := \max \left\{ q(\tilde{X}) \mid \tilde{X} \rightarrow X \text{ quasi-étale cover} \right\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Remark 2.2. If X has rational (e.g. klt) singularities, one has $q(X) = h^1(X, \mathcal{O}_X)$. If additionally X is Kähler, then it follows from [KS19, Corollary 1.8] that

$$q(X) = h^0(Y, \Omega_Y^1) = h^0(X, \Omega_X^{[1]}).$$

Definition 2.3 (Flat sheaves). Let X be an irreducible and reduced complex space and let $\tilde{X} \rightarrow X$ be its universal cover. We say that a rank r vector bundle $E \rightarrow X$ is *flat* if there exists a linear representation $\rho: \pi_1(X) \rightarrow \mathrm{GL}(r, \mathbb{C})$ such that E is isomorphic to the bundle $\tilde{X} \times \mathbb{C}^r / \pi_1(X) \rightarrow X$, where $\pi_1(X)$ acts diagonally.

Definition 2.4. Let X be a complex space. We say that X is *locally algebraic*, or that X has *algebraic singularities*, if there exists an open cover $\{U_i\}_{i \in I}$ of X such that for every $i \in I$, there is a quasi-projective scheme Y_i , an open subset $V_i \subset Y_i^{\mathrm{an}}$ and a biholomorphic map $\varphi_i: U_i \xrightarrow{\sim} V_i$.

Example 2.5. Here are some (non-)examples of algebraic singularities.

- (2.5.1) Every complex manifold, and more generally every complex space with quotient singularities, is locally algebraic [Car57].
- (2.5.2) Every complex space with only isolated singularities is locally algebraic by [Art69, Theorem 3.8] or [Tou68, corollaire 1, §3, chapitre II].
- (2.5.3) If the complex space X admits a locally trivial algebraic approximation, then it is locally algebraic (for notation cf. Section 2.D below).
- (2.5.4) Take a non-isotrivial family of elliptic curves over the unit disc Δ . Restrict it to the punctured disc Δ^* and pull it back along the universal cover $\Delta \rightarrow \Delta^*$. Now take a cone over this family fibrewise. We obtain a log canonical threefold with one-dimensional singular locus which is *not* locally algebraic, because the j -function associated to the exceptional divisor is not algebraic.

2.C. Coverings of complex spaces. We consistently use the following notation.

Definition 2.6 (Covering maps). A *cover* or *covering map* is a finite, surjective morphism $\gamma: Y \rightarrow X$ of normal, connected complex spaces. The covering map γ is called *Galois* if there exists a finite group $G \subset \mathrm{Aut}(Y)$ such that $Y \rightarrow X$ is isomorphic to the quotient map $Y \rightarrow Y/G$.

Definition 2.7 (Quasi-étale maps). A morphism $\gamma: Y \rightarrow X$ between normal complex spaces is called *quasi-étale* if γ is of relative dimension zero and étale in codimension one. In other words, γ is quasi-étale if $\dim Y = \dim X$ and if there exists a closed subset $Z \subset Y$ of codimension $\mathrm{codim}_Y(Z) \geq 2$ such that $\gamma|_{Y \setminus Z}: Y \setminus Z \rightarrow X$ is étale.

By purity of branch locus and the extension theorem of [DG94, Theorem 3.4], we get an equivalence of categories between the quasi-étale covers of X and the étale covers of X_{reg} . We emphasize that with our definitions, an étale or quasi-étale cover is automatically finite.

We will use several times the fact that taking Galois closure also works in the analytic context. Compare [GK20, Lemma 7.4].

Lemma 2.8 (Galois closure). *Let $\gamma: Y \rightarrow X$ be a covering map between normal complex spaces. Then there exists a Galois cover $g: \tilde{Y} \rightarrow Y$ (with \tilde{Y} normal) such that the composed map $f = \gamma \circ g: \tilde{Y} \rightarrow X$ is also Galois and we have an equality of branch loci $\text{Br}(f) = \text{Br}(\gamma)$. In particular, if γ is quasi-étale, then so is f .*

Proof. The morphism γ is étale over a Zariski open set $X^\circ \subset X$ and then it corresponds to a finite index subgroup $H \subset \pi_1(X^\circ)$. This subgroup has a finite number of conjugates in $\pi_1(X^\circ)$ and the intersection of these conjugates is a normal finite index subgroup H° in $\pi_1(X^\circ)$. This subgroup H° gives rise to a Galois cover

$$\tilde{Y}^\circ \rightarrow Y^\circ := \gamma^{-1}(X^\circ) \xrightarrow{\gamma} X^\circ.$$

The finite morphisms $\tilde{Y}^\circ \rightarrow X^\circ$ and $\tilde{Y}^\circ \rightarrow Y^\circ$ can be extended over the whole of X and Y to finite morphisms $f: \tilde{Y} \rightarrow X$ and $g: \tilde{Y} \rightarrow Y$ [DG94, Theorem 3.4]. The factorization $f = \gamma \circ g$ exists (by construction) over X° and it extends naturally. The equality of the branch loci is obvious. \square

As a consequence, quotient singularities can be characterized in terms of smoothness of quasi-étale covers.

Lemma 2.9 (Quotient singularities). *A germ of normal singularity (X, x) is a quotient singularity if and only if it admits a smooth quasi-étale cover.*

Proof. If (X, x) is isomorphic to $(\mathbb{C}^n/G, 0)$, we may assume that G contains no quasi-reflections [ST54, Che55] and then the quotient map $\mathbb{C}^n \rightarrow \mathbb{C}^n/G$ is quasi-étale.

Conversely, if $\gamma: Y \rightarrow X$ is quasi-étale with Y smooth, then we can take a Galois closure as in Lemma 2.8, i.e. a map $g: \tilde{Y} \rightarrow Y$ with $f = \gamma \circ g: \tilde{Y} \rightarrow X$ Galois. Since $f = \gamma \circ g$ is quasi-étale, the map g does not branch over any divisor in Y and since Y is smooth we deduce that the map g is étale by purity of the branch locus. This implies that \tilde{Y} is smooth as well and hence (X, x) is a quotient singularity [Car57]¹. \square

2.D. Deformation theory. In Section 8, the following notions from deformation theory will appear. Compare [GS20].

Definition 2.10 (Deformations of complex spaces). A *deformation* of a (reduced) compact complex space X is a proper flat morphism $\pi: \mathfrak{X} \rightarrow (S, 0)$ from a (not necessarily reduced) complex space \mathfrak{X} to a complex space germ $(S, 0)$, equipped with a fixed isomorphism $\mathfrak{X}_0 := \pi^{-1}(0) \cong X$.

Definition 2.11 (Algebraic approximations). Let X be a compact complex space and $\pi: \mathfrak{X} \rightarrow S$ a deformation of X . Consider the set of projective fibres

$$S^{\text{alg}} := \{s \in S \mid \mathfrak{X}_s \text{ is projective}\} \subset S$$

and its closure $\overline{S^{\text{alg}}} \subset S$. We say that $\mathfrak{X} \rightarrow S$ is an *algebraic approximation of X* if $0 \in \overline{S^{\text{alg}}}$. We say that $\mathfrak{X} \rightarrow S$ is a *strong algebraic approximation of X* if $\overline{S^{\text{alg}}} = S$ as germs, i.e. S^{alg} is dense near $0 \in S$.

¹The reference [Car57] is also available at http://www.numdam.org/article/SHC_1953-1954__6__A12_0.pdf.

Definition 2.12 (Locally trivial deformations). A deformation $\pi: \mathfrak{X} \rightarrow S$ is called *locally trivial* if for every $x \in \mathfrak{X}_0$ there exist open subsets $0 \in S^\circ \subset S$ and $x \in U \subset \pi^{-1}(S^\circ)$ and an isomorphism

$$\begin{array}{ccc} U & \xrightarrow{\sim} & (\mathfrak{X}_0 \cap U) \times S^\circ \\ & \searrow \pi & \swarrow \text{pr}_2 \\ & & S^\circ. \end{array}$$

2.E. Stratifications of complex spaces. Here we just set up some notation to be used when dealing with stratifications of singular spaces. We first recall that a *Whitney stratification* of a closed subspace X of a smooth real manifold M is a locally finite partition \mathcal{S} of X into smooth, connected, locally closed subsets $(X_\lambda)_{\lambda \in \Lambda}$:

$$(\mathcal{S}): \quad X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

satisfying certain incidence conditions. The pair (X, \mathcal{S}) is called a *stratified space* and the X_λ are called the *strata* of (X, \mathcal{S}) . We refer to [GM88, Part I, § 1.2 and 1.3] or to [Mat12] for more background on this classical topic, in particular the existence of Whitney stratifications.

We will mostly be interested in the situation where X is an irreducible and reduced complex space. In this case, the strata are asked to be smooth and connected locally closed *analytic* subsets. In particular, by dimension we will always mean the dimension as a complex space.

A stratification can alternatively be encapsulated in a filtration of X by closed analytic subsets:

$$S_k := \bigsqcup_{\{\lambda \mid \dim(X_\lambda) \leq k\}} X_\lambda.$$

So defined, S_k has dimension at most k and if $n := \dim X$ we have:

$$S_0 \subset S_1 \subset \cdots \subset S_n = X.$$

We will be mainly interested in what we call *maximal strata*.

Definition 2.13 (Maximal strata). Let $\mathcal{S} := (X_\lambda)_{\lambda \in \Lambda}$ be a Whitney stratification of X and $(S_k)_{0 \leq k \leq n}$ be the corresponding filtration. We let $d < n$ be the maximum index such that $S_d \subsetneq X$. The strata indexed by

$$\Lambda_{\max} := \{\lambda \in \Lambda \mid \dim(X_\lambda) = d\}$$

are called the *maximal (dimensional) strata*.

This definition is meaningful in particular when the stratification \mathcal{S} satisfies $S_d = X_{\text{sg}}$.

Definition 2.14 (Refining stratifications). Let \mathcal{S} and \mathcal{S}' be two Whitney stratifications of X . We say that \mathcal{S}' is a *refinement* of \mathcal{S} if the strata of \mathcal{S} are unions of strata of \mathcal{S}' . In this case we write $\mathcal{S}' < \mathcal{S}$.

Since strata are irreducible by definition, in the above situation each \mathcal{S} -stratum contains exactly one \mathcal{S}' -stratum as an open subset.

Definition 2.15 (Stratified maps). A smooth map $f: X \rightarrow Y$ is called *stratified* (with respect to Whitney stratifications \mathcal{S}_X and \mathcal{S}_Y of X and Y , respectively) if for each \mathcal{S}_Y -stratum $A \subset Y$, the preimage $f^{-1}(A)$ is a union of \mathcal{S}_X -strata and f takes each of these strata submersively to A .

Theorem 2.16 (Stratifications of maps, cf. [GM88, Part I, §1.7]). *Let $f: X \rightarrow Y$ be a holomorphic map, X and Y being endowed with Whitney stratifications as above. Then there exist refinements $\mathcal{S}'_X < \mathcal{S}_X$ and $\mathcal{S}'_Y < \mathcal{S}_Y$ such that $f: (X, \mathcal{S}'_X) \rightarrow (Y, \mathcal{S}'_Y)$ is a stratified map. \square*

We finally recall that the transverse structure of a given stratum is topologically locally trivial in a rather strong sense. In the following statement, a *normal (or transverse) slice* N_λ to a stratum X_λ is a smooth submanifold of the ambient space M which is transverse to each stratum of X , intersects X_λ in a single point and satisfies $\dim(N_\lambda) = \text{codim}_M(X_\lambda)$.

Theorem 2.17 (cf. Part I, §1.4 in [GM88]). *Let X_λ be any stratum of (X, \mathcal{S}) . Then there exists a stratified set $L \subset \mathbb{S}^k$ for some integer k such that any point $x_0 \in X_\lambda$ has a neighborhood U in X endowed with a homeomorphism*

$$h: U \longrightarrow (U \cap X_\lambda) \times c(L)$$

where $c(L)$ is the cone over L . The space L is called the *link of the stratum X_λ* and is denoted $\text{Link}(X, X_\lambda)$.

The space $c(L)$ is endowed with a natural stratification whose strata are the vertex of the cone and subspaces of the form $A \times]0, 1]$, with A a stratum of L . The homeomorphism h sends the strata of U into those of $(U \cap X_\lambda) \times c(L)$. In particular, if N is any transverse slice to X_λ at x_0 then $(U \cap N) \setminus \{x_0\}$ has the homotopy type of the link $\text{Link}(X, X_\lambda)$. \square

This local picture can be turned into a global one.

Theorem 2.18 (Existence of tubular neighborhoods, [Mat12]). *Let X_λ be any stratum of (X, \mathcal{S}) . Then there exists a closed neighborhood U_λ of X_λ in X and a continuous retraction*

$$f_\lambda: U_\lambda \longrightarrow X_\lambda$$

that is locally topologically trivial with fibre $c(\text{Link}(X, X_\lambda))$. The natural inclusion $X_\lambda \hookrightarrow U_\lambda$ being a section of f_λ , the punctured open neighborhood $U_\lambda^\circ \setminus X_\lambda$ is also locally topologically trivial over X_λ with fibre homotopy equivalent to $\text{Link}(X, X_\lambda)$.

This topological stability will be used through the following corollary.

Corollary 2.19. *Let $\gamma: Y \rightarrow X$ be a quasi-étale cover between normal complex spaces, and let \mathcal{S} be a Whitney stratification of X such that X_{sg} is a union of strata. Then for any maximal stratum X_λ , the following dichotomy holds: either $X_\lambda \subset \text{Br}(\gamma)$ or $X_\lambda \cap \text{Br}(\gamma) = \emptyset$.*

Proof. This is a direct application of [GKP16a, Corollary 3.12], where $A = X_{\text{sg}}$. Let $\lambda \in \Lambda_{\text{max}}$ be any index. If $X_\lambda \subset A$, then $\overline{X_\lambda}$ equals an irreducible component of A by maximality. Hence we may apply said result. Otherwise, X_λ must be disjoint from A , i.e. contained in X_{reg} . So we are in the second case of the dichotomy. \square

Remark 2.20. The notion of stratum and thus of its normal topological type depends on a given stratification. We will not need it in the sequel, but it has to be noted that any irreducible reduced complex space X has a canonical Whitney stratification \mathcal{S}_{can} . This means that if \mathcal{S} is any other Whitney stratification of X , the strata of \mathcal{S}_{can} are unions of strata of \mathcal{S} . The construction of \mathcal{S}_{can} is explained in [Mat73, Theorem 4.9] (see also [TT81, corollaire 6.1.7] where the canonical stratification is defined in terms of polar varieties).

3. BOCHNER PRINCIPLE FOR REFLEXIVE TENSORS

In this section, we prove the Bochner principle, [Theorem A](#). For generalities about Kähler metrics or currents on normal complex spaces, we refer to [\[Dem85, A.1\]](#), [\[EGZ09, § 5.2\]](#), [\[BEG13, § 4.6.1\]](#) or [\[GK20, § 3\]](#). We work in the following setting.

Setup 3.1. Let X be an n -dimensional complex, compact Kähler space with klt singularities such that K_X is numerically trivial.

We choose a Kähler resolution $\pi: Y \rightarrow X$ such that $\text{Exc}(\pi)$ is an snc divisor $F = \sum_j F_j$, and we fix holomorphic sections $s_j \in H^0(Y, \mathcal{O}_Y(F_j))$ cutting out the smooth component F_j . We set $s_F := \otimes_j s_j$ and $Y^\circ := Y \setminus F$.

