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ASYMPTOTIC ANALYSIS OF THE EDGEWORTH COEFFICIENT FOR A DIFFUSION UNDER THE INFLUENCE OF AN ACCELERATED INCOMPRESSIBLE DRIFT

BRICE FRANKE AND ADEL OULED SAID

ABSTRACT. We analyze the asymptotic behavior of the third coefficient of the Edgeworth expansion of some diffusion under the influence of a drift, which grows to infinity. To do this analysis, we have to extend some result on Edgeworth expansions for discrete time Markov chains from Kontoyiannis and Meyn to diffusion.

INTRODUCTION

The Edgeworth expansion is a refinement of the central limit theorem which has played an important role in many applications. There are numerous works in the literature investigating the Edgeworth expansion in different situations (see for example [6], [12], [3]). For instance, it has been investigated under various dependency assumptions. In particular, Edgeworth expansion for discrete-time Markov chains was studied by Kontoyiannis and Meyn in [12]. One of the goals of this paper is to extend their result to continuous time diffusions. Let $(X_t)_{t \geq 0}$ be a diffusion process on \mathbb{R}^d with invariant probability measure π and infinitesimal generator L . We are interested in the integrated long term behavior of an observable f satisfying $\pi(f) = 0$; i.e :

$$S_t = \int_0^t f(X_t) dt.$$

Under some ergodicity condition, it was established by Bhattacharya in [1] that (S_t) satisfies a central limit theorem of the form

$$(1) \quad \frac{S_t}{\sqrt{t}} \xrightarrow{d} N(0, \sigma^2) \quad \text{as } t \rightarrow \infty.$$

A way to refine this central limit theorem, is to establish an Edgeworth expansion of the following form:

$$(2) \quad \sup_y \left| \mathbb{P} \left\{ \frac{S_t}{\sigma \sqrt{t}} \leq y \right\} - \Phi(y) - \frac{\gamma(y)}{\sigma \sqrt{t}} \left(\frac{\rho_3}{6\sigma^2} (1 - y^2) - \hat{F}(y) \right) \right| = o(t^{-1/2}),$$

where γ is the density of the standard normal distribution and Φ is the corresponding distribution function. In the above formula σ^2 and ρ_3 are suitable constants and \hat{F} is a suitable function. In a first result, Theorem 1.1, we will establish such an Edgeworth expansion for strongly non lattice function $f : \mathbb{R}^d \rightarrow [-1, 1]$ under some V -uniform

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ergodicity condition on the diffusion (X_t) . We will obtain the following expressions for the coefficients of the Edgeworth expansion

$$\sigma^2 = 2 \int_{\mathbb{R}^d} fg d\pi, \quad \rho_3 = 6 \int_{\mathbb{R}^d} fgh d\pi \quad \text{and} \quad \hat{F} = g,$$

where g and h are unique solutions of the two equations $Lg = f$ and $L^*h = f$. Note that we denote by L^* the adjoint of L on $\mathcal{L}^2(\mathbb{R}^d, \pi)$.

In the second part of the manuscript, we investigate the asymptotic behavior of the Edgeworth coefficient of a diffusion subject to growing drift. In this part we will consider a non-reversible \mathbb{R}^d -valued diffusion of the form

$$(3) \quad dX_t^b = -\nabla U(X_t^b) dt + \sqrt{2} dB_t + cb(X_t^b) dt,$$

where $(B_t)_{t \geq 0}$ is a Brownian motion, $U : \mathbb{R}^d \rightarrow \mathbb{R}$ is a given function such that $\pi(dx) = e^{-U(x)} dx$ is a probability measure on \mathbb{R}^d and b is a \mathcal{C}^1 vector field on \mathbb{R}^d satisfying $\operatorname{div}(be^{-U}) = 0$. Note that under some mild assumptions on U and b the resulting diffusion is V -uniformly ergodic and has invariant distribution π (see [9] and ??).

The real number c in front of the vector field b in (3) can be used to tune up the influence of the incompressible drift generated through b on the diffusion (X_t^b) . One can then study the asymptotic behavior of different quantities associated to the diffusion as the parameter c goes to infinity. For example the asymptotic behavior of the semigroup was investigated by Constantin, Kiselev, Ryzhik and Zlotos in [4] in a similar context. They proved that the operator norm of the associated semigroups converge to zero if and only if the drift generator has no non-zero eigenfunctions in \mathcal{H}^1 . Again in a similar context as in equation (3), the limit of the spectral gap of the generator of the diffusions was computed and it can be observed that this limit is infinite if and only if the drift generator has no non-zero eigenfunctions in \mathcal{H}^1 . Note that the equation (3) satisfies a central limit theorem and that the Edgeworth expansion that we obtained in the first part holds under some conditions on U and b . It is then interesting to see how the coefficients σ_c^2 and $\rho_{3,c}$ behave as $c \rightarrow \infty$. The limit of the asymptotic variance σ_c^2 is computed in [9], [5], [15] and [7]. In the second part of this manuscript we compute the limit of the Edgeworth coefficient $\rho_{c,3}$ as the constant c goes to infinity. Our proof is based on a method from Bhattacharya, Gupta and Walker which was used in [2] to compute asymptotic diffusion coefficients in the context of periodic homogenization as a drift-tuning parameter goes to infinity. We will prove in Theorem 2.1 under suitable conditions the following result

$$\lim_{c \rightarrow \infty} \rho_{3,c} = \langle PL_0^{-1} f, (PL_0^{-1} f)^2 \rangle_{\mathcal{H}^1} \quad \text{and} \quad \lim_{c \rightarrow \infty} \hat{F}_c = -PL_0^{-1} f,$$

where P is the orthogonal projection from \mathcal{H}^1 to $N_0 = \ker(b \cdot \nabla) \subset \mathcal{H}^1$. This result is some analogue to the result of Duncan, Lelievre and Pavliotis [5] where they proved that

$$\lim_{|c| \rightarrow \infty} \sigma_{cb}^2(f) = 2 \|PL^{-1} f\|_{\mathcal{H}^1}^2.$$

1. EDGEWORTH EXPANSION FOR DIFFUSION PROCESSE

In this section we extend a result on the Edgeworth expansion for Markov chains from Kontoyiannis and Meyn (see [12]) to continuous time diffusion processes. We essentially

follow the proof that they give for the discrete case, but we have to change to a generator based approach instead of their semi-group based technique at some places in the proof. A good account of the ergodic theory and spectral theory of diffusion processes that we will use is given in Huisinga, Meyn and Schütte (see [10]). We will adopt their setting for this manuscript. Let $(X_t)_{t \geq 0}$ be a diffusion process on \mathbb{R}^d with generator smooth coefficients of the form

$$L_{\text{diff}}h(x) = \sum_i a_i \partial_{x_i} h(x) + \frac{1}{2} \sum_{i,j} a_{ij} \partial_{x_i} \partial_{x_j} h(x), \quad h \in C^2(\mathbb{R}^d).$$

This means that (X_t) is a strong solution to the stochastic differential equation

$$dX_t^i = a_i(X_t)dt + \sum_{j=1}^m \sigma_{ij}(X_t)dB_t^j, \quad X_0 = x_0,$$

where $(B_t)_{t \geq 0}$ is standard Brownian motion on \mathbb{R}^m and $a_{ij} = \sum_{k=1}^m \sigma_{ik} \sigma_{jk}$. Then the canonical semi group $(P_{\text{can}}^t)_{t \geq 0}$, which is defined for bounded continuous functions as

$$P_{\text{can}}^t h(x) := \mathbb{E}_x[h(X_t)]$$

satisfies for all $h \in C^2(\mathbb{R}^d)$ with bounded second order derivatives that

$$\left\| \frac{1}{t}(P_{\text{can}}^t h - h) - L_{\text{diff}}h \right\|_{\text{sup}} \rightarrow 0, \quad \text{as } t \downarrow 0.$$

We assume that the coefficient functions $a_i, a_{ij}; 1 \leq i, j \leq d$ are such that the resulting diffusion is non-explosive. Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel σ -field on \mathbb{R}^d . We will denote by $(P^t)_{t \geq 0}$ the family of transition probability kernels which are related to (X_t) through

$$P^t(x, A) := \mathbb{P}_x(X_t \in A); \quad x \in \mathbb{R}^d, \quad x \in \mathbb{R}^d, \quad t \geq 0, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

It is stated in [10] that if the generator L_{diff} is hypoelliptic, then the resolvent kernel

$$R_\theta := \int_0^\infty e^{-\theta t} P^t dt; \quad \theta > 0$$

is strong Feller and that the existence of a $x_0 \in \mathbb{R}^d$ and a $t_0 > 0$ such that

$$P^t(x, O) > 0 \quad \text{for all } t > t_0, \quad x \in \mathbb{R}^d \quad \text{and all open neighbourhoods } O \text{ of } x_0$$

implies that the Markov process (X_t) is ψ -irreducible with $\psi := R_1(x_0, \cdot)$. Note that this implies that there exists a function $s : \mathbb{R}^d \rightarrow]0, \infty[$, a $\theta > 0$ and a measure ν , which is equivalent to the measure ψ , such that

$$R_\theta(x, A) \geq s(x)\nu(A), \quad x \in \mathbb{R}^d, \quad A \in \mathcal{B}(\mathbb{R}^d); \quad (\text{see Proposition 5.5.5 in [13]}).$$

Besides the differential generator L_{diff} we will eventually have to deal with the extended generator L_{ext} and the strong generator L_V , which is associated to an ψ -almost everywhere finite function $V : \mathbb{R}^d \rightarrow [1, \infty]$. The first one relates two measurable functions $h, g : \mathbb{R}^d \rightarrow \mathbb{R}$ as $L_{\text{ext}}h = g$ if for each starting point $x \in \mathbb{R}^d$ the process

$$M_t := h(X_t) - h(X_0) - \int_0^t g(X_s)ds; \quad t \geq 0$$

is a local martingale with respect to the canonical filtration $\mathcal{F}_t := \sigma(X_s; s \leq t); t \geq 0$. The second one requires the introduction of the space L_V^∞ of measurable, complex valued functions h satisfying

$$\|h\|_V := \sup_{x \in \mathbb{R}^d} \frac{|h(x)|}{V(x)} < \infty.$$

Note that the probability kernel function (P^t) can be identified with a semi groupe on L_V^∞ through

$$P^t h(x) := \int_{\mathbb{R}^d} P^t(x, dy) h(y), \quad t \geq 0, x \in \mathbb{R}^d.$$

