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Self-insurance and Non-concave Distortion Risk Measures.

Sarah Bensalem *

September 4, 2020

Abstract

This article considers an optimization problem for an insurance buyer in the context of proportional insurance and furnishing effort to reduce the size of a potential loss. The buyer's risk preference is given by a distortion risk measure, with risk probabilities being evaluated via a non-concave distortion function. This kind of distortion function reflects potential cognitive biases in the way in which individuals perceive risk probabilities. The buyer will select the optimal level of both insurance coverage and the prevention effort that he/she will furnish to reduce the amount that he/she stands to lose. The distribution of losses is given by a family of stochastically-ordered probability measures, indexed by the prevention effort. Contrary to what is found in the standard economic literature, introducing a non-concave distortion function leads to indeterminacy in the relationship between market insurance and self-insurance. Self-insurance and market insurance may be either substitutes, with a rise in one producing a fall in the other, or complements, so that the two rise or fall together, depending on the price elasticity.

Key words: Prevention, Self-insurance, Distortion risk measures, Distortion function, Non-concave distortion, Cognitive bias.

AMS 2010 subject classification: 91B06, 91B30.

JEL classification: C72, D61, D81, G22.

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1 Introduction

In [16], Ehrlich and Becker developed one of the first theoretical models relating prevention to market insurance, considering a rational agent who maximizes expected utility and who takes out an insurance policy against a potential loss. In addition to the insurance contract, the agent can actively reduce the potential loss via preventive effort. Ehrlich and Becker define two types of prevention that affect different aspects of the risk: the size of the claim can be reduced via self-insurance or the probability of loss via self-protection activities. Their comparative-statics analysis yields the following two conclusions:

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(A) In the general case, market insurance and self-insurance are substitutes, in the sense that a higher insurance price implies more prevention efforts in order to reduce the size of the loss.

(B) In the general case, market insurance and self-protection are either substitutes or complements, so that there is an indeterminacy. Complementarity is found at the actuarial premium, in the sense that more expensive market insurance brings about less effort to reduce the probability of loss.

Although this model is the foundation of prevention in insurance, its results only hold under the Von Neuman-Morgenstern hypothesis, defining Expected Utility (EU) theory, which states that individuals have a completely unbiased view of any risk they face. However, advances in decision-making theory have shown that individuals in risky situations who need to evaluate their risk exposure use rapid mental processes that may produce inaccurate judgments. These fast mental processes yielding less-good outcomes are called cognitive biases. In [22], Kahneman describes the functioning of the human brain in decision-making. Most, if not all, individuals are subject to numerous cognitive biases. The human brain uses two opposing systems. One is a fast and automatic intuition-based system, which operates by associating ideas together via cause and effect to create a flow of thinking. These associations are consistent, but do not represent logic; they are also affected by the subject's past personal experiences. It is this first system that creates biases. On the contrary, the second system is based on assessment: it is slow, calculative and controls the first system. This second system requires concentration and effort. As such, it is inclined to accept the conclusions of the first system, which are biased as they rely on causal rather than statistical judgments.
This functioning of the human brain produces in particular an over- or under-estimation of the actual risk the individual faces, depending on their personal experience and their appetite for risk (see [23] and [33]). This results in a distortion of risk probabilities that can be translated mathematically into a probability-distortion function. This latter is a better description of how individuals evaluate risk probabilities. Different forms of distortion functions have been described in the literature. In [24], the authors design a series of experiments to determine how individuals evaluate probabilities, concluding that there is a tendency to over-estimate the probability of rare events and under-estimate that of common events. [38] describes a survey of different experiments producing evidence in support of this theory. The resulting distortion functions are inverse S-shaped. These functions vary in their concavity and convexity across agents, as individuals may have different attitudes towards risk (see [31], [20] and [1]).

Prevention activities to reduce risk exposure and their economic implications have been analyzed under the assumption that individuals overestimate their risk probabilities (see [36], [10] and, more recently, [3]). The existence of cognitive biases modifies the perception of loss probabilities and challenges the global validity of the EU model as the reference for decision theory. It is therefore natural to test the robustness of the results in [16] with different theories of choice under risk including hypotheses that better reflect the way in which individuals estimate risks.

As EU theory cannot take into account the inverse S-shaped distortion of probabilities, [32] proposes Rank Dependent Expected Utility (RDEU) theory, which allows for the non-linear distortion of probabilities. First, events are ranked by an agent, and then weights are associated to each event, allowing for the over- or under-weighting of good or bad events. For example, the preferences of more risk-averse individuals are defined through the shapes of both the utility and probability-transformation functions. Under RDEU, [26] investigates the relationships between market insurance, self-insurance and self-protection. It is shown that most of the results in the reference model carry over to RDEU. In particular, self-insurance demand increases for risk-averse agents. The ambiguous result for self-protection is also found in this alternate theory. More recently, [18] used the RDEU model to consider the impact of risk perceptions on self-protection in the context of health risks. Their main result underlines the key role of the shape of the probability-transformation function in understanding prevention choices.

The particular RDEU model with a linear utility function is called the Dual Theory (DT), and is presented in [39]. EU theory transforms the value of a prospect by an expectation that is linear in probabilities but non-linear in wealth. DT theory is the opposite: it is linear in wealth but transforms probabilities via a weighting function defined by the cumulative distribution function over wealth. [8] analyzes the relationships between first market insurance and second self-insurance and self-protection under DT. Results (A) and (B) in [16] hold under DT theory: the substitutability of market insurance and self-insurance and the complementarity of self-protection. However in the special case where insurance companies may not price the premium according to effective prevention activities, market insurance and self-protection are substitutes.

More recently, in [3] the robustness of the results in [16] is tested under coherent risk measures. The interaction between an insurance buyer, referred to as IB in the remainder of the paper, and an insurance seller is modeled as a Stackelberg-type game, where both parties' risk preferences are given by convex risk measures. The insurance seller first proposes prices, in the form of safety loadings, and then the insurance buyer chooses both the optimal proportional insurance share and prevention effort to minimize his/her risk measure. In this context, optimal self-insurance and self-protection both exist and are unique. They are also related to the insurance share chosen by the agent as, in both cases, the agent exerts an optimal effort according to the insurance share chosen.

Summary of the approach

The problem. Consider an economic agent who faces a risk and who wishes to take out an insurance policy and who also the opportunity of taking prevention actions. His/her action is given by a vector (α, e), where α is the insurance cover he/she chooses and e is the prevention effort to reduce risk. The IB pays a premium in exchange for the chosen α. In this premium, the actual insurance price is given by the loading factor θ. We here assume moral hazard, as the loading factor θ in the insurance seller's contract with the IB does not depend on the effort e. The prevention effort exerted by the agent is therefore not observable by the insurance company. The problem presented here is of a particular type: the optimization over (α, e) can be separated into two separate problems, the first covering α and the second e. The first shows that the optimal α only takes on two values, 0 or 1. The second is more technical, but can still be solved by finding local solutions and then comparing them carefully in order to have a global view of the results. As in [3] and [15], the positive homogeneity of the risk measure (see Definition 2.1) and the linearity of the insurance contract imply a corner-type solution for optimal coverage. Linear insurance contracts are found in practice, especially in health insurance and in reinsurance with "quota-share" contracts. Note that this analysis readily extends to reinsurance, where the effort of the agent consists for instance in underwriting policy control, or in prevention campaigns for clients.

