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ANALYTIC TORSION, DYNAMICAL ZETA FUNCTION, AND THE FRIED CONJECTURE FOR ADMISSIBLE TWISTS

SHU SHEN

ABSTRACT. We show an equality between the analytic torsion and the absolute value at the zero point of the Ruelle dynamical zeta function on a closed odd dimensional locally symmetric space twisted by an acyclic flat vector bundle obtained by the restriction of a representation of the underlying Lie group. This generalises author's previous result for unitarily flat vector bundles, and the results of Bröcker, Müller, and Wotzke on closed hyperbolic manifolds.

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Introduction

The purpose of this article is to study the relation between the analytic torsion and the value at the zero point of the Ruelle dynamical zeta function associated to a flat vector bundle, which is not necessarily unitary, on a closed odd dimensional locally symmetric space of reductive type.

Let Z be a smooth closed manifold. Let F be a complex flat vector bundle on Z. Let $H^\bullet(Z,F)$ be the cohomology of sheaf of locally constant sections of F. We assume $H^\bullet(Z,F)=0$.

Let g^{TZ},g^F be metrics on TZ and F. The analytic torsion $T(F)$ of Ray-Singer [RaSi71] is a spectral invariant defined by a weighted product of zeta regularised determinants the Hodge Laplacian associated with g^{TZ},g^F. When $\dim Z$ is odd, they showed that $T(F)$ does not depend on the metric data.

Ray and Singer [RaSi71] conjectured, which was proved later by Cheeger [Ch79] and Müller [Mü78], that if F is unitarily flat (i.e., the holonomy representation of F is unitary) the analytic torsion coincides with its topological counterpart, the Reidemeister torsion [Re35, Fr35, dR50]. Bismut-Zhang [BZ92] and Müller [Mü93] simultaneously considered generalisations of this result. Müller [Mü93] extended his result to odd dimensional oriented manifolds where only $\det F$ is required to be unitary. Bismut and Zhang [BZ92] generalised the original Cheeger-Müller theorem to arbitrary flat vector bundles with arbitrary Hermitian metrics on a manifold with arbitrary dimension orientable or not.

Milnor [Mi68] initiated the study of the relation between the torsion invariant and a dynamical system. When Z is an orientable hyperbolic manifold, Fried [F86a, F86b] showed an identity between the analytic torsion of an acyclic unitarily flat vector bundle and the value at the zero point of the Ruelle dynamical zeta function of the geodesic flow of Z. He conjectured [F87, p. 66, Conjecture] that similar results should hold true for more general flows. In [Sh18], following previous contributions by Moscovici-Stanton [MoSt91], using Bismut's orbital integral formula [B11], the author affirmed the Fried conjecture for geodesic flows on closed odd dimensional[1] locally symmetric manifolds equipped with an acyclic unitarily flat vector bundle. In [ShY17], the authors made a further generalisation to closed locally symmetric orbifolds. We refer the reader to [Ma19] for an introduction to the technique used in [Sh18].

[1]The case of even dimension is trivial [Sh19, Remark 5.12] (c.f. Remark 4.5).

When the flat vector bundle is not unitary, Müller [Mü20] and Spilioti [Sp18, Sp20a, Sp20b] related the leading coefficients of the Laurent series of the Ruelle dynamical zeta function at the zero point to a weighted product of zeta regularised determinants of the flat Laplacian of Cappell-Miller [CMi10] on orientable odd dimensional hyperbolic manifolds. When the flat vector bundle is near to an acyclic and unitary one, the authors have shown that the Ruelle dynamical zeta function is regular at the zero point and its value is equal to the complexed valued analytic torsion of Cappell-Miller [CMi10]. In [Sh20], we generalised the above results to odd dimensional locally symmetric spaces.

In this article, we prove the Fried conjecture on odd dimensional locally symmetric spaces for a class[2] of flat vector bundles, which is not necessarily close to a unitary one, and whose holonomy representations are the restrictions of representations of the underlying reductive groups. This generalises the previous results of Bröcker [Brö98], Müller [Mü12], and Wotzke [Wo08] on orientable odd dimensional hyperbolic manifolds.

We refer the reader to [ShY18, DaGRSh20] for the Fried conjecture for the Morse-Smale flow and the Anosov flow, to [Sh19] for a survey on the Fried conjecture.

Now, we will describe our results in more detail, and explain the techniques used in their proofs.

0.1. **The analytic torsion.** Let Z be a smooth closed manifold, and let F be a complex flat vector bundle on Z.

Let g^{TZ} be a Riemannian metric on TZ, and let g^F be a Hermitian metric on F. To g^{TZ} and g^F, we can associate an L^2-metric on $\Omega^\bullet(Z,F)$, the space of differential forms with values in F. Let $\square^Z$ be the Hodge Laplacian acting on $\Omega^\bullet(Z,F)$. By Hodge theory, we have a canonical isomorphism

$$\ker \square^Z \simeq H^\bullet(Z,F). \tag{0.1}$$

The analytic torsion $T(F)$ is a positive real number defined by the following weighted product of the zeta regularised determinants (see Section 1)

$$T(F) = \prod_{i=1}^{\dim Z} \det\Big(\square^Z|_{\Omega^i(Z,F)\cap(\ker\square^Z)^\perp}\Big)^{(-1)^i i/2}. \tag{0.2}$$

By [RaSi71] and [BZ92, Theorem 0.1], if $\dim Z$ is odd and if $H^\bullet(Z,F)=0$, then $T(F)$ is independent of g^{TZ} and g^F. Therefore, it is a topological invariant.

When Z is a closed orbifold, the analytic torsion is still well defined [Ma05, DaiY17]. In [ShY17, Corollary 4.9], the authors show that if Z as well as all the singular strata have odd dimension, then the analytic torsion of an acyclic orbifold flat vector bundle is still a topological invariant.

0.2. **The Ruelle dynamical zeta function.** Let us recall the definition of the Ruelle dynamical zeta function associated to a geodesic flow introduced by Fried [F87, Section 5] (see also [Sh19, Section 2]).

[2]By Margulis' super-rigidity [M91, Section VII.5] (see also [BoW00, Section XIII.4.6]), this is the most interesting case, when the real rank of the locally symmetric space is ≥ 2.

Let (Z, g^{TZ}) be a connected manifold with nonpositive sectional curvature. Let Γ be the fundamental group of Z, and let $[\Gamma]$ be the set of the conjugacy classes of Γ. For $[\gamma] \in [\Gamma]$, let $B_{[\gamma]}$ be the set of closed geodesics in the free homotopy class associated to $[\gamma]$. It is easy to see that all the elements in $B_{[\gamma]}$ have the same length $\ell_{[\gamma]}$.

For simplicity, assume that all the $B_{[\gamma]}$ are smooth finite dimensional submanifolds of the loop space of Z. This is the case if (Z, g^{TZ}) has a negative sectional curvature or if Z is locally symmetric. If $\gamma \neq 1$, the group $\mathbb{S}^1$ acts locally freely on $B_{[\gamma]}$ by rotation, so that $B_{[\gamma]}/\mathbb{S}^1$ is an orbifold. Let $\chi_{\mathrm{orb}}(B_{[\gamma]}/\mathbb{S}^1) \in \mathbf{Q}$ be the orbifold Euler characteristic [Sa57]. Denote by

$$m_{[\gamma]} = \left|\ker\left(\mathbb{S}^1 \to \mathrm{Diff}(B_{[\gamma]})\right)\right| \in \mathbf{N}^* \tag{0.3}$$

the multiplicity of a generic element in $B_{[\gamma]}$. Let $\epsilon_{[\gamma]} = \pm 1$ be the Lefschetz index of the Poincaré return map induced by the geodesic flow (see [Sh19, (2.17)] for a precise definition). If Z is locally symmetric, then $\epsilon_{[\gamma]} = 1$.

If $r \in \mathbf{N}$, let $\rho : \Gamma \to \mathrm{GL}_r(\mathbf{C})$ be a representation of Γ. The formal dynamical zeta function is defined for $\sigma \in \mathbf{C}$ by

$$R_\rho(\sigma) = \exp\left(\sum_{[\gamma]\in[\Gamma_+]} \epsilon_{[\gamma]} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(B_{[\gamma]}/\mathbb{S}^1)}{m_{[\gamma]}} e^{-\sigma \ell_{[\gamma]}}\right), \tag{0.4}$$

where $[\Gamma_+] = [\Gamma] - \{1\}$ is the set of the non trivial conjugacy classes of Γ. We will say that the formal dynamical zeta function is well defined if $R_\rho(\sigma)$ is holomorphic for $\mathrm{Re}(\sigma) \gg 1$ and extends meromorphically to $\sigma \in \mathbf{C}$.

If (Z, g^{TZ}) has negative sectional curvature, the geodesic flow on the sphere bundle of (Z, g^{TZ}) is Anosov. In this case, if ρ is a trivial representation, $R_\rho(\sigma)$ has been shown to be well defined by Giulietti-Liverani-Pollicott [GiLPo13] and Dyatlov-Zworski [DyZ16]. For general ρ, the proof of the meromorphic extension of R_ρ is not particularly difficult. For behaviour of the Ruelle zeta function near $\sigma = 0$, we refer the reader to the work of Dyatlov and Zworski [DyZ17], Dang, Guillarmou, Rivière, and Shen [DaGRSh20], as well as Borns-Weil and Shen [BWSh20].

0.3. **Results of Fried, Bröcker, Wotzke, and Müller on hyperbolic manifolds.** Assume that Z is an odd dimensional connected orientable closed hyperbolic manifold. Let F be the unitarily flat vector bundle on Z with holonomy $\rho : \Gamma \to \mathrm{U}(r)$.

Using the Selberg trace formula, Fried [F86a, Theorem 3] showed that there exist explicit constants $C_\rho \in \mathbf{R}^*$ and $r_\rho \in \mathbf{Z}$ such that as $\sigma \to 0$,

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho+1}). \tag{0.5}$$

Moreover, if $H^\bullet(Z, F) = 0$, then

$$C_\rho = 1, \qquad r_\rho = 0, \tag{0.6}$$

so that

$$R_\rho(0) = T(F)^2. \tag{0.7}$$

When ρ is not unitary but a restriction of a representation of the orientation preserving isometric group of Z, similar results (see Theorem 0.1) have been shown by Bröcker [Brö98] and Wotzke [Wo08], as well as Müller [Mü12, Theorem 1.5].

0.4. **The main result of the article.** Let G be a linear connected real reductive group [Kn86, p. 3], and let θ be the Cartan involution. Let K be the maximal compact subgroup of G of the points of G that are fixed by θ. Let $\mathfrak{k}$ and $\mathfrak{g}$ be the Lie algebras of K and G, and let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition. Let B be a nondegenerate bilinear symmetric form on $\mathfrak{g}$ which is invariant under G and θ. Assume that B is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Set $X = G/K$. Then $B|_{\mathfrak{p}}$ induces a Riemannian metric on X, which has nonpositive sectional curvature.

Let $\Gamma \subset G$ be a discrete torsion free cocompact subgroup of G. Let $Z = \Gamma \backslash X$ be the associated locally symmetric manifold, which is equipped with the induced Riemannian metric g^{TZ}. Let $\rho : \Gamma \to \mathrm{GL}(E)$ be a finite dimensional complex representation of Γ. Let F be the associated flat vector bundle on Z. In [Sh20, Theorem 0.1 i)], we have shown that if $\dim Z$ is odd, the Ruelle zeta function R_ρ has a meromorphic extension to $\mathbf{C}$.

Suppose now that ρ extends to a representation of G, which is still denoted by ρ. We assume also that E has an admissible metric[3] $\langle , \rangle_E$, i.e., $\mathfrak{p}$ acts symmetrically and $\mathfrak{k}$ acts antisymmetrically on $(E, \langle , \rangle_E)$. By a construction due to Matsushima-Murakami [MatMu63], $\langle , \rangle_E$ induces canonically a Hermitian metric g^F on F (see also Section 4.1).

Let $T(F)$ be the analytic torsion of F associated to (g^{TZ}, g^F). The following theorem generalises [Sh18, Theorem 0.1] where ρ is assumed to be unitary, and Bröcker [Brö98], Müller [Mü12], and Wotzke [Wo08] where Z is hyperbolic.

Theorem 0.1. *Assume that* $\dim Z$ *is odd and that* $\rho : G \to \mathrm{GL}(E)$ *is a finite dimensional complex representation of* G *with an admissible metric. Let* (F, g^F) *be the associated Hermitian flat vector bundle. Then there exist constants* $C_\rho \in \mathbf{C}^*$ *and* $r_\rho \in \mathbf{Z}$ *such that when* $\sigma \to 0$*, we have*

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho + 1}). \tag{0.8}$$

Moreover, if $H^\bullet(Z, F) = 0$*, then*

$$|C_\rho| = 1, \qquad r_\rho = 0, \tag{0.9}$$

so that

$$|R_\rho(0)| = T(F)^2. \tag{0.10}$$

Set $\rho^\theta = \rho \circ \theta$. Then ρ^θ is still a representation of G with an admissible metric. If $\rho \simeq \rho^\theta$, we will show in Theorem 4.4 that the constant $C_\rho \in \mathbf{R}^*$ and we can remove the absolute value in (0.9) and (0.10). For general ρ, by [Sh20, Theorem 0.1 ii) iii)], the argument of C_ρ is determined by the argument of a (0.2)-like product of zeta regularised determinants of the flat Laplacian of Cappell and Miller, which is related to the complex valued analytic torsion of Cappell and Miller [CMi10].

[3] If G is semisimple or more generally if G has a compact centre, then all the representations of G has an admissible metric ([MatMu63, Lemma 3.1], Proposition 2.9).

When ρ is irreducible and $\rho \neq \rho^\theta$, thanks to the vanishing of the cohomology $H^\bullet(Z,F)$ [BoW00, Theorem VII.6.7], we have the following corollary.

Corollary 0.2. *Assume that* $\dim Z$ *is odd and that* $\rho : G \to \mathrm{GL}(E)$ *is a finite dimensional complex representation of* G *with an admissible metric. Let* (F, g^F) *be the associated Hermitian flat vector bundle. Suppose that* ρ *is irreducible and* $\rho \neq \rho^\theta$. *Then,* $R_\rho(\sigma)$ *is holomorphic at* $\sigma = 0$ *so that*

$$\left|R_\rho(0)\right| = T(F)^2. \tag{0.11}$$

In Section 4.5, we will show that the unitarily flat vector bundle is contained in the class specified above. Indeed, let $\rho_0 : \Gamma \to \mathrm{U}(r)$ be the holonomy representation of a unitarily flat vector bundle (F_0, g^{F_0}). If $\underline{G} = G \times \mathrm{U}(r)$, $\underline{K} = K \times \mathrm{U}(r)$, and if $\underline{\Gamma} \subset \Gamma \times \mathrm{U}(r)$ is the graph of ρ_0, then we have the identification $Z = \underline{\Gamma} \backslash \underline{G}/\underline{K}$. If $\underline{\rho}_0 : \underline{G} \to \mathrm{U}(r)$ is the projection onto the second component, it is easy to see that $\underline{\rho}_0$ has an admissible metric and the associated Hermitian flat vector bundle is just (F_0, g^{F_0}). In this way, we show :

Corollary 0.3. *Assume that* $\dim Z$ *is odd. If* (F_0, g^{F_0}) *is a unitarily flat vector bundle, and if* (F, g^F) *is the Hermitian flat vector bundle as in Theorem 0.1, then the statements of Theorem 0.1 and Corollary 0.2 hold for* $F_0 \otimes F$.

Remark 0.4. The flat vector bundle $F_0 \otimes F$ in Corollary 0.3 is of particular interest in study of hyperbolic volumes (see e.g. [BéDHP19]).

In Section 7, we will extend all the above results to the case where Γ is not torsion free. Then Z is an orbifold and F is a flat orbifold vector bundle.

Theorem 0.5. *The statement of Theorem 0.1 and Corollaries 0.2 and 0.3 hold for orbifolds.*

0.5. **Proof of Theorem 0.1.** We will first show Theorem 0.1 in the case $\rho^\theta \simeq \rho$, i.e., Theorem 4.4. Using an easy relation $R_{\rho^\theta}(\sigma) = \overline{R_\rho(\overline{\sigma})}$ (Proposition 3.4) and applying Theorem 4.4 to $\rho \oplus \rho^\theta$, we obtain Corollary 0.2. Since ρ has an admissible metric, ρ can be decomposed as a direct sum of irreducible representations which are either $\rho^\theta \simeq \rho$ or $\rho^\theta \not\simeq \rho$. In this way, we get Theorem 0.1 in full generality.

Our proof of Theorem 0.1 in the case $\rho^\theta \simeq \rho$ is inspired by [Sh18] and [Mü12]. Let us explain main steps.

0.5.1. *Moscovici-Stanton's vanishing theorem.* Let $\delta(G) \in \mathbf{N}$ be the fundamental rank of G, i.e., the difference between the complex ranks of G and K. Note that $\delta(G)$ and $\dim Z$ have the same parity.

When F is unitary, by [MoSt91, Corollary 2.2, Remark 3.7], if $\delta(G) \geqslant 3$, we have

$$T(F) = 1, \qquad R_\rho(\sigma) \equiv 1. \tag{0.12}$$

In the current situation, similar results still hold (e.g. [BMaZ11] [BMaZ17, Theorem 8.6, Remark 8.7], [Ma19, Theorem 5.5], [Sh20, Remark 4.2]). Therefore, we can reduce the proof to the case $\delta(G) = 1$.

0.5.2. *Selberg zeta functions.* Assume now $\delta(G)=1$ and $\rho^\theta \simeq \rho$. The proof of Theorem 0.1 in this case is based on the introduction of the Selberg zeta functions. Let us recall its definition and basic properties.

Let $\mathfrak{t}\subset\mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$. Let $\mathfrak{h}\subset\mathfrak{g}$ be the stabiliser of $\mathfrak{t}$ in $\mathfrak{g}$. By [Kn86, p. 129], $\mathfrak{h}\subset\mathfrak{g}$ is a θ-invariant fundamental Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}=\mathfrak{b}\oplus\mathfrak{t}$ be the Cartan decomposition of $\mathfrak{h}$. Note that $\dim\mathfrak{b}=\delta(G)=1$. Let $H\subset G$ be the associated Cartan subgroup of G.

Let $Z(\mathfrak{b})\subset G$ be the stabiliser of $\mathfrak{b}$ in G with Lie algebra $\mathfrak{z}(\mathfrak{b})$. Let $Z^0(\mathfrak{b})$ be the connected component of the identity in $Z(\mathfrak{b})$. Then $\mathfrak{z}(\mathfrak{b}), Z^0(\mathfrak{b})$ split

$$\mathfrak{z}(\mathfrak{b})=\mathfrak{b}\oplus\mathfrak{m}, \qquad Z^0(\mathfrak{b})=\exp(\mathfrak{b})\times M, \tag{0.13}$$

where M is a connected reductive subgroup of G with Lie algebra $\mathfrak{m}$. Let $\mathfrak{m}=\mathfrak{p}_\mathfrak{m}\oplus\mathfrak{k}_\mathfrak{m}$ be the Cartan decomposition of $\mathfrak{m}$. Let $\mathfrak{z}^\perp(\mathfrak{b})\subset\mathfrak{g}$ be the orthogonal space of $\mathfrak{z}^\perp(\mathfrak{b})$ with respect to B.

Let $\eta=\eta^+-\eta^-$ be a virtual representation of M acting on the finite dimensional complex vector spaces $E_\eta=E_\eta^+-E_\eta^-$ such that

i) the Casimir of M acts on $\eta^\pm$ by the same scalar;
ii) the restriction of η to $K_M=K\cap M$ lifts uniquely to a virtual representation of K.

The Selberg zeta function associated to η is defined formally for $\sigma\in\mathbf{C}$ by

$$Z_\eta(\sigma)=\exp\Bigg(-\sum_{\substack{[\gamma]\in[\Gamma_+]\\ \gamma\sim e^a k^{-1}\in H}}\frac{\chi_{\mathrm{orb}}\left(B_{[\gamma]}/\mathbb{S}^1\right)}{m_{[\gamma]}}\frac{\mathrm{Tr}_{\mathrm{s}}^{E_\eta}[k^{-1}]}{\left|\det\left(1-\mathrm{Ad}(e^a k^{-1})\right)|_{\mathfrak{z}^\perp(\mathfrak{b})}\right|^{1/2}}e^{-\sigma\ell_{[\gamma]}}\Bigg), \tag{0.14}$$

where the sum is taken over the non elliptic conjugacy classes $[\gamma]$ of Γ such that γ can be conjugate by element of G into the Cartan subgroup H.

