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Cramér type moderate deviation expansion for a supercritical multi-type branching process in a random environment

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Abstract: Let $Z_n^i = (Z_n^i(1), \dots, Z_n^i(d))$, $n \geq 0$, be a supercritical d -type branching process in an independent and identically distributed random environment $\xi = (\xi_0, \xi_1, \dots)$, starting with one initial particle of type i . We establish a Cramér type moderate deviation expansion for $\log \|Z_n^i\|$. To this end, we first prove uniform results for the existence of harmonic moments and Berry-Esseen type bound under suitably changed measure, using the spectral gap theory on products of random matrices and the fundamental martingale that we found in an earlier paper.

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1. Introduction

Let $Z_n = (Z_n(1), \dots, Z_n(d))$, $n \geq 0$, be a d -type branching process ($d \geq 1$) in an independent and identically distributed (i.i.d.) random environment $\xi = (\xi_0, \xi_1, \dots)$. Denote by M_n be the $d \times d$ random matrix whose components are

$$M_n(i, j) = \mathbb{E}_\xi[Z_{n+1}(j) \mid Z_n = e_i], \quad 1 \leq i, j \leq d,$$

where \mathbb{E}_ξ is the conditional expectation given the environment ξ , e_i is the vector with component 1 in the i -th place and 0 elsewhere. So $M_n(i, j)$ is the conditional mean of the number of particles of type j produced by a particle of type i of n -th generation, given the environment. We define the Lyapunov exponent of the sequence (M_n) as

$$\gamma := \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\|,$$

where $M_{0,n-1} = M_0 \cdots M_{n-1}$ is the product matrix, $\|M_{0,n-1}\|$ denotes its operator norm with respect to the L^1 -vectorial norm (see (2.2) and (2.1)).

The study of the asymptotic behaviour of the branching process (Z_n) is a complex problem which attracts a lot of attention during the last decades. Concerning the critical case $\gamma = 0$ and subcritical case $\gamma < 0$, see for example the works of Peigné, Le Page and Pham [16], Vatutin and Dyakonova [22], and Vatutin and Wachtel [23], who studied the convergence rate of the survival probability of the branching process. For the supercritical case $\gamma > 0$, in [11], [12] and [13] we established asymptotic properties of Z_n such as Kesten-Stigum type theorem, L^p convergence, harmonic moments and Berry-Esseen type theorem.

In this paper, we continue to consider the supercritical case $\gamma > 0$, for which we will give more results on the asymptotic behaviour of (Z_n) . In the sequel, we always assume $\gamma > 0$. Denote by (Z_n^i) the branching process (Z_n) which starts with one initial particle of type i , that is when $Z_0 = e_i$. In [11], under suitable conditions, we established a strong law of large numbers for $\log \|Z_n^i\|$: on the explosion event $\{\|Z_n^i\| \rightarrow +\infty\}$, it holds that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|Z_n^i\| = \gamma \quad \text{a.s.} \quad (1.1)$$

Then, under additional assumptions, we proved in [13] a central limit theorem (CLT) for $\log \|Z_n^i\|$: there exists $\sigma \geq 0$ such that for each $1 \leq i \leq d$,

$$\frac{\log \|Z_n^i\| - n\gamma}{\sqrt{n}} \rightarrow \mathcal{N}(0, \sigma^2) \quad \text{in law,} \quad (1.2)$$

where $\mathcal{N}(0, \sigma^2)$ denotes the normal law with mean 0 and variance σ^2 . We have also established in [13], under further moment conditions, a Berry-Esseen type theorem for $\log \|Z_n^i\|$, which gives the rate of convergence in the CLT: we showed that for all $n \geq 1$,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{\sqrt{n}}, \quad (1.3)$$

where $\sigma^2 = \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}[(\log \|M_{0,n-1}^T x\| - n\gamma)^2]$ is the asymptotic variance independent of x , $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ is the standard normal distribution function, and $C > 0$ is a constant.

The objective of this paper is to establish a Cramér type moderate deviation expansion for $\log \|Z_n^i\|$. We will prove (cf. Theorem 2.1) that uniformly in $0 \leq x \leq o(\sqrt{n})$, as $n \rightarrow +\infty$,

$$\frac{\mathbb{P} \left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x \right)}{1 - \Phi(x)} = e^{\frac{x^3}{\sqrt{n}} \zeta(\frac{x}{\sqrt{n}})} \left[1 + O \left(\frac{1+x}{\sqrt{n}} \right) \right], \quad (1.4)$$

where ζ is the Cramér series (for the precise definition see (2.12)). Notice that a version of this result has been proved by Grama, Liu and Miqueu in [10, Theorem 1.1] for the single type case $d = 1$. The expansion (1.4) is totally new for $d \geq 2$.

Now we explain briefly our approach for the proof of (1.4). For any $s \in (-\eta, \eta)$ with $\eta > 0$ small, we define a transfer operator P_s (see (2.10)) naturally occurring

in the products of random matrices. Using the spectral theory on P_s (see e.g. Buraczewski, Damek, Guivarc'h and Mentemeier [3], Guivarc'h and Le Page [9], Xiao, Grama and Liu [24]), we define the changed measure $\mathbb{P}_s^{e_i}$ (see section 3) for the branching process (Z_n^i) . To prove (1.4), we extend the Berry-Esseen bound (1.3) for the changed measure $\mathbb{P}_s^{e_i}$ uniformly in $s \in (-\eta, \eta)$ (cf. Theorem 4.1 in Section 4): for all $n \geq 1$ and $x \in \mathbb{R}$,

$$\sup_{s \in (-\eta, \eta)} \left| \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{\sqrt{n}}, \quad (1.5)$$

where $\Lambda(s) = \log \kappa(s)$, $\kappa(s)$ is the spectral radius of P_s , $\sigma_s^2 = \Lambda''(s)$, and $C > 0$ is a constant. Then, we combine (1.5) with the standard techniques from Petrov [20] to obtain (1.4).

The Berry-Esseen bound (1.5) plays an important role in our approach. Let us explain its proof. Our method is an adaptation of the arguments that we used in the proof of (1.3) in [13]. The fundamental martingale (W_n^i) associated to the process (Z_n^i) , defined in [11], will play a central role. For each $n, k \geq 0$, let $\rho_{n, n+k}$ be the spectral radius of the product matrix $M_{n, n+k} = M_n \cdots M_{n+k}$. It is well known that, by the Perron-Frobenius theorem, $\rho_{n, n+k}$ is an eigenvalue of $M_{n, n+k}$, and there exists $U_{n, n+k}$ a non negative eigenvector associated to $\rho_{n, n+k}$, with $\|U_{n, n+k}\| = 1$. Using the results of Hennion [14, Lemma 3.3 and Theorem 1], under suitable conditions, for each $n \geq 0$ the limit

$$U_{n, \infty} := \lim_{k \rightarrow \infty} U_{n, n+k} \quad (1.6)$$

exists a.s., with $U_{n, \infty} > 0$ (for a vector or matrix U we write $U > 0$ to mean that each of the components of U is strictly positive) and $\|U_{n, \infty}\| = 1$; in addition, it holds that

$$M_n U_{n+1, \infty} = \lambda_n U_{n, \infty}, \quad (1.7)$$

where $\lambda_n, n \geq 0$ are positive random scalars called the pseudo-spectral radii of the random matrices (M_n) . Iterating (1.7), we get

$$M_{n, n+k} U_{n+k+1, \infty} = \lambda_{n, n+k} U_{n, \infty}, \quad n, k \geq 0, \quad (1.8)$$

where $\lambda_{0, n} := \lambda_0 \cdots \lambda_n$. Then, the martingale (W_n^i) is defined by (see [11]):

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, U_{n, \infty} \rangle}{\lambda_{0, n-1} U_{0, \infty}(i)}, \quad n \geq 1. \quad (1.9)$$

We get from (1.9) the two following relations which make the link between $\log \|Z_n^i\|$ and $\log \|M_{0, n-1}(i, \cdot)\|$:

$$\log \|Z_n^i\| \leq \log \|M_{0, n-1}(i, \cdot)\| + \log W_n^i - \min_{1 \leq j \leq d} \log U_{n, \infty}(j), \quad (1.10)$$

$$\log \|Z_n^i\| \geq \log \|M_{0, n-1}(i, \cdot)\| + \log W_n^i + \min_{1 \leq j \leq d} \log U_{n, \infty}(j). \quad (1.11)$$

We recall that $(U_{n,\infty})$ is a stationary sequence of random variables, and since (W_n^i) is a non-negative martingale, the limit $W^i = \lim_{n \rightarrow +\infty} W_n^i$ exists a.s.. It follows that, when W^i is non degenerate, the terms $\log W_n^i$ and $\log U_{n,\infty}(j)$ in (1.10) and (1.11) will be negligible in the limit properties that we consider. More precisely, we will use the Berry-Esseen bound under the changed measure $\mathbb{P}_s^{e_i}$ proved in Xiao, Grama and Liu [24] to control $\log \|M_{0,n-1}(i, \cdot)\|$; then, by giving a tight control of the quantities $\log W_n^i$ and $\log U_{n,\infty}(j)$ under $\mathbb{P}_s^{e_i}$, we will obtain (1.5) from the two inequalities (1.10) and (1.11). For the term $\log W_n^i$, we will establish a sufficient condition for the existence of harmonic moments of the limit W^i under $\mathbb{P}_s^{e_i}$ uniformly in $s \in (-\eta, \eta)$: we will show that there exists $a > 0$ such that for all $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} (W^i)^{-a} < +\infty, \quad (1.12)$$

(cf. Theorem 3.3). The proof of (1.12) is one of the key arguments to prove (1.5) and (1.4).

The rest of the paper is organized as follows. In Section 2, we introduce some necessary notation and we formulate the Cramér type moderate deviation expansion for $\log \|Z_n^i\|$. We define in Section 3 the changed measure $\mathbb{P}_s^{e_i}$ and we study under $\mathbb{P}_s^{e_i}$ the harmonic moments of W^i . Section 4 is devoted to proof of the Berry-Esseen type theorem for $\log \|Z_n^i\|$ under $\mathbb{P}_s^{e_i}$. Finally we prove in Section 5 the Cramér type moderate deviation expansion for $\log \|Z_n^i\|$.

2. Notation, preliminaries and main results

For $d \geq 1$, let $\mathcal{M}_d(\mathbb{R})$ be the set of $d \times d$ matrices with entries in \mathbb{R} . The d -dimensional space of vectors \mathbb{R}^d will be equipped with the scalar product and the L^1 -norm respectively defined by

$$\langle x, y \rangle := \sum_{i=1}^d x(i) y(i) \quad \text{and} \quad \|x\| := \sum_{i=1}^d |x(i)|, \quad x, y \in \mathbb{R}^d. \quad (2.1)$$

We equip the space $\mathcal{M}_d(\mathbb{R})$ with the operator norm with respect to the L^1 vectoriel norm:

$$\|M\| := \sup_{x \in \mathcal{S}} \|Mx\|, \quad (2.2)$$

where $\mathcal{S} = \{x \in \mathbb{R}^d : x \geq 0, \|x\| = 1\}$ is the intersection of the unit sphere with the positive quadrant. Let $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$ be the vector with all coordinates equal to 0, and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$ the vector with all coordinates equal to 1. Let $\mathbb{N} = \{0, 1, \dots\}$ be the set of non-negative integers. Set $\mathbb{1}_A$ for the indicator function of an event A . We denote by c, C strictly positive constants which may differ from line to line.

Now we define precisely a multi-type branching process in a random environment (MBPRE). The random environment $\xi = (\xi_n)_{n \geq 0}$ is an independent and identically

distributed (i.i.d.) sequence of random variables with values in an abstract space \mathbb{X} ; each realization of ξ_n is associated to d probability generating functions

$$f_n^r(s) = \sum_{k_1, \dots, k_d=0}^{\infty} p_{k_1, \dots, k_d}^r(\xi_n) s_1^{k_1} \cdots s_d^{k_d}, \quad s = (s_1, \dots, s_d) \in [0, 1]^d,$$

$1 \leq r \leq d$. A MBPRE $Z_n = (Z_n(1), \dots, Z_n(d))$, $n \geq 0$ in the random environment ξ is a sequence of random vectors in \mathbb{N}^d such that

$$Z_0 \in \mathbb{N}^d \text{ is fixed, and } Z_{n+1} = \sum_{r=1}^d \sum_{l=1}^{Z_n(r)} N_{l,n}^r \text{ for } n \geq 0, \quad (2.3)$$

where, given the environment ξ , $N_{l,n}^r = (N_{l,n}^r(1), \dots, N_{l,n}^r(d))$ indexed by $l \geq 1$, $n \geq 0$ and $1 \leq r \leq d$ are independent random vectors with probability generating function f_n^r . The random variable $N_{l,n}^r(j)$ represents the offspring of type j at time $n+1$ of the l -th particle of type r in generation n , and $Z_n(j)$ denotes the number of particles of type j in generation n . As explained in the introduction, when $Z_0 = e_i$, we write Z_n^i for Z_n , i.e. (Z_n^i) is the MBPRE which starts with one initial particle of type i .

Given the environment ξ , the underlying probability will be denoted by \mathbb{P}_ξ , which is called the quenched law. Denote by τ the law of the environment ξ . The total probability \mathbb{P} , called annealed law, can be defined as $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$. The expectation with respect to \mathbb{P}_ξ and \mathbb{P} are denoted respectively by \mathbb{E}_ξ and \mathbb{E} . With our notation,

$$f_n^r(s) = \mathbb{E}_\xi \left(\prod_{j=1}^d s_j^{N_{l,n}^r(j)} \right), \quad s = (s_1, \dots, s_d) \in [0, 1]^d,$$

are the quenched probability generating function of $N_{l,n}^r$, which represent the offspring distributions for particles of generation n . For all $n \geq 0$, the mean matrix M_n can be expressed in terms of $f_n = (f_n^1, \dots, f_n^d)$:

$$M_n(i, j) = \frac{\partial f_n^i}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi [Z_{n+1}(j) | Z_n = e_i], \quad 1 \leq i, j \leq d,$$

where $\frac{\partial f}{\partial s_j}(\mathbf{1})$ denotes the left derivative at $\mathbf{1}$ of a d -dimensional probability generating function f with respect to s_j . M_n is the matrix of means of the offspring distributions in the sense that $M_n(i, j)$ represents the conditioned mean of the number of children of type j produced by a particle of type i at time n . The hypothesis that the environment (ξ_n) is i.i.d. implies that $(M_n)_{n \geq 0}$ is a sequence of i.i.d. random matrices. We will use the products of these matrices:

$$M_{k,n} := M_k \cdots M_n, \quad 0 \leq k \leq n.$$

Notice that

$$\mathbb{E}_\xi Z_{n+1}^i(j) = M_{0,n}(i, j), \quad n \geq 0, 1 \leq i, j \leq d. \quad (2.4)$$

Throughout the paper, we assume that the matrix M_0 satisfies the first moment condition

$$\mathbb{E} \log^+ \|M_0\| < +\infty. \quad (2.5)$$

When (2.5) holds, the limit

$$\gamma := \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\|$$

exists and is called the Lyapunov exponent of the sequence of matrices $(M_n)_{n \geq 0}$; moreover, a strong law of large numbers has been established in [8]:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|M_{0,n-1}\| = \gamma \quad \mathbb{P}\text{-a.s.}$$

The Lyapunov exponent γ permits a classification of MBPRE's (see e.g. [11]): a MBPRE is subcritical if $\gamma < 0$, critical if $\gamma = 0$, and supercritical if $\gamma > 0$. We always consider the supercritical case, which means $\gamma > 0$.