Distortion Risk Measures. The standard assumption in behavioral economics with prevention is that the agent maximizes his/her expected utility (see [16], [14], [28]). In this paper, the criterion of the agent is given by a distortion risk measure (see Equation (2.3)). Risk measures are a common tool in actuarial practice, as the European regulatory texts, Solvency II and the Swiss Solvency Test, require insurance and reinsurance companies to calculate the Value-at-Risk or the Expected Shortfall (see [19] for the definitions, and the differences between these measures). These two risk measures are examples of the class of distortion risk measures. This class is a subfamily of law-invariant risk measures. If ρ is a distortion risk measure, then a monetary utility function U can be defined as $U(X) := -\rho(-X)$. The results presented here can be reformulated in terms of law-invariant monetary utility functions. Monetary utility functions are typically assumed to be concave, but in the current framework they can be taken as non-concave. See [17] and [19] for more details on risk measures and insurance applications. The use of a non-concave distortion risk measure takes into account cognitive biases that interfere with the evaluation of risk probabilities. As underlined in [22], various thought mechanisms influence our perception of events, leading in particular to the modification of risk perception. This phenomenon has been studied intensively in the economic literature (see [24], [38] and [1]). It has been shown that the probability of an event differs from the subjective probability that people attach to it in a non-concave way that depends on risk aversion. The risk-averse tend to overestimate small-probability events and underweight medium- and large-probability events; risk lovers behave in the opposite way. Moreover, under the cash-additivity property (see Definition 2.1), the diversification principle is equivalent to the concavity property of a risk measure (see [19] for more details). Using non-concave distortion risk measures allows us to consider situations where there is no diversification of risk sources.

The choice of distributions. In the standard literature, the distributions of losses are discrete, and are concentrated at two points: the IB faces a loss L, defined as $L = \ell B$ with B being a Bernoulli random variable. In the framework of this paper (see Section 2 for more details), a family of random variables $(X_e)_{e\in(0,+\infty)}$ is considered, indexed by a prevention-effort parameter e, and such that the distributions of $(X_e)_{e\in(0,+\infty)}$ form a family of probability measures which is decreasing under first-order stochastic dominance (FSD) (see Assumption 2.1 for more details).

Main contributions.

Our work here extends the problem of self-insurance to the case where the objective function of the IB is characterized by a law-invariant distortion risk measure, where the distortion is non-concave. We prove that there exists a prevention effort that reduces the IB's risk optimally.

The main contribution regarding self-insurance is that either statement (A) or (B) will hold. This last conclusion results from considering how higher prices affect optimal insurance demand and prevention effort, in particular the self-insurance effort, and by focusing on the relative impact of effort on the loss distribution through the risk measure and on prices.

The structure of the paper

In Section 2 the problem and the main assumptions are presented. Section 3 sets out and solves the buyer's problem. Section 4 presents an illustration of the method developed in this paper, in the particular case of the Weibull law and with a specified inverse S-shaped distortion function. Last, Section 5 concludes.

2 Model and assumptions

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a fixed probability space. For a given random variable X, $\overline{F}_X$ denotes the survival function defined by $\overline{F}_X(x) := \mathbb{P}(X > x)$ and $\overline{q}_X$ denotes the tail quantile function, defined as the inverse of $\overline{F}_X$

$$\overline{q}_X(u) := \inf\{x \in \mathbb{R} \mid \overline{F}_X(x) \leq u\}.$$

Assume that the insured can make some effort to reduce their loss, which is modeled by a parameter $e \in [0, \infty)$ that we can consider as prevention. As our analysis is based on the definition of self-insurance in [16], the effort furnished by the insured affects the size of losses by directly affecting their distribution.

In what follows, we introduce the insurance-claims distribution used in this paper and the specific risk measure that is analyzed.

Let $(X_e)_{e\in[0,\infty)}$ be a family of non-negative random variables, representing losses, such that the distributions of X_e are defined by the following equation:

$$P_{X_e} = (1-p)\delta_{\{0\}} + pP_{Y_e}, \tag{2.1}$$

where $0 < p < 1$, $\delta_{\{0\}}$ is the Dirac mass at 0 and P_{Y_e} denotes the distribution of a positive random variable Y_e, which IB can influence through her choice of preventive effort.

Example 2.1. *One example of the distributions of the random variable Y_e is the Pareto distribution, which is often used in standard actuarial models (see [5]). Other well-known possibilities are the Fréchet, Weibull and Log-Normal distributions.*

Assumption 2.1. *The family $(X_e)_{e\in[0,+\infty)}$ is such that the distribution of each X_e is decreasing under first-order stochastic dominance (FSD)*

$$e_1 < e_2 \implies \mathbb{E}[f(X_{e_1})] > \mathbb{E}[f(X_{e_2})], \tag{2.2}$$

for every non-decreasing function $f : \mathbb{R} \to \mathbb{R}$.

$X \underset{\text{(mon)}}{\leq} Y$ denotes that X is dominated by Y for FSD, also called the *monotone* order. The following property of FSD (see Theorem 2.68 in [19]) will also be used

$$X \underset{\text{(mon)}}{\leq} Y \iff \overline{q}_X(u) \leq \overline{q}_Y(u), \quad \forall u \in (0,1).$$

In (2.2), no convexity or concavity assumption about f is made. This implies that the situation in which the distributions of $(X_e)_{e\in(0,+\infty)}$ are decreasing for the monotone convex order is not excluded. Moreover, Assumption 2.1 implies that, for each $u \in (0,1)$, the function $e \mapsto \overline{q}_{X_e}(u)$ is non-increasing.

Since the family of distributions used is indexed by some prevention effort, it is assumed that the effort affects the tail quantile of the loss distribution in such a way that the marginal impact decreases. This classic assumption in economics translates into the following in mathematics:

Assumption 2.2. *For every $u \in (0,1)$, the function $e \mapsto \overline{q}_{X_e}(u)$ is convex.*

The problem of the IB is presented as an optimization problem. When the IB searches for an insurance contract, he/she has to choose optimal insurance coverage and the optimal effort parameter. It is assumed that he/she minimizes a distortion risk measure, where the distortion function is non-concave. Definition 2.1 states the properties of all risk measures.

Definition 2.1. $\rho : L^{\infty}$[1] $\to \mathbb{R}$ *is a risk measure if*

[1] Recall that L^∞ is the space of measurable functions, which take values in $\mathbb{R}$ and are $\mathbb{P}$-essentially bounded, where the essential supremum of the absolute value of the fonctions serves as an appropriate norm.

1. *ρ is monotone: $X \geq Y \Rightarrow \rho(X) \geq \rho(Y)$ almost surely.*
2. *ρ is cash-additive: $\forall m \in \mathbb{R}$, $\rho(X+m) = \rho(X) + m$.*
3. *ρ is positively homogeneous: $\forall \lambda \in \mathbb{R}^+$, $\rho(\lambda X) = \lambda \rho(X)$.*
4. *ρ is law-invariant: $X \stackrel{d}{=} Y \Rightarrow \rho(X) = \rho(Y)$, where $\stackrel{d}{=}$ denotes equality in distribution.*

The first property states that if a loss Y is larger than a loss X in all states of nature, then Y is riskier than X. The second property is a strong mathematical assumption that translates in economics as the risk measure ρ having a unit. In the context of this paper this corresponds to a measure in capital (Euros, Pounds, Dollars etc.), although the fourth property suggests that this risk measure should be independent of this currency. The last property can be interpreted as the dependence of the risk measurement on only the distribution of the losses and not on the underlying assets.

For more details and properties of risk measures, refer to [19] (Chapter 4).

Distortion risk measures are an important class of risk measures defined by

$$\rho(X) := \int_0^1 \overline{q}_X(u) d\psi(u) = \int_0^1 \overline{q}_X(u) d\psi(u) du, \tag{2.3}$$

where ψ is a distortion function, i.e. a non-decreasing function from $[0,1]$ to $[0,1]$ such that $\psi(0) = 0$ and $\psi(1) = 1$.