Fix two integers $p, q \geq 0$. We set $\mathcal{E}_X := (\mathcal{T}_X^{\otimes p} \otimes \Omega_X^{\otimes q})^{\vee\vee}$, where $(-)^{\vee\vee}$ denotes the double dual, and $\mathcal{E}_Y := \mathcal{T}_Y^{\otimes p} \otimes \Omega_Y^{\otimes q}$. One has a natural inclusion of coherent sheaves $\pi_* \mathcal{E}_Y \subset \mathcal{E}_X$ and the quotient sheaf $\mathcal{E}_X / \pi_* \mathcal{E}_Y$ is torsion, being supported on X_{sg} . Thanks to Rückert's Nullstellensatz [\[GR84, Ch. 3, §2\]](#), there exists an integer $k \geq 1$ such that for any reflexive tensor $\tau \in H^0(X, \mathcal{E}_X)$, the section $\pi|_{Y^\circ}^*(\tau|_{X_{\text{reg}}})$ extends to an element $\sigma \in H^0(Y, \mathcal{E}_Y \otimes \mathcal{O}_Y(kF))$.

Thanks to [\[EGZ09\]](#) and [\[Päu08\]](#), there exists in each Kähler class $\alpha \in H^2(X, \mathbb{R})$ a unique closed, positive current $\omega_\alpha \in \alpha$ with bounded potentials, smooth on X_{reg} and satisfying $\text{Ric} \omega_\alpha = 0$ on this locus, cf. [Remark 3.2](#) below.

The Kähler metric ω_α on X_{reg} induces a smooth, hermitian metric on $\mathcal{T}_{X_{\text{reg}}}$ and in turn on $\mathcal{E}_X|_{X_{\text{reg}}}$, with Chern connection $D_{\mathcal{E}_X}$.

Remark 3.2. Given a Kähler metric ω_X on X , it is proved in [\[EGZ09\]](#) that there exists a unique Kähler–Einstein current in the cohomology class $\{\omega_X\} \in H^1(X, \text{PH}_X)$ under the connecting map $H^0(X, L_X^\infty / \text{PH}_X) \rightarrow H^1(X, \text{PH}_X)$. This cohomology group identifies two $(1, 1)$ -currents with local (bounded) dd^c -potentials if and only if they differ by the dd^c of a global (bounded) function. It is however more convenient to view ω_α in the more familiar cohomology space $H^2(X, \mathbb{R})$ using the connecting map arising from the natural exact sequence

$$0 \rightarrow \mathbb{R}_X \rightarrow \mathcal{O}_X \xrightarrow{\text{Im}} \text{PH}_X \rightarrow 0.$$

It was recently proved in [\[GK20, Proposition 3.5\]](#) that the map $H^1(X, \text{PH}_X) \rightarrow H^2(X, \mathbb{R})$ is injective whenever X has rational singularities. Therefore, a Kähler class $\alpha \in H^2(X, \mathbb{R})$ is associated to a unique class of a Kähler metric $\{\omega_X\} \in H^1(X, \text{PH}_X)$. This allows us to consider without any ambiguity the *unique* singular Ricci-flat metric ω_α in $\alpha \in H^2(X, \mathbb{R})$ as stated a few lines above.

Remark 3.3. In [Setup 3.1](#) above, the Abundance Conjecture has been proved in [\[CGP19, Corollary 1.18\]](#) as an application of results of B. Wang [\[Wan16\]](#). So there exists a quasi-étale cover $X' \rightarrow X$ such that $K_{X'}$ is trivial and, in particular, X' has canonical singularities. We do not rely on this fact to prove the Bochner principle.

Quite the opposite is true: one may use the Bochner principle to prove Abundance in [Setup 3.1](#), at least under the slightly stronger assumption that X has canonical (as opposed to klt) singularities. The details of the argument can be found in [Corollary 4.5](#).

Theorem 3.4 (Bochner principle). *Let X be a compact Kähler space with klt singularities such that K_X is numerically trivial and let α be a Kähler class. With the notation of [Setup 3.1](#) above, any holomorphic tensor $\tau \in H^0(X_{\text{reg}}, \mathcal{E}_X)$ is parallel with respect to ω_α , i.e. $D_{\mathcal{E}_X} \tau = 0$ on X_{reg} .*

Remark 3.5 (Comparison with earlier results). In [GGK19], the Bochner principle is proved under the assumption that X is projective. The proof goes as follows: along the lines of [Gue16], one can prove the Bochner principle for *bundles* [GGK19, Theorem 8.1], stating that a saturated subsheaf of slope zero of a tensor bundle $\mathcal{F} \subset \mathcal{E}_X$ is automatically parallel with respect to the Kähler–Einstein metric on the smooth locus. This does not rely on the projectivity assumption.

To go from bundles to tensors, one considers the line bundle generated by a given tensor τ on X_{reg} . As it is parallel, the holonomy group G acts on it, yielding a character of G . The classification of the holonomy [GGK19, Theorem B] shows that up to passing to a quasi-étale cover, G is semi-simple, hence the character is trivial and τ itself is parallel. The above classification result relies on the projectivity assumption through the existence of a maximally quasi-étale cover [GKP16a, Theorem 1.5] and Druel’s integrability result [Dru18, Theorem 1.4].

Proof of Theorem 3.4. We first need to set up some notation.

Hermitian metrics. We choose some smooth hermitian metrics h_j on $\mathcal{O}_Y(F_j)$. We denote by ϑ_j the curvature form of that metric, i.e. $\vartheta_j := i\Theta_{h_j}(F_j)$. Next, we define for any $\varepsilon > 0$ another smooth metric $h_{j,\varepsilon}$ on $\mathcal{O}_Y(F_j)$ by

$$h_{j,\varepsilon} := \frac{1}{|s_j|_{h_j}^2 + \varepsilon^2} \cdot h_j.$$

The curvature form $\vartheta_{j,\varepsilon}$ of that metric, i.e. $\vartheta_{j,\varepsilon} := i\Theta_{h_{j,\varepsilon}}(F_j)$, is given by

$$\vartheta_{j,\varepsilon} = \underbrace{\frac{\varepsilon^2}{(|s_j|^2 + \varepsilon^2)^2} \cdot Ds_j \wedge \overline{Ds_j}}_{=: \beta_{j,\varepsilon}} + \underbrace{\frac{\varepsilon^2}{|s_j|^2 + \varepsilon^2} \cdot \vartheta_j}_{=: \gamma_{j,\varepsilon}}.$$

The symbol D above refers to the Chern connection induced by h_j on $\mathcal{O}_Y(F_j)$. Finally, we set

$$h_F := \prod_j h_j \quad \text{and} \quad h_{F,\varepsilon} = \prod_j h_{j,\varepsilon} = \frac{1}{\prod_j (|s_j|^2 + \varepsilon^2)} \cdot h_F;$$

they define smooth metrics on $\mathcal{O}_Y(F)$.

Approximate Kähler–Einstein metrics. We introduce the rational coefficients $a_i > -1$ such that $K_Y = \pi^*K_X + \sum a_i F_i$. We fix a Kähler reference metric ω_Y on Y , and consider, for each $\varepsilon, t > 0$, the unique Kähler metric $\omega_{t,\varepsilon} \in \pi^*\alpha + t\{\omega_Y\}$ solution of

$$\text{Ric } \omega_{t,\varepsilon} = - \sum_j a_j \vartheta_{j,\varepsilon}$$

Its existence is guaranteed by Yau’s solution of the Calabi conjecture [Yau78]. In terms of Monge–Ampère equations, if ω_X is a smooth representative of α , then $\omega_{t,\varepsilon} = \pi^*\omega_X + t\omega_Y + dd^c\varphi_{t,\varepsilon}$ is solution of

$$(\pi^*\omega_X + t\omega_Y + dd^c\varphi_{t,\varepsilon})^n = \frac{e^{-c_{t,\varepsilon}} dV}{\prod_j (|s_j|^2 + \varepsilon^2)^{a_j}}$$

where dV is a fixed smooth volume form such that $\text{Ric } dV = \sum a_j \vartheta_j$ and $c_{t,\varepsilon} \in \mathbb{R}$ is a normalizing constant defined by $e^{c_{t,\varepsilon}} = \frac{1}{(\pi^*\alpha + t\{\omega_Y\})^n} \int_Y \frac{dV}{\prod_j (|s_j|^2 + \varepsilon^2)^{a_j}}$.

Curvature formula. We set $\mathcal{E}_Y(kF) := \mathcal{E}_Y \otimes \mathcal{O}_Y(kF)$ and we choose $\sigma \in H^0(Y, \mathcal{E}_Y(kF))$ some meromorphic tensor on Y . The Kähler metric $\omega_{t,\varepsilon}$ induces a smooth hermitian metric $h_{\omega_{t,\varepsilon}}$ on \mathcal{E}_Y . We consider the metric

$$h_{t,\varepsilon} := h_{\omega_{t,\varepsilon}} \otimes h_{F,\varepsilon}^{\otimes k} \quad \text{on } \mathcal{E}_Y(kF)$$

with Chern connection D and set $|\sigma| := |\sigma|_{h_{t,\varepsilon}}$. We have the following Poincaré-Lelong type formula

$$(3.5.1) \quad dd^c \log(|\sigma|^2 + 1) = \frac{1}{|\sigma|^2 + 1} \left(|D\sigma|^2 - \frac{|\langle D\sigma, \sigma \rangle|^2}{|\sigma|^2 + 1} - \langle i\Theta_{h_{t,\varepsilon}}(\mathcal{E}_Y(kF))\sigma, \sigma \rangle \right)$$

Wedging this last inequality with $\omega_{t,\varepsilon}^{n-1}$ and integrating it on X yields:

$$\int_Y \frac{\langle i\Theta_{h_{t,\varepsilon}}(\mathcal{E}_Y(kF))\sigma, \sigma \rangle}{|\sigma|^2 + 1} \wedge \omega_{t,\varepsilon}^{n-1} = \int_Y \frac{1}{|\sigma|^2 + 1} \left(|D\sigma|^2 - \frac{|\langle D\sigma, \sigma \rangle|^2}{|\sigma|^2 + 1} \right) \wedge \omega_{t,\varepsilon}^{n-1}$$

As $|\langle D\sigma, \sigma \rangle| \leq |D\sigma| \cdot |\sigma|$, we obtain

$$(3.5.2) \quad \int_Y \frac{\langle i\Theta_{h_{t,\varepsilon}}(\mathcal{E}_Y(kF))\sigma, \sigma \rangle}{|\sigma|^2 + 1} \wedge \omega_{t,\varepsilon}^{n-1} \geq \int_Y \frac{|D\sigma|^2}{(|\sigma|^2 + 1)^2} \wedge \omega_{t,\varepsilon}^{n-1}$$

Now, one has

$$i\Theta_{h_{t,\varepsilon}}(\mathcal{E}_Y(kF)) = i\Theta_{h_{\omega_{t,\varepsilon}}}(\mathcal{E}_Y) \otimes \text{Id}_{\mathcal{O}_Y(kF)} + \text{Id}_{\mathcal{E}_Y} \otimes i\Theta_{h_{F,\varepsilon}^{\otimes k}}(\mathcal{O}_Y(kF)).$$

First let us introduce a notation: let V be a complex vector space of dimension n , let $p \geq 1$ be an integer, and let $f \in \text{End}(V)$. We denote by $f^{\otimes p}$ the endomorphism of $V^{\otimes p}$ defined by

$$f^{\otimes p}(v_1 \otimes \cdots \otimes v_p) := \sum_{i=1}^p v_1 \otimes \cdots \otimes v_{i-1} \otimes f(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_p$$

Let us add that if V has an hermitian structure and if f is hermitian semipositive, then so is $f^{\otimes p}$ with the induced metric, and we have $\text{tr}(f^{\otimes p}) = pn^{p-1} \text{tr}(f)$. Now we can easily check the following identity:

$$ni\Theta(T_Y^{\otimes p}, h_{\omega_{t,\varepsilon}}) \wedge \omega_{t,\varepsilon}^{n-1} = (\# \text{Ric } \omega_{t,\varepsilon})^{\otimes p} \omega_{t,\varepsilon}^n$$

where $\# \text{Ric } \omega$ is the endomorphism of T_Y induced by $\text{Ric } \omega_{t,\varepsilon}$ via $\omega_{t,\varepsilon}$. As $\text{Ric } \omega = -\sum a_j \vartheta_{j,\varepsilon}$, we deduce that

$$\text{tr}_{\omega_{t,\varepsilon}} i\Theta_{h_{\omega_{t,\varepsilon}}}(\mathcal{E}_Y) = -\sum_j a_j \left[(\#\vartheta_{j,\varepsilon})^{\otimes p} \otimes \text{Id}_{T_Y^* \otimes q} - \text{Id}_{T_Y \otimes p} \otimes \overline{(\#\vartheta_{j,\varepsilon})}^{\otimes q} \right]$$

while

$$i\Theta_{h_{F,\varepsilon}^{\otimes k}}(\mathcal{O}_Y(kF)) \wedge \omega_{t,\varepsilon}^{n-1} = k \sum_j \vartheta_{j,\varepsilon} \wedge \omega_{t,\varepsilon}^{n-1}.$$

Locally, one can choose a trivialization e of $\mathcal{O}_Y(kF)$ and write $\sigma = u \otimes e$ where u is a local section of \mathcal{E}_Y . Then

$$(3.5.3) \quad \langle i\Theta_{h_{t,\varepsilon}}(\mathcal{E}_Y(kF))\sigma, \sigma \rangle = \underbrace{\langle i\Theta_{h_{\omega_{t,\varepsilon}}}(\mathcal{E}_Y)u, u \rangle |e|_{h_{F,\varepsilon}^{\otimes k}}^2}_{(I)} + \underbrace{\left(k \sum_j \vartheta_{j,\varepsilon} \right) \cdot |\sigma|^2}_{(II)}$$

Computation of Term (I).

This part is entirely similar to [GGK19, §9]. We have the decomposition

$$(\#\vartheta_{j,\varepsilon})^{\otimes p} = (\#\beta_{j,\varepsilon})^{\otimes p} + (\#\gamma_{j,\varepsilon})^{\otimes p}$$

along with the inequalities

$$(3.5.4) \quad 0 \leq (\#\beta_{j,\varepsilon})^{\otimes p} \leq (\text{tr}_{\text{End}}(\#\beta_{j,\varepsilon})^{\otimes p}) \text{Id}_{T_Y^{\otimes p}} \leq pn^{p-1} \text{tr}_{\omega_{t,\varepsilon}} \beta_{j,\varepsilon} \cdot \text{Id}_{T_Y^{\otimes p}}$$

and

$$(3.5.5) \quad \pm (\#\gamma_{j,\varepsilon})^{\otimes p} \leq \frac{C\varepsilon^2}{|s_j|^2 + \varepsilon^2} \cdot (\#\omega_Y)^{\otimes p} \leq Cpn^{p-1} \cdot \frac{\varepsilon^2}{|s_j|^2 + \varepsilon^2} \cdot \text{tr}_{\omega_{t,\varepsilon}} \omega_Y \cdot \text{Id}_{T_Y^{\otimes p}}$$

as soon as $C > 0$ is large enough so that $\pm\vartheta_j \leq C\omega_Y$ for any j .