Two functions $h, g \in C^V := \{g \in L_V^\infty : \lim_{t \downarrow 0} \|P^t g - g\|_V = 0\}$ satisfy relation $L_V h = g$ if

$$\left\| \frac{1}{t}(P^t h - h) - g \right\|_V \rightarrow 0, \quad \text{as } t \downarrow 0.$$

As is noted in [10], the operator L_{ext} equals L_{diff} when restricted to $C^2(\mathbb{R}^d)$, the operator L_V has domain $\{Rh : h \in C^V\}$ and it satisfies $L_V Rh = Rh - h$ for all $h \in C^V$. A major assumption, that is central in [12] and [10], is also assumed to hold for the rest of this section. It claims the existence of a ψ -almost everywhere finite function $V : \mathbb{R}^d \rightarrow [1, \infty]$, a small function $s : \mathbb{R}^d \rightarrow [0, \infty[$ and two constants $\delta > 0$, $b \in \mathbb{R}$ such that

$$(4) \quad L_{\text{ext}} V(x) \leq -\delta V(x) + bs(x).$$

It is stated in Theorem 2.3 from [10] that if the diffusion (X_t) is ψ -irreducible and aperiodic then the previous condition implies that (X_t) is V -uniform. This means that there exists some $\epsilon > 0$ such that the spectrum $\mathcal{S}(L_V)$ of the operator L_V satisfies

$$\text{Re}(z) \leq -\epsilon \quad \text{for all } z \in \mathcal{S}(L_V) \setminus \{0\}$$

and moreover that the eigenvalue zero is simple; i.e.: its eigenspace is of dimension one. This is equivalent to say that the operator P^1 has simple eigenvalue 1 and the that the spectrum $\mathcal{S}(P^1)$ of P^1 satisfies

$$|z| \leq 1 - \epsilon_0 \quad \text{for all } z \in \mathcal{S}(P^1) \setminus \{1\}.$$

Then Theorem 2.2 of [10] ensures the existence of an invariant probability measure π for the Markov process (X_t) and two constants $\delta, K > 0$ such that

$$\|(P^t - \mathbf{1} \otimes \pi)h\|_V \leq K e^{-\delta t} \|h\|_V, \quad \text{for all } h \in L_V^\infty \text{ and } t \geq 0.$$

For the rest of this section, as in [12], we will assume that

$$\pi(V^2) = \int_{\mathbb{R}^d} V^2(x) \pi(dx) < \infty.$$

Let now $f : \mathbb{R}^d \rightarrow [-1, 1]$ be an observable with zero mean; i.e.: $\pi(f) = 0$. For $\alpha \in \mathbb{C}$, we define the associated Feynman-Kac semi group

$$\hat{P}_\alpha^t g(x) = \mathbb{E}_x \left[e^{\alpha \int_0^t f(X_s) ds} g(X_t) \right].$$

Note that this semi groupe is not probabilistic in the sens that $\hat{P}_\alpha \mathbf{1}$ is in general not constant and equal to one. However, as is mentionned in [10], many concepts of semi groupe theory, as resolvents and generators, can still be defined in this situation. Our

later arguments are based on the strong generator $L_{V,\alpha}$ of the semi group (\hat{P}_α^t) . The relation $L_{V,\alpha}h = g$ holds for $h, g \in C_\alpha^V := \{g \in L_V^\infty : \lim_{t \downarrow 0} \|P_\alpha^t g - g\|_V = 0\}$ if

$$\left\| \frac{1}{t}(P_\alpha^t h - h) - g \right\|_V \rightarrow 0, \text{ as } t \downarrow 0.$$

Moreover, for $\theta > 0$ the resolvent is defined through

$$R_{\alpha,\theta}h := \int_0^\infty e^{-t\theta} P_\alpha^t h dt.$$

It can be seen that the strong generator of the Feynma-Kac semi group factorizes as follows :

$$(5) \quad \hat{L}_{\alpha,V}g = L_V g + \alpha f g, \quad g \in .$$

In the following we will denote the spectral radius of the operator \hat{P}_α^1 on L_V^∞ as

$$\hat{\rho}_\alpha := \lim_{t \rightarrow \infty} \|\hat{P}_\alpha^t\|_V^{1/t} = \sup\{|z|; z \in \mathcal{S}(\hat{P}_\alpha^1)\}.$$

We will say that the semi group (\hat{P}_α^t) is V -uniform, if there exists an eigenvalue λ_α in the spectrum $\mathcal{S}(\hat{P}_\alpha^1)$ of \hat{P}_α^1 satisfying $|\lambda_\alpha| = \hat{\rho}_\alpha$, having multiplicity one and satisfying for some $\epsilon > 0$

$$\mathcal{S}(\hat{P}_\alpha^1) \setminus \{\lambda_\alpha\} \subset \{z \in \mathbb{C}; |z| \leq \hat{\rho}_\alpha - \epsilon_\alpha\}.$$

In the following we will refer to λ_α as the generalized principal eigenvalue. The key to the Edgeworth expansion presented in [12] relies on the fact that for all α in a neighbourhood of zero the semi group (\hat{P}_α^t) also satisfies V -uniformity. We will see in Proposition 1.3 that if the original semi group (P^t) is V -uniform then (\hat{P}_α^t) is also V -uniform for all α in some open neighborhood O of zero in the complex plane. For $\alpha \in O$, let \check{f}_α be the eigenfunction associated to the unique eigenvalues λ_α which satisfies $|\lambda_\alpha| = \hat{\rho}_\alpha$. It follows

$$\hat{P}_\alpha^t \check{f}_\alpha = \lambda_\alpha^t \check{f}_\alpha \quad \text{for all } t \geq 0.$$

The function \check{f}_α is also an eigenfunction for the generator \hat{L}_α ; i.e.:

$$(6) \quad L_{\alpha,V} \check{f}_\alpha = \Lambda(\alpha) \check{f}_\alpha, \quad \text{where } \Lambda(\alpha) = \log \lambda_\alpha.$$

We will see in the proof of Proposition 1.7 that the generalized principal eigenvalue λ_α is an analytic function of α in a neighborhood around zero and so are $\Lambda(\alpha)$ and $\check{f}_\alpha(x)$ for any fixed $x \in \mathbb{R}^d$. Our interest lies in the asymptotic behavior of the integrated observations

$$S_t = \int_0^t f(X_s) ds.$$

We say that (S_t) satisfies an Edgeworth expansion if there exist $\sigma^2 > 0$, $\rho_3 \in \mathbb{R}$ and a function $\hat{F} : \mathbb{R}^d \rightarrow \mathbb{R}$ such that the following asymptotic relation holds

$$(7) \quad \sup_y \left| \mathbb{P} \left\{ \frac{1}{\sigma\sqrt{t}} S_t \leq y \right\} - \Phi(y) - \frac{\gamma(y)}{\sigma\sqrt{t}} \left(\frac{\rho_3}{6\sigma^2} (1 - y^2) - \hat{F}(y) \right) \right| = o(t^{-1/2}),$$

where γ is the density of the standard normal distribution and Φ is the corresponding distribution function. It follows from the above relation that

$$(8) \quad \frac{1}{\sigma\sqrt{t}} S_t \xrightarrow{d} N(0, 1) \quad \text{as } t \rightarrow \infty,$$

which implies that the coefficient σ^2 in (7) equals the asymptotic variance in the central limit theorem (1). For the following we will restrain our analysis to non-lattice functions f . A function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is called lattice if there are $h > 0$ and $0 \leq d < h$, such that for all $x \in \mathbb{R}^d$ the quotient $\frac{1}{h}(f(x) - d)$ is an integer. According to [12] a function f is called almost lattice if it can be decompose as a sum $f = f_l + f_0$, where f_l is lattice and f_0 has zero asymptotic variance, which means that

$$\frac{1}{t} \text{Var} \left(\int_0^t f_0(X_s) ds \right) \longrightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In the following a function is called strongly non-lattice whenever it is not almost lattice. The following is a continuous time version of Theorem 5.1 in [12].

Theorem 1.1. *Assume that $f : \mathbb{R}^d \rightarrow [-1, 1]$ is a strongly nonlattice function with $\pi(f) = 0$. Then the family of random variables $S_t = \int_0^t f(X_s) ds; t \geq 0$ satisfies the Edgeworth expansions (7) for all starting point $x \in \mathbb{R}$ with coefficients*

$$\sigma^2 = 2 \int_{\mathbb{R}^d} f g d\pi, \quad \rho_3 = 6 \int_{\mathbb{R}^d} f h d\pi \quad \text{and} \quad \hat{F} = g,$$

where g and h are the solutions of the two equations $L_V g = -f$ and $L_V h = fg - \pi(fg)$.

