A note on the simultaneous edge coloring
Nicolas Bousquet, Bastien Durain

To cite this version:

HAL Id: hal-02935482
https://hal.science/hal-02935482
Submitted on 7 Mar 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution - NonCommercial| 4.0 International License
A note on the simultaneous edge coloring *

Nicolas Bousquet\(^1\) and Bastien Durain\(^2\)

\(^1\)Univ. Grenoble Alpes, CNRS, Laboratoire G-SCOP, Grenoble-INP, Grenoble, France.\(^\dagger\)
\(^2\)ENS Lyon, Département d’informatique, France. \(^\ddagger\)

December 2, 2019

Abstract

Let \(G = (V, E)\) be a graph. A (proper) \(k\)-edge-coloring is a coloring of the edges of \(G\) such that any pair of edges sharing an endpoint receive distinct colors. A classical result of Vizing [3] ensures that any simple graph \(G\) admits a \((\Delta(G) + 1)\)-edge coloring where \(\Delta(G)\) denotes the maximum degree of \(G\). Recently, Cabello raised the following question: given two graphs \(G_1, G_2\) of maximum degree \(\Delta\) on the same set of vertices \(V\), is it possible to edge-color their (edge) union with \(\Delta + 2\) colors in such a way the restriction of \(G\) to respectively the edges of \(G_1\) and the edges of \(G_2\) are edge-colorings? More generally, given \(\ell\) graphs, how many colors do we need to color their union in such a way the restriction of the coloring to each graph is proper?

In this short note, we prove that we can always color the union of the graphs \(G_1, \ldots, G_\ell\) of maximum degree \(\Delta\) with \(\Omega(\sqrt{\ell} \cdot \Delta)\) colors and that there exist graphs for which this bound is tight up to a constant multiplicative factor. Moreover, for two graphs, we prove that at most \(\frac{3}{2}\Delta + 4\) colors are enough which is, as far as we know, the best known upper bound.

1 Introduction

All along the paper, we only consider simple loopless graphs. In his seminal paper, Vizing proved in [3] that any simple graph \(G\) can be properly edge-colored using \(\Delta(G) + 1\) colors (where \(\Delta(G)\) denotes the maximum degree of \(G\)). The union of two graphs \(G_1\) and \(G_2\) on vertex set \(V\) is the (simple) graph \(G\) with vertex set \(V\) and where \(uv\) is an edge if and only if \(uv\) is an edge of \(G_1\) or an edge of \(G_2\). An edge coloring of \(G\) is simultaneous with respect to \(G_1\) and \(G_2\) if its restrictions to the edge set of \(G_1\) and to the edge set of \(G_2\) are proper edge-colorings. Recently, Cabello raised the following question\(^1\): given two graphs \(G_1, G_2\) of maximum degree \(\Delta\) on the same set of vertices \(V\), does it always exist a simultaneous \((\Delta + 2)\)-edge coloring with respect to \(G_1\) and \(G_2\)? Cabello proved that this property is satisfied if the intersection of \(G_1\) and \(G_2\) is regular [1]. Using Vizing’s theorem, one can easily notice that there exists a simultaneous \((2\Delta + 1)\)-edge coloring. From a lower bound perspective, no graph where \(\Delta + 2\) colors are needed is known.