From (3.5.4), one deduces

$$0 \leq \frac{\langle (\sharp\beta_{j,\varepsilon})^{\otimes p} \otimes \text{Id}_{T_Y^* \otimes q} u, u \rangle |e|_{h_{F,\varepsilon}^{\otimes k}}^2}{|\sigma|^2 + 1} \leq pn^{p-1} \text{tr}_{\omega_{t,\varepsilon}} \beta_{j,\varepsilon} \frac{|\sigma|^2}{|\sigma|^2 + 1} \\ \leq pn^{p-1} \text{tr}_{\omega_{t,\varepsilon}} \beta_{j,\varepsilon}$$

while from (3.5.5), one deduces

$$\pm \frac{\langle (\sharp\gamma_{j,\varepsilon})^{\otimes p} \otimes \text{Id}_{T_Y^* \otimes q} u, u \rangle |e|_{h_{F,\varepsilon}^{\otimes k}}^2}{|\sigma|^2 + 1} \leq Cpn^{p-1} \cdot \frac{\varepsilon^2}{|s_j|^2 + \varepsilon^2} \cdot \text{tr}_{\omega_{t,\varepsilon}} \omega_Y \frac{|\sigma|^2}{|\sigma|^2 + 1} \\ \leq Cpn^{p-1} \cdot \frac{\varepsilon^2}{|s_j|^2 + \varepsilon^2} \cdot \text{tr}_{\omega_{t,\varepsilon}} \omega_Y$$

so in conclusion, one gets

$$\pm \frac{\langle (\sharp\vartheta_{j,\varepsilon})^{\otimes p} \otimes \text{Id}_{T_Y^* \otimes q} u, u \rangle |e|_{h_{F,\varepsilon}^{\otimes k}}^2}{|\sigma|^2 + 1} \leq C \left[\text{tr}_{\omega_{t,\varepsilon}} \beta_{j,\varepsilon} + \frac{\varepsilon^2}{|s_j|^2 + \varepsilon^2} \cdot \text{tr}_{\omega_{t,\varepsilon}} \omega_Y \right]$$

for some C large enough. Performing the same computations with $\text{Id}_{T_Y \otimes p} \otimes \overline{(\sharp\vartheta_{j,\varepsilon})}^{\otimes q}$, one eventually obtains

$$(3.5.6) \quad \pm \frac{\langle i\Theta_{h_{\omega_{t,\varepsilon}}}(\mathcal{E}_Y)u, u \rangle |e|_{h_{F,\varepsilon}^{\otimes k}}^2}{|\sigma|^2 + 1} \wedge \omega_{t,\varepsilon}^{n-1} \leq C \left[(\beta_{j,\varepsilon} + \frac{\varepsilon^2}{|s_j|^2 + \varepsilon^2} \cdot \omega_Y) \wedge \omega_{t,\varepsilon}^{n-1} \right]$$

Computation of Term (II).

By the same token as before, one can decompose

$$\vartheta_{j,\varepsilon} \cdot \frac{|\sigma|^2}{|\sigma|^2 + 1} \wedge \omega_{t,\varepsilon}^{n-1} = \underbrace{\frac{|\sigma|^2}{|\sigma|^2 + 1} \cdot \beta_{j,\varepsilon} \wedge \omega_{t,\varepsilon}^{n-1}}_{\text{(III)}} + \underbrace{\frac{|\sigma|^2}{|\sigma|^2 + 1} \cdot \gamma_{j,\varepsilon} \wedge \omega_{t,\varepsilon}^{n-1}}_{\text{(IV)}}$$

and write $0 \leq \text{(III)} \leq \beta_{j,\varepsilon} \wedge \omega_{t,\varepsilon}^{n-1}$ and $\pm \text{(IV)} \leq \frac{C\varepsilon^2}{|s_j|^2 + \varepsilon^2} \cdot \omega_Y \wedge \omega_{t,\varepsilon}^{n-1}$ to obtain

$$(3.5.7) \quad \vartheta_{j,\varepsilon} \cdot \frac{|\sigma|^2}{|\sigma|^2 + 1} \wedge \omega_{t,\varepsilon}^{n-1} \leq C \left[(\beta_{j,\varepsilon} + \frac{\varepsilon^2}{|s_j|^2 + \varepsilon^2} \cdot \omega_Y) \wedge \omega_{t,\varepsilon}^{n-1} \right]$$

Putting (3.5.6) and (3.5.7) together, one sees from (3.5.3) that

$$(3.5.8) \quad \pm \frac{\langle i\Theta_{h_{t,\varepsilon}}(\mathcal{E}_Y(kF))\sigma, \sigma \rangle}{|\sigma|^2 + 1} \wedge \omega_{t,\varepsilon}^{n-1} \leq C \left[\sum_j \left(\beta_{j,\varepsilon} + \frac{\varepsilon^2}{|s_j|^2 + \varepsilon^2} \cdot \omega_Y \right) \wedge \omega_{t,\varepsilon}^{n-1} \right]$$

Finally, one has

$$\int_Y \beta_{j,\varepsilon} \wedge \omega_{t,\varepsilon}^{n-1} = \int_Y \vartheta_{j,\varepsilon} \wedge \omega_{t,\varepsilon}^{n-1} - \int_Y \gamma_{j,\varepsilon} \wedge \omega_{t,\varepsilon}^{n-1} \\ \leq F_j \cdot (\pi^* \alpha + t\omega_Y)^{n-1} + C \int_Y \frac{\varepsilon^2}{|s_j|^2 + \varepsilon^2} \omega_Y \wedge \omega_{t,\varepsilon}^{n-1}$$

so that (3.5.8) combined with Lemma 3.6 show that

$$(3.5.9) \quad \overline{\lim}_{t \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \int_Y \frac{\langle i\Theta_{h_{t,\varepsilon}}(\mathcal{E}_Y(kF))\sigma, \sigma \rangle}{|\sigma|^2 + 1} \wedge \omega_{t,\varepsilon}^{n-1} = 0.$$

Conclusion.

When ε, t go to zero, $\omega_{t,\varepsilon}$ converge weakly to $\pi^* \omega_\alpha$, and the convergence is smooth on Y° . Inequality (3.5.2) combined with Fatou lemma ensure that σ has zero covariant derivative on Y° with respect to the smooth hermitian metric $h_{\pi^* \omega_\alpha} \otimes h_{F,0}^{\otimes k}$ on $\mathcal{E}_Y(kF)|_{Y^\circ}$. Now, it follows from the definition of $h_{F,0}$ that $|s_F|_{h_{F,0}} \equiv 1$ on Y so

that s_F is parallel with respect to $h_{F,0}$ on Y° . This implies that $\sigma/s_F^{\otimes k}$ is parallel with respect to $h_{\pi^*\omega_\alpha}$ on Y° , hence τ is parallel with respect to h_{ω_α} on X_{reg} . \square

We used the following lemma in the proof above; it can be found e.g. in [GGK19, Claim 9.5].

Lemma 3.6. *For every fixed $t > 0$, and every j , one has that*

$$\int_Y \frac{\varepsilon^2}{|s_j|^2 + \varepsilon^2} \omega_Y \wedge \omega_{t,\varepsilon}^{n-1}$$

converges to 0 when ε goes to 0. \square

Remark 3.7. In Theorem 3.4 above, if $\mathcal{E}_X = \mathcal{T}_X$ or $\mathcal{E}_X = \Omega_X^{[p]}$ for some $1 \leq p \leq n$, then one can choose the resolution π such that the tensor τ pulls back to a holomorphic tensor on the resolution (i.e. it does not acquire any poles along F). This follows from the existence of a functorial resolution of singularities in the first case (see e.g. [Kol07, Proposition 3.9.1]) and from [KS19] in the second one. In such a case, the term (II) in the proof above disappears and the proof reduces to the computations of [Gue16]. We do, however, need the Bochner principle also for more general tensors.

4. STRUCTURE OF THE ALBANESE MAP

For generalities on the Albanese map for singular spaces, see [Gra18, §3]. We prove here the exact analogue of [Kaw85, Theorem 8.3] in the Kähler setting. See also [Gra18, Theorem 1.10] for the three-dimensional case. We then proceed to prove the existence of torus covers (Theorem B) and some immediate corollaries.

Theorem 4.1 (Albanese splits after base change). *Let X be a normal compact Kähler space with canonical singularities such that $c_1(X) = 0$. Let*

$$\alpha: X \longrightarrow A := \text{Alb}(X)$$

be the Albanese map of X . Then there exists a finite étale cover $A_1 \rightarrow A$ such that $X \times_A A_1$ is isomorphic to $F \times A_1$ over A_1 , where F is connected. In particular, α is a surjective analytic fibre bundle with connected fibres, and $q(X) \leq \dim X$.

Remark. Under the stronger assumption that K_X is torsion, the fact that α is surjective with connected fibres has already been shown in [Kaw81, Theorem 24] and [Cam04b, Theorem 5.1 and Proposition 5.3]. Since these results are stated for manifolds, one needs to apply them to a smooth model $Y \rightarrow X$. This is possible since K_Y is a \mathbb{Q} -effective exceptional divisor and so Y has vanishing Kodaira dimension.

On the other hand, we can use Theorem 4.1 to prove that K_X is torsion under the assumptions of that theorem. See Corollary 4.5.

Proof of Theorem 4.1. By the universal property of the Albanese torus, for every $g \in \text{Aut}(X)$ there exists a unique $\varphi_g: A \rightarrow A$ such that $\varphi_g \circ \alpha = \alpha \circ g$. If $g \in \text{Aut}^\circ(X)$, the identity component, then g acts trivially on $H^1(X, \mathbb{C})$ and in particular on $H^0(X, \Omega_X^{[1]})$. Consequently, the linear part of φ_g is the identity. That is, φ_g is a translation by some element $\varphi(g) \in A$. We have thus defined a homomorphism of complex Lie groups $\varphi: \text{Aut}^\circ(X) \rightarrow A$, which we will now show to be the desired cover $A_1 \rightarrow A$. First of all, since X has canonical singularities, it is not uniruled. Hence $\text{Aut}^\circ(X)$ is a complex torus by [Fuj78, Proposition 5.10]. It is thus sufficient to show that the induced Lie algebra map $d\varphi: H^0(X, \mathcal{T}_X) \rightarrow$

$\mathrm{H}^0(X, \Omega_X^{[1]})^\vee$ is an isomorphism. One checks easily that $d\varphi$ is given by the natural contraction pairing

$$(4.1.1) \quad \mathrm{H}^0(X, \mathcal{T}_X) \times \mathrm{H}^0(X, \Omega_X^{[1]}) \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X) = \mathbb{C}.$$

Fix a Kähler class $a \in \mathrm{H}^2(X, \mathbb{R})$ and consider the associated Ricci-flat metric ω_a as in Section 3. Let $0 \neq \vec{v} \in \mathrm{H}^0(X, \mathcal{T}_X)$ be a nonzero holomorphic vector field. Due to the Bochner principle, Theorem 3.4, \vec{v} is parallel with respect to the Chern connection D induced by ω_a . Dualizing using this metric, \vec{v} gives rise to a parallel (hence holomorphic, as $D^{0,1} = \bar{\partial}$) 1-form α on X . Clearly the contraction $\iota_{\vec{v}}\alpha \neq 0$. The argument can also be read backwards, hence (4.1.1) is a perfect pairing and $d\varphi$ is an isomorphism.

Next, we show that the family $f_1: X \times_A A_1 \rightarrow A_1$ is trivial. To this end, let $g \in A_1 = \mathrm{Aut}^\circ(X)$ be any point. Since $\varphi_g \circ \alpha = \alpha \circ g$, the automorphism g restricts to an isomorphism $\alpha^{-1}(0) \xrightarrow{\simeq} \alpha^{-1}(\varphi_g(0)) = \alpha^{-1}(\varphi(g))$. This is the same thing as an isomorphism $f_1^{-1}(0) \xrightarrow{\simeq} f_1^{-1}(g)$, independent of any choices. We have constructed a section of $\mathrm{Isom}_{A_1}(X \times_A A_1, F \times A_1) \rightarrow A_1$. The claimed triviality follows.

What we have observed so far already implies that $X \rightarrow A$ is a surjective analytic fibre bundle, because the base change to A_1 has these properties. It remains to see that the fibre F is connected. For this, consider the Stein factorization $X \xrightarrow{\beta} B \xrightarrow{\gamma} A$ of α . We claim that γ is étale. The question is local on A , so we may pass to a small open subset $U \subset A$ such that $\alpha^{-1}(U) \cong U \times F$ over U . Then there is an isomorphism $\alpha_* \mathcal{O}_{\alpha^{-1}(U)} \cong \mathcal{O}_U^{\oplus N}$ in the category of finitely presented \mathcal{O}_U -algebras, where N is the number of connected components of F . By construction of the Stein factorization, it follows that $\gamma^{-1}(U)$ is a disjoint union of N copies of U . In particular, γ is étale.

Consequently, B is a complex torus too. By the universal property of A , the map $X \xrightarrow{\beta} B$ factors via A and we get a section $A \rightarrow B$ of γ . This means that γ is an isomorphism, i.e. $B = A$. By the definition of Stein factorization, $X \rightarrow A = B$ has connected fibres.

The last statement, $q(X) \leq \dim X$, follows from the surjectivity of α as follows:

$$q(X) = h^1(Y, \mathcal{O}_Y) = h^0(Y, \Omega_Y^1) = \dim \mathrm{Alb}(Y) = \dim A \leq \dim X,$$

where $Y \rightarrow X$ is a resolution of singularities. \square

Corollary 4.2 (Torus covers). *Let X be a normal compact Kähler space with klt singularities such that $c_1(X) = 0$. Then:*

(4.2.1) *The augmented irregularity $\tilde{q}(X) \leq \dim X$. In particular, it is finite.*

Furthermore, there exist normal compact Kähler spaces T and Z with canonical singularities together with a quasi-étale cover $\gamma: T \times Z \rightarrow X$ such that:

(4.2.2) *T is a complex torus of dimension $\tilde{q}(X)$.*

(4.2.3) *The canonical sheaf of Z is trivial, $\omega_Z \cong \mathcal{O}_Z$.*

(4.2.4) *The augmented irregularity of Z vanishes, $\tilde{q}(Z) = 0$.*

Any cover γ as above will be called a torus cover of X .

Proof. The proof closely follows along the lines of [GGK19, Proposition 7.5], but with different references. For later reference, and also as a courtesy to the reader, we give the argument here.

To begin with, note that by [CGP19, Corollary 1.18], the canonical divisor K_X is torsion. We may thus consider an index one cover $X_1 \rightarrow X$. This has the

additional property that K_{X_1} is trivial (in particular, Cartier) and hence X_1 has canonical singularities. That is, X_1 satisfies the assumptions of [Theorem 4.1](#).

For (4.2.1), note that $\tilde{q}(X) = \tilde{q}(X_1)$ by [Lemma 4.3](#). It is therefore sufficient to prove $\tilde{q}(X_1) \leq \dim X_1 = \dim X$. But any quasi-étale cover $X' \rightarrow X_1$ reproduces the assumptions of [Theorem 4.1](#), hence $q(X') \leq \dim X' = \dim X_1$ by that result. The claim follows by taking the supremum over all quasi-étale covers of X_1 .

Next, we construct a torus cover γ as a sequence of quasi-étale covers as follows:

$$T \times Z = X_3 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X.$$

The first map, $X_1 \rightarrow X$, is still the index one cover. We have already seen that $\tilde{q}(X_1)$ is finite. Choose now a quasi-étale cover $X_2 \rightarrow X_1$ that realizes this augmented irregularity, $q(X_2) = \tilde{q}(X_1)$. Finally, apply [Theorem 4.1](#) to X_2 in order to obtain a further cover $X_3 \rightarrow X_2$ that splits off a torus as $X_3 = T \times Z$. We need to show that (4.2.2)–(4.2.4) hold.