The goal of the present paper is to establish a Cramér type moderate deviation expansion for $\log \|Z_n^i\|$ in the supercritical case. The asymptotic behaviour of the MBPRE (Z_n^i) , when it is supercritical, is determined by that of the product of random matrices $M_{0,n-1}$ and the fundamental martingale (W_n^i) that we mentioned in the introduction. Set $\rho_{n,n+k}$ the spectral radius of $M_{n,n+k}$. By the Perron-Frobenius theorem (see e.g. [1]), we know that $\rho_{n,n+k}$ is a positive eigenvalue of $M_{n,n+k}$, and there exist positive right and left eigenvectors $U_{n,n+k}$ and $V_{n,n+k}$ associated to $\rho_{n,n+k}$ with the normalizations $\|U_{n,n+k}\| = 1$ and $\langle V_{n,n+k}, U_{n,n+k} \rangle = 1$. Let \mathcal{G}_+^0 be the set of matrices whose entries are strictly positive. Assuming that M_0 is a.s. allowable in the sense that every row and column contains a strictly positive element, and that

$$\mathbb{P} \left(\bigcup_{n \geq 0} \{M_{0,n} \in \mathcal{G}_+^0\} \right) > 0, \quad (2.6)$$

Hennion [14, Lemma 3.3 and Theorem 1] proved that the random vectors $U_{n,\infty}$ and the random scalars λ_n defined by (1.6) and (1.7) exist. It is easily seen that the relation (1.8) holds and that the sequences $(U_{n,\infty})$ and (λ_n) are stationary and ergodic. Under the same conditions, we proved in [11, Theorem 1] that the sequence (W_n^i) defined by (1.9) is a non-negative martingale under the probability measures \mathbb{P}_ξ and \mathbb{P} , w.r.t. the filtration

$$\mathcal{F}_0 = \sigma(\xi), \quad \mathcal{F}_n = \sigma(\xi, N_{l,k}^r(j), 0 \leq k \leq n-1, 1 \leq r, j \leq d, l \geq 1) \text{ for } n \geq 1.$$

Define $W^i := \lim_{n \rightarrow +\infty} W_n^i$, the a.s. limit of the martingale (W_n^i) .

Under the supercritical condition $\gamma > 0$, we proved in [11, Theorem 2.6 and Corollary 2.8] that the condition

$$\mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)} \right) < +\infty \quad \forall 1 \leq i, j \leq d \quad (2.7)$$

is sufficient for the non-degeneracy of each W^i in the sense that $\mathbb{P}(W^i > 0) > 0$, with

$$\mathbb{E}_\xi W^i = 1 \quad \text{and} \quad \mathbb{P}_\xi(W^i > 0) = \mathbb{P}_\xi(\|Z_n^i\| \xrightarrow[n \rightarrow +\infty]{} +\infty) = 1 - q^i(\xi) > 0 \text{ a.s.}, \quad (2.8)$$

where $q^i(\xi)$ is the probability of extinction of the process (Z_n^i) .

Now we introduce some conditions to formulate the Cramér type moderate deviation expansion for $\log \|Z_n^i\|$. For $n \geq 0$, define the vector $p_0(\xi_n)$ whose components are

$$p_0(\xi_n)(i) := f_n^i(\mathbf{0}) = \mathbb{P}_\xi(\|Z_1^i\| = 0), \quad 1 \leq i \leq d.$$

Throughout the paper, we will assume that each individual of the population gives birth to at least one child :

H1. *The vector $p_0(\xi_0) = (f_0^1(\mathbf{0}), \dots, f_0^d(\mathbf{0}))$ satisfies*

$$p_0 = \mathbf{0} \quad \mathbb{P}\text{-a.s.} \quad (2.9)$$

Notice that when **H1** and (2.7) hold, we have $q^i(\xi) = 0$ a.s. and $\|Z_n^i\| \rightarrow +\infty$ a.s. as $n \rightarrow +\infty$ by (2.8). For all $n \geq 0$ and $p > 1$ denote by

$$\theta_n(p) := \max_{1 \leq i, j \leq d} \mathbb{E}_\xi \left| \frac{N_{1,n}^i(j)}{M_n(i,j)} - 1 \right|^p.$$

We need the following moment conditions on the offspring distributions :

H2. *There exist two constants $p \in (1, 2]$ and $\eta_0 \in (0, 1)$ such that*

$$\mathbb{E}\|M_0\|^{\eta_0} < +\infty, \quad \max_{1 \leq i, j \leq d} \mathbb{E}M_0(i, j)^{-\eta_0} < +\infty \quad \text{and} \quad \mathbb{E}\theta_0(p)^{\eta_0} < +\infty.$$

Clearly, **H2** implies that M_0 is a.s. allowable and (2.6) holds. We proved in [13, Lemma 3.11] that **H2** implies (2.7). Therefore when $\gamma > 0$ and **H2** holds, each W^i is non-degenerate. By [24, Proposition 3.14], we know that under the condition **H2**, the asymptotic variance

$$\sigma^2 = \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}[(\log \|M_{0,n-1}^T x\| - n\gamma)^2]$$

exists uniformly in $x \in \mathcal{S}$, with $0 \leq \sigma^2 < +\infty$. We will need the assumption

H3. *The asymptotic variance σ^2 satisfies*

$$\sigma^2 > 0.$$

For $x \in \mathcal{S}$ and $M \in \mathcal{G}_+^0$, define the projective action of M on \mathcal{S} by $M \cdot x := \frac{Mx}{\|Mx\|}$. Denote by μ the law of M_0 , and $\Gamma_\mu = [\text{supp } \mu]$ the semi-group generated by the support of μ . Under **H2**, each $M \in \Gamma_\mu$ is strictly positive, hence by the Perron-Frobenius theorem the spectral radius ρ_M of M is the unique eigenvalue

with the largest modulus, and it is simple. Denote by u_M the associated unique right eigenvector with unit norm. Set $V(\Gamma_\mu) = \{\pm u_M, \overline{M \in \Gamma_\mu}\}$, where \overline{A} denotes the closure of the set A . We say that μ is arithmetic if there exist $t > 0$, $\theta \in [0, 2\pi)$ and a function $h : \mathcal{S} \rightarrow \mathbb{R}$ such that for all $M \in \Gamma_\mu$ and $x \in V(\Gamma_\mu)$,

$$\exp[it \log \|Mx\| - i\theta + ih(M \cdot x) - ih(x)] = 1.$$

By [4, Lemma 7.2], condition **H3** holds when the probability measure μ is non-arithmetic.

We need some additional notation. Let $\mathcal{C}(\mathcal{S})$ be the space of continuous functions on \mathcal{S} with real values. We equip $\mathcal{C}(\mathcal{S})$ with the L^∞ -norm

$$\|\varphi\|_\infty := \sup_{x \in \mathcal{S}} \|\varphi x\|, \quad \varphi \in \mathcal{C}(\mathcal{S}).$$

Under condition **H2**, for any $s \in [-\eta_0, \eta_0]$, define the transfer operator P_s as follows : for all $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s \varphi(x) := \mathbb{E}[\|M_0 x\|^s \varphi(M_0 \cdot x)], \quad x \in \mathcal{S}. \quad (2.10)$$

By [3, Proposition 3.1] and [12, Proposition 3.1], under **H2**, for $s \in [-\eta_0, \eta_0]$ the limit

$$\kappa(s) := \lim_{n \rightarrow +\infty} (\mathbb{E}\|M_{0,n-1}\|^s)^{1/n} \quad (2.11)$$

exists, with $0 < \kappa(s) < +\infty$, and is the spectral radius of P_s . Moreover, by [2, Lemma 10.17] the function $s \mapsto \kappa(s)$ is analytic in $(-\eta, \eta)$, for $\eta > 0$ small enough. Set $\Lambda(s) := \log \kappa(s)$ and $\gamma_k := \Lambda^{(k)}(0)$, $k \geq 1$. Then $\gamma_1 = \gamma$ and $\gamma_2 = \sigma^2$ (see [4, Corollary 7.3]).

Denote by ζ the Cramér series associated to Λ (see [5] and [20]):

$$\zeta(t) := \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3}t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}}t^2 + \dots, \quad (2.12)$$

which converges for $|t|$ small enough.

The following theorem gives a Cramér type moderate deviation expansion for $\log \|Z_n^i\|$. Recall that $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$, $x \in \mathbb{R}$.

Theorem 2.1. *Assume conditions **H1**, **H2**, **H3** and $\gamma > 0$. Then, for $0 \leq x \leq o(\sqrt{n})$ and any $1 \leq i \leq d$, as $n \rightarrow +\infty$,*

$$\frac{\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\right)}{1 - \Phi(x)} = e^{\frac{x^3}{\sqrt{n}}\zeta\left(\frac{x}{\sqrt{n}}\right)} \left[1 + O\left(\frac{1+x}{\sqrt{n}}\right)\right], \quad (2.13)$$

and

$$\frac{\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} < -x\right)}{\Phi(-x)} = e^{-\frac{x^3}{\sqrt{n}}\zeta\left(-\frac{x}{\sqrt{n}}\right)} \left[1 + O\left(\frac{1+x}{\sqrt{n}}\right)\right]. \quad (2.14)$$

In the single type case $d = 1$, Grama, Liu and Miqueu established a version of this result in [10, Theorem 1.3]. Notice that when $d = 1$, we have $\gamma = \mathbb{E} \log m_0$, $\sigma^2 = \mathbb{E}(\log m_0 - \gamma)^2$, where $m_0 = \mathbb{E}_\xi Z_1$, and the condition **H2** reduces to the following: there exist two constants $p \in (1, 2]$ and $\eta_0 \in (0, 1)$ such that

$$\mathbb{E} m_0^{\eta_0} < +\infty \quad \text{and} \quad \mathbb{E} \theta_0(p)^{\eta_0} < +\infty, \quad \text{where} \quad \theta_0(p) = \mathbb{E}_\xi \left| \frac{Z_1}{m_0} - 1 \right|^p.$$

From Theorem 2.1, we obtain the moderate deviation expansion for $\log \|Z_n^i\|$ for $x = o(n^{1/6})$, as $n \rightarrow +\infty$:

Corollary 2.2. *Assume the conditions of Theorem 2.1. Then, for all $0 \leq x \leq o(n^{1/6})$ and $1 \leq i \leq d$, as $n \rightarrow +\infty$,*

$$\frac{\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\right)}{1 - \Phi(x)} = 1 + O\left(\frac{1+x}{\sqrt{n}}\right), \quad (2.15)$$

and

$$\frac{\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} < -x\right)}{\Phi(-x)} = 1 + O\left(\frac{1+x}{\sqrt{n}}\right). \quad (2.16)$$

3. Harmonic moments of W^i

In this section, we prove the existence of harmonic moments of W^i under a new measure $\mathbb{P}_s^{e_i}$, uniformly in $s \in (-\eta, \eta)$ for a $\eta > 0$ small enough.

3.1. Definition of the change of measure \mathbb{P}_s^x

We define a new probability measure called \mathbb{P}_s^x , for $x \in \mathcal{S}$, $s \in (-\eta, \eta)$ with $\eta > 0$ small enough. The construction of \mathbb{P}_s^x is based on several properties of the transfer operator P_s defined by (2.10). For $s \in [-\eta_0, \eta_0]$, we introduce the conjugate operator P_s^* on $\mathcal{C}(\mathcal{S})$ defined by:

$$P_s^* \varphi(x) := \mathbb{E}[\|M_0^T x\|^s \varphi(M_0^T \cdot x)], \quad x \in \mathcal{S}, \quad \forall \varphi \in \mathcal{C}(\mathcal{S}). \quad (3.1)$$

A lot of results have been established on these operators P_s and P_s^* in recent years; in the following proposition, we list some of them established in [3, Proposition 3.1], [9, Corollary 3.20], [12, Proposition 3.1] and [24, Proposition 3.1], which will be used in the proofs of our results.

Proposition 3.1. *Assume condition **H2**. Then, for $\eta > 0$ small enough and $s \in (-\eta, \eta)$, the following assertions hold:*

- (1) *the spectral radii of P_s and P_s^* are both equal to $\kappa(s)$;*
- (2) *there exists a unique strictly positive function $r_s \in \mathcal{C}(\mathcal{S})$ with norm $\|r_s\|_\infty = 1$ such that*

$$P_s r_s = \kappa(s) r_s;$$

(3) there exists a unique strictly positive function $r_s^* \in \mathcal{C}(\mathcal{S})$ with norm $\|r_s^*\|_\infty = 1$ such that

$$P_s^* r_s^* = \kappa(s) r_s^*;$$

(4) $\kappa(0) = 1$ and $r_0 = r_0^* = 1$, where 1 denotes the constant function equal to 1 on \mathcal{S} ;

(5) the function $s \mapsto \kappa(s)$ is analytic on $(-\eta, \eta)$;

(6) the mappings $s \mapsto r_s$ and $s \mapsto r_s^*$ are analytic on $(-\eta, \eta)$.

For $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $A \in \mathcal{G}_+^0$, set

$$q_n^s(x, A) := \frac{\|Ax\|^s r_s(A \cdot x)}{\kappa(s)^n r_s(x)}. \quad (3.2)$$

Notice that the family (q_n^s) satisfies the following cocycle property: for any $n, m \geq 1$ and $A_1, A_2 \in \mathcal{G}_+^0$,

$$q_n^s(x, A_1) q_m^s(A_1 \cdot x, A_2) = q_{n+m}^s(x, A_2 A_1). \quad (3.3)$$

Denote by μ the law of the environment ξ_0 on \mathbb{X} . It is clear that for any $x \in \mathcal{S}$, by the assertion (2) of Proposition 3.1 and since $\kappa(s)$ and r_s are strictly positive, $q_n^s(x, M_{0,n-1}^T) \mu(d\xi_0) \cdots \mu(d\xi_{n-1})$, $n \geq 1$, is a sequence of probability measures which forms a projective system on $\mathbb{X}^{\mathbb{N}}$. Therefore, by the Kolmogorov extension theorem, there is a unique probability measure τ_s^x on $\mathbb{X}^{\mathbb{N}}$ with marginals $q_n^s(x, M_{0,n-1}^T) \mu(d\xi_0) \cdots \mu(d\xi_{n-1})$. Denote by $\mathbb{P}_s^x(dy, d\xi) = \mathbb{P}_\xi^x(dy) \tau_s^x(d\xi)$ the corresponding annealed probability, and by \mathbb{E}_s^x the expectation with respect to \mathbb{P}_s^x . For $x \in \mathcal{S}$ define the process

$$X_0^x = x, \quad \text{and} \quad X_n^x = M_{0,n-1}^T \cdot x, \quad n \geq 1,$$

which forms a Markov chain on \mathcal{S} . Then, by definition of \mathbb{P}_s^x , for $n \geq 1$ and any bounded measurable function h on \mathbb{X}^n , we have

$$\mathbb{E} \left[\frac{\|M_{0,n-1}^T x\|^s r_s(X_n^x)}{\kappa(s)^n r_s(x)} h(\xi_0, \dots, \xi_{n-1}) \right] = \mathbb{E}_s^x [h(\xi_0, \dots, \xi_{n-1})]. \quad (3.4)$$

3.2. Existence of harmonic moments of W^i under $\mathbb{P}_s^{e_i}$

It is clear that, under the new measure $\mathbb{P}_s^{e_i}$, the sequence (W_n^i) is a non negative martingale w.r.t. the filtration (\mathcal{F}_n) , hence converges $\mathbb{P}_s^{e_i}$ -a.s. to a non negative and finite random variable W^i . Denote by ϕ_ξ^i the quenched Laplace transform of W^i , and by ϕ_s^i its annealed Laplace transform under $\mathbb{P}_s^{e_i}$: for all $t \geq 0$, $s \in (-\eta, \eta)$, $\eta > 0$ small, and $1 \leq i \leq d$,

$$\phi_\xi^i(t) = \mathbb{E}_\xi e^{-tW^i} \quad \text{and} \quad \phi_s^i(t) = \mathbb{E}_s^{e_i} \phi_\xi^i(t) = \mathbb{E}_s^{e_i} e^{-tW^i}.$$

In a last article [13, Theorem 3.10] we established the following result which gives a bound for ϕ_0^i , and the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ for all $1 \leq i \leq d$ and $a > 0$ small enough.