A distortion risk measure is not convex in general. If the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has no atoms, then we can prove that a risk measure of the form given by Equation (2.3) is convex if and only if the distortion function is concave (see [19] for the distribution without atoms case, and [4] for the atom-distribution case). The convexity of the risk measure incites economic agents to use diversification. The risk measure is then called coherent. Note that non-coherent risk measures are used in insurance and reinsurance, as insurance and reinsurance companies need to calculate a risk measure called Value-at-Risk according to the Solvency II rules.

One important example of a non-concave distortion function is the inverse S-shaped distortion function. Figure 1 depicts that given by $\psi(u) = \exp(-(-\log(u))^\alpha)$, with $0 < \alpha < 1$.

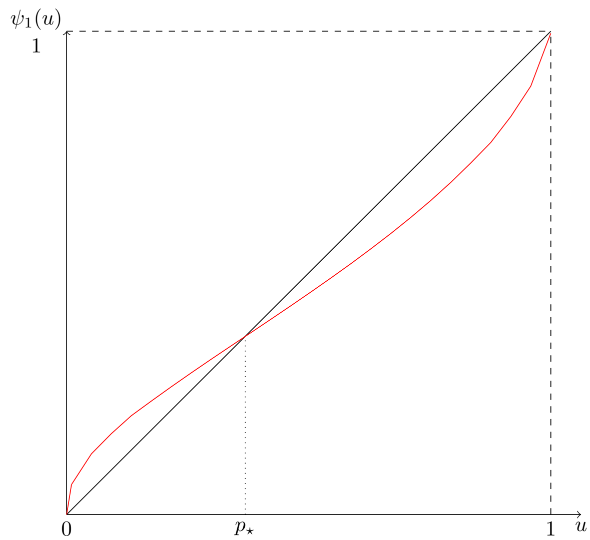

Figure 1: An example of an inverse S-shaped distortion function: $\psi(u) = \exp(-(-\log(u))^\alpha)$, with $0 < \alpha < 1$.

As noted in the Introduction, choosing a non-concave, and in particular an inverse S-shaped, probability-distortion more accurately represents how individuals evaluate their risk probabilities (see [20], [24], [34] and [38]). This function divides the probability interval into two: a first part of small probabilities, where the function is concave, and a second part of medium and high probabilities, where the distortion function is convex. The value of the boundary between those two intervals is denoted by $p_\star$. As described in [34], [37] and [31], the point $p_\star$ is an invariant fixed point of the distortion function ψ. Moreover, [1] and [35] underline the relationship between the form of this type of distortion function and individual optimism regarding the best outcomes and pessimism regarding the worst outcomes. The concavity below $p_\star$ reflects the overweighting of low probabilities, and the convexity after $p_\star$ the underweighting of high probabilities. This change in the shape of the curve reveals economic agents' sensitivity to extreme outcomes and the change in sensitivity moving towards medium probabilities. For an optimistic agent, the distortion function will be quite steep away from 0 and quite flat away from 1, meaning that a small increase in the risk probability has a greater impact on low than on high probabilities. On the contrary, for a pessimistic agent, the inverse S-shaped distortion curve is relatively flat away from 0 and relatively steep away from 1: high probabilities are more distorted than are small probabilities. Last, as noted in [34], the inverse S-shaped curve implies that people are insensitive to probabilities in the middle of the range. For more examples of this type of distortion function, see [1] and [31].

Proposition 2.1. *Let ρ be a distortion risk measure defined by Equation* (2.3). *If the distributions of $(X_e)_{e\in[0,\infty)}$ satisfy Assumption 2.2 then*

i) The function $e \mapsto \rho(X_e)$ is convex.

ii) The function $e \mapsto \rho(X_e)$ is continuous and non-increasing.

Proof. i) Since ψ is non-decreasing and non-negative, the distortion risk measure ρ depends on e only through the function $e \mapsto \overline{q}_{X_e}$, which is convex by Assumption 2.2. Hence $e \mapsto \rho(X_e)$ is convex.

ii) Convexity entails that $e \mapsto \rho(X_e)$ is continuous. The assumption that the distribution of the family $(X_e)_{e\in[0,\infty)}$ is decreasing by FSD implies that the tail quantile function $\overline{q}_{X_e}$ is a non-increasing function of effort. Equation (2.3) then implies that $e \mapsto \rho(X_e)$ is non-increasing. □

Remark 2.1. *As the convexity of the risk measure in* e *comes entirely from the convexity of the tail quantile function, an example for* ρ *non convex in* e *would be a distortion function* ψ *that is sometimes non-increasing. This means that the cumulative distribution function of* X_e *would be non-increasing for some intervals of* $[0, \infty)$ *and the density of* X_e *would allow negative probabilities in the sense that the economic agent who distorts objective probabilities using* ψ *perceives some probabilities as being negative.*

Remark 2.2. *The last point of the proof states that for* $\psi(u) = u$, $\rho(X_e) = \mathbb{E}[X_e] \geq 0$, *as* X_e *is non-negative. Hence, by Proposition 2.1,* $e \mapsto \mathbb{E}[X_e]$ *is a continuous, non-increasing, non-negative and convex function.*

From Equation (2.3), the risk measure $\rho(X_e)$ can be written as

$$\rho(X_e) = \int_0^1 \overline{q}_{X_e}(u) d\psi(u) du$$

which, from Equation (2.1), is equal to

$$\rho(X_e) = \int_0^p \overline{q}_{X_e}(u) d\psi(u) du.$$

Remark 2.3. *Note that since* ψ *is non-decreasing and non-negative, and as the random variable* X_e *is non-negative, the risk measure* ρ *is non-negative.*

Remark 2.4. *In the case where* ψ *is differentiable,* $d\psi(u) = \psi^{'}(u)$ *for all* $u \in (0, 1)$, *then the risk measure* ρ *can be written as* $\rho(X_e) = \int_0^p \overline{q}_{X_e}(u)\psi^{'}(u)du$.

Assume that the IB has initial wealth denoted by ω_0 and faces a risk with loss X_e, against which he/she takes out a proportional insurance contract with an insurance company. In this insurance contract, he/she chooses her level of insurance, denoted by $\alpha \in [0, 1]$, and pays a premium denoted by Π. The prevention effort $e \in [0, \infty)$ made by the IB has a monetary cost $c(e)$, where c is a non-decreasing and strictly convex function.[2] Common examples of the functions for the cost of the effort can be found in [11]. Suppose that the IB uses a distortion risk measure ρ. Her goal is to minimize the risk measure associated with her total loss, which is given by

$$L(\alpha, e) := \rho\Big((1 - \alpha)X_e + \Pi(\alpha X_e) + c(e) - \omega_0\Big).$$

Using the positive homogeneity and cash-additivity properties of ρ, the IB's objective function then simplifies to

$$L(\alpha, e) = (1 - \alpha)\rho(X_e) + \Pi(\alpha X_e) + c(e) - \omega_0. \tag{2.4}$$

The IB therefore solves the following optimization problem

$$\inf_{(\alpha,e)\in[0,1]\times[0,\infty)} L(\alpha, e). \tag{2.5}$$

Remark 2.5. *As* ρ *is a cash-additive risk measure, without loss of generality initial wealth* ω_0 *is considered to be 0 throughout the remainder of the paper: the initial wealth of the agent,* ω_0, *plays no role here. However, in Economics (see [6] for health insurance) initial wealth is known to affect insurance choices. This economic consideration could be included by considering cash-subadditive risk measures, or analyzed using expected utility. This point is left for further research.*

[2] If c is convex but not strictly convex, all of our analysis holds by considering the smallest effort that minimizes the IB's objective function.