1.1. Proof Theorem 1.1. The following proof is an adaptation of the proof from Kontoyiannis and Meyn [12] to the cas of a continuous time process. We add it here for convenience of the reader. Its starting point is the well known Berry-Esseen inequality which can be found in Feller's book (see formula (13.3) on page 538 in [6]). It states that for any choice $m > 0$ and $C > 0$ and whenever F and G are distribution functions satisfying $|G'(y)| \leq m$, then one has

$$(9) \quad \left| F(y) - G(y) \right| \leq \frac{1}{\pi} \int_C^{\infty} \left| \chi_F(\omega) - \chi_G(\omega) \right| \frac{d\omega}{|\omega|} + \frac{24m}{C\pi}; \quad y \in \mathbb{R}.$$

Here χ_F and χ_G are the characteristic functions which are associated with F and G . In our situation we will choose

$$\begin{aligned} F(y) &= F_t(y) = P_x \left\{ \frac{1}{\sigma\sqrt{t}} (S_t - \mathbb{E}_x[S_t]) \leq y \right\} \\ G(y) &= G_t(y) = \Phi(y) - \frac{\rho_3}{6\sigma^2\sqrt{t}} (y^2 - 1) \gamma(y) \end{aligned}$$

and $C = A\sqrt{t}$, where $A > 24(\epsilon\pi)^{-1} |G'_t(y)|$ for a given small $\epsilon > 0$. Then (9) becomes

$$(10) \quad \left| F_t(y) - G_t(y) \right| \leq \frac{1}{\pi} \int_{-A\sqrt{t}}^{A\sqrt{t}} \left| \eta_{F_t} \left(\frac{i\omega}{\sigma\sqrt{t}} \right) - \chi_{G_t}(\omega) \right| \frac{d\omega}{|\omega|} + \frac{\epsilon}{\sqrt{t}},$$

where

$$(11) \quad \eta_{F_t}(\alpha) := \mathbb{E}_x \left[\exp(\alpha \bar{S}_t) \right] = m_t(\alpha) \exp \left(-\alpha \mathbb{E}_x(S_t) \right),$$

with $m_t(\alpha) = \mathbb{E}_x \left[\exp(\alpha S_t) \right]$ and $\bar{S}_t = S_t - \mathbb{E}_x[S_t]$. Note that the characteristic function associated with the distribution function to $G_t(y)$ is given by

$$\chi_{G_t}(\omega) = e^{-\frac{\omega^2}{2}} \left(1 + \frac{\rho_3(i\omega)^3}{6\sigma^3\sqrt{t}} \right).$$

To show that the Edgeworth expansion (7) holds, it suffices to show that the integral I in inequality (10) is of order $o(t^{-1/2})$. To do this we divide the integration domain in three parts and show that all of them are $o(T^{-1/2})$. We use the following decomposition

$$]-A\sqrt{t}, A\sqrt{t}[=]-A\sqrt{t}, -B\sqrt{t}[\cup]-B\sqrt{t}, B\sqrt{t}[\cup]B\sqrt{t}, A\sqrt{t}[$$

to decompose the integral of (10) as a sum of three integrals $I = I_1 + I_2 + I_3$. The constant B is supposed to satisfy $B < A$ and will be fixed later. As a consequence, we have to analyze the asymptotic behavior of the integrals

$$I_1 = \frac{1}{\pi} \int_{B\sqrt{t}}^{A\sqrt{t}} \left| \eta_{F_t} \left(\frac{i\omega}{\sigma\sqrt{t}} \right) - \chi_{G_t}(\omega) \right| \frac{d\omega}{|\omega|}.$$

Applying the change of variable $u = \frac{\omega}{\sigma\sqrt{t}}$, this leads to the following inequality

$$\begin{aligned} I_1 &= \frac{1}{\pi} \int_{B/\sigma}^{A/\sigma} \left| \eta_{F_t}(iu) - \chi_{G_t}(\sigma\sqrt{t}u) \right| \frac{du}{|u|} \\ &= \frac{1}{\pi} \int_{B/\sigma}^{A/\sigma} \left| m_t(iu) e^{\mathbb{E}_x[iuS_t]} - \chi_{G_t}(\sigma\sqrt{t}u) \right| \frac{du}{|u|} \\ &\leq \frac{\sigma}{B\pi} \int_{B/\sigma}^{A/\sigma} |m_t(iu)| du + \frac{\sigma}{B\pi} \int_{B/\sigma}^{A/\sigma} |\chi_{G_t}(\sigma\sqrt{t}u)| du \\ &= \frac{\sigma}{B\pi} \int_{B/\sigma}^{A/\sigma} |e^{t\Lambda(0)} \epsilon_t'| du + \frac{\sigma}{B\pi} \int_{B/\sigma}^{A/\sigma} \left| e^{-\frac{u^2 t \sigma^2}{2}} \left(1 + \frac{\rho_3 t (iu)^3}{6} \right) \right| du. \end{aligned}$$

Using part i) from Proposition 1.5 both integrals in the previous expression converge toward zero exponentially fast, as long as B stays bounded away from zero and as long as A stays bounded. Therefore the integral I_1 is of order $o(t^{-1/2})$. The integral I_3 can be treated in the same way. We now have to deal with the integral I_2 . After the change of variables $u = \omega/\sigma\sqrt{t}$, we obtain

$$\begin{aligned} I_2 &= \frac{1}{\pi} \int_{-B\sqrt{t}}^{B\sqrt{t}} \left| \eta_{F_t} \left(\frac{i\omega}{\sigma\sqrt{t}} \right) - \chi_{G_t}(\omega) \right| \frac{d\omega}{|\omega|} \\ &= \frac{1}{\pi} \int_{-B/\sigma}^{B/\sigma} \left| \eta_{F_t}(iu) - \chi_{G_t}(u\sigma\sqrt{t}) \right| \frac{du}{|u|}. \end{aligned}$$

In order to understand the behavior of the integral I_2 as t goes to infinity, we first investigate it's integrand :

$$K_t(u) = \left| \eta_{F_t}(iu) - \chi_{G_t}(u\sigma\sqrt{t}) \right|.$$

With the Propositions 1.5 and 1.7 we can see that η_{F_t} equals

$$\eta_{F_t}(iu) = e^{-iu\mathbb{E}_x[S_t]} e^{t\Lambda(iu)} \left[\check{f}_{iu}(x) + |iu|\epsilon_t \right].$$

Expanding $\Lambda(iu)$ in a Taylor series around zero yields.

$$\Lambda(iu) = iu\Lambda'(0) - \frac{1}{2}u^2\Lambda''(0) + \frac{1}{6}(iu)^3\Lambda'''(is), \quad \text{for some suitable } s \in]0, u[.$$

By Proposition 1.7, we have $\Lambda'(0) = 0$ and $\Lambda''(0) = \sigma^2$, and it follows that

$$\eta_{F_t}(iu) = e^{-iu\mathbb{E}_x[S_t]} \exp \left(t \left(-\frac{1}{2}u^2\sigma^2 + \frac{1}{6}(iu)^3\Lambda'''(is) \right) \right) \left[\check{f}_{iu}(x) + |iu|\epsilon_t \right].$$

According to the definition of d_t (see Proposition 1.8) this can be written as

$$\eta_{F_t}(iu) = e^{-iud_t - iug(x)} \exp\left(t\left(-\frac{1}{2}u^2\sigma^2 + \frac{1}{6}(iu)^3\Lambda'''(is)\right)\right)[\check{f}_{iu}(x) + iu\epsilon_t],$$

where g is the solution of the Poisson equation $L_{\text{diff}}g = -f$. From this we deduce the following expression for the integrand $K_t(u)$:

$$K_t(u) = e^{-\frac{1}{2}tu^2\sigma^2} \left| \exp\left(-iud_t + \frac{t}{6}(iu)^3\Lambda'''(is) + \log(\check{f}_{iu}(x) + iu\epsilon_t) - iug(x)\right) - 1 - t\frac{\rho_3}{6}(iu)^3 \right|.$$

Note that

$$\begin{aligned} \log(\check{f}_{iu}(x) + iu\epsilon_t) - iug(x) &= \log\left(1 + \frac{i u \epsilon_t}{\check{f}_{iu}}\right) + \log(\check{f}_{iu}(x)) - iug(x) \\ &= \delta(iu) + \log\left(1 + \frac{i u \epsilon_t}{\check{f}_{iu}}\right), \end{aligned}$$

where $\delta(\cdot)$ is as in Proposition 1.8. The second exponential in the above expression for $K_t(u)$ then becomes

$$K_t(u) = \exp\left(\frac{t}{6}(iu)^3\Lambda'''(is) - iud_t + iu\epsilon_t^\#(iu) + \delta(iu)\right),$$

where

$$\epsilon_t^\#(iu) := \frac{1}{iu} \log\left(1 + iu \frac{\epsilon_t}{\check{f}_{iu}}\right).$$

Since according to Proposition 1.5 the function ϵ_t converges to zero as $t \rightarrow \infty$, it follows that

$$(12) \quad |\epsilon_t^\#(iu)| \rightarrow 0 \quad \text{as } t \rightarrow \infty \text{ uniformly for all } |u| \leq B/\sigma.$$

Therefore, the expression for $K_t(u)$ becomes

$$(13) \quad K_t(u) = e^{-\frac{1}{2}tu^2\sigma^2} \left| \exp\left(-iud_t + \frac{t}{6}(iu)^3\Lambda'''(is) + \delta(iu) + iu\epsilon_t^\#(iu)\right) - 1 - t\frac{\rho_3}{6}(iu)^3 \right|.$$

To show that this expression is $o(t^{-1/2})$ we will apply the following inequality from Feller's book (see page 534 in [6]). It states that for arbitrary complex numbers α, β and $\gamma \geq \max\{|\alpha|, |\beta|\}$ one has

$$(14) \quad \left| e^\alpha - 1 - \beta \right| \leq \left(|\alpha - \beta| + \frac{1}{2}|\beta|^2 \right) e^\gamma.$$

In our case we choose

$$\alpha := -iud_t + \frac{t}{6}(iu)^3\Lambda'''(is) + iu\epsilon_t^\#(iu) + \delta(iu)$$

and

$$\beta := \frac{t\rho_3}{6}(iu)^3.$$

With this choice of α and β we obtain the following bound for K_t in (14)

$$K_t(u) \leq \left(|\alpha - \beta| + \frac{1}{2}|\beta|^2 \right) e^\gamma.$$

We now seek bounds for $|\alpha - \beta|, |\beta|$ and $\gamma \geq \max\{|\alpha|, |\beta|\}$. By proposition 1.7, Λ is analytic in α and $\Lambda'''(0) = \rho_3$. It follows that Λ''' is continuous. Therefore, we can choose B small enough such that

$$(15) \quad |\Lambda'''(is) - \rho_3| < 6\epsilon, \quad \text{for } |s| < B/\sigma.$$

