BERRY-ESSEEN BOUND AND PRECISE MODERATE DEVIATIONS FOR BRANCHING RANDOM WALKS WITH PRODUCTS OF RANDOM MATRICES

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ABSTRACT. We consider a branching random walk where particles give birth to children as a Galton-Watson process, which move in $\mathbb{R}^d$ according to products of independent and identically distributed random matrices. We establish a Berry-Esseen bound and a Cramér type moderate deviation expansion for the counting measure which counts the number of particles in generation n situated in a region, as $n \to \infty$. In the proof, we construct a new martingale, and establish its uniform convergence as well as that of the fundamental martingale.

1. INTRODUCTION

A branching random walk in $\mathbb{R}^d$ is a system of particles, where particles behave independently, and each particle gives birth to a random number of children which move in $\mathbb{R}^d$ with independent and identically distributed (i.i.d.) displacements. One of the fundamental problems on this model is the study of the counting measure which counts the number of particles of generation n situated in a Borel set of $\mathbb{R}^d$. This problem has been studied by many authors, see e.g. [21, 30, 1, 2, 6, 7, 16, 18, 19, 12], where central limit theorems and large deviations have been considered. For other important topics and closely related models, see for example the recent papers [4, 13, 24, 27, 3], the recent books [29, 14, 25] and many references therein.

In the classical branching random walk, a particle whose parent is at position y, moves to position $y + l$ with i.i.d. increments l's for different particles, so that the moving is a simple random translation. Recently, in [10] the authors consider a branching random walk in $\mathbb{R}^d$ with products of random matrices, in which the position of a particle is obtained by the action of a matrix A on the position of its parent, where the matrices A's corresponding to different particles are i.i.d. In other words, the positions of particles are obtained by the action of products of random matrices on

Date: September 8, 2020.

2000 *Mathematics Subject Classification.* 60J80, 60J85, 60F05, 60F10, 60B20, 60J10.

Key words and phrases. Branching random walk, central limit theorem, Berry-Esseen bound, moderate deviations, products of random matrices, martingale.

the position of the initial particle. This permits us to extend significantly the domains of applications of the theory of branching random walks, but the study of the model becomes much more involved. In [10], a central limit theorem and a large deviation asymptotic expansion of Bahadur-Rao type for the counting measure have been proved. In this paper, we will establish the Berry-Esseen bound about the rate of convergence in the central limit theorem, and a moderate deviation expansion of Cramér type.

For a precise description of the model we need some notation. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$. Set $\mathbb{U} := \cup_{n=0}^{\infty} (\mathbb{N}^*)^n$, where by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. A particle of generation n will be denoted by a sequence $u = u_1 \cdots u_n = (u_1, \cdots, u_n) \in (\mathbb{N}^*)^n$ of length n; the initial particle will be denoted by the null sequence $\emptyset$. Assume that on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a set of independent identically distributed random variables $(N_u)_{u\in\mathbb{U}}$ of the same law $p = \{p_k : k \in \mathbb{N}\}$, and a set of independent identically distributed $d \times d$ random matrices $(A_u)_{u\in\mathbb{U}}$ of the same law μ on the set of $d \times d$ matrices $M(d, \mathbb{R})$, where $d \geqslant 2$. The two families $(N_u)_{u\in\mathbb{U}}$ and $(A_u)_{u\in\mathbb{U}}$ are also assumed to be independent.

A branching random walk in $\mathbb{R}^d$ with products of random matrices is defined as follows. At time 0, there is one initial particle $\emptyset$ of generation 0, with initial position $Y_\emptyset := x \in \mathbb{R}^d \setminus \{0\}$. At time 1, the initial particle $\emptyset$ is replaced by $N = N_\emptyset$ new particles $i = \emptyset i$ of generation 1, located at $Y_i = A_i Y_\emptyset, 1 \leqslant i \leqslant N$. In general, at time $n+1$, each particle $u = u_1 \dots u_n$ of generation n, located at $Y_u \in \mathbb{R}^d$, is replaced by N_u new particles ui of generation $n+1$, located at $Y_{ui} = A_{ui} Y_u, 1 \leqslant i \leqslant N_u$. Namely, the position of the particle ui is obtained from the position of u by the action of the matrix A_{ui} on the vector Y_u. Consequently the position Y_u of a particle u in generation $n \geqslant 1$ is given by the action of products of random matrices on the position x of the initial particle $\emptyset$:

$$Y_u = G_u x, \quad \text{where} \quad G_u = A_{u_1 \dots u_n} \dots A_{u_1}. \tag{1.1}$$

Denote by $\mathbb{T}$ the genealogical tree associated to the elements $\{N_u : u \in \mathbb{U}\}$, defined by the following properties: 1) $\emptyset \in \mathbb{T}$; 2) when $u \in \mathbb{T}$, then for $i \in \mathbb{N}$, $ui \in \mathbb{T}$ if and only if $1 \leqslant i \leqslant N_u$; 3) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$. Let

$$\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$$

be the set of particles of generation n, where $|u|$ denotes the length of the sequence u and represents the number of generation to which u belongs; by convention $|\emptyset| = 0$.

The space $\mathbb{R}^d$ is equipped with the Euclidean norm $|\cdot|$. The position $G_u x$ of the particle u is completely described by two components: its norm $|G_u x|$

and its projection on the unit sphere $\mathbb{S}^{d-1} := \{y \in \mathbb{R}^d, |y| = 1\}$ denoted by

$$X_u^x := \frac{G_u x}{|G_u x|}.$$

Accordingly, we consider the following counting measure of particles of generation n which describes the configuration of the branching random walk at time n: for measurable sets $B_1 \subset \mathbb{S}^{d-1}$ and $B_2 \subset \mathbb{R}$,

$$Z_n^x(B_1, B_2) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{X_u^x \in B_1,\, \log |G_u x| \in B_2\}}, \tag{1.2}$$

where for a set D, $\mathbb{1}_D$ denotes its indicator function.

In [10], a central limit theorem for the counting measure Z_n^x (with the starting point $x \in \mathbb{S}^{d-1}$) was established for both the case where the matrices A_u are nonnegative, and the case where the matrices A_u are invertible. It implies that, under suitable conditions, for some constants γ, σ explicitly defined (see (2.1) and (2.2)) , the counting measure $B_2 \mapsto Z_n^x(\mathbb{S}^{d-1}, n\gamma + \sigma\sqrt{n}B_2)$ on $\mathbb{R}$ with a suitable norming converges to the standard normal law. In [10], a precise large deviation result of Bahadur-Rao type was also established, which gives in particular the exact asymptotic of $Z_n^x\Big(\mathbb{S}^{d-1}, [na, +\infty)\Big)$ for $a > \gamma$.

In this paper, our first objective is to strengthen the central limit theorem in [10] to a Berry-Esseen bound for the counting measure Z_n^x with a target function φ on X_u^x: see Theorem 2.1. With $\varphi = 1$, it implies that, under suitable conditions, for any $x \in \mathbb{S}^{d-1}$ and $n \geqslant 1$, we have, a.s.

$$\sup_{y \in \mathbb{R}} \left| \frac{1}{m^n} Z_n^x\Big(\mathbb{S}^{d-1}, n\gamma + \sigma\sqrt{n}(-\infty, y]\Big) - W\Phi(y) \right| \leqslant \frac{M}{\sqrt{n}}, \tag{1.3}$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt$ is the distribution function of the standard normal law and M is a finite and positive random variable.

Our second objective is to establish Cramér type moderate deviation expansion for Z_n^x with a target function φ on X_u^x: see Theorem 2.2. From this theorem with $\varphi = 1$, we know that, under suitable conditions, for any $x \in \mathbb{S}^{d-1}$ and $0 \leqslant y = o(\sqrt{n})$, as $n \to \infty$, a.s.,

$$\frac{Z_n^x\Big(\mathbb{S}^{d-1}, n\gamma + \sigma\sqrt{n}(-\infty, y]\Big)}{m^n W[1 - \Phi(y)]} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} \Big[1 + O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \tag{1.4}$$

where $t \mapsto \zeta(t)$ is the Cramér series (see (2.5)).

An important step in attaining these two objectives is to establish a Berry-Esseen bound for the Cramér type changed measure $Z_{s,n}^x$ (see (2.14)). This will be done in Theorem 2.3. Theorem 2.1 will be obtained from Theorem 2.3 by taking $s = 0$, and Theorem 2.2 will be established by using Theorem 2.3 and by adapting the techniques from Petrov [28].

To facilitate the comprehension, let us present some ideas in the proof of Theorem 2.3. As in [11] where the one dimensional case is considered, we need to study the asymptotic of the characteristic function of the changed measure $Z_{s,n}^x$. Inspired by the approach in [11], we would like to express the characteristic function of $Z_{s,n}^x$ in terms of a martingale and a quantity that can be controlled by the theory of products of random matrices. However, in contrast to the one dimensional case, we cannot obtain directly an expression of the characteristic function in terms of a martingale. Fortunately, using the spectral gap theory for products of random matrices established in [20, 13, 15] and recently developed in [31], we have been able to define a new martingale which is similar to the fundamental martingale and which can be used for a suitable approximation of the characteristic function of $Z_{s,n}^x$. We conclude by proving the uniform convergence and analyticity with respect to a complex parameter of the new martingale, and by using the asymptotic properties of the eigenvalue of the pertubed transfer operator related to the products of random matrices. See Theorem 4.3 and Lemma 5.6 for details.

The rest of the paper is organized as follows. In Section 2, we fix some notation, introduce our assumptions on the branching products of random matrices, and state the main results. In Section 3, we recall some spectral gap properties on products of random matrices stated in [31]. In Section 4, the uniform convergence and analyticity of the constructed martingale are established. Sections 5 and 6 are devoted to the proofs of the main results.

2. Main results

2.1. **Notation and assumptions on products of random matrices.** Note that in our model, along each branch we encounter a product of random matrices. In this section, we introduce some notation and the necessary assumptions on products of random matrices in order to formulate our main results. We shall consider two cases, the case when the matrices are nonnegative and the case when the matrices are invertible.

The set $M(d,\mathbb{R})$ of $d \times d$ real matrices is equipped with the operator norm: $\|\mathbf{a}\| = \sup_{x\in\mathbb{S}^{d-1}} |\mathbf{a}x|$ for $\mathbf{a} \in M(d,\mathbb{R})$, where $|\cdot|$ is a given vectorial norm on $\mathbb{R}^d$, and $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere in $\mathbb{R}^d$. A matrix $\mathbf{a} \in M(d,\mathbb{R})$ is said to be *proximal* if it has an algebraic simple dominant eigenvalue. Denote by $\mathcal{M}_+$ the set of matrices with nonnegative entries. A nonnegative matrix $\mathbf{a} \in \mathcal{M}_+$ is said to be *allowable* if every row and every column of $\mathbf{a}$ has a strictly positive entry.

Let μ be a probability measure on $M(d,\mathbb{R})$. Denote by $\Gamma_\mu := [\text{supp } \mu]$ the smallest closed semigroup of $M(d,\mathbb{R})$ generated by the support of μ. We say that the measure μ is *arithmetic* if there are $t > 0$, $\theta \in [0, 2\pi)$ and a function $\vartheta : \mathbb{S}_+^{d-1} \to \mathbb{R}$ such that

$$\forall \mathbf{a} \in \Gamma, \forall x \in V(\Gamma) : \exp[it \log |\mathbf{a}x| - i\theta + i(\vartheta(\mathbf{a}{\cdot}x) - \vartheta(x))] = 1,$$

where $\mathbb{S}_+^{d-1} = \{x \geqslant 0 : |x| = 1\}$ is the intersection of the unit sphere with the positive quadrant. Notice that when $d = 1$, we have $\mathbb{S}_+^{d-1} = \{1\}$, and the above arithmetic condition reduces to the following more usual form: $\log a$ is almost surely (a.s.) concentrated on an arithmetic progression $a_0 + a_1\mathbb{N}$ for some $a_0, a_1 > 0$.

We will need the following assumptions on the law μ.

C1.

(1) *For invertible matrices:*
   (a) *(Strong irreducibility)There is no finite union $\mathcal{W} = \bigcup_{i=1}^n W_i$ of proper subspaces $0 \neq W_i \subsetneq \mathbb{R}^d$ which is Γ_μ-invariant (in the sense that $a\mathcal{W} = \mathcal{W}$ for each $a \in \Gamma_\mu$).*
   (b) *(Proximality) Γ_μ contains at least one proximal matrix.*

(2) *For nonnegative matrices:*
   (a) *(Allowability) Every $\mathbf{a} \in \Gamma_\mu$ is allowable.*
   (b) *(Positivity) Γ_μ contains at least one matrix belonging to int($\mathcal{M}_+$).*
   (c) *(Non-arithmeticity) The measure μ is non-arithmetic.*

For both invertible matrices and nonnegative matrices, we will need a moment condition. For $\mathbf{a} \in M(d, \mathbb{R})$, set

$$\iota(\mathbf{a}) := \inf_{x \in \mathcal{S}} |\mathbf{a}x|, \quad \text{and} \quad \mathbf{a} \cdot x := \frac{\mathbf{a}x}{|\mathbf{a}x|} \quad \text{when } \mathbf{a}x \neq 0,$$

where $\mathbf{a} \cdot x$ is called the projective action of the matrix $\mathbf{a}$ on the vector $x \in \mathbb{S}^{d-1}$. Then $\iota(\mathbf{a}) > 0$ for both invertible matrices and allowable nonnegative matrices. Set, for an invertible or nonnegative matrix $\mathbf{a}$,

$$N(\mathbf{a}) = \max\{\|\mathbf{a}\|, \iota(\mathbf{a})^{-1}\}.$$

For invertible matrices we have $\iota(\mathbf{a}) = \|\mathbf{a}^{-1}\|^{-1}$ and $N(\mathbf{a}) = \max\{\|\mathbf{a}\|, \|\mathbf{a}^{-1}\|\}$.

C2. *(Moment condition) There exists $\eta_0 \in (0, 1)$ such that*

$$\mathbb{E}[N(A_1)^{\eta_0}] < \infty.$$

We will consider the action of invertible matrices on the projective space $\mathbb{P}^{d-1}$ which is obtained from $\mathbb{S}^{d-1}$ by identifying x and $-x$, and the action of nonnegative matrices on $\mathbb{S}_+^{d-1}$. For convenience we identify $x \in \mathbb{P}^{d-1}$ with one of its representants in $\mathbb{S}^{d-1}$. To unify the exposition, we use the symbol $\mathcal{S}$ to denote $\mathbb{P}^{d-1}$ for invertible matrices, and $\mathbb{S}_+^{d-1}$ for nonnegative matrices. The space $\mathcal{S}$ will be equipped with the metric $\mathbf{d}$, which is the angular distance (see [9]) for invertible matrices, and the Hilbert cross-ratio metric (see [22]) for nonnegative matrices. Moreover, $\mathcal{S}$ is a separable metric space equipped with Borel σ-field.