For tractability reasons, and to facilitate reading, suppose that the insurance premium Π is defined as follows:

$$\Pi(X_e) := (1+\theta)\mathbb{E}[X_e]. \tag{2.6}$$

This form of premium with a safety loading is a standard choice in the actuarial-science literature (see [13] and [27]).

3 The optimization problem

This section solves the IB's problem. First, optimal insurance coverage is determined, and due to the special form of the IB's objective function, the optimal insurance coverage can be written as a function of the IB's prevention effort. Optimal self-insurance effort is then outlined, depending on both insurance coverage and the safety loading of the insurance premium, to obtain the solution to the IB's problem.

3.1 Optimal insurance coverage

The resolution of the IB's problem begins by the calculation of her optimal insurance coverage.

Proposition 3.1. *Optimal insurance coverage, $\alpha^\star$, takes only two values, 0 or 1, depending on the value of effort e.*

Proof. In Equation (2.6), note that the pricing functional is positively homogeneous as it is defined by an expectation, so that for all $\alpha \in [0,1]$

$$(1+\theta)\mathbb{E}[\alpha X_e] = \alpha(1+\theta)\mathbb{E}[X_e].$$

Equation (2.4) can be rewritten as

$$\begin{aligned} L(\alpha, e) &= (1-\alpha)\rho(X_e) + \alpha(1+\theta)\mathbb{E}[X_e] + c(e), \\ &= \rho(X_e) + \alpha\left[(1+\theta)\mathbb{E}[X_e] - \rho(X_e)\right] + c(e). \end{aligned}$$

The value of $\alpha^\star$ hence depends only on the sign of $(1+\theta)\mathbb{E}[X_e] - \rho(X_e)$. Optimal insurance coverage can then be written as a function of e as

$$\alpha^\star(e) = \begin{cases} 1 & \text{if } e \in \mathcal{I} \\ 0 & \text{if } e \in \mathcal{N} := \mathcal{I}^c, \end{cases} \tag{3.1}$$

where $\mathcal{I} := \left\{ e \in [0,\infty) | \; 1+\theta \leq \frac{\rho(X_e)}{\mathbb{E}[X_e]} \right\}$. □

This proposition means that the insurance buyer will either take out full insurance or no insurance. This type of choice is found in other models of prevention with insurance (see [8]), and also empirically (see [15]).

3.2 Optimal self-insurance effort

It is important to recall that the IB makes an effort to reduce the size of the potential loss that is faced. This translates into the choice of the distribution of this loss.

As was the case for fixed e, the Problem (2.5) is already solved in α; it remains to minimize the functional L in effort for fixed α. Suppose α is fixed, the function $e \mapsto L(\alpha^\star(e), e)$ is not convex so the minimization is not straightforward. From Equation (3.1), the optimal insurance coverage only takes on values 0 or 1 in separate sets, so $e \mapsto L(\alpha^\star(e), e)$ can be analyzed in these respective sets where the function is now convex. We can then compare the local minima in each domain to identify the global minimum.

For $e \in [0, \infty)$, define

$$L_{\mathcal{N}}(e) := \rho(X_e) + c(e), \text{ and} \tag{3.2}$$
$$L_{\mathcal{I}}^{\theta}(e) := (1+\theta)\mathbb{E}[X_e] + c(e), \tag{3.3}$$

such that

$$L(\alpha^\star(e), e) = \begin{cases} L_{\mathcal{N}}(e), & \text{if } e \in \mathcal{N}, \\ L_{\mathcal{I}}^{\theta}(e), & \text{if } e \in \mathcal{I}. \end{cases}$$

Define also

$$G(e) := \frac{\rho(X_e)}{\mathbb{E}[X_e]}, \tag{3.4}$$

which reveals how effort affects the risk and the price of insurance.

The sets $\mathcal{I}$ and $\mathcal{N}$ can be rewritten as

$$\mathcal{I} := \{e \in [0, \infty) | \, 1+\theta \le G(e)\},$$
$$\mathcal{N} := \{e \in [0, \infty) | \, 1+\theta > G(e)\}.$$

As the two intervals $\mathcal{I}$ and $\mathcal{N}$ are defined according to the function G, establishing the monotonicity of G will help to solve the IB's problem.

Assumption 3.1. *The function G is non-monotonic.*

Assumption 3.1 implies that, for a fixed θ, G can potentially cross the horizontal line at $1+\theta$ multiple times if G is not monotonic, as illustrated in Figure 2.

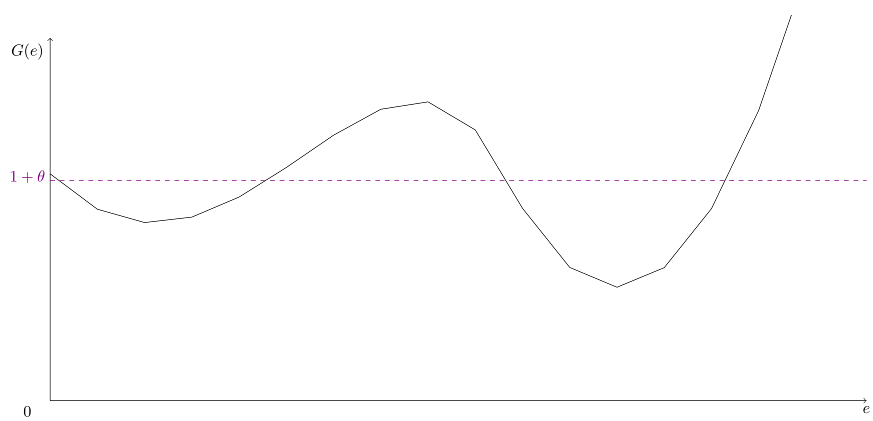

Figure 2: Illustration of a non-monotonic function G crossing $1+\theta$ multiple times for a fixed θ.

As the sets $\mathcal{I}$ and $\mathcal{N}$ depend on θ by definition, their shapes will change as θ rises. As θ rises so does $1+\theta$, so that the set $\mathcal{N}$ that satisfies the inequality $1+\theta > G(e)$ increases in size and its complement, the set $\mathcal{I}$, shrinks. Figure 3 shows how these two sets change as θ rises.

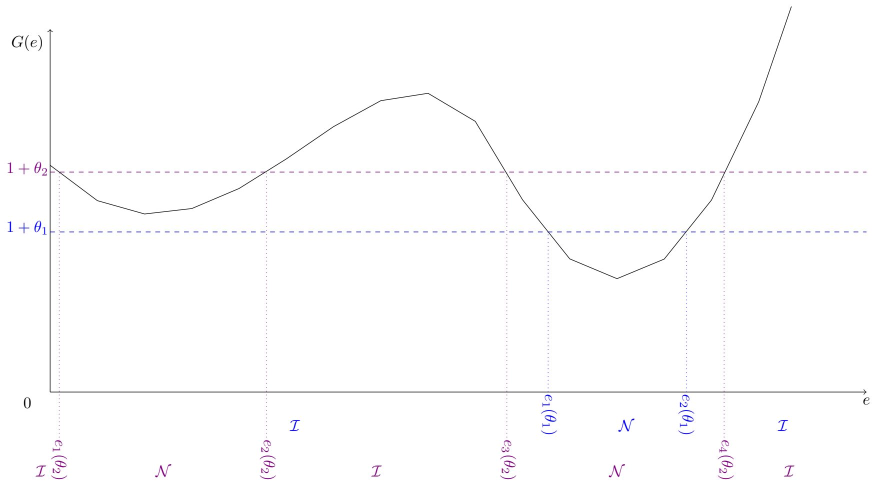

Figure 3: Illustration of the change in the $\mathcal{I}$ and $\mathcal{N}$ sets as θ rises.