In [Sh18, Section 6] and [Sh20, Section 3.4], we have shown that the adjoint action of K_M on $\mathfrak{p}_{\mathfrak{m},\mathbf{C}}$ lifts uniquely to a virtual representation of K. Let $\widehat{\eta}=\widehat{\eta}^+-\widehat{\eta}^-$ be the unique virtual representation of K such that

$$\widehat{\eta}|_{K_M}=\Lambda^\bullet(\mathfrak{p}_{\mathfrak{m},\mathbf{C}}^*)\widehat{\otimes}\eta_{|K_M}. \tag{0.15}$$

The Casimir operator of $\mathfrak{g}$ acts as a generalised Laplacian $C^{\mathfrak{g},Z,\widehat{\eta}^\pm}$ on the smooth sections over Z of the locally homogenous vector bundle induced by $\widehat{\eta}^\pm$ (see (3.6)). By the general theory on elliptic differential operators, the regularised determinant $\det\left(C^{\mathfrak{g},Z,\widehat{\eta}^\pm}+\sigma\right)$ is holomorphic on $\sigma\in\mathbf{C}$.

In [Sh18, Section 7] and [Sh20, Section 5], we show that $Z_\eta(\sigma)$ has a meromorphic extension to $\sigma\in\mathbf{C}$. Moreover, up to a multiplication by a non zero entire function, $Z_\eta(\sigma)$ is just the graded regularised determinant

$$\frac{\det\left(C^{\mathfrak{g},Z,\widehat{\eta}^+}+\sigma_\eta+\sigma^2\right)}{\det\left(C^{\mathfrak{g},Z,\widehat{\eta}^-}+\sigma_\eta+\sigma^2\right)}, \tag{0.16}$$

where $\sigma_\eta\in\mathbf{R}$ is some constant.

One of main steps in our proof of Theorem 0.1 is to construct a family of virtual M-representations η_β satisfying the above assumptions i) ii), parametrised by

finite elements $\beta \in \mathfrak{b}^*$, such that

$$\frac{\sum_{\beta\in\mathfrak{b}^*} e^{\langle\beta,a\rangle}\operatorname{Tr}_{\mathrm{s}}\left[\eta_\beta\left(k^{-1}\right)\right]}{\left|\det\left(1-\operatorname{Ad}(e^a k^{-1})\right)|_{\mathfrak{z}^\perp(\mathfrak{b})}\right|^{1/2}} = \operatorname{Tr}\left[\rho\left(e^a k^{-1}\right)\right], \tag{0.17}$$

and such that the following identity of virtual K-representations holds,

$$\bigoplus_{\beta\in\mathfrak{b}^*}\widehat{\eta}_\beta = \bigoplus_{i=1}^{m}(-1)^{i-1} i\Lambda^i\left(\mathfrak{p}^*_{\mathbf{C}}\right)\otimes\rho_{|K}. \tag{0.18}$$

Using (0.17), we can write R_ρ as an alternating product of Z_{η_β}. Thanks to (0.16), we get a relation between the Ruelle zeta function and the Casimir operator, which is the Hodge Laplacian of (g^{TZ}, g^F) by (0.18) (c.f. Proposition 4.2). In this way, we get (0.8).

0.5.3. *Dirac cohomology*. The construction of η_β is based on the Dirac cohomology [HuPa06]. Recall that in [Sh18], we have shown that $(\mathfrak{g},\mathfrak{z}(\mathfrak{b}))$ is a symmetric pair. Let $(\mathfrak{u},\mathfrak{u}(\mathfrak{b}))$ be the associated compact symmetric pair. The Dirac cohomology $H^\pm_D(\rho)$ of the $\mathfrak{u}$-representation ρ with respect to the symmetric pair $(\mathfrak{u},\mathfrak{u}(\mathfrak{b}))$ is a $(\mathfrak{b}_{\mathbf{C}}\oplus\mathfrak{m}_{\mathbf{C}},K_M)$-modules. We define $\eta^\pm_\beta$ to be the $(\mathfrak{m}_{\mathbf{C}},K_M)$-modules such that

$$H^\pm_D(\rho)\simeq\bigoplus_{\beta\in\mathfrak{b}^*}\mathbf{C}_\beta\boxtimes\eta^\pm_\beta, \tag{0.19}$$

where $\mathbf{C}_\beta$ is the one dimensional representation of $\mathfrak{b}$ such that $a\in\mathfrak{b}$ acts as $\langle\beta,a\rangle\in\mathbf{R}$. The virtual M-representation of η_β is defined by $\eta^+_\beta-\eta^-_\beta$. Now, the assumption i)[4], (0.17), and (0.18) are easy consequences of properties of the Dirac cohomology. In Section 5.3, we show the assumption ii) as well, so that the Selberg zeta function of η_β is well defined.

Let us remark that the Dirac cohomology is more or less equivalent to the $\mathfrak{n}$-cohomology used in [Mü12]. However, Dirac cohomology is closer to the spin construction used in [Sh18, Section 6].

0.5.4. *Infinitesimal character and the vanishing of* $(\mathfrak{g},K)$*-cohomology*. The proof of (0.9) is based on a relation between the infinitesimal character and the vanishing of $(\mathfrak{g},K)$-cohomology of a unitary Harish-Chandra $(\mathfrak{g}_{\mathbf{C}},K)$-module, which is due to Vogan-Zuckermann [VZu84], Vogan [V84], and Salamanca-Riba [SR99]. The idea of the proof is very similar to the one given in [Sh18, Section 8]. We refer the reader to [Sh18, Section 1H] for an introduction.

0.5.5. *Final remark*. In the case where G have a noncompact centre, we have $\mathfrak{n}=0$. It is unnecessary to use the Dirac cohomology and the results Vogan, Zuckermann, and Salamanca-Riba. For greater clarity, we single out this case in Section 4.6.

[4]More precisely, we need assume that the Casimir of $\mathfrak{g}$ acts on ρ as a scalar.

0.6. **The organisation of the article.** This article is organised as follows. In Section 1, we recall the definition of the Ray-Singer analytic torsion of a flat vector bundle.

In Section 2, we introduce the real reductive group G and the admissible metrics on finite dimension representations of G.

In Section 3, we recall the definition and the proprieties of the zeta functions of Ruelle and Selberg established in [Sh18, Sh20].

In Section 4, we state our main Theorem 4.4, from which we deduce Theorem 0.1, Corollaries 0.2 and 0.3. Also, we prove Theorem 4.4 when $\delta(G) \neq 1$ or Z_G is non compact.

In Sections 5 and 6, we establish Theorem 4.4 when $\delta(G) = 1$ and Z_G is compact.

Finally, in Section 7, we extends the previous results to orbifolds and we show Theorem 0.5.

0.7. **Notation.** Throughout the paper, we use the superconnection formalism of [Q85] and [BeGeVe04, Section 1.3]. If A is a $\mathbf{Z}_2$-graded algebra and if $a, b \in A$, the supercommutator $[a, b]$ is given by

$$ab - (-1)^{\deg a \deg b} ba. \tag{0.20}$$

If B is another $\mathbf{Z}_2$-graded algebra, we denote by $A \widehat{\otimes} B$ the super tensor product algebra of A and B. If $E = E^+ \oplus E^-$ is a $\mathbf{Z}_2$-graded vector space, the algebra $\mathrm{End}(E)$ is $\mathbf{Z}_2$-graded. If $\tau = \pm 1$ on $E^\pm$ and if $a \in \mathrm{End}(E)$, the supertrace $\mathrm{Tr}_s[a]$ is defined by $\mathrm{Tr}[\tau a]$.

If M is a topological group, we will denote by M^0 the connected component of the identity in M. If V is a real vector space, we will use the notation $V_{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C}$ for its complexification. We make the convention that $\mathbf{N} = \{0, 1, 2, \ldots\}$, $\mathbf{N}^* = \{1, 2, \ldots\}$, $\mathbf{R}_+^* = (0, \infty)$.

1. The analytic torsion

Let Z be a closed smooth manifold of dimension m. Let (F, ∇^F) be a flat complex vector bundle on Z with flat connection ∇^F. Let $(\Omega^\bullet(Z, F), d^Z)$ be the de Rham complex of smooth sections of $\Lambda^\bullet(T^*Z) \otimes_{\mathbf{R}} F$ on Z. Let $H^\bullet(Z, F)$ be the de Rham cohomology.

We define the Euler characteristic number $\chi(Z, F)$ and the derived Euler characteristic number $\chi'(Z, F)$ by

$$\chi(Z, F) = \sum_{i=0}^{m} (-1)^i \dim H^i(Z, F), \qquad \chi'(Z, F) = \sum_{i=1}^{m} (-1)^i i \dim H^i(Z, F). \tag{1.1}$$

If F is trivial, we write $\chi(Z)$ and $\chi'(Z)$.

Let g^{TZ} be a Riemannian metric on Z. Let g^F be a Hermitian metric on F. The metrics g^{TZ}, g^F induce a scalar product $\langle , \rangle_{\Lambda^\bullet(T^*Z) \otimes_{\mathbf{R}} F}$ on $\Lambda^\bullet(T^*Z) \otimes_{\mathbf{R}} F$. Let $\langle , \rangle_{L^2}$ be an L^2-product on $\Omega^\bullet(Z, F)$ defined for $s_1, s_2 \in \Omega^\bullet(Z, F)$ by

$$\langle s_1, s_2 \rangle_{L^2} = \int_{z \in Z} \langle s_1(z), s_2(z) \rangle_{\Lambda^\bullet(T^*Z) \otimes_{\mathbf{R}} F} dv_Z, \tag{1.2}$$

where dv_Z is the Riemannian volume form of (Z, g^{TZ}).

Let $d^{Z,*}$ be the formal adjoint of d^Z. Set

$$\square^Z = \left[d^Z, d^{Z,*}\right]. \tag{1.3}$$

Then, $\square^Z$ is a second order self-adjoint elliptic differential operator acting on $\Omega^\bullet(Z,F)$. By Hodge theory, we have

$$\ker \square^Z = H^\bullet(Z,F). \tag{1.4}$$

Let $\left(\ker \square^Z\right)^\perp$ be the orthogonal vector space to $\ker \square^Z$ in $\Omega^\bullet(X,F)$. Then $\square^Z$ acts as an invertible operator on $\left(\ker \square^Z\right)^\perp$. Let $\left(\square^Z\right)^{-1}$ denote the inverse of $\square^Z$ acting on $\left(\ker \square^Z\right)^\perp$. If $0 \leqslant i \leqslant m$, for $s \in \mathbf{C}$ and $\mathrm{Re}(s) > \frac{m}{2}$, set

$$\theta_i(s) = \mathrm{Tr_s}\left[\left(\square^Z\right)^{-s}_{|\Omega^i(Z,F)}\right]. \tag{1.5}$$

By [Se67] and [BeGeVe04, Proposition 9.35], $\theta_i(s)$ extends to a meromorphic function of $s \in \mathbf{C}$, which is holomorphic at $s = 0$. The regularised determinant is defined by

$$\det{}^*\left(\square^Z_{|\Omega^i(Z,F)}\right) = \exp\left(-\theta_i'(0)\right). \tag{1.6}$$

Formally, it is the product of non zero eigenvalues counted with multiplicities.

Definition 1.1. The Ray-Singer analytic torsion [RaSi71] of F is defined by

$$T(F) = \prod_{i=1}^{m} \det{}^*\left(\square^Z_{|\Omega^i(Z,F)}\right)^{(-1)^i i/2} \in \mathbf{R}_+^*. \tag{1.7}$$

By [RaSi71] and [BZ92, Theorem 0.1], if $\dim Z$ is odd and if $H^\bullet(Z,F) = 0$, then $T(F)$ does not depend on the metrics g^{TZ}, g^F. It becomes a topological invariant.

For $0 \leqslant i \leqslant m$, if $\sigma > 0$, the operator $\sigma + \square^Z_{|\Omega^i(Z,F)}$ does not contain the zero spectrum, we denote its regularised determinant by $\det\left(\sigma + \square^Z_{|\Omega^i(Z,F)}\right)$. By [Vo87] and [Sh20, Theorem 1.5], the function $\det\left(\sigma + \square^Z_{|\Omega^i(Z,F)}\right)$ extends to a holomorphic function of $\sigma \in \mathbf{C}$, whose zeros are located at $\sigma = -\lambda$ with order $\dim\ker\left(\square^Z_{|\Omega^i(Z,F)} - \lambda\right)$, where $\lambda \in \mathrm{Sp}\left(\square^Z_{|\Omega^i(Z,F)}\right)$.

Set

$$T(\sigma) = \prod_{i=1}^{m} \det\left(\sigma + \square^Z_{|\Omega^i(Z,F)}\right)^{(-1)^i i}. \tag{1.8}$$

Then, $T(\sigma)$ is meromorphic. By (1.1), (1.7), and (1.8), as $\sigma \to 0$, we have

$$T(\sigma) = T(F)^2 \sigma^{\chi'(Z,F)} + \mathscr{O}(\sigma^{\chi'(Z,F)+1}). \tag{1.9}$$

2. Reductive groups and finite dimensional representations

The purpose of this section is to recall some basic facts about real reductive groups and their finite dimensional representations.

This section is organised as follows. In Sections 2.1-2.4, we introduce the real reductive group G, its Lie algebra $\mathfrak{g}$, the enveloping algebra $\mathscr{U}(\mathfrak{g})$, the Casimir operator, the Dirac operator, as well as the semisimple elements.

In Sections 2.5 and 2.6, we introduce the fundamental Cartan subalgebra of $\mathfrak{g}$ and some related constructions. We recall a key lifting proprieties established in [Sh20, Section 3.4].

In Section 2.7, we recall the definition of admissible metrics on finite dimensional representations of G introduced by [MatMu63]. We show that if G has a compact centre, then all the finite dimensional representations have admissible metrics.

2.1. **Real reductive groups.** Let G be a linear connected real reductive group [Kn86, p. 3], and let $\theta \in \mathrm{Aut}(G)$ be the Cartan involution. That means G is a closed connected group of real matrices that is stable under transpose, and θ is the composition of transpose and inverse of matrices. If $\mathfrak{g}$ is the Lie algebra of G, then θ acts as an automorphism on $\mathfrak{g}$.

Let $K \subset G$ be the fixed point set of θ in G. Then K is a compact connected subgroup of G, which is a maximal compact subgroup. If $\mathfrak{k}$ is the Lie algebra of K, then $\mathfrak{k}$ is the eigenspace of θ associated with the eigenvalue 1. Let $\mathfrak{p}$ be the eigenspace of θ associated with the eigenvalue -1, so that we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{2.1}$$

Set

$$m = \dim \mathfrak{p}, \qquad n = \dim \mathfrak{k}. \tag{2.2}$$

By [Kn86, Proposition 1.2], we have the diffeomorphism

$$(Y, k) \in \mathfrak{p} \times K \to e^{Y} k \in G. \tag{2.3}$$

Let B be a nondegenerate bilinear real symmetric form on $\mathfrak{g}$ which is invariant under the adjoint action Ad of G, and also under θ. Then (2.1) is an orthogonal splitting of $\mathfrak{g}$ with respect to B. We assume B to be positive-definite on $\mathfrak{p}$, and negative-definite on $\mathfrak{k}$. Then, $\langle \cdot, \cdot \rangle = -B(\cdot, \theta \cdot)$ defines an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{g}$ such that the splitting (2.1) is still orthogonal. We denote by $|\cdot|$ the corresponding norm.

Let $Z_G \subset G$ be the centre of G with Lie algebra $\mathfrak{z}_{\mathfrak{g}} \subset \mathfrak{g}$. By [Kn86, Corollary 1.3], Z_G is a (possibly non connected) reductive group with maximal compact subgroup $Z_G \cap K$ with the Cartan decomposition

$$\mathfrak{z}_{\mathfrak{g}} = \mathfrak{z}_{\mathfrak{p}} \oplus \mathfrak{z}_{\mathfrak{k}}. \tag{2.4}$$

Since $\mathfrak{z}_{\mathfrak{p}}$ commutes with $Z_G \cap K$, by (2.3), we have an identification of the groups

$$Z_G = \exp(\mathfrak{z}_{\mathfrak{p}}) \times (Z_G \cap K). \tag{2.5}$$

Let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification of $\mathfrak{g}$ and let $\mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}$ be the compact form of $\mathfrak{g}$. By $\mathbf{C}$-linearity, the bilinear form B extends to a complex symmetric bilinear form on $\mathfrak{g}_{\mathbf{C}}$. The restriction $B|_{\mathfrak{u}}$ to $\mathfrak{u}$ is real and negative-definite.

Let $\mathscr{U}(\mathfrak{g})$ and $\mathscr{U}(\mathfrak{g}_{\mathbf{C}})$ be the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbf{C}}$. Let $\mathscr{Z}(\mathfrak{g})$ and $\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$ be respectively the centres of $\mathscr{U}(\mathfrak{g})$ and $\mathscr{U}(\mathfrak{g}_{\mathbf{C}})$. Clearly,

$$\mathscr{U}(\mathfrak{g}_{\mathbf{C}}) = \mathscr{U}(\mathfrak{g}) \otimes_{\mathbf{R}} \mathbf{C}, \qquad \mathscr{Z}(\mathfrak{g}_{\mathbf{C}}) = \mathscr{Z}(\mathfrak{g}) \otimes_{\mathbf{R}} \mathbf{C}. \tag{2.6}$$

If V is a complex vector space, and if $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of $\mathfrak{g}$, then the map ρ extends to a morphism $\rho : \mathscr{U}(\mathfrak{g}_{\mathbf{C}}) \to \mathrm{End}(V)$ of algebras.

If $\rho : G \to \mathrm{GL}(V)$ is a finite dimensional complex representation of G, then the induced morphism $\rho : \mathscr{U}(\mathfrak{g}_{\mathbf{C}}) \to \mathrm{End}(V)$ of algebras is K-equivalent. In this way, V is a finite dimensional $(\mathfrak{g}_{\mathbf{C}}, K)$-module, i.e., a $\mathscr{U}(\mathfrak{g}_{\mathbf{C}})$-module, equipped with a compatible K-action. By [KnV95, Proposition 4.46], it is equivalent to consider finite dimensional representations of G and finite dimensional $(\mathfrak{g}_{\mathbf{C}}, K)$-modules. In the sequel, we will not distinguish these two objets.

Remark 2.1. If $\rho : G \to \mathrm{GL}(V)$ is a finite dimensional complex representation of G, and if $W \subset V$ is a $\mathfrak{g}$-invariant subspace, by taking the exponential of the action of $\mathfrak{g}$, we see that the group G preserves W. In particular, the set of $\mathfrak{g}$-subrepresentations of ρ coincides with the set of G-subrepresentations of ρ.