Let $\mathcal{C}(\mathcal{S})$ be the space of continuous complex-valued functions on $\mathcal{S}$. For $\beta > 0$ sufficiently small, we introduce the Banach space

$$\mathcal{B}_\beta = \{f \in \mathcal{C}(\mathcal{S}) : \|f\|_\beta < +\infty\},$$

equipped with the norm

$$\|f\|_\beta := \|f\|_\infty + |f|_\beta,$$

where

$$\|f\|_\infty := \sup_{x\in\mathcal{S}} |f(x)|, \quad |f|_\beta := \sup_{x,y\in\mathcal{S},x\neq y} \frac{|f(x) - f(y)|}{\mathbf{d}^\beta(x,y)}.$$

Let $G_n = A_n \dots A_2 A_1$ be the product of i.i.d. $d \times d$ real random matrices A_i, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with common law μ. Let $x \in \mathcal{S}$ be a starting point. As mentioned in the introduction, the random walk $G_n x$ is completely determined by its log norm and its projection on $\mathcal{S}$, denoted respectively by

$$S_n^x := \log |G_n x|, \quad X_n^x := G_n{\cdot}x = \frac{G_n x}{|G_n x|}, \quad n \geqslant 0,$$

with the convention that $G_0 x = x$. Since $S_{n+1}^x = \log |A_{n+1} X_n^x| + S_n^x$ and $X_{n+1}^x = A_{n+1} \cdot X_n^x$, the sequence $(S_n^x, X_n^x)_{n\geqslant 0}$ is a Markov chain.

Denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. By the law of large numbers of Furstenberg [17], under conditions **C1** and **C2**, we have

$$\lim_{n\to\infty} \frac{1}{n} S_n^x = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[S_n^x] = \gamma \quad \mathbb{P}\text{-a.s.}, \tag{2.1}$$

where $\gamma = \inf_{n\in\mathbb{N}} \frac{1}{n} \mathbb{E} \log \|G_n\|$ is the upper Lyapunov exponent associated with the product sequence (G_n). Le Page [26] and Henion [22] showed that

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \mathbb{E} \left(S_n^x - n\gamma\right)^2 \tag{2.2}$$

exists and is independent of x for invertible matrices and nonnegative matrices, respectively. Moreover, there exists a unique μ-stationary probability measure ν on $\mathcal{S}$ (see [20, 13]): $\mu * \nu = \nu$, that is, for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$(\mu * \nu)(\varphi) := \int_{\mathcal{S}} \int_{\Gamma_\mu} \varphi(\mathbf{a}{\cdot}x) \mu(d\mathbf{a}) \nu(dx) = \nu(\varphi),$$

where $\nu(\varphi) = \int_{\mathcal{S}} \varphi(x)\nu(dx)$, and this notation for the integral will be used for any function and any measure. Define the transfer operator on $\mathcal{C}(\mathcal{S})$ as follows: for any $s \in (-\eta_0, \eta_0)$, and $f \in \mathcal{C}(\mathcal{S})$,

$$P_s f(x) = \mathbb{E}[|A_1 x|^s f(A_1{\cdot}x)], \quad \text{for all } x \in \mathcal{S}. \tag{2.3}$$

It is known that under conditions **C1** and **C2**, there exists a small constant $0 < \eta_1 < \eta_0$ such that for any $s \in (-\eta_1, \eta_1)$, there are a unique probability measure ν_s and a unique Hölder continuous function r_s on $\mathcal{S}$ satisfying $\nu(r_s) = 1$ and

$$\nu_s P_s = \kappa(s)\nu_s \quad \text{and} \quad P_s r_s = \kappa(s) r_s, \tag{2.4}$$

where $\kappa(s)$ is the unique dominant eigenvalue of P_s, $\nu_s P_s$ is the mesure on $\mathcal{S}$ such that $(\nu_s P_s)(f) = \nu_s(P_s f)$ for all $f \in \mathcal{C}(\mathcal{S})$. In particular, $r_0 = 1$ and $\kappa(0) = 1$. For $s \in [0, \eta_1)$, the property (2.4) is proved in [13, Proposition 3.1] and [15, Corollary 7.3] for positive matrices, and in [20, Theorem 2.6 and Corollary 3.20] for invertible matrices. For both positive matrices and invertible matrices, the existence of $\eta_1 > 0$ and the property (2.4) for $s \in (-\eta_1, \eta_1)$ are proved in [31, Proposition 3.1], where the following properties are also established: the functions $s \mapsto \kappa(s)$ and $s \mapsto r_s(x)$ are strictly positive and analytic in $(-\eta_1, \eta_1)$, for $x \in \mathcal{S}$. Moreover, it is proved (see [20, Lemma 3.5], [13, Lemma 6.2], [31, Propositions 3.12 and 3.14]) that, under conditions **C1** and **C2**, the function $\Lambda(s) = \log \kappa(s)$ is finite and analytic on $(-\eta_1, \eta_1)$, and satisfies

$$\Lambda'(0) = \gamma, \quad \Lambda''(0) = \sigma^2 > 0, \quad \text{and} \quad \Lambda''(s) > 0 \quad \forall s \in (-\eta_1, \eta_1).$$

Denote $\gamma_k = \Lambda^k(0), k \geqslant 1$. Throughout the paper, we write ζ for the Cramér series associated to Λ (see [28, Theorem VIII.2.2] for details):

$$\zeta(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3} t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}} t^2 + \dots \tag{2.5}$$

which converges for $|t|$ small enough.

2.2. **Main results.** Let $Z_n = Z_n^x(S, \mathbb{R})$ be the population size at time n, which does not depend on the starting point x, and which forms a Galton-Watson process with $Z_0 = 1$ and $Z_1 = N$. Denote by $m = \mathbb{E}N$ the expected value of the offspring distribution. Throughout the paper, we assume that

$$m \in (1, \infty) \quad \text{and} \quad \mathbb{P}(N = 0) = 0.$$

Therefore the branching process (Z_n) is supercritical, and $Z_n \to \infty$ a.s. as $n \to \infty$. It is well known that $\mathbb{E}Z_n = m^n$. Let

$$W = \lim_{n\to\infty} W_n, \quad \text{where} \quad W_n = \frac{Z_n}{m^n}, \quad n \geqslant 0,$$

is the fundamental martingale for the Galton-Watson process (Z_n), and the limit exists a.s. by the martingale convergence theorem. An important ingredient in studying Berry-Esseen bound and moderate deviation expansion is the fundamental martingale associated to branching random walks with products of random matrices, defined for $s \in (-\eta_1, \eta_1)$ and $x \in \mathcal{S}$

$$W_n^x(s) := \frac{\sum_{u \in \mathbb{T}_n} e^{s S_u^x} r_s(X_u^x)}{[m\kappa(s)]^n r_s(x)}, \quad n \geqslant 0. \tag{2.6}$$

This is a positive martingale with respect to the natural filtration

$$\mathscr{F}_0 = \{\emptyset, \Omega\} \text{ and } \mathscr{F}_n = \sigma(N_u, A_{ui} : i \geqslant 1, |u| < n) \text{ for } n \geqslant 1.$$

By the martingale convergence theorem, the limit

$$W^x(s) := \lim_{n \to \infty} W_n^x(s) \quad \text{ exists in } \mathbb{R} \quad \mathbb{P}\text{-a.s.}$$

Set $\Lambda^*(q_s) = sq_s - \Lambda(s)$ with $q_s = \Lambda'(s)$. It is proved in [10] that under conditions **C1** and **C2**, if

$$\Lambda^*(q_s) - \log m < 0 \tag{2.7}$$

and

$$\mathbb{E}[\max_{x \in \mathcal{S}} W_{s,1}^x \log_+ \max_{x \in \mathcal{S}} W_{s,1}^x] < \infty, \tag{2.8}$$

where $\log_+ x = \max\{0, \log x\}$ denotes the positive part of $\log x$, then for all $x \in \mathcal{S}$, $W^x(s)$ is non-degenerate with

$$\mathbb{E}[W^x(s)] = 1.$$

Set

$$J = \{s \in (-\eta_1, \eta_1) : \Lambda^*(q_s) - \log m < 0\}, \tag{2.9}$$

which is an open interval containing 0. We assume the following moment condition slightly stronger than (2.8):

C3. *There are constants $\gamma_0 > 1$ and $0 < \eta_2 < \frac{\eta_1}{2}$ with $[-\eta_2, \eta_2] \subset J$ such that*

$$\mathbb{E}\Big[\max_{x \in \mathcal{S}} \Big(W_1^x(s)\Big)^{\gamma_0}\Big] < \infty \quad \forall\ s \in [-\eta_2, \eta_2].$$

It is clear that conditions **C1**-**C3** (together with the hypothesis $\mathbb{P}(N = 0) = 0$ that we assume always), imply that for all $x \in \mathcal{S}$, $W^x(s) > 0$ a.s. and $\mathbb{E}[W^x(s)] = 1$; in particular (when $s = 0$), $W > 0$ a.s. and $\mathbb{E}[W] = 1$.

Our first result is the Berry-Esseen bound for the counting measure Z_n^x:

Theorem 2.1. *Assume conditions **C1**-**C3**. Then, for any $x \in \mathcal{S}$, $\varphi \in \mathcal{B}_\beta$ and $n \geqslant 1$, we have, a.s.,*

$$\sup_{y \in \mathbb{R}} \left| \frac{1}{m^n} \sum_{u \in \mathbb{T}_n} \varphi(X_u^x) \mathbb{1}_{\left\{\frac{S_u^x - n\gamma}{\sigma\sqrt{n}} \leqslant y\right\}} - W\nu(\varphi)\Phi(y) \right| \leqslant \frac{M}{\sqrt{n}}, \tag{2.10}$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-t^2/2} dt$ is the distribution function of the standard normal law and M is a finite and positive random variable.

This is a Berry-Esseen type bound for the counting measure Z_n^x with suitable norming because the sum in (2.10) is an integral with respect to Z_n^x:

$$\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)\mathbb{1}_{\left\{\frac{S_u^x-n\gamma}{\sigma\sqrt{n}}\leqslant y\right\}}=\int_{\mathcal{S}\times\mathbb{R}}\varphi(z_1)\mathbb{1}_{\left\{\frac{z_2-n\gamma}{\sigma\sqrt{n}}\leqslant y\right\}}Z_n^x(dz_1,dz_2). \tag{2.11}$$

Our second result is the Cramér's moderate deviation expansion for Z_n^x.

Theorem 2.2. *Assume conditions C1-C3. Then, we have for any $x\in\mathcal{S}$, $\varphi\in\mathcal{B}_\beta, 0\leqslant y=o(\sqrt{n})$, as $n\to\infty$, a.s.,*

$$\frac{\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)\mathbb{1}_{\{S_u^x-n\gamma\geqslant\sqrt{n}\sigma y\}}}{m^nW[1-\Phi(y)]}=e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi)+O\Big(\frac{y+1}{\sqrt{n}}\Big)\Big], \tag{2.12}$$

and

$$\frac{\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)\mathbb{1}_{\{S_u^x-n\gamma\leqslant-\sqrt{n}\sigma y\}}}{m^nW\Phi(-y)}=e^{-\frac{y^3}{\sqrt{n}}\zeta(-\frac{y}{\sqrt{n}})}\left[\nu(\varphi)+O\left(\frac{y+1}{\sqrt{n}}\right)\right]. \tag{2.13}$$

An important step in the proof of the moderate deviation expansion is to establish a Berry-Esseen bound for the changed measure $Z_{s,n}^x$ defined by for measurable sets $B_1\subset\mathbb{S}^{d-1}$ and $B_2\subset\mathbb{R}$,

$$\begin{aligned}Z_{s,n}^x(B_1,B_2)&=\int_{B_1\times B_2}\frac{e^{sz_2}r_s(z_1)}{[m\kappa(s)]^nr_s(x)}Z_n^x(dz_1,dz_2)\\&=\sum_{u\in\mathbb{T}_n}\frac{e^{sS_u^x}r_s(X_u^x)}{[m\kappa(s)]^nr_s(x)}\mathbb{1}_{\{X_u^x\in B_1,S_u^x\in B_2\}}.\end{aligned} \tag{2.14}$$

Our third result is a Berry-Esseen bound for the changed measure $Z_{s,n}^x$:

Theorem 2.3. *Assume conditions C1-C3. Then, for any $x\in\mathcal{S}$ and $\varphi\in\mathcal{B}_\beta$ there exists a constant $0<\eta<\eta_2$ such that a.s., for $n\geqslant 1$,*

$$\sup_{s\in(-\eta,\eta)}\sup_{y\in\mathbb{R}}\left|\sum_{u\in\mathbb{T}_n}\frac{e^{sS_u^x}r_s(X_u^x)\varphi(X_u^x)}{[m\kappa(s)]^nr_s(x)}\mathbb{1}_{\left\{\frac{S_u^x-n\Lambda'(s)}{\sigma_s\sqrt{n}}\leqslant y\right\}}-W^x(s)\pi_s(\varphi)\Phi(y)\right|\leqslant\frac{M}{\sqrt{n}}, \tag{2.15}$$

where M is a positive and finite random variable.

This is a Berry-Esseen type bound for $Z_{s,n}^x$ because, similar to the case of Theorem 2.1, the sum in (2.15) is an integral with respect to $Z_{s,n}^x$:

$$\begin{aligned}&\sum_{u\in\mathbb{T}_n}\frac{e^{sS_u^x}r_s(X_u^x)\varphi(X_u^x)}{[m\kappa(s)]^nr_s(x)}\mathbb{1}_{\left\{\frac{S_u^x-n\Lambda'(s)}{\sigma_s\sqrt{n}}\leqslant y\right\}}\\&=\int_{\mathcal{S}\times\mathbb{R}}\frac{e^{sz_2}r_s(z_1)\varphi(z_1)}{[m\kappa(s)]^nr_s(x)}\mathbb{1}_{\left\{\frac{z_2-n\Lambda'(s)}{\sigma_s\sqrt{n}}\leqslant y\right\}}Z_{s,n}^x(dz_1,dz_2).\end{aligned} \tag{2.16}$$

3. Preliminary results on products of random matrices

In this section we recall some spectral gap properties stated in [31] which will be used for the proofs of main results.

Define the operator P_z on $\mathcal{C}(\mathcal{S})$ by

$$P_z f(x) = \mathbb{E}[|A_1 x|^z f(A_1 \cdot x)], \quad \text{for all } x \in \mathcal{S}, z \in \mathbb{C}. \tag{3.1}$$

Denote by $\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$ the set of all bounded linear operators from $\mathcal{B}_\beta$ to $\mathcal{B}_\beta$ equipped with the operator norm

$$\|P\|_{\mathcal{B}_\beta \to \mathcal{B}_\beta} := \sup_{f \neq 0} \frac{\|Pf\|_\beta}{\|f\|_\beta}, \quad \forall P \in \mathcal{L}(\mathcal{B}_\beta, B_\beta).$$

We write $\mathcal{B}'_\beta$ for the topological dual of $\mathcal{B}_\beta$ endowed with the norm $\|\nu\|_{\mathcal{B}'_\beta} = \sup_{\|\varphi\|_\beta = 1} |\nu(\varphi)|$ for any linear functional $\nu \in \mathcal{B}'_\beta$. For any $\eta > 0$, set $B_\eta(0) = \{z \in \mathbb{C} : |z| < \eta\}$ for the ball with center 0 and radius η in the complex plane $\mathbb{C}$.

