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A note on the simultaneous edge coloring *

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Abstract

Let $G = (V, E)$ be a graph. A *(proper) k-edge-coloring* is a coloring of the edges of G such that any pair of edges sharing an endpoint receive distinct colors. A classical result of Vizing [3] ensures that any simple graph G admits a $(\Delta(G) + 1)$-edge coloring where $\Delta(G)$ denotes the maximum degreee of G. Recently, Cabello raised the following question: given two graphs G_1, G_2 of maximum degree Δ on the same set of vertices V, is it possible to edge-color their (edge) union with $\Delta + 2$ colors in such a way the restriction of G to respectively the edges of G_1 and the edges of G_2 are edge-colorings? More generally, given ℓ graphs, how many colors do we need to color their union in such a way the restriction of the coloring to each graph is proper?

In this short note, we prove that we can always color the union of the graphs $G_1, \dots, G_\ell$ of maximum degree Δ with $\Omega(\sqrt{\ell} \cdot \Delta)$ colors and that there exist graphs for which this bound is tight up to a constant multiplicative factor. Moreover, for two graphs, we prove that at most $\frac{3}{2}\Delta + 4$ colors are enough which is, as far as we know, the best known upper bound.

1 Introduction

All along the paper, we only consider simple loopless graphs. In his seminal paper, Vizing proved in g [3] that any simple graph G can be properly edge-colored using $\Delta(G) + 1$ colors (where $\Delta(G)$ denotes the maximum degreee of G). The *union* of two graphs G_1 and G_2 on vertex set V is the (simple) graph G with vertex set V and where uv is an edge if and only if uv is an edge of G_1 or an edge of G_2. An edge coloring of G is *simultaneous with respect to G_1 and G_2* if its restrictions to the edge set of G_1 and to the edge set of G_2 are proper edge-colorings. Recently, Cabello raised the following question[1]: given two graphs G_1, G_2 of maximum degree Δ on the same set of vertices V, does it always exist a simultaneous $(\Delta + 2)$-edge coloring with respect to G_1 and G_2? Cabello proved that this property is satisfied if the intersection of G_1 and G_2 is regular [1]. Using Vizing's theorem, one can easily notice that there exists a simultaneous $(2\Delta + 1)$-edge coloring. From a lower bound perspective, no graph where $\Delta + 2$ colors are needed is known.

Cabello introduced a generalization of this notion. Let ℓ graphs $G_1, G_2, \dots, G_\ell$ and G be their (edge) union. In other words, uv is an edge of G if and only if uv is an edge of at least one graph G_i with $i \leq \ell$. An edge-coloring of G is *simultaneous with respect to $G_1, \dots, G_\ell$* if its restriction to each graph G_i is a proper edge-coloring. Cabello asked how many colors are needed to ensure the existence of a simultaneous coloring of G with respect to each G_i. Let us denote by $\chi'(G_1, \dots, G_\ell)$ the minimum number of colors needed to obtain a simultaneous coloring. And let $\chi'(\ell, \Delta)$ be the largest integer k such that $k = \chi'(G_1, \dots, G_\ell)$ for some graphs $G_1, \dots, G_\ell$ of maximum degree (at most) Δ. Vizing's theorem ensures that $\chi'(\ell, \Delta) \leq \ell\Delta + 1$ and Cabello exhibit a graph for which $\chi'(3, \Delta) \geq \Delta + 5$ (with $\Delta = 10$) [1]. In this note, we prove that the order of magnitude of $\chi'(\ell, \Delta)$ is $\Theta(\sqrt{\ell}\Delta)$. More precisely, we prove that the following statement holds:

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Theorem 1

$$\chi'(\ell, \Delta) \leq 2\sqrt{2\ell}\Delta - \sqrt{2\ell} + 2.$$

We claim that this bound is tight up to a constant multiplicative factor. Let $\ell \in \mathbb{N}$ and Δ be an even value. Let $G := S_{1,k\Delta}$ be the star with $k\Delta$ leaves, where $k = \lfloor\sqrt{\frac{\ell}{2}}\rfloor$. Partition the edges of G into $2k$ sets $A_1, \ldots, A_{2k}$ of size $\frac{\Delta}{2}$. For every pair i, j, create the graph $G_{i,j}$ with edge set $A_i \cup A_j$. Note that each graph $G_{i,j}$ has maximum degree Δ since by construction the set of edges A_i induces a graph of maximum degree $\Delta/2$. Moreover the total number of graphs $G_{i,j}$ is $2k(2k-1)/2 \leq \ell$. Finally, by construction, every pair of edges of G appears in at least one graph $G_{i,j}$. So in order to obtain a simultaneous coloring, we need to color all the edges of G with different colors (since all the edges are incident to the center of the star). So:

Proposition 2

$$\chi'(\ell, \Delta) \geq \left\lfloor\sqrt{\frac{\ell}{2}}\right\rfloor\Delta.$$

Note that, for $\ell = 3$ and the graph $S_{1,3\lfloor\Delta/2\rfloor}$, a similar construction ensures that $\chi'(3, \Delta) \geq 3\lfloor\frac{\Delta}{2}\rfloor$, improving the lower bound of $\Delta + 5$. Indeed, let us partition the edges of the star into three sets A_1, A_2, A_3 of size $\lfloor\Delta/2\rfloor$. We similarly define for every $i \neq j$ the graph $G_{i,j}$ with edge set $A_i \cup A_j$. Each graph $G_{i,j}$ has maximum degree Δ and every pair of edges appears in at least one graph $G_{i,j}$. So an edge coloring of $S_{1,3\lfloor\Delta/2\rfloor}$ simultaneous with respect to $G_{1,2}, G_{1,3}$ and $G_{2,3}$ is a proper edge coloring of $S_{1,3\lfloor\Delta/2\rfloor}$.

When $\ell = 2$, a careful reading of the proof of Theorem 1 permits to remark that we can improve the trivial $(2\Delta + 1)$ upper bound into 2Δ.We prove the following much better upper bound with a different technique:

Theorem 3

$$\chi'(2, \Delta) \leq \left\lfloor\frac{3}{2}\Delta + 4\right\rfloor.$$

As far as we know, it is the best known upper bound.

2 Proof of Theorem 1

Let $G_1, \ldots, G_\ell$ be ℓ graphs of maximum degree Δ. Let us partition the set of edges of $G = \cup_{i=1}^{\ell} G_i$ into two sets (all along the paper, the notation $\cup$ stands for edge union). The *multiplicity* of an edge e is the number of graphs G_i with $i \leq \ell$ on which e appears. For some fixed k, the set E_1 is the set of edges with multiplicity at least k and E_2 is the set of edges with multiplicity less than k. We will optimize the value of k later. (Note that we do not necessarily assume that k is an integer). For every $i \in \{1, 2\}$, let us denote by H_i the graph G restricted to the edges of E_i. Note that $G = H_1 \cup H_2$.

We claim that the graph H_1 has degree at most $\ell\Delta/k$. Indeed, let u be a vertex and $E_1(u)$ be the set of edges of H_1 incident to it. Since every edge of H_1 has multiplicity at least k and at most $\ell\Delta$ edges (with multiplicity) are incident to u in G, at most $\frac{\ell}{k}\Delta$ different edges are in $E_1(u)$. So H_1 has maximum degree $\frac{\ell}{k}\Delta$. By Vizing's theorem, H_1 can be properly edge-colored with $(\frac{\ell}{k}\Delta + 1)$ colors.

Let us now prove that H_2 can be simultaneously edge-colored with $2k(\Delta - 1) + 1$ colors. Let us prove it by induction on the number of edges of H_2. The empty graph can indeed be edge-colored with $2k(\Delta-1)+1$ colors. Let $e = uv$ be an edge of H_2. By induction, there exists a simultaneous coloring c' of $H_2 \setminus e$ with $2k(\Delta - 1) + 1$ colors. Let us prove that c' can be extended into a simultaneous coloring of H_2. Without loss of generality, we can assume that e is an edge of the graphs $G_1, \ldots, G_r$ with $r < k$ and is not an edge of $G_{r+1}, \ldots, G_\ell$. Let F be the set of edges of $G_1, \ldots, G_r$ incident to u or to v distinct from e. By assumption, there are at most $2r(\Delta-1)$ such edges $(2(\Delta-1)$ in each graph). Since $r < k$, at most $2k(\Delta - 1)$ edges are in F. So there exists a color a that does not appear in F. The edge e can be colored with a without violating any constraints. It holds by choice of a for G_i with $i \leq r$ and it holds since $e \notin G_i$ for $i > r$.

So $\chi'(\ell,\Delta) \leq \frac{\ell}{k}\Delta + 2k(\Delta-1) + 2$ colors. We finally optimize the integer k which minimize the number of colors. We want to minimize $\frac{\ell}{k} + 2k$ which is minimal when $k = \sqrt{\frac{\ell}{2}}$. It finally ensures that $\chi'(\ell,\Delta) \leq 2\sqrt{2\ell}\Delta - \sqrt{2\ell} + 2$, which completes the proof of Theorem 1.