2.2. **The Casimir operator.** Let $C^{\mathfrak{g}} \in \mathscr{Z}(\mathfrak{g})$ be the Casimir element associated to B. If $e_1, \cdots, e_m$ is an orthonormal basis of $(\mathfrak{p}, B|_{\mathfrak{p}})$, and if $e_{m+1}, \cdots, e_{m+n}$ is an orthonormal basis of $(\mathfrak{k}, -B|_{\mathfrak{k}})$, then

$$C^{\mathfrak{g}} = -\sum_{i=1}^{m} e_i^2 + \sum_{i=m+1}^{n+m} e_i^2. \tag{2.7}$$

If $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a complex representation of $\mathfrak{g}$, we denote by $C^{\mathfrak{g},V}$ or $C^{\mathfrak{g},\rho} \in \mathrm{End}(V)$ the corresponding Casimir operator acting on V, i.e.,

$$C^{\mathfrak{g},V} = C^{\mathfrak{g},\rho} = \rho(C^{\mathfrak{g}}). \tag{2.8}$$

Similarly, the Casimir of $\mathfrak{u}$ (with respect to B) acts on V, so that

$$C^{\mathfrak{u},V} = C^{\mathfrak{g},V}. \tag{2.9}$$

2.3. **The Dirac operator.** Let $c(\mathfrak{p})$ be the Clifford algebra of $(\mathfrak{p}, B|_{\mathfrak{p}})$. That is an algebra over $\mathbf{R}$ generated by $1 \in \mathbf{R}$, $a \in \mathfrak{p}$ with the commutation relation for $a_1, a_2 \in \mathfrak{p}$,

$$a_1 a_2 + a_2 a_1 = -2B(a_1, a_2). \tag{2.10}$$

Let $S^{\mathfrak{p}}$ be the spinor of $(\mathfrak{p}, B|_{\mathfrak{p}})$. If $a \in \mathfrak{p}$, the action of a on $S^{\mathfrak{p}}$ is denoted by $c(a)$. If $a \in \mathfrak{k}$, $\mathrm{ad}(a)|_{\mathfrak{p}}$ acts as an antisymmetric endomorphism on $\mathfrak{p}$. It acts on $S^{\mathfrak{p}}$ by

$$c\left(\mathrm{ad}(a)|_{\mathfrak{p}}\right) = \frac{1}{4} \sum_{i,j=1}^{m} \langle [a, e_i], e_j \rangle c(e_i) c(e_j). \tag{2.11}$$

Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a complex representation of $\mathfrak{g}$. Let $D^{S^{\mathfrak{p}} \otimes V}$ be the Dirac operator acting on $S^{\mathfrak{p}} \otimes V$, i.e.,

$$D^{S^{\mathfrak{p}} \otimes V} = \sum_{i=1}^{m} c(e_i) \rho(e_i). \tag{2.12}$$

Recall that $\mathfrak{k}$ acts on $\mathfrak{p}$ by adjoint action. The operator $C^{\mathfrak{k},\mathfrak{p}}$ is defined in (2.8).

Proposition 2.2. *The following identity of operators on $S^{\mathfrak{p}} \otimes V$ holds,*

$$\left(D^{S^{\mathfrak{p}} \otimes V}\right)^2 = C^{\mathfrak{g},V} + \frac{1}{8} \mathrm{Tr}\left[C^{\mathfrak{k},\mathfrak{p}}\right] - C^{\mathfrak{k},S^{\mathfrak{p}} \otimes V}. \tag{2.13}$$

Proof. This is a consequence of [BoW00, Lemma II.6.11] and of Kostant's strange formula [Ko76] (see also [B11, (2.6.11), (7.5.4)]).

There is another way proving this result, by imitating the proof of [B11, Theorem 7.2.1] which uses Kostant's cubic Dirac operator [Ko76, Ko97]. □

In the sequel, we will also consider the Dirac operator associated to the compact symmetric pair $(\mathfrak{u},\mathfrak{k})$ with respect to the positive bilinear form $-B|_{\mathfrak{u}}$. Using the identification $a\in\mathfrak{p}\to\sqrt{-1}a\in\sqrt{-1}\mathfrak{p}$, we can identify $c(\mathfrak{p})$ with the Clifford algebra of the Euclidean space $(\sqrt{-1}\mathfrak{p},-B)$. Also, $S^{\mathfrak{p}}$ can be identified with the spinor $S^{\sqrt{-1}\mathfrak{p}}$ of $(\sqrt{-1}\mathfrak{p},-B)$. Then, the Dirac operator $D^{S^{\sqrt{-1}\mathfrak{p}}\otimes V}$ is just $\sqrt{-1}D^{S^{\mathfrak{p}}\otimes V}$. By (2.9) and (2.13), we have

$$-\left(D^{S^{\sqrt{-1}\mathfrak{p}}\otimes V}\right)^2=C^{\mathfrak{u},V}+\frac{1}{8}\operatorname{Tr}\left[C^{\mathfrak{k},\sqrt{-1}\mathfrak{p}}\right]-C^{\mathfrak{k},S^{\sqrt{-1}\mathfrak{p}}\otimes V}. \tag{2.14}$$

2.4. **Semisimple elements.** If $\gamma\in G$, we denote by $Z(\gamma)\subset G$ the centraliser of γ in G, and by $\mathfrak{z}(\gamma)\subset\mathfrak{g}$ its Lie algebra. If $a\in\mathfrak{g}$, let $Z(a)\subset G$ be the stabiliser of a in G, and let $\mathfrak{z}(a)\subset\mathfrak{g}$ be its Lie algebra. If $\mathfrak{a}\subset\mathfrak{g}$ is a subset, we define $Z(\mathfrak{a})$ and $\mathfrak{z}(\mathfrak{a})$ similarly.

Let $\gamma\in G$ be a semisimple element, i.e., there is $g_\gamma\in G$ such that $\gamma=g_\gamma e^a k^{-1}g_\gamma^{-1}$ and

$$a\in\mathfrak{p},\qquad k\in K,\qquad \operatorname{Ad}(k)a=a. \tag{2.15}$$

The norm $|a|$ depends only on the conjugacy class of γ in G. Write

$$\ell_{[\gamma]}=|a|. \tag{2.16}$$

A semisimple element γ is called elliptic, if $\ell_{[\gamma]}=0$.

If γ is semisimple, by [Kn02, Proposition 7.25], $Z(\gamma)$ is a (possibly non connected) reductive group with Cartan involution[5] $g_\gamma\theta g_\gamma^{-1}$. Let $K(\gamma)\subset Z(\gamma)$ be the associated maximal compact subgroup of $Z(\gamma)$.

2.5. **The fundamental Cartan subalgebra.** Let $T\subset K$ be a maximal torus of K. Let $\mathfrak{t}\subset\mathfrak{k}$ be the Lie algebra of T. If $N_K(T)$ is the normaliser of T in K, let $W(T:K)=N_K(T)/T$ be associated Weyl group.

Set

$$\mathfrak{b}=\{a\in\mathfrak{p}:[a,\mathfrak{t}]=0\},\qquad \mathfrak{h}=\mathfrak{b}\oplus\mathfrak{t}. \tag{2.17}$$

By [Kn86, p. 129], $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $H=Z(\mathfrak{h})$ be the associated Cartan subgroup of G. By [Kn86, Theorem 5.22], H is a connected abelian reductive subgroup of G, so that

$$H=\exp(\mathfrak{b})\times T. \tag{2.18}$$

We will call $\mathfrak{h}$ and H respectively the fundamental Cartan subalgebra of $\mathfrak{g}$ and the fundamental Cartan subgroup of G.

Recall that complex ranks of G and K are defined respectively by the dimensions of Cartan subalgebras of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{k}_{\mathbf{C}}$.

[5] By [BSh19, Theorem 2.3], this is indeed independent of the choice of g_γ.

Definition 2.3. The fundamental rank $\delta(G)$ of G is defined by the difference of complex ranks of G and K, i.e.,

$$\delta(G) = \dim \mathfrak{b}. \tag{2.19}$$

Note that m and $\delta(G)$ have the same parity.

2.6. **A splitting of $\mathfrak{g}$ according to the $\mathfrak{b}$-action.** By [Kn02, Proposition 7.25], $Z(\mathfrak{b})$ is a (possibly non connected) reductive subgroup of G, so that we have the Cartan decomposition

$$\mathfrak{z}(\mathfrak{b}) = \mathfrak{p}(\mathfrak{b}) \oplus \mathfrak{k}(\mathfrak{b}). \tag{2.20}$$

Let $\mathfrak{m} \subset \mathfrak{z}(\mathfrak{b})$ be the orthogonal space (with respect to B) of $\mathfrak{b}$ in $\mathfrak{z}(\mathfrak{b})$. Then $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{g}$, and θ acts on $\mathfrak{m}$ so that we have the Cartan decomposition

$$\mathfrak{m} = \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}. \tag{2.21}$$

Let $M \subset G$ be the connected Lie subgroup associated to the Lie algebra $\mathfrak{m}$. By [B11, (3.3.11) and Theorem 3.3.1], M is closed in G and is a connected reductive subgroup of G with maximal compact subgroup

$$K_M = M \cap K. \tag{2.22}$$

Moreover, we have

$$Z^0(\mathfrak{b}) = \exp(\mathfrak{b}) \times M, \qquad \mathfrak{z}(\mathfrak{b}) = \mathfrak{b} \oplus \mathfrak{m}, \qquad \mathfrak{p}(\mathfrak{b}) = \mathfrak{b} \oplus \mathfrak{p}_{\mathfrak{m}}, \qquad \mathfrak{k}(\mathfrak{b}) = \mathfrak{k}_{\mathfrak{m}}. \tag{2.23}$$

Since $\mathfrak{h}$ is also a Cartan subalgebra for $\mathfrak{z}(\mathfrak{b})$, we have

$$\delta(M) = 0. \tag{2.24}$$

Let $\mathfrak{p}^{\perp}(\mathfrak{b}), \mathfrak{k}^{\perp}(\mathfrak{b}), \mathfrak{z}^{\perp}(\mathfrak{b})$ be respectively the orthogonal spaces (with respect to B) of $\mathfrak{p}(\mathfrak{b}), \mathfrak{k}(\mathfrak{b}), \mathfrak{z}(\mathfrak{b})$ in $\mathfrak{p}, \mathfrak{k}, \mathfrak{g}$. Clearly,

$$\mathfrak{z}^{\perp}(\mathfrak{b}) = \mathfrak{p}^{\perp}(\mathfrak{b}) \oplus \mathfrak{k}^{\perp}(\mathfrak{b}). \tag{2.25}$$

And also

$$\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{p}^{\perp}(\mathfrak{b}), \qquad \mathfrak{k} = \mathfrak{k}_{\mathfrak{m}} \oplus \mathfrak{k}^{\perp}(\mathfrak{b}), \qquad \mathfrak{g} = \mathfrak{b} \oplus \mathfrak{m} \oplus \mathfrak{z}^{\perp}(\mathfrak{b}). \tag{2.26}$$

The group K_M acts trivially on $\mathfrak{b}$. It also acts on $\mathfrak{p}_{\mathfrak{m}}$, $\mathfrak{p}^{\perp}(\mathfrak{b})$, $\mathfrak{k}_{\mathfrak{m}}$ and $\mathfrak{k}^{\perp}(\mathfrak{b})$, and preserves the splittings (2.26). Similarly, the groups M and $Z^0(\mathfrak{b})$ act trivially on $\mathfrak{b}$, act on $\mathfrak{m}, \mathfrak{z}^{\perp}(\mathfrak{b})$, and preserves the third splitting in (2.26).

Remark 2.4. We can define similar objects associated to the action of $\sqrt{-1}\mathfrak{b} \subset \mathfrak{u}$ on $\mathfrak{u}$. Let $\mathfrak{u}_{\mathfrak{m}}$ and $\mathfrak{u}(\mathfrak{b})$ be the compact forms of $\mathfrak{m}$ and $\mathfrak{z}(\mathfrak{b})$. Then,

$$\mathfrak{u}(\mathfrak{b}) = \sqrt{-1}\mathfrak{b} \oplus \mathfrak{u}_{\mathfrak{m}}. \tag{2.27}$$

Let $\mathfrak{u}^{\perp}(\mathfrak{b})$ be the orthogonal space of $\mathfrak{u}(\mathfrak{b})$ in $\mathfrak{u}$. Then,

$$\mathfrak{u}^{\perp}(\mathfrak{b}) = \sqrt{-1}\mathfrak{p}^{\perp}(\mathfrak{b}) \oplus \mathfrak{k}^{\perp}(\mathfrak{b}), \qquad \mathfrak{u} = \sqrt{-1}\mathfrak{b} \oplus \mathfrak{u}_{\mathfrak{m}} \oplus \mathfrak{u}^{\perp}(\mathfrak{b}). \tag{2.28}$$

Elements of $\mathfrak{b}$ act on $\mathfrak{z}^{\perp}(\mathfrak{b})$ with semisimple real eigenvalues. We fix an element $f_{\mathfrak{b}} \in \mathfrak{b}$, called positive, such that $\mathrm{ad}(f_{\mathfrak{b}})|_{\mathfrak{z}^{\perp}(\mathfrak{b})}$ is invertible. The choice of $f_{\mathfrak{b}}$ is irrelevant. Let $\mathfrak{n} \subset \mathfrak{z}^{\perp}(\mathfrak{b})$ (resp. $\overline{\mathfrak{n}} \subset \mathfrak{z}^{\perp}(\mathfrak{b})$) be the direct sum of the eigenspaces of $\mathrm{ad}(f_{\mathfrak{b}})|_{\mathfrak{z}^{\perp}(\mathfrak{b})}$ associated to the positive (resp. negative) eigenvalues. Then,

$$\mathfrak{z}^{\perp}(\mathfrak{b}) = \mathfrak{n} \oplus \overline{\mathfrak{n}}, \qquad \overline{\mathfrak{n}} = \theta\mathfrak{n}. \tag{2.29}$$

Clearly, $Z^0(\mathfrak{b})$ acts on $\mathfrak{n}$, $\overline{\mathfrak{n}}$ and preserves the first decomposition in (2.29).

Proposition 2.5. *The following statements hold.*

i) The vector spaces $\mathfrak{n},\overline{\mathfrak{n}} \subset \mathfrak{g}$ are Lie subalgebras of $\mathfrak{g}$, which have the same even dimension.

ii) The bilinear form B vanishes on $\mathfrak{n},\overline{\mathfrak{n}}$ and induces a $Z^0(\mathfrak{b})$-isomorphism,

$$\overline{\mathfrak{n}}^* \simeq \mathfrak{n}. \tag{2.30}$$

iii) The actions of M on $\mathfrak{n},\overline{\mathfrak{n}},\mathfrak{n}^*,\overline{\mathfrak{n}}^*$ are equivalent. For $0 \leqslant j \leqslant \dim \mathfrak{n}$, we have isomorphisms of representations of M,

$$\Lambda^j(\mathfrak{n}^*) \simeq \Lambda^{\dim \mathfrak{n}-j}(\mathfrak{n}^*). \tag{2.31}$$

iv) The projections on $\mathfrak{p},\mathfrak{k}$ map $\mathfrak{n},\overline{\mathfrak{n}}$ into $\mathfrak{p}_{\mathfrak{m}}^{\perp},\mathfrak{k}_{\mathfrak{m}}^{\perp}$ isomorphically.

v) The actions of K_M on $\mathfrak{n},\overline{\mathfrak{n}},\mathfrak{p}_{\mathfrak{m}}^{\perp},\mathfrak{k}_{\mathfrak{m}}^{\perp}$ are equivalent.

Proof. This is [Sh20, Proposition 3.2, Corollary 3.3] (see also [BSh19, Proposition 3.10]). □

Let $R(K)$ be the representation ring of K. We can identify $R(K)$ with the subring of the $\mathrm{Ad}(K)$-invariant smooth functions on K which is generated by the characters of finite dimensional complex representations of K.

Similarly, we can define $R(T)$. The Weyl group $W(T:K) = N_K(T)/T$ acts on $R(T)$. By [BrDi85, Proposition VI.2.1], the restriction induces an isomorphism of rings

$$R(K) \simeq R(T)^{W(T:K)}. \tag{2.32}$$

Since K_M and K have the same maximal torus T, the restriction induces an injective morphism $R(K) \to R(K_M)$ of rings. Recall a key result established in [Sh20, Theorem 3.5, Corollary 3.6] (see also [Sh18, Theorem 6.1, Corollary 6.12]).

Theorem 2.6. *For $i,j \in \mathbf{N}$, the adjoint representations of K_M on $\Lambda^i(\mathfrak{p}_{\mathfrak{m},\mathbf{C}}^*)$ and $\Lambda^j(\mathfrak{n}_{\mathbf{C}}^*)$ have unique lifts in $R(K)$.*

2.7. **Admissible metrics.** Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional complex representation of G. Set

$$\rho^\theta = \rho \circ \theta. \tag{2.33}$$

Then $\rho^\theta : G \to \mathrm{GL}(V)$ is still a representation of G.

Proposition 2.7. *If $\delta(G) = 0$, we have an isomorphism of representations of G,*

$$\rho \simeq \rho^\theta. \tag{2.34}$$

Proof. When $\delta(G) = 0$, by [Kn86, Problem XII.10.14], there is $k_0 \in K$, such that $\mathrm{Ad}(k_0) = -1$ on $\mathfrak{p}$ and $\mathrm{Ad}(k_0) = 1$ on $\mathfrak{k}$. By (2.3), for $g \in G$, we have $\theta(g) = k_0 g k_0^{-1}$. Therefore, $\rho(k_0) : V \to V$ is the required isomorphism (2.34). □

Definition 2.8. A Hermitian metric $\langle , \rangle_V$ on V is called admissible, if for $u,v \in V$, $Y_1 \in \mathfrak{p}$, $Y_2 \in \mathfrak{k}$, we have

$$\langle \rho(Y_1)u,v\rangle_V = \langle u,\rho(Y_1)v\rangle_V, \qquad \langle \rho(Y_2)u,v\rangle_V = -\langle u,\rho(Y_2)v\rangle_V. \tag{2.35}$$

Assume that V has an admissible metric. If $\overline{\rho}^*$ denotes the anti-dual representation of ρ, the admissible metric induces an isomorphism of G-representations,

$$\rho^\theta \simeq \overline{\rho}^*. \tag{2.36}$$

If $W \subset V$ is a $(\mathfrak{g}_{\mathbf{C}}, K)$-submodule of V, then the orthogonal space $W^\perp \subset V$ is still a $(\mathfrak{g}_{\mathbf{C}}, K)$-submodule. Moreover, by restrictions, W and $W^\perp$ still have admissible metrics. In this way, we see that any finite dimensional G-representation with an admissible metric is completely reducible, i.e., it can be decomposed as a direct sum of irreducible G-representations.

By [MatMu63, Lemma 3.1], if G is semisimple, any finite dimensional representation of G has an admissible Hermitian metric. When G is reductive and has a compact centre, we have a similar result.

Proposition 2.9. *If G has a compact centre Z_G, then any finite dimensional complex representation $\rho : G \to \mathrm{GL}(V)$ has an admissible Hermitian metric.*

Proof. Let $G_{\mathrm{ss}} \subset G$ be the connected Lie subgroup of G associated to the Lie algebra $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$. By [Kn02, Corollary 7.11], G_{ss} is a closed subgroup of G, which is semisimple and $G = G_{\mathrm{ss}} \cdot Z_G^0$. Let U_{ss} be the compact form of G_{ss}. By Weyl's Theorem [Kn02, Theorem 4.69], the universal cover $\widetilde{U}_{\mathrm{ss}}$ of U_{ss} is still compact. Since $\widetilde{U}_{\mathrm{ss}}$ is simply connected, by Weyl's unitary trick, the group $\widetilde{U}_{\mathrm{ss}}$ acts on V which is compatible with the action of $[\mathfrak{g}, \mathfrak{g}]$. Moreover, the $\widetilde{U}_{\mathrm{ss}}$-action commutes with the Z_G^0-action. Thus, the group $\widetilde{U}_{\mathrm{ss}} \times Z_G^0$ acts on V. Since $\widetilde{U}_{\mathrm{ss}} \times Z_G^0$ is compact, there is a $\widetilde{U}_{\mathrm{ss}} \times Z_G^0$-invariant Hermitian metric on V, which is the desired admissible Hermitian metric. □

Remark 2.10. If G has a noncompact centre, Proposition 2.9 does not hold. For example, when $G = \mathbf{R}$, the representation $x \in \mathbf{R} \to \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{C})$ is not completely reducible, so it does not have an admissible metric.

3. The zeta functions of Ruelle and Selberg

The purpose of this section is to introduce the zeta functions of Ruelle and of Selberg on locally symmetric spaces.

This section is organised as follows. In Section 3.1, we introduce the symmetric space $X = G/K$, the K-principal bundle $p : G \to X$, and a Hermitian vector bundle associated to a finite dimensional unitary representation of K.

In Section 3.2, we introduce a discrete cocompact subgroup $\Gamma \subset G$ of G, the corresponding locally symmetric space $Z = \Gamma \backslash X$, and a flat vector bundle associated to a finite dimensional representation of Γ.

Finally, in Sections 3.3 and 3.4, we introduce the zeta functions of Ruelle and of Selberg. We recall their properties established in [Sh18, Section 7] and [Sh20, Section 5].

We use the notation in Section 2.

3.1. Symmetric spaces.

Set $X = G/K$. Let $p : G \to X$ be the natural projection. Then $p : G \to X$ is a K-principal bundle.