Furthermore, we choose B small enough such that

$$(16) \quad \frac{1}{6}|\Lambda'''(is)| \leq \frac{\sigma^3}{4B} \quad \text{and} \quad \frac{B\rho_3}{6\sigma^3} \leq \frac{1}{4}.$$

Using Proposition 1.8 and (15) we obtain

$$\begin{aligned} |\alpha - \beta| &\leq \frac{t}{6}|u|^3|\Lambda'''(is) - \rho_3| + \frac{|u|}{t} + Cu^2 \\ &\leq \epsilon t|u|^3 + \frac{|u|}{t} + Cu^2. \end{aligned}$$

Further, if we use (16), (12) and the fact that $|u| \leq B/\sigma$ we obtain that for large t

$$\begin{aligned} |\alpha| &\leq t|u|^3 \frac{1}{6} \sup_{|u| < B/\sigma} |\Lambda'''(iu)| + |u|(|d_t| + |\epsilon_t^\#(iu)|) + |\delta(iu)| \\ &\leq t|u|^3 \frac{\sigma^3}{4B} + \frac{|u|}{t} + \frac{\epsilon}{2} \\ &\leq \frac{1}{4}tu^2\sigma^2 + \epsilon. \end{aligned}$$

On the other hand using again (16) and the fact that $|u| \leq B/\sigma$, we obtain

$$|\beta| = \frac{t\rho_3}{6}|u|^3 \leq \frac{u\sigma}{B} \frac{1}{4}t\sigma^2u^2 \leq \frac{1}{4}t\sigma^2u^2.$$

From those bounds for α and β , we see that the choice $\gamma := \frac{1}{4}tu^2\sigma^2 + \epsilon$ satisfies $\gamma \geq \max\{|\alpha|, |\beta|\}$. It now follows from inequality (14) that

$$K_t(u) \leq e^{-\frac{1}{4}tu^2\sigma^2 + \epsilon} \left(\epsilon t|u|^3 + Cu^2 + \frac{|u|}{t} + \frac{t\rho_3|u|^3}{12} \right).$$

This yields that I_2 is bounded by

$$I_2 \leq \frac{1}{\pi} \int_{-B/\sigma}^{B/\sigma} e^{-\frac{1}{4}tu^2\sigma^2 + \epsilon} \left(\epsilon t|u|^3 + Cu^2 + \frac{|u|}{t} + \frac{t\rho_3|u|^3}{12} \right) \frac{du}{|u|}.$$

We still have to prove that the integral in the previous formula is of order $o(t^{-1/2})$. With the change of variable $\omega = u(\sigma\sqrt{t})$ we obtain

$$\begin{aligned} I_2 &\leq \frac{e^\epsilon}{\pi} \int_{-B\sqrt{t}}^{B\sqrt{t}} e^{-\frac{\omega^2}{4}} \left(\epsilon \frac{\omega^2}{\sqrt{t}\sigma^3} + C \frac{|\omega|}{t\sigma^2} + \frac{1}{t^{3/2}\sigma} + \frac{\rho_3\omega^2}{12\sqrt{t}\sigma^3} \right) d\omega \\ &\leq \frac{\epsilon}{\sqrt{t}} \frac{e^\epsilon}{\pi} \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{4}} \left(\omega^2 + \frac{\rho_3\omega^2}{12\sigma^3} \right) d\omega + O(t^{-1}). \end{aligned}$$

This shows that the integral I_2 is of order $o(t^{-1/2})$ and the proof is archived. \square

1.2. Auxillary results for the proof of Theorem 1.1. The following results have been used in the proof of the previous theorem.

Lemma 1.2. *Suppose that the semigroup $(P^t)_{t \geq 0}$ satisfies assumption (4). Then for all $t \geq 0$ the family $\alpha \mapsto \hat{P}_\alpha^t; \alpha \in \mathbb{C}$ of bounded operators on L_V^∞ is analytic in the complex plane.*

Proof. The analyticity of the exponential function together with the boundedness of the function f yield the existence of a constant $K_f > 0$ such that

$$\left| \int_0^t f(X_s) ds e^{\alpha_0 \int_0^t f(X_s) ds} - \frac{1}{\alpha_0 - \alpha} \left(e^{\alpha_0 \int_0^t f(X_s) ds} - e^{\alpha \int_0^t f(X_s) ds} \right) \right| \leq K_f |\alpha_0 - \alpha|.$$

Note that since f is bounded, the operator

$$P'h(x) := \mathbb{E}_x \left[\int_0^t f(X_s) ds e^{\alpha_0 \int_0^t f(X_s) ds} h(X_t) \right]$$

is a bounded operator on L_V^∞ . Using assumption (4), we then have for all $h \in L_V^\infty$ satisfying $\|h\|_V \leq 1$ that

$$\begin{aligned} & \frac{1}{V(x)} \left| P'h(x) - \frac{1}{\alpha_0 - \alpha} (\hat{P}_{\alpha_0}^t h(x) - \hat{P}_\alpha^t h(x)) \right| \\ & \leq \frac{K_f}{V(x)} |\alpha_0 - \alpha| \mathbb{E}_x[V(X_t)] \\ & = \frac{K_f}{V(x)} |\alpha_0 - \alpha| \int_0^t P^s L_{\text{ext}} V(x) ds \\ & \leq \frac{K_f}{V(x)} |\alpha_0 - \alpha| \left(-\delta \int_0^t P^s V(x) ds + b \int_0^t P^s s(x) ds \right) = O(|\alpha_0 - \alpha|). \end{aligned}$$

This shows that

$$\sup_{\|h\|_V=1} \left\| P'h(x) - \frac{1}{\alpha_0 - \alpha} (\hat{P}_{\alpha_0}^t h(x) - \hat{P}_\alpha^t h(x)) \right\|_V = O(|\alpha_0 - \alpha|).$$

Thus $\alpha \mapsto \hat{P}_\alpha^t$ is complex differentiable in $\alpha_0 \in \mathbb{C}$. This proves our result. \square

Proposition 1.3. *Suppose that the semi group $(P^t)_{t \geq 0}$ is V -uniform and satisfies assumption (4). Then the Feynman-Kac semi group $(\hat{P}_\alpha^t)_{t \geq 0}$ is V -uniform for all α from an open neighbourhood of zero in the complex plane.*

Proof. This follows from the spectral characterization of V -uniformity and spectral continuity with respect to closed curves as described in Kato's book (see [11] p.379). Note that Lemma 1.2 shows that for fixed $t \geq 0$ the family of operators (\hat{P}_α^t) is a type A holomorphic family of operators. \square

The following proposition corresponds to Proposition 4.5 in [12] for the discrete case. The proof that we present follows the line of arguments presented there with minor modifications. We add it here for the convenience of the reader, since its proof introduces many notions, which are used later.

Proposition 1.4. *Suppose that the semi group $(\hat{P}_\alpha^t)_{t \geq 0}$ is V -uniform for all α in a neighborhood of zero and that the semi group $(P^t)_{t \geq 0}$ satisfies assumption (4). Then the generalized principal eigenvalue λ_α is an analytic function of α in a neighborhood of zero and so is $\check{f}_\alpha(x)$ for any fixed $x \in \mathbb{R}^d$.*

Proof. We know from Lemma 1.2 that the function $\alpha \mapsto \hat{P}_\alpha^1$ is analytic. For $\alpha = 0$, we have $\lambda_0 = 1$ and $\lambda_0 \notin \mathcal{S}(\hat{P}_0^1 - s_0 \otimes \nu_0)$, where $s_0 \otimes \nu_0$ is the linear form, which corresponds to the projection to the one dimensional eigenspace associated with λ_0 . It

follows from spectral continuity that for α in a sufficiently small neighborhood of zero, we also have $\lambda_\alpha \notin \mathcal{S}(\hat{P}_\alpha^1 - s_0 \otimes \nu_0)$. We define the potential operator

$$\hat{U}_{z,\alpha} := (Iz - \hat{P}_\alpha^1 + s_0 \otimes \nu_0)^{-1}$$

Then, since

$$\hat{U}_{z,0} := (Iz - \hat{P}_0^1 + s_0 \otimes \nu_0)^{-1}$$

is a bounded linear operator for z in a neighborhood of λ_0 , it follows that $\hat{U}_{z,\alpha}$ is a bounded linear operator for z in a sufficiently small neighbourhood around λ_0 . Since $\alpha \mapsto \hat{P}_\alpha^1$ is analytic and since taken the inverse is an analytic operation, it follows that $\hat{U}_{z,\alpha}$ is analytic for (z, α) in a neighborhood O of $(\lambda_0, 0)$. On the other hand, by Proposition 5.2 of [14] we have the following equality

$$(17) \quad [Iz - \hat{P}_\alpha^1]^{-1} = \hat{U}_{z,\alpha} + \frac{(\hat{U}_{z,\alpha}s_0) \otimes (\nu_0\hat{U}_{z,\alpha})}{1 - \nu_0\hat{U}_{z,\alpha}s_0}.$$

Since the left side of (17) is known to have a singularity in $z = \lambda_\alpha$, it follows that

$$\nu_0\hat{U}_{\lambda_\alpha} s_0 = 1.$$

In particular it follows from this that $\hat{U}_{\lambda_\alpha} s_0 \neq 0$. Applying $(Iz - \hat{P}_\alpha^1)$ to (17) from the left and s_0 from the right, we obtain

$$(18) \quad s_0 = (Iz - \hat{P}_\alpha^1)\hat{U}_{z,\alpha}s_0 + (Iz - \hat{P}_\alpha^1) \frac{(\hat{U}_{z,\alpha}s_0)(\nu_0\hat{U}_{z,\alpha}s_0)}{1 - \nu_0\hat{U}_{z,\alpha}s_0}.$$

Multiplying (18) by $(1 - \nu_0\hat{U}_{z,\alpha}s_0)$, it follows that

$$(19) \quad (1 - \nu_0\hat{U}_{z,\alpha}s_0)s_0 = (1 - \nu_0\hat{U}_{z,\alpha}s_0)(Iz - \hat{P}_\alpha^1)\hat{U}_{z,\alpha}s_0 + (Iz - \hat{P}_\alpha^1)(\hat{U}_{z,\alpha}s_0)(\nu_0\hat{U}_{z,\alpha}s_0).$$