3 Proof of Theorem 3

Let G_1, G_2 be two graphs of maximum degree Δ and let G be their union. Let E_2 be the edges that appear in both graphs and, for every $i \in \{1,2\}$, let E_1^i be the set of edges that appears only in G_i. Let us denote by H_2 (resp. H_1^i) the graph restricted to the edges of E_2 (resp. E_1^i).

For every vertex v and every graph H, we denote by $degH(v)$ the degree of v in H. Let H be a graph and f, g be two functions from $V(H)$ to $\mathbb{R}^+$. A (g,f)-*factor* of H is an edge-subgraph H' of H such that every vertex v satisfies $g(v) \leq deg_{H'}(v) \leq f(v)$. Kano and Saito proved in [2] that the graph H admits a (g,f)-factor if

(i) f and g are two integer valued functions, and

(ii) for every vertex v, $g(v) < f(v)$, and

(iii) there exists a real number θ such that $0 \leq \theta \leq 1$ and for every vertex v, $g(v) \leq \theta \cdot deg_H(v) \leq f(v)$.

Let $1 \leq i \leq 2$. We will extract from H_1^i a (g,f)-factor where $g(v) = \lceil \frac{deg_{H_1^i}(v)}{2} - 1 \rceil$ and $f(v) = \lceil \frac{deg_{H_1^i}(v)}{2} \rceil$. The points (i) and (ii) are satisfied. Moreover, by choosing $\theta = \frac{1}{2}$, (iii) is also satisfied. Thus by [2], the graphs H_1^1 and H_1^2 admit (g,f)-factors. For $i \leq 2$, let K_1^i be a (g,f)-factor of H_1^i. For every i, let $L_1^i = H_1^i \setminus K_1^i$. Let $L = L_1^1 \cup L_1^2$ and $R = H_2 \cup K_1^1 \cup K_1^2$. Note that $G = L \cup R$. Let us now color these two graphs.

Let us first prove that L can be colored with $\lfloor \frac{\Delta}{2} \rfloor + 2$ colors. For every i, the graph L_1^i has maximum degree at most $\lfloor (\Delta/2) + 1 \rfloor$ since every vertex v of K_1^i has degree at least $\lceil (deg_{H_1^i}(v)/2) - 1 \rceil$. By Vizing's theorem, the graph L_1^i can be colored with at most $\lfloor \frac{\Delta}{2} + 1 \rfloor + 1 = \lfloor \frac{\Delta}{2} \rfloor + 2$ colors. Since the edges of L_1^1 and L_1^2 are disjoint, $L = L_1^1 \cup L_1^2$ can be simulteaneously colored with $\lfloor \frac{\Delta}{2} \rfloor + 2$ colors (the same set of colors can be re-used for each graph).

Let us now color the graph R. Let v be a vertex of R. Let us denote by d the degree of v in H_2. Since edges of H_2 are in both G_1 and G_2, the vertex v has degree at most $\Delta - d$ in both graphs H_1^1 and H_1^2. Since the graphs K_1^1 and K_1^2 are (g,f)-factors of respectively H_1^1 and H_1^2, the degree of the vertex v is at most $\lceil \frac{\Delta - d}{2} \rceil$ in each graph. So the degree of v in the graph R is at most $d + 2\lceil \frac{\Delta - d}{2} \rceil \leq \Delta + 1$. By Vizing's theorem, the graph R can be colored using at most $\Delta + 2$ colors.

Since $G = L \cup R$, we can find a simultaneous edge-coloring with respect to G_1 and G_2 using at most $\lfloor \frac{\Delta}{2} \rfloor + 2 + \Delta + 2 = \lfloor \frac{3}{2}\Delta \rfloor + 4$ colors.

4 Conclusion

Theorem 1 and Proposition 2 ensures that the following holds:

$$\left\lfloor \sqrt{\frac{\ell}{2}} \right\rfloor \Delta \leq \chi'(\ell,\Delta) \leq 2\sqrt{2\ell}\Delta - \sqrt{2\ell} + 2.$$

Closing the multiplicative gap of 4 between lower and upper bound is an interesting open problem. For $\ell = 2$, we still do not know any graph for which $\chi'(2,\Delta) > \Delta + 1$. Cabello asked the following question that is still widely open despite the progress obtained in Theorem 3:

Question 4 (Cabello) *Is it true that*

$$\chi'(2,\Delta) \leq \Delta + 2 \quad ?$$

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