Lemma 3.2. *Assume conditions **H1**, **H2** and $\gamma > 0$. Then there exist two constants $a > 0$ and $C > 0$ such that for all $t > 0$, all $x > 0$ and $1 \leq i \leq d$,*

$$\phi_0^i(t) \leq \frac{C}{t^a}, \quad (3.5)$$

$$\mathbb{P}(W^i \leq x) \leq Cx^a \quad \text{and} \quad \mathbb{E}(W^i)^{-a} \leq C. \quad (3.6)$$

Now, we prove the corresponding results of Lemma 3.2 under the probability measure $\mathbb{P}_s^{e_i}$. The following theorem gives a control of ϕ_s^i uniformly in $s \in (-\eta, \eta)$, and that the harmonic moments $\mathbb{E}_s^{e_i}(W^i)^{-a}$ are uniformly bounded in $s \in (-\eta, \eta)$, for small $\eta > 0$ and $a > 0$.

Theorem 3.3. *Assume conditions **H1**, **H2** and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exist two constants $a > 0$ and $C > 0$ such that for all $t > 0$, all $x > 0$ and $1 \leq i \leq d$,*

$$\sup_{s \in (-\eta, \eta)} \phi_s^i(t) \leq \frac{C}{t^a}, \quad (3.7)$$

$$\sup_{s \in (-\eta, \eta)} \mathbb{P}_s^{e_i}(W^i \leq x) \leq Cx^a \quad \text{and} \quad \sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i}(W^i)^{-a} \leq C. \quad (3.8)$$

3.3. Preliminaries to the proof of Theorem 3.3

To prove Theorem 3.3, we need some preliminary results. The first lemma below combines two results of a previous article [13, Lemmata 3.1 and 3.11]. It gives a link between the expectation of $\varphi(W_n^i)$ and that of $\varphi(W^i)$ under \mathbb{P} and \mathbb{P}_ξ , where φ is a positive convex function on \mathbb{R}_+ .

Lemma 3.4. *Assume $\gamma > 0$ and either condition **H2** or (2.7). Then for all $1 \leq i \leq d$ and any convex function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$,*

$$\lim_{n \rightarrow +\infty} \mathbb{E}_\xi \varphi(W_n^i) = \sup_{n \geq 0} \mathbb{E}_\xi \varphi(W_n^i) = \mathbb{E}_\xi \varphi(W^i), \quad (3.9)$$

and

$$\lim_{n \rightarrow +\infty} \mathbb{E} \varphi(W_n^i) = \sup_{n \geq 0} \mathbb{E} \varphi(W_n^i) = \mathbb{E} \varphi(W^i). \quad (3.10)$$

The second lemma is a direct consequence of the Marcinkiewicz-Zygmund inequality in [6, Theorem 1.5], as stated in [19, Lemma 1.4]. It allows us to control the L^p -moments of the martingale (W_n^i) under \mathbb{P}_ξ .

Lemma 3.5. *Let $(X_k)_{k \in \mathbb{N}^*}$ be a sequence of i.i.d. random centered variables. Then for all $n \in \mathbb{N}^*$ and $p > 1$:*

$$\mathbb{E} \left| \sum_{k=1}^n X_k \right|^p \leq \begin{cases} (B_p)^p \mathbb{E} |X_k|^p n, & \text{if } 1 < p \leq 2, \\ (B_p)^p \mathbb{E} |X_k|^p n^{\frac{p}{2}}, & \text{if } p > 2, \end{cases}$$

where $B_p = 2 \min\{k^{1/2} : k \in \mathbb{N}, k \geq \frac{p}{2}\}$.

The following result gives the convergence in L^α of W_n^i to W^i under $\mathbb{P}_s^{e_i}$ with an exponential speed, uniformly in $s \in (-\eta, \eta)$.

Proposition 3.6. *Assume conditions **H1**, **H2** and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exist constants $C > 0$, $\delta \in (0, 1)$ and $\varepsilon > 0$ such that for all $n \geq 0$ and $1 \leq i \leq d$,*

$$\sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} \left[\left(\mathbb{E}_\xi |W_n^i - W^i|^p \right)^\varepsilon \right] \leq C \delta^n. \quad (3.11)$$

Moreover, with $\alpha = p\varepsilon$, $W_n^i \xrightarrow[n \rightarrow +\infty]{L^\alpha} W^i$ in L^α under $\mathbb{P}_s^{e_i}$ uniformly in $s \in (-\eta, \eta)$, such that for all $n \geq 0$ and $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} |W_n^i - W^i|^\alpha \leq C \delta^n. \quad (3.12)$$

Proof. By (2.3) and (1.9), for all $n \geq 0$ and $1 \leq i \leq d$, we have

$$\begin{aligned} W_{n+1}^i - W_n^i &= \sum_{j=1}^d \frac{U_{n+1, \infty}(j)}{\lambda_{0, n} U_{0, \infty}(i)} \sum_{r=1}^d \sum_{l=1}^{Z_n^i(r)} N_{l, n}^r(j) - W_n^i \\ &= \sum_{r=1}^d \frac{U_{n, \infty}(r)}{\lambda_{0, n-1} U_{0, \infty}(i)} \sum_{l=1}^{Z_n^i(r)} \sum_{j=1}^d \frac{U_{n+1, \infty}(j) N_{l, n}^r(j)}{\lambda_n U_{n, \infty}(r)} - W_n^i \\ &= \sum_{r=1}^d \frac{U_{n, \infty}(r)}{\lambda_{0, n-1} U_{0, \infty}(i)} \sum_{l=1}^{Z_n^i(r)} (W_{l, n}^r - 1), \end{aligned} \quad (3.13)$$

where

$$W_{l, n}^r := \frac{\langle N_{l, n}^r, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(r)}.$$

Clearly, given the environment ξ , the random variables $W_{l, n}^r$, $l \geq 1$, are i.i.d., and they are independent of ξ_0, \dots, ξ_{n-1} and Z_n^i . Therefore, applying (3.13), the convexity of the function $x \mapsto x^p$ on \mathbb{R}_+ (together with the fact that $\sum_{r=1}^d \frac{M_{0, n-1}(i, r) U_{n, \infty}(r)}{\lambda_{0, n-1} U_{0, \infty}(i)} =$

1 a.s. by (1.8)) and Lemma 3.5, we get that for all $n \geq 0$ and $1 \leq i \leq d$, \mathbb{P} -a.s.,

$$\begin{aligned}
\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p &\leq \mathbb{E}_\xi \left(\sum_{r=1}^d \frac{U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \left| \sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1) \right| \right)^p \\
&= \mathbb{E}_\xi \left(\sum_{r=1}^d \frac{M_{0,n-1}(i,r) U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \frac{1}{M_{0,n-1}(i,r)} \left| \sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1) \right| \right)^p \\
&\leq \sum_{r=1}^d \frac{M_{0,n-1}(i,r) U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \frac{1}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi \left| \sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1) \right|^p \\
&\leq B_p^p \max_{1 \leq r \leq d} \left\{ \frac{\mathbb{E}_\xi Z_n^i(r)}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi |W_{1,n}^r - 1|^p \right\} \\
&= B_p^p \max_{1 \leq r \leq d} \mathbb{E}_\xi |W_{1,n}^r - 1|^p \max_{1 \leq j \leq d} (M_{0,n-1}(i,j))^{1-p}. \tag{3.14}
\end{aligned}$$

Using again the convexity of $x \mapsto x^p$, the same argument yields, for all $n \geq 0$ and $1 \leq r \leq d$, \mathbb{P} -a.s., we have

$$\begin{aligned}
\mathbb{E}_\xi |W_{1,n}^r - 1|^p &= \mathbb{E}_\xi \left| \frac{\langle N_{1,n}^r, U_{n+1,\infty} \rangle}{\lambda_n U_{n,\infty}(r)} - 1 \right|^p \\
&= \mathbb{E}_\xi \left| \sum_{j=1}^d \frac{M_n(r,j) U_{n+1,\infty}(j)}{\lambda_n U_{n,\infty}(r)} \left(\frac{N_{1,n}^r(j)}{M_n(r,j)} - 1 \right) \right|^p \\
&\leq \max_{1 \leq i,j \leq d} \mathbb{E}_\xi \left| \frac{N_{1,n}^i(j)}{M_n(i,j)} - 1 \right|^p = \theta_n(p). \tag{3.15}
\end{aligned}$$

This, together with (3.14), implies that for all $n \geq 0$ and $1 \leq r \leq d$, \mathbb{P} -a.s.,

$$\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p \leq B_p^p \theta_n(p) \max_{1 \leq j \leq d} M_{0,n-1}(i,j)^{1-p}. \tag{3.16}$$

Let $\varepsilon > 0$, and $\eta > 0$ small enough. Taking the moment of order ε under $\mathbb{P}_s^{e_i}$ in (3.16), by (3.4) and the fact that $\theta_n(p)$ and $M_{0,n-1}$ depend only on the environments ξ_k for $k \leq n$, we obtain that for $n \geq 0$, $s \in (-\eta, \eta)$ and $1 \leq i \leq d$,

$$\begin{aligned}
&\mathbb{E}_s^{e_i} \left[\left(\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p \right)^\varepsilon \right] \\
&\leq (B_p)^{\varepsilon p} \mathbb{E}_s^{e_i} \left[\theta_n(p)^\varepsilon \max_{1 \leq j \leq d} M_{0,n-1}(i,j)^{\varepsilon(1-p)} \right] \\
&= (B_p)^{\varepsilon p} \mathbb{E} \left[q_{n+1}^s(e_i, M_{0,n}^T) \theta_n(p)^\varepsilon \max_{1 \leq j \leq d} M_{0,n-1}(i,j)^{\varepsilon(1-p)} \right]. \tag{3.17}
\end{aligned}$$

By Proposition 3.1 we know that $s \mapsto r_s$ is a continuous map on $(-\eta, \eta)$, and that r_s is a strictly positive function in $\mathcal{C}(\mathcal{S})$ with norm $\|r_s\|_\infty = 1$. This implies that

$$D := \sup_{s \in (-\eta, \eta)} \left\{ \frac{\sup_{x \in \mathcal{S}} r_s(x)}{\inf_{x \in \mathcal{S}} r_s(x)} \right\} = \sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} r_s^{-1}(x) < +\infty, \tag{3.18}$$

where $r_s^{-1}(x) := [r_s(x)]^{-1}$ for $x \in \mathcal{S}$. Moreover, we know by **H1** that $\|M_{0,n}^T\| \geq 1$ a.s., and κ is a strictly positive increasing function on $(-\eta, \eta)$, so that $\kappa(s) \geq \kappa(-\eta) > 0$ for all $s \in (-\eta, \eta)$. This, together with (3.18) and the definition of $q_1^s(x, A)$ (see (3.2)), implies that for all $x \in \mathcal{S}$ and $s \in (-\eta, \eta)$,

$$q_1^s(x, M_0^T) \leq \frac{D}{\kappa(-\eta)} \|M_0^T x\|^s \leq \frac{D}{\kappa(-\eta)} \|M_0^T\|^\eta \leq \frac{d^\eta D}{\kappa(-\eta)} \|M_0\|^\eta \quad \mathbb{P}\text{-a.s.} \quad (3.19)$$

Therefore, combining the relations (3.17), (3.3), (3.4) and (3.18), we get that for $n \geq 2$, $s \in (-\eta, \eta)$ and $1 \leq i \leq d$,

$$\begin{aligned} & \mathbb{E}_s^{e_i} \left[\left(\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p \right)^\varepsilon \right] \\ & \leq (B_p)^{\varepsilon p} \mathbb{E} \left[q_{n-1}^s(e_i, M_{0,n-2}^T) q_1^s(X_{n-1}^{e_i}, M_{n-1}^T) q_1^s(X_n^{e_i}, M_n^T) \times \right. \\ & \quad \left. \theta_n(p)^\varepsilon \max_{1 \leq j \leq d} M_{n-1}(i, j)^{\varepsilon(1-p)} \|M_{0,n-2}(i, \cdot)\|^{\varepsilon(1-p)} \right] \\ & \leq (B_p)^{\varepsilon p} \left(\frac{d^\eta D}{\kappa(-\eta)} \right)^2 \mathbb{E} [\|M_0\|^\eta \theta_0(p)^\varepsilon] \mathbb{E} \left[\|M_0\|^\eta \max_{1 \leq i, j \leq d} M_0(i, j)^{\varepsilon(1-p)} \right] \times \\ & \quad \mathbb{E}_s^{e_i} \left[\|M_{0,n-2}(i, \cdot)\|^{\varepsilon(1-p)} \right]. \end{aligned} \quad (3.20)$$

Now we control the three expectations in the right side of (3.20). For the two first expectations, by Cauchy Schwarz's inequality and condition **H2**, for $\eta > 0$ and $\varepsilon > 0$ both sufficiently small such that $\eta \leq \frac{\eta_0}{2}$ and $\varepsilon \leq \frac{\eta_0}{2}$, we have

$$\mathbb{E} [\|M_0\|^\eta \theta_0(p)^\varepsilon] \leq (\mathbb{E} \|M_0\|^{2\eta})^{\frac{1}{2}} (\mathbb{E} \theta_0(p)^{2\varepsilon})^{\frac{1}{2}} < +\infty, \quad (3.21)$$

and

$$\begin{aligned} \mathbb{E} \left[\|M_0\|^\eta \max_{1 \leq i, j \leq d} M_0(i, j)^{\varepsilon(1-p)} \right] & \leq (\mathbb{E} \|M_0\|^{2\eta})^{\frac{1}{2}} \left(\mathbb{E} \max_{1 \leq i, j \leq d} M_0(i, j)^{2\varepsilon(1-p)} \right)^{\frac{1}{2}} \\ & < +\infty. \end{aligned} \quad (3.22)$$

For the third expectation, using again (3.4) and (3.18), we get that for all $n \geq 2$ and $s \in (-\eta, \eta)$,

$$\begin{aligned} & \mathbb{E}_s^{e_i} \left[\|M_{0,n-2}(i, \cdot)\|^{\varepsilon(1-p)} \right] \\ & = \mathbb{E} \left[\frac{\|M_{0,n-2}(i, \cdot)\|^{s+\varepsilon(1-p)} r_s(X_{n-1}^{e_i})}{\kappa(s)^{n-1} r_s(e_i)} \right] \\ & \leq D^2 \left(\frac{\kappa(s+\varepsilon(1-p))}{\kappa(s)} \right)^{n-1} \mathbb{E} \left[\frac{\|M_{0,n-2}(i, \cdot)\|^{s+\varepsilon(1-p)} r_{s+\varepsilon(1-p)}(X_{n-1}^{e_i})}{\kappa(s+\varepsilon(1-p))^{n-1} r_{s+\varepsilon(1-p)}(e_i)} \right] \\ & = D^2 e^{(n-1)[\Lambda(s+\varepsilon(1-p))-\Lambda(s)]}, \end{aligned} \quad (3.23)$$

where $\Lambda(s) = \log \kappa(s)$ (and the last equality follows from (3.4) with $h = 1$). By Proposition 3.1, for $\eta > 0$ small enough, the function Λ is analytic on $(-\eta, \eta)$, with

$\Lambda(0) = 0$ and $\Lambda'(0) = \gamma$ by [4, Corollary 7.3]. By hypothesis we have $\gamma > 0$, so Λ is strictly increasing on $[-\eta, \eta]$ for $\eta > 0$ small enough. Therefore, taking $\eta > 0$ and $\varepsilon > 0$ both sufficiently small, since Λ is continuous and strictly increasing on $[-\eta, \eta]$, we obtain

$$\sup_{s \in [-\eta, \eta]} \{\Lambda(s + \varepsilon(1-p)) - \Lambda(s)\} < 0. \quad (3.24)$$

It follows that $\delta := e^{\sup_{s \in (-\eta, \eta)} \{\Lambda(s + \varepsilon(1-p)) - \Lambda(s)\}} \in (0, 1)$, and we deduce from (3.23) that for all $n \geq 2$ and $s \in (-\eta, \eta)$,

$$\mathbb{E}_s^{e_i} \left[\|M_{0, n-2}(i, \cdot)\|^{\varepsilon(1-p)} \right] \leq D^2 \delta^{n-1}. \quad (3.25)$$

Now, combining the inequalities (3.20), (3.21), (3.22) and (3.25), there exists a constant $C > 0$ such that for all $n \geq 2$ and $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} \left[(\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p)^\varepsilon \right] \leq C \delta^n. \quad (3.26)$$