By construction, $\dim(T) = q(X_3) = q(X_2) = \tilde{q}(X_1) = \tilde{q}(X)$, which proves (4.2.2). For (4.2.3), since ω_{X_1} is already trivial, the same is true of $\omega_{T \times Z}$ and hence also of ω_Z . Finally, if there was a cover $Z' \rightarrow Z$ with $q(Z') > 0$, then $T \times Z' \rightarrow X$ would be a cover with irregularity $\dim T + q(Z') > \tilde{q}(X)$, which contradicts the definition of $\tilde{q}(X)$. This shows (4.2.4). \square

Lemma 4.3 (Invariance of \tilde{q}). *Let $Y \rightarrow X$ be a quasi-étale cover of normal compact complex spaces. Then $\tilde{q}(Y) = \tilde{q}(X)$.*

Proof. Every quasi-étale cover $Y' \rightarrow Y$ is, by composition, also a quasi-étale cover of X . The inequality $\tilde{q}(Y) \leq \tilde{q}(X)$ is therefore obvious. For the other direction, let $X' \rightarrow X$ be any quasi-étale cover of X , and consider the normalized fibre product diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X. \end{array}$$

Since all the maps in this diagram are quasi-étale covers, we obtain

$$\tilde{q}(Y) \geq q(Z) \geq q(X').$$

The claim follows by taking the supremum over all quasi-étale $X' \rightarrow X$. \square

Proposition 4.4 (Uniqueness of torus cover). *In the setting of [Corollary 4.2](#), the spaces T and Z are unique up to quasi-étale cover, in the following sense: Suppose that $\gamma': T' \times Z' \rightarrow X$ is another torus cover of X . Then:*

(4.4.1) *The complex tori T and T' are isogeneous.*

(4.4.2) *There is a common quasi-étale cover $Z \leftarrow Z'' \rightarrow Z'$, where Z'' is likewise compact, connected and has canonical singularities.*

Proof. Consider the normalized fibre product diagram

$$\begin{array}{ccc} Y & \xrightarrow{p'} & T' \times Z' \\ p \downarrow & & \downarrow \gamma' \\ T \times Z & \xrightarrow{\gamma} & X. \end{array}$$

Since $q(T \times Z)$ is already maximal, $q(Y) = q(T \times Z)$ and hence

$$\text{alb}(p): \text{Alb}(Y) \longrightarrow \text{Alb}(T \times Z) = T$$

is finite, i.e. an isogeny. The same argument applies to $T' \times Z'$. So both T and T' are isogeneous to $\text{Alb}(Y)$. This implies (4.4.1) because isogeny is an equivalence relation.

We turn to (4.4.2). Consider an arbitrary $s \in \text{Alb}(Y)$. Set $t := \text{alb}(p)(s) \in T$ and $Z_t := \{t\} \times Z \subset T \times Z$. We claim that we can take $Z'' := \text{alb}_{\bar{Y}}^{-1}(s)$. By Theorem 4.1, Z'' is connected and has canonical singularities. It is clear that $p(Z'') \subset Z_t$. Then they are equal, because they have the same dimension and Z_t is irreducible. Hence the restriction of p is a quasi-étale cover $Z'' \rightarrow Z_t \cong Z$. The same argument also shows that p' restricts to a quasi-étale cover $Z'' \rightarrow Z'$. This proves the claim. \square

Corollary 4.5 (Special case of Abundance). *Let X be a normal compact Kähler space with canonical singularities and $c_1(X) = 0$. Then ω_X is torsion, i.e. $\omega_X^{[m]} \cong \mathcal{O}_X$ for some $m > 0$.*

Proof. It follows from Theorem 4.1 that $\tilde{q}(X) \leq \dim X$ is finite (cf. the proof of Corollary 4.2). Choose a quasi-étale cover $X_1 \rightarrow X$ such that $q(X_1) = \tilde{q}(X)$, and apply Theorem 4.1 to X_1 . This yields an étale cover $X_2 \rightarrow X_1$ that splits as $X_2 = T \times Z$, where T is a complex torus of dimension $q(X_1)$. As $q(T) + q(Z) = q(X_2) = q(X_1) = q(T)$, we see that $q(Z) = 0$.

Now note that $h^1(Z, \mathcal{O}_Z) \leq q(Z)$ by the Leray spectral sequence. (In fact, equality holds because Z has rational singularities.) Thus $q(Z) = 0$ implies, via the exponential sequence, that any line bundle on Z with vanishing integral first Chern class is trivial. Consequently, for any \mathbb{Q} -Cartier reflexive rank one sheaf \mathcal{A} on Z with vanishing real first Chern class, there is $m > 0$ such that $\mathcal{A}^{[m]} \cong \mathcal{O}_Z$. We may apply this observation to $\mathcal{A} = \omega_Z$ and combine it with the fact that ω_T is trivial. The conclusion is that $\omega_{X_2} = \omega_{T \times Z}$ is torsion as well. Then the same is true of ω_X . \square

Corollary 4.6 (Characterization of torus quotients via \tilde{q}). *Let X be a normal compact Kähler space with klt singularities such that $c_1(X) = 0$. Then the following conditions are equivalent:*

- (4.6.1) *The augmented irregularity attains its maximum possible value, namely $\tilde{q}(X) = \dim X$.*
- (4.6.2) *There exists a complex torus T and a holomorphic action of a finite group $G \subset T$, free in codimension one, such that $X \cong T/G$.*

Proof. “(4.6.1) \Rightarrow (4.6.2)”: Consider a torus cover $\gamma: T \times Z \rightarrow X$. The dimension of Z equals $\dim X - \dim T = \dim X - \tilde{q}(X) = 0$, so Z is a point. This means that γ is a finite quasi-étale cover of X by the complex torus T . The claim follows by taking Galois closure, cf. Lemma 2.8 and [GK20, Lemma 7.4].

“(4.6.2) \Rightarrow (4.6.1)”: Obvious, as $\tilde{q}(X) \geq q(T) = \dim T = \dim X$. \square

5. SPACES WITH FLAT TANGENT SHEAF ON THE SMOOTH LOCUS

The aim of this section is to prove the following result, which comes in a local as well as a global version. This is an important step towards Theorem D.

Theorem 5.1 (Compact spaces with flat tangent sheaf). *Let X be a normal compact complex space with klt singularities. Assume that the tangent sheaf of the smooth locus $\mathcal{T}_{X_{\text{reg}}}$ is flat in the sense of Definition 2.3. Then X admits a quasi-étale Galois cover $\tilde{X} \rightarrow X$ with \tilde{X} smooth.*

Theorem 5.2 (Germs with flat tangent sheaf). *Let (X, x) be a germ of a klt singularity such that $\mathcal{T}_{X_{\text{reg}}}$ is flat. Then (X, x) is a quotient singularity.*

The notion of *maximally quasi-étale covers*, defined next, is central to the proof.

Definition 5.3. Let X be a normal complex space. A *maximally quasi-étale cover* of X is a quasi-étale Galois cover $\gamma: \tilde{X} \rightarrow X$ satisfying the following equivalent conditions:

(5.3.1) Any étale cover of \tilde{X}_{reg} extends to an étale cover of \tilde{X} .

(5.3.2) Any quasi-étale cover of \tilde{X} is étale.

(5.3.3) The natural map of étale fundamental groups $\hat{\pi}_1(\tilde{X}_{\text{reg}}) \rightarrow \hat{\pi}_1(\tilde{X})$ induced by the inclusion $\tilde{X}_{\text{reg}} \hookrightarrow \tilde{X}$ is an isomorphism.

The equivalence of conditions (5.3.1)–(5.3.3) is discussed in [GKP16a, Theorem 1.5] and its proof.

In general, a normal space X will not admit a maximally quasi-étale cover (for an easy example, take a cone over an elliptic curve). It was shown in [GKP16a, Theorem 1.1] that such a cover exists if X is an *algebraic* variety with klt singularities. Also, it is relatively easy to see that existence of the cover would imply Theorem 5.1, cf. Proposition 5.4 below. Our Proposition 5.8 is a weaker version of [GKP16a] for complex spaces: we construct a maximally quasi-étale cover “generically”, i.e. only over general points of the singular locus X_{sg} . This, however, is sufficient to prove the main result.

5.A. Technical preparations. As mentioned above, Theorem 5.1 is easy to prove if a maximally quasi-étale cover is already known to exist.

Proposition 5.4. *Let X be a normal complex space with log canonical singularities, and assume that the tangent sheaf of the smooth locus $\mathcal{T}_{X_{\text{reg}}}$ is flat. Then any maximally quasi-étale cover of X is smooth (if it exists).*

Proof. Let $\gamma: \tilde{X} \rightarrow X$ be a maximally quasi-étale cover. The pullback $\gamma^* \mathcal{T}_{X_{\text{reg}}}$ is a flat sheaf on $\gamma^{-1}(X_{\text{reg}})$, which satisfies $\gamma^{-1}(X_{\text{reg}}) \subset \tilde{X}_{\text{reg}}$ and the complement has codimension at least two. Hence $\gamma^* \mathcal{T}_{X_{\text{reg}}}$ extends to a flat sheaf on \tilde{X}_{reg} . The arguments in the proof of [GKP16a, Theorem 1.14] show that we can extend this sheaf further, to a flat sheaf \mathcal{F} on all of \tilde{X} . By reflexivity, $\mathcal{F} \cong \mathcal{T}_{\tilde{X}}$ and in particular $\mathcal{T}_{\tilde{X}}$ is locally free. The solution of the Lipman–Zariski conjecture for log canonical spaces [GK14, Corollary 1.3] then shows that \tilde{X} is smooth. \square

The following lemma describes the étale fundamental group of the link of maximal dimensional strata of a klt space.

Lemma 5.5. *Let X be a klt complex space and let us choose a Whitney stratification \mathcal{S} of X . The maximum dimensional strata are denoted by X_λ , $\lambda \in \Lambda_{\text{max}}$, as in Definition 2.13. For any $\lambda \in \Lambda_{\text{max}}$, the group $\hat{\pi}_1(\text{Link}(X, X_\lambda))$ is finite.*

Proof. Pick a normal slice N of X_λ cutting the stratum in the unique point p . Set $N_X := N \cap X$ and $N^* := N_X \setminus \{p\}$. From Theorem 2.17 we know that N^* has the homotopy type of $\text{Link}(X, X_\lambda)$ and in particular $\hat{\pi}_1(\text{Link}(X, X_\lambda)) = \hat{\pi}_1(N^*)$. But now according to [KM98, Lemma 5.12], (N_X, p) is an isolated klt singularity, hence algebraic (2.5.2). We can now appeal to [Xu14, Theorem 1] and conclude that the algebraic local fundamental group $\hat{\pi}_1(N_X \setminus \{p\})$ is finite. \square

Remark 5.6. The lemma above should apply to any stratum, not only to the maximal dimensional ones. Unfortunately, Xu’s result is only available in the algebraic setting, see also Remark 6.10. This is why we stick to maximal dimensional strata: they are the ones whose slices have only isolated singularities.

We now observe that quasi-étale covers are well understood over the maximal dimensional strata.

Lemma 5.7. *Let $\gamma: Y \rightarrow X$ be a quasi-étale Galois cover between klt complex spaces, X being endowed with a Whitney stratification \mathcal{S} . Fix an index $\lambda \in \Lambda_{\max}$ with respect to \mathcal{S} . Then:*

(5.7.1) *The cover γ naturally induces a subgroup $G_\lambda(\gamma) \subset \hat{\pi}_1(\text{Link}(X, X_\lambda))$, well-defined up to conjugation.*

(5.7.2) *If $g: Z \rightarrow Y$ is any further quasi-étale cover such that the composition $\gamma \circ g: Z \rightarrow X$ is also Galois, then $G_\lambda(\gamma \circ g) \subset G_\lambda(\gamma)$.*

Furthermore, there is a dense open subset $U_\lambda \subset X_\lambda$, which depends on γ but not on g , such that $X_\lambda \setminus U_\lambda$ is analytic and the following holds:

(5.7.3) *We have equality $G_\lambda(\gamma \circ g) = G_\lambda(\gamma)$ if and only if g is étale over $\gamma^{-1}(U_\lambda)$.*

Proof. By [Theorem 2.16](#), we can choose Whitney stratifications $\mathcal{S}' < \mathcal{S}$ and \mathcal{S}_Y of X and Y , respectively, such that $\gamma: (Y, \mathcal{S}_Y) \rightarrow (X, \mathcal{S}')$ is a stratified map. We denote by $X'_\lambda \subset X_\lambda$ the unique stratum of \mathcal{S}' contained in X_λ as an open subset.

The map γ being stratified, $\gamma^{-1}(X'_\lambda)$ is a union of strata of \mathcal{S}_Y that are of maximum dimension. Let us fix a normal slice N of X'_λ such that $\gamma^{-1}(N)$ is still a union of normal slices. We pick a component T of $\gamma^{-1}(X'_\lambda)$ and denote by N_T the corresponding slice. The map induced on the punctured slices (cut with X and Y , respectively) $N_T^* \rightarrow N^*$ is finite étale and thus corresponds to a finite index subgroup

$$G_\lambda(\gamma) \subset \hat{\pi}_1(N^*) = \hat{\pi}_1(\text{Link}(X, X'_\lambda)) = \hat{\pi}_1(\text{Link}(X, X_\lambda)).$$

The morphism γ being Galois, this subgroup is independent of the choice of the component T . It does however depend on the choice of a basepoint in N_T^* , which as usual we have suppressed. This is why $G_\lambda(\gamma)$ is only well-defined up to conjugation. This proves the first assertion, and the second one is then clear by construction.

Concerning the third assertion, we claim that we can take $U_\lambda = X'_\lambda$. Indeed, it is clear that the groups in question are equal if g is étale over $\gamma^{-1}(U_\lambda)$. For the other direction, pick stratifications \mathcal{S}_Z of Z and \mathcal{S}'_Y of Y such that \mathcal{S}'_Y is a refinement of \mathcal{S}_Y and g becomes stratified, as above. Write $\gamma^{-1}(U_\lambda)$ as a union of \mathcal{S}_Y -strata,

$$\gamma^{-1}(U_\lambda) = \bigcup_{\mu \in M} Y_\mu,$$

and let Y'_μ be the unique \mathcal{S}'_Y -stratum that is open in Y_μ . It follows from [Corollary 3.16](#) in the preprint version of [\[GKP16a\]](#) that g is étale over Y'_μ , for each $\mu \in M$. But then g is even étale over all of Y_μ , by [Corollary 2.19](#) applied to the original stratification \mathcal{S}_Y . In other words, g is étale over $\gamma^{-1}(U_\lambda)$, as desired. \square

With the previous results at hand, we can prove the existence of maximally quasi-étale covers, at least after discarding a sufficiently small analytic subset.

Proposition 5.8. *Let X be a klt complex space. Assume that X is either compact or a germ. Then there exists an analytic subset $Z \subset X_{\text{sg}}$ with $\dim Z \leq \dim X_{\text{sg}} - 1$ such that $X^\circ := X \setminus Z$ admits a maximally quasi-étale cover.*

If X is compact with only *isolated* klt singularities, then we have $\dim Z = -1$ in the above statement, which means $Z = \emptyset$. Hence X itself admits a maximally quasi-étale cover. In fact, more is true:

Proposition 5.9. *Let X be a compact klt complex space which is locally algebraic in the sense of [Definition 2.4](#). Then X admits a maximally quasi-étale cover.*

Proof of Proposition 5.8. Fix a Whitney stratification \mathcal{S} of X such that $S_d = X_{\text{sg}}$. If X is compact, then Λ and in particular Λ_{\max} are finite sets. If X is a germ, we may assume Λ to be finite after shrinking X , by the local finiteness of \mathcal{S} . First we show the following.

Claim 5.10. There exists a quasi-étale Galois cover $\gamma: Y \rightarrow X$ such that for every further cover $g: Z \rightarrow X$ with $\gamma \circ g$ Galois and any $\lambda \in \Lambda_{\max}$, we have $G_\lambda(\gamma \circ g) = G_\lambda(\gamma)$.