The finite boundary points of the sets $\mathcal{I}$ and $\mathcal{N}$ are denoted by $e_i(\theta)$, with $i \in \mathbb{N}$ satisfying $e_i(\theta) \leq e_j(\theta)$ for $i < j$. Note that for all $i \in \mathbb{N}$, $L^\theta_{\mathcal{I}}(e_i(\theta)) = L_{\mathcal{N}}(e_i(\theta))$.

Assumption 3.2. *If Assumption 3.1 does not hold then the function G is non-decreasing.*

In the remainder of the paper, the function G satisfies Assumption 3.1. The case of Assumption 3.2 will be discussed in Remark 3.1.

The next steps focus on each set $\mathcal{I}$ and $\mathcal{N}$ and establish the local minima of each for fixed θ. In the set $\mathcal{I}$, the analysis of the function $e \mapsto L(\alpha^\star(e), e)$ reduces to that of

$$L^\theta_{\mathcal{I}}(e) = (1+\theta)\mathbb{E}[X_e] + c(e).$$

Recall that $e \mapsto \mathbb{E}[X_e]$ and $e \mapsto c(e)$ are differentiable almost everywhere. In the following proposition, it is assumed that they are differentiable everywhere.

Proposition 3.2. *$L^\theta_{\mathcal{I}}$ has a unique minimizer, denoted by $e_I(\theta)$ on $[0,\infty)$. Moreover, the function $\theta \mapsto e_I(\theta)$ is non-decreasing and there exists a positive constant θ_0 such that if $\theta \leq \theta_0$ then $e_I(\theta) = 0$.*

Proof. From Equation (3.3), $L^\theta_{\mathcal{I}}$ is continuous, strictly convex and goes to $+\infty$ as e goes to $+\infty$; it then has a unique minimizer on $[0,\infty)$ denoted by $e_I(\theta)$, defined as the solution to the equation

$$(1+\theta) = \frac{-c^{'}(e)}{\partial_e \mathbb{E}[X_e]}. \tag{3.5}$$

The remainder of this proof follows that in Proposition 3.2 in [3]; for completeness, we show the arguments below.

As the random variables $(X_e)_{e\in\mathbb{R}^+}$ have finite expected values, the function $L^\theta_{\mathcal{I}}$ is finite at $e=0$. Its right-hand derivative $\frac{d}{de}L^\theta_{\mathcal{I}}(0)$ is well-defined and

$$\frac{d}{de}L^\theta_{\mathcal{I}}(0) \geq 0 \iff \theta \leq -\frac{c'(0)}{E'(0)} - 1 =: \theta_0. \tag{3.6}$$

In this case, $L^\theta_\mathcal{I}$ is non-decreasing at 0, and by convexity it is non-decreasing everywhere, so its minimum is attained at $e = 0$. Note that for the constant θ_0 to be well-defined, the expectation needs to satisfy $E'(0) \neq 0$. If this is not the case, then $\frac{d}{de} L^\theta_\mathcal{I}(0) = c'(0) \geq 0$ and the function $L^\theta_\mathcal{I}$ is non-decreasing for any $\theta \in \mathbb{R}$.

Now suppose that $\theta > \theta_0$. This means that the function $L^\theta_\mathcal{I}$ is decreasing at 0. Therefore, the minimizer $e_I(\theta)$ of $L^\theta_\mathcal{I}$ on $\mathbb{R}^+$ is at an interior point characterized by

$$1 + \theta = -\frac{c'(e_I(\theta))}{E'(e_I(\theta))}. \tag{3.7}$$

$\theta \mapsto e_I(\theta)$ is non-decreasing, as the function $e \mapsto c'(e) \frac{-1}{E'(e)}$ is non-decreasing (as it is the product of two non-negative non-decreasing functions). As $1 + \theta$ obviously rises with θ, (3.7) implies that $e_I(\theta)$ is non-decreasing in θ. □

Proposition 3.3. *Suppose $\theta > \theta_0$. If $e_I(\theta)$ belongs to $\mathcal{I}$ then it is the local minimizer of $L^\theta_\mathcal{I}$ on this set.*

Proof. If $e_I(\theta)$ is in the set $\mathcal{I}$, then by Proposition 3.2 it is the local minimizer of $L^\theta_\mathcal{I}$ on $\mathcal{I}$. Otherwise, the restriction of $L^\theta_\mathcal{I}$ to $\mathcal{I}$ is first non-increasing and then non-decreasing, and so the local minimizer is given by a boundary point of the set $\mathcal{I}$. □

Recall that on the set $\mathcal{N}$, $e \mapsto L_\mathcal{N}(e)$ is the function to be analyzed.

Proposition 3.4. *$L_\mathcal{N}$ has a unique minimizer, denoted by e_N on $[0, \infty)$.*

Proof. Note that $e \mapsto L_\mathcal{N}(e)$ is continuous and non-negative, as the risk measure ρ and the cost function c are continuous and non-negative functions of e. In addition, $L_\mathcal{N}$ is strictly convex as ρ is non-increasing and convex, and $c(e)$ is non-decreasing and strictly convex.

We have to consider two cases to prove this proposition. First, assume that the risk measure ρ is finite at 0. Then the right-hand derivative of $L_\mathcal{N}$ at 0 is well-defined. $L_\mathcal{N}$ is therefore non-decreasing at 0 if and only if $-\partial_e \rho(X_0) \leq c'(0)$. The convexity of $L_\mathcal{N}$ leads to its monotonicity on $[0, \infty)$ and there exists a unique minimizer $e_N = 0$. In the case where $-\partial_e \rho(X_0) > c'(0)$, $L_\mathcal{N}$ starts by being decreasing at 0 and its strict convexity establishes the existence of a unique minimizer $e_N > 0$ on $[0, \infty)$.

Now assume that $\rho(X_e) \to \infty$ as $e \to 0^+$. Since $c(e) \to \infty$ as $e \to \infty$, $L_\mathcal{N}$ is a strictly convex and coercive function. Hence, it has a unique minimizer on $[0, \infty)$. □

Proposition 3.5. *There exists a unique positive contant θ_N such that*

> *If $\theta \leq \theta_N$, then e_N belongs to the set $\mathcal{I}$.*
>
> *If $\theta > \theta_N$, then e_N belongs to the set $\mathcal{N}$.*

Moreover, when e_N is in $\mathcal{N}$ then it is the local minimizer of $L_\mathcal{N}$ on $\mathcal{N}$.

Proof. Recall the definition of $\mathcal{N} := \{e \in [0, \infty) \mid 1 + \theta > G(e)\}$. Since e_N is independent of θ, as θ rises there exists a constant denoted by θ_N such that for $\theta \geq \theta_N$ e_N belongs to the set $\mathcal{N}$. Hence the definition $\theta_N := \inf\{\theta \geq 0 \mid 1 + \theta > G(e_N)\}$. This constant is unique by definition, as e_N is independent of θ. Moreover when e_N is on $\mathcal{I}$, the concavity of $e \mapsto L_\mathcal{N}$ in e_N proves that its restriction to the set $\mathcal{N}$ is non-decreasing. Hence the local minimizer on $\mathcal{N}$ is given by a boundary point of $\mathcal{N}$. □

The remainder of our analysis of self-insurance effort aims to find the global minimum by comparing the local minima on each set $\mathcal{N}$ and $\mathcal{I}$, depending on the value of θ. In particular, $e_I(\theta)$ can switch from one set to another as θ rises. To do so, the values $L_{\mathcal{I}}^{\theta}(e_I(\theta))$ and $L_{\mathcal{N}}(e_N)$ have to be compared to each other. As the bounds of $\mathcal{I}$ and $\mathcal{N}$, and $e_I(\theta)$ cannot be clearly compared, all possible orders of $e_I(\theta)$ and e_N need to be considered. For a fixed θ, four cases can be distinguished

(a) $e_I(\theta) \in \mathcal{I}$ and $e_N \in \mathcal{I}$,

(b) $e_I(\theta) \in \mathcal{N}$ and $e_N \in \mathcal{N}$,

(c) $e_I(\theta) \in \mathcal{I}$ and $e_N \in \mathcal{N}$,

(d) $e_I(\theta) \in \mathcal{N}$ and $e_N \in \mathcal{I}$

The next Lemma proves that case (d) is not possible.