Lemma 3.1. *Assume conditions **C1** and **C2**. There exists a small $\eta_1 \in (0, \eta_0)$ such that for any $z \in B_{\eta_1}(0)$ and $n \geqslant 1$, we have the decomposition*

$$P_z^n = \kappa^n(z) M_z + L_z, \tag{3.2}$$

where the operator M_z is a rank one projection on $\mathcal{B}_\beta$, the mappings on $B_{\eta_1}(0)$,

$$z \mapsto \kappa(z) \in \mathbb{C}, \quad z \mapsto r_z \in \mathcal{B}_\beta, \quad z \mapsto \nu_z \in \mathcal{B}'_\beta, \quad z \mapsto L_z \in \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$$

are well-defined under the normalizing conditions $\nu_z(1) = \nu(r_z) = 1$. All these mappings are analytic in $B_{\eta_1}(0)$, and possess the following properties:

1. *for any $z \in B_{\eta_1}(0)$, it holds that $M_z L_z = L_z M_z = 0$;*
2. *for any $z \in B_{\eta_1}(0)$, $P_z r_z = \kappa(z) r_z$ and $\nu_z P_z = \kappa(z) \nu_z$;*
3. *$\kappa(s)$ and r_s are real-valued and satisfy $\kappa(s) > 0$ and $r_s(x) > 0$ for any $s \in (-\eta_1, \eta)$ and $x \in \mathcal{S}$;*
4. *there exist two constants $0 < a_1 < a_2 < 1$ such that for all $z \in B_{\eta_1}(0)$ and all $n \in \mathbb{N}^*$, $|\kappa(z)| > 1 - a_1$ and $\|L_z^n\|_{\mathcal{B}_\beta \to \mathcal{B}_\beta} \leqslant c(1 - a_2)^n$.*

For fixed $s \in (-\eta_1, \eta_1)$ and $x \in \mathcal{S}$, the spectral gap property (2.4) allows to define a probability measure $\mathbb{Q}_s^x$ on $(\Omega, \mathcal{F})$ such that for any $n \in \mathbb{N}$ and any bounded and measurable function h on $(\mathcal{S} \times \mathbb{R})^{n+1}$,

$$\begin{aligned} &\mathbb{E}\Big[\frac{e^{sS_n^x} r_s(X_n^x)}{\kappa^n(s) r_s(x)} h(X_0^x, S_0^x, \dots, X_n^x, S_n^x)\Big] \\ &= \mathbb{E}_{\mathbb{Q}_s^x}\left[h(X_0^x, S_0^x, \dots, X_n^x, S_n^x)\right], \end{aligned} \tag{3.3}$$

where $\mathbb{E}_{\mathbb{Q}_s^x}$ denotes the expectation with respect to $\mathbb{Q}_s^x$. See [13, 15, 20] for $s \geqslant 0$, and [31] for $s < 0$.

Under the changed measure $\mathbb{Q}_s^x$, the process $(X_n^x)_{n\in\mathbb{N}}$ is a Markov chain with the transition operator Q_s defined by, for any $s \in (-\eta_1, \eta_1)$ and $\varphi \in \mathcal{B}_\beta$,

$$Q_s\varphi(x) = \frac{1}{\kappa(s) r_s(x)} P_s(\varphi r_s)(x), \quad x \in \mathcal{S}.$$

It has been proved in [31, Proposition 3.4] that Q_s has a unique stationary probability measure defined by $\pi_s(\varphi) := \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}$, $\varphi \in \mathcal{B}_\beta$, and there exist two constants $0 < a < 1$, $c_1 > 0$ such that

$$\sup_{s\in(-\eta_1,\eta_1)} \sup_{x\in\mathcal{S}} |\mathbb{E}_{\mathbb{Q}_s^x}[\varphi(X_n^x)] - \pi_s(\varphi)| \leqslant c_1 a^n. \tag{3.4}$$

Moreover, the perturbed operator $R_{s,it}$ defined by

$$R_{s,it}\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\Big[e^{it[S_1^x - \Lambda'(s)]}\varphi(X_1^x)\Big], \quad s \in (-\eta_1, \eta_1,) \text{ and } t \in \mathbb{R}, \tag{3.5}$$

satisfies for any compact $K \subset \mathbb{R}\backslash\{0\}, n \geqslant 1$ and $\varphi \in \mathcal{B}_\beta$,

$$\sup_{s\in(-\eta_1,\eta_1)} \sup_{t\in K} \sup_{x\in\mathcal{S}} |R_{s,it}^n \varphi(x)| \leqslant \|\varphi\|_\beta a_K^n, \quad 0 < a_K < 1. \tag{3.6}$$

The operator $R_{s,it}$ has eigenvalue $\lambda_{s,it}$ satisfying for $s \in (-\eta_1, \eta_1)$ and $t \in (-\delta, \delta) \subset (-\eta_1, \eta_1)$,

$$\lambda_{s,it} = e^{\Lambda(s+it) - \Lambda(s) - \Lambda'(s)it}. \tag{3.7}$$

4. Associated martingales

In this section, for the fundamental martingale $(W_n^x(s))$ we first reveal a relationship between the moments of $W_1^x(s)$ and $W_*^x(s) := \sup_{n\geqslant 0} W_n^x(s)$. We next prove the uniform convergence of $W_n^x(z)$ for $z \in B_{\eta_2}(0)$. We finally introduce a new martingale and establish its similar properties; this martingale will play a key role in the proof of the main results.

Theorem 4.1. *Assume conditions* ***C1-C3****. Then there is a constant $\eta \in (0, \eta_2)$ such that*

$$\sup_{s\in(-\eta,\eta)} \sup_{x\in\mathcal{S}} \mathbb{E}[W_*^x(s)]^{\gamma_0} < \infty. \tag{4.1}$$

Proof. In [10, Lemma 5.6], it is proved that if $\mathbb{E}[W^x(s)] = 1$, then $W_*^x(s)$ and $W^x(s)$ have similar tail behaviour for $s \in (-\eta_2, , \eta_2)$ and for all $x \in \mathcal{S}$, i.e. for $s \in (-\eta_2, , \eta_2)$ and for any $a \in (0, 1)$, for $s \in (-\eta_2, , \eta_2)$, there is a constant $b > 0$ such that for all $t > 0$, for all $x \in \mathcal{S}$

$$\mathbb{P}(W_s^x \geqslant at) \geqslant b\mathbb{P}(W_{s,*}^x \geqslant t) \geqslant b\mathbb{P}(W_s^x \geqslant t).$$

A slight modification in the proof of [10, Lemmas 5.5 and 5.6] shows that we can choose b independent of $s \in (-\eta_2, , \eta_2)$. (To see this, we just need to check the proof therein, and replace W_s^y by $\sup_{s\in(-\eta_2,\eta_2)} W_s^y$ in the formula

$\sup_{y\in S}\mathbb{E}\Big[W_s^y 1_{\{W_s^y>T\}}\Big] \overset{T\to+\infty}{\longrightarrow} 0$ of the proof of Lemma 5.5, at the last line of page 34.) Recall that $\mathbb{E}[W^x(s)] = 1$ under the hypothesis of Theorem 4.1. Thus, in order to prove (4.1), it suffices to show that there is a constant $\eta \in (0,\eta_2)$ such that

$$\sup_{s\in(-\eta,\eta)}\sup_{x\in\mathcal{S}}\mathbb{E}[W^x(s)]^{\gamma_0} < \infty. \tag{4.2}$$

Set $h(x) = x^\delta$ where $\delta = \gamma_0 - 1 \in (0,1]$. Observe that

$$W_{n+1}^x(s) = \sum_{u\in\mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s), \text{ where } H_{n,u}^x = \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^n r_s(x)}. \tag{4.3}$$

Using (4.3) and the subadditivity of h, we have

$$\begin{aligned}\mathbb{E}_n\Big[W_{n+1}^x(s)\ h\Big(W_{n+1}^x(s)\Big)\Big] &\leqslant \mathbb{E}_n\Big[\sum_{u\in\mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s)\ h\Big(H_{n,u}^x W_1^{X_u^x}(s)\Big)\Big] \\ &\quad + \mathbb{E}_n\Big[\sum_{u\in\mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s)\ h\Big(\sum_{\substack{v\in\mathbb{T}_n\\ v\neq u}} H_{n,v}^x W_1^{X_v^x}(s)\Big)\Big].\end{aligned}$$

Using Jensen's inequality for the conditional expectation and the facts that $\mathbb{E}_n\Big[W_1^{X_u^x}(s)\Big] = 1$ and h is an increasing function, the second term in the inequality above is less than $W_n^x(s)\ h(W_n^x(s))$. Then taking expectations in the two sides of the inequality above, we get

$$\begin{aligned}\mathbb{E}\Big[W_{n+1}^x(s)\ h\Big(W_{n+1}^x(s)\Big)\Big] &\leqslant \mathbb{E}\Big[\sum_{u\in\mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s)\ h\Big(H_{n,u}^x W_1^{X_u^x}(s)\Big)\Big] \\ &\quad + \mathbb{E}\Big[W_n^x(s)\ h\Big(W_n^x(s)\Big)\Big].\end{aligned}$$

So by recurrence on n and Fatou's lemma, we obtain

$$\begin{aligned}&\mathbb{E}\Big[W^x(s)\ h\Big(W^x(s)\Big)\Big] \leqslant \liminf_{n\to\infty}\mathbb{E}\Big[W_n^x(s)\ h\Big(W_n^x(s)\Big)\Big] \\ &\leqslant \mathbb{E}\Big[W_1^x(s)\ h\Big(W_1^x(s)\Big)\Big] + \sum_{n=1}^{\infty}\mathbb{E}\Big[\sum_{u\in\mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s)\ h\Big(H_{n,u}^x W_1^{X_u^x}(s)\Big)\Big].\end{aligned}$$

To prove (4.2), it suffices to show that there is a constant $\eta \in (0,\eta_2)$ such that

$$\sup_{s\in(-\eta,\eta)}\sup_{x\in\mathcal{S}}\mathbb{E}\Big[W_1^x(s)\ h\Big(W_1^x(s)\Big)\Big] \leqslant \sup_{s\in(-\eta_2,\eta_2)}\mathbb{E}[\sup_{x\in\mathcal{S}} W_1^x(s)]^{\gamma_0} < \infty, \tag{4.4}$$

and

$$\sup_{s\in(-\eta,\eta)}\sup_{x\in\mathcal{S}}\sum_{n=0}^{\infty}\mathbb{E}\Big[\sum_{u\in\mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s)\ h\Big(H_{n,u}^x W_1^{X_u^x}(s)\Big)\Big] < \infty. \tag{4.5}$$

For (4.4), we see that for all $s \in (-\eta_1, \eta_1)$,

$$W_1^x(s) = \frac{1}{m\kappa(s)} \sum_{i=1}^{N} e^{s \log |A_i x|} r_s(A_i \cdot x)$$
$$\leqslant \frac{\max_{x \in \mathcal{S}} |r_s(x)|}{m\kappa(s)} \Big(\sum_{i=1}^{N} e^{-\eta_2 \log |A_i x|} + \sum_{i=1}^{N} e^{\eta_2 \log |A_i x|} \Big). \tag{4.6}$$

Since the functions $s \mapsto r_s$ and $s \mapsto \kappa(s)$ are strictly positive and analytic on $(-\eta_1, \eta_1)$ and $r_0 = 1, \kappa(0) = 1$, there are two constants $d_1, d_2 > 0$ such that

$$\frac{\max_{x \in \mathcal{S}} r_s(x)}{\min_{x \in \mathcal{S}} r_s(x)} \leqslant d_1 \quad \text{for all} \quad s \in (-\eta_1, \eta_1) \tag{4.7}$$

and

$$\frac{\sup_{s \in (-\eta_1, \eta_1)} \kappa(s)}{\inf_{s \in (-\eta_1, \eta_1)} \kappa(s)} \leqslant d_2. \tag{4.8}$$

Hence, from (4.6), (4.7) and (4.8), for all $s \in (-\eta_1, \eta_1)$,

$$W_1^x(s) \leqslant d_1 d_2 \Big(W_1^x(-\eta_2) + W_1^x(\eta_2) \Big).$$

Therefore, by the inequality

$$(a+b)^{\gamma_0} \leqslant 2^{\gamma_0 - 1}(a^{\gamma_0} + b^{\gamma_0}), \quad a, b \in \mathbb{R}, \tag{4.9}$$

and condition **C3**,

$$\sup_{s \in [-\eta_2, \eta_2]} \mathbb{E}[\sup_{x \in \mathcal{S}} W_1^x(s)]^{\gamma_0}$$
$$\leqslant (d_1 d_2)^{\gamma_0} 2^{\gamma_0 - 1} \Big(\mathbb{E} \sup_{x \in \mathcal{S}} [W_1^x(-\eta_2)]^{\gamma_0} + \mathbb{E} \sup_{x \in \mathcal{S}} [W_1^x(\eta_2)]^{\gamma_0} \Big) \quad < \infty. \tag{4.10}$$

For (4.5), we consider the general term in its series. Since $h(x) = x^\delta$, we have, by (4.7), for all $s \in (-\eta_2, \eta_2)$

$$\mathbb{E}\Big[\sum_{u \in \mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s) \ h\Big(H_{n,u}^x W_1^{X_u^x}(s) \Big) \Big]$$
$$= \mathbb{E}\Big[\sum_{u \in \mathbb{T}_n} \Big(H_{n,u}^x W_1^{X_u^x}(s) \Big)^{\gamma_0} \Big]$$
$$\leqslant \mathbb{E}\Big[\sum_{u \in \mathbb{T}_n} (H_{n,u}^x)^{\gamma_0} \Big] \mathbb{E} \sup_{x \in \mathcal{S}} W_1^x(s)^{\gamma_0}$$
$$\leqslant d_1^{2\gamma_0} \Big(\frac{m\kappa(s\gamma_0)}{[m\kappa(s)]^{\gamma_0}} \Big)^n \mathbb{E}[W_n^x(s\gamma_0)] \mathbb{E} \sup_{x \in \mathcal{S}} W_1^x(s)^{\gamma_0}. \tag{4.11}$$

Set $f(s) = \frac{m\kappa(s\gamma_0)}{[m\kappa(s)]^{\gamma_0}}$, $s \in (-\eta_2, \eta_2)$. We see that $f(0) = m^{1-\gamma_0} < 1$ and f is continous on $(-\eta_2, \eta_2)$ by the continuity of κ. Hence there is a small constant $\eta > 0$ with $(-\eta, \eta) \subset (-\eta_2, \eta_2)$ such that

$$c_1 := \sup_{s \in (-\eta, \eta)} \frac{m\kappa(s\gamma_0)}{[m\kappa(s)]^{\gamma_0}} < 1. \tag{4.12}$$

We can choose $\eta > 0$ sufficiently small so that $s\gamma_0 \in (-\eta_1, \eta_1)$. Then $W_n^x(s\gamma_0)$ is well-defined and a martingale, so $\mathbb{E}[W_n^x(s\gamma_0)] = 1$. Therefore, from (4.11), (4.12) and (4.10), we obtain

$$\begin{aligned} &\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} \mathbb{E}\Big[\sum_{u \in \mathbb{T}_n} H_{n,u}^x W_1^{X_u^x}(s)\ h\Big(H_{n,u}^x W_1^{X_u^x}(s)\Big)\Big] \\ &\leqslant d_1^{2\gamma_0} \sup_{s \in [-\eta_2, \eta_2]} \mathbb{E} \sup_{x \in \mathcal{S}} W_1^x(s)^{\gamma_0} \sum_{n=1}^{\infty} c_1^n < \infty. \end{aligned}$$

This completes the proof of (4.5). Thus (4.2) is proved.

□

Now we consider the martingale with complex parameter:

$$W_n^x(z) := \frac{\sum_{u \in \mathbb{T}_n} e^{zS_u^x} r_z(X_u^x)}{[m\kappa(z)]^n r_z(x)}, \quad n \geqslant 0, \quad z \in B_{\eta_1}(0). \tag{4.13}$$

For each fixed $z \in B_{\eta_1}(0)$, it can be easily checked that $(W_n^x(z))$ remains a martingale with respect to $(\mathscr{F}_n)$. Throughout, the real par of $z \in \mathbb{C}$ will be denoted by s, so that $z = s + iIm(z)$.