Moreover, by similar calculation as in (3.20), for all $s \in (-\eta, \eta)$ and $1 \leq i \leq d$ we have

$$\mathbb{E}_s^{e_i} \left[(\mathbb{E}_\xi |W_1^i - W_0^i|^p)^\varepsilon \right] \leq (B_p)^{\varepsilon p} \frac{d^n D}{\kappa(-\eta)} \mathbb{E} [\|M_0\|^\eta \theta_0(p)^\varepsilon], \quad (3.27)$$

and

$$\begin{aligned} & \mathbb{E}_s^{e_i} \left[(\mathbb{E}_\xi |W_2^i - W_1^i|^p)^\varepsilon \right] \\ & \leq (B_p)^{\varepsilon p} \left(\frac{d^n D}{\kappa(-\eta)} \right)^2 \mathbb{E} [\|M_0\|^\eta \theta_0(p)^\varepsilon] \mathbb{E} \left[\|M_0\|^\eta \max_{1 \leq i, j \leq d} M_0(i, j)^{\varepsilon(1-p)} \right]. \end{aligned} \quad (3.28)$$

Therefore, putting together the inequalities (3.21), (3.22), (3.29), (3.27) and (3.28), by taking $C > 0$ sufficiently large, it holds that for all $n \geq 0$ and $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} \left[(\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p)^\varepsilon \right] \leq C \delta^n. \quad (3.29)$$

By the triangular inequality and the sub-additivity of the function $x \mapsto x^\varepsilon$ on \mathbb{R}_+ , it follows that for all $n, k \geq 0$ and $1 \leq i \leq d$,

$$\begin{aligned} \sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} \left[(\mathbb{E}_\xi |W_{n+k}^i - W_n^i|^p)^\varepsilon \right] & \leq \sum_{r=n}^{n+k-1} \sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} \left[(\mathbb{E}_\xi |W_{r+1}^i - W_r^i|^p)^\varepsilon \right] \\ & \leq C \sum_{r=n}^{n+k-1} \delta^r \\ & \leq \frac{C}{1-\delta} \delta^n. \end{aligned}$$

So, by letting $k \rightarrow +\infty$, (3.11) holds. Let $\alpha = p\varepsilon > 0$. Using Hölder's inequality with $\varepsilon \in (0, 1)$, we obtain from (3.11) that for all $n \geq 0$ and $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} |W^i - W_n^i|^\alpha \leq \sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} \left[(\mathbb{E}_\xi |W^i - W_n^i|^p)^\varepsilon \right] \leq \frac{C}{1-\delta} \delta^n.$$

Therefore, (3.12) holds. This concludes the proof of Proposition 3.6. \square

3.4. Proof of Theorem 3.3

Now we proceed to prove Theorem 3.3.

Proof of Theorem 3.3. First, we prove the implication (3.7) \Rightarrow (3.8). Assume that $\eta > 0$ and $a > 0$ are constants such that (3.7) holds. Let $b \in (0, a)$. We know that for all $s \in (-\eta, \eta)$ and $1 \leq i \leq d$,

$$\mathbb{E}_s^{e_i} (W^i)^{-b} = \frac{1}{\Gamma(b)} \int_0^{+\infty} \phi_s^i(t) t^{b-1} dt,$$

where Γ is the Gamma function. So, by (3.7) we get that for all $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} (W^i)^{-b} \leq \frac{C}{\Gamma(b)} \int_0^{+\infty} t^{a-b-1} dt < +\infty. \quad (3.30)$$

Moreover, by Markov's inequality we have that for all $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{P}_s^{e_i} (W^i \leq x) \leq x^{-b} \sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} (W^i)^{-b}. \quad (3.31)$$

It is clear that (3.30) and (3.31) imply (3.8).

Now we prove (3.7), which will conclude the proof of Theorem 3.3. Let $\eta > 0$ be small enough, and $\varepsilon \in (0, 1)$. For all $n \geq 0$, $s \in (-\eta, \eta)$, $t \geq 0$ and $1 \leq i \leq d$, we have

$$\begin{aligned} \phi_s^i(t) &= \mathbb{E}_s^{e_i} \left[e^{-tW^i} \mathbb{1}_{\{|W_n^i - W^i| \leq \varepsilon^n\}} \right] + \mathbb{E}_s^{e_i} \left[e^{-tW^i} \mathbb{1}_{\{|W_n^i - W^i| > \varepsilon^n\}} \right] \\ &\leq \mathbb{E}_s^{e_i} \left[e^{-t(W_n^i - \varepsilon^n)} \right] + \mathbb{E}_s^{e_i} \left[\mathbb{1}_{\{|W_n^i - W^i| > \varepsilon^n\}} \right] \\ &= e^{t\varepsilon^n} \mathbb{E}_s^{e_i} \left[e^{-tW_n^i} \right] + \mathbb{P}_s^{e_i} (|W_n^i - W^i| > \varepsilon^n). \end{aligned} \quad (3.32)$$

By Markov's inequality and Proposition 3.6, for $\eta > 0$ small enough, there exist constants $\delta_0 \in (0, 1)$ and $\alpha > 0$ such that for all $n \geq 0$, $s \in (-\eta, \eta)$ and $1 \leq i \leq d$,

$$\mathbb{P}_s^{e_i} (|W_n^i - W^i| > \varepsilon^n) \leq \varepsilon^{-\alpha n} \mathbb{E}_s^{e_i} |W_n^i - W^i|^\alpha \leq C \left(\frac{\delta_0}{\varepsilon^\alpha} \right)^n.$$

Taking $\varepsilon \in (0, 1)$ such that $\varepsilon > \delta_0^{1/\alpha}$, we get that for all $n \geq 0$ and $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{P}_s^{e_i} (|W_n^i - W^i| > \varepsilon^n) \leq C \delta_1^n. \quad (3.33)$$

where $\delta_1 := \frac{\delta_0}{\varepsilon^\alpha} \in (0, 1)$.

We now give a bound of $\mathbb{E}_s^{e_i}[e^{-tW_n^i}]$ uniformly in s . By definition of W_n^i , for all $n \geq 1$ and $1 \leq i \leq d$ we have

$$\frac{\|Z_n^i\|}{\|M_{0,n-1}(i, \cdot)\|} \min_{1 \leq r \leq d} U_{n,\infty}(r) \leq W_n^i \leq \frac{\|Z_n^i\|}{\|M_{0,n-1}(i, \cdot)\|} \frac{1}{\min_{1 \leq r \leq d} U_{n,\infty}(r)}. \quad (3.34)$$

Moreover, by (1.7) we get that for all $n \geq 0$,

$$1 \geq \min_{1 \leq r \leq d} U_{n,\infty}(r) = \min_{1 \leq r \leq d} \frac{\langle M_n(r, \cdot), U_{n+1,\infty} \rangle}{\|M_n U_{n+1,\infty}\|} \geq \min_{1 \leq r, j \leq d} \frac{M_n(r, j)}{\|M_n\|}. \quad (3.35)$$

Combining (3.34) and (3.35), it holds that for all $n \geq 1$ and $1 \leq i \leq d$,

$$W_n^i \geq \frac{\|Z_n^i\|}{\|M_{0,n-1}(i, \cdot)\|} \min_{1 \leq r, j \leq d} \frac{M_n(r, j)}{\|M_n\|} =: Y_n^i. \quad (3.36)$$

The interesting point here is that Y_n^i is independent of the future $(\xi_{n+1}, \xi_{n+2}, \dots)$. Set $\beta > 0$. Recall that $\Lambda(s) = \log \kappa(s)$. By (3.36), (3.4) and (3.18), we obtain that for all $n \geq 0$, $s \in (-\eta, \eta)$, $t \geq 0$ and $1 \leq i \leq d$,

$$\begin{aligned} \mathbb{E}_s^{e_i}[e^{-tW_n^i}] &\leq \mathbb{E}_s^{e_i}[e^{-tY_n^i} (\mathbb{1}_{\{\log \|M_{0,n}(i, \cdot)\| \leq \beta(n+1)\}} + \mathbb{1}_{\{\log \|M_{0,n}(i, \cdot)\| > \beta(n+1)\}})] \\ &\leq \mathbb{E}\left[\frac{\|M_{0,n}(i, \cdot)\|^s r_s(X_{n+1}^{e_i})}{\kappa(s)^{n+1} r_s(e_i)} e^{-tY_n^i} \mathbb{1}_{\{\log \|M_{0,n}(i, \cdot)\| \leq \beta(n+1)\}}\right] \\ &\quad + \mathbb{P}_s^{e_i}(\log \|M_{0,n}(i, \cdot)\| > \beta(n+1)) \\ &\leq D e^{(n+1)[\beta s - \Lambda(s)]} \mathbb{E}[e^{-tY_n^i}] + \mathbb{P}_s^{e_i}(\log \|M_{0,n}(i, \cdot)\| > \beta(n+1)). \end{aligned} \quad (3.37)$$

We have to control all the terms on the right side of (3.37).

First, we give a suitable bound of $\mathbb{E}[e^{-tY_n^i}]$. For all $n \geq 0$, $t > 0$ and $1 \leq i \leq d$, we have

$$\mathbb{E}[e^{-tY_n^i}] = \int_{u=0}^1 \mathbb{P}(e^{-tY_n^i} \geq u) du = \int_{u=0}^1 \mathbb{P}\left(Y_n^i \leq -\frac{\log u}{t}\right) du. \quad (3.38)$$

We will give a suitable bound of $\mathbb{P}(Y_n^i \leq x)$ for any $x > 0$, to obtain the decay rate of $\mathbb{E}[e^{-tY_n^i}]$. To this end we will estimate the harmonic moments of Y_n^i . By [13, Lemma 3.11], condition **H2** implies (2.7). Therefore, applying Lemma 3.4 with the convex function $x \mapsto x^{-a}$ on \mathbb{R}_+ and Lemma 3.2, there exists a constant $a > 0$ such that for all $1 \leq i \leq d$,

$$\sup_{n \geq 0} \mathbb{E}(W_n^i)^{-a} = \mathbb{E}(W^i)^{-a} < +\infty. \quad (3.39)$$

Using (3.34), (3.35) and Hölder's inequality, we obtain that for all $b > 0$, $n \geq 0$ and $1 \leq i \leq d$,

$$\begin{aligned} \mathbb{E}(Y_n^i)^{-b} &\leq \mathbb{E}\left[(W_n^i)^{-b} \|M_n\|^b \max_{1 \leq r, j \leq d} M_n(r, j)^{-b}\right] \\ &\leq \left(\mathbb{E}(W_n^i)^{-3b}\right)^{\frac{1}{3}} \left(\mathbb{E}\|M_0\|^{3b}\right)^{\frac{1}{3}} \left(\mathbb{E} \max_{1 \leq r, j \leq d} M_0(r, j)^{-3b}\right)^{\frac{1}{3}}. \end{aligned} \quad (3.40)$$

Applying (3.40) with $b = \frac{1}{3} \min\{a, \eta_0\} > 0$, by (3.39) and condition **H2** we get that for all $1 \leq i \leq d$,

$$\sup_{n \geq 0} \mathbb{E}(Y_n^i)^{-b} \leq C.$$

By Markov's inequality, this implies that for all $x > 0$ and $1 \leq i \leq d$,

$$\sup_{n \geq 0} \mathbb{P}(Y_n^i \leq x) \leq x^b \sup_{n \geq 0} \mathbb{E}(Y_n^i)^{-b} \leq Cx^b. \quad (3.41)$$

Combining the inequalities (3.38) and (3.41), we deduce that for all $t > 0$ and $1 \leq i \leq d$,

$$\sup_{n \geq 0} \mathbb{E}[e^{-tY_n^i}] \leq Ct^{-b} \int_{u=0}^1 (-\log u)^b du \leq Ct^{-b}. \quad (3.42)$$

We next control the probability term in (3.37). Let $q > 0$ be a small constant. By Markov's inequality, (3.18) and (3.4), for all $n \geq 1$, $s \in (-\eta, \eta)$ and $1 \leq i \leq d$, we have

$$\begin{aligned} & \mathbb{P}_s^{e_i}(\log \|M_{0,n}(i, \cdot)\| > \beta(n+1)) \\ & \leq e^{-\beta q(n+1)} \mathbb{E}_s^{e_i} \|M_{0,n}(i, \cdot)\|^q \\ & = e^{-\beta q(n+1)} \mathbb{E} \left[\frac{\|M_{0,n}(i, \cdot)\|^{s+q} r_s(X_{n+1}^{e_i})}{\kappa(s)^{n+1} r_s(e_i)} \right] \\ & \leq D^2 \frac{\kappa(s+q)^{n+1}}{\kappa(s)^{n+1}} e^{-\beta q(n+1)} \mathbb{E} \left[\frac{\|M_{0,n}(i, \cdot)\|^{s+q} r_{s+q}(X_{n+1}^{e_i})}{\kappa(s+q)^{n+1} r_{s+q}(e_i)} \right] \\ & = D^2 e^{(n+1)(\Lambda(s+q) - \Lambda(s) - \beta q)}. \end{aligned} \quad (3.43)$$

We know by Proposition 3.1 that Λ is analytic on $(-\eta, \eta)$ for $\eta > 0$ small. From now, we choose $\beta > \sup_{s \in (-\eta, \eta)} \Lambda'(s)$. Then, by the mean value theorem, it holds that for all $s, q \in (-\eta/2, \eta/2)$,

$$\Lambda(s+q) - \Lambda(s) - \beta q = (\Lambda'(c) - \beta)q \leq q \sup_{s \in (-\eta, \eta)} \{\Lambda'(s) - \beta\} < 0, \quad (3.44)$$

where c is a point between s and $s+q$. Combining this with (3.43), we get that for $\eta > 0$ small enough (half of the previous value), there exists a constant $\delta_2 \in (0, 1)$ such that for all $n \geq 1$ and $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{P}_s^{e_i}(\log \|M_{0,n}(i, \cdot)\| > \beta(n+1)) \leq C\delta_2^n. \quad (3.45)$$

We then deal with the term $e^{(n+1)[\beta s - \Lambda(s)]}$ in (3.37). Using the mean value theorem with $\Lambda(0) = 0$, we get that for all $s \in (-\eta, \eta)$, $|\Lambda(s)| = |\Lambda(s) - \Lambda(0)| \leq \sup_{c \in (-\eta, \eta)} \Lambda'(c)|s| < \beta|s|$, so that $\beta s - \Lambda(s) \leq 2\beta|s|$. It follows that for all $n \geq 1$ and $s \in (-\eta, \eta)$,

$$e^{(n+1)[\beta s - \Lambda(s)]} \leq e^{2\beta|s|(n+1)} \leq e^{2\beta\eta(n+1)}. \quad (3.46)$$

Putting together the inequalities (3.32), (3.33), (3.42), (3.45) and (3.46), we obtain that for all $n \geq 1$, $t \geq 0$ and $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \phi_s^i(t) \leq C e^{t\varepsilon^n} \left(e^{2\beta\eta(n+1)} t^{-b} + \delta_2^n \right) + C\delta_1^n.$$

Taking $\delta = \max \{ e^{2\beta\eta} \varepsilon^b \delta_1, \delta_2, \delta_1 \}$, this implies that for all $n \geq 1$, $t \geq \varepsilon^{-n}$ and $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \phi_s^i(t) \leq C\delta^n, \tag{3.47}$$

with $\delta \in (0, 1)$ for $\eta > 0$ small enough. Define

$$N(t) := \left\lfloor -\frac{\log t}{\log \varepsilon} \right\rfloor + 1, \quad t \geq \frac{1}{\varepsilon}.$$

It is clear that when $t \geq \frac{1}{\varepsilon}$, we have $N(t) \geq 1$, $t \geq \varepsilon^{-N(t)}$ and $N(t) \geq -\log t / \log \varepsilon$. Therefore, using the inequality (3.47) with $n = N(t)$, we get that for all $t \geq \frac{1}{\varepsilon}$ and $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \phi_s^i(t) \leq C\delta^{N(t)} \leq C\delta^{-\frac{\log t}{\log \varepsilon}} = Ct^{-\frac{\log \delta}{\log \varepsilon}},$$

where $\frac{\log \delta}{\log \varepsilon} > 0$. So (3.7) holds. This concludes the proof of Theorem 3.3. \square

4. Berry-Esseen bound for $\log \|Z_n^i\|$ under $\mathbb{P}_s^{e_i}$

In this section, we establish a Berry-Esseen bound for $\log \|Z_n^i\|$, the logarithm of the population size $\|Z_n^i\| = Z_n^i(1) + \dots + Z_n^i(d)$, under the changed measure $\mathbb{P}_s^{e_i}$, uniformly in $s \in (-\eta, \eta)$.