The group K acts isometrically on $\mathfrak{p}$. The tangent bundle of X is given by

$$TX = G \times_K \mathfrak{p}. \tag{3.1}$$

By (3.1), the scalar product $B|_{\mathfrak{p}}$ on $\mathfrak{p}$ induces a Riemannian metric g^{TX} on X. Classically, (X, g^{TX}) has a parallel and nonpositive sectional curvature.

Let τ be an orthogonal (resp. unitary) representation of K acting on a finite dimensional Euclidean (resp. Hermitian) space E_τ. Set

$$\mathscr{E}_\tau = G \times_K E_\tau. \tag{3.2}$$

Then $\mathscr{E}_\tau$ is a Euclidean (resp. Hermitian) vector bundle on X.

By (3.2), we have an identification

$$C^\infty(X, \mathscr{E}_\tau) = C^\infty(G, E_\tau)^K. \tag{3.3}$$

The group G acts on the left on $C^\infty(X, \mathscr{E}_\tau)$. Denote by $C^{\mathfrak{g},X,\tau}$ the Casimir element of G on $C^\infty(X, \mathscr{E}_\tau)$. By (2.7), $C^{\mathfrak{g},X,\tau}$ is a generalised Laplacian on X in the sense of [BeGeVe04, Definition 2.2], which is self adjoint with respect to the standard L^2-product (c.f. (1.2)).

3.2. **Locally symmetric spaces.** Let $\Gamma \subset G$ be a discrete cocompact subgroup of G. By [S60, Lemma 1] (see also [Ma19, Proposition 3.9]), Γ contains only semisimple elements. Let $\Gamma_e \subset \Gamma$ be the subset of elliptic elements, and let $\Gamma_+ = \Gamma - \Gamma_e$ be the subset of nonelliptic elements.

The group Γ acts isometrically on the left on X. Take

$$Z = \Gamma \backslash X = \Gamma \backslash G / K. \tag{3.4}$$

Then, Z is a compact orbifold with Riemannian metric g^{TZ}. We denote by $\widehat{p} : \Gamma \backslash G \to Z$ and $\widehat{\pi} : X \to Z$ the natural projections, so that the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "p"] & \Gamma \backslash G \arrow[d, "\widehat{p}"] \\ X \arrow[r, "\widehat{\pi}"] & Z \end{tikzcd} \tag{3.5}$$

commutes.

From now on until Section 6, we assume that Γ is torsion free, i.e., $\Gamma_e = \{\mathrm{id}\}$. Then Z is a connected closed orientable Riemannian locally symmetric manifold with nonpositive sectional curvature. Since X is contractible, $\pi_1(Z) = \Gamma$ and X is the universal cover of Z. In Section 7, we will consider the case where Γ is not torsion free.

The Γ-action on X lifts to all the homogeneous Euclidean or Hermitian vector bundles $\mathscr{E}_\tau$ on X constructed in (3.2). Then $\mathscr{E}_\tau$ descends to a Euclidean or Hermitian vector bundle on Z,

$$\mathscr{F}_\tau = \Gamma \backslash \mathscr{E}_\tau = \Gamma \backslash G \times_K E_\tau. \tag{3.6}$$

If $r \in \mathbf{N}^*$, and if $\rho : \Gamma \to \mathrm{GL}_r(\mathbf{C})$ is a representation of Γ, let F be the associated flat vector bundle on Z,

$$F = \Gamma \backslash (X \times \mathbf{C}^r). \tag{3.7}$$

The group Γ acts on $C^\infty(G,E_\tau)^K$, as well as on $\mathbf{C}^r$ by ρ. We have the identification

$$C^\infty(Z,\mathscr{F}_\tau\otimes F)=\left(C^\infty(X,\mathscr{E}_\tau)\otimes\mathbf{C}^r\right)^\Gamma. \tag{3.8}$$

The Casimir operator $C^{\mathfrak{g},X,\tau}\otimes\mathrm{id}$ preserves the above invariant space. Its action on $C^\infty(Z,\mathscr{F}_\tau\otimes F)$ will be denoted by $C^{\mathfrak{g},Z,\tau,\rho}$. If ρ is unitary, $C^{\mathfrak{g},Z,\tau,\rho}$ is self-adjoint with respect to the L^2-product on $C^\infty(Z,\mathscr{F}_\tau\otimes F)$ induced by the Hermitian metric on $\mathscr{E}_\tau$, the standard Hermitian metric on $\mathbf{C}^r$, as well as the Riemannian volume of (Z,g^{TZ}) (c.f. (1.2)). When ρ is the trivial representation, we denote it by $C^{\mathfrak{g},Z,\tau}$.

3.3. **The Ruelle zeta function.** Let us recall the definition of the Ruelle dynamical zeta function introduced by Fried [F87, Section 5].

For $\gamma\in\Gamma$, set

$$\Gamma(\gamma)=Z(\gamma)\cap\Gamma. \tag{3.9}$$

By [S60, Lemma2] (see also [Sh18, Proposition 4.9], [Ma19, Proposition 3.9]), $\Gamma(\gamma)$ is cocompact in $Z(\gamma)$.

Let $[\Gamma_+]$ and $[\Gamma]$ be the sets of conjugacy classes in Γ_+ and Γ. If $\gamma\in\Gamma$, the associated conjugacy class in Γ is denoted by $[\gamma]\in[\Gamma]$.[6] If $[\gamma]\in[\Gamma]$, for all $\gamma'\in[\gamma]$, the locally symmetric spaces

$$\Gamma(\gamma')\backslash Z(\gamma')/K(\gamma') \tag{3.10}$$

are canonically diffeomorphic, and will be denoted by $B_{[\gamma]}$.

By [DuKVa79, Proposition 5.15], the set of nontrivial closed geodesics on Z consists of a disjoint union

$$\coprod_{[\gamma]\in[\Gamma_+]}B_{[\gamma]}. \tag{3.11}$$

If $[\gamma]\in[\Gamma_+]$, all the elements of $B_{[\gamma]}$ have the same length $\ell_{[\gamma]}>0$.

If $[\gamma]\in[\Gamma_+]$, the geodesic flow induces a locally free action of $\mathbb{S}^1$ on $B_{[\gamma]}$, so that $B_{[\gamma]}/\mathbb{S}^1$ is a closed orbifold. Let $\chi_{\mathrm{orb}}\left(B_{[\gamma]}/\mathbb{S}^1\right)\in\mathbf{Q}$ be the orbifold Euler characteristic number [Sa57]. We refer the reader to [Sh18, Proposition 5.1] for an explicit formula for $\chi_{\mathrm{orb}}\left(B_{[\gamma]}/\mathbb{S}^1\right)$. In particular, if $\delta(G)\geqslant 2$, or if $\delta(G)=1$ and γ can not be conjugate by an element of G into the fundamental Cartan subgroup H, then

$$\chi_{\mathrm{orb}}\left(B_{[\gamma]}/\mathbb{S}^1\right)=0. \tag{3.12}$$

The $\mathbb{S}^1$-action on $B_{[\gamma]}$ is not necessarily effective. Let

$$m_{[\gamma]}=\left|\ker\left(\mathbb{S}^1\to\mathrm{Diff}(B_{[\gamma]})\right)\right|\in\mathbf{N}^* \tag{3.13}$$

be the generic multiplicity.

Recall that $\rho:\Gamma\to\mathrm{GL}_r(\mathbf{C})$ is a representation of Γ. By [Sh20, (4.4)], there is $\sigma_0>0$ such that

$$\sum_{[\gamma]\in[\Gamma_+]}\frac{\left|\chi_{\mathrm{orb}}\left(B_{[\gamma]}/\mathbb{S}^1\right)\right|}{m_{[\gamma]}}\left|\mathrm{Tr}\left[\rho(\gamma)\right]\right|e^{-\sigma_0\ell_{[\gamma]}}<\infty. \tag{3.14}$$

[6] The quantity $\ell_{[\gamma]}$ depends only on the conjugacy class of γ in G. So they are well defined on the conjugacy classes of Γ.

Definition 3.1. For $\mathrm{Re}(\sigma) \geqslant \sigma_0$, the Ruelle dynamical zeta function is defined by

$$R_\rho(\sigma) = \exp\Big(\sum_{[\gamma]\in[\Gamma_+]} \frac{\chi_{\mathrm{orb}}\left(B_{[\gamma]}/\mathbb{S}^1\right)}{m_{[\gamma]}} \mathrm{Tr}\left[\rho(\gamma)\right] e^{-\sigma \ell_{[\gamma]}} \Big). \tag{3.15}$$

Remark 3.2. By (3.12), if $\delta(G) \geqslant 2$, the Ruelle zeta function $R_\rho(\sigma)$ is the constant function 1. Moreover, if $\delta(G) = 1$, then the sum on the right-hand side of (3.15) can be reduced to a sum over $[\gamma] \in [\Gamma_+]$ such that γ can be conjugate into H.

Theorem 3.3. *If* $\dim Z$ *is odd, the Ruelle zeta function* R_ρ *has a meromorphic extension to* $\sigma \in \mathbf{C}$.

Proof. This is [Sh20, Theorem 0.1 i)]. □

Let $\overline{R}_\rho$ be the meromorphic function defined for $\sigma \in \mathbf{C}$ by

$$\overline{R}_\rho(\sigma) = \overline{R_\rho(\overline{\sigma})}. \tag{3.16}$$

By [Sh20, Proposition 4.4], we have

$$R_{\rho^*} = R_\rho, \qquad R_{\overline{\rho}} = \overline{R}_\rho. \tag{3.17}$$

If ρ is a finite dimensional complex representation of G, the restriction $\rho_{|\Gamma}$ is a representation of Γ. We write $R_\rho = R_{\rho_{|\Gamma}}$ to ease the notation.

Proposition 3.4. *If* ρ *is a finite dimensional complex representation of* G *with an admissible metric, then the following identity of meromorphic functions on* $\mathbf{C}$ *holds,*

$$R_{\rho^\theta} = \overline{R}_\rho. \tag{3.18}$$

Proof. This is a consequence of (2.36) and (3.17). □

3.4. **The Selberg zeta function.** In this subsection, we assume $\delta(G) = 1$ and we use the notation in Section 2.6. Recall that K_M is defined in (2.22). We have seen that the morphism $R(K) \to R(K_M)$ of rings is injective.

Assumption 3.5. *Assume that* $\eta = \eta^+ - \eta^-$ *is a virtual* M*-representation on the finite dimensional complex vector space* $E^+_\eta - E^-_\eta$ *such that*

(1) $\eta_{|K_M} = \eta^+_{|K_M} - \eta^-_{|K_M} \in R(K_M)$ *has a unique lift in* $R(K)$.
(2) the Casimir $C^{\mathfrak{u}_\mathfrak{m}}$ *of* $\mathfrak{u}_\mathfrak{m}$ *acts on* $\eta^\pm$ *by the same scalar* $C^{\mathfrak{u}_\mathfrak{m},\eta} \in \mathbf{R}$.

Following [Sh18, Definition 7.4] and [Sh20, Defintion 5.7], let us define the Selberg zeta function associated[7] to η. Recall that $H = \exp(\mathfrak{b}) \times T$ is the fundamental Cartan subgroup of G. For $e^a k^{-1} \in H$, we write $\gamma \sim e^a k^{-1} \in H$ if there is $g_\gamma \in G$ such that $\gamma = g_\gamma e^a k^{-1} g_\gamma^{-1}$. By [Sh18, (7-62)], there is $\sigma_1 > 0$ such that

$$\sum_{\substack{[\gamma]\in[\Gamma_+] \\ \gamma\sim e^a k^{-1}\in H}} \frac{\left|\chi_{\mathrm{orb}}\left(B_{[\gamma]}/\mathbb{S}^1\right)\right|}{m_{[\gamma]}} \frac{e^{-\sigma_1 \ell_{[\gamma]}}}{\left|\det\left(1 - \mathrm{Ad}(e^a k^{-1})\right)|_{\mathfrak{z}^\perp(\mathfrak{b})}\right|^{1/2}} < \infty. \tag{3.19}$$

[7] A more general construction for the Selberg zeta function is given in [Sh20], which is associated to η and to an arbitrary representation of $\rho : \Gamma \to \mathrm{GL}_r(\mathbf{C})$.

Definition 3.6. For $\mathrm{Re}(\sigma) \geqslant \sigma_1$, set

$$Z_\eta(\sigma) = \exp\Bigg(- \sum_{\substack{[\gamma]\in[\Gamma_+] \\ \gamma \sim e^a k^{-1} \in H}} \frac{\chi_{\mathrm{orb}}\left(B_{[\gamma]}/\mathbb{S}^1\right)}{m_{[\gamma]}} \frac{\mathrm{Tr}_{\mathrm{s}}^{E_\eta}[k^{-1}]}{\left|\det\left(1-\mathrm{Ad}(e^a k^{-1})\right)|_{\mathfrak{z}^\perp(\mathfrak{b})}\right|^{1/2}} e^{-\sigma \ell_{[\gamma]}} \Bigg). \tag{3.20}$$

Recall that by Corollary 2.6, $\Lambda^\bullet(\mathfrak{p}^*_{\mathfrak{m},\mathbf{C}})$ has a unique lift in $R(K)$.

Definition 3.7. Let $\widehat{\eta} \in R(K)$ be the unique virtual representation of K on $E_{\widehat{\eta}} = E^+_{\widehat{\eta}} - E^-_{\widehat{\eta}}$ such that the following identity in $R(K_M)$ holds,

$$E_{\widehat{\eta}|K_M} = \Lambda^\bullet(\mathfrak{p}^*_{\mathfrak{m},\mathbf{C}}) \widehat{\otimes} E_{\eta|K_M} \in R(K_M). \tag{3.21}$$

Let $C^{\mathfrak{g},Z,\widehat{\eta}}$ be the self adjoint generalised Laplacian acting on $C^\infty(Z,\mathscr{F}_{\widehat{\eta}})$ introduced below (3.8). For $\lambda \in \mathbf{C}$, set

$$m_\eta(\lambda) = \dim\ker\left(C^{\mathfrak{g},Z,\widehat{\eta}^+} - \lambda\right) - \dim\ker\left(C^{\mathfrak{g},Z,\widehat{\eta}^-} - \lambda\right). \tag{3.22}$$

Let

$$\det_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta}} + \sigma\right) = \frac{\det\left(C^{\mathfrak{g},Z,\widehat{\eta}^+} + \sigma\right)}{\det\left(C^{\mathfrak{g},Z,\widehat{\eta}^-} + \sigma\right)} \tag{3.23}$$

be a graded determinant of $C^{\mathfrak{g},Z,\widehat{\eta}} + \sigma$. As in (1.8), by [Vo87] (see also [Sh20, Theorem 1.5]), the function (3.23) is meromorphic on $\sigma \in \mathbf{C}$. Moreover, its zeros and poles belong to the set $\left\{-\lambda : \lambda \in \mathrm{Sp}\left(C^{\mathfrak{g},Z,\widehat{\eta}}\right)\right\}$. If $\lambda \in \mathrm{Sp}\left(C^{\mathfrak{g},Z,\widehat{\eta}}\right)$, the order of the zero at $\sigma = -\lambda$ is $m_\eta(\lambda)$.

Following [Sh18, (7-60)] and [Sh20, (5.18), (5.19)], we set

$$\sigma_\eta = \frac{1}{8}\mathrm{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right] - C^{\mathfrak{u}_{\mathfrak{m}},\eta}. \tag{3.24}$$

Remark 3.8. When Z_G is non compact, by [Sh18, (4-52)], we have

$$G = \exp(\mathfrak{b}) \times M. \tag{3.25}$$

Therefore, $\mathfrak{u}^\perp(\mathfrak{b}) = 0$. By (3.24), we have

$$\sigma_\eta = -C^{\mathfrak{u}_{\mathfrak{m}},\eta}. \tag{3.26}$$

Let $P_\eta(\sigma)$ be the odd polynomial defined in [Sh18, (7-61)] and [Sh20, (5.20), Remark 5.9].

Theorem 3.9. *The Selberg zeta function $Z_\eta(\sigma)$ has a meromorphic extension to $\sigma \in \mathbf{C}$ such that the following identity of meromorphic functions on $\mathbf{C}$ holds,*

$$Z_\eta(\sigma) = \det_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta}} + \sigma_\eta + \sigma^2\right)\exp\left(\mathrm{vol}(Z)P_\eta(\sigma)\right). \tag{3.27}$$

The zeros and poles of $Z_\eta(\sigma)$ belong to the set $\left\{\pm i\sqrt{\lambda+\sigma_\eta} : \lambda \in \mathrm{Sp}\left(C^{\mathfrak{g},Z,\widehat{\eta}}\right)\right\}$. If $\lambda \in \mathrm{Sp}\left(C^{\mathfrak{g},Z,\widehat{\eta}}\right)$ and $\lambda \neq -\sigma_\eta$, the order of zero at $\sigma = \pm i\sqrt{\lambda+\sigma_\eta}$ is $m_\eta(\lambda)$. The order of zero at $\sigma = 0$ is $2m_\eta(-\sigma_\eta)$. Also,

$$Z_\eta(\sigma) = Z_\eta(-\sigma)\exp\left(2\,\mathrm{vol}(Z)P_\eta(\sigma)\right). \tag{3.28}$$

Proof. This is [Sh20, Theorem 5.10] for the trivial twist (c.f. Footnote 7). □

4. The Fried conjecture and admissible metrics

In this section, we introduce a class of Hermitian flat vector bundles on locally symmetric spaces associated to representations of G with an admissible metrics. We state the main result (Theorem 4.4) of this article, which confirms the Fried conjecture for this class of Hermitian flat vector bundles.

This section is organised as follows. In Section 4.1, we introduce a Hermitian metric on a flat vector bundle whose holonomy representation is the restriction $\rho_{|\Gamma}$ of a representation ρ of G with an admissible metric.

In Section 4.2, we state Theorem 4.4.

In Sections 4.3-4.5, we deduce Theorem 0.1, Corollaries 0.2 and 0.3 from Theorem 4.4.

Finally, in Section 4.6, we show Theorem 4.4 when $\delta(G)=1$ and Z_G is compact.

4.1. **Hermitian metrics on flat vector bundles.** Let $\rho : G \to \mathrm{GL}(E)$ be a finite dimensional complex representation of G with admissible Hermitian metric $\langle , \rangle_E$.

Let F be the flat vector bundle associated to $\rho_{|\Gamma}$ defined in (3.7). Let us construct a Hermitian metric g^F on F following [Mü12, Section 2.5] and [BMaZ17, Section 8.1]. By the second identity of (2.35), the restriction $\rho_{|K}$ of ρ to K is unitary. Let

$$\mathscr{E}_{\rho_{|K}} = G \times_K E \tag{4.1}$$

be the Hermitian vector bundle on X defined in (3.2). We have a canonical G-equivariant identification

$$[g,v] \in G \times_K E \to (pg, gv) \in X \times E. \tag{4.2}$$

In this way, the G-invariant Hermitian metric on $\mathscr{E}_{\rho_{|K}}$ induces a G-invariant Hermitian metric $g^{\widehat{\pi}^* F}$ on the trivial vector bundle $\widehat{\pi}^* F = X \times E$. It descends to a Hermitian metric g^F on $F = \Gamma \backslash (X \times E)$.

Definition 4.1. We will call such (F, g^F) an admissible Hermitian flat vector bundle. The g^F will be called an admissible Hermitian metric on F.

By (4.2), as in (3.3) and (3.8), we have the identifications

$$\Omega^\bullet\left(X, \widehat{\pi}^* F\right) = C^\infty\left(G, \Lambda^\bullet(\mathfrak{p}^*) \otimes_{\mathbf{R}} E\right)^K, \quad \Omega^\bullet(Z, F) = C^\infty\left(\Gamma \backslash G, \Lambda^\bullet(\mathfrak{p}^*) \otimes_{\mathbf{R}} E\right)^K. \tag{4.3}$$

Let $\square^X$ be the Hodge Laplacian on X acting on $\Omega^\bullet(X, \widehat{\pi}^* F)$ with respect to the metrics $(g^{TX}, g^{\widehat{\pi}^* F})$. Let $\square^Z$ be the Hodge Laplacian on Z acting on $\Omega^\bullet(Z, F)$ with respect to the metrics (g^{TZ}, g^F). Recall that $C^{\mathfrak{u},\rho} \in \mathrm{End}(E)$ is the Casimir operator of $\mathfrak{u}$ acting on E (see (2.9)). The following proposition is classical (see [BMaZ17, Proposition 8.4]).