Cabello introduced a generalization of this notion. Let \(\ell\) graphs \(G_1, G_2, \ldots, G_\ell\) and \(G\) be their (edge) union. In other words, \(uv\) is an edge of \(G\) if and only if \(uv\) is an edge of at least one graph \(G_i\) with \(i \leq \ell\). An edge-coloring of \(G\) is simultaneous with respect to \(G_1, \ldots, G_\ell\) if its restriction to each graph \(G_i\) is a proper edge-coloring. Cabello asked how many colors are needed to ensure the existence of a simultaneous coloring of \(G\) with respect to each \(G_i\). Let us denote by \(\chi'(G_1, \ldots, G_\ell)\) the minimum number of colors needed to obtain a simultaneous coloring. And let \(\chi'(\ell, \Delta)\) be the largest integer \(k\) such that \(k = \chi'(G_1, \ldots, G_\ell)\) for some graphs \(G_1, \ldots, G_\ell\) of maximum degree (at most) \(\Delta\). Vizing’s theorem ensures that \(\chi'(\ell, \Delta) \leq \ell \Delta + 1\) and Cabello exhibit a graph for which \(\chi'(3, \Delta) \geq \Delta + 5\) (with \(\Delta = 10\)) [1]. In this note, we prove that the order of magnitude of \(\chi'(\ell, \Delta)\) is \(\Theta(\sqrt{\ell} \cdot \Delta)\). More precisely, we prove that the following statement holds:

\(^1\)This work was supported by the ANR Project GrR (ANR-18-CE40-0032).
\(^1\)nicolas.bousquet@grenoble-inp.fr
\(^1\)bastien.durain@ens-lyon.fr
\(^1\)BIRS Workshop: Geometric and Structural Graph Theory (2017). See e.g. https://sites.google.com/site/sophiespirkl/birs-open-problems.pdf

© 2019 published by Elsevier. This manuscript is made available under the CC BY NC user license https://creativecommons.org/licenses/by-nc/4.0/
Theorem 1

\[ \chi'(\ell, \Delta) \leq 2\sqrt{2\ell \Delta} - \sqrt{2\ell} + 2. \]

We claim that this bound is tight up to a constant multiplicative factor. Let \( \ell \in \mathbb{N} \) and \( \Delta \) be an even value. Let \( G := S_{1,k\Delta} \) be the star with \( k\Delta \) leaves, where \( k = \lfloor \sqrt{\frac{3}{2}} \rfloor \). Partition the edges of \( G \) into \( 2k \) sets \( A_1, \ldots, A_{2k} \) of size \( \frac{\Delta}{2} \). For every pair \( i, j \), create the graph \( G_{i,j} \) with edge set \( A_i \cup A_j \). Note that each graph \( G_{i,j} \) has maximum degree \( \Delta \) since by construction the set of edges \( A_i \) induces a graph of maximum degree \( \Delta/2 \). Moreover the total number of graphs \( G_{i,j} \) is \( 2k(2k-1)/2 \leq \ell \). Finally, by construction, every pair of edges of \( G \) appears in at least one graph \( G_{i,j} \). So in order to obtain a simultaneous coloring, we need to color all the edges of \( G \) with different colors (since all the edges are incident to the center of the star). So:

Proposition 2

\[ \chi'(\ell, \Delta) \geq \left\lceil \sqrt{\frac{\ell}{2}} \right\rceil \Delta. \]

Note that, for \( \ell = 3 \) and the graph \( S_{1,3\lfloor \Delta/2 \rfloor} \), a similar construction ensures that \( \chi'(3, \Delta) \geq 3\lfloor \Delta/2 \rfloor \), improving the lower bound of \( \Delta + 5 \). Indeed, let us partition the edges of the star into three sets \( A_1, A_2, A_3 \) of size \( |\Delta/2| \). We similarly define for every \( i \neq j \) the graph \( G_{i,j} \) with edge set \( A_i \cup A_j \). Each graph \( G_{i,j} \) has maximum degree \( \Delta \) and every pair of edges appears in at least one graph \( G_{i,j} \). So an edge coloring of \( S_{1,3\lfloor \Delta/2 \rfloor} \) simultaneous with respect to \( G_{1,2}, G_{1,3} \) and \( G_{2,3} \) is a proper edge coloring of \( S_{1,3\lfloor \Delta/2 \rfloor} \).