Then setting $z = \lambda_\alpha$ in (19) we obtain

$$(I\lambda_\alpha - \hat{P}_\alpha^1)\hat{U}_{\lambda_\alpha,\alpha}s_0 = 0.$$

It follows that the eigenfunction \check{f}_α can be written as

$$\check{f}_\alpha = \hat{U}_{\lambda_\alpha,\alpha}s_0.$$

By applying ν_0 on the left to (17) and doing the same argument we can show that

$$\check{\mu}_\alpha = \nu_0\hat{U}_{\lambda_\alpha,\alpha},$$

is a left eigenmeasure to the eigenvalue λ_α of P_α^1 . If we define the analytic function $\mathcal{J}(z, \alpha) = \nu_0(\hat{U}_{z,\alpha}s_0)$, $z, \alpha \in \mathbb{C}$, we have

$$\mathcal{J}(\lambda_\alpha, \alpha) = \nu_0(\hat{U}_{\lambda_\alpha} s_0) = 1$$

and by differentiation we obtain

$$\frac{\partial}{\partial z} \mathcal{J}(z, \alpha) \Big|_{z=\lambda_\alpha} = \nu_0(-\hat{U}_{\lambda_\alpha,\alpha}^2 s_0) = -\check{\mu}_\alpha(\check{f}_\alpha).$$

Note that $\check{\mu}(\check{f}_0)$ is positive by Perron-Frobenius theorem and the irreducibility of P^1 . Thus, we can find a small neighborhood such that $\check{\mu}_\alpha(\check{f}_\alpha) \neq 0$. It follows from the implicit function theorem that λ_α is an analytic function of α . From the expansion

$$\check{f}_\alpha = \hat{U}_{\lambda_\alpha,\alpha}s_0 = \sum_{n=0}^{\infty} \lambda_\alpha^{-n-1} (\hat{P}_\alpha - s_0 \otimes \nu_0)^n s_0$$

we also can deduce that \check{f}_α is analytic as a composition of two analytic functions. \square

The following proposition corresponds to the statements P1 and P2 in the proof of the discrete Edgeworth expansion in [12]. In the discrete case, they rely on Theorem 4.1 and 4.2 from [12]. Continuous counterparts of those two theorems are stated in [12] as Theorem 4.16 and 4.17.

Proposition 1.5. *For $x \in \mathbb{R}^d$ let $m_t(\alpha) := \mathbb{E}_x[e^{\alpha S_t}]$, $t \geq 1, \alpha \in \mathbb{C}$. i) There exists an open neighbourhood O of zero in the complex plane and a family of functions ϵ_t satisfying*

$$m_t(\alpha) = \exp(t\Lambda(\alpha))(\check{f}_\alpha(x) + |\alpha|\epsilon_t(\alpha)),$$

and $\sup_{\alpha \in O} |\epsilon_t(\alpha)| \rightarrow 0$ exponentially fast as $t \rightarrow \infty$.

ii) *If f is strongly nonlattice, there exist $0 < \omega_0 < \omega_1 < \infty$, $a > 0$ and a family of function ϵ'_t satisfying*

$$m_t(\alpha) = \exp(t\Lambda(\operatorname{Re}(\alpha)))\epsilon'_t(\alpha),$$

and $\sup_{\alpha \in [-a, a] \times [i\omega_0, i\omega_1]} |\epsilon'_t(\alpha)| \rightarrow 0$ exponentially fast as $t \rightarrow \infty$.

Proof. It is proved in Theorem 4.16 from [12] that there exist constants $B_0 > 0$ and $b_0 > 0$ such that for all $x \in \mathbb{R}^d$ and $t > 0$ one has

$$\left| \mathbb{E}_x \left[\exp \left(\alpha S(t) - t\Lambda(\alpha) \right) \right] - \check{f}_\alpha \right| \leq B_0 |\alpha| \exp(-b_0 t) V(x),$$

where \check{f}_α is the eigenvectors of the operator \hat{P}_α^1 associated to the eigenvalue λ_α having absolute value equal to the spectral radius $\hat{\rho}_\alpha$. Multiplying both sides with $\exp(t\Lambda(\alpha))$ yields

$$\left| \mathbb{E}_x [\exp(\alpha S(t))] - \exp(t\Lambda(\alpha))\check{f}_\alpha \right| \leq |\alpha|\epsilon_t(\alpha) \exp(t\Lambda(\alpha)),$$

where the function $|\epsilon_t(\alpha)| := B_0 e^{-b_0 t} V(x)$ converges exponentially fast as $t \rightarrow \infty$.

It is also proved in Theorem 4.17 from [12] that for any $0 < \omega_0 < \omega_1 < \infty$ there exist $b_0 > 0$ and $B_0 < \infty$ such that for all α satisfying $-\bar{\alpha} \leq \operatorname{Re}(\alpha) \leq \bar{\alpha}$ and $\omega_0 \leq |\operatorname{Im}(\alpha)| \leq \omega_1$.

$$\left| \mathbb{E}_x \left[\exp \left(\alpha S(t) - t\Lambda(\operatorname{Re}(\alpha)) \right) \right] \right| \leq B_0 V(x) \exp(-b_0 t).$$

Multiplying both sides with $\exp(t\Lambda(\operatorname{Re}(\alpha)))$ yields

$$\begin{aligned} m_t(\alpha) &= \left| \mathbb{E}_x \left[\exp \left(\alpha S(t) \right) \right] \right| \leq B_0 V(x) \exp(-b_0 t) \exp(t\Lambda(\operatorname{Re}(\alpha))) \\ &= \exp(t\Lambda(\operatorname{Re}(\alpha)))\epsilon'_t(\alpha), \end{aligned}$$

where the function $\epsilon'_t(\alpha) = B_0 V(x) \exp(-b_0 t)$ converges to zero exponentially fast uniformly for all α satisfying $-\bar{\alpha} \leq \operatorname{Re}(\alpha) \leq \bar{\alpha}$ and $\omega_0 \leq |\operatorname{Im}(\alpha)| \leq \omega_1$. \square

The following proposition corresponds to Proposition 4.9 in [12]. Note however, that the proof, given there for the discrete case, relies on some particularity of the discrete time Feynman-Kac semi group

$$\hat{P}_\alpha^n g(x) := \mathbb{E}_x \left[e^{\alpha \sum_{i=1}^n f(X_i)} g(X_n) \right],$$

which has the advantage of factorizing at single time step level; i.e.:

$$(20) \quad \hat{P}_\alpha^1 g(x) = \mathbb{E}_x [e^{\alpha f(X_1)} g(X_1)] = P^1(g e^{\alpha f}).$$

To prove the result in the continuous time case, we will use the factorization of the generator $\hat{L}_{\alpha,V}$ of the Feynman-Kac semi group (\hat{P}_α^t) , which was given in (5).

Proposition 1.6. *There exist a neighborhood O of zero in \mathbb{R} satisfying the following properties :*

i) *For all $a \in O$ we have $\frac{d}{da}\Lambda(a) = \check{\pi}_a(f)$, where $\check{\pi}_a$ is the invariant probability measure for $\check{L}_{a,V}$.*

ii) *For all $a \in O$, the function $\hat{F}_a = \frac{d}{da} \log \check{f}_a$ is a solution to the equation*

$$(21) \quad \check{L}_{a,V} \hat{F}_a = \check{\pi}_a(f) - f.$$

iii) *In particular, for $a = 0$, we have $\check{\pi}_0 = \pi$, $\check{L}_0 = L$ and we obtain the Poisson equation*

$$L_V \hat{F}_0 = \pi(f) - f.$$

Proof. We will see later in Proposition 1.8 that for real a in a small neighborhood of zero the eigenfunction \check{f}_a is strictly positive. Therefore, we can define the following auxiliary operator

$$\check{L}_{a,V} = (I_{\check{f}_a})^{-1} (\hat{L}_{a,V} - \Lambda(a)) I_{\check{f}_a},$$

where for a given function g we introduce the multiplication operator $I_g h(x) := g(x)h(x)$. The differentiation of the eigenvalue equation (6) gives

$$f \check{f}_a + \hat{L}_{a,V} \check{f}_a' = \Lambda'(a) \check{f}_a + \Lambda_a \check{f}_a'.$$

Dividing this identity by \check{f}_a , we obtain

$$f + \check{f}_a^{-1} \hat{L}_{a,V} \check{f}_a' = \Lambda'(a) \check{f}_a^{-1} \check{f}_a + \Lambda(a) \check{f}_a^{-1} \check{f}_a'.$$

This can be written as

$$(I_{\check{f}_a})^{-1} (\hat{L}_{a,V} - \Lambda(a)) I_{\check{f}_a} \frac{\check{f}_a'}{\check{f}_a} = \Lambda'(a) \check{f}_a^{-1} \check{f}_a - f.$$

This implies

$$(22) \quad \check{L}_{a,V} \hat{F}_a = \Lambda'(a) - f.$$

By choice of a suitable norming constant k_a , we obtain the probability measure

$$\check{\pi}_a = k_a \check{\mu}_a I_{\check{f}_a},$$

where $\check{\mu}_a$ is the eigenmeasure associated to \hat{P}_a^1 which for all $t \geq 0$ also satisfies :

$$\check{\mu}_a \hat{P}_a^t = \lambda_a^t \check{\mu}_a.$$

We will prove in Proposition 1.8 that \check{f}_a is strictly positive for real a in small neighborhood of zero. We have

$$\check{\pi}_a(|\hat{F}_a|) = \check{\pi}_a(|\check{f}_a'/\check{f}_a|) = k_a \check{\mu}_a(|\check{f}_a'|) < \infty$$

and

$$\begin{aligned}
\check{\pi}_a \check{L}_{a,V} 1_{\mathbb{R}^d} &= k_a \check{\mu}_a I_{\check{f}_a} \left((I_{\check{f}_a})^{-1} (\hat{L}_a - \Lambda(a)) I_{\check{f}_a} \right) 1_{\mathbb{R}^d} \\
&= k_a \check{\mu}_a I_{\check{f}_a} \left((I_{\check{f}_a})^{-1} \hat{L}_a \check{f}_a \right) - k_a \check{\mu}_a I_{\check{f}_a} \left((I_{\check{f}_a})^{-1} \Lambda(a) \check{f}_a \right) \\
&= k_a \check{\mu}_a (\hat{L}_a \check{f}_a) - k_a \check{\mu}_a (\Lambda(a) \check{f}_a) \\
&= 0.
\end{aligned}$$

It follows that the probability measure $\check{\pi}_a$ is invariant. Since \hat{F}_a is a solution to the Poisson equation it follows that the right side of (22) is of π measure zero. We deduce from this that $\Lambda'(a) = \check{\pi}_a(f)$. This proves the points (i) and (ii). The proof of (iii) follows by considering the special case $a = 0$. \square

The following proposition corresponds to statements P4 and P5 in [12] for the discrete case. The proof follows the same line of arguments, while substituting the discrete generator by the continuous time generator. However, instead of expressing the coefficients as series, we express them in terms of integrated solutions of suitable Poisson equations.