Proposition 4.2. *Under the identifications* (4.3)*, we have*

$$\square^X = C^{\mathfrak{g}, X, \Lambda^\bullet(\mathfrak{p}^*) \otimes_{\mathbf{R}} E_{|K}} - C^{\mathfrak{u},\rho}, \qquad \square^Z = C^{\mathfrak{g}, Z, \Lambda^\bullet(\mathfrak{p}^*) \otimes_{\mathbf{R}} E_{|K}} - C^{\mathfrak{u},\rho}. \tag{4.4}$$

Let $T(F)$ be the analytic torsion of F associated to (g^{TZ}, g^F). Let $N^{\Lambda^\bullet(T^*Z)}$ be the number operator, i.e. $N^{\Lambda^\bullet(T^*Z)}$ acts by multiplication by k on $\Omega^k(Z, F)$.

Theorem 4.3. *Assume $\delta(G) \neq 1$. For any $t > 0$, we have*

$$\mathrm{Tr_s}\left[\left(N^{\Lambda^\bullet(T^*Z)} - \frac{m}{2}\right) \exp\left(-t\square^Z\right)\right] = 0. \tag{4.5}$$

In particular,

$$T(F) = 1. \tag{4.6}$$

Proof. This is [BMaZ11] and [BMaZ17, Theorem 8.6, Remark 8.7] (see also [Ma19, Theorem 5.5]), which generalises a vanishing theorem originally due to [MoSt91, Corollary 2.2] (see also [B11, Theorem 7.9.3]) where F is assumed to be unitarily flat. □

Recall that $\rho^\theta = \rho \circ \theta$. Clearly, $\langle , \rangle_E$ is still an admissible metric for ρ^θ. Let (F^θ, g^{F^θ}) be the associated admissible Hermitian flat vector bundle on Z. Since $\rho_{|K} = \rho^\theta_{|K}$, by (4.3), the Hodge Laplacians of (F, g^F) and (F^θ, g^{F^θ}) act on the same space. By (4.4) and by $C^{\mathrm{u},\rho} = C^{\mathrm{u},\rho^\theta}$, these two Laplacians coincide. In particular,

$$H^\bullet(Z,F) \simeq H^\bullet\left(Z, F^\theta\right), \qquad T(F) = T\left(F^\theta\right). \tag{4.7}$$

4.2. **The statement of the main result.** The main result of this article is the following.

Theorem 4.4. *Assume that* $\dim Z$ *is odd and that* $\rho : G \to \mathrm{GL}(E)$ *is a finite dimensional complex representation of* G *with an admissible metric. Let* (F, g^F) *be the associated admissible Hermitian flat vector bundle. If* $\rho \simeq \rho^\theta$*, then there exist explicit constants* $C_\rho \in \mathbf{R}^*$ *and* $r_\rho \in \mathbf{Z}$ *(see* (4.36) *and Remark 5.11) such that when* $\sigma \to 0$*, we have*

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho + 1}). \tag{4.8}$$

Moreover, if $H^\bullet(Z,F) = 0$*, then*

$$C_\rho = 1, \qquad r_\rho = 0, \tag{4.9}$$

so that

$$R_\rho(0) = T(F)^2. \tag{4.10}$$

Proof. Since $\dim Z$ is odd, $\delta(G)$ is odd. If $\delta(G) \geqslant 3$, by Remark 3.2 and Theorem 4.3, our theorem follows easily. If $\delta(G) = 1$, we will consider the case where Z_G is non compact in Section 4.6 and the case where Z_G is compact in Sections 5 and 6. □

Remark 4.5. Assume $\dim Z$ is even. Then $\delta(G)$ is even as well. If $\delta(G) \geqslant 2$, then by Remark 3.2 and Theorem 4.3, we have $R_\rho(0) = T(F)^2 = 1$. If $\delta(G) = 0$, by the Theorem of Gauss-Bonnet-Chern and by [Sh18, Proposition 4.1], we have

$$\chi(Z,F) = \dim E \cdot \chi(Z) \neq 0. \tag{4.11}$$

In particular, there are no acyclic flat vector bundles.

4.3. **Proof of Corollary 0.2.** Let us restate Corollary 0.2.

Theorem 4.6. *Assume that* $\dim Z$ *is odd and that* $\rho : G \to \mathrm{GL}(E)$ *is a finite dimensional complex representation of* G *with an admissible metric. Let* (F, g^F) *be the associated admissible Hermitian flat vector bundle. If* ρ *is irreducible and* $\rho \neq \rho^\theta$*, then*

$$H^\bullet(Z,F) = 0, \tag{4.12}$$

and R_ρ is regular at $\sigma = 0$, so that

$$\left|R_\rho(0)\right| = T(F)^2. \tag{4.13}$$

Proof. The vanishing of the cohomology $H^\bullet(Z,F)$ is a consequence of [BoW00, Theorem VII.6.7] (see (6.20) for a proof).

Take

$$\rho' = \rho \oplus \rho^\theta. \tag{4.14}$$

By (3.18) and (4.14), we have

$$R_{\rho'} = R_\rho \overline{R}_\rho. \tag{4.15}$$

Note that ρ' has an admissible metric. Let F' be the admissible Hermitian flat vector bundle associated to ρ'. By (4.7) and (4.12), we have

$$H^\bullet\left(Z,F'\right) = H^\bullet(Z,F) \oplus H^\bullet\left(Z,F^\theta\right) = 0, \qquad T(F') = T(F)T\left(F^\theta\right) = T(F)^2. \tag{4.16}$$

Since $\rho' = \rho \oplus \rho^\theta$ is invariant by θ, we can apply Theorem 4.4 to ρ'. Therefore, $R_{\rho'}$ is regular at $\sigma = 0$ and

$$R_{\rho'}(0) = T(F')^2. \tag{4.17}$$

By (4.15)-(4.17), we get (4.13). □

4.4. **Proof of Theorem 0.1.** Since any G-representation with an admissible metric can be decomposed as a direct sum of irreducible G-representations, which still have admissible metrics, by Theorems 4.4 and 4.6, we get Theorem 0.1. □

4.5. **Proof of Corollary 0.3.** Let $\rho_0 : \Gamma \to \mathrm{U}(r)$ be a unitary representation of Γ. Let F_0 be the associated flat vector bundle . Since ρ_0 is unitary, F_0 admits a flat metric g^{F_0}. We will show that (F_0, g^{F_0}) is indeed an admissible Hermitian flat vector bundle associated to a larger reductive group.

Let

$$\underline{G} = G \times \mathrm{U}(r), \qquad \underline{K} = K \times \mathrm{U}(r). \tag{4.18}$$

Then, $\underline{G}$ is a connected real reductive group with maximal compact subgroup $\underline{K}$. We have an identification

$$\underline{G}/\underline{K} \simeq X. \tag{4.19}$$

Let $\underline{\Gamma}$ be the graph of ρ_0, i.e.,

$$\underline{\Gamma} = \left\{(\gamma, \rho_0(\gamma)) \in \underline{G} : \gamma \in \Gamma\right\}. \tag{4.20}$$

Then $\underline{\Gamma}$ is a discrete torsion free and cocompact subgroup of $\underline{G}$, so that

$$\underline{\Gamma}\backslash\underline{G}/\underline{K} \simeq Z. \tag{4.21}$$

Let $\underline{\rho}_0 : \underline{G} \to \mathrm{U}(r)$ be the representation of $\underline{G}$ defined by the projection onto the second component. The standard Hermitian metric on $\mathbf{C}^r$ is admissible for the representation $\underline{\rho}_0$. Also, we have an identification of Hermitian flat vector bundles

$$\underline{\Gamma}\backslash(\underline{G}/\underline{K} \times \mathbf{C}^r) \simeq F_0. \tag{4.22}$$

In particular, (F_0, g^{F_0}) is an admissible Hermitian flat vector bundle associated to the representation $\underline{\rho}_0$ of $\underline{G}$.

More generally, the admissible Hermitian flat vector bundle (F, g^F) associated to the G-representation ρ is also an admissible Hermitian flat vector bundle associated to the $\underline{G}$-representation $\underline{\rho} = \rho \boxtimes \mathbf{1}$. Therefore, $F_0 \otimes F$ with the induced Hermitian metric is an admissible Hermitian flat vector bundle associated to $\underline{\rho}_0 \otimes \underline{\rho}$. By these considerations, Corollary 0.3 follows from Theorem 0.1 and Corollary 0.2. □

4.6. **Proof of Theorem 4.4 when $\delta(G) = 1$ and Z_G is noncompact.** Suppose now $\delta(G) = 1$ and Z_G is non compact. Let ρ be a representation of G with an admissible metric such that $\rho \simeq \rho^\theta$. We can and we will assume that the Casimir $C^{\mathfrak{u}}$ acts on ρ as a scalar $C^{\mathfrak{u},\rho} \in \mathbf{R}$.

For $\beta \in \mathfrak{b}^*_{\mathbf{C}}$, denote by $\mathbf{C}_\beta$ the one dimensional representation of $\exp(\mathfrak{b})$ such that $a \in \mathfrak{b}$ acts as the scalar $\langle \beta, a \rangle \in \mathbf{C}$. Clearly, $\mathbf{C}_\beta$ has an admissible metric if and only if $\beta \in \mathfrak{b}^*$.

Recall that since Z_G is not compact, we have $G = \exp(\mathfrak{b}) \times M$ (see (3.25)). Since a representation with an admissible metric is completely reducible, we can write

$$\rho = \bigoplus_{\beta \in \mathfrak{b}^*} \mathbf{C}_\beta \boxtimes \eta_\beta, \tag{4.23}$$

where η_β is a representation of M.

Proposition 4.7. *The following statements hold.*

i) For $\beta \in \mathfrak{b}^*$, we have isomorphisms of representations of M,

$$\eta_{-\beta} \simeq \eta_\beta. \tag{4.24}$$

ii) For $\beta \in \mathfrak{b}^*$, the representation η_β of M satisfies Assumption 3.5, so that

$$C^{\mathfrak{u}_{\mathfrak{m}},\eta_\beta} = C^{\mathfrak{u},\rho} + |\beta|^2 \in \mathbf{R}, \tag{4.25}$$

Proof. Since $\rho^\theta \simeq \rho$, by (4.23), we have isomorphisms of representations of M, $\eta^\theta_\beta \simeq \eta_{-\beta}$. Since $\delta(M) = 0$, by Proposition 2.7, we have isomorphisms of representations of M, $\eta^\theta_\beta \simeq \eta_\beta$. From these considerations, (4.24) follows.

By (3.25), we have $K_M = K$, so Assumption 3.5 (1) is trivial. Since $C^{\mathfrak{u},\rho}$ is a scalar, by (3.25) and (4.23), the Casimir of $\mathfrak{u}_{\mathfrak{m}}$ acts on η_β as a scalar given in (4.25). In particular, η_β satisfies also Assumption 3.5 (2). □

Recall that in Section 2.6, we have fixed a positive element $f_{\mathfrak{b}} \in \mathfrak{b}$ in $\mathfrak{b}$. Set

$$\mathfrak{b}^*_+ = \{\alpha \in \mathfrak{b}^* : \langle \alpha, f_{\mathfrak{b}} \rangle > 0\}. \tag{4.26}$$

By (4.24), we can rewrite (4.23) as

$$\rho = \mathbf{1} \boxtimes \eta_0 \oplus \bigoplus_{\beta \in \mathfrak{b}^*_+} \left(\mathbf{C}_\beta \oplus \mathbf{C}_{-\beta}\right) \boxtimes \eta_\beta. \tag{4.27}$$

Note that η_0 can be zero.

Let $Z_{\eta_\beta}(\sigma)$ be Selberg zeta function associated to η_β.

Proposition 4.8. *The following identity of meromorphic functions on* $\mathbf{C}$ *holds,*

$$R_\rho(\sigma) = \left(Z_{\eta_0}(\sigma)\right)^{-1} \prod_{\beta\in\mathfrak{b}_+^*} \left(Z_{\eta_\beta}\left(\sigma+|\beta|\right) Z_{\eta_\beta}\left(\sigma-|\beta|\right)\right)^{-1}. \tag{4.28}$$

Proof. By (4.27), for $e^a k^{-1} \in H$, we have

$$\mathrm{Tr}\left[\rho\left(e^a k^{-1}\right)\right] = \mathrm{Tr}\left[\eta_0\left(k^{-1}\right)\right] + \sum_{\beta\in\mathfrak{b}_+^*} \mathrm{Tr}\left[\eta_\beta\left(k^{-1}\right)\right]\left(e^{|\beta||a|} + e^{-|\beta||a|}\right). \tag{4.29}$$

By (3.25), we have $\mathfrak{z}^\perp(\mathfrak{b}) = 0$, so that the denominator $\left|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}^\perp(\mathfrak{b})}\right|^{1/2}$ in (3.20) disappears. By (3.15), (3.20), and (4.29), we get (4.28). □

By (3.26), (3.27), and (4.25), we see that

$$Z_{\eta_\beta}\left(-\sqrt{\sigma^2+|\beta|^2}\right) Z_{\eta_\beta}\left(\sqrt{\sigma^2+|\beta|^2}\right) = \left[\det{}_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta}_\beta} - C^{\mathfrak{u},\rho} + \sigma^2\right)\right]^2 \tag{4.30}$$

is a meromorphic function on $\mathbf{C}$. We have a generalisation of [Sh18, Theorem 5.6]. Recall that $T(\sigma)$ is defined in (1.8).

Theorem 4.9. *The following identity of meromorphic functions on* $\mathbf{C}$ *holds,*

$$R_\rho(\sigma) = T\left(\sigma^2\right)\exp\left(-\mathrm{vol}(Z)P_{\eta_0}(\sigma)\right) \prod_{\beta\in\mathfrak{b}_+^*} \frac{Z_{\eta_\beta}\left(-\sqrt{\sigma^2+|\beta|^2}\right) Z_{\eta_\beta}\left(\sqrt{\sigma^2+|\beta|^2}\right)}{Z_{\eta_\beta}\left(\sigma+|\beta|\right) Z_{\eta_\beta}\left(\sigma-|\beta|\right)}. \tag{4.31}$$

Proof. By (3.25), we have $\mathfrak{p} = \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{b}$. By (3.21), we have an identity in $R(K)$,

$$\widehat{\eta}_\beta = \sum_{i=1}^{m} (-1)^{i-1} i\Lambda^i(\mathfrak{p}_{\mathbf{C}}^*) \otimes \eta_{\beta|K}. \tag{4.32}$$

By (4.4), (4.27), and (4.32), we have an identity of meromorphic functions,

$$T(\sigma) = \det{}_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta}_0} - C^{\mathfrak{u},\rho} + \sigma\right)^{-1} \prod_{\beta\in\mathfrak{b}_+^*} \det{}_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta}_\beta} - C^{\mathfrak{u},\rho} + \sigma\right)^{-2}. \tag{4.33}$$

By (3.26), (3.27), (4.25), (4.30), (4.33), we have

$$\begin{aligned} T\left(\sigma^2\right) = {} & Z_{\eta_0}(\sigma)^{-1}\exp\left(\mathrm{vol}(Z)P_{\eta_0}(\sigma)\right) \\ & \times \prod_{\beta\in\mathfrak{b}_+^*} \left(Z_{\eta_\beta}\left(-\sqrt{\sigma^2+|\beta|^2}\right) Z_{\eta_\beta}\left(\sqrt{\sigma^2+|\beta|^2}\right)\right)^{-1}. \end{aligned} \tag{4.34}$$

By (4.28) and (4.34), we get (4.31). □

Set

$$r_{\eta_\beta} = \dim\ker\left(C^{\mathfrak{g},Z,\widehat{\eta}_\beta^+} - C^{\mathfrak{u},\rho}\right) - \dim\ker\left(C^{\mathfrak{g},Z,\widehat{\eta}_\beta^-} - C^{\mathfrak{u},\rho}\right). \tag{4.35}$$

Following [Sh18, (7-75)], put

$$C_\rho = \prod_{\beta\in\mathfrak{b}_+^*} \left(-4|\beta|^2\right)^{-r_{\eta_\beta}}, \qquad r_\rho = -2\sum_{\beta\in\{0\}\cup\mathfrak{b}_+^*} r_{\eta_\beta}. \tag{4.36}$$

Proceeding as [Sh18, (7-76)-(7-78)], using Theorem 4.9 instead of [Sh18, Theorem 7.8], we get (4.8).

Let F_β be the admissible Hermitian flat subbundle of F associated to the G-representation $\mathbf{C}_\beta \boxtimes \eta_\beta$. By (1.1), (4.4), (4.32), and (4.35), we have

$$r_{\eta_\beta} = -\chi'(Z, F_\beta). \tag{4.37}$$

If $H^{\bullet}(Z,F)=0$, then for all $\beta\in\mathfrak{b}^*$, $H^{\bullet}(Z,F_\beta)=0$, so $r_{\eta_\beta}=0$. By (4.36), we get (4.9).

The proof of Theorem 4.4 in the case when $\delta(G)=1$ and Z_G is non compact is completed. □

5. The proof of (4.8) when $\delta(G)=1$ and Z_G is compact

The purpose of this section is to show (4.8) when $\delta(G)=1$ and Z_G is compact by generalising the arguments given in Section 4.6. One of the difficulties is to construct virtual representations η_β of M satisfying Assumption 3.5 such that an analogue of (4.28) holds. In the case of hyperbolic manifolds, such representations are constructed using $\mathfrak{n}$-cohomology [Mü12]. Here, we construct η_β via Dirac cohomology. These two methods are equivalent. We adopt the latter since it is closer to certain constructions given in [Sh18, Section 6].

This section is organised as follows. In Section 5.1, we recall some facts on the structure of real reductive groups with $\delta(G)=1$ established in [Sh18, Section 6].

In Section 5.2, we decompose ρ according to the action of $\mathfrak{b}$. We show certain representations of K_M obtained in this way can be lifted in $R(K)$.

In Section 5.3, we introduce virtual representations η_β of M satisfying Assumption 3.5.

Finally, in Section 5.4, we establish analogues of Proposition 4.8 and Theorem 4.9, and we show (4.8).

In this section, we assume that $\delta(G)=1$ and Z_G is compact. Suppose also that the G-representation (ρ,E) has an admissible metric and is such that $\rho\simeq\rho^\theta$. As in Section 4.6, we can and we will assume that Casimir operator $C^{\mathfrak{u}}$ acts as a scalar $C^{\mathfrak{u},\rho}\in\mathbf{R}$.

5.1. **The structure of the reductive group G with $\delta(G)=1$.** Since G has a compact centre and $\dim\mathfrak{b}=1$, we have $\mathfrak{b}\not\subset\mathfrak{z}_{\mathfrak{g}}$, so $\mathfrak{n}\neq 0$. By Proposition 2.5 i), $\dim\mathfrak{n}$ is a positive even number. Set

$$\ell=\frac{1}{2}\dim\mathfrak{n}\in\mathbf{N}^*. \tag{5.1}$$

Proposition 5.1. *Elements of $\mathfrak{b}$ act on $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ as a scalar, i.e., there is $\alpha_0\in\mathfrak{b}^*$ such that for $a\in\mathfrak{b}$, $f\in\mathfrak{n}$, $\overline{f}\in\overline{\mathfrak{n}}$, we have*

$$[a,f]=\langle\alpha_0,a\rangle f,\qquad \left[a,\overline{f}\right]=-\langle\alpha_0,a\rangle\overline{f}. \tag{5.2}$$

In particular,

$$\left[\mathfrak{n},\overline{\mathfrak{n}}\right]\subset\mathfrak{z}(\mathfrak{b}),\qquad [\mathfrak{n},\mathfrak{n}]=\left[\overline{\mathfrak{n}},\overline{\mathfrak{n}}\right]=0, \tag{5.3}$$

and

$$\begin{aligned}&[\mathfrak{z}(\mathfrak{b}),\mathfrak{z}(\mathfrak{b})]\subset\mathfrak{z}(\mathfrak{b}),&&\left[\mathfrak{z}(\mathfrak{b}),\mathfrak{z}^{\perp}(\mathfrak{b})\right]\subset\mathfrak{z}^{\perp}(\mathfrak{b}),&&\left[\mathfrak{z}^{\perp}(\mathfrak{b}),\mathfrak{z}^{\perp}(\mathfrak{b})\right]\subset\mathfrak{z}(\mathfrak{b}),\\&[\mathfrak{u}(\mathfrak{b}),\mathfrak{u}(\mathfrak{b})]\subset\mathfrak{u}(\mathfrak{b}),&&\left[\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^{\perp}(\mathfrak{b})\right]\subset\mathfrak{u}^{\perp}(\mathfrak{b}),&&\left[\mathfrak{u}^{\perp}(\mathfrak{b}),\mathfrak{u}^{\perp}(\mathfrak{b})\right]\subset\mathfrak{u}(\mathfrak{b}).\end{aligned} \tag{5.4}$$

Proof. This is [Sh18, Propositions 6.2, 6.3, and (6-29)]. □

Let $\alpha_0\in\mathfrak{b}$ be such that

$$\langle\alpha_0,\alpha_0\rangle=1. \tag{5.5}$$

By (5.4), $(\mathfrak{u},\mathfrak{u}(\mathfrak{b}))$ is a compact symmetric pair. Recall that G has a compact centre. Let U be the compact form of G [Kn86, Proposition 5.3]. Let $U(\mathfrak{b})$ be the centraliser of $\mathfrak{b}$ in U. Then, $U(\mathfrak{b})$ is a connected Lie group [Kn02, Corollary 4.51] with Lie algebra $\mathfrak{u}(\mathfrak{b})$.