Lemma 3.1. *There is no positive θ such that both $e_I(\theta) \in \mathcal{N}$ and $e_N \in \mathcal{I}$.*

Proof. By contradiction, suppose there exists $\theta \geq 0$ such that both $e_I(\theta) \in \mathcal{N}$ and $e_N \in \mathcal{I}$. Recall that

$$\forall\, e \in \mathcal{I},\, L_{\mathcal{I}}^{\theta}(e) \leq L_{\mathcal{N}}(e),$$
$$\forall\, e \in \mathcal{N},\, L_{\mathcal{N}}(e) < L_{\mathcal{I}}^{\theta}(e),$$

then $L_{\mathcal{I}}^{\theta}(e_N) \leq L_{\mathcal{N}}(e_N)$ and $L_{\mathcal{N}}(e_I(\theta)) \leq L_{\mathcal{I}}^{\theta}(e_I(\theta))$. As $e_I(\theta)$ is the global minimum of $L_{\mathcal{I}}^{\theta}$, $L_{\mathcal{N}}(e_I(\theta)) < L_{\mathcal{N}}(e_N)$, which contradicts the global minimality of e_N for $L_{\mathcal{N}}$. □

The next Proposition deals with case (a).

Proposition 3.6. *If e_N belongs to $\mathcal{I}$, then the global minimum of L is $e_I(\theta)$.*

Proof. If e_N belongs to $\mathcal{I}$, then by Lemma 3.1 $e_I(\theta)$ also belongs to the set $\mathcal{I}$, and from the definition of this set and the global optimality of $e_I(\theta)$ for $e \mapsto L_{\mathcal{I}}^{\theta}(e)$, $L_{\mathcal{I}}^{\theta}(e_I(\theta)) \leq L_{\mathcal{I}}^{\theta}(e_N) < L_{\mathcal{N}}(e_N)$.

In the following Proposition, we consider case (b).

Proposition 3.7. *If $e_I(\theta)$ belongs to $\mathcal{N}$, then the global minimum of L is e_N.*

Proof. By Lemma 3.1, $e_I(\theta) \in \mathcal{N}$ implies that e_N is in $\mathcal{N}$. From the definition of $\mathcal{N}$ and the global optimality of e_N for $e \mapsto L_{\mathcal{N}}(e)$, $L_{\mathcal{N}}(e_N) < L_{\mathcal{N}}(e_I(\theta)) \leq L_{\mathcal{I}}^{\theta}(e_I(\theta))$. □

The Theorem that follows addresses case (c).

Theorem 3.1. *There exists a positive constant θ_M such that*

If $\theta \leq \theta_M$, then $e_I(\theta)$ is the global minimum of the objective function L.

If $\theta > \theta_M$, then e_N is the global minimum of the objective function L.

Moreover θ_M is larger than θ_N.

Proof. The proof of the Theorem follows the argument of the proof of Theorem 3.2 in [3]. For completeness, it is reproduced in the following.

First note that, for all effort e, the objective function $\theta \mapsto L_{\mathcal{I}}^{\theta}$ is non-decreasing. For all e, if $\theta < \theta'$ then

$$(1+\theta)\mathbb{E}[X_e] + c(e) < (1+\theta')\mathbb{E}[X_e] + c(e) \Longleftrightarrow L_{\mathcal{I}}^{\theta}(e) < L_{\mathcal{I}}^{\theta'}(e).$$

From Proposition 3.2, $\theta \mapsto e_I(\theta)$ is non-decreasing, so that $\theta \mapsto L^{\theta}_{\mathcal{I}}(e_I(\theta))$ is also non-decreasing.

Since $L_{\mathcal{N}}(e_N)$ is independent of θ, and since at the actuarial premium it is better to have full insurance coverage than no insurance at all, meaning $L^0_{\mathcal{I}}(e_I(0)) < L_{\mathcal{N}}(e_N)$, the non-decreasing mapping $\theta \mapsto L^{\theta}_{\mathcal{I}}(e_I(\theta))$ leads to the definition of a constant $\theta_M \geq 0$ such that if $\theta \leq \theta_M$ then $L^{\theta}_{\mathcal{I}}(e_I(\theta)) < L_{\mathcal{N}}(e_N)$, and if $\theta > \theta_M$ then $L^{\theta}_{\mathcal{I}}(e_I(\theta)) > L_{\mathcal{N}}(e_N)$.

We can prove by contradiction that $\theta_M \geq \theta_N$. Suppose that $\theta_N > \theta_M$, and take a θ such that $\theta_N > \theta > \theta_M$. From the definition of θ_M, $L^{\theta}_{\mathcal{I}}(e_I(\theta)) < L_{\mathcal{N}}(e_N)$. From the definition of θ_N, e_N belongs to $\mathcal{I}$ and so does $e_I(\theta)$, so that $L^{\theta}_{\mathcal{I}}(e_I(\theta)) > L_{\mathcal{N}}(e_N)$, which is impossible. □

Remark 3.1. *Note that the case of monotonic non-decreasing G is simpler. Here the function G will cross $1+\theta$ only once, dividing the sets $\mathcal{I}$ and $\mathcal{N}$ into only two intervals of $\mathbb{R}^+$ at a single boundary effort denoted by e_θ. The examination of the global minimum of the value function of the agent remains similar to that developed above, as effort e_I and boundary effort e_θ are both non-decreasing functions of the insurance price θ. As there is no obvious way to compare these two effort levels with respect to the values of θ, all of the possibilities need to be evaluated.*

To conclude this Section, the following Corollary characterizes the behavior of the optimal effort $e^\star(\theta)$, which is very similar to that described in [29].

Corollary 3.1. *The map $\theta \mapsto e^\star(\theta)$ is non-decreasing for $\theta \leq \theta_M$. It becomes constant, and equal to e_N, for $\theta > \theta_M$. It exhibits a jump at $\theta = \theta_M$ equal to $e_N - e_I(\theta_M)$. As $e_I(\theta)$ can be either larger or smaller than e_N, this jump can be negative.*

Corollary 3.1 can be interpreted in economic terms as follows. An increase in the price of insurance can either imply greater prevention effort to reduce risk or less prevention effort to reduce his/her loss. Hence this result reflects an indeterminacy in the relationship between market insurance and self-insurance. Substitutability between market insurance and self-insurance may result: a higher insurance price leads to less insurance demand and increases the demand for prevention. This is the most-commonly discussed case in the literature, as it is the main conclusion of the reference model [16], as well as in the two extensions under RDEU [8] and coherent risk measures [3]. This result is also found in the experimental analysis in [30].