Proposition 1.7. *The function $\alpha \mapsto \Lambda(\alpha)$ is analytic in an open neighbourhood O of the origin in the complex plane. Further, one has $\Lambda(0) = \Lambda'(0) = 0$ and*

$$\begin{aligned}
\sigma^2 &:= \Lambda''(0) = 2 \int_{\mathbb{R}^d} f g d\pi, \\
\rho_3 &:= \Lambda'''(0) = 6 \int_{\mathbb{R}^d} f h d\pi,
\end{aligned}$$

where g and h are solutions to the two equations $L_V g = -f$ and $L_V h = fg - \pi(fg)$.

Proof. We saw in Proposition 1.4 that λ_α is analytic in a neighbourhood O of zero in the complex plane. Since we have $\Lambda(\alpha) = \log(\lambda_\alpha)$, this implies $\Lambda(\alpha)$ is analytic in O . It follows that

$$\Lambda'(\alpha) = \frac{\lambda'_\alpha}{\lambda_\alpha} \quad \text{and} \quad \Lambda''(\alpha) = \frac{\lambda''_\alpha \lambda_\alpha - \lambda'_\alpha \lambda'_\alpha}{\lambda_\alpha^2}.$$

For $\alpha = 0$ the operator $\hat{P}_0^1 = P^1$ is stochastic and it follows that $\lambda_0 = 1$, which implies $\Lambda(0) = 0$. Further, by Proposition 1.6 we have $\Lambda'(0) = \pi(f) = 0$. The following computation is in its spirit the same as the one given in [12] for the time discrete case. However, instead of using the semi groupe \hat{P}_α we use its generator $\hat{L}_{\alpha,V}$ in the argument. Since \check{f}_α is eigenfunction of \hat{P}_α associated to λ_α it follows that it is eigenfunction of $\hat{L}_{\alpha,V}$ associated to the eigenvalues $\Lambda(\alpha)$; i.e :

$$(23) \quad \hat{L}_{\alpha,V} \check{f}_\alpha = \Lambda(\alpha) \check{f}_\alpha.$$

Without loss of generality, we will suppose that \check{f}_α is normalized to satisfy $\pi(\check{f}_\alpha) = 1$. This implies that for $H_\alpha = \Lambda(\alpha) Id - \hat{L}_{\alpha,V} + 1 \otimes \pi$, we obtain

$$H_\alpha \check{f}_\alpha = \Lambda(\alpha) \check{f}_\alpha - \hat{L}_{\alpha,V} \check{f}_\alpha + 1_{\mathbb{R}^d} \pi(\check{f}_\alpha) = 1_{\mathbb{R}^d} \pi(\check{f}_\alpha) = 1_{\mathbb{R}^d},$$

This yields the equation

$$\check{f}_\alpha = H_\alpha^{-1} 1_{\mathbb{R}^d}.$$

Moreover, formula (5) for the Feynman-Kac generator $\hat{L}_{\alpha,V}$ yields

$$\frac{d}{d\alpha}H_\alpha g = \Lambda'(\alpha)g - fg \quad \text{and} \quad \frac{d^k}{d^k\alpha}H_\alpha g = \Lambda^{(k)}(\alpha)g, \quad \text{for } k \geq 2.$$

It follows from this that

$$\frac{d}{d\alpha}\Big|_{\alpha=0}H_\alpha g = \Lambda'(0)g - fg = -fg \quad \text{and} \quad \frac{d^k}{d^k\alpha}\Big|_{\alpha=0}H_\alpha g = \Lambda^{(k)}(0)g, \quad \text{for } k \geq 2.$$

Moreover, for all $k \geq 1$ we have

$$(24) \quad \pi\left(\frac{d^k \check{f}_\alpha}{d^k \alpha}\right) = \frac{d^k}{d^k \alpha}\pi(\check{f}_\alpha) = \frac{d^k}{d^k \alpha}1 = 0, \quad \text{for } \alpha \text{ in some neighborhood of zero.}$$

By differentiation of (23), we obtain

$$\begin{aligned} \frac{d^2}{d^2\alpha}\Big|_{\alpha=0}\check{f}_\alpha &= H_0^{-2}(H_0')^2H_0^{-1}1_{\mathbb{R}^d} - H_0^{-1}H_0''H_0^{-1}1_{\mathbb{R}^d} + H_0^{-1}(H_0')^2H_0^{-2}1_{\mathbb{R}^d} \\ &= 2H_0^{-1}H_0'H_0^{-1}H_0'H_0^{-1}1_{\mathbb{R}^d} - H_0^{-1}H_0''H_0^{-1}1_{\mathbb{R}^d}. \end{aligned}$$

Note that we have $H_0^{-1} = [-L_V + \Pi]^{-1}$ where $\Pi = 1_{\mathbb{R}^d} \otimes \pi$ is a rank one operator. Since $H_0^{-1}1_{\mathbb{R}^d} = \check{f}_0 = 1_{\mathbb{R}^d}$ (see Proposition 1.8), it follows that

$$(25) \quad \frac{d^2}{d^2\alpha}\Big|_{\alpha=0}\pi(\check{f}_\alpha) = 2\Pi H_0^{-1}(fH_0^{-1}f) - \Pi(H_0^{-1}\Lambda''(0)).$$

On the other hand, for $\psi \in \mathcal{H}$ the equation $H_0^{-1}(g) = \psi$ is equivalent to

$$g = -L_V\psi + \Pi(\psi).$$

It follows from the invariance of π with respect the semi group (P^t) that

$$\begin{aligned} \Pi(g) &= \Pi(-L_V\psi) + \Pi(\psi)\Pi(1) = \Pi\left(-\frac{d}{dt}P^t\psi\right) + \Pi(\psi) \\ &= -\frac{d}{dt}\Pi(\psi) + \Pi(\psi) = \Pi(\psi) = \Pi(H_0^{-1}g). \end{aligned}$$

Therefore, we see from (25) that

$$\frac{d^2}{d^2\alpha}\Big|_{\alpha=0}\pi(\check{f}_\alpha) = 2\Pi(fH_0^{-1}f) - \Lambda''(0) = -2\int_{\mathbb{R}^d}fL_V^{-1}fd\pi - \Lambda''(0).$$

The last equality follows from the fact that $\Pi(f) = 0$ implies $H_0^{-1}f = -L_V^{-1}f$. It follows from (24) that

$$\sigma^2 = \Lambda''(0) = -2\int_{\mathbb{R}^d}fL_V^{-1}fd\pi = 2\int_{\mathbb{R}^d}fgd\pi$$

where g is the solution of the equation $L_Vg = -f$. This finish the proof of i).

Again by differentiation of $\check{f}_\alpha = H_\alpha^{-1}1$ we obtain :

$$\begin{aligned} \frac{d^3}{d^3\alpha}\Big|_{\alpha=0}\check{f}_\alpha &= -6H_0^{-1}H_0'H_0^{-1}H_0'H_0^{-1}H_0'H_0^{-1}1_{\mathbb{R}^d} + 3H_0^{-1}H_0'H_0^{-1}H_0''H_0^{-1}1_{\mathbb{R}^d} \\ &\quad + 3H_0^{-1}H_0''H_0^{-1}H_0'H_0^{-1}1_{\mathbb{R}^d} - H_0^{-1}H_0'''H_0^{-1}1_{\mathbb{R}^d}. \end{aligned}$$

Since $\Pi(fH_0^{-1}1_{\mathbb{R}^d}) = \Pi f = \pi(f) = 0$ and $\Pi(H_0^{-1}f) = \Pi f = 0$, it follows that

$$\begin{aligned} \frac{d^3}{d^3\alpha} \Big|_{\alpha=0} \pi(\check{f}_\alpha) &= -6\Pi(fH_0^{-1}(fH_0^{-1}f)) + 3\Lambda''(0)\Pi(fH_0^{-1}1_{\mathbb{R}^d}) \\ &\quad + 3\Lambda''(0)\Pi(H_0^{-1}f) - \Lambda'''(0)\Pi(1_{\mathbb{R}^d}) \\ &= 6 \int_{\mathbb{R}^d} fH_0^{-1}(fL_V^{-1}f)d\pi - \Lambda'''(0) \\ &= -6 \int_{\mathbb{R}^d} fH_0^{-1}(fg)d\pi - \Lambda'''(0) \\ &= 6 \int_{\mathbb{R}^d} fhd\pi - \Lambda'''(0), \end{aligned}$$

where we put $h := -H_0^{-1}(fg)$. Note that it follows that

$$fg = -H_0h = L_Vh - \Pi h = L_Vh + \Pi(H_0^{-1}fg) = L_Vh + \Pi(fg).$$

Thus g and h are the solutions of the equations $L_Vg = -f$ and $L_Vh = fg - \pi(fg)$. It follows from (24) that

$$\rho^3 = \Lambda'''(0) = 6 \int_{\mathbb{R}^d} fhd\pi.$$

The proof is complete. \square

The part (i) of the following proposition corresponds to statement P6 in [12]. The parts (ii) and (iii) corresponds to statement P7 in [12]. Its proof carries over without difficulties to the continuous case