In [Sh18, (6-31)], we have shown that $J = \mathrm{ad}\left(\sqrt{-1}a_0\right) \in \mathrm{End}(\mathfrak{u}^{\perp}(\mathfrak{b}))$ defines a $U(\mathfrak{b})$-invariant complex structure on $\mathfrak{u}^{\perp}(\mathfrak{b})$. Moreover, the associated holomorphic and anti-holomorphic subspaces are $\mathfrak{n}_{\mathbf{C}}$ and $\overline{\mathfrak{n}}_{\mathbf{C}}$, so that we have a $U(\mathfrak{b})$-equivariant splitting

$$\mathfrak{u}^{\perp}(\mathfrak{b}) \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{n}_{\mathbf{C}} \oplus \overline{\mathfrak{n}}_{\mathbf{C}}. \tag{5.6}$$

Let $S^{\mathfrak{u}^{\perp}(\mathfrak{b})}$ be the spinor of $(\mathfrak{u}^{\perp}(\mathfrak{b}), -B|_{\mathfrak{u}^{\perp}(\mathfrak{b})})$. Classically ([Hi74], see also [Sh18, (6-33) and (6-34)]), we have an isomorphism of $U(\mathfrak{b})$-representations,

$$S^{\mathfrak{u}^{\perp}(\mathfrak{b})}_{\pm} \simeq \Lambda^{\mathrm{even/odd}}\left(\overline{\mathfrak{n}}^{*}_{\mathbf{C}}\right) \otimes \det(\mathfrak{n}_{\mathbf{C}})^{-1/2}. \tag{5.7}$$

In the sequel, there are representations which do not always lift to $U(\mathfrak{b})$. Therefore, it is more convenient to consider $S^{\mathfrak{u}^{\perp}(\mathfrak{b})}$ as a $(\mathfrak{b}_{\mathbf{C}} \oplus \mathfrak{m}_{\mathbf{C}}, K_M)$-module.

For $0 \leqslant j \leqslant 2\ell$, let η_j be the $(\mathfrak{m}_{\mathbf{C}}, K_M)$-module $\Lambda^j(\mathfrak{n}^{*}_{\mathbf{C}})$. By (2.31), we have an isomorphism of $(\mathfrak{m}_{\mathbf{C}}, K_M)$-modules,

$$\eta_{\ell-j} \simeq \eta_{\ell+j}. \tag{5.8}$$

Proposition 5.2. *We have an isomorphism of* $(\mathfrak{b}_{\mathbf{C}} \oplus \mathfrak{m}_{\mathbf{C}}, K_M)$*-modules,*

$$S^{\mathfrak{u}^{\perp}(\mathfrak{b})} \simeq \bigoplus_{j=-\ell}^{\ell} \mathbf{C}_{j\alpha_0} \boxtimes \eta_{\ell-j}. \tag{5.9}$$

For $e^a k^{-1} \in H$*, we have*

$$\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{u}^{\perp}(\mathfrak{b})}}\left[e^a k^{-1}\right] = \left|\det\left(1 - \mathrm{Ad}(e^a k^{-1})\right)|_{\mathfrak{z}^{\perp}(\mathfrak{b})}\right|^{1/2}. \tag{5.10}$$

Proof. By (5.2), (5.7), and (5.8), we get (5.9). By (5.9) and [Sh18, Proposition 6.5], we get (5.10). □

Let B^* be the bilinear form on $\mathfrak{g}^*$ induced by B.

Proposition 5.3. *We have*

$$\frac{1}{8}\mathrm{Tr}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^{\perp}(\mathfrak{b})}\right] = -\ell^2 B^*(\alpha_0, \alpha_0). \tag{5.11}$$

Proof. This is [Sh18, Proposition 6.13] with $j = 0$. A direct proof of this result can be obtained by applying Kostant's stranger formula (see also [B11, (2.6.11), (7.5.4)]), which is left to reader. □

5.2. **A splitting of ρ according to the $\mathfrak{b}$-action.** Recall that $Z^0(\mathfrak{b}) = \exp(\mathfrak{b}) \times M$. Since ρ has an admissible metric and since $\rho \simeq \rho^{\theta}$, as in (4.23) and (4.27), we can write

$$\rho_{|\exp(\mathfrak{b})\times M} = \bigoplus_{\beta \in \mathfrak{b}^*} \mathbf{C}_{\beta} \boxtimes \rho_{\beta} = \mathbf{1} \boxtimes \rho_0 \oplus \bigoplus_{\beta \in \mathfrak{b}^*_{+}} \left(\mathbf{C}_{\beta} \oplus \mathbf{C}_{-\beta}\right) \boxtimes \rho_{\beta}. \tag{5.12}$$

where ρ_{β} are representations of M such that $\rho_{-\beta} \simeq \rho_{\beta}$.

Recall that the restriction induces an injective morphism $R(K) \to R(K_M)$ of rings.

Theorem 5.4. *For all $\beta \in \mathfrak{b}^*$, the restriction $\rho_{\beta|K_M}$ has a unique lift in $R(K)$.*

Proof. By (2.32), it is enough to show that the character of $\rho_{\beta|T}$ is invariant under the Weyl group $W(T:K)$.

For $e^a k^{-1} \in H$, by in (5.12), as (4.29), we have

$$\mathrm{Tr}\left[\rho\left(e^a k^{-1}\right)\right] = \mathrm{Tr}\left[\rho_0\left(k^{-1}\right)\right] + \sum_{\beta\in\mathfrak{b}_+^*}\left(e^{|\beta||a|}+e^{-|\beta||a|}\right)\mathrm{Tr}\left[\rho_\beta\left(k^{-1}\right)\right]. \tag{5.13}$$

Let $w \in N_K(T)$. Since ρ is a G representation, for $e^a k^{-1} \in H$, we have

$$\mathrm{Tr}\left[\rho\left(e^a k^{-1}\right)\right] = \mathrm{Tr}\left[\rho\left(we^a k^{-1}w^{-1}\right)\right] = \mathrm{Tr}\left[\rho\left(e^{\mathrm{Ad}(w)a} wk^{-1}w^{-1}\right)\right]. \tag{5.14}$$

By (2.17), $\mathrm{Ad}(w)$ preserves $\mathfrak{t}$ and $\mathfrak{b}$ (see [Sh20, Proposition 3.4]). Since K preserves $B|_{\mathfrak{p}}$, and since $\dim\mathfrak{b}=1$, we see that $\mathrm{Ad}(w)a = a$ or $-a$. By (5.13), we get

$$\begin{aligned}\mathrm{Tr}\left[\rho\left(e^{\mathrm{Ad}(w)a} wk^{-1}w^{-1}\right)\right] &= \mathrm{Tr}\left[\rho_0\left(wk^{-1}w^{-1}\right)\right]\\ &\quad+\sum_{\beta\in\mathfrak{b}_+^*}\left(e^{|\beta||a|}+e^{-|\beta||a|}\right)\mathrm{Tr}\left[\rho_\beta\left(wk^{-1}w^{-1}\right)\right].\end{aligned} \tag{5.15}$$

By (5.13)-(5.15), since $\dim\mathfrak{b}=1$, for $\beta\in\mathfrak{b}^*$, we have

$$\mathrm{Tr}\left[\rho_\beta\left(wk^{-1}w^{-1}\right)\right] = \mathrm{Tr}\left[\rho_\beta\left(k^{-1}\right)\right], \tag{5.16}$$

i.e., the character of $\rho_{\beta|T}$ is invariant under the Weyl group $W(T:K)$. □

5.3. **The representation η_β.** In [Sh18], we have shown that η_j satisfies Assumption 3.5 and we consider the Selberg zeta function associated to η_j. A naive way to generalise the arguments in Section 4.6 is to consider the Selberg zeta function associated to $\eta_j\otimes\rho_\beta$. However, $\eta_j\otimes\rho_\beta$ satisfies Assumption 3.5 (1), while Assumption 3.5 (2) fails in general. We need consider all the j together to produce a virtual representation η_β.

Recall that $\rho: G\to \mathrm{GL}(E)$ is a representation of G, and that $S^{\mathfrak{u}^\perp(\mathfrak{b})}$ is the spinor of $\left(\mathfrak{u}^\perp(\mathfrak{b}), -B|_{\mathfrak{u}^\perp(\mathfrak{b})}\right)$. Let $D^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}$ be the Dirac operator defined in Section 2.3. Let $D_\pm^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}$ be the restriction of $D^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}$ to $S_\pm^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E$.

By (2.14), we have

$$-\left(D^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}\right)^2 = C^{\mathfrak{u},\rho} + \frac{1}{8}\mathrm{Tr}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right] - C^{\mathfrak{u}(\mathfrak{b}),S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}. \tag{5.17}$$

Since $\mathfrak{u}$ acts unitarily on $(E,\langle,\rangle_E)$, we have

$$\ker D^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E} = \ker\left(D^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}\right)^2. \tag{5.18}$$

If the $\mathfrak{u}(\mathfrak{b})$-action on E lifts to $U(\mathfrak{b})$, then $\ker D_\pm^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}$ are representations[8] of $U(\mathfrak{b})$. In general, $\ker D_\pm^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}$ are $(\mathfrak{b}_{\mathbf{C}}\oplus\mathfrak{m}_{\mathbf{C}}, K_M)$-modules. Note that by (5.9) and (5.12), $\mathfrak{b}$ acts semisimplely on $\ker D_\pm^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}$.

[8]They are called Dirac cohomology of E (see [HuPa06]).

Definition 5.5. Define the $(\mathfrak{m}_{\mathbf{C}},K_M)$-modules η_β^+ and η_β^-, so that

$$\ker D_+^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}=\bigoplus_{\beta\in\mathfrak{b}^*}\mathbf{C}_\beta\boxtimes\eta_\beta^+,\qquad \ker D_-^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}=\bigoplus_{\beta\in\mathfrak{b}^*}\mathbf{C}_\beta\boxtimes\eta_\beta^-. \tag{5.19}$$

Set

$$\eta_\beta=\eta_\beta^+-\eta_\beta^-. \tag{5.20}$$

Recall that $C^{\mathfrak{u},\rho}\in\mathbf{R}$ is a scalar.

Proposition 5.6. *The following statements hold.*

i) We have isomorphisms of $(\mathfrak{m}_{\mathbf{C}},K_M)$*-modules,*

$$\eta_\beta^\pm\simeq\eta_{-\beta}^\pm. \tag{5.21}$$

ii) The virtual $(\mathfrak{m}_{\mathbf{C}},K_M)$*-modules* η_β *satisfy Assumption 3.5, so that the Casimir of* $\mathfrak{u}_{\mathfrak{m}}$ *acts on* $\eta_\beta^\pm$ *by the same scalar*

$$C^{\mathfrak{u}_{\mathfrak{m}},\eta_\beta}=|\beta|^2+C^{\mathfrak{u},\rho}+\frac{1}{8}\mathrm{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right]\in\mathbf{R}. \tag{5.22}$$

Proof. By (5.17)-(5.19), on $\mathbf{C}_\beta\boxtimes\eta_\beta^\pm$, we have

$$C^{\mathfrak{u},\rho}+\frac{1}{8}\mathrm{Tr}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right]-C^{\mathfrak{u}(\mathfrak{b}),S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}=0. \tag{5.23}$$

On $\mathbf{C}_\beta\boxtimes\eta_\beta^\pm$, we have

$$C^{\mathfrak{u}(\mathfrak{b}),S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}=-|\beta|^2+C^{\mathfrak{u}_{\mathfrak{m}},\eta_\beta^\pm}. \tag{5.24}$$

By (5.23) and (5.24), we see that $C^{\mathfrak{u}_{\mathfrak{m}},\eta_\beta^\pm}$ coincide and are given by (5.22).

By (5.8), (5.9), and (5.12), we see that $S_\pm^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E$ admits a decomposition like (5.12). By (5.17) and (5.18), $\ker D_\pm^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}$ consists of the components on which (5.22) holds. Form these two considerations, i) follows.

It remains to show each $\eta_{\beta|K_M}$ lifts to $R(K)$. By (5.19), for $e^a k^{-1}\in H$, we have

$$\sum_{\beta\in\mathfrak{b}^*}e^{\langle a,\beta\rangle}\mathrm{Tr}_{\mathrm{s}}\left[\eta_\beta(k^{-1})\right]=\mathrm{Tr}_{\mathrm{s}}^{\ker\left(D^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}\right)}\left[e^a k^{-1}\right]. \tag{5.25}$$

We claim that for $e^a k^{-1}\in H$, we have

$$\mathrm{Tr}_{\mathrm{s}}^{\ker\left(D^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}\right)}\left[e^a k^{-1}\right]=\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}\left[e^a k^{-1}\right]. \tag{5.26}$$

Indeed, since $D^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}$ commutes with $e^a k^{-1}$, using the fact that the super trace varnishes on the super commutator, we see that

$$\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}\left[e^a k^{-1}\exp\left(-t\left(D^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}\right)^2\right)\right] \tag{5.27}$$

does not depend on $t\in\mathbf{R}$, from which we get (5.26).

From (5.25) and (5.26), for $e^a k^{-1}\in H$, we have

$$\sum_{\beta\in\mathfrak{b}^*}e^{\langle\beta,a\rangle}\mathrm{Tr}_{\mathrm{s}}\left[\eta_\beta(k^{-1})\right]=\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{u}^\perp(\mathfrak{b})}}\left[e^a k^{-1}\right]\mathrm{Tr}\left[\rho\left(e^a k^{-1}\right)\right]. \tag{5.28}$$

By Theorems 2.6, 5.4, (5.9), (5.12), the right-hand side of (5.28) is a sum of products of $e^{\langle\beta,a\rangle}$ with elements in $R(K)\simeq R(T)^{W(T:K)}$. Thus, η_β has a lift in $R(K)$. □

The following two propositions are analogues of [Sh18, Propositions 6.5, 6.10].

Proposition 5.7. *We have an isomorphism of virtual* $(\mathfrak{b}_{\mathbf{C}} \oplus \mathfrak{m}_{\mathbf{C}}, K_M)$*-modules,*

$$\left(S_+^{\mathfrak{u}^\perp(\mathfrak{b})} - S_-^{\mathfrak{u}^\perp(\mathfrak{b})}\right) \otimes \rho_{|\exp(\mathfrak{b})\times M} = \bigoplus_{\beta\in\mathfrak{b}^*} \mathbf{C}_\beta \boxtimes \eta_\beta = \mathbf{1}\boxtimes\eta_0 \oplus \bigoplus_{\beta\in\mathfrak{b}_+^*} \left(\mathbf{C}_\beta \oplus \mathbf{C}_{-\beta}\right)\boxtimes\eta_\beta. \tag{5.29}$$

For $e^a k^{-1} \in H$*, we have*

$$\begin{aligned} \mathrm{Tr}_{\mathrm{s}}\left[\eta_0\left(k^{-1}\right)\right] + \sum_{\beta\in\mathfrak{b}_+^*} \left(e^{|a||\beta|} + e^{-|a||\beta|}\right) \mathrm{Tr}_{\mathrm{s}}\left[\eta_\beta\left(k^{-1}\right)\right] \\ = \left|\det\left(1 - \mathrm{Ad}(e^a k^{-1})\right)|_{\mathfrak{z}^\perp(\mathfrak{b})}\right|^{1/2} \mathrm{Tr}\left[\rho\left(e^a k^{-1}\right)\right] \end{aligned} \tag{5.30}$$

Proof. This is a consequence of (5.10), (5.21), and (5.28). □

Proposition 5.8. *The following identity in* $R(K)$ *holds,*

$$\bigoplus_{\beta\in\mathfrak{b}^*} \widehat{\eta}_\beta = \bigoplus_{i=1}^{m} (-1)^{i-1} i \Lambda^i\left(\mathfrak{p}_{\mathbf{C}}^*\right) \otimes \rho_{|K}. \tag{5.31}$$

Proof. By (5.7) and (5.29), we have an identity in $R(K_M)$,

$$\bigoplus_{\beta\in\mathfrak{b}^*} \eta_{\beta|K_M} = \Lambda^\bullet\left(\mathfrak{n}_{\mathbf{C}}^*\right)_{|K_M} \otimes \rho_{|K_M}. \tag{5.32}$$

By (2.26), (3.21), (5.32) and Proposition 2.5 v), we get

$$\bigoplus_{\beta\in\mathfrak{b}^*} \widehat{\eta}_{\beta|K_M} = \bigoplus_{\beta\in\mathfrak{b}^*} \Lambda^\bullet\left(\mathfrak{p}_{\mathfrak{m},\mathbf{C}}^*\right) \widehat{\otimes} \eta_\beta = \bigoplus_{i=1}^{m} (-1)^{i-1} i \Lambda^i\left(\mathfrak{p}_{\mathbf{C}}^*\right)_{|K_M} \otimes \rho_{|K_M}. \tag{5.33}$$

Since the restriction $R(K) \to R(K_M)$ is injective, from (5.33), we get (5.31). □

5.4. **The Selberg zeta function** Z_{η_β}**.** By Proposition 5.6 ii), η_β satisfies Assumption 3.5. Let Z_{η_β} be the associated Selberg zeta function. We have an analogue of [Sh18, Theorem 7.7] and of Proposition 4.8.

Proposition 5.9. *The following identity of meromorphic function on* $\mathbf{C}$ *holds,*

$$R_\rho(\sigma) = \left(Z_{\eta_0}(\sigma)\right)^{-1} \prod_{\beta\in\mathfrak{b}_+^*} \left(Z_{\eta_\beta}\left(\sigma + |\beta|\right) Z_{\eta_\beta}\left(\sigma - |\beta|\right)\right)^{-1}. \tag{5.34}$$

Proof. This is a consequence of (3.15), (3.20), and (5.30). □

As in (4.30), by (3.24), (3.27), and (5.22),

$$Z_{\eta_\beta}\left(-\sqrt{\sigma^2 + |\beta|^2}\right) Z_{\eta_\beta}\left(\sqrt{\sigma^2 + |\beta|^2}\right) = \left[\det_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta}_\beta} - C^{\mathfrak{u},\rho} + \sigma^2\right)\right]^2. \tag{5.35}$$

is a meromorphic function on $\mathbf{C}$. The following proposition is an analogue of [Sh18, Theorem 7.8] and of Theorem 4.9.

Theorem 5.10. *The following identity of meromorphic functions on* $\mathbf{C}$ *holds,*

$$R_\rho(\sigma) = T\left(\sigma^2\right) \exp\left(-\mathrm{vol}(Z) P_{\eta_0}(\sigma)\right) \prod_{\beta\in\mathfrak{b}_+^*} \frac{Z_{\eta_\beta}\left(-\sqrt{\sigma^2 + |\beta|^2}\right) Z_{\eta_\beta}\left(\sqrt{\sigma^2 + |\beta|^2}\right)}{Z_{\eta_\beta}\left(\sigma + |\beta|\right) Z_{\eta_\beta}\left(\sigma - |\beta|\right)}. \tag{5.36}$$

Proof. By (1.8), (4.4), (5.21), and (5.31), the statement of (4.33) still holds in the current situation. The rest part of the proof is exactly the same as in the proof of Theorem 4.9. □

Remark 5.11. Define r_{η_β}, C_ρ, and r_ρ by the same formula as in (4.35) and (4.36). Proceeding as [Sh18, (7-76)-(7-78)], using Theorem 5.10 instead of [Sh18, Theorem 7.8], we get (4.8) when $\delta(G)=1$ and Z_G is compact.