Recall that by Proposition 3.1, under condition H2, the function $s \mapsto \Lambda(s) = \log \kappa(s)$ is analytic on $(-\eta, \eta)$, hence $\Lambda'(s)$ and $\sigma_s := \Lambda''(s)$ are well defined and analytic on $s \in (-\eta, \eta)$. From [24, Proposition 3.12], we have a strong law of large numbers for $\log \|M_{0,n-1}^T x\|$ under the changed measure \mathbb{P}_s^x : for $s \in (-\eta, \eta)$, and $x \in \mathcal{S}$,

$$\frac{1}{n} \log \|M_{0,n-1}^T x\| \xrightarrow[n \rightarrow +\infty]{} \Lambda'(s) \quad \mathbb{P}_s^x\text{-a.s.}$$

Moreover, by [24, Proposition 3.14], uniformly in $s \in (-\eta, \eta)$ and $x \in \mathcal{S}$, we have

$$\sigma_s^2 = \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}_s^x [(\log \|M_{0,n-1}^T x\| - n\gamma)^2] \in [0, \infty).$$

When condition H3 holds, since the function $s \mapsto \sigma_s$ is continuous on $(-\eta, \eta)$ and $\sigma_0 = \sigma > 0$, it follows that for $\eta > 0$ small enough,

$$\inf_{s \in (-\eta, \eta)} \sigma_s > 0. \tag{4.1}$$

Now, we formulate a Berry-Esseen bound for $\log \|Z_n^i\|$ under the changed measure $\mathbb{P}_s^{e_i}$, uniformly in $s \in (-\eta, \eta)$.

Theorem 4.1. *Assume conditions **H1**, **H2**, **H3** and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exists a constant $C > 0$ such that for all $n \geq 1$ and $x \in \mathbb{R}$,*

$$\sup_{s \in (-\eta, \eta)} \left| \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{\sqrt{n}}.$$

Notice that when $s = 0$, Theorem 4.1 reduces to the Berry-Essen bound (1.3) under the initial measure \mathbb{P} , which has been proved in a previous article [13, Theorem 2.4].

For the proof of Theorem 4.1, we need several preliminary results. We start by the following lemma which gives the convergence in L^1 of $\log W_n^i$ to $\log W^i$ under $\mathbb{P}_s^{e_i}$ with an exponential rate, uniformly in s .

Lemma 4.2. *Assume conditions **H1**, **H2** and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exist two constants $C > 0$ and $\delta \in (0, 1)$ such that for all $n \geq 0$ and $1 \leq i \leq d$,*

$$\sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} |\log W_n^i - \log W^i| \leq C\delta^n.$$

Proof. For all $n \geq 0$ and $1 \leq i \leq d$, we have

$$\log W_n^i - \log W^i = \log(1 + R_n^i), \quad (4.2)$$

where

$$R_n^i := \frac{W_n^i}{W^i} - 1.$$

Let $\eta > 0$ small, and $K \in (0, 1)$. Then, taking the L^1 -norm under the changed measure $\mathbb{P}_s^{e_i}$ in (4.2), we get that for all $n \geq 0$, $s \in (-\eta, \eta)$ and $1 \leq i \leq d$,

$$\begin{aligned} & \mathbb{E}_s^{e_i} |\log W_n^i - \log W^i| \\ &= \mathbb{E}_s^{e_i} |\log(1 + R_n^i) \mathbb{1}_{\{R_n^i \geq -K\}}| + \mathbb{E}_s^{e_i} |\log(1 + R_n^i) \mathbb{1}_{\{R_n^i < -K\}}|. \end{aligned} \quad (4.3)$$

Let $\varepsilon \in (0, 1]$ be small enough. Notice that $x \mapsto x^{-\varepsilon} \log(1+x)$ is a bounded function on $[-K, +\infty)$. So, for all $n \geq 0$, $s \in (-\eta, \eta)$ and $1 \leq i \leq d$,

$$\mathbb{E}_s^{e_i} |\log(1 + R_n^i) \mathbb{1}_{\{R_n^i \geq -K\}}| \leq C \mathbb{E}_s^{e_i} |R_n^i|^\varepsilon. \quad (4.4)$$

On the other hand, by Theorem 3.3 there exist constants $\eta > 0$ and $a > 0$ such that $\mathbb{E}_s^{e_i} (W^i)^{-a} \leq C$ for any $1 \leq i \leq d$ and uniformly in $s \in (-\eta, \eta)$. Therefore, using **H2** and Lemma 3.4 with the convex function $x \mapsto x^{-a}$, we obtain that for all $s \in (-\eta, \eta)$ and $1 \leq i \leq d$,

$$\sup_{n \geq 0} \mathbb{E}_s^{e_i} (W_n^i)^{-a} \leq \mathbb{E}_s^{e_i} \left[\sup_{n \geq 0} \mathbb{E}_\zeta (W_n^i)^{-a} \right] = \mathbb{E}_s^{e_i} (W^i)^{-a} \leq C. \quad (4.5)$$

We know that $|\log x|^2 \leq C(x + x^{-a})$ for all $x > 0$. So from (4.2), Fatou's lemma and (4.5), we get that for all $s \in (-\eta, \eta)$ and $1 \leq i \leq d$,

$$\begin{aligned} \sup_{n \geq 0} (\mathbb{E}_s^{e_i} |\log(1 + R_n^i)|^2)^{\frac{1}{2}} &\leq \sup_{n \geq 0} (\mathbb{E}_s^{e_i} |\log W_n^i|^2)^{\frac{1}{2}} + (\mathbb{E}_s^{e_i} |\log W^i|^2)^{\frac{1}{2}} \\ &\leq 2 \sup_{n \geq 0} (\mathbb{E}_s^{e_i} |\log W_n^i|^2)^{\frac{1}{2}} \\ &\leq C \sup_{n \geq 0} (\mathbb{E}_s^{e_i} W_n^i + \mathbb{E}_s^{e_i} (W_n^i)^{-a})^{\frac{1}{2}} \\ &\leq C. \end{aligned} \tag{4.6}$$

Therefore, by Cauchy-Schwarz's inequality, (4.6) and Markov's inequality, we obtain that for all $n \geq 0$, $s \in (-\eta, \eta)$ and $1 \leq i \leq d$,

$$\begin{aligned} \mathbb{E}_s^{e_i} |\log(1 + R_n^i) \mathbb{1}_{\{R_n^i < -K\}}| &\leq (\mathbb{E}_s^{e_i} |\log(1 + R_n^i)|^2)^{\frac{1}{2}} (\mathbb{E}_s^{e_i} \mathbb{1}_{\{R_n^i < -K\}})^{\frac{1}{2}} \\ &\leq \left[\sup_{k \geq 0} (\mathbb{E}_s^{e_i} |\log(1 + R_k^i)|^2)^{\frac{1}{2}} \right] \mathbb{P}_s^{e_i} (|R_n^i| > K)^{\frac{1}{2}} \\ &\leq C (\mathbb{E}_s^{e_i} |R_n^i|^\varepsilon)^{\frac{1}{2}}. \end{aligned} \tag{4.7}$$

Putting together the relations (4.3), (4.4) and (4.7), we get that for all $n \geq 0$, $s \in (-\eta, \eta)$ and $1 \leq i \leq d$,

$$\mathbb{E}_s^{e_i} |\log W_n^i - \log W^i| \leq C \mathbb{E}_s^{e_i} |R_n^i|^\varepsilon + C (\mathbb{E}_s^{e_i} |R_n^i|^\varepsilon)^{\frac{1}{2}}. \tag{4.8}$$

By the definition of R_n^i and Cauchy-Schwarz's inequality, for all $n \geq 0$, $s \in (-\eta, \eta)$ and $1 \leq i \leq d$, we have

$$\mathbb{E}_s^{e_i} |R_n^i|^\varepsilon = \mathbb{E}_s^{e_i} [(W^i)^{-\varepsilon} |W_n^i - W^i|^\varepsilon] \leq (\mathbb{E}_s^{e_i} (W^i)^{-2\varepsilon})^{\frac{1}{2}} (\mathbb{E}_s^{e_i} |W_n^i - W^i|^{2\varepsilon})^{\frac{1}{2}}.$$

Therefore, by (4.5) and (3.12) in Proposition 3.6, for $\varepsilon > 0$ and $\eta > 0$ small enough, there exists a constant $\delta \in (0, 1)$ such that for all $n \geq 0$ and $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} |R_n^i|^\varepsilon \leq C \delta^{2n}. \tag{4.9}$$

Combining (4.8) and (4.9), we obtain that for all $n \geq 0$ and $1 \leq i \leq d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} |\log W_n^i - \log W^i| \leq C \delta^n.$$

This concludes the proof of Lemma 4.2. \square

Now we formulate the Berry-Esseen bound for $\log \|M_{0,n-1}^T y\|$ under the changed measure \mathbb{P}_s^y , for any $y \in \mathcal{S}$ and uniformly in $s \in (-\eta, \eta)$. This result was established by Xiao, Grama and Liu in [24, Theorem 5.1], and will play a crucial role in proving Theorem 4.1. Recall that $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ is the standard normal distribution function.

Lemma 4.3. *Assume conditions **H2** and **H3**. Then, for $\eta > 0$ small enough, there exists a constant $C > 0$ such that for all $n \geq 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$,*

$$\sup_{s \in (-\eta, \eta)} \left| \mathbb{P}_s^y \left(\frac{\log \|M_{0,n-1}^T y\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{\sqrt{n}}.$$

The next lemma gives, for any $1 \leq i \leq d$, a control of the joint law $(\log \|Z_n^i\|, \log \|M_{0,n-1}(i, \cdot)\|)$ under $\mathbb{P}_s^{e_i}$, uniformly in $s \in (-\eta, \eta)$.

Lemma 4.4. *Assume conditions **H1**, **H2**, **H3** and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exists a constant $C > 0$ such that for all $n \geq 1$, $s \in (-\eta, \eta)$ and $x \in \mathbb{R}$,*

$$\mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x \right) \leq \frac{C}{\sqrt{n}}, \quad (4.10)$$

and

$$\mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \leq \frac{C}{\sqrt{n}}. \quad (4.11)$$

Proof. Since the proof of (4.11) is similar to that of (4.10), we will only prove (4.10).

Let $s \in (-\eta, \eta)$, where $\eta > 0$ is small enough such that (4.1) holds. For all $n \geq 1$, $x \in \mathbb{R}$ and $1 \leq i \leq d$, set

$$F_n^i(x) := \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x \right).$$

As before, we write that $C > 0$ for a constant independent of s and n , which may differ from line to line. For $0 \leq m < n$, $y \in \mathcal{S}$ and $1 \leq i \leq d$, set

$$S_{m,n}^y := \frac{\log \|M_{m,n-1}^T y\| - (n-m)\Lambda'(s)}{\sigma_s \sqrt{n}} \quad \text{and} \quad L_{m,n}^i := \frac{\log W_m^i}{\sigma_s \sqrt{n}}. \quad (4.12)$$

By (3.34), for all $n \geq 1$, $x \in \mathbb{R}$ and $1 \leq i \leq d$, we have

$$F_n^i(x) \leq \mathbb{P}_s^{e_i} \left(S_{0,n}^{e_i} + L_{n,n}^i + \min_{1 \leq r \leq d} \frac{\log U_{n,\infty}(r)}{\sigma_s \sqrt{n}} \leq x, S_{0,n}^{e_i} > x \right). \quad (4.13)$$

Set $m := m(n) = \lfloor \sqrt{n} \rfloor$, where $\lfloor x \rfloor$ is the integer part of x . By Markov's inequality and Lemma 4.2, for $\eta > 0$ small enough, there exists a constant $\delta \in (0, 1)$ such that for all $n \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned} & \mathbb{P}_s^{e_i} \left(|L_{n,n}^i - L_{m,n}^i| > \frac{1}{\sqrt{n}} \right) \\ & \leq \sqrt{n} \mathbb{E}_s^{e_i} |L_{n,n}^i - L_{m,n}^i| \\ & = \frac{1}{\sigma_s} \mathbb{E}_s^{e_i} |\log W_n^i - \log W_m^i| \\ & \leq \frac{1}{\sigma_s} \left(\mathbb{E}_s^{e_i} |\log W_n^i - \log W^i| + \mathbb{E}_s^{e_i} |\log W_m^i - \log W^i| \right) \\ & \leq \frac{C}{\sigma_s} (\delta^n + \delta^m). \end{aligned} \quad (4.14)$$

Notice that $\delta^n + \delta^m = o(\frac{1}{\sqrt{n}})$ as $n \rightarrow +\infty$. Combining this with (4.1) and (4.14), we get that for all $n \geq 1$ and $1 \leq i \leq d$,

$$\mathbb{P}_s^{e_i} \left(|L_{n,n}^i - L_{m,n}^i| > \frac{1}{\sqrt{n}} \right) \leq \frac{C}{\sqrt{n}}.$$

This, together with (4.13), implies that for all $n \geq 1$, $x \in \mathbb{R}$ and $1 \leq i \leq d$,

$$\begin{aligned} F_n^i(x) &\leq \mathbb{P}_s^{e_i} \left(S_{0,n}^{e_i} + L_{m,n}^i + \min_{1 \leq r \leq d} \frac{\log U_{n,\infty}(r)}{\sigma_s \sqrt{n}} \leq x + \frac{1}{\sqrt{n}}, S_{0,n}^{e_i} > x \right) \\ &\quad + \mathbb{P}_s^{e_i} \left(|L_{n,n}^i - L_{m,n}^i| > \frac{1}{\sqrt{n}} \right) \\ &\leq \mathbb{P}_s^{e_i} \left(S_{0,n}^{e_i} + L_{m,n}^i + \min_{1 \leq r \leq d} \frac{\log U_{n,\infty}(r)}{\sigma_s \sqrt{n}} \leq x + \frac{1}{\sqrt{n}}, S_{0,n}^{e_i} > x \right) + \frac{C}{\sqrt{n}}. \end{aligned} \quad (4.15)$$

For any $n \geq 1$ and $1 \leq i \leq d$, we have the following decomposition:

$$\begin{aligned} S_{0,n}^{e_i} &= \frac{\log \|M_{m+1,n-1}^T(M_{0,m}^T e_i)\| - n\gamma}{\sigma_s \sqrt{n}} \\ &= \frac{\log \|M_{0,m}^T e_i\| + \log \|M_{m+1,n-1}^T(M_{0,m}^T \cdot e_i)\| - n\gamma}{\sigma_s \sqrt{n}} \\ &= \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}}. \end{aligned} \quad (4.16)$$

By (3.34) and (3.35), for all $n \geq 1$ and $1 \leq i \leq d$ we have

$$\begin{cases} L_{m,n}^i \leq \frac{1}{\sigma_s \sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i, \cdot)\|} - \frac{1}{\sigma_s \sqrt{n}} \min_{1 \leq r,j \leq d} \log \frac{M_m(r,j)}{\|M_m\|}, \\ L_{m,n}^i \geq \frac{1}{\sigma_s \sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i, \cdot)\|} + \frac{1}{\sigma_s \sqrt{n}} \min_{1 \leq r,j \leq d} \log \frac{M_m(r,j)}{\|M_m\|}. \end{cases} \quad (4.17)$$