Proposition 1.8. *Let g be the unique solution of the Poisson equation $L_Vg = -f$. Then the two following statements hold :*

- i) *For $d_t = \mathbb{E}_x[S_t] - g$, we have $|d_t| \rightarrow 0$ exponentially fast as $t \rightarrow \infty$ uniformly in x .*
- ii) *The eigenfunction \check{f}_α is analytic with respect to α in a neighborhood of zero in the complex plane. It satisfies $\check{f}_0 = 1$ and it is strictly positive for all real α in a neighborhood of zero.*
- iii) *There is some $\omega_0 > 0$ (depending on x) and a constant C such that*

$$(26) \quad \delta(i\omega) := |\log \check{f}_{i\omega}(x) - i\omega g| \leq C\omega^2, \quad \text{for all } |\omega| \leq \omega_0.$$

Proof. : From the definition of S_t we have

$$\mathbb{E}_x[S_t] = \mathbb{E}_x \left[\int_0^t f(X_s) ds \right] = \int_0^t \mathbb{E}_x[f(X_s)] ds = \int_0^t P^s f(x) ds.$$

Also note that we have the following representation of the solution of the Poisson equation

$$g = \int_0^\infty P^s f(x) ds.$$

When restricted to functions with mean zero with respect to π , the semi group (P_t) has spectral radius $\rho^* < \infty$ on L^∞ . Since f is of π -measure zero and since the absolute value of f is bounded by one, we have $\|P_t f\|_{\text{sup}} = \|P^t f\|_\infty \leq K e^{t \log \rho^*}$ for all $t > 0$. This yields

$$|d_t(x)| = \left| \mathbb{E}_x[S_t] - g \right| \leq \int_t^\infty |P^s f(x)| ds \leq K \int_t^\infty e^{s \log \rho^*} ds = -\frac{K}{\log \rho^*} e^{t \log \rho^*}.$$

Hence $\sup_x |d_t(x)| \rightarrow 0$ exponentially as $t \rightarrow \infty$.

In the following we will use the notation $(\hat{P}_a^n)_{n \in \mathbb{N}}$ to denote the discrete semigroup, which results from observing $(\hat{P}_a^t)_{t \geq 0}$ at integer time. For real $a \in [-\bar{a}, \bar{a}]$ and $\bar{\lambda} \geq \max(\lambda_{\bar{a}}, \lambda_{-\bar{a}})$, we define the operator \hat{G}_a by

$$\hat{G}_a = \sum_{n=0}^{\infty} \bar{\lambda}^{-n-1} \hat{P}_a^n.$$

Note that if the semi group $(P^t)_{t \geq 0}$ is ψ -irreducible then the discrete semi group $(\hat{P}_a^n)_{n \in \mathbb{N}}$ also is. For any $\theta > 0$ we define the resolvent associated with the discrete semi group $(\hat{P}_a^n)_{n \in \mathbb{N}}$ as

$$(27) \quad R_\theta := \sum_{n=0}^{\infty} (1 - e^{-\theta}) e^{-n\theta} \hat{P}_a^n.$$

It is proved in Proposition 5.5.5 from [13] that for a ψ -irreducible discrete semi group $(\hat{P}_a^n)_{n \in \mathbb{N}}$ on can always find a $\theta > 0$, a function $s : \mathbb{R}^d \rightarrow]0, \infty]$ and a signed measures ν on \mathcal{B} such that

$$R_\theta(x, A) > s(x)\nu(A), x \in \mathbb{R}^d, A \in \mathcal{B}.$$

Since by assumption $|f| \leq 1$, this implies for $x \in \mathbb{R}^d$ and $A \in \mathcal{B}$

$$\begin{aligned} \hat{G}_a(x, A) &= \sum_{n=0}^{\infty} \bar{\lambda}^{-n-1} \hat{P}_a^n 1_A = \frac{1}{\bar{\lambda}} \sum_{n=0}^{\infty} e^{-n(\log \bar{\lambda} + \bar{a})} \mathbb{E}_x \left[e^{-a \int_0^n f(X_s) ds} 1_A(X_n) \right] \\ &\geq \frac{1}{\bar{\lambda}} \sum_{n=0}^{\infty} e^{-n(\log \bar{\lambda} + \bar{a})} P_a^n 1_A(x) \geq \frac{1}{\bar{\lambda}} R_{\log \bar{\lambda} + \bar{a}} 1_A(x) \geq \frac{1}{\bar{\lambda}} s(x)\nu(A). \end{aligned}$$

It follows that for all $a \in [-\bar{a}, \bar{a}]$ and for all $x \in \mathbb{R}^d$

$$(\bar{\lambda} - \lambda_a)^{-1} \check{f}_a(x) = \sum_{n=0}^{\infty} \bar{\lambda}^{-n-1} \lambda_a^n \check{f}_a(x) = \sum_{n=0}^{\infty} \bar{\lambda}^{-n-1} \hat{P}_a^n \check{f}_a(x) = \hat{G}_a \check{f}_a(x) \geq s(x)\nu(\check{f}_a) > 0.$$

We deduce that $\check{f}_a(x)$ is strictly positive. We know from (i) in Proposition 1.5 that $\check{f}_0 = 1_{\mathbb{R}^d}$. Expanding $\log \check{f}_{i\omega}$ in a Taylor series around zero yields $\log \check{f}_{i\omega} = i\omega \hat{F} - \omega^2 \hat{F}'$. This implies that there exist a constant C such that

$$|\log \check{f}_{i\omega}(x) - i\omega \hat{F}| \leq C\omega^2.$$

This finises the proof. \square

2. ASYMPTOTIC ANALYSIS OF EDGEWORTH COEFFICIENT

In this section we want to analyze the asymptotic behavior of the Edgeworth coefficient as some drift enhancement parameter grows to infinity. We first fix the setting for our study. for this let $U : \mathbb{R}^d \rightarrow \mathbb{R}$ be a given \mathcal{C}^1 function such that $\pi(dx) := e^{-U(x)} dx$ is a the probability measure on \mathbb{R}^d . Let $(X_t)_{t \geq 0}$ be the strong solution of the stochastic differential equation

$$(28) \quad dX_t = -\nabla U(X_t) dt + \sqrt{2} dB_t,$$

where $(B_t)_{t \geq 0}$ is a standard Brownian motion on \mathbb{R}^d . Under some assumptions on the fonction U , the diffusion (X_t) has equilibrium distribution π . The generator of the diffusion (X_t) is the differential operator L_0 , given by the following differential operator

$$(29) \quad L_0 f = \Delta f - \nabla U \cdot \nabla f.$$

This generator generates a semi group on $L^2(\mathbb{R}^d, \pi)$ through

$$P^t f(x) = \mathbb{E}_x[f(X(t))]; \quad x \in \mathbb{R}^d, t \geq 0.$$

Perturbing the reversible diffusion (28) by adding a drift term, which preserves the measure π , results in a stochastic equation of the form

$$(30) \quad dX_t^c = -\nabla U(X_t^c) dt + \sqrt{2} dB_t + cb(X_t^c)dt,$$

where b is a \mathcal{C}^1 -vectorfield on \mathbb{R}^d satisfying $\operatorname{div}(be^{-U}) = 0$. Then under some suitable assumptions the resulting diffusion has equilibrium distribution π . The generator of the diffusion (X_t^c) is the differential operator

$$(31) \quad L_{cb}f = L_0f + cb \cdot \nabla f.$$

Let \mathcal{H} be the Hilbert space of mean zero functions given by

$$\mathcal{H} = \left\{ \psi \in \mathcal{L}^2(\mathbb{R}^d, \pi), \int_{\mathbb{R}^d} \psi d\pi = 0 \right\},$$

with the inner product

$$\langle \psi, \phi \rangle := \int_M \bar{\psi} \phi d\pi.$$

Note that the operator L_0 is self adjoint in \mathcal{H} . In the following we assume that L_0 has discrete spectrum with eigenvalues $\lambda_0 = 0 \geq -\lambda_1 \geq -\lambda_2 \geq \dots$ and associated eigenfunctions $\phi_0, \phi_1, \phi_2, \dots$. It is well known that those functions can be chosen to be orthonormal with respect to the scalar product on \mathcal{H} . For all $m \in \mathbb{R}$ we introduce the Hilbert space

$$\mathcal{H}^m := \left\{ \psi = \sum_{i=1}^{\infty} \alpha_i \phi_i \in \mathcal{H}; \sum_{i=1}^{\infty} |\alpha_i|^2 \lambda_i^m < \infty \right\}$$

with scalar product

$$\langle \psi, \phi \rangle_{\mathcal{H}^m} = \sum_{i=1}^{\infty} \bar{\alpha}_i \beta_i \lambda_i^m, \quad \text{where } \psi = \sum_{i=1}^{\infty} \alpha_i \phi_i \text{ and } \phi = \sum_{i=1}^{\infty} \beta_i \phi_i.$$

From the above condition on the spectrum of L_0 for all $f \in \mathcal{H}$ there exists a solution $g \in \mathcal{H}^1$ to the Poisson equation

$$L_0 g = -f.$$

It follows, under suitable conditions on b that for all $f \in \mathcal{H}$ there exists a solution $g \in \mathcal{H}^1$ to the Poisson equation

$$L_{cb}g = -f.$$

Suppose that the diffusion (X_t^c) is V-uniform and that $f : \mathbb{R}^d \rightarrow [-1, 1]$ is strongly nonlattice function with zero mean, then it turns out that

$$S_t := \int_0^t f(X_s^b) ds$$

satisfies the Edgeworth expansion from Theorem 1.1 for all starting point $x \in \mathbb{R}$ with

$$\sigma_c^2 = 2 \int_{\mathbb{R}^d} f g d\pi, \rho_{3,c} = 6 \int_{\mathbb{R}^d} f g h d\pi \quad \text{and} \quad \hat{F}_c = g,$$

where $L_{cb}g = -f$ and $L_{-cb}h = -f$.