Proof. Assuming that the claim is false, we can construct an infinite tower of quasi-étale covers

$$X = Y_0 \xleftarrow{\gamma_1} Y_1 \xleftarrow{\gamma_2} Y_2 \xleftarrow{\gamma_3} \dots$$

such that for each $i \geq 1$, the map $\eta_i = \gamma_1 \circ \dots \circ \gamma_i: Y_i \rightarrow X$ is Galois and for some index $\lambda(i) \in \Lambda_{\max}$, the inclusion $G_{\lambda(i)}(\eta_{i+1}) \subset G_{\lambda(i)}(\eta_i)$ is strict. Because Λ_{\max} is finite, some $\lambda_0 \in \Lambda_{\max}$ has to appear as $\lambda(i)$ for infinitely many values of i . This yields a contradiction to the finiteness of $\hat{\pi}_1(\text{Link}(X, X_{\lambda_0}))$ given by Lemma 5.5. \square

Applying (5.7.3) to the cover γ given by Claim 5.10 and to all $\lambda \in \Lambda_{\max}$, we obtain dense open subsets $U_\lambda \subset X_\lambda$ with the following property: any quasi-étale cover $W \rightarrow Y$ such that $W \rightarrow X$ is Galois is étale over each $\gamma^{-1}(U_\lambda)$. If $W \rightarrow X$ is not Galois, the conclusion still holds because we may replace W by its Galois closure, cf. Lemma 2.8. We now consider $X^\circ = X \setminus Z$, with

$$Z := S_{d-1} \cup \bigcup_{\lambda \in \Lambda_{\max}} (X_\lambda \setminus U_\lambda),$$

and claim that $Y^\circ := \gamma^{-1}(X^\circ) \rightarrow X^\circ$ is a maximally quasi-étale cover. Noting that $Z \subset X_{\text{sg}}$ has dimension less than $\dim X_{\text{sg}}$, this will finish the proof.

We aim to verify condition (5.3.2) for Y° . That is, we need to show that any quasi-étale cover $g^\circ: W^\circ \rightarrow Y^\circ$ is in fact étale. Given such a cover, we may extend it to a quasi-étale cover $g: W \rightarrow Y$. As we have seen above, g is étale over each $\gamma^{-1}(U_\lambda)$. But we can write Y as a disjoint union

$$Y = \gamma^{-1}(Z) \cup \bigcup_{\lambda \in \Lambda_{\max}} \gamma^{-1}(U_\lambda) \cup \gamma^{-1}(X_{\text{reg}})$$

and hence also $Y^\circ = \bigcup_{\lambda} \gamma^{-1}(U_\lambda) \cup \gamma^{-1}(X_{\text{reg}})$. Recalling from purity of branch locus that $\gamma^{-1}(X_{\text{reg}}) \subset Y_{\text{reg}}$ and that g is étale over Y_{reg} , we conclude that g is étale over all of Y° . In other words, g° is étale. This ends the proof. \square

Proof of Proposition 5.9. Consider an infinite tower of quasi-étale covers

$$X = Y_0 \xleftarrow{\gamma_1} Y_1 \xleftarrow{\gamma_2} Y_2 \xleftarrow{\gamma_3} \dots$$

such that for each $i \geq 1$, the map $\eta_i = \gamma_1 \circ \dots \circ \gamma_i: Y_i \rightarrow X$ is Galois. We need to show that almost all γ_i are étale.

For any point $x \in X$, the regional fundamental group $\pi_1^{\text{reg}}(X, x)$ is finite thanks to [Bra20, Theorem 1] and the local algebraicity assumption. In particular, the image of the natural map $\pi_1^{\text{reg}}(X, x) \rightarrow \pi_1(X_{\text{reg}})$ is finite. Let $U_x \subset X$ be a sufficiently small open neighborhood of x . The above finiteness implies that if we restrict the given tower to $(U_x)_{\text{reg}}$, then almost all maps γ_i will become trivial, i.e. a disjoint union of copies of $\eta_{i-1}^{-1}((U_x)_{\text{reg}})$. Equivalently, if we restrict the tower to U_x , then almost all γ_i will become étale. Since X is compact, we can cover it by finitely many open sets U_x and hence there exists a uniform bound M such that γ_i is étale for $i \geq M$, as desired. \square

5.B. Proof of Theorem 5.1. We proceed by induction on the dimension of the singular locus X_{sg} . If $\dim X_{\text{sg}} = -1$, i.e. X_{sg} is empty, then we may simply take $\tilde{X} = X$. Otherwise we apply Proposition 5.8 to obtain an open subset $X_{\text{reg}} \subset X^\circ \subset X$ which has a maximally quasi-étale cover $\gamma^\circ: \tilde{X}^\circ \rightarrow X^\circ$ and satisfies $\dim(X \setminus X^\circ) \leq \dim X_{\text{sg}} - 1$. We may restrict γ° to an étale cover γ' of X_{reg} ,

and then in turn extend γ' to a quasi-étale cover of X , say $\gamma: \tilde{X} \rightarrow X$. By the uniqueness part of [DG94, Theorem 3.4], the map γ will be an extension of γ° .

$$\begin{array}{ccccc} X' & \hookrightarrow & \tilde{X}^\circ & \hookrightarrow & \tilde{X} \\ \gamma' \downarrow & & \gamma^\circ \downarrow & & \downarrow \gamma \\ X_{\text{reg}} & \hookrightarrow & X^\circ & \hookrightarrow & X \end{array}$$

The space \tilde{X}° is smooth thanks to Proposition 5.4. Thus we see that $\dim \tilde{X}_{\text{sg}} \leq \dim X_{\text{sg}} - 1$. Since $\mathcal{T}_{\tilde{X}_{\text{reg}}}$ is still flat, the induction hypothesis applied to \tilde{X} yields a quasi-étale cover $\hat{X} \rightarrow \tilde{X}$ with \hat{X} smooth. We arrive at the desired smooth cover of X by taking the Galois closure of the composed map $\hat{X} \rightarrow X$. \square

5.C. Proof of Theorem 5.2. We still argue by induction on $\dim((X, x)_{\text{sg}})$ as above. If (X, x) is smooth, there is nothing to show. We just have to check that we can apply the induction hypothesis: to start with, we choose X as a small neighborhood of x with a finite Whitney stratification. We can then find a quasi-étale cover $f: \tilde{X} \rightarrow X$ with $\dim(\tilde{X}_{\text{sg}}) < \dim(X_{\text{sg}})$. It is enough to pick a point $\tilde{x} \in f^{-1}(x) \subset \tilde{X}$: this \tilde{x} has a neighborhood \tilde{U} that is a quotient singularity. The open neighborhood $U := f(\tilde{U})$ of x has a smooth quasi-étale cover and (X, x) is then a quotient singularity according to Lemma 2.9. \square

6. HOLONOMY OF SINGULAR RICCI-FLAT METRICS

The goal of this section is to prove Theorem C, which combined with the Bochner principle leads to Corollary E.

6.A. A vanishing lemma. The following lemma will be used repeatedly. The upshot is that a vanishing assumption on X implies a certain vanishing result on X_{reg} .

Lemma 6.1 (Vanishing lemma). *Let X be a compact Kähler space with klt singularities such that $q(X) = 0$. Then one has*

$$H^1(X_{\text{reg}}, \mathbb{C}) = 0.$$

If X additionally satisfies $\tilde{q}(X) = 0$, then any representation $\rho: \pi_1(X_{\text{reg}}) \rightarrow G$ with virtually abelian image actually has finite image.

Example 6.2. The example of the (simply connected) singular Kummer surface $X = T/\pm 1$ where T is a complex 2-torus shows that one can have $q(X) = 0$ while $\pi_1(X_{\text{reg}})$ is virtually abelian, yet infinite. Indeed, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}^4 \longrightarrow \pi_1(X_{\text{reg}}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Of course, in this example $\tilde{q}(X) = 2 > 0$.

Remark 6.3. The proof of the Lemma combined with [KS19] shows more generally that if X is compact Kähler with klt singularities and $\hat{X} \rightarrow X$ is a resolution, then one has a (non-canonical) isomorphism

$$H^1(X_{\text{reg}}, \mathbb{C}) = H^1(\hat{X} \setminus E, \mathbb{C}) \cong H^0(X, \Omega_X^{[1]}) \oplus H^1(X, \mathcal{O}_X).$$

On the other hand, since X has only rational singularities, one also has

$$H^1(X, \mathbb{C}) \cong H^1(\hat{X}, \mathbb{C}) = H^0(\hat{X}, \Omega_{\hat{X}}^1) \oplus H^1(\hat{X}, \mathcal{O}_{\hat{X}})$$

canonically. Hence the restriction map $H^1(X, \mathbb{C}) \rightarrow H^1(X_{\text{reg}}, \mathbb{C})$ is an isomorphism. Equivalently, the *abelianization* of the natural surjection $\pi_1(X_{\text{reg}}) \rightarrow \pi_1(X)$ has finite kernel. Cf. [Example 6.2](#) above.

Proof of Lemma 6.1. Let $\widehat{X} \rightarrow X$ be a strong projective log resolution of X , and let $E = \sum_i E_i$ be the exceptional divisor. In what follows, one will identify X_{reg} with $\widehat{X} \setminus E$. One has

$$(6.3.1) \quad H^1(\widehat{X}, \mathcal{O}_{\widehat{X}}) = 0$$

since $q(X) = 0$, cf. [Definition 2.1](#). Since \widehat{X} is a compact Kähler manifold, one also has $H^0(\widehat{X}, \Omega_{\widehat{X}}^1) = 0$. From [\[GK14, Theorem 3.1\]](#), it then follows that even

$$(6.3.2) \quad H^0(\widehat{X}, \Omega_{\widehat{X}}^1(\log E)) = 0.$$

Note that one could also have invoked [Remark 2.2](#) to derive the same conclusion. By Deligne's theorem [\[Voi02, Theorem 8.35\(b\)\]](#), there is a short exact sequence

$$0 \rightarrow H^0(\widehat{X}, \Omega_{\widehat{X}}^1(\log E)) \rightarrow H^1(\widehat{X} \setminus E, \mathbb{C}) \rightarrow H^1(\widehat{X}, \mathcal{O}_{\widehat{X}}) \rightarrow 0,$$

so that [\(6.3.1\)](#) and [\(6.3.2\)](#) yield the requested vanishing

$$(6.3.3) \quad H^1(X_{\text{reg}}, \mathbb{C}) = H^1(\widehat{X} \setminus E, \mathbb{C}) = 0.$$

Using the universal coefficient theorem, this implies that the finitely generated abelian group $H_1(X_{\text{reg}}, \mathbb{Z})$ has rank zero, hence is finite.

Let us now move on to the second part of the lemma. Without loss of generality, one can assume that ρ is surjective, i.e. $\rho(\pi_1(X_{\text{reg}})) = G$. Under the assumptions, there exists a finite index normal subgroup $H \trianglelefteq \pi_1(X_{\text{reg}})$ such that $\rho(H) \subset G$ is abelian. The group H is realized as the fundamental group of a Kähler manifold Y° equipped with an étale cover $f^\circ: Y^\circ \rightarrow X_{\text{reg}}$, i.e. $f_*^\circ(\pi_1(Y^\circ)) = H$. One can extend f° to a quasi-étale cover $f: Y \rightarrow X$ with Y a compact Kähler space with klt singularities. The inclusion $Y^\circ \subset Y_{\text{reg}}$ induces an isomorphism $\pi_1(Y^\circ) \cong \pi_1(Y_{\text{reg}})$ since $Y_{\text{reg}} \setminus Y^\circ \subset f^{-1}(X_{\text{sg}})$ has codimension at least two in Y_{reg} .

Now, one has $q(Y) = \tilde{q}(X) = 0$, hence $H^1(Y_{\text{reg}}, \mathbb{Z})$ is finite by the first part of the lemma. Since the image of the representation $\tau := \rho \circ f_*^\circ$ of $\pi_1(Y^\circ) \cong \pi_1(Y_{\text{reg}})$ is abelian, it factors through $H^1(Y_{\text{reg}}, \mathbb{Z})$ and is therefore finite. The following diagram visualizes the argument.

$$\begin{array}{ccc} \pi_1(Y^\circ) & & \\ \downarrow f_*^\circ & \searrow \tau & \\ \pi_1(X_{\text{reg}}) & \xrightarrow{\rho} & G \end{array}$$

The lemma follows since $f_*^\circ(\pi_1(Y^\circ)) \trianglelefteq \pi_1(X_{\text{reg}})$ has finite index. \square

6.B. Holonomy of the non-flat factors. We fix the following setup.

Setup 6.4. Let X be a n -dimensional complex, compact Kähler space with canonical singularities and trivial canonical sheaf, $K_X \sim 0$. Next, we fix a Kähler class $\alpha \in H^2(X, \mathbb{R})$ and consider the unique singular Ricci-flat metric $\omega_\alpha \in \alpha$ as in [Setup 3.1](#).

It follows from [\[Gue16, Theorem A\]](#) (see also [\[GGK19, Theorem 8.1\]](#), where the assumption on the projectivity of X is not used, cf. [Remark 3.5](#)) that one can

decompose the tangent sheaf as

$$(6.4.1) \quad \mathcal{T}_X = \bigoplus_{i \in I} \mathcal{E}_i$$

where the \mathcal{E}_i are stable bundles with slope zero (with respect to α) and are such that $\mathcal{E}_i|_{X_{\text{reg}}}$ is a parallel subbundle of $\mathcal{T}_{X_{\text{reg}}}$ with respect to ω_α . We set $n_i := \text{rk}(\mathcal{E}_i)$. The holonomy group $G := \text{Hol}(\mathcal{T}_{X_{\text{reg}}}, \omega_\alpha)$ can be decomposed as a product $G = \prod_i G_i$ where $G_i = \text{Hol}(\mathcal{E}_i|_{X_{\text{reg}}}, \omega_\alpha)$ is irreducible.

By [GGK19, Proposition 7.3], there exists a quasi-étale cover $f: Y \rightarrow X$ such that the decomposition induced by the standard representation of the holonomy group $G_Y := \text{Hol}(\mathcal{T}_{Y_{\text{reg}}}, f^*\omega_\alpha)$ is a refinement of the one of the identity component G_Y° . Note that the construction of this so-called *weak holonomy cover* above is independent of the projectivity assumption in [GGK19]. From now on, we will replace X by Y and work with the following

Additional Assumption 6.5. The restricted holonomy G_i° of each factor \mathcal{E}_i in (6.4.1) is either trivial or irreducible.

By [GGK19, Proposition 5.3], there are only three possibilities for G_i° : one has

$$G_i^\circ = \begin{cases} \{1\} & \text{or} \\ \text{SU}(n_i), & n_i \geq 3, & \text{or} \\ \text{Sp}(n_i/2), & n_i \geq 2 \text{ even.} \end{cases}$$

We set $J := \{i \in I \mid G_i^\circ = \{1\}\}$, $K := I \setminus J$ and $\mathcal{F} := \bigoplus_{j \in J} \mathcal{E}_j$. In other words, one has

$$(6.5.1) \quad \mathcal{T}_X = \mathcal{F} \oplus \bigoplus_{k \in K} \mathcal{E}_k$$

with $\text{Hol}^\circ(\mathcal{F}|_{X_{\text{reg}}}, \omega_\alpha) = \{1\}$ while the full holonomy group $G_k = \text{Hol}(\mathcal{E}_k|_{X_{\text{reg}}}, \omega_\alpha)$ satisfies $G_k^\circ = \text{SU}(n_k)$ or $G_k^\circ = \text{Sp}(n_k/2)$. In either case, one has

$$(6.5.2) \quad G_k/G_k^\circ \subset \text{U}(1)$$

by the proof of [GGK19, Lemma 7.8]. Consider a torus cover $\gamma: T \times Z \rightarrow X$; then $\text{pr}_T^* \mathcal{T}_T$ is a direct summand of $\gamma^{[*]} \mathcal{F}$, so that $\gamma^{[*]} \mathcal{E}_k$ is canonically identified with a direct summand of $\text{pr}_Z^* \mathcal{T}_Z$. Hence one can realize G_k as an irreducible factor of the holonomy group of Z_{reg} . Since $\tilde{q}(Z) = 0$, the canonical surjection

$$\rho: \pi_1(Z_{\text{reg}}) \rightarrow G_k/G_k^\circ$$

combined with Lemma 6.1 leads to the following

Proposition 6.6. *For any $k \in K$, the group G_k/G_k° is finite.* \square

Corollary 6.7. *In Setup 6.4 and up to performing a quasi-étale cover, the full holonomy group of a non-flat, irreducible factor \mathcal{E}_k in the decomposition (6.5.1) is either $\text{SU}(n_k)$ or $\text{Sp}(n_k/2)$.* \square

We can now move on to the

Proof of Theorem C. Starting from Setup 6.4, we pass to an additional cover of X so that the conclusion of Corollary 6.7 holds.