Therefore, combining the relations (4.15)-(4.17), we obtain that for all $n \geq 1$, $x \in \mathbb{R}$ and $1 \leq i \leq d$,

$$\begin{aligned} F_n^i(x) &\leq \mathbb{P}_s^{e_i} \left(\sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} + B_{m,n}^i \leq x + \frac{1}{\sqrt{n}}, \right. \\ &\quad \left. \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} > x \right) + \frac{C}{\sqrt{n}}, \end{aligned} \quad (4.18)$$

where

$$\begin{aligned} B_{m,n}^i &:= \frac{1}{\sigma_s \sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i, \cdot)\|} + \frac{1}{\sigma_s \sqrt{n}} \min_{1 \leq r,j \leq d} \log \frac{M_m(r,j)}{\|M_m\|} \\ &\quad + \frac{1}{\sigma_s \sqrt{n}} \min_{1 \leq r,j \leq d} \log \frac{M_n(r,j)}{\|M_n\|}. \end{aligned} \quad (4.19)$$

Since $S_{0,m+1}^{e_i}$, $S_{m+1,n}^{X_{m+1}^{e_i}}$ and $B_{m,n}^i$ depend only on the environments $\xi_0, \xi_1, \dots, \xi_n$ (but not on $\xi_{n+1}, \xi_{n+2}, \dots$), by (4.18) and (3.4) we get that for all $n \geq 1$, $x \in \mathbb{R}$ and $1 \leq i \leq d$,

$$\begin{aligned} F_n^i(x) &\leq \\ &\mathbb{E} \left[q_{n+1}^s(e_i, M_{0,n}^T) \mathbb{1} \left\{ \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} + B_{m,n}^i \leq x + \frac{1}{\sqrt{n}}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} > x \right\} \right] \\ &+ \frac{C}{\sqrt{n}}. \end{aligned} \quad (4.20)$$

For all $n \geq 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$, set

$$G_{m,n}^y(x) = \mathbb{P}_s^{e_i}(S_{m,n}^y \leq x).$$

For each $n \geq 1$, denote by h_n the function on $\mathcal{S} \times \mathbb{R} \times \mathbb{R}$ defined as follows: for all $y \in \mathcal{S}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$h_n(y, z, t) := \mathbb{P}_s^y \left(S_{m+1,n}^y + z + t \leq \frac{1}{\sqrt{n}}, S_{m+1,n}^y + z > 0 \right). \quad (4.21)$$

Notice that $X_{m+1}^{e_i}$, $S_{0,m+1}^{e_i}$ and $B_{m,n}^i$ are independent of the environments $\xi_{m+1} \cdots \xi_{n-1}$, so they are independent of $S_{m+1,n}^y$ for any $y \in \mathcal{S}$. Therefore, by (4.20) and (3.3), we see that for all $n \geq 1$, $x \in \mathbb{R}$ and $1 \leq i \leq d$,

$$\begin{aligned} F_n^i(x) &\leq \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) q_{n-m-1}^s(X_{m+1}^{e_i}, M_{m+1,n-1}^T) q_1^s(X_n^{e_i}, M_n^T) \right. \\ &\quad \left. \mathbb{1} \left\{ \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} + B_{m,n}^i \leq x + \frac{1}{\sqrt{n}}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} > x \right\} \right] + \frac{C}{\sqrt{n}} \\ &\leq \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) \sup_{u \in \mathcal{S}} q_1^s(u, M_n^T) h_n \left(X_{m+1}^{e_i}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} - x, B_{m,n}^i \right) \right] \\ &\quad + \frac{C}{\sqrt{n}}. \end{aligned} \quad (4.22)$$

Now we give a bound of the function h_n . It is clear that for all $n \geq 1$, $y \in \mathcal{S}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$0 \leq h_n(y, z, t) = \left[G_{m+1,n}^y \left(\frac{1}{\sqrt{n}} - z - t \right) - G_{m+1,n}^y(-z) \right] \mathbb{1}_{\{t \leq \frac{1}{\sqrt{n}}\}}. \quad (4.23)$$

Since the matrices M_n , $n \geq 0$, are i.i.d., for all $n \geq 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$ we have

$$\begin{aligned} G_{m+1,n}^y(x) &= \mathbb{P}_s^y \left(\frac{\log \|M_{0,n-m-1}^T y\| - (n-m-1)\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \\ &= G_{0,n-m-1}^y(a_n x), \end{aligned}$$

where $a_n = \sqrt{\frac{n}{n-m-1}}$. It is clear that $a_n = (1 - \frac{m+1}{n})^{-1/2} = 1 + O(\frac{m}{n}) = 1 + O(\frac{1}{\sqrt{n}})$ as $n \rightarrow +\infty$. Therefore, using Lemma 4.3 we obtain that, for $\eta > 0$ small enough and all $n \geq 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$,

$$\begin{aligned} |G_{m+1,n}^y(x) - \Phi(a_n x)| &= |G_{0,n-m-1}^y(a_n x) - \Phi(a_n x)| \\ &\leq \frac{C}{\sqrt{n-m-1}} \\ &= \frac{C a_n}{\sqrt{n}} \leq \frac{C}{\sqrt{n}}. \end{aligned} \quad (4.24)$$

Moreover, applying the mean value theorem on $t \mapsto \Phi(tx)$, for all $n \geq 1$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} |\Phi(a_n x) - \Phi(x)| &\leq |a_n - 1| \sup_{t \geq 1} |x \Phi'(tx)| \\ &\leq \frac{C}{\sqrt{n}} \sup_{z \in \mathbb{R}} |z \Phi'(z)| \\ &\leq \frac{C}{\sqrt{n}}, \end{aligned} \quad (4.25)$$

where we have used the fact that $z \mapsto |z \Phi'(z)|$ is a bounded function on \mathbb{R} . Combining the relations (4.23)-(4.25), we get that for all $n \geq 1$, $y \in \mathcal{S}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$h_n(y, z, t) \leq \left| \Phi\left(\frac{1}{\sqrt{n}} - z - t\right) - \Phi(-z) \right| \mathbb{1}_{\{t \leq \frac{1}{\sqrt{n}}\}} + \frac{C}{\sqrt{n}}. \quad (4.26)$$

Using again the mean value theorem, since $\sup_{x \in \mathbb{R}} |\Phi'(x)| \leq 1$, for all $x, z \in \mathbb{R}$ we have

$$|\Phi(x+z) - \Phi(x)| \leq |z|. \quad (4.27)$$

This, together with (4.26), implies that for all $n \geq 1$, $y \in \mathcal{S}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$h_n(y, z, t) \leq \left| \frac{1}{\sqrt{n}} + t \right| + \frac{C}{\sqrt{n}} \leq |t| + \frac{C}{\sqrt{n}}. \quad (4.28)$$

By (4.22) and (4.28), we obtain that for all $n \geq 1$, $x \in \mathbb{R}$ and $1 \leq i \leq d$,

$$\begin{aligned} F_n^i(x) &\leq \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) \sup_{u \in \mathcal{S}} q_1^s(u, M_n^T) \left(|B_{m,n}^i| + \frac{C}{\sqrt{n}} \right) \right] + \frac{C}{\sqrt{n}} \\ &= \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) \sup_{u \in \mathcal{S}} q_1^s(u, M_n^T) |B_{m,n}^i| \right] \\ &\quad + \frac{1}{\sqrt{n}} \mathbb{E} \left[\sup_{u \in \mathcal{S}} q_1^s(u, M_0^T) \right] + \frac{C}{\sqrt{n}}. \end{aligned} \quad (4.29)$$

Then, combining (4.29), (3.19) and the definition of $B_{m,n}^i$ (see (4.19)), we get that for all $n \geq 1$, $x \in \mathbb{R}$ and $1 \leq i \leq d$,

$$\begin{aligned} F_n^i(x) &\leq \frac{C}{\sigma_s \sqrt{n}} \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) \|M_n\|^\eta \left| \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i, \cdot)\|} \right| \right] \\ &\quad + \frac{C}{\sigma_s \sqrt{n}} \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) \|M_n\|^\eta \max_{1 \leq r, j \leq d} \left| \log \frac{M_m(r, j)}{\|M_m\|} \right| \right] \\ &\quad + \frac{C}{\sigma_s \sqrt{n}} \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) \|M_n\|^\eta \max_{1 \leq r, j \leq d} \left| \log \frac{M_n(r, j)}{\|M_n\|} \right| \right] \\ &\quad + \frac{C}{\sqrt{n}}. \end{aligned}$$

By condition **H2**, (4.1) and (4.17), this implies that for all $n \geq 1$, $x \in \mathbb{R}$ and $1 \leq i \leq d$,

$$\begin{aligned} F_n^i(x) &\leq \frac{C}{\sqrt{n}} \mathbb{E}_s^{e_i} \left| \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i, \cdot)\|} \right| + \frac{C}{\sqrt{n}} \mathbb{E}_s^{e_i} \left[\max_{1 \leq r, j \leq d} \left| \log \frac{M_m(r, j)}{\|M_m\|} \right| \right] \\ &\quad + \frac{C}{\sqrt{n}} \mathbb{E} \left[\|M_0\|^\eta \max_{1 \leq r, j \leq d} \left| \log \frac{M_0(r, j)}{\|M_0\|} \right| \right] + \frac{C}{\sqrt{n}} \\ &\leq \frac{C}{\sqrt{n}} \mathbb{E}_s^{e_i} |\log W_m^i| + \frac{C}{\sqrt{n}} \mathbb{E}_s^{e_i} \left[\max_{1 \leq r, j \leq d} \left| \log \frac{M_m(r, j)}{\|M_m\|} \right| \right] \\ &\quad + \frac{C}{\sqrt{n}} \mathbb{E} \left[\|M_0\|^\eta \max_{1 \leq r, j \leq d} \left| \log \frac{M_0(r, j)}{\|M_0\|} \right| \right] + \frac{C}{\sqrt{n}}. \end{aligned} \tag{4.30}$$

Now we give a bound of the three expectations in (4.30). First, by Lemma 4.2 we get that, for $\eta > 0$ small enough and all $n \geq 1$ and $1 \leq i \leq d$,

$$\mathbb{E}_s^{e_i} |\log W_m^i| \leq C. \tag{4.31}$$

Next, using (3.3) and (3.19), we obtain that for all $n \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned} &\mathbb{E}_s^{e_i} \left[\max_{1 \leq r, j \leq d} \left| \log \frac{M_m(r, j)}{\|M_m\|} \right| \right] \\ &= \mathbb{E} \left[q_m^s(e_i, M_{0,m-1}^T) q_1^s(M_{0,m-1}^T \cdot e_i, M_m^T) \max_{1 \leq r, j \leq d} \left| \log \frac{M_m(r, j)}{\|M_m\|} \right| \right] \\ &\leq C \mathbb{E} \left[q_m^s(e_i, M_{0,m-1}^T) \|M_m\|^\eta \max_{1 \leq r, j \leq d} \left| \log \frac{M_m(r, j)}{\|M_m\|} \right| \right] \\ &= C \mathbb{E} \left[\|M_0\|^\eta \max_{1 \leq r, j \leq d} \left| \log \frac{M_0(r, j)}{\|M_0\|} \right| \right]. \end{aligned} \tag{4.32}$$

Then, by the inequality $|\log x| \leq C(x^\eta + x^{-\eta})$ for $x > 0$ and Cauchy-Schwarz's

inequality, we have

$$\begin{aligned}
& \mathbb{E} \left[\left\| M_0 \right\|^\eta \max_{1 \leq r, j \leq d} \left| \log \frac{M_0(r, j)}{\|M_0\|} \right| \right] \\
& \leq C \mathbb{E} \left[\max_{1 \leq r, j \leq d} M_0(r, j)^\eta \right] + C \mathbb{E} \left[\|M_0\|^{2\eta} \max_{1 \leq r, j \leq d} M_0(r, j)^{-\eta} \right] \\
& \leq C \mathbb{E} \|M_0\|^\eta + C \left(\mathbb{E} \|M_0\|^{4\eta} \right)^{\frac{1}{2}} \left(\mathbb{E} \max_{1 \leq r, j \leq d} M_0(r, j)^{-2\eta} \right)^{\frac{1}{2}}. \tag{4.33}
\end{aligned}$$

Taking $\eta > 0$ small enough, by **H2** it follows that

$$\mathbb{E} \left[\left\| M_0 \right\|^\eta \max_{1 \leq r, j \leq d} \left| \log \frac{M_0(r, j)}{\|M_0\|} \right| \right] \leq C. \tag{4.34}$$

Therefore, combining the inequalities (4.30)-(4.34), we get (4.10). This concludes the proof of Lemma 4.4. \square

Now we shall prove Theorem 4.1.

Proof of Theorem 4.1. Let $\eta > 0$ be sufficiently small such that (4.1) holds. For all $n \geq 1$, $s \in (-\eta, \eta)$, $x > 0$ and $1 \leq i \leq d$, we have

$$\begin{aligned}
& \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \\
& = \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x, \frac{\log \|M_{0, n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \\
& \quad + \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x, \frac{\log \|M_{0, n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x \right) \\
& = \mathbb{P}_s^{e_i} \left(\frac{\log \|M_{0, n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \\
& \quad - \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x, \frac{\log \|M_{0, n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \\
& \quad + \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x, \frac{\log \|M_{0, n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x \right).
\end{aligned}$$

Therefore, using the Berry-Essen bound in lemma 4.3 and Lemma 4.4, by taking

$\eta > 0$ small enough, we get that for all $n \geq 1$, $s \in (-\eta, \eta)$, $x > 0$ and $1 \leq i \leq d$,

$$\begin{aligned}
& \left| \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) - \Phi(x) \right| \\
& \leq \left| \mathbb{P}_s^{e_i} \left(\frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) - \Phi(x) \right| \\
& \quad + \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \\
& \quad + \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x \right) \\
& \leq \frac{C}{\sqrt{n}}.
\end{aligned}$$

This concludes the proof of Theorem 4.1. \square

5. Proof of Cramér's moderate deviation expansion

In this section, we prove Theorem 2.1. The proof is based on a control of the joint law of $(\log \|Z_n^i\|, \log \|M_{0,n-1}(i, \cdot)\|)$ under $\mathbb{P}_s^{e_i}$, uniformly in $s \in (-\eta, \eta)$. We already have a control in Lemma 4.4. Unfortunately this is not sufficient, and we need additional information. For $0 < x < y$, set $\Phi([x, y]) = \Phi(y) - \Phi(x)$. The first result below about the convergence to the normal distribution is a consequence of Theorem 4.1 and Lemma 4.4.

Lemma 5.1. *Assume conditions H1, H2, H3 and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exists a constant $C > 0$ such that for all $n \geq 1$, $s \in (-\eta, \eta)$, $x > 0$ and $1 \leq i \leq d$,*

$$\begin{aligned}
& \left| \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > 0, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) - \Phi([0, x]) \right| \\
& \leq \frac{C}{\sqrt{n}}, \quad (5.1)
\end{aligned}$$

and

$$\begin{aligned}
& \left| \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} < 0, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \geq -x \right) \right. \\
& \quad \left. - \Phi([-x, 0]) \right| \leq \frac{C}{\sqrt{n}}. \quad (5.2)
\end{aligned}$$

Proof. Let $\eta > 0$ be small enough such that (4.1) holds. For all $n \geq 1$, $s \in (-\eta, \eta)$,

$x > 0$ and $1 \leq i \leq d$, we have

$$\begin{aligned}
& \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > 0, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \\
&= \mathbb{P}_s^{e_i} \left(0 < \frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \\
&\quad + \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \\
&= \mathbb{P}_s^{e_i} \left(0 < \frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \\
&\quad - \mathbb{P}_s^{e_i} \left(0 < \frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x \right) \\
&\quad + \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right).
\end{aligned}$$

Then, applying the Berry-Essen bound in Theorem 4.1 and the inequalities in Lemma 4.4, when $\eta > 0$ is sufficiently small, we obtain that for all $n \geq 1$, $s \in (-\eta, \eta)$, $x > 0$ and $1 \leq i \leq d$,

$$\begin{aligned}
& \left| \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > 0, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) - \Phi([0, x]) \right| \\
&\leq \left| \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) - \Phi([0, x]) \right| \\
&\quad + \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x \right) \\
&\quad + \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > x, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq x \right) \\
&\leq \frac{C}{\sqrt{n}}.
\end{aligned}$$

Therefore, (5.1) holds. It is clear that (5.2) can be proved by similar calculations. This concludes the proof of Lemma 5.1. \square

The second result gives a control of the probabilities in Lemma 5.1 when $x < 0$, uniformly in $s \in (-\eta, \eta)$.