The next theorem gives the uniform convergence of $W_n^x(z)$. Let

$$\Omega_\alpha^1 = \operatorname{int}\left\{ z \in B_{\eta_2}(0) : \frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} < 1 \right\} \quad \text{and} \quad \Omega_{\gamma_0} = \bigcup_{1 < \alpha \leqslant \gamma_0} \Omega_\alpha^1. \tag{4.14}$$

Since the derivative at 1 of the function $\alpha \mapsto \frac{m\kappa(\alpha s)}{[m\kappa(s)]^\alpha}$ is equal to $\Lambda^*(s) - \log m$ which is negative for $s \in (-\eta_2, \eta_2)$, we have, for these values of s, $\frac{m\kappa(\alpha s)}{[m\kappa(s)]^\alpha} < 1$ when $\alpha > 1$ is close to 1. This shows that the open set Ω_{γ_0} contains the segment $(-\eta_2, \eta_2)$, so that $(-\eta_2, \eta_2)$ is the intersection of Ω_{γ_0} with the real axis.

Theorem 4.2. *Assume conditions **C1**-**C3**. Then the sequence $(W_n^x(z))_{n \geqslant 0}$ converges a.s. to some complex valued random variable $W^x(z)$, uniformly in z on any compact subset $K \subset \Omega_{\gamma_0}$. Moreover, we have a.s., for all $n \geqslant 0$,*

$$\sup_{z \in K} |W_n^x(z) - W^x(z)| \leqslant M\delta^n, \tag{4.15}$$

where M is a positive and finite random variable and $\delta \in (0, 1)$, and $W^x(z)$ is analytic on Ω_{γ_0}.

Proof. The basic ideas here are the same as those used in the proof of Theorem 2 in Biggins [8]. To prove the uniform convergence on a compact subset $K \subset \Omega_{\gamma_0}$, it suffices to show that for each $z_0 \in \Omega_{\gamma_0}$, the uniform convergence holds in a disc centred at z_0. Given any $z_0 \in \Omega_{\gamma_0}$, we can find $1 < \alpha \leqslant \min\{2, \gamma_0\}$ and a small η such that $B_{2\eta}(z_0) \subset \Omega^1_\alpha$ and

$$c_1 = \sup_{z \in B_{2\eta}(z_0)} \frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} < 1. \tag{4.16}$$

For any $N \geqslant n$, $W^x_{N+1}(z) - W^x_n(z)$ is analytic in z on $B_{2\eta}(z_0)$, so by [8, Lemme 3], we deduce that for all $n \geqslant 0$,

$$\begin{aligned} \sup_{N \geqslant n} \sup_{z \in B_\eta(z_0)} |W^x_{N+1}(z) - W^x_n(z)| &\leqslant \sum_{k=n}^{\infty} \sup_{z \in B_\eta(z_0)} |W^x_{k+1}(z) - W^x_k(z)| \\ &\leqslant \frac{1}{\pi} \int_0^{2\pi} \sum_{k=n}^{\infty} |W^x_{k+1}(z(t)) - W^x_k(z(t))| dt, \end{aligned} \tag{4.17}$$

where $z(t) = z_0 + 2\eta e^{it}$, $0 \leqslant t \leqslant 2\pi$. (This can be easily proved by Cauchy's formula.) Note that, by Fubini's theorem, for $n \geqslant 0$,

$$\mathbb{E} \int_0^{2\pi} \sum_{k=n}^{\infty} |W^x_{k+1}(z(t)) - W^x_k(z(t))| dt \leqslant 2\pi \sup_{z \in \partial B_{2\eta}(z_0)} \sum_{k=n}^{\infty} \mathbb{E}|W^x_{k+1}(z) - W^x_k(z)|, \tag{4.18}$$

where $\partial B_{2\eta}(z_0) = \{z \in \mathbb{C} : |z - z_0| = 2\eta\}$. Therefore, if the right hand side of (4.18) is finite for all $n \geqslant 0$, then the right-hand side of (4.17) goes to 0 a.s. as $n \to \infty$, so that a.s. the sequence $(W^x_n(z))$ converges uniformly on $B_\eta(z_0)$.

Now we prove that the right hand side of (4.18) is finite. Notice that

$$W^x_{k+1}(z) - W^x_k(z) = \sum_{u \in \mathbb{T}_k} \frac{e^{zS^x_u} r_z(X^x_u)}{[m\kappa(z)]^k r_z(x)} \Big(W_1^{X^x_u}(z) - 1 \Big). \tag{4.19}$$

Taking the α-th absolute moment at both sides of (4.19) conditional on $\mathscr{F}_k$ and applying Lemma 1 of Biggins [8], we obtain

$$\mathbb{E}_k |W^x_{k+1}(z) - W^x_k(z)|^\alpha \leqslant 2^\alpha \sum_{u \in \mathbb{T}_k} \left| \frac{e^{zS^x_u} r_z(X^x_u)}{[m\kappa(z)]^k r_z(x)} \right|^\alpha \mathbb{E}_k |W_1^{X^x_u}(z) - 1|^\alpha. \tag{4.20}$$

Since the function $z \mapsto r_z$ is analytic on $B_{\eta_1}(0)$ and $r_0 = 1$, there is a constant $d_3 > 0$ such that

$$\frac{\max_{x \in \mathcal{S}} |r_z(x)|}{\min_{x \in \mathcal{S}} |r_z(x)|} \leqslant d_3 \quad \text{for all} \quad z \in B_{\eta_1}(0). \tag{4.21}$$

Recall that s is the real part of z. Because $B_{3\eta}(z_0) \subset \Omega^1_\alpha \subset B_{\eta_2}(0) \subset B_{\frac{\eta_1}{2}}(0)$, we have $z, \frac{\alpha s}{2} \in B_{\frac{\eta_1}{2}}(0)$ for $z \in \partial B_{2\eta}(z_0)$. It follows from (4.21) that for all $z \in \partial B_{2\eta}(z_0)$,

$$\begin{aligned}\left|\frac{e^{zS^x_u} r_z(X^x_u)}{[m\kappa(z)]^k r_z(x)}\right|^\alpha &\leqslant \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha}\right)^k \frac{e^{\alpha s S^x_u} r_{\alpha s(X^x_u)}}{[m\kappa(\alpha s)]^k r_{\alpha s}(x)} \frac{|r_z(X^x_u)|^\alpha r_{\alpha s}(x)}{|r_z(x)|^\alpha r_{\alpha s}(X^x_u)} \\ &\leqslant d_3^{\alpha+1} \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha}\right)^k \frac{e^{\alpha s S^x_u} r_{\alpha s(X^x_u)}}{[m\kappa(\alpha s)]^k r_{\alpha s}(x)}. \qquad (4.22)\end{aligned}$$

On the other hand, from (4.9) and (4.21), we obtain the following estimation, for all $z \in \partial B_{2\eta}(z_0)$,

$$\begin{aligned}&\mathbb{E}_k|W_1^{X^x_u}(z) - 1|^\alpha \\ &\leqslant 2^{\alpha-1}\Big(\mathbb{E}_k|W_1^{X^x_u}(z)|^\alpha + 1\Big) = 2^{\alpha-1}\mathbb{E}_k\Big|\sum_{v\in\mathbb{T}_1(u)} \frac{e^{zS_v^{X^x_u}} r_z(X_v^{X^x_u})}{m\kappa(z) r_z(X^x_u)}\Big|^\alpha + 2^{\alpha-1} \\ &\leqslant 2^{\alpha-1}\left(\frac{\kappa(s)}{|\kappa(z)|}\right)^\alpha \mathbb{E}_k\Big[\sum_{v\in\mathbb{T}_1(u)} \frac{|r_z(X_v^{X^x_u})| r_s(X^x_u)}{|r_z(X^x_u)| r_s(X_v^{X^x_u})} \frac{e^{sS_v^{X^x_u}} r_s(X_v^{X^x_u})}{m\kappa(s) r_s(X^x_u)}\Big]^\alpha + 2^{\alpha-1} \\ &\leqslant d_3^{2\alpha} 2^{\alpha-1}\left(\frac{\kappa(s)}{|\kappa(z)|}\right)^\alpha \mathbb{E}\sup_{x\in\mathcal{S}}(W_1^x(s))^\alpha + 2^{\alpha-1}.\end{aligned}$$

Combining this with (4.20) and (4.22) gives, for all $z \in \partial B_{2\eta}(z_0)$,

$$\begin{aligned}&\mathbb{E}_k|W^x_{k+1}(z) - W^x_k(z)|^\alpha \\ &\leqslant c\Big(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha}\Big)^k W^x_n(\alpha s)\Big[\Big(\frac{\kappa(s)}{|\kappa(z)|}\Big)^\alpha \mathbb{E}\sup_{x\in\mathcal{S}}(W^x_1(s))^\alpha + 1\Big].\end{aligned}$$

Taking expectation at both sides of this inequality and using Jensen's inequality, we obtain for all $z \in \partial B_{2\eta}(z_0)$,

$$\mathbb{E}|W^x_{k+1}(z) - W^x_k(z)| \leqslant c^{\frac{1}{\alpha}}\left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha}\right)^{\frac{k}{\alpha}} \Big[\Big(\frac{\kappa(s)}{|\kappa(z)|}\Big)^\alpha \mathbb{E}\sup_{x\in\mathcal{S}}(W^x_1(s))^\alpha + 1\Big]^{\frac{1}{\alpha}}.$$

From (4.16), (4.10), the analyticity of $\kappa(z)$ on $\partial B_{2\eta}(z_0) \subset B_{\eta_1}(0)$ and the fact that $|\kappa(z)| > 0$ for all $z \in B_{\eta_1}(0)$, we obtain

$$\sup_{z\in\partial B_{2\eta}(z_0)} \mathbb{E}|W^x_{k+1}(z) - W^x_k(z)| \leqslant Cc_1^{\frac{k}{\alpha}}, \qquad (4.23)$$

This concludes that (4.18) is finite for all $n \geqslant 0$. We have therefore proved that it is a.s. that the sequence $(W^x_n(z))$ converges uniformly on $B_\eta(z_0)$ for each $z_0 \in \Omega_{\gamma_0}$, which implies the uniform convergence on each compact subset $K \subset \Omega_{\gamma_0}$.

We now come to the speed of convergence (4.15). Clearly, it is enough to prove that there is a $\delta \in (0,1)$ such that on each compact subset $K \subset \Omega_{\gamma_0}$,

$$\delta^{-n} \sup_{z\in K} |W_{n+1}^x(z) - W_n^x(z)| \overset{n\to\infty}{\to} 0 \quad \text{a.s.} \tag{4.24}$$

From (4.17), (4.18) and (4.23), we have for each $z_0 \in \Omega_{\gamma_0}$, there is $\eta > 0$ small enough such that for all $n \geqslant 0$,

$$\mathbb{E} \sup_{z\in B_\eta(z_0)} |W_{n+1}^x(z) - W_n^x(z)| \leqslant 2\sum_{k=n}^{\infty} C c_1^{\frac{k}{\alpha}},$$

where C and c_1 are constants which may depend on z_0. Since K is compact, by Borel's theorem, K can be covered by a finite number of open balls $B_{\eta_i}(z_i)$, $i = 1, \dots, n_0$, so that there exist two constants $C_1 > 0$ and $c_2 \in (0,1)$ which may depend on K, such that for $n \geqslant 0$,

$$\mathbb{E} \sup_{z\in K} |W_{n+1}^x(z) - W_n^x(z)| \leqslant 2\sum_{k=n}^{\infty} C_1 c_2^k \leqslant C_2 c_2^n. \tag{4.25}$$

Taking $\delta \in (c_2, 1)$ and using Fubini's theorem we see that

$$\mathbb{E} \sum_{n=0}^{\infty} \delta^{-n} \sup_{z\in K} |W_{n+1}^x(z) - W_n^x(z)| \leqslant C_2 \sum_{n=0}^{\infty} \big(\frac{c_2}{\delta}\big)^n < \infty,$$

so that

$$\sum_{n=0}^{\infty} \delta^{-n} \sup_{z\in K} |W_{n+1}^x(z) - W_n^x(z)| < \infty \quad \text{a.s.}$$

Therefore, (4.24) is proved. This ends the proof of (4.15).

Finally, since a.s. each $W_n^x(z)$ is analytic on Ω_{γ_0} and the sequence $(W_n^x(z))$ converges uniformly on each compact set of Ω_{γ_0}, a standard result of complex analysis (see e.g. Corollary 2.2.4 in Hörmander [23]) gives the analyticity of $W^x(z)$ on Ω_{γ_0}. $\square$

In the following we introduce a new martingale and prove its uniform convergence and the analyticity of its limit. This is an important ingredient in the proof of Theorem 2.3 about the Berry-Esseen bound for the changed measure $Z_{s,n}^x$, which is crucial in establishing the main results of this paper. For $z \in B_{\eta_1}(0),\ x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\beta$, set

$$\widetilde{W}_n^x(z) = \sum_{u\in\mathbb{T}_n} \frac{e^{zS_u^x} M_z(r_s\varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)}, \quad n \geqslant 0,$$

where M_z is defined in (3.2) and $(r_s\varphi)(X_u^x) := r_s(X_u^x)\varphi(X_u^x)$.

Theorem 4.3. *Assume conditions **C1**-**C3**. Then the sequence $(\widetilde{W}_n^x(z))_{n\geqslant 0}$ is a martingale with respect to the filtration $(\mathscr{F}_n)$ and converges a.s. to* *some complex valued random variable* $\widetilde{W}^x(z)$, *uniformly in* z *on any compact subset* $K \subset \Omega_{\gamma_0}$, *and the limit* $\widetilde{W}^x(z)$ *is analytic on* Ω_{γ_0}.

Proof. The fact that $(\widetilde{W}^x_n(z), \mathscr{F}_n)_{n\geqslant 0}$ is a martingale can be easily shown: it suffices to notice that

$$\mathbb{E}_n \widetilde{W}^x_{n+1}(z) = \sum_{u\in\mathbb{T}_n} \frac{e^{zS_u^x} M_z(r_s\varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)} \mathbb{E}_n \Big[\sum_{v\in\mathbb{T}_1(u)} \frac{e^{zS_v^{X_u^x}} M_z(r_s\varphi)(X_v^{X_u^x})}{m\kappa(z) M_z(r_s\varphi)(X_u^x)} \Big],$$

where $\mathbb{T}_1(u)$ represents the descendants of $u \in \mathbb{T}_n$ at time $n+1$. Moreover, by the branching property, the definition of P_z (3.1) and Lemma 3.1(1), we have for $u \in \mathbb{T}_n$,

$$\begin{aligned}\mathbb{E}_n \Big[\sum_{v\in\mathbb{T}_1(u)} \frac{e^{zS_v^{X_u^x}} M_z(r_s\varphi)(X_v^{X_u^x})}{m\kappa(z) M_z(r_s\varphi)(X_u^x)} \Big] &= \frac{\mathbb{E}_n \Big[|A_{u1} X_u^x|^z M_z(r_s\varphi)(X_{u1}^{X_u^x}) \Big]}{\kappa(z) M_z(r_s\varphi)(X_u^x)} \\ &= \frac{P_z(M_z(r_s\varphi))(X_u^x)}{\kappa(z) M_z(r_s\varphi)(X_u^x)} = 1.\end{aligned}$$

The proof of the uniform convergence and the analyticity of the limit is the same as in the proof of Theorem 4.2, whose details are omitted. □

5. Proof of Theorems 2.1 and 2.3

Theorem 2.1 is a particular case of Theorem 2.3 with $s = 0$. Thus we only prove Theorem 2.3. Our proof is based on Petrov's method [28] for the proof of the Cramér's moderate deviation asymptotic on sums of i.i.d. real random variables. We split the proof of Theorem 2.3 into two theorems: Theorems 5.1 and 5.2, whose combination gives Theorem 2.3.