6. A COHOMOLOGICAL FORMULA FOR r_{η_β}

The purpose of this section is to show (4.9) when $\delta(G)=1$ and Z_G is compact. Its proof relies on some deep results from the classification of unitary representations of real reductive groups.

This section is organised as follows. In Section 6.1, we recall the definition of the infinitesimal characters of a $\mathscr{U}(\mathfrak{g}_{\mathbf{C}})$-modules, the Harish-Chandra $(\mathfrak{g}_{\mathbf{C}},K)$-modules, and a relation between the infinitesimal character and the vanishing of $(\mathfrak{g},K)$-cohomology of a unitary Harish-Chandra $(\mathfrak{g}_{\mathbf{C}},K)$-module, which is due to Vogan-Zuckermann [VZu84], Vogan [V84], and Salamanca-Riba [SR99]. The latter is our essential algebraic input in the proof of (4.9).

In Section 6.2, we obtain a formula relating $H^\bullet(Z,F)$ and the $(\mathfrak{g},K)$-cohomology of certain Harish-Chandra $(\mathfrak{g}_{\mathbf{C}},K)$-modules.

Finally, in Section 6.3, we deduce a similar formula for r_{η_β} and we prove (4.9).

We use the notation in Sections 2 and 3. In Sections 6.1 and 6.2, we assume neither $\delta(G)=1$ nor Z_G is compact.

6.1. **Some results from representation theory.** We recall some basic facts on the representation theory of real reductive groups.

6.1.1. *Infinitesimal characters.* A morphism of algebras $\chi:\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})\to\mathbf{C}$ will be called a character of $\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$. Clearly, for $a\in\mathbf{C}$, we have

$$\chi(a)=a. \tag{6.1}$$

By (2.6), $\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$ is equipped with a complex conjugation. Moreover, the Cartan involution θ extends to complex automorphism on $\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$. Also, the anti-automorphism $z\to z^{\mathrm{tr}}$ of $\mathscr{U}(\mathfrak{g}_{\mathbf{C}})$ [Kn02, Proposition 3.7], induced by $a\in\mathfrak{g}\to -a\in\mathfrak{g}$, descends to a complex automorphisms of $\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$. For $z\in\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$, set

$$\overline{\chi}(z)=\overline{\chi(\overline{z})},\qquad \chi^\theta(z)=\chi(\theta z),\qquad \chi^{\mathrm{tr}}(z)=\chi\left(z^{\mathrm{tr}}\right). \tag{6.2}$$

Then, $\overline{\chi}$, χ^θ, and χ^{tr} are characters of $\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$.

Definition 6.1. A complex representation of $\mathfrak{g}_{\mathbf{C}}$ is said to have infinitesimal character χ, if $z\in\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$ acts as a scalar $\chi(z)\in\mathbf{C}$.

A complex representation of $\mathfrak{g}_{\mathbf{C}}$ is said to have generalised infinitesimal character χ, if for there is $i\gg 1$ such that for all $z\in\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$, $(z-\chi(z))^i$ acts like 0.

If W is a complex representation of $\mathfrak{g}$ with infinitesimal character χ_W, then $\overline{W}$, W^θ (defined in an obvious way as (2.33)), and W^* have infinitesimal characters $\chi_{\overline{W}}$, χ_{W^θ}, and χ_{W^*}, so that

$$\chi_{\overline{W}}=\overline{\chi}_W,\qquad \chi_{W^\theta}=\chi_W^\theta,\qquad \chi_{W^*}=\chi_W^{\mathrm{tr}}. \tag{6.3}$$

Therefore, if W is a unitary representation of $\mathfrak{g}$, we have

$$\overline{\chi}_W^{\mathrm{tr}}=\chi_W. \tag{6.4}$$

If W is a representation of $\mathfrak{g}$ which is equipped with an admissible Hermitian metric, then $\overline{W}^* \simeq W^\theta$, so that

$$\overline{\chi}_W^{\mathrm{tr}} = \chi_W^\theta. \tag{6.5}$$

Let us recall the definition of the Harsh-Chandra parameter for a character of $\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$. Let $\mathfrak{h}_{\mathbf{C}} \subset \mathfrak{g}_{\mathbf{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Let $S(\mathfrak{h}_{\mathbf{C}})$ be the symmetric algebra of $\mathfrak{h}_{\mathbf{C}}$. If $W(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$ denotes the algebraic Weyl group, let $S(\mathfrak{h}_{\mathbf{C}})^{W(\mathfrak{h}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})} \subset S(\mathfrak{h}_{\mathbf{C}})$ be the $W(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$-invariant subalgebra of $S(\mathfrak{h}_{\mathbf{C}})$. Let

$$\phi_{\mathrm{HC}} : \mathscr{Z}(\mathfrak{g}_{\mathbf{C}}) \simeq S(\mathfrak{h}_{\mathbf{C}})^{W(\mathfrak{h}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})} \tag{6.6}$$

be the Harish-Chandra isomorphism [Kn02, Section V.5]. For $\Lambda \in \mathfrak{h}_{\mathbf{C}}^*$, we can associate to it a character χ_Λ of $\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$ as follows: for $z \in \mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$,

$$\chi_\Lambda(z) = \langle \phi_{\mathrm{HC}}(z), \Lambda \rangle. \tag{6.7}$$

By [Kn02, Theorem 5.62], every character of $\mathscr{Z}(\mathfrak{g}_{\mathbf{C}})$ is of the form χ_Λ, for some $\Lambda \in \mathfrak{h}_{\mathbf{C}}^*$. Also, Λ is uniquely determined up to an action of $W(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$. Such an element $\Lambda \in \mathfrak{h}_{\mathbf{C}}^*$ is called the Harish-Chandra parameter of the character.

6.1.2. *Harish-Chandra* $(\mathfrak{g}_{\mathbf{C}}, K)$*-module and its* $(\mathfrak{g}, K)$*-cohomology.*

Definition 6.2. A complex $\mathscr{U}(\mathfrak{g}_{\mathbf{C}})$-module V, equipped with an action of K, is called a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module, if

(1) the space V is a finitely generated $\mathscr{U}(\mathfrak{g}_{\mathbf{C}})$-module;
(2) every $v \in V$ is K-finite, i.e., $\{k \cdot v\}_{k \in K}$ spans a finite dimensional vector space;
(3) the actions of $\mathfrak{g}_{\mathbf{C}}$ and K are compatible;
(4) each irreducible K-module occurs only for a finite number of times in V.

Let $\widehat{G}_u$ be the unitary dual of G, that is the set of equivalence classes of complex irreducible unitary representations π of G on Hilbert spaces V_π. For $(\pi, V_\pi) \in \widehat{G}_u$, let $V_{\pi,K}$ be the space of K-finite vectors. By [Kn86, Theorem 8.1, Proposition 8.5], $\mathfrak{g}_{\mathbf{C}}$ acts on $V_{\pi,K}$ such that $V_{\pi,K}$ is a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module.

If V is a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module, let $H^\bullet(\mathfrak{g}, K; V)$ be the $(\mathfrak{g}, K)$-cohomology of V [BoW00, Section I.1.2].

Theorem 6.3. *Let V be a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module with generalised infinitesimal character χ. Let W be a finite dimensional $(\mathfrak{g}_{\mathbf{C}}, K)$-module with infinitesimal character χ_W. If $\chi \neq \chi_W^{\mathrm{tr}}$, then*

$$H^\bullet(\mathfrak{g}, K; V \otimes W) = 0. \tag{6.8}$$

Proof. If V has an infinitesimal character, the theorem is due to [BoW00, Theorem I.5.3(ii)]. If V has a generalised infinitesimal character, a proof can be found in [Sh18, Theorem 8.8]. □

The converse of (6.8) is not true in general. But it still holds if V is unitary.

Theorem 6.4. *Let W be a finite dimensional $(\mathfrak{g}_{\mathbf{C}}, K)$-module with infinitesimal character χ_W. If $(\pi, V_\pi) \in \widehat{G}_u$, then*

$$\chi_\pi \neq \chi_W^{\mathrm{tr}} \iff H^\bullet(\mathfrak{g}, K; V_{\pi,K} \otimes W) = 0. \tag{6.9}$$

Proof. The direction $\Longrightarrow$ of (6.9) is (6.8). The direction $\Longleftarrow$ of (6.9) is a consequence of Vogan-Zuckerman [VZu84], Vogan [V84], and Salamanca-Riba [SR99]. Indeed, if $\chi_\pi = \chi_W^{\mathrm{tr}}$, then the Harish-Chandra parameter of χ_π is stronger regular in the sense of [SR99, p. 525]. Such representation π is classified in [SR99], which has non vanishing $(\mathfrak{g},K)$-cohomology by [VZu84, V84]. □

6.1.3. *Root system and* $\mathfrak{n}$*-homology.* We use the notation in Section 2.5. Let $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{t}$ be the fundamental Cartan subalgebra of $\mathfrak{g}$. Let $R \subset \mathfrak{b}^* \oplus \sqrt{-1}\mathfrak{t}^*$ be a root system of $(\mathfrak{h},\mathfrak{g})$. By [Kn02, Proposition 11.16] (see also [BSh19, Proposition 3.7]), there are no real roots in R. Let R^{im} and R^{c} be the systems of imaginary roots and complex roots, so that

$$R = R^{\mathrm{im}} \sqcup R^{\mathrm{c}}. \tag{6.10}$$

Then, R^{im} is a root system of $(\mathfrak{h},\mathfrak{z}(\mathfrak{b}))$. Also, $R^{\mathrm{im}}|_{\mathfrak{t}}$ is a root system of $(\mathfrak{t},\mathfrak{m})$.

We fix a positive root system $R_+ \subset R$. Set

$$R_+^{\mathrm{im}} = R^{\mathrm{im}} \cap R_+, \qquad R_+^{\mathrm{c}} = R^{\mathrm{c}} \cap R_+. \tag{6.11}$$

As explained in [BSh19, Section 3.5], we can choose R_+ such that R_+^{c} is stable under complex conjugation.

Set

$$\varrho^{\mathfrak{u}} = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in \mathfrak{b}^* \oplus \sqrt{-1}\mathfrak{t}^*. \tag{6.12}$$

By Kostant's strange formula [Ko76] or [B11, Proposition 7.5.1], we have

$$\left|\varrho^{\mathfrak{u}}\right|^2 = -\frac{1}{24} \mathrm{Tr}^{\mathfrak{u}}\left[C^{\mathfrak{u},\mathfrak{u}}\right]. \tag{6.13}$$

Define $\varrho^{\mathfrak{u}(\mathfrak{b})} \in \mathfrak{b}^* \oplus \sqrt{-1}\mathfrak{t}^*$ and $\varrho^{\mathfrak{u}_{\mathfrak{m}}} \in \sqrt{-1}\mathfrak{t}^*$ in the same way, which are associated to R_+^{im} and $R_{+|\mathfrak{t}}^{\mathrm{im}}$. Then,

$$\varrho^{\mathfrak{u}(\mathfrak{b})} = \left(0, \varrho^{\mathfrak{u}_{\mathfrak{m}}}\right), \qquad \varrho_{|\mathfrak{t}}^{\mathfrak{u}} = \varrho^{\mathfrak{u}_{\mathfrak{m}}}. \tag{6.14}$$

If V is a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}},K)$-module, denote by $H_\bullet(\mathfrak{n},V)$ its $\mathfrak{n}$-homology. By [HeSc83, Proposition 2.24], $H_\bullet(\mathfrak{n},V)$ is a Harish-Chandra $(\mathfrak{m}_{\mathbf{C}} \oplus \mathfrak{b}_{\mathbf{C}},K_M)$-module. If V possesses an infinitesimal character with Harish-Chandra parameter $\Lambda \in \mathfrak{h}_{\mathbf{C}}^*$, by [HeSc83, Corollary 3.32], $H_\bullet(\mathfrak{n},V)$ can be decomposed into a finite direct sum of Harish-Chandra $(\mathfrak{m}_{\mathbf{C}} \oplus \mathfrak{b}_{\mathbf{C}},K_M)$-modules whose generalised infinitesimal characters are given by

$$\chi_{w\Lambda+\varrho^{\mathfrak{u}}-\varrho^{\mathfrak{u}(\mathfrak{b})}}, \tag{6.15}$$

for some $w \in W(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$.

6.2. **The cohomology of $H^\bullet(Z,F)$.** We use the notation in Section 4.1. Recall that $\Gamma \subset G$ is a discrete cocompact torsion free subgroup of G and that $\rho : G \to \mathrm{GL}(E)$ is a representation of G with an admissible metric. Let (F,g^F) be the associated Hermitian flat vector bundle.

By [GelGrPS69, p. 23, Theorem], we can decompose $L^2(\Gamma\backslash G)$ into a direct Hilbert sum of countable irreducible unitary representations of G,

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi\in\widehat{G}_u}^{\mathrm{Hil}} n(\pi)V_\pi, \tag{6.16}$$

with $n(\pi) < \infty$.

For any unitary representation (τ, E_τ) of K, since $C^{\mathfrak{g},Z,\tau}$ is elliptic and self-adjoint and since Z is compact, we have a finite sum

$$\ker\left(C^{\mathfrak{g},Z,\tau} - \lambda\right) = \bigoplus_{\pi\in\widehat{G}_u,\chi_\pi(C^{\mathfrak{g}})=\lambda} n(\pi)\left(V_{\pi,K}\otimes E_\tau\right)^K. \tag{6.17}$$

Let $\mathscr{X}(\rho^*)$ be set of the infinitesimal characters of all irreducible subrepresentations of ρ^*. Note that by Remark 2.1, the sets of all irreducible subrepresentations of ρ^* of the group G and of the Lie algebra $\mathfrak{g}$ coincide.

Theorem 6.5. *We have*

$$H^\bullet(Z,F) = \bigoplus_{\pi\in\widehat{G}_u,\chi_\pi\in\mathscr{X}(\rho^*)} n(\pi)H^\bullet\left(\mathfrak{g},K;V_{\pi,K}\otimes E\right). \tag{6.18}$$

If $H^\bullet(Z,F)=0$, then for any $\pi\in\widehat{G}_u$ such that $\chi_\pi\in\mathscr{X}(\rho^*)$, we have

$$n(\pi) = 0. \tag{6.19}$$

If ρ is irreducible such that $\rho^\theta \neq \rho$, then

$$H^\bullet(Z,F) = 0. \tag{6.20}$$

Proof. Since the G-representation with an admissible metric is completely reducible, we can assume that ρ is irreducible with the infinitesimal character χ_ρ.

By (6.3), we have $\chi_\rho^{\mathrm{tr}}(C^{\mathfrak{g}}) = \chi_\rho(C^{\mathfrak{g}}) = C^{\mathfrak{g},\rho}$. By (1.4), (4.4) and (6.17), we have

$$H^\bullet(Z,F) = \bigoplus_{\pi\in\widehat{G}_u,\chi_\pi(C^{\mathfrak{g}})=\chi_\rho^{\mathrm{tr}}(C^{\mathfrak{g}})} n(\pi)\left(V_{\pi,K}\otimes\Lambda^\bullet(\mathfrak{p}_{\mathbf{C}}^*)\otimes E\right)^K. \tag{6.21}$$

When $\chi_\pi(C^{\mathfrak{g}}) = \chi_\rho^{\mathrm{tr}}(C^{\mathfrak{g}})$, by Hodge theory for Lie algebras [BoW00, Proposition II.3.1], we have

$$\left(V_{\pi,K}\otimes\Lambda^\bullet(\mathfrak{p}_{\mathbf{C}}^*)\otimes E\right)^K = H^\bullet(\mathfrak{g},K;V_{\pi,K}\otimes E). \tag{6.22}$$

By (6.8), (6.21), and (6.22), we get (6.18). By (6.9) and (6.18), we get (6.19).

To show (6.20), it is enough to show that if $\rho^\theta \neq \rho$, then for all $\pi\in\widehat{G}_u$, we have

$$\chi_\pi \neq \chi_\rho^{\mathrm{tr}}. \tag{6.23}$$

Otherwise there is $\pi\in\widehat{G}_\pi$ such that $\chi_\pi = \chi_\rho^{\mathrm{tr}}$. Using $\overline{\pi}^* \simeq \pi$ and $\overline{\rho}^* \simeq \rho^\theta$, by (6.4) and (6.5), we have

$$\chi_\rho = \chi_\pi^{\mathrm{tr}} = \overline{\chi}_\pi = \overline{\chi}_\rho^{\mathrm{tr}} = \chi_\rho^\theta. \tag{6.24}$$

Since ρ and ρ^θ are irreducible and have finite dimensions, Equation (6.24) implies $\rho \simeq \rho^\theta$, which is a contradiction with our assumption. □

Remark 6.6. Equations (6.18) and (6.20) are [BoW00, Theorems VII.6.1 and VII.6.7]. Equation (6.18) is originally due to Matsushima [Mat67] where ρ is supposed to be trivial.

6.3. **A formula for r_{η_β}.** Assume now that $\delta(G)=1$ and Z_G is compact, and that $\rho: G\to \mathrm{GL}(E)$ is a G-representation with an admissible metric such that $\rho\simeq\rho^\theta$ and that $C^{\mathfrak{u},\rho}\in\mathbf{R}$ is a scalar.

By [Sh18, Corollary 8.15], we have

$$r_{\eta_\beta}=\frac{1}{\chi(K/K_M)}\sum_{\substack{\pi\in\widehat{G}_u,\chi_\pi(C^{\mathfrak{g}})=C^{\mathfrak{u},\rho}\\ 0\leqslant i\leqslant\dim\mathfrak{p}_{\mathfrak{m}}\\ 0\leqslant j\leqslant 2\ell}}(-1)^{i+j}n(\pi)\Big(\dim H^i\big(\mathfrak{m},K_M;H_j(\mathfrak{n},V_{\pi,K})\otimes E^+_{\eta_\beta}\big)\\ -\dim H^i\big(\mathfrak{m},K_M;H_j(\mathfrak{n},V_{\pi,K})\otimes E^-_{\eta_\beta}\big)\Big). \tag{6.25}$$

Proposition 6.7. *Let $(\pi,V_\pi)\in\widehat{G}_u$. Assume that $\chi_\pi(C^{\mathfrak{g}})=C^{\mathfrak{u},\rho}$ and*

$$\bigoplus_{\substack{0\leqslant i\leqslant\dim\mathfrak{p}_{\mathfrak{m}}\\ 0\leqslant j\leqslant 2\ell\\ s\in\{\pm\}}} H^i\big(\mathfrak{m},K_M;H_j(\mathfrak{n},V_{\pi,K})\otimes E^s_{\eta_\beta}\big)\neq 0. \tag{6.26}$$

Then,

$$\chi_\pi\in\mathscr{X}(\rho^*). \tag{6.27}$$

Proof. We use the notation in Section 6.1.3. Let $\Lambda(\pi^*)\in\mathfrak{h}^*_{\mathbf{C}}$ be a Harish-Chandra parameter of $V_{\pi^*,K}$. We need to show that there is $w\in W(\mathfrak{h}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})$ and a Harish-Chandra parameter $\Lambda(\rho)\in\mathfrak{b}^*\oplus\sqrt{-1}\mathfrak{t}^*$ of an irreducible $\mathfrak{g}$-submodule of ρ, such that

$$w\Lambda(\pi^*)=\Lambda(\rho). \tag{6.28}$$

Recall that B^* is the bilinear form on $\mathfrak{g}^*$ induced by B. It extends to $\mathfrak{g}^*_{\mathbf{C}}$ in an obvious way. Since $\chi_\pi(C^{\mathfrak{g}})=C^{\mathfrak{u},\rho}$, using Harish-Chandra isomorphism (see [Kn02, Example 5.64]), we have

$$B^*\big(\Lambda(\pi^*),\Lambda(\pi^*)\big)-B^*\big(\varrho^{\mathfrak{u}},\varrho^{\mathfrak{u}}\big)=C^{\mathfrak{u},\rho}. \tag{6.29}$$

By (6.8), (6.14), (6.15), and (6.26), there exist $w\in W(\mathfrak{h}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})$, $w'\in W(\mathfrak{t}_{\mathbf{C}}:\mathfrak{m}_{\mathbf{C}})\subset W(\mathfrak{h}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})$ and the highest weight $\mu_\beta\in\sqrt{-1}\mathfrak{t}^*$ of an irreducible $(\mathfrak{m}_{\mathbf{C}},K_M)$-submodule of $\eta^+_\beta\oplus\eta^-_\beta$ such that

$$w\Lambda(\pi^*)_{|\mathfrak{t}_{\mathbf{C}}}=w'\big(\mu_\beta+\rho^{\mathfrak{u}_{\mathfrak{m}}}\big). \tag{6.30}$$

By Proposition 5.1 and (6.12), we have

$$\varrho^{\mathfrak{u}}=\varrho^{\mathfrak{u}(\mathfrak{b})}+(\ell\alpha_0,0). \tag{6.31}$$

By (5.11), (5.22), (6.14), (6.29)-(6.31), there exists $w''\in W(\mathfrak{h}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})$ such that

$$w''\Lambda(\pi^*)=\big(\pm\beta,\mu_\beta+\rho^{\mathfrak{u}_{\mathfrak{m}}}\big)=\big(\pm\beta,\mu_\beta\big)+\rho^{\mathfrak{u}(\mathfrak{b})}. \tag{6.32}$$

In particular, $\Lambda(\pi^*)\in\mathfrak{b}^*\oplus\sqrt{-1}\mathfrak{t}^*$.