Lemma 5.2. *Assume conditions H1, H2, H3 and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exist constants $C > 0$, $\alpha > 0$, $\beta > 0$ and $\delta \in (0, 1)$ such that for all $n \geq 1$, $s \in (-\eta, \eta)$, $x > 0$ and $1 \leq i \leq d$,*

$$\begin{aligned}
& \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > 0, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq -x \right) \\
&\leq \frac{C}{\sqrt{n}} e^{-\alpha x \sqrt{n}} + C \min \{ e^{-\beta x \sqrt{n}}, x^{-\frac{1}{2}} n^{-\frac{1}{4}} \delta \sqrt{n} \}, \quad (5.3)
\end{aligned}$$

and

$$\begin{aligned} \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} < 0, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \geq x \right) \\ \leq \frac{C}{\sqrt{n}} e^{-\alpha x \sqrt{n}} + C \min \{ e^{-\beta x \sqrt{n}}, x^{-\frac{1}{2}} n^{-\frac{1}{4}} \delta \sqrt{n} \}. \end{aligned} \quad (5.4)$$

Proof. We only prove (5.3), since the second assertion (5.4) can be proved in the same way.

We use the same notation as in the proof of Lemma 5.1. Let $\eta > 0$ be small enough such that (4.1) holds. Let $s \in (-\eta, \eta)$. As before, $C > 0$ will be a constant independent of s and n , which may differ from line to line.

By (3.34), we get that for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$\begin{aligned} & \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > 0, \frac{\log \|M_{0,n-1}(i, \cdot)\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq -x \right) \\ & \leq \mathbb{P}_s^{e_i} \left(S_{0,n}^{e_i} + L_{n,n}^i - \min_{1 \leq r \leq d} \frac{\log U_{n,\infty}(r)}{\sigma_s \sqrt{n}} > 0, S_{0,n} \leq -x \right) \\ & \leq \mathbb{P}_s^{e_i} \left(S_{0,n}^{e_i} + L_{m,n}^i - \min_{1 \leq r \leq d} \frac{\log U_{n,\infty}(r)}{\sigma_s \sqrt{n}} > -\frac{x}{2}, S_{0,n} \leq -x \right) \\ & \quad + \mathbb{P}_s^{e_i} \left(|L_{n,n}^i - L_{m,n}^i| > \frac{x}{2} \right) \\ & =: A_1^i(x, n) + A_2^i(x, n). \end{aligned} \quad (5.5)$$

Now, we give a bound for the two terms $A_1^i(x, n)$ and $A_2^i(x, n)$.

Control of $A_1^i(x, n)$. Using the relations (4.16) and (4.17), we obtain that for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$\begin{aligned} A_1^i(x, n) \leq \mathbb{P}_s^{e_i} \left(\sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} + K_{m,n}^i > -\frac{x}{2}, \right. \\ \left. \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} \leq -x \right), \end{aligned}$$

where

$$\begin{aligned} K_{m,n}^i &:= \frac{1}{\sigma_s \sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i, \cdot)\|} - \frac{1}{\sigma_s \sqrt{n}} \min_{1 \leq r, j \leq d} \log \frac{M_m(r, j)}{\|M_m\|} \\ &\quad - \frac{1}{\sigma_s \sqrt{n}} \min_{1 \leq r, j \leq d} \log \frac{M_n(r, j)}{\|M_n\|}. \end{aligned}$$

For each $n \geq 1$ and $x > 0$, let $h_{n,x}$ be the function on $\mathcal{S} \times \mathbb{R} \times \mathbb{R}$ defined by: for all $y \in \mathcal{S}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$h_{n,x}(y, z, t) := \mathbb{P}_s^y \left(S_{m+1,n}^y + z + t > -\frac{x}{2}, S_{m+1,n}^y + z \leq -x \right). \quad (5.6)$$

By an argument similar to the proof of (4.20) and (4.22), we obtain that for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$\begin{aligned}
 & A_1^i(x, n) \\
 \leq & \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) q_{n-m-1}^s(X_{m+1}^{e_i}, M_{m+1,n-1}^T) q_1^s(X_n^{e_i}, M_n^T) \right. \\
 & \left. \mathbb{1}_{\left\{ \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} + K_{m,n}^i > -\frac{x}{2}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} \leq -x \right\}} \right] \\
 \leq & \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) \sup_{u \in \mathcal{S}} q_1^s(u, M_n^T) h_{n,x} \left(X_{m+1}^{e_i}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i}, K_{m,n}^i \right) \right]. \tag{5.7}
 \end{aligned}$$

Using (4.24), (4.25) and (4.27), we have that for all $n \geq 1$, $x > 0$, $y \in \mathcal{S}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$\begin{aligned}
 0 \leq h_{n,x}(y, z, t) &= \left[G_{m+1,n}^y(-x-z) - G_{m+1,n}^y\left(-\frac{x}{2} - z - t\right) \right] \mathbb{1}_{\{t \geq \frac{x}{2}\}} \\
 &\leq \left(\left| t - \frac{x}{2} \right| + \frac{C}{\sqrt{n}} \right) \mathbb{1}_{\{t \geq \frac{x}{2}\}} \\
 &\leq \left(t + \frac{C}{\sqrt{n}} \right) \mathbb{1}_{\{t \geq \frac{x}{2}\}}.
 \end{aligned}$$

Combining this with (5.7), we get that for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$A_1^i(x, n) \leq \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) \sup_{u \in \mathcal{S}} q_1^s(u, M_n^T) \left(K_{m,n}^i + \frac{C}{\sqrt{n}} \right) \mathbb{1}_{\{K_{m,n}^i \geq \frac{x}{2}\}} \right]. \tag{5.8}$$

We will slightly change the expression of the above expectation in order to facilitate the passage to the expectation with respect to the new measure $\mathbb{P}_s^{e_i}$. For any $n \geq 1$ and $1 \leq i \leq d$, set

$$\begin{aligned}
 \tilde{K}_{m,n}^i &:= \frac{1}{\sigma_s \sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i, \cdot)\|} - \frac{1}{\sigma_s \sqrt{n}} \min_{1 \leq r, j \leq d} \log \frac{M_m(r, j)}{\|M_m\|} \\
 &\quad - \frac{1}{\sigma_s \sqrt{n}} \min_{1 \leq r, j \leq d} \log \frac{M_{m+1}(r, j)}{\|M_{m+1}\|}.
 \end{aligned}$$

Notice that the expectation in (5.8) remains the same if the environment ξ_n is replaced by ξ_{m+1} due to the independence structure. So in (5.8) we can replace $(M_n, K_{m,n}^i)$ by $(M_{m+1}, \tilde{K}_{m,n}^i)$. This, together with (3.18), yields that for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$\begin{aligned}
 A_1^i(x, n) &\leq \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) \sup_{u \in \mathcal{S}} q_1^s(u, M_{m+1}^T) \left(\tilde{K}_{m,n}^i + \frac{C}{\sqrt{n}} \right) \mathbb{1}_{\{\tilde{K}_{m,n}^i \geq \frac{x}{2}\}} \right] \\
 &\leq D^2 \mathbb{E} \left[q_{m+1}^s(e_i, M_{0,m}^T) q_1^s(X_{m+1}^{e_i}, M_{m+1}^T) \left(\tilde{K}_{m,n}^i + \frac{C}{\sqrt{n}} \right) \mathbb{1}_{\{\tilde{K}_{m,n}^i \geq \frac{x}{2}\}} \right] \\
 &\leq C \mathbb{E}_s^{e_i} \left[\tilde{K}_{m,n}^i \mathbb{1}_{\{\tilde{K}_{m,n}^i \geq \frac{x}{2}\}} \right] + \frac{C}{\sqrt{n}} \mathbb{P}_s^{e_i} \left(\tilde{K}_{m,n}^i \geq \frac{x}{2} \right). \tag{5.9}
 \end{aligned}$$

Let $\varepsilon \in (0, 1)$ be arbitrary fixed. By Markov's inequality and (3.19), for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$ we have

$$\begin{aligned}
& \mathbb{P}_s^{e_i} \left(\tilde{K}_{m,n}^i \geq \frac{x}{2} \right) \\
&= \mathbb{P}_s^{e_i} \left(e^{\varepsilon \sigma_s \sqrt{n} \tilde{K}_{m,n}^i} \geq e^{\frac{\varepsilon \sigma_s}{2} x \sqrt{n}} \right) \\
&\leq e^{-\frac{\varepsilon \sigma_s}{2} x \sqrt{n}} \mathbb{E} \left[q_m^s(e_i, M_{0,m-1}^T) q_1^s(X_m^{e_i}, M_m^T) q_1^s(X_{m+1}^{e_i}, M_{m+1}^T) e^{\varepsilon \sigma_s \sqrt{n} \tilde{K}_{m,n}^i} \right] \\
&\leq C e^{-\frac{\varepsilon \sigma_s}{2} x \sqrt{n}} \mathbb{E} \left[q_m^s(e_i, M_{0,m-1}^T) \left(\frac{\|Z_m^i\|}{\|M_{0,m-1}(i, \cdot)\|} \right)^\varepsilon \right] \times \\
&\quad \mathbb{E} \left[\frac{\|M_m\|^{\eta+\varepsilon}}{\min_{1 \leq r, j \leq d} M_m(r, j)^\varepsilon} \right] \mathbb{E} \left[\frac{\|M_{m+1}\|^{\eta+\varepsilon}}{\min_{1 \leq r, j \leq d} M_{m+1}(r, j)^\varepsilon} \right] \\
&\leq C e^{-\frac{\varepsilon \sigma_s}{2} x \sqrt{n}} \mathbb{E}_s^{e_i} \left(\frac{\|Z_m^i\|}{\|M_{0,m-1}(i, \cdot)\|} \right)^\varepsilon \left(\mathbb{E} \left[\|M_0\|^{\eta+\varepsilon} \min_{1 \leq r, j \leq d} M_0(r, j)^{-\varepsilon} \right] \right)^2.
\end{aligned} \tag{5.10}$$

Notice that $\mathbb{E}_s^{e_i} \left[\frac{\|Z_m^i\|}{\|M_{0,m-1}(i, \cdot)\|} \right] = 1$. Therefore, using Jensen's and Cauchy-Shwartz's inequalities and condition **H2**, by taking $\eta > 0$ and $\varepsilon \in (0, 1)$ sufficiently small, we get from (5.10) that for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$\begin{aligned}
\mathbb{P}_s^{e_i} \left(\tilde{K}_{m,n}^i \geq \frac{x}{2} \right) &\leq C e^{-\frac{\varepsilon \sigma_s}{2} x \sqrt{n}} \left(\mathbb{E}_s^{e_i} \left[\frac{\|Z_m^i\|}{\|M_{0,m-1}(i, \cdot)\|} \right] \right)^\varepsilon \times \\
&\quad \left(\mathbb{E} \|M_0\|^{2(\eta+\varepsilon)} \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\min_{1 \leq r, j \leq d} M_0(r, j)^{-2\varepsilon} \right] \right)^{\frac{1}{2}} \\
&\leq C e^{-\frac{\varepsilon \sigma_s}{2} x \sqrt{n}}.
\end{aligned} \tag{5.11}$$

Then, using again Cauchy-Shwartz's inequality and (5.11), for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$ we have

$$\begin{aligned}
\mathbb{E}_s^{e_i} \left[\tilde{K}_{m,n}^i \mathbb{1}_{\{\tilde{K}_{m,n}^i \geq \frac{x}{2}\}} \right] &\leq \left(\mathbb{E}_s^{e_i} |\tilde{K}_{m,n}^i|^2 \right)^{\frac{1}{2}} \mathbb{P}_s^{e_i} \left(\tilde{K}_{m,n}^i \geq \frac{x}{2} \right)^{\frac{1}{2}} \\
&\leq C e^{-\frac{\varepsilon \sigma_s}{4} x \sqrt{n}} \left(\mathbb{E}_s^{e_i} |\tilde{K}_{m,n}^i|^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{5.12}$$

Then, by the triangular inequality in L^2 under $\mathbb{P}_s^{e_i}$ and (4.17), we obtain that for all $n \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned}
\left(\mathbb{E}_s^{e_i} |\tilde{K}_{m,n}^i|^2 \right)^{\frac{1}{2}} &\leq \frac{1}{\sigma_s \sqrt{n}} \left[\left(\mathbb{E}_s^{e_i} |\log W_m^i|^2 \right)^{\frac{1}{2}} + 2 \left(\mathbb{E}_s^{e_i} \max_{1 \leq r, j \leq d} \left| \log \frac{M_m(r, j)}{\|M_m\|} \right|^2 \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(\mathbb{E}_s^{e_i} \max_{1 \leq r, j \leq d} \left| \log \frac{M_{m+1}(r, j)}{\|M_{m+1}\|} \right|^2 \right)^{\frac{1}{2}} \right].
\end{aligned} \tag{5.13}$$

Notice that we have proved in (4.6) that for all $n \geq 1$ and $1 \leq i \leq d$,

$$\left(\mathbb{E}_s^{e_i} |\log W_n^i|^2 \right)^{\frac{1}{2}} \leq C. \tag{5.14}$$

By an argument similar to the proof of (4.32)-(4.34) with the inequality $|\log x|^2 \leq C(x^\eta + x^{-\eta})$ for $x > 0$, we get that for all $n \geq 1$ and $1 \leq i \leq d$,

$$\left(\mathbb{E}_s^{e_i} \max_{1 \leq r, j \leq d} \left| \log \frac{M_n(r, j)}{\|M_n\|} \right|^2 \right)^{\frac{1}{2}} \leq C. \quad (5.15)$$

Therefore, combining the inequalities (5.12)-(5.15) and (4.1), we see that for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$\mathbb{E}_s^{e_i} \left[\tilde{K}_{m,n}^i \mathbb{1}_{\{\tilde{K}_{m,n}^i \geq \frac{x}{2}\}} \right] \leq \frac{C}{\sqrt{n}} e^{-\frac{\varepsilon \sigma_s}{4} x \sqrt{n}}. \quad (5.16)$$

Putting together (5.9), (5.11), (5.16) and (4.1), we obtain that, with $\alpha = \frac{\varepsilon}{4} \inf_{s \in (-\eta, \eta)} \sigma_s > 0$, for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$A_1^i(x, n) \leq \frac{C}{\sqrt{n}} e^{-\alpha x \sqrt{n}}. \quad (5.17)$$

Control of $A_2^i(x, n)$. First, by Markov's and Jensen's inequalities, and using (4.1) and Lemma 4.2, we see that for $\eta > 0$ small enough, there exists a constant $\delta_0 \in (0, 1)$ such that for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$\begin{aligned} A_2^i(x, n) &= \mathbb{P}_s^{e_i} \left(\left| \log W_n^i - \log W_m^i \right|^{\frac{1}{2}} > \left(\frac{\sigma_s}{2} \right)^{\frac{1}{2}} x^{\frac{1}{2}} n^{\frac{1}{4}} \right) \\ &\leq \left(\frac{\sigma_s}{2} \right)^{-\frac{1}{2}} x^{-\frac{1}{2}} n^{-\frac{1}{4}} \left(\mathbb{E}_s^{e_i} \left| \log W_n^i - \log W_m^i \right| \right)^{\frac{1}{2}} \\ &\leq C x^{-\frac{1}{2}} n^{-\frac{1}{4}} \left(\mathbb{E}_s^{e_i} \left| \log W_n^i - \log W^i \right| + \mathbb{E}_s^{e_i} \left| \log W_m^i - \log W^i \right| \right)^{\frac{1}{2}} \\ &\leq C x^{-\frac{1}{2}} n^{-\frac{1}{4}} \delta_0^{\frac{m}{2}}. \end{aligned}$$

Taking $\delta = \delta_0^{\frac{1}{2}} \in (0, 1)$, since $m = \lfloor \sqrt{n} \rfloor$, it follows that for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$A_2^i(x, n) \leq C x^{-\frac{1}{2}} n^{-\frac{1}{4}} \delta \sqrt{n}. \quad (5.18)$$

On the other hand, by Markov's inequality we have that for any $a \in (0, 1)$, and for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$\begin{aligned} A_2^i(x, n) &= \mathbb{P}_s^{e_i} \left(e^{a \left| \log W_n^i - \log W_m^i \right|} > e^{\frac{a \sigma_s}{2} x \sqrt{n}} \right) \\ &\leq e^{-\frac{a \sigma_s}{2} x \sqrt{n}} \mathbb{E}_s^{e_i} \left(e^{a \left| \log W_n^i - \log W_m^i \right|} \right) \\ &\leq e^{-\frac{a \sigma_s}{2} x \sqrt{n}} \left[\mathbb{E}_s^{e_i} \left(\frac{W_n^i}{W_m^i} \right)^a + \mathbb{E}_s^{e_i} \left(\frac{W_m^i}{W_n^i} \right)^a \right]. \end{aligned} \quad (5.19)$$

By Cauchy-Schwarz's and Jensen's inequalities, and using (4.5), for $\eta > 0$ and

$a \in (0, \frac{1}{2})$ small enough we get that for all $n, k \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned} \mathbb{E}_s^{e_i} \left(\frac{W_n^i}{W_k^i} \right)^a &\leq (\mathbb{E}_s^{e_i} (W_n^i)^{2a})^{\frac{1}{2}} (\mathbb{E}_s^{e_i} (W_k^i)^{-2a})^{\frac{1}{2}} \\ &\leq (\mathbb{E}_s^{e_i} W_n^i)^a (\mathbb{E}_s^{e_i} (W^i)^{-2a})^{\frac{1}{2}} \\ &\leq C. \end{aligned}$$

This, together with (5.19) and (4.1), implies that with $\beta = \frac{a}{2} \inf_{s \in (-\eta, \eta)} \sigma_s > 0$, for all $n \geq 1$, $x > 0$ and $1 \leq i \leq d$,

$$A_2^i(x, n) \leq C e^{-\beta x \sqrt{n}}. \quad (5.20)$$

Combining (5.5), (5.17) and (5.20), we get (5.3). This concludes the proof of Lemma 5.2. \square

Now we proceed to the proof of Theorem 2.1. It is based on the control of the joint law of $(\log \|Z_n^i\|, \log \|M_{0, n-1}(i, \cdot)\|)$ in Lemmas 5.1 and 5.2, together with standard techniques from Petrov [20].