Theorem 5.1. *Under the conditions of Theorem 2.3. Then, for any* $x \in \mathcal{S}$ *and* $\varphi \in \mathcal{B}_\beta$ *there exists a constant* $\eta \in (0, \eta_2)$ *such that a.s., for* $n \geqslant 1$,

$$\sup_{s\in(-\eta,\eta)} \Big| \sum_{u\in\mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x)\varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} - W^x(s)\pi_s(\varphi) \Big| \leqslant M\delta^n,$$

where M *is a positive and finite random variable and* $\delta \in (0, 1)$.

Theorem 5.2. *Under the conditions of Theorem 2.3. Then, for any* $x \in \mathcal{S}$ *and* $\varphi \in \mathcal{B}_\beta$ *there exists a constant* $\eta \in (0, \eta_2)$ *such that uniformly in* $s \in (-\eta, \eta)$ *and* $y \in \mathbb{R}$, *a.s., for* $n \geqslant 1$,

$$\Big| \sum_{u\in\mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x)\varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\Big\{ \frac{S_u^x - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant y \Big\}} - \sum_{u\in\mathbb{T}_n} \frac{e^{sS_u^x} r_s(X_u^x)\varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \Phi(y) \Big| \leqslant \frac{M}{\sqrt{n}},$$

where M *is a positive and finite random variable (independent of* s*).*

5.1. **Proof of Theorem 5.1.** The following decomposition which follows from the branching property will play a key role in our approach with a delicate choice of k for $0 < k \leqslant n$,

$$\sum_{u\in\mathbb{T}_n} \frac{e^{sS_u^x}(r_s\varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} = \sum_{u\in\mathbb{T}_k} \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^k r_s(x)} \sum_{v\in\mathbb{T}_{n-k}(u)} \frac{e^{sS_v^{X_u^x}}(r_s\varphi)(X_v^{X_u^x})}{[m\kappa(s)]^{n-k} r_s(X_u^x)}. \tag{5.1}$$

Recall that by our definition, for $u \in \mathbb{T}_k$, $\mathbb{T}_{n-k}(u)$ represents the descendants of u at time n.

For each n, we choose an integer $k_n = \lceil \frac{n}{2} \rceil$, which is the least integer greater than or equal to $\frac{n}{2}$. For brevity, we denote for $u \in \mathbb{T}_{k_n}$,

$$Y_{n-k_n}^u(s) = \sum_{v\in\mathbb{T}_{n-k_n}(u)} \frac{e^{sS_v^{X_u^x}}(r_s\varphi)(X_v^{X_u^x})}{[m\kappa(s)]^{n-k_n} r_s(X_u^x)}.$$

Then by (5.1), the following decomposition holds:

$$\sum_{u\in\mathbb{T}_n} \frac{e^{sS_u^x}(r_s\varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} - W^x(s)\pi_s(\varphi) = A_n(s) + B_n(s) + C_n(s), \tag{5.2}$$

where

$$\begin{aligned}
A_n(s) &= \sum_{u\in\mathbb{T}_{k_n}} \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} \Big[Y_{n-k_n}^u(s) - \mathbb{E}_{k_n} Y_{n-k_n}^u(s) \Big], \\
B_n(s) &= \sum_{u\in\mathbb{T}_{k_n}} \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} \Big[\mathbb{E}_{k_n} Y_{n-k_n}^u(s) - \pi_s(\varphi) \Big], \\
C_n(s) &= [W_{k_n}^x(s) - W^x(s)]\pi_s(\varphi).
\end{aligned}$$

By virtue of the decomposition (5.2), we shall divide the proof of Theorem 5.1 into three lemmas.

Lemma 5.3. *Under the conditions of Theorem 2.3, there exist two constants $\eta \in (0, \eta_2)$ and $\delta \in (0,1)$ such that*

$$\delta^{-n} \sup_{s\in(-\eta,\eta)} |A_n(s)| \overset{n\to\infty}{\to} 0, \quad a.s.$$

Proof. To prove Lemma 5.3, we will use the Borel-Cantelli Lemma. We can obtain the required result once we prove that there exist a small $\eta > 0$ and a constant $\delta \in (0,1)$ such that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(\delta^{-n} \sup_{s\in(-\eta,\eta)} |A_n(s)| > \varepsilon) < \infty. \tag{5.3}$$

By Markov's inequality,

$$\sum_{n=1}^{\infty} \mathbb{P}(\delta^{-n} \sup_{s\in(-\eta,\eta)} |A_n(s)| > \varepsilon) \leqslant \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \delta^{-n} \mathbb{E} \sup_{s\in(-\eta,\eta)} |A_n(s)|. \tag{5.4}$$

Because Ω_{γ_0} is an open set containing 0, we can find a small $\rho > 0$ such that $B_\rho(0) \subset \Omega^1_\alpha$ for some $1 < \alpha \leqslant \min\{2, \gamma_0\}$. Let $\eta \in (0, \frac{\rho}{3})$ whose value will be fixed later. Then $B_{3\eta}(0) \subset B_\rho(0)$. We see that for every $n \in \mathbb{N}$, the function

$$z \mapsto A_n(z) = \sum_{u\in\mathbb{T}_{k_n}} \frac{e^{zS_u^x} r_z(X_u^x)}{[m\kappa(z)]^{k_n} r_z(x)} \Big[Y^u_{n-k_n}(z) - \mathbb{E}_{k_n} Y^u_{n-k_n}(z) \Big]$$

is well-defined as an analytic function on $B_{\eta_1}(0)$. Recall that s is the real part of z. By Lemma 3 of Biggins [8], we have

$$\sup_{s\in(-\eta,\eta)} |A_n(s)| \leqslant \sup_{z\in B_\eta(0)} |A_n(z)| \leqslant \frac{1}{\pi} \int_0^{2\pi} |A_n(z(t))| dt,$$

where $z(t) = 2\eta e^{it},\ 0 \leqslant t \leqslant 2\pi$. Note that, by Fubini's theorem,

$$\mathbb{E} \sup_{s\in(-\eta,\eta)} |A_n(s)| \leqslant \int_0^{2\pi} \mathbb{E}|A_n(z(t))| dt \leqslant 2\pi \sup_{|z|=2\eta} \mathbb{E}|A_n(z)|. \tag{5.5}$$

Consider now $\mathbb{E}|A_n(z)|$ for $|z| = 2\eta$. Taking the α-th absolute moment of $A_n(z)$ conditional on $\mathscr{F}_k$ and applying Lemma 1 of Biggins [8], we obtain

$$\mathbb{E}_{k_n} |A_n(z)|^\alpha \leqslant 2^\alpha \sum_{u\in\mathbb{T}_{k_n}} \Big| \frac{e^{zS_u^x} r_z(X_u^x)}{[m\kappa(z)]^{k_n} r_z(x)} \Big|^\alpha \mathbb{E}_{k_n} |Y^u_{n-k_n}(z) - \mathbb{E}_{k_n} Y^u_{n-k_n}(z)|^\alpha. \tag{5.6}$$

Because $B_{3\eta}(0) \subset B_\rho(0) \subset B_{\eta_2}(0) \subset B_{\frac{\eta_1}{2}}(0)$, we see that if $|z| = 2\eta$, then $z, \frac{\alpha s}{2} \in B_{\frac{\eta_1}{2}}(0)$. Hence, by (4.21), we get for $|z| = 2\eta$,

$$\begin{aligned} \Big| \frac{e^{zS_u^x} (r_z\varphi)(X_u^x)}{[m\kappa(z)]^{k_n} r_z(x)} \Big|^\alpha &\leqslant \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^{k_n} \frac{e^{\alpha s S_u^x} r_{\alpha s}(X_u^x)}{[m\kappa(\alpha s)]^{k_n} r_{\alpha s}(x)} \frac{|r_z(X_u^x)|^\alpha r_{\alpha s}(x)}{|r_z(x)|^\alpha r_{\alpha s}(X_u^x)} \\ &\leqslant d_3^{1+\alpha} \left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha} \right)^{k_n} \frac{e^{\alpha s S_u^x} r_{\alpha s}(X_u^x)}{[m\kappa(\alpha s)]^{k_n} r_{\alpha s}(x)}. \end{aligned} \tag{5.7}$$

We now estimate the expectation in (5.6). Using $|a+b|^\alpha \leqslant 2^{\alpha-1}(|a|^\alpha+|b|^\alpha) \leqslant 2(|a|^\alpha + |b|^\alpha)$ and (4.21), we have for $|z| = 2\eta$,

$$\begin{aligned}
&\mathbb{E}_{k_n}|Y^u_{n-k_n}(z) - \mathbb{E}_{k_n} Y^u_{n-k_n}(z)|^\alpha \leqslant 2\mathbb{E}_{k_n}|Y^u_{n-k_n}(z)|^\alpha \\
&\leqslant 2\mathbb{E}_{k_n}\Bigg[\sum_{v\in\mathbb{T}_{n-k_n}(u)} \frac{e^{sS_v^{X_u^x}} r_s(X_v^{X_u^x})}{[m\kappa(s)]^{n-k_n} r_s(X_u^x)} \frac{|r_z\varphi(X_v^{X_u^x})| r_s(X_u^x)}{|r_z(X_u^x)| r_s(X_v^{X_u^x})}\Big(\frac{\kappa(s)}{|\kappa(z)|}\Big)^{n-k_n}\Bigg]^\alpha \\
&\leqslant 2(d_3^2\|\varphi\|_\beta)^\alpha \Big(\frac{\kappa(s)}{|\kappa(z)|}\Big)^{\alpha(n-k_n)} \mathbb{E}_{k_n}\Big[W^{X_u^x}_{n-k_n}(s)\Big]^\alpha \\
&\leqslant 2(d_3^2\|\varphi\|_\beta)^\alpha \Big(\frac{\kappa(s)}{|\kappa(z)|}\Big)^{\alpha(n-k_n)} \sup_{x\in\mathcal{S}} \mathbb{E}(W^x_*(s))^\alpha. \qquad (5.8)
\end{aligned}$$

From (5.6), (5.7) and (5.8), we have for all $\eta > 0$ small enough and $|z| = 2\eta$,

$$\mathbb{E}_{k_n}|A_n(z)|^\alpha \leqslant c\left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha}\right)^{k_n}\left(\frac{\kappa(s)}{|\kappa(z)|}\right)^{\alpha(n-k_n)} W^x_{k_n}(\alpha s) \sup_{x\in\mathcal{S}}\mathbb{E}(W^x_*(s))^\alpha. \tag{5.9}$$

Since $\alpha s \in (-\eta_1, \eta_1)$, $(W_n^x(\alpha s))$ is a martingale, so $\mathbb{E}[W_n^x(\alpha s)] = 1$. Taking expectations at both sides of (5.9), we obtain for $|z| = 2\eta$,

$$\mathbb{E}|A_n(z)|^\alpha \leqslant c\left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha}\right)^{k_n}\left(\frac{\kappa(s)}{|\kappa(z)|}\right)^{\alpha(n-k_n)} \sup_{x\in\mathcal{S}}\mathbb{E}(W^x_*(s))^\alpha. \tag{5.10}$$

From (5.5), Jensen's inequality and (5.10), we get that

$$\begin{aligned}
&\mathbb{E}\sup_{s\in(-\eta,\eta)}|A_n(s)| \\
&\leqslant c^{\frac{1}{\alpha}} \sup_{|z|=2\eta}\left\{\Big(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha}\Big)^{\frac{k_n}{\alpha}}\left|e^{(n-k_n)[\Lambda(s)-\Lambda(z)]}\right|\Big[\sup_{x\in\mathcal{S}}\mathbb{E}(W^x_*(s))^\alpha\Big]^{\frac{1}{\alpha}}\right\}. \qquad (5.11)
\end{aligned}$$

From the facts that $B_{3\eta}(0) \subset B_\rho(0) \subset \Omega^1_\alpha$ and the definition of Ω^1_α, we obtain

$$\sup_{|z|=2\eta}\left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha}\right)^{\frac{1}{\alpha}} \leqslant \sup_{z\in B_\rho(0)}\left(\frac{m\kappa(\alpha s)}{|m\kappa(z)|^\alpha}\right)^{\frac{1}{\alpha}} =: c_1 < 1. \tag{5.12}$$

From (5.11), (5.12) and the choice of k_n which implies that $k_n \geqslant n - k_n$, we get

$$\mathbb{E}\sup_{s\in(-\eta,\eta)}|A_n(s)| \leqslant cc_1^{n-k_n}\sup_{|z|=3\eta}\left\{\left|e^{(n-k_n)[\Lambda(s)-\Lambda(z)]}\right|\Big[\sup_{x\in\mathcal{S}}\mathbb{E}(W^x_*(s))^\alpha\Big]^{\frac{1}{\alpha}}\right\}. \tag{5.13}$$

By Theorem 4.1, for $\eta > 0$ small enough,

$$\sup_{s\in(-\eta,\eta)}\sup_{x\in\mathcal{S}}\mathbb{E}(W^x_*(s))^\alpha < \infty. \tag{5.14}$$

Note that $c_1 < 1$ is independent of η. Let $c_2 \in (1, \frac{1}{c_1})$. Since Λ is continuous on $B_{\eta_1}(0)$ and $\Lambda(0) = 0$, there exists a small $\eta_3 > 0$ such that

$$\sup_{z \in B_{\eta_3}(0)} \left| e^{[\Lambda(s) - \Lambda(z)]} \right| \leqslant c_2. \tag{5.15}$$

Take η small enough such that $\eta < \eta_3$. Since $k_n = \lceil \frac{n}{2} \rceil$, we have $n - k_n \geqslant \frac{n}{2} - 1$. So combining (5.13), (5.14), (5.15) we obtain for all $\eta > 0$ small enough,

$$\mathbb{E} \sup_{s \in (-\eta, \eta)} |A_n(s)| \leqslant c(c_1 c_2)^{n-k_n} \leqslant c(c_1 c_2)^{\frac{n}{2}-1}.$$

Therefore, using (5.4) and taking $\delta \in \left((c_1 c_2)^{\frac{1}{2}}, 1\right)$, we get that

$$\sum_{n=1}^{\infty} \mathbb{P}(\delta^{-n} \sup_{s \in (-\eta, \eta)} |A_n(s)| > \varepsilon) \leqslant \frac{c}{\varepsilon c_1 c_2} \sum_{n=1}^{\infty} \left(\frac{(c_1 c_2)^{\frac{1}{2}}}{\delta} \right)^n < \infty.$$

This completes the proof of Lemma 5.3. □

Lemma 5.4. *Under the conditions of Theorem 2.3, there exist two constants $\eta \in (0, \eta_2)$ and $\delta \in (0,1)$ such that*

$$\delta^{-n} \sup_{s \in (-\eta, \eta)} |B_n(s)| \overset{n \to \infty}{\to} 0 \quad a.s.$$

Proof. Using the branching property and the definition of $\mathbb{Q}_s^x$ (3.3), we have for $u \in \mathbb{T}_{k_n}$,

$$\begin{aligned}
\mathbb{E}_{k_n} Y^u_{n-k_n}(s) &= \mathbb{E}_{k_n} \sum_{v \in \mathbb{T}_{n-k_n}(u)} \frac{e^{s S_v^{X_u^x}} (r_s \varphi)(X_v^{X_u^x})}{[m\kappa(s)]^{n-k_n} r_s(X_u^x)} \\
&= \frac{\mathbb{E}_{k_n} \left[e^{s S_{n-k_n}^{X_u^x}} (r_s \varphi)(X_{n-k_n}^{X_u^x}) \right]}{\kappa^{n-k_n}(s) r_s(X_u^x)} \\
&= \mathbb{E}_{\mathbb{Q}_s^{X_u^x}} [\varphi(X_{n-k_n}^{X_u^x})].
\end{aligned}$$

Hence

$$\begin{aligned}
|B_n(s)| &\leqslant \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{s S_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} \sup_{x \in \mathcal{S}} |\mathbb{E}_{\mathbb{Q}_s^x} [\varphi(X_{n-k_n}^x)] - \pi_s(\varphi)| \\
&\leqslant W_{k_n}^x(s) \sup_{x \in \mathcal{S}} |\mathbb{E}_{\mathbb{Q}_s^x} [\varphi(X_{n-k_n}^x)] - \pi_s(\varphi)|.
\end{aligned}$$

By Theorem 4.2 and the bound (3.4), for $\eta \in (0, \eta_2)$, there exist a constant $c \in (0,1)$ and a positive finite random variable M such that for all $n \geqslant 0$,

$$\sup_{s \in (-\eta, \eta)} |B_n(s)| \leqslant M c^{n-k_n} \leqslant M c^{\frac{n}{2}-1}.$$

Therefore the conclusion of Lemma 5.4 holds for each $\delta \in (c^{\frac{1}{2}}, 1)$. □

Lemma 5.5. *Under the conditions of Theorem 2.3, there exist two constants $\eta \in (0, \eta_2)$ and $\delta \in (0,1)$ such that*

$$\delta^{-n} \sup_{s \in (-\eta, \eta)} |C_n(s)| \overset{n\to\infty}{\to} 0 \quad a.s.$$

Proof. This is an immediate consequence of Theorem 4.2 and the fact that $|\pi_s(\varphi)| \leqslant \|\varphi\|_\infty$.