First item. Recall that the sheaves \mathcal{E}_i from (6.4.1) are such that $\mathcal{E}_i|_{X_{\text{reg}}}$ are parallel with respect to ω . Since the Chern connection D is torsion-free, we have $[u, v] = D_u v - D_v u$ for any local smooth vector fields u, v on X_{reg} . Applying this formula to local sections of $\mathcal{E}_i|_{X_{\text{reg}}}$ and keeping in mind that the latter subbundle

is parallel, we see that $\mathcal{E}_i|_{X_{\text{reg}}}$ is preserved under Lie bracket. Moreover, \mathcal{E}_i is saturated in \mathcal{T}_X , being a direct summand of \mathcal{T}_X . Combining those two facts, we get that the image of the Lie bracket $[-, -]: \bigwedge^2 \mathcal{E}_i \rightarrow \mathcal{T}_X/\mathcal{E}_i$ is a torsion-free sheaf supported on X_{sg} , hence it vanishes. This shows that \mathcal{E}_i defines indeed a foliation on X . Although a direct sum of foliations needs not be integrable in general, the argument above still applies and shows that $\mathcal{F} = \bigoplus_{j \in J} \mathcal{E}_j$ is a foliation as well.

Next, we know that K_X is a trivial line bundle and that for any $k \in K$, one has $\det \mathcal{E}_k \cong \mathcal{O}_X$ since the holonomy of \mathcal{E}_k lies in $\text{SU}(n_k)$ by [Corollary 6.7](#). This implies that $\det \mathcal{F} \cong \mathcal{O}_X$.

Second item. This is clear from the definition of \mathcal{F} and the first item.

Third item. The statement on the holonomy groups is contained in [Corollary 6.7](#). Now, we claim that \mathcal{E}_k is strongly stable with respect to α . Since restricted holonomy does not change after passing to a quasi-étale cover, it is enough to show that \mathcal{E}_k is stable with respect to α . But if it were not the case, any saturated destabilizing subsheaf would be parallel with respect to ω_α on X_{reg} by Bochner's principle for bundles, cf. [Remark 3.5](#). Therefore, the group $\text{Hol}(\mathcal{E}_k|_{X_{\text{reg}}}, \omega_\alpha)$ would not act irreducibly on \mathbb{C}^{n_k} , which is a contradiction.

Now, let $\beta \in H^2(X, \mathbb{R})$ be a Kähler class and let us show that \mathcal{E}_k is strongly stable with respect to β . As before, it is enough to show stability. Let $\omega_\beta \in \beta$ be the singular Ricci-flat metric. Since $c_1(\mathcal{E}_k) = 0$, a (saturated) β -destabilizing subsheaf $0 \neq \mathcal{G} \subsetneq \mathcal{E}_k$ satisfies $c_1(\mathcal{G}) \cdot \beta^{n-1} = 0$. By Bochner's principle, $\mathcal{G}|_{X_{\text{reg}}}$ is parallel with respect to ω_β and we have a holomorphic splitting $\mathcal{T}_X = \mathcal{G} \oplus \mathcal{G}^\perp$ over X_{reg} which extends over X by reflexivity. Here, \perp is meant with respect to ω_β . Since \mathcal{T}_X is semistable with respect to α , we have necessarily $c_1(\mathcal{G}) \cdot \alpha^{n-1} = 0$. This contradicts the α -stability of \mathcal{E}_k established earlier.

Fourth item. Assume that $X = T \times Z$, where T is a complex torus and $\tilde{q}(Z) = 0$. We need to show that the decomposition $\mathcal{T}_X = \text{pr}_T^* \mathcal{T}_T \oplus \text{pr}_Z^* \mathcal{T}_Z$ is such that $\text{pr}_T^* \mathcal{T}_T$ is a direct summand of \mathcal{F} . The Bochner principle for bundles (cf. [Remark 3.5](#)) shows that over $X_{\text{reg}} = T \times Z_{\text{reg}}$, both summands are parallel subbundles and, in particular, ω can be decomposed as $\omega_\alpha = \text{pr}_T^* \omega_T \oplus \text{pr}_Z^* \omega_Z$ where ω_T (resp. ω_Z) is a Kähler Ricci-flat metric on T (resp. Z_{reg}). Since a Ricci-flat Kähler metric on a torus is necessarily flat, we have $\text{Hol}^\circ(\text{pr}_T^* \mathcal{T}_T|_{X_{\text{reg}}}, \omega_\alpha) = \text{Hol}^\circ(T, \omega_T) = \{1\}$. The first part of the statement now follows easily. The second part follows from [Proposition 6.9](#) below. \square

6.C. On the flat factor. Since \mathcal{T}_T is a direct summand of \mathcal{F} , it is clear that $\tilde{q}(X) \leq \text{rk } \mathcal{F}$. We conjecture that equality always holds. In particular, if $\tilde{q}(X) = 0$ then the flat factor \mathcal{F} in the decomposition [\(6.5.1\)](#) should be zero. In the projective case, the conjecture was established by [\[GGK19, Corollary 7.2\]](#) as a consequence of Druel's algebraic integrability result for flat sheaves [\[Dru18\]](#). We are able to prove two partial results in this direction:

- (6.7.1) If $\text{rk } \mathcal{F} = n := \dim X$, then also $\tilde{q}(X) = n$. This is a direct consequence of [Proposition 6.8](#) below, since $\text{rk } \mathcal{F} = n$ means that $\mathcal{T}_{X_{\text{reg}}}$ is flat and torus quotients obviously have $\tilde{q}(X) = n$.
- (6.7.2) The conjecture can be derived from the complex space version of [\[GKP16a\]](#), cf. [Proposition 6.9](#) and [Remark 6.10](#).

Proposition 6.8. *Let X be a normal compact Kähler space that has only klt singularities. If $\mathcal{T}_{X_{\text{reg}}}$ is flat, then X is a torus quotient.*

Proof. According to Theorem 5.1, X has a smooth quasi-étale cover \tilde{X} . This compact space \tilde{X} is still Kähler [Var89, Corollary 3.2.2] and, by construction, the tangent bundle of \tilde{X} is flat. We can then apply the classical characterization of torus quotients derived from Yau's theorem [Yau78] and the uniformization theorem, cf. e.g. [Tia00, Theorem 2.13], to conclude that \tilde{X} is a torus quotient. Hence, so is X . \square

Proposition 6.9. *In Setup 6.4, assume that X admits a maximally quasi-étale cover. Then the equality $\mathrm{rk} \mathcal{F} = \tilde{q}(X)$ holds.*

Proof. After replacing X with a maximally quasi-étale cover, we can assume that $\hat{\pi}_1(X_{\mathrm{reg}}) \cong \hat{\pi}_1(X)$. We may furthermore replace X by a torus cover $T \times Z \rightarrow X$. It is then sufficient to show that \mathcal{T}_Z has no flat factor. In other words, we may replace X by Z and we need to show that $\tilde{q}(X) = 0$ implies $\mathrm{rk} \mathcal{F} = 0$.

The flat factor \mathcal{F} is given by a finite dimensional representation $\rho: \pi_1(X_{\mathrm{reg}}) \rightarrow \mathrm{SU}(r)$, with $r := \mathrm{rk} \mathcal{F}$. This representation factors thus through the fundamental group of X and it yields $\rho_X: \pi_1(X) \rightarrow \mathrm{SU}(r)$. Let us consider $\hat{X} \rightarrow X$ a desingularization of X : this compact Kähler manifold has vanishing Kodaira dimension and is thus special (in the sense of [Cam04b]). Since X has canonical singularities, we know that its fundamental group is unchanged² when passing to a smooth model by [Tak03, Theorem 1.1] hence we can interpret the representation ρ_X as a representation $\rho_{\hat{X}}: \pi_1(\hat{X}) \rightarrow \mathrm{SU}(r)$. The manifold \hat{X} being special, we know that none of its étale covers dominate a variety of general type and [CCE15, Theorem 6.5] implies that the image of $\rho_{\hat{X}}$ is virtually abelian and, in particular, $\mathrm{im} \rho$ is virtually abelian as well. By Lemma 6.1, ρ has finite image. This means that one can construct a quasi-étale cover $f: Y \rightarrow X$ such that the direct summand $f^{[*]} \mathcal{F}$ of \mathcal{T}_Y satisfies $f^{[*]} \mathcal{F} \cong \mathcal{O}_Y^{\oplus r}$. As $q(Y) = 0$, this can only occur if $r = 0$. \square

Remark 6.10. The existence of maximally quasi-étale covers should be true for any compact complex space with klt singularities (or even for Zariski open subsets of such spaces). In [GKP16a], the algebraicity assumption is mainly used when appealing to [Xu14] where it is shown that $\hat{\pi}_1(\mathrm{Link}(X, x))$ is finite for (X, x) klt. In that article, it is proven that it is possible to extract a *Kollár component*, i.e. a birational morphism $\mu: Y \rightarrow X$ such that the fiber $\mu^{-1}(x)$ has a natural structure of \mathbb{Q} -Fano variety. This is achieved by considering a (projective) desingularization $\hat{\mu}: \hat{X} \rightarrow X$ and running a well-chosen MMP $\hat{X} \dashrightarrow Y$ over X . It is thus an urgent task to generalize the known results [BCHM10] to obtain a relative MMP for projective morphisms between normal complex spaces.

6.D. Varieties with strongly stable tangent sheaf. The following definition originates in the projective setting from [GKP16b].

Definition 6.11 (CY and IHS varieties). Let X be a compact Kähler space of dimension $n \geq 2$ with canonical singularities and $\omega_X \cong \mathcal{O}_X$.

(6.11.1) We call X *Calabi–Yau (CY)* if $H^0(Y, \Omega_Y^{[p]}) = 0$ for all integers $0 < p < n$ and all quasi-étale covers $Y \rightarrow X$, in particular for X itself.

(6.11.2) We call X *irreducible holomorphic symplectic (IHS)* if there exists a holomorphic symplectic two-form $\sigma \in H^0(X, \Omega_X^{[2]})$ such that for all quasi-étale

²[Tak03, Theorem 1.1] is only stated for projective varieties but it holds in the complex analytic setting, its proof being completely local (in the Euclidean topology). It can also be noted that the corresponding result for the étale fundamental group is [Kol93, Theorem 7.5] and that this result is explicitly stated for normal analytic spaces.

covers $\gamma: Y \rightarrow X$, the exterior algebra of global reflexive differential forms is generated by $\gamma^{[*]}\sigma$.

Definition 6.12 (Strong stability). Let X be a compact Kähler space of dimension $n \geq 2$ with klt singularities and trivial first Chern class. We say that \mathcal{T}_X is *strongly stable* if for any quasi-étale cover $f: Y \rightarrow X$, the tangent sheaf \mathcal{T}_Y is stable with respect to any Kähler class $a \in H^2(Y, \mathbb{R})$.

The following lemma shows that the notion of (strong) stability can be tested on a single Kähler class.

Lemma 6.13 (Independence of polarization). *Let X be a compact Kähler space with dimension $n \geq 2$ with klt singularities and trivial first Chern class. If \mathcal{T}_X is stable with respect to a given Kähler class, then it is stable with respect to any Kähler class.*

*In particular, \mathcal{T}_X is strongly stable if and only if there exists a Kähler class $a \in H^2(X, \mathbb{R})$ such that for any quasi-étale cover $f: Y \rightarrow X$, the tangent sheaf \mathcal{T}_Y is stable with respect to f^*a .*

Proof. Assume that \mathcal{T}_X is stable with respect to a Kähler class a but not stable with respect to another Kähler class a' . By polystability of \mathcal{T}_X with respect to the class a' , one can decompose $\mathcal{T}_X = \mathcal{F}_1 \oplus \mathcal{F}_2$ where \mathcal{F}_i have zero slope with respect to a' . Since $0 = c_1(\mathcal{T}_X) = c_1(\mathcal{F}_1) + c_1(\mathcal{F}_2)$, there exists $i \in \{1, 2\}$ such that $c_1(\mathcal{F}_i) \cdot a^{n-1} \geq 0$. Such a subsheaf $\mathcal{F}_i \subset \mathcal{T}_X$ then destabilizes \mathcal{T}_X with respect to a , contradiction. The lemma is proved. \square

Putting together [Theorem 3.4](#), [Corollary 6.7](#) and [Proposition 6.8](#), we get the following result.

Corollary 6.14 (Spaces with strongly stable tangent sheaf). *Let X be a compact Kähler space with klt singularities and trivial first Chern class of dimension $n \geq 2$. If \mathcal{T}_X is strongly stable, then X admits a quasi-étale cover that is either a CY or an IHS variety.*

Proof. By the Abundance Conjecture (cf. [Remark 3.3](#)), one can assume that X has trivial canonical bundle and canonical singularities. Given that \mathcal{T}_X is strongly stable, the decomposition [\(6.5.1\)](#) on a cover $Y \rightarrow X$ reduces to a single factor. If that factor is the flat factor \mathcal{F} , then X is a torus quotient by [Proposition 6.8](#), which is impossible since \mathcal{T}_X is strongly stable and $\dim X \geq 2$. Therefore, we can apply [Corollary 6.7](#) to see that the holonomy of Y_{reg} is either $\text{SU}(n)$ or $\text{Sp}(n/2)$. By standard results of representation theory of the latter groups, the statement follows from the Bochner principle, i.e. [Theorem 3.4](#). \square

Remark 6.15. For the proof of [Corollary 6.14](#) above, we do not need the full Bochner principle since we only need to understand reflexive differential forms, cf. [Remark 3.7](#). The Bochner principle for more general tensors will however be applied in the proof of [Proposition 7.1](#).

7. FUNDAMENTAL GROUPS

This section is devoted to obtaining consequences about the fundamental groups of X and of X_{reg} , where X has vanishing first Chern class. In a first step, we provide a sufficient criterion for $\pi_1(X)$ to be finite. This is the analogue of [[GKP16b](#), Proposition 8.23] for Kähler spaces, and as in the projective case the proof is an application of [[Cam95](#), Corollary 5.3]. However, instead of using Miyaoka's Generic Semipositivity Theorem, we rely on the Bochner principle, [Theorem 3.4](#). This idea has already appeared in the proof of [[Cam20](#), Theorem 3.5] (for X projective).

Proposition 7.1 (Finiteness criterion for π_1). *Let X be a compact Kähler space of dimension $n \geq 1$ with klt singularities and $c_1(X) = 0$. Assume moreover that $\chi(X, \mathcal{O}_X) \neq 0$. Then $\pi_1(X)$ is finite, of cardinality*

$$|\pi_1(X)| \leq \frac{2^{n-1}}{|\chi(X, \mathcal{O}_X)|}.$$

In terms of spaces with strongly stable tangent sheaf (Corollary 6.14), this criterion already applies to all even-dimensional spaces.

Theorem 7.2 (Fundamental groups in even dimensions). *Let X be a compact Kähler space with klt singularities and $c_1(X) = 0$.*

(7.2.1) *If $\dim X$ is even and \mathcal{T}_X is strongly stable, then $\pi_1(X)$ is finite.*

(7.2.2) *If X is IHS or an even-dimensional CY, then X is simply connected.*

For odd-dimensional CYs, we have no results about $\pi_1(X)$ itself, but only about its representation theory. These results are actually valid for all spaces with vanishing augmented irregularity. In addition, assuming the existence of maximally quasi-étale covers we also obtain information about $\pi_1(X_{\text{reg}})$.