By (5.19), $\big(\beta,\mu_\beta\big)\in\mathfrak{b}^*\oplus\sqrt{-1}\mathfrak{t}^*$ is a highest weight of an irreducible $(\mathfrak{m}_{\mathbf{C}}\oplus\mathfrak{b}_{\mathbf{C}},K_M)$-submodule of $\ker D^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}$. By [HuPa06, Theorem 4.2.2], there exists $w_1\in W(\mathfrak{h}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})$ and a Harish-Chandra parameter $\Lambda(\rho)\in\mathfrak{h}^*_{\mathbf{C}}$ of an irreducible $\mathfrak{g}$-submodule of ρ, such that

$$\big(\beta,\mu_\beta\big)=w_1\Lambda(\rho)-\rho^{\mathfrak{u}(\mathfrak{b})}. \tag{6.33}$$

By (5.19) and (5.21), $(-\beta,\mu_\beta)\in\mathfrak{b}^*\oplus\sqrt{-1}\mathfrak{t}^*$ is also the highest weight of an irreducible $(\mathfrak{m}_{\mathbf{C}}\oplus\mathfrak{b}_{\mathbf{C}},K_M)$-submodule of $\ker D^{S^{\mathfrak{u}^\perp(\mathfrak{b})}\otimes E}$. As before, there exists $w_2\in W(\mathfrak{h}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})$ and a Harish-Chandra parameter $\Lambda(\rho)\in\mathfrak{h}_{\mathbf{C}}^*$ of an irreducible $\mathfrak{g}$-submodule of ρ, such that

$$\left(-\beta,\mu_\beta\right)=w_2\Lambda(\rho)-\rho^{\mathfrak{u}(\mathfrak{b})}. \tag{6.34}$$

By (6.32)-(6.34), we get (6.28). □

Corollary 6.8. *For $\beta\in\mathfrak{b}^*$, we have*

$$r_{\eta_\beta}=\frac{1}{\chi(K/K_M)}\sum_{\substack{\pi\in\widehat{G}_u,\chi_\pi\in\mathscr{X}(\rho^*)\\ 0\leqslant i\leqslant\dim\mathfrak{p}_{\mathfrak{m}}\\ 0\leqslant j\leqslant 2\ell}}(-1)^{i+j}n(\pi)\Big(\dim H^i\left(\mathfrak{m},K_M;H_j(\mathfrak{n},V_{\pi,K})\otimes E^+_{\eta_\beta}\right) - \dim H^i\left(\mathfrak{m},K_M;H_j(\mathfrak{n},V_{\pi,K})\otimes E^-_{\eta_\beta}\right)\Big). \tag{6.35}$$

In particular, if $H^\bullet(Z,F)=0$, then for all $\beta\in\mathfrak{b}^*$, we have

$$r_{\eta_\beta}=0. \tag{6.36}$$

Proof. By (6.25) and Proposition 6.7, we get (6.35). From (6.19) and (6.35), we get (6.36). □

Remark 6.9. By (4.36), (6.36), and by Remark 5.11, we get (4.9) in the case where $\delta(G)=1$ and Z_G is compact. We finish the proof of Theorem 4.4 in full generality.

7. An extension to orbifolds

In this section, we no longer assume $\Gamma\subset G$ is torsion free. Then $Z=\Gamma\backslash G/K$ is a closed Riemannian orbifold with Riemannian metric g^{TZ}. Let us indicate the essential steps in generalising the previous results to orbifolds.

If $\gamma\in\Gamma$, $\Gamma(\gamma)$ is not always torsion free. The cardinality

$$\left|\ker\left(\Gamma(\gamma)\to\mathrm{Diffeo}(Z(\gamma)/K(\gamma))\right)\right| \tag{7.1}$$

depends only on the conjugacy class $[\gamma]$ and will be denoted by $n_{[\gamma]}$. We define $B_{[\gamma]}$ by the same formula as in (3.10). By [ShY17, Proposition 5.3], we have

$$\frac{\mathrm{vol}(\Gamma(\gamma)\backslash Z(\gamma))}{\mathrm{vol}(K(\gamma))}=\frac{\mathrm{vol}\left(B_{[\gamma]}\right)}{n_{[\gamma]}}. \tag{7.2}$$

By [ShY17, Remark 5.6, (5.59)], as in (3.11), the closed geodesics (see [GuHa06] or [ShY17, Remark 2.26]) on the orbifold Z with positive length are given by

$$\coprod_{[\gamma]\in[\Gamma_+]}B_{[\gamma]}. \tag{7.3}$$

For $[\gamma]\in[\Gamma_+]$, all the elements of $B_{[\gamma]}$ have the same length $\ell_{[\gamma]}>0$.

For $[\gamma]\in[\Gamma_+]$, the group $\mathbb{S}^1$ acts locally freely on the orbifold $B_{[\gamma]}$ by rotation, so that $B_{[\gamma]}/\mathbb{S}^1$ is still a closed orbifold. Set

$$m_{[\gamma]}=n_{[\gamma]}\left|\ker\left(\mathbb{S}^1\to\mathrm{Diffeo}(\mathrm{B}_{[\gamma]})\right)\right|\in\mathbf{N}^*. \tag{7.4}$$

If $\rho : \Gamma \to \mathrm{GL}_r(\mathbf{C})$ is a representation of Γ, for $\mathrm{Re}(\sigma) \gg 1$ large enough, we define Ruelle's dynamical zeta function $R_\rho(\sigma)$ by the same formula (3.15) with $m_{[\gamma]}$ defined by (7.4). As before, when $\delta(G) \geqslant 2$,

$$R_\rho(\sigma) \equiv 1. \tag{7.5}$$

By [Sh20, Theorem 7.3], if $\dim Z$ is odd, $R_\rho(\sigma)$ has a meromorphic extension to $\sigma \in \mathbf{C}$.

Let $\rho : G \to \mathrm{GL}(E)$ be a finite dimensional complex representation of G with an admissible metric. Let F be the orbifold flat vector bundle on Z associated to $\rho_{|\Gamma}$. As in Section 4.1, F is equipped canonically with a Hermitian metric g^F. The analytic torsion of F associated to (g^{TZ}, g^F) is defined in [DaiY17], [ShY17, Section 4.2] (see also [Ma05]).

Theorem 7.1. *The statements of Theorems 4.3, 4.4, and 4.6 still hold for orbifolds. In particular, we get Theorem 0.5.*

Proof. Using the orbifold trace formula [ShY17, Theorem 5.4] and [Ma19, Theorem 5.4], we get the orbifold version of Theorem 4.3.

The proof of Theorem 4.4 in the case of orbifold is similar as before and we need only consider the case $\delta(G) = 1$. We can define the Selberg zeta function by the same formula (3.20) with $m_{[\gamma]}$ defined by (7.4). By [Sh20, Section 7.2], the statement of Theorem 3.9 still holds for orbifolds. By exactly the same method, the statements of Proposition 4.8, Theorem 4.9, Proposition 5.9, and Theorem 5.10 hold for orbifold. Using the orbifold Hodge theory (c.f. [ShY17, Theorem 4.1]), we can deduce that the statements of Theorem 6.5 and Corollary 6.8 hold as well. In this way, we get Theorem 4.4 for orbifolds.

As in the proof of Theorem 4.6 given in Section 4.3, the orbifold version of Theorem 4.6 is a consequence of the orbifold version of Theorem 4.4. The proof of our Theorem is completed. □

References

[BéDHP19] L. Bénard, J. Dubois, M. Heusener, and J. Porti, *Asymptotics of twisted Alexander polynomials and hyperbolic volume*, arXiv:1912.12946 (2019).

[BeGeVe04] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004, Corrected reprint of the 1992 original. MR 2273508 (2007m:58033)

[B11] J.-M. Bismut, *Hypoelliptic Laplacian and orbital integrals*, Annals of Mathematics Studies, vol. 177, Princeton University Press, Princeton, NJ, 2011. MR 2828080

[BMaZ11] J.-M. Bismut, X. Ma, and W. Zhang, *Opérateurs de Toeplitz et torsion analytique asymptotique*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 17-18, 977–981. MR 2838248 (2012k:58055)

[BMaZ17] J.-M. Bismut, X. Ma, and W. Zhang, *Asymptotic torsion and Toeplitz operators*, J. Inst. Math. Jussieu **16** (2017), no. 2, 223–349. MR 3615411

[BSh19] J.-M. Bismut and S. Shen, *Geometric orbital integrals and the center of the enveloping algebra*, arXiv: 1910.11731 (2019).

[BZ92] J.-M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque (1992), no. 205, 235, With an appendix by François Laudenbach. MR 1185803 (93j:58138)

[BoW00] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000. MR 1721403 (2000j:22015)

[BWSh20] Y. Borns-Weil and S. Shen, *Dynamical zeta functions in the nonorientable case*, arXiv:2007.08043 (2020).

[BrDi85] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1985. MR 781344 (86i:22023)

[Brö98] U. Bröcker, *Die Ruellesche Zetafunktion für G-induzierte Anosov-Flüsse*, Ph.D. thesis, Humboldt-Universität Berlin, Berlin (1998).

[CMi10] S.E. Cappell and E.Y. Miller, *Complex-valued analytic torsion for flat bundles and for holomorphic bundles with* $(1,1)$ *connections*, Comm. Pure Appl. Math. **63** (2010), no. 2, 133–202. MR 2588459

[Ch79] J. Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. (2) **109** (1979), no. 2, 259–322. MR 528965 (80j:58065a)

[DaiY17] X. Dai and J. Yu, *Comparison between two analytic torsions on orbifolds*, Math. Z. **285** (2017), no. 3-4, 1269–1282. MR 3623749

[DaGRSh20] N. V. Dang, C. Guillarmou, G. Rivière, and S. Shen, *The Fried conjecture in small dimensions*, Invent. Math. **220** (2020), no. 2, 525–579. MR 4081137

[dR50] G. de Rham, *Complexes à automorphismes et homéomorphie différentiable*, Ann. Inst. Fourier Grenoble **2** (1950), 51–67 (1951). MR 0043468 (13,268c)

[DuKVa79] J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan, *Spectra of compact locally symmetric manifolds of negative curvature*, Invent. Math. **52** (1979), no. 1, 27–93. MR 532745 (82a:58050a)

[DyZ16] S. Dyatlov and M. Zworski, *Dynamical zeta functions for Anosov flows via microlocal analysis*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 3, 543–577. MR 3503826

[DyZ17] S. Dyatlov and M. Zworski, *Ruelle zeta function at zero for surfaces*, Invent. Math. **210** (2017), no. 1, 211–229. MR 3698342

[Fr35] W. Franz, *Über die Torsion einer Überdeckung.*, J. Reine Angew. Math. **173** (1935), 245–254 (German).

[F86a] D. Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. Math. **84** (1986), no. 3, 523–540. MR 837526 (87g:58118)

[F86b] D. Fried, *Fuchsian groups and Reidemeister torsion*, The Selberg trace formula and related topics (Brunswick, Maine, 1984), Contemp. Math., vol. 53, Amer. Math. Soc., Providence, RI, 1986, pp. 141–163. MR 853556 (88e:58098)

[F87] D. Fried, *Lefschetz formulas for flows*, The Lefschetz centennial conference, Part III (Mexico City, 1984), Contemp. Math., vol. 58, Amer. Math. Soc., Providence, RI, 1987, pp. 19–69. MR 893856 (88k:58138)

[GelGrPS69] I. M. Gel′fand, M. I. Graev, and I. I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*, Translated from the Russian by K. A. Hirsch, W. B. Saunders Co., Philadelphia, Pa.-London-Toronto, Ont., 1969. MR 0233772 (38 #2093)

[GiLPo13] P. Giulietti, C. Liverani, and M. Pollicott, *Anosov flows and dynamical zeta functions*, Ann. of Math. (2) **178** (2013), no. 2, 687–773. MR 3071508

[GuHa06] K. Guruprasad and A. Haefliger, *Closed geodesics on orbifolds*, Topology **45** (2006), no. 3, 611–641. MR 2218759

[HeSc83] H. Hecht and W. Schmid, *Characters, asymptotics and n-homology of Harish-Chandra modules*, Acta Math. **151** (1983), no. 1-2, 49–151. MR 716371 (84k:22026)

[Hi74] N. Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55. MR 0358873 (50 #11332)

[HuPa06] J.-S. Huang and P. Pandžić, *Dirac operators in representation theory*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 2006. MR 2244116

[Kn86] A. W. Knapp, *Representation theory of semisimple groups*, Princeton Mathematical Series, vol. 36, Princeton University Press, Princeton, NJ, 1986, An overview based on examples. MR 855239 (87j:22022)

[Kn02] A. W. Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002. MR 1920389 (2003c:22001)

[KnV95] A. W. Knapp and D. A. Vogan, Jr., *Cohomological induction and unitary representations*, Princeton Mathematical Series, vol. 45, Princeton University Press, Princeton, NJ, 1995. MR 1330919 (96c:22023)

[Ko76] B. Kostant, *On Macdonald's η-function formula, the Laplacian and generalized exponents*, Advances in Math. **20** (1976), no. 2, 179–212. MR 0485661 (58 #5484)

[Ko97] B. Kostant, *Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the ρ-decomposition $C(\mathfrak{g}) = \operatorname{End} V_\rho \otimes C(P)$, and the $\mathfrak{g}$-module structure of $\bigwedge \mathfrak{g}$*, Adv. Math. **125** (1997), no. 2, 275–350. MR 1434113

[Ma05] X. Ma, *Orbifolds and analytic torsions*, Trans. Amer. Math. Soc. **357** (2005), no. 6, 2205–2233 (electronic). MR 2140438

[Ma19] X. Ma, *Geometric hypoelliptic Laplacian and orbital integrals [after Bismut, Lebeau and Shen]*, no. 407, 2019, Séminaire Bourbaki. Vol. 2016/2017. Exposés 1120–1135, pp. Exp. No. 1130, 333–389. MR 3939281

[M91] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991. MR 1090825

[Mat67] Y. Matsushima, *A formula for the Betti numbers of compact locally symmetric Riemannian manifolds*, J. Differential Geometry **1** (1967), 99–109. MR 0222908 (36 #5958)

[MatMu63] Y. Matsushima and S. Murakami, *On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds*, Ann. of Math. (2) **78** (1963), 365–416. MR 0153028

[Mi68] J. Milnor, *Infinite cyclic coverings*, Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967), Prindle, Weber & Schmidt, Boston, Mass., 1968, pp. 115–133. MR 0242163 (39 #3497)

[MoSt91] H. Moscovici and R. J. Stanton, *R-torsion and zeta functions for locally symmetric manifolds*, Invent. Math. **105** (1991), no. 1, 185–216. MR 1109626 (92i:58199)

[Mü78] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. in Math. **28** (1978), no. 3, 233–305. MR 498252 (80j:58065b)

[Mü93] W. Müller, *Analytic torsion and R-torsion for unimodular representations*, J. Amer. Math. Soc. **6** (1993), no. 3, 721–753. MR 1189689 (93m:58119)

[Mü11] W. Müller, *A Selberg trace formula for non-unitary twists*, Int. Math. Res. Not. IMRN (2011), no. 9, 2068–2109. MR 2806558

[Mü12] W. Müller, *The asymptotics of the Ray-Singer analytic torsion of hyperbolic 3-manifolds*, Metric and differential geometry, Progr. Math., vol. 297, Birkhäuser/Springer, Basel, 2012, pp. 317–352. MR 3220447

[Mü20] W. Müller, *On Fried's conjecture for compact hyperbolic manifolds*, arXiv:2005.01450 (2020).

[Q85] D. Quillen, *Superconnections and the Chern character*, Topology **24** (1985), no. 1, 89–95. MR 790678 (86m:58010)

[RaSi71] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971), 145–210. MR 0295381 (45 #4447)

[Re35] K. Reidemeister, *Homotopieringe und Linsenräume*, Abh. Math. Sem. Univ. Hamburg **11** (1935), no. 1, 102–109. MR 3069647

[SR99] S. A. Salamanca-Riba, *On the unitary dual of real reductive Lie groups and the $A_{\mathfrak{q}}(\lambda)$ modules: the strongly regular case*, Duke Math. J. **96** (1999), no. 3, 521–546. MR 1671213 (2000a:22023)

[Sa57] I. Satake, *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan **9** (1957), 464–492. MR 0095520 (20 #2022)

[Se67] R. T. Seeley, *Complex powers of an elliptic operator*, Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966), Amer. Math. Soc., Providence, R.I., 1967, pp. 288–307. MR 0237943 (38 #6220)

[S60] A. Selberg, *On discontinuous groups in higher-dimensional symmetric spaces*, Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960), Tata Institute of Fundamental Research, Bombay, 1960, pp. 147–164. MR 0130324

[Sh18] S. Shen, *Analytic torsion, dynamical zeta functions, and the Fried conjecture*, Anal. PDE **11** (2018), no. 1, 1–74. MR 3707290

[Sh19] S. Shen, *Analytic torsion and dynamical flow: a survey on the Fried conjecture*.

[Sh20] S. Shen, *Complex valued analytic torsion and dynamical zeta function on locally symmetric spaces*, arXiv:2009.03427 (2020).

[ShY17] S. Shen and J. Yu, *Flat vector bundles and analytic torsion on orbifolds*, to appear in Comm. Anal. Geom., arXiv: 1704.08369 (2017).

[ShY18] S. Shen and J. Yu, *Morse-Smale flow, Milnor metric, and dynamical zeta function*, arXiv: 1806.00662 (2018).

[Sp18] P. Spilioti, *Selberg and Ruelle zeta functions for non-unitary twists*, Ann. Global Anal. Geom. **53** (2018), no. 2, 151–203. MR 3766581

[Sp20a] P. Spilioti, *Functional equations of Selberg and Ruelle zeta functions for non-unitary twists*, Ann. Global Anal. Geom. **58** (2020), no. 1, 35–77. MR 4117920

[Sp20b] P. Spilioti, *Twisted ruelle zeta function and complex-valued analytic torsion*, arXiv:2004.13474 (2020).

[V84] D. A. Vogan, Jr., *Unitarizability of certain series of representations*, Ann. of Math. (2) **120** (1984), no. 1, 141–187. MR 750719 (86h:22028)

[VZu84] D. A. Vogan, Jr. and G. J. Zuckerman, *Unitary representations with nonzero cohomology*, Compositio Math. **53** (1984), no. 1, 51–90. MR 762307 (86k:22040)

[Vo87] A. Voros, *Spectral functions, special functions and the Selberg zeta function*, Comm. Math. Phys. **110** (1987), no. 3, 439–465. MR 891947

[Wo08] A. Wotzke, *Die Ruellsche Zetafunktion und die analytische Torsion hyperbolischer Mannigfaltigkeiten*, Ph.D. thesis, Bonn, Bonner Mathematische Schriften (2008), no. Nr. 389.

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