□

5.2. **Proof of Theorem 5.2.** To prove Theorem 5.2, we need the following result.

Lemma 5.6. *Under the conditions of Theorem 5.2, there is a constant $\eta \in (0, \eta_2)$ such that*

$$\sup_{z \in B_\eta(0)} \left| \sum_{u \in \mathbb{T}_n} \frac{e^{zS_u^x}(r_s\varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)} - \widetilde{W}^x(z) \right| \overset{n\to\infty}{\to} 0 \quad a.s. \tag{5.16}$$

Moreover, $\sum_{u\in\mathbb{T}_n} \frac{e^{zS_u^x}(r_s\varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)}$ is a.s. bounded by a positive and finite random variable uniformly in $z \in B_\eta(0)$ and $n \geqslant 0$.

Proof. By the branching property, for $k \leqslant n$,

$$\sum_{u\in\mathbb{T}_n} \frac{e^{zS_u^x}(r_s\varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)} = \sum_{u\in\mathbb{T}_k} \frac{e^{zS_u^x} r_s(X_u^x)}{[m\kappa(z)]^k r_s(x)} \sum_{v\in\mathbb{T}_{n-k}(u)} \frac{e^{zS_v^{X_u^x}}(r_s\varphi)(X_v^{X_u^x})}{[m\kappa(z)]^{n-k} r_s(X_u^x)}. \tag{5.17}$$

As before, for each n, we take $k_n = \lceil \frac{n}{2} \rceil$. For brevity, we denote for $u \in \mathbb{T}_{k_n}$,

$$\tilde{Y}^u_{n-k_n}(z) = \sum_{v\in\mathbb{T}_{n-k_n}(u)} \frac{e^{zS_v^{X_u^x}}(r_s\varphi)(X_v^{X_u^x})}{[m\kappa(z)]^{n-k_n} r_s(X_u^x)}.$$

Then by (5.17), the following decomposition holds:

$$\sum_{u\in\mathbb{T}_n} \frac{e^{zS_u^x}(r_s\varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)} - \widetilde{W}^x(s) = A_n(z) + B_n(z) + C_n(z), \tag{5.18}$$

where

$$\begin{aligned}
A_n(z) &= \sum_{u\in\mathbb{T}_{k_n}} \frac{e^{zS_u^x} r_s(X_u^x)}{[m\kappa(z)]^{k_n} r_s(x)} \Big[\tilde{Y}^u_{n-k_n}(z) - \mathbb{E}_{k_n} \tilde{Y}^u_{n-k_n}(z) \Big], \\
B_n(z) &= \sum_{u\in\mathbb{T}_{k_n}} \frac{e^{zS_u^x} r_s(X_u^x)}{[m\kappa(z)]^{k_n} r_s(x)} \Big[\mathbb{E}_{k_n} \tilde{Y}^u_{n-k_n}(z) - \frac{M_z(r_s\varphi)(X_u^x)}{r_s(X_u^x)} \Big], \\
C_n(z) &= \widetilde{W}^x_{k_n}(z) - \widetilde{W}^x(z).
\end{aligned}$$

By virtue of the decomposition (5.18), in order to prove (5.16), it suffices to show that there is a constant $\eta \in (0, \eta_2)$ such that

$$\sup_{z \in B_\eta(0)} |A_n(z)| \overset{n\to\infty}{\to} 0, \quad \text{a.s.}, \tag{5.19}$$

$$\sup_{z \in B_\eta(0)} |B_n(z)| \overset{n\to\infty}{\to} 0, \quad \text{a.s.}, \tag{5.20}$$

$$\sup_{z \in B_\eta(0)} |C_n(z)| \overset{n\to\infty}{\to} 0, \quad \text{a.s.}. \tag{5.21}$$

The proof of (5.19) is similar to that of Lemma 5.3, and is omitted here. It is clear that (5.21) is an immediate consequence of Theorem 4.3. It remains to prove (5.20). By the branching property and the definition of the operator P_z (see (3.1)), we have

$$\begin{aligned}
\mathbb{E}_{k_n} \tilde{Y}^u_{n-k_n}(z) &= \mathbb{E}_{k_n} \sum_{v \in \mathbb{T}_{n-k_n}(u)} \frac{e^{z S_v^{X_u^x}} (r_s\varphi)(X_v^{X_u^x})}{[m\kappa(z)]^{n-k_n} r_s(X_u^x)} \\
&= \frac{\mathbb{E}_{k_n}\Big[e^{z S_{n-k_n}^{X_u^x}} (r_s\varphi)(X_{n-k_n}^{X_u^x})\Big]}{\kappa^{n-k_n}(z) r_s(X_u^x)} \\
&= \frac{P_z^{n-k_n}(r_s\varphi)(X_u^x)}{\kappa^{n-k_n}(z) r_s(X_u^x)}.
\end{aligned}$$

Hence, by the decomposition (3.2) and Lemma 3.1(4), for any $z \in B_{\eta_1}(0)$, we have

$$\begin{aligned}
|B_n(z)| &\leqslant \left| \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{z S_u^x} r_s(X_u^x)}{[m\kappa(z)]^{k_n} r_s(x)} \left[\frac{P_z^{n-k_n}(r_s\varphi)(X_u^x)}{\kappa^{n-k_n}(z) r_s(X_u^x)} - \frac{M_z(r_s\varphi)(X_u^x)}{r_s(X_u^x)} \right] \right| \\
&\leqslant \sum_{u \in \mathbb{T}_{k_n}} \left| \frac{e^{z S_u^x} r_s(X_u^x)}{[m\kappa(z)]^{k_n} r_s(x)} \frac{L_z^{n-k_n}(r_s\varphi)(X_u^x)}{\kappa^{n-k_n}(z) r_s(X_u^x)} \right| \\
&\leqslant \frac{\|L_z^{n-k_n}\|_{\mathcal{B}_\beta \to \mathcal{B}_\beta}}{|k(z)|^{n-k_n}} \sum_{u \in \mathbb{T}_{k_n}} \frac{e^{s S_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} \Big(\frac{\kappa(s)}{|\kappa(z)|} \Big)^{k_n} \frac{\|r_s\varphi\|_\beta}{\min_{y \in \mathcal{S}} r_s(y)} \\
&\leqslant c \left(\frac{1-a_2}{1-a_1} \right)^{n-k_n} \left| e^{k_n[\Lambda(s) - \Lambda(z)]} \right| W_{k_n}^x(s),
\end{aligned} \tag{5.22}$$

where $0 < a_1 < a_2 < 1$ is defined in Lemma 3.1(4). In the last step we use the fact that $\|r_s\varphi\|_\beta \leqslant 3\|r_s\|_\beta \|\varphi\|_\beta \leqslant c$ and that the map $s \mapsto r_s$ is analytic with $r_0 = 1$. Since $k_n = \lceil \frac{n}{2} \rceil$, we have $n - k_n \geqslant \frac{n}{2} - 1 \geqslant k_n - 2$, so $\left(\frac{1-a_2}{1-a_1}\right)^{n-k_n} \leqslant \left(\frac{1-a_2}{1-a_1}\right)^{\frac{n}{2}-1} \leqslant \left(\frac{1-a_2}{1-a_1}\right)^{k_n-2}$. Let $c_1 \in \left(1, \frac{1-a_1}{1-a_2}\right)$. Using the facts that the function Λ is continuous on $B_{\eta_1}(0)$ and $\Lambda(0) = 0$, there exist a small $\eta \in (0, \eta_1)$ such that

$$\sup_{z \in B_\eta(0)} \left| e^{[\Lambda(s) - \Lambda(z)]} \right| \leqslant c_1. \tag{5.23}$$

By Theorem 4.2, for $\eta \in (0, \eta_2)$ small enough, $\sup_{s \in (-\eta,\eta)} W^x_{k_n}(s) \leqslant M$, where M is a positive and finite random variable. This together with (5.22) and (5.23) implies that for $\eta \in (0, \eta_2)$ small enough,

$$\sup_{z \in B_\eta(0)} |B_n(z)| \leqslant c_2 M \Big[\frac{c_1(1-a_2)}{1-a_1}\Big]^{k_n} \overset{n\to\infty}{\to} 0 \quad \text{a.s.}$$

This completes the proof of (5.20). So the proof of (5.16) is finished.

The uniform bound of $\sum_{u \in \mathbb{T}_n} \frac{e^{zS^x_u}(r_s\varphi)(X^x_u)}{[m\kappa(z)]^n r_s(x)}$ is an immediate consequence of (5.16) and the fact that $\widetilde{W}^x(z)$ is analytic in z (by Theorem 4.3). $\square$

Proof of Theorem 5.2. For simplicity, we suppose that $\varphi \geqslant 0$; otherwise we can consider the positive and negative parts of φ to conclude. Consider the distribution functions of finite measures:

$$F_{s,n}(y) = \sum_{u \in \mathbb{T}_n} \frac{e^{sS^x_u}(r_s\varphi)(X^x_u)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\left\{\frac{S^x_u - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant y\right\}}, \quad y \in \mathbb{R},$$

$$H_{s,n}(y) = \sum_{u \in \mathbb{T}_n} \frac{e^{sS^x_u}(r_s\varphi)(X^x_u)}{[m\kappa(s)]^n r_s(x)} \Phi(y), \quad y \in \mathbb{R},$$

and their characteristic functions at $-t$:

$$f_{s,n}(t) = \int_{\mathbb{R}} e^{-ity} dF_{s,n}(y), \quad h_{s,n}(t) = \int_{\mathbb{R}} e^{-ity} dH_{s,n}(y), \quad t \in \mathbb{R}.$$

By straightforward calculations we have

$$h_{s,n}(t) = \sum_{u \in \mathbb{T}_n} \frac{e^{sS^x_u}(r_s\varphi)(X^x_u)}{[m\kappa(s)]^n r_s(x)} e^{-\frac{t^2}{2}} \tag{5.24}$$

$$\begin{aligned} f_{s,n}(t) &= \sum_{u \in \mathbb{T}_n} \frac{e^{sS^x_u}(r_s\varphi)(X^x_u)}{[m\kappa(s)]^n r_s(x)} e^{-it\frac{S^x_u - n\Lambda'(s)}{\sigma_s\sqrt{n}}} \\ &= \sum_{u \in \mathbb{T}_n} \frac{e^{(s - \frac{it}{\sigma_s\sqrt{n}})S^x_u}(r_s\varphi)(X^x_u)}{[m\kappa(s - \frac{it}{\sigma_s\sqrt{n}})]^n r_s(x)} \left(\frac{\kappa(s - \frac{it}{\sigma_s\sqrt{n}})}{\kappa(s)}\right)^n e^{\frac{itn\Lambda'(s)}{\sigma_s\sqrt{n}}} \\ &= \sum_{u \in \mathbb{T}_n} \frac{e^{(s - \frac{it}{\sigma_s\sqrt{n}})S^x_u}(r_s\varphi)(X^x_u)}{[m\kappa(s - \frac{it}{\sigma_s\sqrt{n}})]^n r_s(x)} \lambda^n_{s, \frac{-it}{\sigma_s\sqrt{n}}}, \end{aligned} \tag{5.25}$$

where the last equality holds by the definition of $\lambda_{s,it}$ (see (3.7)).

Notice that $F_{s,n}(-\infty) = H_{s,n}(-\infty) = 0$, $F_{s,n}(+\infty) = H_{s,n}(+\infty) = \sum_{u \in \mathbb{T}_n} \frac{e^{sS^x_u}(r_s\varphi)(X^x_u)}{[m\kappa(s)]^n r_s(x)}$, $F_{s,n}$ and $H_{s,n}$ are non-decreasing on $\mathbb{R}$, and $H_{s,n}$ is differentiable on $\mathbb{R}$. So by Esseen's smoothing inequality (see [28, Theorem V.2.2.]), for all $T > 0$ and $s \in (-\eta_1, \eta_1)$,

$$\sup_{y\in\mathbb{R}} |F_{s,n}(y) - H_{s,n}(y)| \leqslant \frac{1}{\pi} \int_{-T}^{T} \left| \frac{f_{s,n}(t) - h_{s,n}(t)}{t} \right| dt + \frac{c_0}{T} \sum_{u\in\mathbb{T}_n} \frac{e^{sS_u^x}(r_s\varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)},$$

where c_0 is a positive constant. Therefore, to prove Theorem 5.2, it suffices to show that there exists a small $\eta \in (0, \eta_2)$ such that as $n \to \infty$, a.s.,

$$\sup_{s\in(-\eta,\eta)} \int_{-T}^{T} \left| \frac{f_{s,n}(t) - h_{s,n}(t)}{t} \right| dt = O\Big(\frac{1}{\sqrt{n}}\Big), \tag{5.26}$$

and

$$\sup_{s\in(-\eta,\eta)} \frac{c_0}{T} \sum_{u\in\mathbb{T}_n} \frac{e^{sS_u^x}(r_s\varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} = O\Big(\frac{1}{\sqrt{n}}\Big). \tag{5.27}$$

In the following, we denote by M_i a positive and finite random variable. Let $T := \eta\underline{\sigma}\sqrt{n}$ with $\eta > 0$ small enough such that the conclusion in Lemma 5.6 holds, where $\underline{\sigma} := \inf_{s\in(-\eta,\eta)} \sigma_s > 0$. By Lemma 5.6, we have

$$\sup_{s\in(-\eta,\eta)} \sum_{u\in\mathbb{T}_n} \frac{e^{sS_u^x}(r_s\varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} \leqslant M_1.$$

Hence (5.27) is proved since

$$\sup_{s\in(-\eta,\eta)} \frac{c_0}{T} \sum_{u\in\mathbb{T}_n} \frac{e^{sS_u^x}(r_s\varphi)(X_u^x)}{[m\kappa(s)]^n r_s(x)} \leqslant \frac{c_0 M_1}{\eta\underline{\sigma}\sqrt{n}}.$$

It remains to prove (5.26). We will prove this by showing that there exists a small $\eta \in (0, \eta_2)$ such that as $n \to \infty$, a.s.,

$$I_1(n) + I_2(n) = O\Big(\frac{1}{\sqrt{n}}\Big), \tag{5.28}$$

where

$$I_1(n) = \sup_{s\in(-\eta,\eta)} \int_{|t|<\delta_1\underline{\sigma}\sqrt{n}} \left| \frac{f_{s,n}(t) - h_{s,n}(t)}{t} \right| dt,$$

$$I_2(n) = \sup_{s\in(-\eta,\eta)} \int_{\delta_1\underline{\sigma}\sqrt{n}\leqslant|t|\leqslant\eta\underline{\sigma}\sqrt{n}} \left| \frac{f_{s,n}(t) - h_{s,n}(t)}{t} \right| dt,$$

with $\delta_1 \in (0, \eta)$ whose value will be fixed later.