When \( \ell = 2 \), a careful reading of the proof of Theorem 1 permits to remark that we can improve the trivial \( (2\Delta + 1) \) upper bound into \( 2\Delta \). We prove the following much better upper bound with a different technique:

Theorem 3

\[ \chi'(2, \Delta) \leq \left\lfloor \frac{3}{2} \Delta + 4 \right\rfloor. \]

As far as we know, it is the best known upper bound.

2 Proof of Theorem 1

Let \( G_1, \ldots, G_\ell \) be \( \ell \) graphs of maximum degree \( \Delta \). Let us partition the set of edges of \( G = \bigcup_{i=1}^{\ell} G_i \) into two sets (all along the paper, the notation \( \cup \) stands for edge union). The multiplicity of an edge \( e \) is the number of graphs \( G_i \) with \( i \leq \ell \) on which \( e \) appears. For some fixed \( k \), the set \( E_1 \) is the set of edges with multiplicity at least \( k \) and \( E_2 \) is the set of edges with multiplicity less than \( k \). We will optimize the value of \( k \) later. (Note that we do not necessarily assume that \( k \) is an integer). For every \( i \in \{1, 2\} \), let us denote by \( H_i \) the graph \( G \) restricted to the edges of \( E_i \). Note that \( G = H_1 \cup H_2 \).

We claim that the graph \( H_1 \) has degree at most \( \ell \Delta/k \). Indeed, let \( u \) be a vertex and \( E_1(u) \) be the set of edges of \( H_1 \) incident to it. Since every edge of \( H_1 \) has multiplicity at least \( k \) and at most \( \ell \Delta \) edges (with multiplicity) are incident to \( u \) in \( G \), at most \( \frac{\ell}{k} \Delta \) different edges are in \( E_1(u) \). So \( H_1 \) has maximum degree \( \frac{\ell}{k} \Delta \). By Vizing’s theorem, \( H_1 \) can be properly edge-colored with \( (\frac{\ell}{k} \Delta + 1) \) colors.

Let us now prove that \( H_2 \) can be simultaneously edge-colored with \( 2k(\Delta - 1) + 1 \) colors. Let us prove it by induction on the number of edges of \( H_2 \). The empty graph can indeed be edge-colored with \( 2k(\Delta - 1) + 1 \) colors. Let \( e = uv \) be an edge of \( H_2 \). By induction, there exists a simultaneous coloring \( c' \) of \( H_2 \setminus e \) with \( 2k(\Delta - 1) + 1 \) colors. Let us prove that \( c' \) can be extended into a simultaneous coloring of \( H_2 \). Without loss of generality, we can assume that \( e \) is an edge of the graphs \( G_1, \ldots, G_r \) with \( r < k \). If it is an edge of \( G_{r+1}, \ldots, G_\ell \), then it holds since \( e \notin G_i \) for \( i > r \).
these two graphs. We denote by \( H \) (resp. \( H_i \)) the graph restricted to the edges of \( E_2 \) (resp. \( E^*_i \)).

For every vertex \( v \) and every graph \( H \), we denote by \( deg_H(v) \) the degree of \( v \) in \( H \). Let \( H \) be a graph and \( f, g \) be two functions from \( V(H) \) to \( \mathbb{R}^+ \). A \((g, f)\)-factor of \( H \) is an edge-subgraph \( H' \) of \( H \) such that every vertex \( v \) satisfies \( g(v) \leq deg_{H'}(v) \leq f(v) \).

Kano and Saito proved in [2] that the graph \( H \) admits a \((g, f)\)-factor if

(i) \( f \) and \( g \) are two integer valued functions, and

(ii) for every vertex \( v \), \( g(v) < f(v) \), and

(iii) there exists a real number \( \theta \) such that \( 0 \leq \theta \leq 1 \) and for every vertex \( v \), \( g(v) \leq \theta \cdot deg_{H}(v) \leq f(v) \).