Proof of Theorem 2.1. Notice that, when $x \in (0, 1]$, Theorem 2.1 is a direct consequence of the Berry-Esseen bound for $\log \|Z_n^i\|$ (Theorem 4.1 with $s = 0$). So, it remains to prove Theorem 2.1 for $x \geq 1$ such that $x = o(\sqrt{n})$, as $n \rightarrow +\infty$.

We first prove (2.13). Let $\eta > 0$ be a small constant. Using the changed measure $\mathbb{P}_s^{e_i}$, for all $n \geq 1$, $s \in (-\eta, \eta)$ and $1 \leq i \leq d$ we have

$$\begin{aligned} &\mathbb{P} \left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma \sqrt{n}} > x \right) \\ &= r_s(e_i) \kappa(s)^n \mathbb{E}_s^{e_i} \left[r_s^{-1}(X_n^{e_i}) e^{-s \log \|M_{0, n-1}(i, \cdot)\|} \mathbb{1}_{\{\log \|Z_n^i\| - n\gamma > \sigma x \sqrt{n}\}} \right]. \quad (5.21) \end{aligned}$$

Since $\Lambda = \log \kappa$, we get from (5.21), (4.12) and (4.1) that, for $\eta > 0$ small enough and all $n \geq 1$, $s \in (-\eta, \eta)$ and $1 \leq i \leq d$,

$$\begin{aligned} &\mathbb{P} \left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma \sqrt{n}} > x \right) = r_s(e_i) e^{-n[s\Lambda'(s) - \Lambda(s)]} \times \\ &\quad \mathbb{E}_s^{e_i} \left[r_s^{-1}(X_n^{e_i}) e^{-s\sigma_s \sqrt{n} S_{0, n}^{e_i}} \mathbb{1}_{\left\{ \frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > \frac{\sqrt{n}[\gamma - \Lambda'(s)] + \sigma x}{\sigma_s} \right\}} \right]. \quad (5.22) \end{aligned}$$

Recall that, by Proposition 3.1, the function Λ is analytic on $(-\eta, \eta)$ for $\eta > 0$ small enough, so that $\Lambda(s) = \sum_{k=1}^{+\infty} \frac{\gamma_k}{k!} s^k$ for $s \in (-\eta, \eta)$, where $\gamma_k := \Lambda^{(k)}(0)$, $k \geq 1$. From [20], we know that for $x = o(\sqrt{n})$ as $n \rightarrow +\infty$, $x \geq 1$, and $n \geq 1$ sufficiently large, the equation

$$\sqrt{n}[\Lambda'(s) - \gamma] = \sigma x, \quad (5.23)$$

has a unique root $s(x, n) \in (0, \eta)$ which has the expression

$$s(x, n) = \frac{t}{\sqrt{\gamma_2}} - \frac{\gamma_3}{2\gamma_2^2} t^2 - \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{6\gamma_2^{7/2}} t^3 + \dots, \quad \text{with } t = \frac{x}{\sqrt{n}}. \quad (5.24)$$

From now, let $s = s(x, n)$. Again from [20], we have the equality:

$$s\Lambda'(s) - \Lambda(s) = \frac{x^2}{2n} - \frac{x^3}{n^{3/2}} \zeta\left(\frac{x}{\sqrt{n}}\right), \quad (5.25)$$

where $\zeta(t)$ is the Cramér series defined in (2.12) (entirely determined by the function Λ), which converges for $|t|$ small enough. Therefore, combining (5.22), (5.23) and (5.25), we get that for $n \geq 1$ large enough and $1 \leq i \leq d$,

$$\begin{aligned} & \mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\right) \\ &= r_s(e_i) e^{-\frac{x^2}{2} + \frac{x^3}{\sqrt{n}} \zeta\left(\frac{x}{\sqrt{n}}\right)} \mathbb{E}_s^{e_i} \left[r_s^{-1}(X_n^{e_i}) e^{-s\sigma_s \sqrt{n} S_{0,n}^{e_i}} \mathbb{1}_{\left\{\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > 0\right\}} \right] \\ &= e^{-\frac{x^2}{2} + \frac{x^3}{\sqrt{n}} \zeta\left(\frac{x}{\sqrt{n}}\right)} J^i(n), \end{aligned} \quad (5.26)$$

where

$$J^i(n) := r_s(e_i) \mathbb{E}_s^{e_i} \left[r_s^{-1}(X_n^{e_i}) e^{-s\sigma_s \sqrt{n} S_{0,n}^{e_i}} \mathbb{1}_{\left\{\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > 0\right\}} \right].$$

By Proposition 3.1, for $\eta > 0$ small enough the map $s \mapsto r_s$ is analytic on $(-\eta, \eta)$, with $r_0 = 1$. Since $s = O\left(\frac{x}{\sqrt{n}}\right)$ as $n \rightarrow +\infty$ by (5.24), we obtain

$$\|r_s - 1\|_\infty \leq C|s| \leq \frac{Cx}{\sqrt{n}}.$$

This, together with (3.18), implies that for all $y_1, y_2 \in \mathcal{S}$,

$$\begin{aligned} \left| \frac{r_s(y_1)}{r_s(y_2)} - 1 \right| &\leq r_s^{-1}(y_2) (|r_s(y_1) - 1| + |r_s(y_2) - 1|) \\ &\leq 2D \|r_s - 1\|_\infty \\ &\leq \frac{Cx}{\sqrt{n}}. \end{aligned} \quad (5.27)$$

Therefore, using (5.27) and the definition of $J^i(n)$, we deduce that for all $1 \leq i \leq d$, as $n \rightarrow +\infty$,

$$J^i(n) = J_1^i(n) \left[1 + O\left(\frac{x}{\sqrt{n}}\right) \right], \quad (5.28)$$

where

$$J_1^i(n) := \mathbb{E}_s^{e_i} \left[e^{-s\sigma_s \sqrt{n} S_{0,n}^{e_i}} \mathbb{1}_{\left\{\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s \sqrt{n}} > 0\right\}} \right].$$

Then, using Fubini's theorem and the integration by parts, we obtain the following decomposition for $J_1^i(n)$:

$$\begin{aligned}
J_1^i(n) &= \mathbb{E}_s^{e_i} \left[\int_{\mathbb{R}} s\sigma_s\sqrt{n} e^{-s\sigma_s\sqrt{n}u} \mathbb{1}_{\left\{\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > 0, S_{0,n}^{e_i} \leq u\right\}} du \right] \\
&= s\sigma_s\sqrt{n} \int_{\mathbb{R}} e^{-s\sigma_s\sqrt{n}u} \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > 0, S_{0,n}^{e_i} \leq u \right) du \\
&= s\sigma_s\sqrt{n} \int_0^{+\infty} e^{-s\sigma_s\sqrt{n}u} \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > 0, S_{0,n}^{e_i} \leq u \right) du \\
&\quad + s\sigma_s\sqrt{n} \int_{-\infty}^0 e^{-s\sigma_s\sqrt{n}u} \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > 0, S_{0,n}^{e_i} \leq u \right) du \\
&=: J_2^i(n) + J_3^i(n). \tag{5.29}
\end{aligned}$$

Control of $J_2^i(n)$. For any $n \geq 1$, set

$$I(n) := s\sigma_s\sqrt{n} \int_0^{+\infty} e^{-s\sigma_s\sqrt{n}u} \Phi([0, u]) du. \tag{5.30}$$

By (5.1) in Lemma 5.1, we get that for all $n \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned}
|J_2^i(n) - I(n)| &\leq s\sigma_s\sqrt{n} \int_0^{+\infty} e^{-s\sigma_s\sqrt{n}u} \\
&\quad \times \left| \mathbb{P}_s^{e_i} \left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > 0, S_{0,n}^{e_i} \leq u \right) - \Phi([0, u]) \right| du \\
&\leq \frac{C}{\sqrt{n}} s\sigma_s\sqrt{n} \int_0^{+\infty} e^{-s\sigma_s\sqrt{n}u} du \\
&= \frac{C}{\sqrt{n}}. \tag{5.31}
\end{aligned}$$

So, it remains to estimate $I(n)$. Applying an integration by parts, for all $n \geq 1$ we have

$$I(n) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-s\sigma_s\sqrt{n}u - \frac{u^2}{2}} du. \tag{5.32}$$

Since Λ is analytic on $(-\eta, \eta)$ with $\Lambda'(0) = \gamma$ and $\sigma_s^2 = \Lambda''(s) > 0$ by (4.1), by Taylor's formula we have $\Lambda'(s) - \gamma = s\sigma^2 + O(s^2)$ and $\sigma_s^2 = \sigma^2 + O(s)$. Using (5.23) and the fact that $s = O\left(\frac{x}{\sqrt{n}}\right)$, we obtain

$$s\sigma_s = \frac{\Lambda'(s) - \gamma}{\sigma} + o(s) = \frac{x}{\sqrt{n}} + o\left(\frac{x}{\sqrt{n}}\right). \tag{5.33}$$

Using standard methods from Petrov [20], from (5.32) and (5.33) we get

$$I(n) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-xu - \frac{u^2}{2}} du + O\left(\frac{1}{\sqrt{n}}\right).$$

By a simple calculation,

$$\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-xu - \frac{u^2}{2}} du = \frac{e^{\frac{x^2}{2}}}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\frac{u^2}{2}} du = e^{\frac{x^2}{2}} (1 - \Phi(x)).$$

So we deduce that, as $n \rightarrow +\infty$,

$$I(n) = e^{\frac{x^2}{2}} (1 - \Phi(x)) + O\left(\frac{x}{\sqrt{n}}\right). \quad (5.34)$$

Therefore, from (5.31), we get that, as $n \rightarrow +\infty$,

$$J_2^i(n) = e^{\frac{x^2}{2}} (1 - \Phi(x)) + O\left(\frac{1+x}{\sqrt{n}}\right). \quad (5.35)$$

Control of $J_3^i(n)$. By the definition of $J_3^i(n)$ (see (5.29)) and the bound (5.3) in Lemma 5.2, there exist some constants $\alpha > 0$, $\beta > 0$ and $\delta \in (0, 1)$ such that for all $n \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned} J_3^i(n) &\leq C s \sigma_s \int_{-\infty}^0 e^{(s\sigma_s - \alpha)\sqrt{n}|u|} du \\ &\quad + C s \sigma_s \sqrt{n} \int_{-\infty}^0 e^{s\sigma_s \sqrt{n}|u|} \min\{e^{-\beta\sqrt{n}|u|}, |u|^{-\frac{1}{2}} n^{-\frac{1}{4}} \delta \sqrt{n}\} du \\ &\leq C s \sigma_s \int_{-\infty}^0 e^{(s\sigma_s - \alpha)\sqrt{n}|u|} du + C s \sigma_s \sqrt{n} \int_{-\infty}^{-1} e^{(s\sigma_s - \beta)\sqrt{n}|u|} du \\ &\quad + C s \sigma_s n^{1/4} \delta \sqrt{n} \int_{-1}^0 \frac{e^{s\sigma_s \sqrt{n}|u|}}{\sqrt{|u|}} du. \end{aligned} \quad (5.36)$$

Let $\varepsilon \in (0, \min\{\alpha, \beta\})$. By (5.33) we have $s\sigma_s \rightarrow 0$ as $n \rightarrow +\infty$, hence $s\sigma_s \leq \varepsilon$ for n large enough. Implementing this in (5.36), we obtain that for n sufficiently large,

$$\begin{aligned} J_3^i(n) &\leq C \int_{-\infty}^0 e^{(\varepsilon - \alpha)\sqrt{n}|u|} du + C \sqrt{n} \int_{-\infty}^{-1} e^{(\varepsilon - \beta)\sqrt{n}|u|} du \\ &\quad + C n^{1/4} \delta \sqrt{n} \int_{-1}^0 \frac{e^{\varepsilon \sqrt{n}|u|}}{\sqrt{|u|}} du \\ &\leq C \left(\frac{1}{(\alpha - \varepsilon)\sqrt{n}} + \frac{e^{(\varepsilon - \beta)\sqrt{n}}}{(\beta - \varepsilon)} + n^{1/4} (\delta e^\varepsilon)^{\sqrt{n}} \int_{-1}^0 \frac{du}{\sqrt{|u|}} \right). \end{aligned}$$

Taking ε small enough such that $\delta e^\varepsilon < 1$, it follows that for n sufficiently large,

$$J_3^i(n) \leq \frac{C}{\sqrt{n}}. \quad (5.37)$$

Now, combining (5.29), (5.35) and (5.37) we get that, as $n \rightarrow +\infty$,

$$J_1^i(n) = e^{\frac{x^2}{2}} (1 - \Phi(x)) + O\left(\frac{1+x}{\sqrt{n}}\right).$$

Therefore, using (5.26) and (5.28), we obtain that, as $n \rightarrow +\infty$,

$$\begin{aligned} & \mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\right) \\ &= e^{-\frac{x^2}{2} + \frac{x^3}{\sqrt{n}}\zeta(\frac{x}{\sqrt{n}})} \left[e^{\frac{x^2}{2}}(1 - \Phi(x)) + O\left(\frac{1+x}{\sqrt{n}}\right) \right] \left(1 + O\left(\frac{x}{\sqrt{n}}\right)\right) \\ &= e^{\frac{x^3}{\sqrt{n}}\zeta(\frac{x}{\sqrt{n}})}(1 - \Phi(x)) \left[1 + O\left(\frac{1+x}{\sqrt{n}}\right)\right]. \end{aligned}$$

This concludes the proof of (2.13).

The proof of (2.14) is very similar to that of (2.13). We just need to consider the unique root of the equation $\sqrt{n}[\Lambda'(s) - \gamma] = -\sigma x$ instead of (5.23), and apply the inequality (5.2) instead of (5.1), and (5.4) instead of (5.3). This ends the proof of Theorem 2.1. \square

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