Control of $I_1(n)$. Denote for $z = s + it$ with $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$,

$$U_n(z) = \sum_{u\in\mathbb{T}_n} \frac{e^{zS_u^x}(r_s\varphi)(X_u^x)}{[m\kappa(z)]^n r_s(x)}.$$

With this notation and using (5.24) and (5.25), we have

$$I_1(n) \leqslant I_{11}(n) + I_{12}(n),$$

where

$$I_{11}(n) = \sup_{s\in(-\eta,\eta)} \int_{|t|<\delta_1\underline{\sigma}\sqrt{n}} \left| \frac{\lambda^n_{s,\frac{-it}{\sigma_s\sqrt{n}}}}{t} \Big(U_n(s - \frac{it}{\sigma_s\sqrt{n}}) - U_n(s) \Big) \right| dt$$

$$I_{12}(n) = \sup_{s\in(-\eta,\eta)} \int_{|t|<\delta_1\underline{\sigma}\sqrt{n}} \left| \frac{\Big(\lambda^n_{s,\frac{-it}{\sigma_s\sqrt{n}}} - e^{-\frac{t^2}{2}}\Big) U_n(s)}{t} \right| dt.$$

For $I_{11}(n)$, by Taylor's formula and the fact that $\Lambda''(s) = \sigma_s^2$, we have

$$\begin{aligned} \lambda^n_{s,\frac{-it}{\sigma_s\sqrt{n}}} &= e^{n[\Lambda(s-\frac{it}{\sigma_s\sqrt{n}})-\Lambda(s)+\Lambda'(s)\frac{it}{\sigma_s\sqrt{n}}]} \\ &= e^{n\sum_{k=2}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}(\frac{-it}{\sigma_s\sqrt{n}})^k} \\ &= e^{-\frac{t^2}{2}} e^{n\sum_{k=3}^{\infty}\frac{\Lambda^{(k)}(s)}{k!}(\frac{-it}{\sigma_s\sqrt{n}})^k}. \end{aligned} \tag{5.29}$$

By choosing δ_1 small enough, we have for all $s \in (-\eta, \eta)$ and $|t| < \delta_1\underline{\sigma}\sqrt{n}$,

$$\Big| \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \Big(\frac{-it}{\sigma_s\sqrt{n}}\Big)^k \Big| \leqslant \frac{t^2}{4n}, \tag{5.30}$$

and so, from (5.29),

$$\Big|\lambda^n_{s,\frac{-it}{\sigma_s\sqrt{n}}}\Big| \leqslant e^{-\frac{t^2}{4}}. \tag{5.31}$$

Therefore, for η and δ_1 small enough,

$$I_{11}(n) \leqslant \sup_{s\in(-\eta,\eta)} \int_{|t|<\delta_1\underline{\sigma}\sqrt{n}} \frac{e^{-\frac{t^2}{4}}}{|t|} \left| U_n\Big(s - \frac{it}{\sigma_s\sqrt{n}}\Big) - U_n(s) \right| dt. \tag{5.32}$$

By Lemma 5.6, there is a constant η_4 small enough such that for all $n \geqslant 0$,

$$\sup_{|z|=\frac{\eta_4}{2}} |U_n(z)| \leqslant M_2. \tag{5.33}$$

Notice that U_n is a.s. analytic on $B_{\eta_1}(0)$. Let $\eta, \delta_1 > 0$ be small enough such that $\eta + i\delta_1 \in B_{\frac{\eta_4}{3}}(0)$. By the mean value theorem, for $s \in (-\eta, \eta)$ and

$\frac{t}{\sigma_s\sqrt{n}} \in (-\delta_1, \delta_1)$, we have

$$\begin{aligned}\Big|U_n(s - \frac{it}{\sigma_s\sqrt{n}}) - U_n(s)\Big| &\leqslant \frac{|t|}{\sigma_s\sqrt{n}} \sup_{t\in(-\delta_1,\delta_1)} \Big|U_n'(s - \frac{it}{\sigma_s\sqrt{n}})\Big| \\ &\leqslant \frac{|t|}{\sigma_s\sqrt{n}} \sup_{z\in B_{\frac{\eta_4}{3}}(0)} |U_n'(z)|. \end{aligned} \tag{5.34}$$

By the Cauchy's formula, when $z \in B_{\frac{\eta_4}{2}}(0)$,

$$U_n'(z) = \frac{1}{2\pi i}\int_{|w|=\frac{\eta_4}{2}} \frac{U_n(w)}{(w-z)^2}dw.$$

Hence, by (5.33) and the fact that $|w-z| \geqslant \frac{\eta_4}{6}$ for $z \in B_{\frac{\eta_4}{3}}(0)$ and $|w| = \frac{\eta_4}{2}$, we have

$$\sup_{z\in B_{\frac{\eta_4}{3}}(0)} |U_n'(z)| \leqslant \frac{18M_2}{\eta_4}.$$

Combining this with (5.32), (5.34) and the fact that $\sigma_s > \underline{\sigma}$ for all $s \in (-\eta,\eta)$, we obtain

$$I_{11}(n) \leqslant \sup_{s\in(-\eta,\eta)} \frac{18M_2}{\sigma_s\sqrt{n}\eta_4}\int_{|t|<\delta_1\underline{\sigma}\sqrt{n}} e^{-\frac{t^2}{4}}dt \leqslant \frac{M_3}{\sqrt{n}}. \tag{5.35}$$

For $I_{12}(n)$, using (5.29), the inequality $|e^z - 1| \leqslant |z|e^{|z|}$ for all $z \in \mathbb{C}$ and (5.30), we obtain

$$\begin{aligned}&\Big|\lambda^n_{s,\frac{-it}{\sigma_s\sqrt{n}}} - e^{-\frac{t^2}{2}}\Big| \leqslant e^{-\frac{t^2}{2}}\Big|e^{n\sum_{k=3}^\infty \frac{\Lambda^{(k)}(s)}{k!}(\frac{-it}{\sigma_s\sqrt{n}})^k} - 1\Big| \\ &\leqslant e^{-\frac{t^2}{2}+\left|n\sum_{k=3}^\infty \frac{\Lambda^{(k)}(s)}{k!}\left(\frac{-it}{\sigma_s\sqrt{n}}\right)^k\right|}\left|n\sum_{k=3}^\infty \frac{\Lambda^{(k)}(s)}{k!}\Big(\frac{-it}{\sigma_s\sqrt{n}}\Big)^k\right| \\ &\leqslant e^{-\frac{t^2}{4}}\left|n\sum_{k=3}^\infty \frac{\Lambda^{(k)}(s)}{k!}\Big(\frac{-it}{\sigma_s\sqrt{n}}\Big)^k\right|. \end{aligned} \tag{5.36}$$

By choosing δ_1 small enough, we have for all $s \in (-\eta,\eta)$ and $|t| < \delta_1\underline{\sigma}\sqrt{n}$,

$$\left|n\sum_{k=3}^\infty \frac{\Lambda^{(k)}(s)}{k!}\Big(\frac{-it}{\sigma_s\sqrt{n}}\Big)^k\right| \leqslant C\Big(\frac{|t|^3}{\sqrt{n}}\Big). \tag{5.37}$$

From (5.36) and (5.37), we have for all $s \in (-\eta,\eta)$ and $|t| < \delta_1\underline{\sigma}\sqrt{n}$,

$$\left|\frac{\lambda^n_{s,\frac{-it}{\sigma_s\sqrt{n}}} - e^{-\frac{t^2}{2}}}{t}\right| \leqslant \frac{C}{\sqrt{n}}t^2e^{-\frac{t^2}{4}}. \tag{5.38}$$

By Lemma 5.6, $U_n(s)$ is a.s. bounded uniformly in $s \in (-\eta, \eta)$. This together with (5.38) and the fact that $\int_0^\infty t^2 e^{-\frac{t^2}{4}} dt < \infty$ implies that

$$I_{12}(n) \leqslant \frac{C \sup_{s\in(-\eta,\eta)} |U_n(s)|}{\sqrt{n}} \int_0^\infty t^2 e^{-\frac{t^2}{4}} dt \leqslant \frac{M_4}{\sqrt{n}}. \tag{5.39}$$

Putting together (5.35) and (5.39), we get $I_1(n) = O\Big(\frac{1}{\sqrt{n}}\Big)$.

Control of $I_2(n)$. Using the constraint $|t| \geqslant \delta_1 \underline{\sigma} \sqrt{n}$, we have

$$\begin{aligned} I_2(n) &\leqslant \frac{1}{\delta_1 \underline{\sigma}\sqrt{n}} \sup_{s\in(-\eta,\eta)} \int_{\delta_1\underline{\sigma}\sqrt{n} \leqslant |t| \leqslant \eta\underline{\sigma}\sqrt{n}} |f_{s,n}(t) - h_{s,n}(t)| dt \\ &\leqslant \frac{1}{\delta_1 \underline{\sigma}\sqrt{n}} \sup_{s\in(-\eta,\eta)} \int_{\delta_1\underline{\sigma}\sqrt{n} \leqslant |t| \leqslant \eta\underline{\sigma}\sqrt{n}} (|f_{s,n}(t)| + |h_{s,n}(t)|) dt \end{aligned} \tag{5.40}$$

By (5.24) and Lemma 5.6, for $\eta > 0$ small enough and $\delta_1 \underline{\sigma}\sqrt{n} \leqslant |t| \leqslant \eta \underline{\sigma}\sqrt{n}$,

$$\sup_{s\in(-\eta,\eta)} |h_{s,n}(t)| \leqslant e^{-\frac{t^2}{2}} \sup_{s\in(-\eta,\eta)} |U_n(s)| \leqslant M_5 e^{-\frac{t^2}{2}}.$$

This implies that

$$\frac{1}{\delta_1 \underline{\sigma}\sqrt{n}} \sup_{s\in(-\eta,\eta)} \int_{\delta_1\underline{\sigma}\sqrt{n} \leqslant |t| \leqslant \eta\underline{\sigma}\sqrt{n}} |h_{s,n}(t)| dt \leqslant \frac{M_6}{\sqrt{n}}.$$

Hence, from (5.40), to prove that $I_2(n) = O\Big(\frac{1}{\sqrt{n}}\Big)$, it remains to show that there exist a small $\eta \in (0, \eta_2)$ such that as $n \to \infty$, a.s.,

$$\frac{1}{\delta_1 \underline{\sigma}\sqrt{n}} \sup_{s\in(-\eta,\eta)} \int_{\delta_1\underline{\sigma}\sqrt{n} \leqslant |t| \leqslant \eta\underline{\sigma}\sqrt{n}} |f_{s,n}(t)| dt = O\Big(\frac{1}{\sqrt{n}}\Big). \tag{5.41}$$

By the branching property, we have the following decomposition: for $n \geqslant 0$ and $k_n = \lceil \frac{n}{2} \rceil$,

$$f_{s,n}(t) = A_{s,n}(t) + B_{s,n}(t), \tag{5.42}$$

where

$$A_{s,n}(t) = \sum_{u\in\mathbb{T}_{k_n}} \frac{e^{(s - \frac{it}{\sigma_s\sqrt{n}})S_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} e^{\frac{itk_n\Lambda'(s)}{\sigma_s\sqrt{n}}} \Big[\hat{Y}^u_{s,n-k_n}(t) - \mathbb{E}_{k_n} \hat{Y}^u_{s,n-k_n}(t)\Big],$$

$$B_{s,n}(t) = \sum_{u\in\mathbb{T}_{k_n}} \frac{e^{(s - \frac{it}{\sigma_s\sqrt{n}})S_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)} e^{\frac{itk_n\Lambda'(s)}{\sigma_s\sqrt{n}}} \mathbb{E}_{k_n} \hat{Y}^u_{s,n-k_n}(t),$$

with

$$\hat{Y}^u_{s,n-k_n}(t) = \sum_{v\in\mathbb{T}_{n-k_n}(u)} \frac{e^{sS_v^{X_u^x}} (r_s\varphi)(X_v^{X_u^x})}{[m\kappa(s)]^{n-k_n} r_s(X_u^x)} e^{\frac{-it}{\sigma_s\sqrt{n}}[S_v^{X_u^x} - (n-k_n)\Lambda'(s)]}.$$

For $A_{s,n}(t)$, using the same argument as in the proof of Lemma 5.3, we can prove that for $\eta > 0$ small enough, there exists $\delta \in (0,1)$ such that

$$\sup_{s\in(-\eta,\eta)} \sup_{\delta_1\underline{\sigma}\sqrt{n}\leqslant|t|\leqslant\eta\underline{\sigma}\sqrt{n}} \delta^{-n}|A_{s,n}(t)| \overset{n\to\infty}{\to} 0 \quad \text{a.s.}$$

Therefore,

$$\sup_{s\in(-\eta,\eta)} \sup_{\delta_1\underline{\sigma}\sqrt{n}\leqslant|t|\leqslant\eta\underline{\sigma}\sqrt{n}} |A_{s,n}(t)| \leqslant M_{11}\delta^n. \tag{5.43}$$

For $B_{s,n}(t)$, using the branching property and the definitions of $\mathbb{Q}_s^x$ (see (3.3)) and $R_{s,it}$ (see (3.5)), we have for $u \in \mathbb{T}_{k_n}$,

$$\begin{aligned}
\mathbb{E}_{k_n}\hat{Y}^u_{s,n-k_n}(t) &= \mathbb{E}_{k_n} \sum_{v\in\mathbb{T}_{n-k_n}(u)} \frac{e^{sS_v^{X_u^x}}(r_s\varphi)(X_v^{X_u^x})}{[m\kappa(s)]^{n-k_n}r_s(X_u^x)} e^{\frac{-it}{\sigma_s\sqrt{n}}[S_v^{X_u^x}-(n-k_n)\Lambda'(s)]} \\
&= \frac{\mathbb{E}_{k_n}\left[e^{sS^{X_u^x}_{n-k_n}}(r_s\varphi)(X^{X_u^x}_{n-k_n})e^{\frac{-it}{\sigma_s\sqrt{n}}[S^{X_u^x}_{n-k_n}-(n-k_n)\Lambda'(s)]}\right]}{[\kappa(s)]^{n-k_n}r_s(X_u^x)} \\
&= \mathbb{E}_{\mathbb{Q}_s^{X_u^x}}\left[\varphi(X^{X_u^x}_{n-k_n})e^{\frac{-it}{\sigma_s\sqrt{n}}[S^{X_u^x}_{n-k_n}-(n-k_n)\Lambda'(s)]}\right] \\
&= R^{n-k_n}_{s,\frac{-it}{\sigma_s\sqrt{n}}}\varphi(X_u^x).
\end{aligned}$$

Therefore, by (3.6) and Theorem 4.2, there is a constant $a \in (0,1)$ such that for $k_n = \lceil \frac{n}{2} \rceil$,

$$\sup_{s\in(-\eta,\eta)} \sup_{\delta_1\underline{\sigma}\sqrt{n}\leqslant|t|\leqslant\eta\underline{\sigma}\sqrt{n}} |B_{s,n}(t)| \leqslant \|\varphi\|_\beta a^{n-k_n} \sup_{s\in(-\eta,\eta)} W^x_{k_n}(s) \leqslant M_7 a^{\frac{n}{2}-1}. \tag{5.44}$$

From (5.42), (5.43) and (5.44), we obtain for $c_1 = \max\{\delta, a^{\frac{1}{2}}\}$,

$$\sup_{s\in(-\eta,\eta)} \sup_{\delta_1\underline{\sigma}\sqrt{n}\leqslant|t|\leqslant\eta\underline{\sigma}\sqrt{n}} |f_{s,n}(t)| \leqslant M_8 c_1^n.$$

Thus

$$\frac{1}{\delta_1\underline{\sigma}\sqrt{n}} \sup_{s\in(-\eta,\eta)} \int_{\delta_1\underline{\sigma}\sqrt{n}\leqslant|t|\leqslant\eta\underline{\sigma}\sqrt{n}} |f_{s,n}(t)|dt \leqslant \frac{2(\eta-\delta_1)M_8c_1^n}{\delta_1},$$

which implies (5.41). This concludes that $I_2(n) = O\Big(\frac{1}{\sqrt{n}}\Big)$, which ends the proof of (5.28) and (5.26). So Theorem 5.2 is proved.