Let \( 1 \leq i \leq 2 \). We will extract from \( H_i \) a \((g, f)\)-factor where \( g(v) = \left\lfloor \frac{deg_{H_i}(v)}{2} \right\rfloor - 1 \) and \( f(v) = \left\lceil \frac{deg_{H_i}(v)}{2} \right\rceil \). The points (i) and (ii) are satisfied. Moreover, by choosing \( \theta = \frac{1}{2} \), (iii) is also satisfied. Thus by [2], the graphs \( H_1^i \) and \( H_2^i \) admit \((g, f)\)-factors. For \( i \leq 2 \), let \( K_i \) be a \((g, f)\)-factor of \( H_i \). For every \( i \), let \( L_1^i = H_1^i \setminus K_1^i \). Let \( L = L_1^1 \cup L_1^2 \) and \( R = H_2 \cup K_1^1 \cup K_2^2 \). Note that \( G = L \cup R \). Let us now color these two graphs.

Let us first prove that \( L \) can be colored with \( \left\lceil \frac{\Delta}{2} \right\rceil + 2 \) colors. For every \( i \), the graph \( L_1^i \) has maximum degree at most \( \left\lfloor (\Delta/2) + 1 \right\rfloor \) since every vertex \( v \) of \( K_1^i \) has degree at least \( \left\lceil (deg_{H_i}(v)/2) - 1 \right\rceil \). By Vizing’s theorem, the graph \( L_1^i \) can be colored with at most \( \left\lceil \frac{\Delta}{2} \right\rceil + 1 = \left\lceil \frac{\Delta}{2} \right\rceil + 2 \) colors. Since the edges of \( L_1^1 \) and \( L_1^2 \) are disjoint, \( L = L_1^1 \cup L_1^2 \) can be simultaneously colored with \( \left\lceil \frac{\Delta}{2} \right\rceil + 2 \) colors (the same set of colors can be re-used for each graph).

Let us now color the graph \( R \). Let \( v \) be a vertex of \( R \). Let us denote by \( d \) the degree of \( v \) in \( H_2 \). Since edges of \( H_2 \) are in both \( G_1 \) and \( G_2 \), the vertex \( v \) has degree at most \( \Delta - d \) in both graphs \( H_1^1 \) and \( H_2^2 \). Since the graphs \( K_1^1 \) and \( K_2^2 \) are \((g, f)\)-factors of respectively \( H_1^1 \) and \( H_2^2 \), the degree of the vertex \( v \) is at most \( \left\lfloor \frac{\Delta - d}{2} \right\rfloor \) in each graph. So the degree of \( v \) in the graph \( R \) is at most \( d + 2\left\lfloor \frac{\Delta - d}{2} \right\rfloor \leq \Delta + 1 \). By Vizing’s theorem, the graph \( R \) can be colored using at most \( \Delta + 2 \) colors.

Since \( G = L \cup R \), we can find a simultaneous edge-coloring with respect to \( G_1 \) and \( G_2 \) using at most \( \left\lceil \frac{\Delta}{2} \right\rceil + 2 + \Delta + 2 = \left\lceil \frac{\Delta}{2} \right\rceil + 4 \) colors.

4 Conclusion

Theorem 1 and Proposition 2 ensures that the following holds:

\[
\left\lceil \frac{\sqrt{\ell}}{2} \right\rceil \Delta \leq \chi'(\ell, \Delta) \leq 2\sqrt{2\ell}\Delta - \sqrt{2\ell} + 2.
\]

Closing the multiplicative gap of 4 between lower and upper bound is an interesting open problem. For \( \ell = 2 \), we still do not know any graph for which \( \chi'(2, \Delta) > \Delta + 1 \). Cabello asked the following question that is still widely open despite the progress obtained in Theorem 3:

Question 4 (Cabello) Is it true that

\[
\chi'(2, \Delta) \leq \Delta + 2 ?
\]

Acknowledgements. The authors want to thank Louis Esperet for fruitful discussions and suggestions.
References