Theorem 7.3 (Fundamental groups if $\tilde{q} = 0$). *Let X be a compact Kähler space with klt singularities, $c_1(X) = 0$ and $\tilde{q}(X) = 0$. Then:*

(7.3.1) *Any complex linear representation $\pi_1(X) \rightarrow \text{GL}(r, \mathbb{C})$ has finite image.*

Assume moreover that X admits a maximally quasi-étale cover. Then:

(7.3.2) *For each $r \in \mathbb{N}$, there are only finitely many r -dimensional complex linear representations of $\pi_1(X)$, up to conjugation.*

(7.3.3) *The above claims (7.3.1) and (7.3.2) remain true if one replaces $\pi_1(X)$ by $\pi_1(X_{\text{reg}})$, the fundamental group of the smooth locus of X .*

Restricting ourselves to dimension four, we have a completely unconditional result:

Theorem 7.4 (Fundamental groups in dimension four, I). *Let X be a compact Kähler space of dimension ≤ 4 with klt singularities and $c_1(X) = 0$. Then:*

(7.4.1) *$\pi_1(X)$ is virtually abelian, i.e. it contains an abelian (normal) subgroup $\Gamma \trianglelefteq \pi_1(X)$ of finite index.*

(7.4.2) *All finite index abelian subgroups of $\pi_1(X)$ have even rank at most $2\tilde{q}(X) \leq 8$. In particular, if $\tilde{q}(X) = 0$ then $\pi_1(X)$ is finite.*

This in turn has the following consequence for Campana's Abelianity Conjecture.

Corollary 7.5 (Fundamental groups in dimension four, II). *Let X be a compact Kähler manifold of dimension 4 and Kodaira dimension zero, admitting a good minimal model. Then $\pi_1(X)$ is virtually abelian.*

7.A. Non-vanishing Euler characteristic. Recall the following notion, which was introduced by Campana.

Definition 7.6 (Refined Kodaira dimension). Let X be a normal compact complex space. The *refined Kodaira dimension* $\kappa^+(X)$ of X is defined to be

$$\kappa^+(X) := \max \left\{ \kappa(\det \mathcal{F}) \mid \exists p: \mathcal{F} \subset \Omega_X^{[p]} \text{ coherent} \right\} \in \{-\infty, 0, \dots, \dim X\},$$

where $\det \mathcal{F} := (\bigwedge^r \mathcal{F})^{\vee\vee}$ with $r = \text{rk } \mathcal{F}$.

The proof of the following lemma is standard, and hence omitted.

Lemma 7.7. *Let $f: Y \rightarrow X$ be a proper bimeromorphic map of normal compact complex spaces. Then $\kappa^+(Y) \leq \kappa^+(X)$. \square*

Proof of Proposition 7.1. Let $f: Y \rightarrow X$ be a projective resolution of singularities. Then $\chi(Y, \mathcal{O}_Y) = \chi(X, \mathcal{O}_X) \neq 0$ because X has rational singularities. Under this assumption, Campana has shown in [Cam95, Corollary 5.1, 5.3] that $\pi_1(Y)$ is finite, of cardinality at most $2^{n-1} \cdot |\chi(Y, \mathcal{O}_Y)|^{-1}$ if we additionally have $\kappa^+(Y) \leq 0$. As f has connected fibers, the natural map $\pi_1(Y) \rightarrow \pi_1(X)$ is surjective, so the finiteness statement (and the bound) for $\pi_1(Y)$ will imply the sought result for $\pi_1(X)$. By Lemma 7.7, it is therefore sufficient to show that $\kappa^+(X) \leq 0$.

So let $0 \leq p \leq n$ be any integer and let $\mathcal{F} \subset \Omega_X^{[p]}$ be a coherent subsheaf, say of rank $r \geq 1$. As a subsheaf of a torsion-free sheaf, \mathcal{F} itself is torsion-free and therefore locally free in codimension one. Recall that $\Omega_X^{[p]} \subset ((\Omega_X^1)^{\otimes p})^{\vee\vee}$. Hence outside a closed subvariety of codimension ≥ 2 , we have an inclusion of locally free sheaves

$$\det \mathcal{F} \subset (\mathcal{F}^{\otimes r})^{\vee\vee} \subset ((\Omega_X^1)^{\otimes pr})^{\vee\vee},$$

which remains valid on X by Hartogs' theorem. This gives an inclusion

$$(\det \mathcal{F})^{[m]} \subset ((\Omega_X^1)^{\otimes mpr})^{\vee\vee}$$

for every $m \geq 1$. Hence by the Bochner principle, Theorem 3.4, any global section of $(\det \mathcal{F})^{[m]}$ is parallel with respect to the Ricci-flat Kähler metric ω_α on X_{reg} . Thus evaluating sections at a point $x \in X_{\text{reg}}$ gives an inclusion

$$H^0(X, (\det \mathcal{F})^{[m]}) \hookrightarrow (\det \mathcal{F})_x^{\otimes m}.$$

Since the right-hand side is one-dimensional, we infer that $\kappa(\det \mathcal{F}) \leq 0$. \square

7.B. Even and odd dimensions. Theorem 7.2 is a more or less direct consequence of Proposition 7.1, hence the proof is omitted. The incredulous reader wishing to verify the details is advised to digest [GGK19, Theorem 13.1 and Corollary 13.3].

Proof of Theorem 7.3. We may assume that X has canonical singularities. Let $\tilde{X} \rightarrow X$ be a projective resolution of singularities. Then \tilde{X} has Kodaira dimension zero, hence is special in the sense of Campana and thus every complex linear representation of $\pi_1(\tilde{X}) = \pi_1(X)$ has virtually abelian image (cf. the proof of Proposition 6.9). By Lemma 6.1 and the assumption $\tilde{q}(X) = 0$, every such representation actually has finite image, proving (7.3.1).

Assume that X admits a maximally quasi-étale cover. Then (after a cover), the full holonomy group of X is a product of $\text{SU}(n_k)$ and $\text{Sp}(n_k/2)$, for various values of n_k . It is well-known that these groups do not have any nonzero polynomial invariants, see e.g. [GGK19, Lemma 11.6]. By the Bochner principle, Theorem 3.4, we conclude that $H^0(\tilde{X}, \text{Sym}^m \Omega_{\tilde{X}}^1) = 0$ for all $m > 0$. We can deduce (7.3.2) from this in the same way as in the proof of [GGK19, (13.6.2)].

The $\pi_1(X_{\text{reg}})$ version of (7.3.1) can be derived from (7.3.1) as in the proof of [GGK19, (13.10.1)], because we assume the existence of a maximally quasi-étale cover. Ditto for (7.3.2), using the argument of [GGK19, (13.10.2)]. \square

7.C. Dimension four. The statements remaining to be proven concern Kähler fourfolds.

Proof of Theorem 7.4. Recall from [Gra18, Corollary 1.8] that all statements are well-known if $\dim X \leq 3$. We can therefore assume for the remainder that X is of dimension four. Let $\gamma: T \times Z \rightarrow X$ be a torus cover (Corollary 4.2). By e.g. [Cam91, Proposition 1.3], the image of $\pi_1(T \times Z) = \pi_1(T) \times \pi_1(Z)$ in $\pi_1(X)$ has finite index, and $\pi_1(T)$ is free abelian of rank $2\tilde{q}(X)$. It is therefore sufficient to show that $\pi_1(Z)$ is finite.

If $\dim Z \leq 3$, this holds by [Gra18, Corollary 1.8] again. It remains to consider the case where $\dim Z = 4$, i.e. where $X = Z$ and $\omega_X \cong \mathcal{O}_X$ and $\tilde{q}(X) = 0$. By Remark 2.2, the last property implies $H^1(X, \mathcal{O}_X) = 0$ and then also $H^3(X, \mathcal{O}_X) \cong H^3(X, \omega_X) = H^1(X, \mathcal{O}_X)^\vee = 0$ by Serre duality [BS76, Chapter VII, Theorem 3.10]. So

$$\chi(X, \mathcal{O}_X) = 1 + h^2(X, \mathcal{O}_X) + \underbrace{h^4(X, \mathcal{O}_X)}_{=1} \geq 2.$$

By Proposition 7.1, $\pi_1(X)$ is finite (of cardinality at most 4). This settles (7.4.1).

We have already exhibited a finite index abelian subgroup of $\pi_1(X)$ of rank at most $2\tilde{q}(X)$, namely the image of $\pi_1(T) \rightarrow \pi_1(X)$. It is well-known that all such subgroups have the same rank. Claim (7.4.2) follows easily. \square

Proof of Corollary 7.5. By [Tak03], the fundamental group is unchanged when passing to a minimal model $X \dashrightarrow X_{\min}$. We can apply Theorem 7.4 to X_{\min} , since a good minimal model in Kodaira dimension zero satisfies $c_1(X_{\min}) = 0$. \square

8. KODAIRA PROBLEM AND DECOMPOSITION THEOREM

In this last section, we apply the results obtained so far to the study of the Kodaira problem, i.e. the question of finding small deformations of a given Kähler space that are algebraic varieties. Since most of our results only work after performing a finite cover, we introduce the following notation.

Notation 8.1. Let X be a normal compact complex space. We say that X is *approximable up to a cover* if there exists a quasi-étale Galois cover $\tilde{X} \rightarrow X$ such that \tilde{X} admits a strong locally trivial algebraic approximation.

We also fix the following setup:

Setup 8.2. Let X be a compact Kähler space with klt singularities and trivial first Chern class. Let $\gamma: T \times Z \rightarrow X$ be a torus cover of X , as given by Corollary 4.2.

A closely related question is whether X admits a BB (= Beauville–Bogomolov) decomposition, that is, whether the splitting (6.5.1) of \mathcal{T}_X actually comes from a splitting of X . We prove that having the BB decomposition settles the Kodaira problem in the above sense. Of course, this applies in particular if \mathcal{T}_X itself is already strongly stable.

Theorem 8.3 (BB decomposition implies Kodaira problem). *Assume that Z decomposes (possibly after a further quasi-étale cover) into a product $\prod_i Z_i$, where each Z_i has strongly stable tangent sheaf. Then X is approximable up to a cover.*

In dimension four, we obtain a much stronger statement: assuming only that X has algebraic singularities, we can solve the Kodaira problem for X and this in turn implies the BB decomposition.

Theorem 8.4 (Kodaira problem in dimension four). *Assume that $\dim X \leq 4$ and X has algebraic singularities (or more generally, assume that X admits a maximally quasi-étale cover). Then:*

(8.4.1) *X is approximable up to a cover.*

(8.4.2) *X admits a Beauville–Bogomolov decomposition.*

In the special case of isolated singularities, we have the following more precise structural result.

Proposition 8.5 (Isolated singularities). *In Setup 8.2, assume additionally that X has only isolated singularities and that X is approximable up to a cover. Then either X admits a smooth quasi-étale cover or \mathcal{T}_X is strongly stable in the sense of Definition 6.12.*

Remark 8.6. As a corollary of [Theorem 8.4](#), the conclusion of [Proposition 8.5](#) above holds as soon as $\dim X \leq 4$ and X has isolated singularities.

8.A. A sufficient criterion. All results in this section are based on the following criterion for approximability.

Proposition 8.7 (Approximation criterion). *In [Setup 8.2](#), assume that for some smooth subgerm $S \subset \text{Def}^{\text{lt}}(Z)$ and some Kähler class $[\omega_Z] \in \mathbf{H}^1(Z, \Omega_Z^1)$ on Z , the map*

$$T_0S \xrightarrow{\kappa} \mathbf{H}^1(Z, \mathcal{T}_Z) \xrightarrow{[\omega_Z] \cup -} \mathbf{H}^2(Z, \mathcal{O}_Z)$$

is surjective. Here κ is the restriction of the Kodaira–Spencer map of Z . Then X is approximable up to a cover.

Proof. For any complex torus T , the semiuniversal deformation space $\text{Def}(T)$ is a smooth germ [[Voi03](#), Chapter 5, Example 1]. (Of course, this also follows from the much more general Bogomolov–Tian–Todorov theorem, since T is smooth.) Hence also $B := \text{Def}(T) \times S$ is smooth, with tangent space

$$T_0B = \mathbf{H}^1(T, \mathcal{T}_T) \oplus T_0S.$$

Consider a Kähler class $[\omega_T]$ on T and the induced Kähler class $[\omega] := \text{pr}_T^*[\omega_T] + \text{pr}_Z^*[\omega_Z]$ on $T \times Z$, where pr_T and pr_Z are the projections. We know that $[\omega_T] \cup - : \mathbf{H}^1(T, \mathcal{T}_T) \rightarrow \mathbf{H}^2(T, \mathcal{O}_T)$ is surjective [[Voi03](#), Chapter 5, Example 1]. Also, since $q(Z) = 0$ the Künneth formula yields

$$(8.7.1) \quad \mathbf{H}^2(T \times Z, \mathcal{O}_{T \times Z}) = \mathbf{H}^2(T, \mathcal{O}_T) \oplus \mathbf{H}^2(Z, \mathcal{O}_Z).$$

Consider now the product deformation of $T \times Z$ over B , i.e. the fibre over a point $(t, s) \in B$ is $T_t \times Z_s$. Letting κ be Kodaira–Spencer map of this deformation, from (8.7.1) we see that

$$T_0B \xrightarrow{\kappa} \mathbf{H}^1(T \times Z, \mathcal{T}_{T \times Z}) \xrightarrow{[\omega] \cup -} \mathbf{H}^2(T \times Z, \mathcal{O}_{T \times Z})$$

is surjective. By [[GS20](#), Theorem 3.1], it follows that $T \times Z$ has a locally trivial algebraic approximation. We conclude by taking the Galois closure of γ , using [Lemma 8.8](#) below. \square

Lemma 8.8. *Let X be a normal compact complex space and $\gamma: \tilde{X} \rightarrow X$ a quasi-étale cover. If X admits a locally trivial algebraic approximation, then so does \tilde{X} .*

Proof. Let $\mathfrak{X} \rightarrow S$ be a locally trivial algebraic approximation of X over a smooth germ S . Fix a Whitney stratification \mathcal{S} of X such that the open stratum equals X_{reg} . By Thom’s first isotopy lemma, \mathfrak{X} is homeomorphic to $X \times S$ over S via a stratum-preserving homeomorphism [[GM88](#), Part I, Theorem in §1.5]. In particular, $\mathfrak{X}_{\text{reg}}$ is homeomorphic to $X_{\text{reg}} \times S$. Since S is contractible, this implies that $\pi_1(\mathfrak{X}_{\text{reg}}) \cong \pi_1(X_{\text{reg}})$. Using the classification of étale covers via (conjugacy classes of) finite index subgroups of the fundamental group, we see that γ extends to an étale cover of $\mathfrak{X}_{\text{reg}}$, and then also to a quasi-étale cover $\tilde{\mathfrak{X}} \rightarrow \mathfrak{X}$ of \mathfrak{X} itself. The composed map $\tilde{\mathfrak{X}} \rightarrow S$ is a locally trivial deformation of its central fibre, which of course is \tilde{X} . \square

8.B. Proof of [Theorem 8.3](#). Consider any factor Z_i . We will define a smooth subgerm $S_i \subset \text{Def}^{\text{lt}}(Z_i)$. By [Corollary 6.14](#), we may assume that Z_i is either CY or IHS.

- If Z_i is CY, we set $S_i = \{0\}$.
- If Z_i is IHS, we set $S_i = \text{Def}^{\text{lt}}(Z_i)$. This makes sense because $\text{Def}^{\text{lt}}(Z_i)$ is smooth by [[BL20](#), Theorem 4.7].