$\square$

6. Proof of Theorem 2.2

For $y \in [0,1]$, Theorem 2.2 is a direct consequence of Theorem 2.1, as we will see in the following. For $n \geqslant 1$,

$$\begin{aligned}
&\Big|\frac{\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)\mathbb{1}_{\{S_u^x-n\gamma\geqslant\sqrt{n}\sigma y\}}}{m^nW[1-\Phi(y)]e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}}-\nu(\varphi)\Big| \\
&=\frac{1}{W[1-\Phi(y)]e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}}\Big|\frac{1}{m^n}\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)-\frac{1}{m^n}\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)\mathbb{1}_{\{\frac{S_u^x-n\gamma}{\sigma\sqrt{n}}\leqslant y\}} \\
&\qquad\qquad -W\nu(\varphi)(1-\Phi(y))e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big|. \qquad (6.1)
\end{aligned}$$

Since $\sup_{y\in[0,1]}|\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})|\to 0$, there exists n_0 large enough such that for all $y\in[0,1]$ and $n\geqslant n_0$, $e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\geqslant 1/2$. Using this and the fact that $1-\Phi(y)\geqslant c:=1-\Phi(1)$ for all $y\in[0,1]$, from (6.1) we get for all $n\geqslant n_0$,

$$\begin{aligned}
&\Big|\frac{\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)\mathbb{1}_{\{S_u^x-n\gamma\geqslant\sqrt{n}\sigma y\}}}{m^nW[1-\Phi(y)]e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}}-\nu(\varphi)\Big| \\
&\leqslant\frac{2}{cW}\Big|\frac{1}{m^n}\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)-W\nu(\varphi)\Big| \\
&\quad+\frac{2}{cW}\Big|-\frac{1}{m^n}\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)\mathbb{1}_{\{\frac{S_u^x-n\gamma}{\sigma\sqrt{n}}\leqslant y\}}+W\nu(\varphi)\Phi(y)\Big| \\
&\quad+\frac{2}{cW}\Big|W\nu(\varphi)(1-\Phi(y))\Big(1-e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big)\Big|. \qquad (6.2)
\end{aligned}$$

In the last display, by Theorem 2.1, when $n\to\infty$, the two first terms are $O\Big(\frac{1}{\sqrt{n}}\Big)$. We will show below that the third term is also $O\Big(\frac{1}{\sqrt{n}}\Big)$. In fact, using the inequality $|1-e^t|\leqslant|t|e^t$ for $t\in\mathbb{R}$ and the fact that $\sup_{y\in[0,1]}|\zeta(\frac{y}{\sqrt{n}})|$ is bounded for $n\geqslant n_0$, we obtain for $y\in[0,1]$, as $n\to\infty$,

$$\Big|1-e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big|\leqslant\Big|\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})\Big|e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}=O\Big(\frac{1}{\sqrt{n}}\Big).$$

Since $|\nu(\varphi)|\leqslant\|\varphi\|_\infty$, this implies that the third term in (6.2) is $O\Big(\frac{1}{\sqrt{n}}\Big)$.

From (6.2) and the above estimations, we see that for $y\in[0,1]$, as $n\to\infty$,

$$\Big|\frac{\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)\mathbb{1}_{\{S_u^x-n\gamma\geqslant\sqrt{n}\sigma y\}}}{m^nW[1-\Phi(y)]e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}}-\nu(\varphi)\Big|=O\Big(\frac{1}{\sqrt{n}}\Big),$$

which implies

$$\frac{\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)\mathbb{1}_{\{S_u^x-n\gamma\geqslant\sqrt{n}\sigma y\}}}{m^nW[1-\Phi(y)]}=e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}\Big[\nu(\varphi)+O\Big(\frac{1}{\sqrt{n}}\Big)\Big].$$

We now deal with the case $1<y=o(\sqrt{n})$. We can suppose that $\varphi\geqslant 0$ by considering the positive and negative parts of φ. We will focus on the proof of (2.12), as the proof of (2.13) is similar. For $u\in(\mathbb{N}^*)^n$, set

$$V_u^x=\frac{S_u^x-n\Lambda'(s)}{\sigma_s\sqrt{n}}.$$

Then we have

$$\begin{aligned} I &:= \frac{1}{m^n}\sum_{u\in\mathbb{T}_n}\varphi(X_u^x)\mathbb{1}_{\{S_u^x-n\gamma>y\sigma\sqrt{n}\}} \\ &= r_s(x)\frac{e^{-ns\Lambda'(s)}}{\kappa^{-n}(s)}\sum_{u\in\mathbb{T}_n}e^{-s\sigma_s\sqrt{n}V_u^x}\frac{e^{sS_u^x}\varphi(X_u^x)}{[m\kappa(s)]^nr_s(x)}\mathbb{1}_{\{V_u^x>\frac{\sigma y}{\sigma_s}+\frac{\sqrt{n}[\gamma-\Lambda'(s)]}{\sigma_s}\}}. \end{aligned} \tag{6.3}$$

Because $\Lambda(s)$ is analytic on $(-\eta_1,\eta_1)$ with $\Lambda(0)=0$, it has the Taylor expansion

$$\Lambda(s)=\sum_{k=1}^{\infty}\frac{\gamma_k}{k!}s^k,\quad \text{where}\quad \gamma_k=\Lambda^{(k)}(0),\ s\in(-\eta_1,\eta_1), \tag{6.4}$$

which implies that

$$\Lambda'(s)-\gamma=\sum_{k=2}^{\infty}\frac{\gamma_k}{(k-1)!}s^{k-1}. \tag{6.5}$$

Consider the equation

$$\sqrt{n}[\Lambda'(s)-\gamma]=\sigma y. \tag{6.6}$$

Set $t=\frac{y}{\sqrt{n}}$. Using (6.5), we get

$$\sigma t=\sum_{k=2}^{\infty}\frac{\gamma_k}{(k-1)!}s^{k-1}. \tag{6.7}$$

Since $\gamma_2=\sigma^2>0$, the equation (6.7) has the unique solution given by

$$s=\frac{t}{\gamma_2^{1/2}}-\frac{\gamma_3}{2\gamma_2^2}t^2-\frac{\gamma_4\gamma_2-3\gamma_3^2}{6\gamma_2^{7/2}}t^3+\dots, \tag{6.8}$$

which converges for $|t|$ small enough (see [28, Theorem VIII.2.2] for details). From (6.4) and (6.5), we see that

$$s\Lambda'(s)-\Lambda(s)=\sum_{k=2}^{\infty}\frac{k-1}{k!}\gamma_k s^k.$$

Choosing s given by (6.8), we obtain

$$s\Lambda'(s) - \Lambda(s) = \frac{t^2}{2} - t^3\zeta(t) = \frac{y^2}{2n} - \frac{y^3}{n^{3/2}}\zeta\left(\frac{x}{\sqrt{n}}\right), \tag{6.9}$$

where ζ is the Cramér series defined in (2.5), which converges for $|t|$ small enough (see [28, Theorem VIII.2.2] for details). Coming back to the expression of I (cf. (6.3)), using (6.9) together with (6.6) and the fact that $\frac{e^{-ns\Lambda'(s)}}{\kappa^{-n}(s)} = e^{-n[s\Lambda'(s)-\Lambda(s)]}$, we have

$$\begin{aligned} I &= r_s(x) e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} \sum_{u\in\mathbb{T}_n} e^{-s\sigma_s\sqrt{n}V_u^x} \frac{e^{sS_u^x}\varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\{V_u^x>0\}} \\ &= r_s(x) e^{-\frac{y^2}{2}+\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} \int_0^\infty e^{-s\sigma_s\sqrt{n}y} \overline{Z}_{s,n}^x(dy), \end{aligned} \tag{6.10}$$

where $\overline{Z}_{s,n}^x$ is the finite measure on $\mathbb{R}$ defined by:

$$\overline{Z}_{s,n}^x(B_2) = \sum_{u\in\mathbb{T}_n} \frac{e^{sS_u^x}\varphi(X_u^x)}{[m\kappa(s)]^n r_s(x)} \mathbb{1}_{\{V_u^x\in B_2\}}, \quad B_2\subset\mathbb{R}.$$

Its mass satisfies $\mathbb{E}[\overline{Z}_{s,n}^x(\mathbb{R})] \leqslant \left\|\frac{\varphi}{r_s}\right\|_\infty$.

Since $t = \frac{y}{\sqrt{n}} \to 0$ as $n\to\infty$, by (6.8) we have $s\to 0^+$ as $n\to\infty$. Hence, for sufficiently large n_0 and all $n \geqslant n_0$, we have $s\in(0,\eta)$ where η is defined in Theorem 2.3. Therefore, denoting

$$l_{n,s}(y) = \overline{Z}_{s,n}^x((-\infty,y]) - W^x(s)\pi_s(\frac{\varphi}{r_s})\Phi(y), \quad y\in\mathbb{R},$$

we get from Theorem 2.3 that for all $n\geqslant n_0$,

$$\sup_{y\in\mathbb{R}} |l_{n,s}(y)| \leqslant \frac{M}{\sqrt{n}}, \tag{6.11}$$

where M is a positive and finite random variable independent of n and s. In the following, we write M_i for a positive and finite random variable. Notice that

$$\begin{aligned} &\int_0^\infty e^{-s\sigma_s\sqrt{n}y}\overline{Z}_{s,n}^x(dy) \\ &= \int_0^\infty e^{-s\sigma_s\sqrt{n}y} dl_{n,s}(y) + \frac{W^x(s)\pi_s(\varphi r_s^{-1})}{\sqrt{2\pi}} \int_0^\infty e^{-s\sigma_s\sqrt{n}y-\frac{y^2}{2}} dy \\ &=: I_1 + W^x(s)\pi_s(\varphi r_s^{-1}) I_2. \end{aligned} \tag{6.12}$$

Estimate of I_1. Using the integration by parts and (6.11), we get for $n\geqslant n_0$,

$$|I_1| \leqslant |l_{n,s}(0)| + s\sigma_s\sqrt{n}\int_0^\infty e^{-s\sigma_s\sqrt{n}y}|l_{n,s}(y)|dy \leqslant \frac{2M}{\sqrt{n}}. \tag{6.13}$$

Estimate of I_2. The integral I_2 appears in the proof of Cramér's large deviation expansion theorem for sums of i.i.d random variables (see [28, Theorem VIII.2.2]), where the following results have been proved:

(i) there exist some positive constants c_1, c_2 such that for all $s \in (-\eta, \eta)$ and n large enough,

$$c_1 \leqslant s\sigma_s \sqrt{n} I_2 \leqslant c_2;$$

(ii) the integral I_2 admits the following asymptotic expansion:

$$I_2 = e^{\frac{y^2}{2}}[1 - \Phi(y)]\Big[1 + O\left(\frac{y}{\sqrt{n}}\right)\Big]. \tag{6.14}$$

By the definition of σ_s, the mapping $s \mapsto \sigma_s$ is strictly positive and continuous on $(-\eta, \eta)$. Hence, there exist constants $c_3, c_4 > 0$ such that

$$c_3 \leqslant s\sqrt{n} I_2 \leqslant c_4. \tag{6.15}$$

Notice that for all $s \in (-\eta, \eta), W^x(s) > 0$ a.s. Moreover, $W^x(s)$ is a.s. continuous in $(-\eta, \eta)$ by the continuity and uniform convergence of $W_n^x(s)$ on $(-\eta, \eta)$. Combining this with (6.15), we get

$$M_3 \leqslant s\sqrt{n} W^x(s) I_2 \leqslant M_4. \tag{6.16}$$

We now come back to (6.12), and let s be defined by (6.8). Recall that for $n \geqslant n_0, s \in (0, \eta)$. From (6.12),(6.13) and (6.16), we have, as $n \to \infty$,

$$\begin{aligned}
\int_0^\infty e^{-s\sigma_s\sqrt{n}y} \overline{Z}_{s,n}^x(dy) &= W^x(s) I_2 \Big[\pi_s(\varphi r_s^{-1}) + \frac{I_1}{W^x(s) I_2}\Big] \\
&= W^x(s) I_2 \Big[\pi_s(\varphi r_s^{-1}) + \frac{s\sqrt{n} I_1}{s\sqrt{n} W^x(s) I_2}\Big] \\
&= W^x(s) I_2 \Big[\pi_s(\varphi r_s^{-1}) + O(s)\Big].
\end{aligned}$$

Substituting this into (6.10) and using (6.14), we obtain

$$I = r_s(x) W^x(s) e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}[1 - \Phi(y)]\Big[1 + O\left(\frac{y}{\sqrt{n}}\right)\Big]\Big[\pi_s(\varphi r_s^{-1}) + O(s)\Big]. \tag{6.17}$$

According to Theorem 4.2, $W^x(s)$ is analytic on $(-\eta, \eta)$ and using the mean theorem we see that $|W^x(s) - W^x| = |W^x(s) - W^x(0)| \leqslant M_5 s$. On the other hand, by [31, Lemma 6.1], we have $\|r_s - 1\|_\infty \leqslant cs$ and $|\pi_s(\varphi r_s^{-1}) - \nu(\varphi)| = |\frac{\nu_s(\varphi)}{\nu_s(r_s)} - \nu(\varphi)| \leqslant cs\|\varphi\|_\beta$. Since $s = O\Big(\frac{y}{\sqrt{n}}\Big)$ by (6.8), it follows from (6.17) that

$$I = \Big[1 + O(s)\Big]\Big[W^x + O(s)\Big]e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}[1 - \Phi(y)]\Big[1 + O\left(\frac{y}{\sqrt{n}}\right)\Big]\Big[\nu(\varphi) + O(s)\Big]$$

$$= W^x e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}[1 - \Phi(y)]\Big[\nu(\varphi) + O\Big(\frac{y}{\sqrt{n}}\Big)\Big],$$

which concludes the proof of (2.12).

The proof of (2.13) can be carried out in a similar way as that of (2.12). The only difference is that, instead of using (6.6), we consider the equation

$$\sqrt{n}[\Lambda'(s) - \Lambda'(0)] = -\sigma y,$$

where $1 < y = o(\sqrt{n})$ and $s \in (-\eta, 0)$. Since the rest of the argument is the same as that in the proof of (2.12), we omit the details.

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