Remark : The Theorem 1.1 actually states that

$$\rho_{3,c} = 6 \int_{\mathbb{R}^d} f \tilde{h} d\pi,$$

where \tilde{h} is the solutions to the equations $L_{cb}\tilde{h} = fg - \pi(fg)$. Using the relation $(L_{cb})^* = L_{-cb}$ the expression for $\rho_{3,c}$ can be reexpressed as

$$\begin{aligned} \rho_{3,c} &= 6 \int_{\mathbb{R}^d} f L_{cb}^{-1}(fg - \pi(fg)) d\pi = 6 \int_{\mathbb{R}^d} (fg - \pi(fg)) L_{-cb}^{-1} f d\pi \\ &= 6 \int_{\mathbb{R}^d} f g L_{-cb}^{-1} f d\pi = 6 \int_{\mathbb{R}^d} f g h d\pi. \end{aligned}$$

Note that ρ_c^3, σ_c^2 and \hat{F}_c depend on c . The behavior of the asymptotic variance σ_c^2 as c goes to infinity was studied in [5] and [9]. The next theorem investigates the behavior of $\rho_{3,c}$ and \hat{F}_c as c goes to infinity.

Theorem 2.1. *Suppose that the vector field b is bounded. For all $f : \mathbb{R}^d \rightarrow [-1, 1]$ in \mathcal{H} , we have*

$$\lim_{c \rightarrow \infty} \rho_{3,c} = \langle PL_0^{-1}f, (PL_0^{-1}f)^2 \rangle_{\mathcal{H}^1} \quad \text{and} \quad \lim_{c \rightarrow \infty} \hat{F}_c = -PL_0^{-1}f$$

where P is the orthogonal projection from \mathcal{H}^1 to $N_0 = \ker(b \cdot \nabla) \subset \mathcal{H}^1$.

Proof. We introduce the operator $B = L_0^{-1}b \cdot \nabla$ which turns out to be a compact and antiself-adjoint operator on \mathcal{H}^1 . Then B has eigenfunctions $\{\psi_k\}_{k \geq 1}$ and corresponding eigenvalues $i\mu_k, k \in \mathbb{N}$ where each μ_k is real and $\lim_{k \rightarrow \infty} \mu_k = 0$. For all $f \in \mathcal{H}$ the Poisson equation $L_{cb}g = -f$ has unique solution $g \in \mathcal{H}^1$. This implies that the equation $(L_0 - cb \cdot \nabla)g = f$ can be changed to

$$(32) \quad g + cL_0^{-1}b \cdot \nabla g = -L^{-1}f.$$

We have the following representations in \mathcal{H}^1 :

$$g = Pg + \sum_{k=1}^{\infty} \beta_k \psi_k \quad \text{with} \quad \beta_k \in \mathbb{C}$$

and

$$(33) \quad L_0^{-1}f = PL_0^{-1}f + \sum_{k=1}^{\infty} \alpha_k \psi_k,$$

where $\alpha_k = \langle L_0^{-1}f, \psi_k \rangle_{\mathcal{H}^1} = \langle f, \psi_k \rangle_{\mathcal{H}}$ for all $k \in \mathbb{N}$. If we replace f and $L_0^{-1}f$ by their respective representations in (32), this yields

$$Pg + \sum_{k=1}^{\infty} \beta_k \psi_k + cB \sum_{k=1}^{\infty} \beta_k \psi_k = - \sum_{k=1}^{\infty} \alpha_k \psi_k - PL_0^{-1}f,$$

which is equivalent to

$$Pg + \sum_{k=1}^{\infty} (\beta_k + ic\mu_k\beta_k)\psi_k = - \sum_{k=1}^{\infty} \alpha_k\psi_k - PL_0^{-1}f.$$

By identification of the coefficients in the series we obtain

$$Pg = -PL_0^{-1}f \quad \text{and} \quad \beta_k = -\frac{\alpha_k}{1 + ic\mu_k}.$$

It is clear that when $c \rightarrow \infty$ we have $\beta_k \rightarrow 0$, and it follows that

$$g = L_{cb}^{-1}f \rightarrow Pg = -PL_0^{-1}f \quad \text{in } \mathcal{H}^1.$$

The same argument shows also that

$$g = \hat{F}_c \rightarrow Pg = -PL_0^{-1}f \quad \text{in } \mathcal{H}^1 \text{ as } c \rightarrow \infty.$$

if we apply the same argument equation and it follows that

$$L_{-cb}h = -f,$$

we obtain $h = -L_{-cb}^{-1}f \rightarrow Ph = -PL_0^{-1}f$ in \mathcal{H}^1 as $c \rightarrow \infty$. This implies that

$$\begin{aligned} \lim_{c \rightarrow \infty} \rho_{3,c} &= \lim_{c \rightarrow \infty} 6\langle f, hg \rangle_{\mathcal{H}} \\ &= \lim_{c \rightarrow \infty} 6\langle f, (L_{-cb}^{-1}f)L_{cb}^{-1}f \rangle_{\mathcal{H}} \\ &= \lim_{c \rightarrow \infty} 6\langle L_0^{-1}f, (L_{-cb}^{-1}f)L_{cb}^{-1}f \rangle_{\mathcal{H}^1} \\ &= 6\langle L_0^{-1}f, (PL_0^{-1}f)^2 \rangle_{\mathcal{H}^1} \\ &= 6\langle PL_0^{-1}f, (PL_0^{-1}f)^2 \rangle_{\mathcal{H}^1}, \end{aligned}$$

where in the last equality we used the fact that the product of two elements in \mathcal{N} is again in \mathcal{N} . The theorem is established. \square

2.1. The case of the perturbed Ornstein-Uhlenbeck diffusion. We now want to study an example, where the conditions of Theorem 2.1 are satisfied. Let $U(x) = -\frac{1}{2}\langle x, Dx \rangle$ where D is a symmetric and strictly negative definite real $d \times d$ -matrix. Then the measure $\pi(dx) = Z^{-1}e^{-U(x)} = Z^{-1}e^{\frac{1}{2}\langle x, Dx \rangle}$ is a centered gaussian distribution with covariance matrix $-D^{-1}$. We want to study the linear drift $b(x) = SDx$, where D is a symmetric and strictly positive-definite real matrix. Note that this vectorfield is weighted divergence-free with respect $e^{-U(x)}$; i.e.: $\text{div}(e^{-U}b) = 0$. The unperturbed Ornstein-Uhlenbeck process (X_t^0) has generator

$$(34) \quad L_0f = \Delta f - \langle Dx, \nabla f \rangle.$$

and invariant measure π . It is well established, that L_0 is a self-adjoint operator on $L^2(\mathbb{R}^d, \pi)$ and that it has discrete spectrum. Let

$$L_c f = \Delta f - \langle Dx, \nabla f \rangle + c\langle SDx, \nabla f \rangle$$

be the the generator of the perturbed Ornstein Uhlenbeck process $(X_t^c)_{t \geq 0}$. We now will see that (X_t^c) is V -uniform, if we choose

$$V : \mathbb{R}^d \rightarrow [1, \infty[; \quad x \mapsto -\langle x, Dx \rangle + 1.$$

If we denote by $\nu_i, 1 \leq i \leq d$ the eigenvalues of the matrix D , then we have since they are all strictly negative that

$$\begin{aligned} L_{cb}V(x) &= \Delta V - \langle Dx, \nabla V \rangle + \langle SDx, \nabla V \rangle \\ &= \Delta V - 2\langle Dx, Dx \rangle + 2\langle SDx, Dx \rangle \\ &= -tr(D) - 2\langle Dx, Dx \rangle \\ &\leq -tr(D) + 2 \max_i(\nu_i)V(x) - 2 \max_i(\nu_i) \\ &= b_0 - \delta V(x), \end{aligned}$$

where $b_0 = -tr(D) - 2 \max_i(\nu_i) > 0$ and $\delta = -2 \max_i(\nu_i) > 0$.

Since the generator of the perturbed Ornstein Uhlenbeck semigroupe is elliptic, we know that for all choice of $x \in \mathbb{R}^d$ and $t > 0$ the transition density $y \mapsto p(t, x, y)$ is positive and continuous as a function of x and y . Let ν be the Lebesgue measure restricted to $[0, 1]^d$. We then have for any $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$ that

$$\begin{aligned} R_1(x, A) &= \int_0^\infty e^{-t} P^t(x, A) dt = \int_0^\infty e^{-t} \int_A p(t, x, y) dy dt \\ &\geq \int_0^\infty e^{-t} \int_{A \cap [0, 1]^d} p(t, x, y) dy dt \geq s(x) \nu(A). \end{aligned}$$

with the positive function

$$s(x) := \inf_{y \in [0, 1]^d} \int_0^\infty e^{-t} p(t, x, y) dt.$$

We now show that the diffusion (X_t^c) is ψ -irreducible for $\psi = \ell$, where ℓ is the Lebesgue measure on \mathbb{R}^d . Let $u : \mathbb{R}^d \rightarrow [0, 1]$ be a function satisfying $\ell(u) = \int_{\mathbb{R}^d} u(x) \ell(dx) > 0$. Then we have

$$\begin{aligned} R(x, u) &= \int_{\mathbb{R}^d} R(x, dy) u(y) \\ &= \int_{\mathbb{R}^d} \int_0^\infty \theta e^{-\theta t} P^t(x, dy) u(y) dt \\ &= \int_{\mathbb{R}^d} \int_0^\infty \theta e^{-\theta t} p(t, x, dy) u(y) dt dx > 0. \end{aligned}$$

This shows that the diffusion process (X_t^c) is ψ -irreducible. Similarly, we have

$$\begin{aligned} P^t(x, u) &= \int_{\mathbb{R}^d} P^t(x, dy) u(y) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(x, y, t) u(y) dx dy > 0, \end{aligned}$$

which implies that (X_t^c) is aperiodic. It follows that (X_t^c) is V -uniform. This shows that the perturbed Ornstein Uhlenbeck process (X_t^c) satisfies all conditions for Theorem 1.1 to hold.

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