We now set

$$S := \prod_i S_i \subset \prod_i \text{Def}^{\text{lt}}(Z_i) \subset \text{Def}^{\text{lt}}(Z)$$

and we take a Kähler class $[\omega_Z]$ on Z of the form $\sum_i \text{pr}_i^*[\omega_{Z_i}]$, where the $[\omega_{Z_i}]$ are Kähler classes on Z_i . By the Künneth formula,

$$\mathrm{H}^2(Z, \mathcal{O}_Z) = \bigoplus_i \mathrm{H}^2(Z_i, \mathcal{O}_{Z_i}) = \bigoplus_{Z_i \text{ IHS}} \mathrm{H}^2(Z_i, \mathcal{O}_{Z_i})$$

and $T_0 S_i \rightarrow \mathrm{H}^2(Z_i, \mathcal{O}_{Z_i})$ is surjective for each value of i , by [GS20, Theorem 4.1]. It follows that we can apply Proposition 8.7 with our choice of S and $[\omega_Z]$. \square

8.C. Proof of Theorem 8.4. By [Gra18, Theorem 1.4], we may assume that $\dim X = 4$ and $\tilde{q}(X) = 0$. After performing a quasi-étale cover, consider the decomposition of \mathcal{T}_X given by (6.5.1). By Propositions 5.9 and 6.9, the flat factor \mathcal{F} is trivial. Furthermore, by Theorem 8.3 we may assume that \mathcal{T}_X is not strongly stable. The only possibility we are left with is

$$\mathcal{T}_X = \mathcal{E}_1 \oplus \mathcal{E}_2,$$

where both \mathcal{E}_i have $\mathrm{SU}(2) = \mathrm{Sp}(1)$ holonomy. Therefore X carries a holomorphic symplectic form and $\text{Def}^{\text{lt}}(X)$ is smooth by [BL20, Theorem 4.7].³ We conclude by Proposition 8.7 that X is approximable up to a cover.

Concerning the existence of a BB decomposition, let X_t be a small projective deformation of X . Clearly, X_t likewise carries a symplectic form, which is non-unique. Therefore X_t splits as $S'_1 \times S'_2$, where the S'_i are (possibly singular) projective K3 surfaces, by the BB decomposition for projective varieties [HP19]. As klt surface singularities are quotient singularities, it follows that X_t has only quotient singularities [KM98, Proposition 4.18]. Then the same is true of X , by local triviality of the deformation. But the BB decomposition for orbifolds is known by [Cam04a, Theorem 6.4]. Hence $X = S_1 \times S_2$ splits into a product of singular K3 surfaces. \square

8.D. Proof of Proposition 8.5. If a quasi-étale cover $Y \rightarrow X$ splits, then any factor of Y has to be smooth since it would otherwise yield a positive dimensional singular locus for Y , hence for X too. In particular, Y is smooth.

From now on, we assume that no quasi-étale cover of X splits. By Corollary 4.2, this implies that $\tilde{q}(X) = 0$. Since X admits a maximally quasi-étale cover by Proposition 5.9, there is no flat factor in the decomposition of \mathcal{T}_X by Proposition 6.9. Now, Corollary 6.7 implies that up to performing a quasi-étale cover $Y \rightarrow X$, the full holonomy of Y is a product of SU and Sp of maximal rank. We claim that only one factor occurs, which is the content of the corollary.

First of all, consider an algebraic approximation $Z = Y_t$ of Y . That is, Z is a projective variety with klt singularities and $c_1(Z) = 0$ arising as a fiber of a locally trivial deformation of Y . In particular, Z has non-empty isolated singularities and no quasi-étale cover of Z splits, since it would otherwise admit a smooth quasi-étale cover, hence Y would too. By [HP19], this implies that \mathcal{T}_Z is strongly stable; hence by [GGK19, Theorem E], it admits a quasi-étale cover that is either a CY variety or an IHS variety. In particular, the proof of Lemma 8.8 shows that a quasi-étale cover of Y can be deformed locally trivially to a CY or IHS variety. In such a family, the number of holomorphic p -forms is constant for any p (as one can see

³Note that strictly speaking, that result is not applicable because X is not primitive symplectic (using their terminology). Indeed, $\mathrm{H}^0(X, \Omega_X^{[2]})$ is two-dimensional and so the symplectic form is not unique. An inspection of the proof, however, reveals that primitivity is in fact not needed: all that matters is the existence of an isomorphism $\mathcal{T}_X \cong \Omega_X^{[1]}$ given by a symplectic form.

e.g. by performing a simultaneous resolution and using [KS19]). Now one can check easily that the algebra of invariants

$$A_n(G) := \left(\bigoplus_{p=0}^n \bigwedge^p \mathbb{C}^n \right)^G$$

for a group $G = \prod_{i \in I} G_i$ such that $G_i = \mathrm{SU}(n_i)$ or $G_i = \mathrm{Sp}(n_i/2)$ with $\sum_{i \in I} n_i = n$ satisfies $A_n(G) = A_n(\mathrm{SU}(n))$ (resp. $A_n(G) = A_n(\mathrm{Sp}(n/2))$) if and only if $G = \mathrm{SU}(n)$ (resp. $G = \mathrm{Sp}(n/2)$). By the Bochner principle, this implies that the holonomy of (a quasi-étale cover of) Y is either $\mathrm{SU}(n)$ or $\mathrm{Sp}(n/2)$, hence \mathcal{T}_X is strongly stable. \square

REFERENCES

- [ABC⁺96] J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo. *Fundamental Groups of Compact Kähler Manifolds*, volume 44 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1996. \uparrow 5
- [Art69] M. Artin. Algebraic approximation of structures over complete local rings. *Inst. Hautes Études Sci. Publ. Math.*, (36):23–58, 1969. \uparrow 6
- [BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23:405–468, 2010. \uparrow 25
- [Bea83] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782 (1984), 1983. \uparrow 1
- [BEG13] Sébastien Boucksom, Philippe Eyssidieux, and Vincent Guedj. Introduction. In *An introduction to the Kähler-Ricci flow*, volume 2086 of *Lecture Notes in Math.*, pages 1–6. Springer, Cham, 2013. \uparrow 10
- [BL20] Benjamin Bakker and Christian Lehn. The global moduli theory of symplectic varieties. [arXiv:1812.09748 \[math.AG\]](https://arxiv.org/abs/1812.09748), version 3, June 2020. \uparrow 30, 31
- [Bra20] Lukas Braun. The local fundamental group of a Kawamata log terminal singularity is finite. [arXiv:2004.00522 \[math.AG\]](https://arxiv.org/abs/2004.00522), version 2, May 2020. \uparrow 20
- [BS76] Constantin Bănică and Octavian Stănăşilă. *Algebraic Methods in the Global Theory of Complex Spaces*. John Wiley & Sons, 1976. \uparrow 29
- [Cam91] F. Campana. On twistor spaces of the class \mathcal{C} . *J. Differential Geom.*, 33(2):541–549, 1991. \uparrow 28
- [Cam95] Frédéric Campana. Fundamental group and positivity of cotangent bundles of compact Kähler manifolds. *J. Algebraic Geom.*, 4(3):487–502, 1995. \uparrow 26, 28
- [Cam04a] Frédéric Campana. Orbifolds à première classe de Chern nulle. In *The Fano Conference*, pages 339–351. Univ. Torino, Turin, 2004. \uparrow 31
- [Cam04b] Frédéric Campana. Orbifolds, special varieties and classification theory. *Ann. Inst. Fourier (Grenoble)*, 54(3):499–630, 2004. \uparrow 4, 14, 25
- [Cam20] Frédéric Campana. The Bogomolov–Beauville–Yau decomposition for klt projective varieties with trivial first Chern class –without tears–. [arXiv:2004.08261 \[math.AG\]](https://arxiv.org/abs/2004.08261), version 2, May 2020. \uparrow 26
- [Car57] Henri Cartan. Quotient d’un espace analytique par un groupe d’automorphismes. In *Algebraic geometry and topology*, pages 90–102. Princeton University Press, Princeton, N. J., 1957. A symposium in honor of S. Lefschetz. \uparrow 6, 7
- [CCE15] Frédéric Campana, Benoît Claudon, and Philippe Eyssidieux. Représentations linéaires des groupes kähleriens: factorisations et conjecture de Shafarevich linéaire. *Compos. Math.*, 151(2):351–376, 2015. \uparrow 25
- [CGP19] Junyan Cao, Henri Guenancia, and Mihai Păun. Variation of singular Kähler-Einstein metrics: Kodaira dimension zero. Preprint [arXiv:1908.08087](https://arxiv.org/abs/1908.08087), to appear in *J. Eur. Math. Soc.*, 2019. \uparrow 10, 15
- [Che55] Claude Chevalley. Invariants of finite groups generated by reflections. *Amer. J. Math.*, 77:778–782, 1955. \uparrow 7
- [CHL19] Benoît Claudon, Andreas Höring, and Hsueh-Yung Lin. The fundamental group of compact Kähler threefolds. *Geometry & Topology*, 23:3233–3271, 2019. \uparrow 5
- [Dem85] Jean-Pierre Demailly. Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines. *Mém. Soc. Math. France (N.S.)*, (19):124, 1985. \uparrow 10
- [DG94] G. Dethloff and H. Grauert. Seminormal complex spaces. In *Several complex variables, VII*, volume 74 of *Encyclopaedia Math. Sci.*, pages 183–220. Springer, Berlin, 1994. \uparrow 7, 21

- [Dru18] Stéphane Druel. A decomposition theorem for singular spaces with trivial canonical class of dimension at most five. *Invent. Math.*, 211(1):245–296, 2018. ↑ 2, 4, 11, 24
- [EGZ09] Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi. Singular Kähler-Einstein metrics. *J. Amer. Math. Soc.*, 22:607–639, 2009. ↑ 2, 10
- [Fuj78] Akira Fujiki. On Automorphism Groups of Compact Kähler Manifolds. *Invent. Math.*, 44:225–258, 1978. ↑ 14
- [GGK19] Daniel Greb, Henri Guenancia, and Stefan Kebekus. Klt varieties with trivial canonical class: holonomy, differential forms, and fundamental groups. *Geom. Topol.*, 23:2051–2124, 2019. ↑ 2, 4, 11, 12, 14, 15, 22, 23, 24, 28, 31
- [GK14] Patrick Graf and Sándor J. Kovács. An optimal extension theorem for 1-forms and the Lipman–Zariski Conjecture. *Documenta Math.*, 19:815–830, 2014. ↑ 4, 18, 22
- [GK20] Patrick Graf and Tim Kirschner. Finite quotients of three-dimensional complex tori. *Ann. Inst. Fourier (Grenoble)*, 70(2):881–914, 2020. ↑ 7, 10, 17
- [GKKP11] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell. Differential forms on log canonical spaces. *Publications Mathématiques de L’IHÉS*, 114:1–83, 2011. ↑ 2
- [GKP16a] Daniel Greb, Stefan Kebekus, and Thomas Peternell. Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties. *Duke Math. J.*, 165(10):1965–2004, 2016. ↑ 2, 4, 9, 11, 18, 19, 24, 25
- [GKP16b] Daniel Greb, Stefan Kebekus, and Thomas Peternell. Singular spaces with trivial canonical class. In *Minimal Models and Extremal Rays, Kyoto, 2011*, volume 70 of *Adv. Stud. Pure Math.*, pages 67–113. Mathematical Society of Japan, Tokyo, 2016. ↑ 25, 26
- [GM88] Mark Goresky and Robert MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. ↑ 8, 9, 30
- [GR84] Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. ↑ 10
- [Gra18] Patrick Graf. Algebraic approximation of Kähler threefolds of Kodaira dimension zero. *Math. Annalen*, 371:487–516, 2018. ↑ 5, 14, 28, 29, 31
- [GS20] Patrick Graf and Martin Schwald. The Kodaira problem for Kähler spaces with vanishing first Chern class. [arXiv:2001.06339 \[math.AG\]](https://arxiv.org/abs/2001.06339), January 2020. ↑ 5, 7, 30, 31
- [Gue16] Henri Guenancia. Semistability of the tangent sheaf of singular varieties. *Algebraic Geometry*, 3(5):508–542, November 2016. ↑ 11, 14, 22
- [HP19] Andreas Höring and Thomas Peternell. Algebraic integrability of foliations with numerically trivial canonical bundle. *Invent. Math.*, 216(2):395–419, 2019. ↑ 2, 31
- [Kaw81] Yujiro Kawamata. Characterization of abelian varieties. *Compos. Math.*, 43(2):253–276, 1981. ↑ 14
- [Kaw85] Yujiro Kawamata. Minimal models and the Kodaira dimension of algebraic fiber spaces. *J. reine angew. Math.*, 363:1–46, 1985. ↑ 2, 14
- [KM98] János Kollár and Shigefumi Mori. *Birational Geometry of Algebraic Varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. ↑ 18, 31
- [Kol93] János Kollár. Shafarevich maps and plurigenera of algebraic varieties. *Invent. Math.*, 113:177–215, 1993. ↑ 25
- [Kol07] János Kollár. *Lectures on resolution of singularities*, volume 166 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2007. ↑ 14
- [KS19] Stefan Kebekus and Christian Schnell. Extending holomorphic forms from the regular locus of a complex space to a resolution of singularities. [arXiv:1811.03644 \[math.AG\]](https://arxiv.org/abs/1811.03644), version 3, February 2019. ↑ 6, 14, 21, 32
- [Mat73] John N. Mather. Stratifications and mappings. In *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pages 195–232, 1973. ↑ 9
- [Mat12] John Mather. Notes on topological stability. *Bull. Amer. Math. Soc. (N.S.)*, 49(4):475–506, 2012. ↑ 8, 9
- [Pău08] Mihai Păun. Regularity properties of the degenerate Monge-Ampère equations on compact Kähler manifolds. *Chin. Ann. Math., Ser. B*, 29(6):623–630, 2008. ↑ 10
- [ST54] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canad. J. Math.*, 6:274–304, 1954. ↑ 7
- [Tak03] Shigeharu Takayama. Local simple connectedness of resolutions of log-terminal singularities. *Internat. J. Math.*, 14(8):825–836, 2003. ↑ 25, 29
- [Tia00] Gang Tian. *Canonical metrics in Kähler geometry*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2000. Notes taken by Meike Akveld. ↑ 25

- [Tou68] Jean-Claude Tougeron. Idéaux de fonctions différentiables. I. *Ann. Inst. Fourier (Grenoble)*, 18(fasc., fasc. 1):177–240, 1968. ↑ 6
- [TT81] Lê Dũng Tráng and Bernard Teissier. Variétés polaires locales et classes de Chern des variétés singulières. *Ann. of Math. (2)*, 114(3):457–491, 1981. ↑ 9
- [Var89] Jean Varouchas. Kähler spaces and proper open morphisms. *Math. Ann.*, 283(1):13–52, 1989. ↑ 25
- [Voi02] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2002. ↑ 22
- [Voi03] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2003. ↑ 30
- [Voi04] Claire Voisin. On the homotopy types of compact Kähler and complex projective manifolds. *Invent. Math.*, 157:329–343, 2004. ↑ 5
- [Wan16] Botong Wang. Torsion points on the cohomology jump loci of compact Kähler manifolds. *Math. Res. Lett.*, 23(2):545–563, 2016. ↑ 10
- [Xu14] Chenyang Xu. Finiteness of algebraic fundamental groups. *Compos. Math.*, 150(3):409–414, 2014. ↑ 2, 18, 25
- [Yau78] Shing-Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978. ↑ 1, 11, 25

UNIV RENNES, CNRS, IRMAR — UMR 6625, F-35000 RENNES, FRANCE ET INSTITUT UNIVERSITAIRE DE FRANCE

E-mail address: benoit.claudon@univ-rennes1.fr

URL: perso.univ-rennes1.fr/benoit.claudon/

LEHRSTUHL FÜR MATHEMATIK I, UNIVERSITÄT BAYREUTH, 95440 BAYREUTH, GERMANY

E-mail address: patrick.graf@uni-bayreuth.de

URL: www.graficland.uni-bayreuth.de

INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UNIVERSITÉ PAUL SABATIER, 31062 TOULOUSE CEDEX 9, FRANCE

E-mail address: henri.guenancia@math.cnrs.fr

URL: hguenancia.perso.math.cnrs.fr/

LEHRSTUHL FÜR MATHEMATIK VIII, UNIVERSITÄT BAYREUTH, 95440 BAYREUTH, GERMANY

E-mail address: philipp.naumann@uni-bayreuth.de

URL: www.pnaumann.uni-bayreuth.de