However, market insurance and self-insurance may also be complementary: a higher insurance price implies less insurance demand and a lower demand for prevention. To the best of my knowledge, this result is new in the prevention literature and only appears in the theoretical model in [29], which considers individual behavior in the context of mandatory complementary health insurance in France where effort is compensated either financially or through advice given via a prevention program. This complementarity could be explained by individual risk aversion, as in advantageous-selection (as opposed to adverse-selection) models. Advantageous selection is based on individual preferences and risk aversion: less risk-averse agents purchase less insurance and carry out less prevention, with the opposite holding for the more risk-averse (see [12] and [21]). These results have been confirmed empirically (see [9] and [25]).

4 Case study

Consider the loss given by a random variable X_e, with distribution as in Equation (2.1), with Y_e following a Weibull distribution. The Weibull distribution is chosen as it is often used in insurance and reinsurance.

Depending on how the shape parameter of the Weibull distribution is defined, it is possible to have either light tail claims (e.g. small claims) or heavy tail claims (e.g. large claims) (see [2] for further discussion).

As we are analyzing self-insurance, the IB's prevention action translates into the shape parameter denoted by τ. $e \mapsto \tau(e)$ is assumed to be a positive, non-decreasing and concave function. The inverse S-shaped distortion function of the agent is chosen as $\psi(u) = \exp(-(-\log(u))^{\eta})$, where $\eta \in (0, 1)$.

The survival function of X_e is

$$\mathbb{P}[X_e > u] = p \exp\left(-\lambda u^{\tau(e)}\right). \tag{4.1}$$

where $\lambda > 0$ is the scale parameter of the distribution.

One useful representation of the risk measure ρ for this case study will be the following

$$\rho(X_e) = \int_0^{\infty} \psi\left(\mathbb{P}[X_e > u]\right) du. \tag{4.2}$$

First, note that the Weibull distribution is decreasing by FSD, as for $e_1 < e_2$,

$$pe^{-\lambda u^{\tau(e_1)}} > pe^{-\lambda u^{\tau(e_2)}},$$

and hence $\mathbb{P}[X_{e_1} > u] > \mathbb{P}[X_{e_2} > u]$. Using the representation for any function g such that $g(0) = 0$

$$\mathbb{E}[g(X_e)] = \int_0^{\infty} g^{'}(u)\mathbb{P}[X_e > u]du,$$

we have

$$\mathbb{E}[g(X_{e_1})] \geq \mathbb{E}[g(X_{e_2})].$$

From Equations (4.1) and (4.2), the risk measure ρ is

$$\begin{aligned}
\rho(X_e) &= \int_0^{\infty} \exp\left(-\left(-\log\left(p\exp\left(-\lambda u^{\tau(e)}\right)\right)\right)^{\eta}\right) du, \\
&= \int_0^{\infty} \exp\left(-\left(-\log(p) + \lambda u^{\tau(e)}\right)^{\eta}\right) du,
\end{aligned}$$

by taking $v := -\log(p) + \lambda u^{\tau(e)}$, $du = \frac{(v+\log(p))^{1/\tau(e)-1}}{\tau(e)\lambda^{1/\tau(e)}}dv$

$$\rho(X_e) = \frac{1}{\tau(e)\lambda^{1/\tau(e)}} \int_{-\log(p)}^{\infty} (v + \log(p))^{1/\tau(e)-1} \exp\left(-v^{\eta}\right) dv. \tag{4.3}$$

Moreover the expectation of X_e is given by

$$\mathbb{E}[X_e] = \frac{p}{\lambda}\Gamma\left(1 + \frac{1}{\tau(e)}\right), \tag{4.4}$$

where Γ denotes the Gamma function defined as $\Gamma(u) = \int_0^{\infty} t^{u-1}\exp(-t)dt$ for $u > 0$.

The function G can be calculated from Equations (3.4), (4.3) and (4.4):

$$G(e) = \frac{p\lambda^{1-1/\tau(e)}}{\tau(e)\Gamma\left(1+\frac{1}{\tau(e)}\right)} \int_{-\log(p)}^{\infty} (v+\log(p))^{1/\tau(e)-1} \exp\left(-v^{\eta}\right) dv,$$
$$= \frac{p\lambda^{1-1/\tau(e)}}{\Gamma\left(\frac{1}{\tau(e)}\right)} \int_{-\log(p)}^{\infty} (v+\log(p))^{1/\tau(e)-1} \exp\left(-v^{\eta}\right) dv. \tag{4.5}$$

In Equation (4.5), the integral is not easy to calculate, so we cannot further describe the function G without specifying some parameters. In the following, four graphical possibilities are presented with particular parameter choices, to provide an idea of what may result.

First consider small claims, where the shape parameter $\tau(e)$ is larger than 1. For the set of parameters $p = 0.06$, $\eta = 0.4$ and $\lambda = 10$, and taking $\tau(e) = \sqrt{e + \frac{1}{100}}$, the function G is graphically non-increasing convex at first then non-decreasing concave as show in Figure 4. It can be seen that the function G is rapidly decreasing then increases at a slower pace in a concave way.

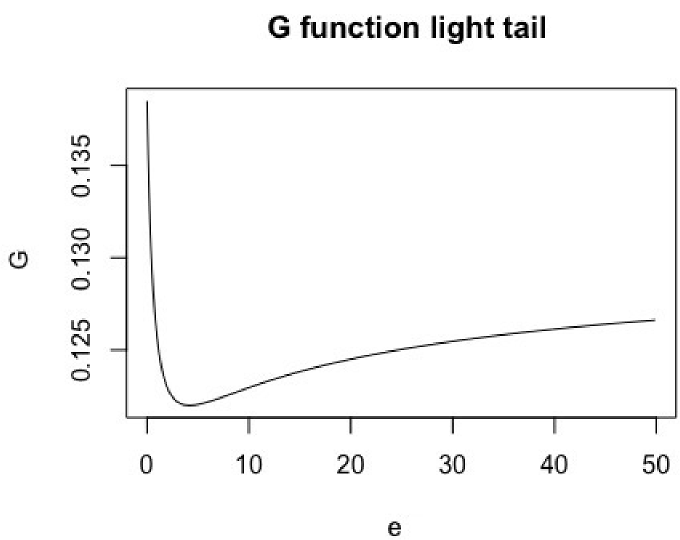

Figure 4: Illustration of G for small claims with $p = 0.06$, $\eta = 0.4$, $\lambda = 10$ and $\tau(e) = \sqrt{e + \frac{1}{100}}$.

Figure 5, shows the function G for the set of parameters $p = 0.08$, $\eta = 0.2$, $\lambda = 150$ and for a shape parameter function $\tau(e) = \sqrt{e}$. G decreases almost instantly and then increases only slowly.

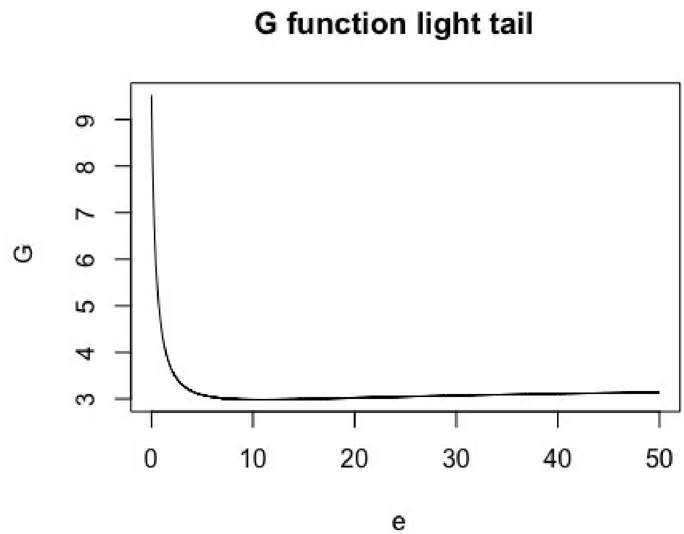

Figure 5: Illustration of G for small claims with $p = 0.08$, $\eta = 0.2$, $\lambda = 150$ and $\tau(e) = \sqrt{e}$.

Second, consider large claims, where the shape parameter $\tau(e)$ is such that $1 > \tau(e) > 0$. By taking $p = 0.05$, $\eta = 0.5$, $\lambda = 8$ and $\tau(e) = \sqrt{e}$, the G function is non-decreasing and convex, as shown in Figure 6.

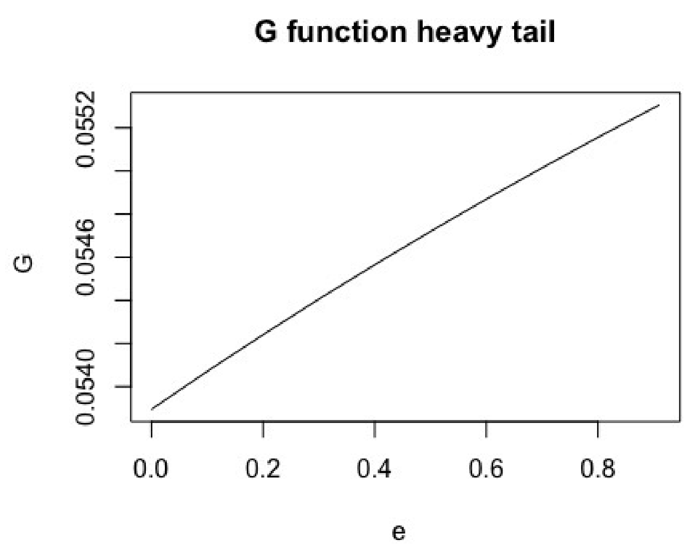

Figure 6: Illustration of G for small claims with $p = 0.05$, $\eta = 0.4$, $\lambda = 8$ and $\tau(e) = \sqrt{e}$.

By playing with the parameters used in the calculation of G, different results can be obtained for small- and large-claims insurance. The function G can be non-decreasing or not monotonic, as assumed in Section 3.2.

With these specified parameters, it is also possible to calculate $e_I(\theta)$ and e_N. The value of $e_I(\theta)$ is given by Proposition 3.3 as the minimizer of $L^{\theta}_{\mathcal{I}}$. From Equation (4.4), and assuming that the cost function is given by $c(e) := \frac{e^2}{2}$, $e_I(\theta)$ is the solution of the following equation

$$-\frac{(1+\theta)p\tau^{'}(e)}{\tau(e)^2}\partial_e\Gamma\left(1+\frac{1}{\tau(e)}\right)+e=0.$$

The value of e_N is given by Proposition 3.5, which is numerically solvable. Figure 7 shows the evolution of $e_I(\theta)$ and e_N, as functions of θ, for the set of parameters $p = 0.06$, $\eta = 0.4$ and $\lambda = 10$, and taking $\tau(e) = \sqrt{e+\frac{1}{100}}$. The distortion used for this illustration is given by $\psi(u) = \exp(-(-\log(u))^\eta)$. The effort $e_I(\theta)$, as a function of θ, evolves in a concave way, and e_N is independent of θ and equal to 1. This can be interpreted as the IB making less effort to reduce risk as the insurance price rises, while being insured.

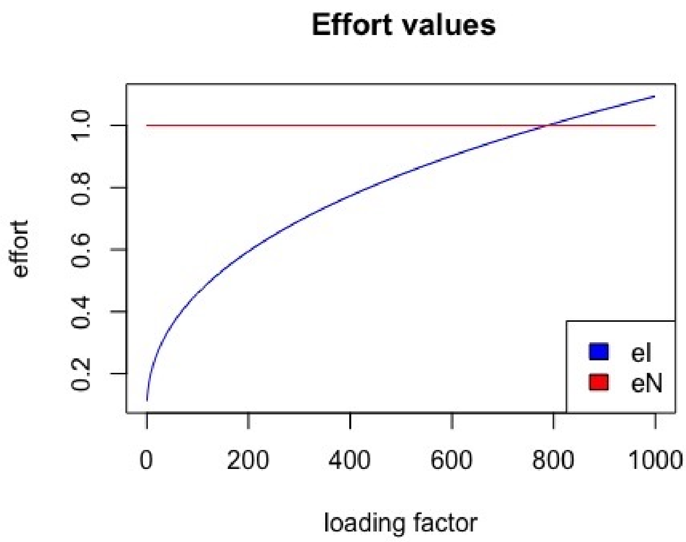

Figure 7: Illustration of the values of $e_I(\theta)$ and e_N for small claims with $p = 0.06$, $\eta = 0.4$, $\lambda = 10$ and $\tau(e) = \sqrt{e+\frac{1}{100}}$.

$L^\theta_\mathcal{I}(e_I(\theta))$ and $L_\mathcal{N}(e_N)$ can also be compared graphically, to find the maximum value of the insurance price acceptable to the IB, θ_M. Figure 8 illustrates the changes with respect to θ of the minimum value of the objective functions $L^\theta_\mathcal{I}$ and $L_\mathcal{N}$, with the set of parameters $p = 0.06$, $\eta = 0.4$ and $\lambda = 10$, taking $\tau(e) = \sqrt{e+\frac{1}{100}}$, and with the distortion function $\psi(u) = \exp(-(-\log(u))^\eta)$.

As $L_\mathcal{N}(e_N)$ is the objective function of the IB while being uninsured, it is independent of θ and equals 2.31. $L^\theta_\mathcal{I}(e_I(\theta))$ is increasing in θ and equals $L_\mathcal{N}(e_N)$ at θ_M, which latter is estimated to be 313.

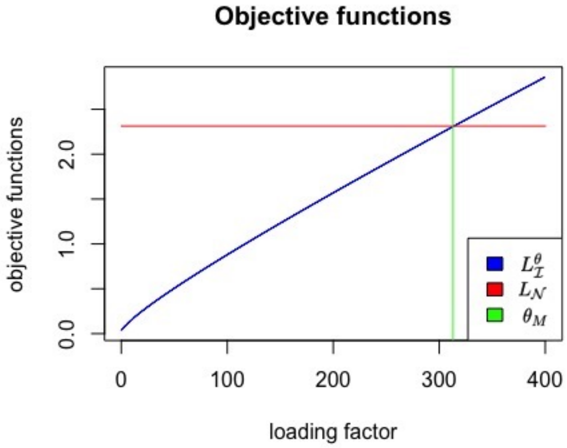

Figure 8: Illustration of the values of $e_I(\theta)$ and e_N for small claims with $p = 0.06$, $\eta = 0.4$, $\lambda = 10$ and $\tau(e) = \sqrt{e + \frac{1}{100}}$.

5 Conclusion

This paper has modeled the standard problem of self-insurance as an optimization problem for an insurance buyer. The criterion of the insurance buyer comes from a distortion risk measure, where the distortion is non-concave. Non-concave distortion functions yield new conclusions for market insurance and self-insurance: these are either substitutes or complements depending on the elasticity of the price of insurance. Either the insurance buyer enters the insurance contract, and by doing so takes out full insurance, or does not. In either case the IB chooses the corresponding optimal effort.

Due to the cash-additive property of the risk measures used here, the initial wealth of the agent does not play any role. However, initial wealth can influence insurance choice (as shown in [7] for health insurance and [28] for self-insurance). The present model could be extended to the case of cash-subaddtive risk measures, or analyzed using expected utility. Both of these aspects are left for future research.

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