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Convergence in L^p for a supercritical multi-type branching process in a random environment

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Abstract: Consider a d -type supercritical branching process $Z_n^i = (Z_n^i(1), \dots, Z_n^i(d))$, $n \geq 0$, in an independent and identically distributed random environment $\xi = (\xi_0, \xi_1, \dots)$, starting with one initial particle of type i . In a previous paper we have established a Kesten-Stigum type theorem for Z_n^i , which implies that for any $1 \leq i, j \leq d$, $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j) \rightarrow W^i$ in probability as $n \rightarrow +\infty$, where $\mathbb{E}_\xi Z_n^i(j)$ is the conditional expectation of $Z_n^i(j)$ given the environment ξ , and W^i is a non-negative and finite random variable. The goal of this paper is to obtain a necessary and sufficient condition for the convergence in L^p of $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$, and to prove that the convergence rate is exponential. To this end, we first establish the corresponding results for the fundamental martingale (W_n^i) associated to the branching process (Z_n^i) .

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1. Introduction

A significant advancement in the theory of branching processes was made with the introduction of a random environment such that the offspring distribution of generation n depends on some random environment ξ_n at time n , in contrast to a constant distribution assumed in the Galton-Watson process. This allows a more adequate modeling, and turned out to be very fruitful in theoretical as well as in practical senses. For the first fundamental results on branching processes in random environments, see Athreya and Karlin [1, 2]. The importance of the study of branching processes in random environments is mainly due to its wide application background, both in theory and in practical problems. For example, Kesten, Kozlov and Spitzer [20] used such a process to study limit properties of for random walks in random environments; biologists are currently paying special attention to the problems of genetic transformation, and such problems can be studied via a multi-type branching process in a random environment; see Bansaye [4] for application in cell contamination. Due to huge applications and important technique challenge, in recent years, there is a great progress in the study of branching processes in random environments, see e.g. the recent papers [26, 30, 31], the recent book by Kersting and Vatutin [23] and many references therein. In an earlier work [11], for a supercritical

multi-type branching process in an independent and identically distributed random environment, we have studied the convergence of the normalized population size and the non-degeneracy of its limit. In this paper, we will consider its convergence in L^p .

Let $Z_n = (Z_n(1), \dots, Z_n(d))$, $n \geq 0$, be a d -type branching process in an independent and identically distributed (i.i.d.) random environment $\xi = (\xi_0, \xi_1, \dots)$. For $n \geq 0$, denote by M_n the matrix of the conditioned means of the offspring distribution of n -th generation given the environment: the (i, j) -th entry of M_n is

$$M_n(i, j) = \mathbb{E}_\xi[Z_{n+1}(j) \mid Z_n = e_i],$$

where \mathbb{E}_ξ denotes the conditional expectation given the environment ξ . Let $M_{0,n} = M_0 \cdots M_n$ be the product matrix. The process $(Z_n)_{n \geq 0}$ will be denoted $(Z_n^i)_{n \geq 0}$ when it starts with one initial particle of type i , that is when $Z_0 = e_i$, where e_i is the unit vector whose i -th component is 1 and all the others are 0. In [11] we obtained an extension of the famous Kesten-Stigum result on the Galton-Watson process to the multi-type branching process in random environment (MBPRE). Assume that the MBPRE $(Z_n^i)_{n \geq 0}$ is in the supercritical regime, in the sense that

$$\gamma := \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\| > 0,$$

where $\|M_{0,n-1}\|$ is the L_1 -norm of the matrix $M_{0,n-1}$. Under the Furstengerg-Kesten condition **H1** (see Section 2), we proved in [11, Theorem 2.11] that for all $1 \leq i, j \leq d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i, j)} \rightarrow W^i \quad \text{in probability,} \quad (1.1)$$

where W^i is a non-negative random variable independent of j ; moreover, W^i is non-degenerate for all i if and only if

$$\mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i, j)} \log^+ \frac{Z_1^i(j)}{M_0(i, j)} \right) < +\infty \quad \forall i, j = 1, \dots, d; \quad (1.2)$$

in addition $\mathbb{E}_\xi W^i = 1$ almost surely (a.s.) when (1.2) holds. By Sheffé's theorem, it follows that $\frac{Z_n^i(j)}{M_{0,n-1}(i, j)} \rightarrow W^i$ in L^1 if and only if (1.2) holds.

The main objective in this paper is to find a necessary and sufficient condition under which the normalized population size $Z_n^i(j)/M_{0,n-1}(i, j)$ converges to W^i in L^p , $p > 1$, and to prove that the convergence rate is exponential, for all $1 \leq i, j \leq d$. In the single type case, it is known that such kind of results play an important role in the study of asymptotic properties of large deviations and Berry Esseen bounds in the central limit theorem on the process (Z_n^i) , see [16, 10]. The situation is the same in the multi-type case, as can be seen in the preprints [12, 13].

For a single type branching process in a random environment $(Z_n)_{n \geq 0}$, Guivarc'h and Liu [14, Theorem 1.3] established the (annealed) L^p convergence criterion: they

showed that when $d = 1$, for each given $p > 1$, $(Z_n/m_{0,n-1})_{n \geq 0}$ converges in L^p to a non negative random variable W if and only if

$$\mathbb{E} \left(\frac{Z_1}{m_0} \right)^p < +\infty \quad \text{and} \quad \mathbb{E} m_0^{1-p} < 1, \quad (1.3)$$

where $m_{0,n-1} = m_0 \cdots m_{n-1}$, and m_k denotes the conditioned mean of the offspring distribution at time k given the environment. Huang and Liu [17, Theorem 1.5] proved that the L^p convergence rate is exponential: if (1.3) holds, then

$$\lim_{n \rightarrow +\infty} \delta^{-n} \left(\mathbb{E} \left| \frac{Z_n}{m_{0,n-1}} - W \right|^p \right)^{1/p} = 0 \quad \forall \delta > \delta_c(p), \quad (1.4)$$

with

$$\delta_c(p) = \begin{cases} (\mathbb{E} m_0^{1-p})^{1/p} & \text{if } p \in (1, 2), \\ \max \left\{ (\mathbb{E} m_0^{1-p})^{1/p}, (\mathbb{E} m_0^{-p/2})^{1/p} \right\} & \text{if } p \geq 2. \end{cases} \quad (1.5)$$

For the MBPRE's case, the only result about the annealed L^p convergence is a claim by Cohn [8] which concerns the L^2 convergence. Assume that the supercritical condition $\gamma > 0$ holds, that each entry of M_0 is bounded a.s. from below and above by two positive constants, and that all the conditional second moments of the offspring distributions given the environment are bounded a.s. by a constant. Assume also the integrability condition $\mathbb{E} |\log \sum_{i=1}^d (1 - \mathbb{P}(\|Z_1^i\| = 0))| < \infty$. Under these conditions Cohn [8] claimed that for each $j = 1, \dots, d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \rightarrow W^i \quad \text{in } L^2 \quad (1.6)$$

under the annealed law \mathbb{P} , where W^i is a non degenerate random variable satisfying $\mathbb{E} W^i = 1$. However, the claim of Cohn [8] is false. To see this, it suffices to notice that when $d = 1$, (1.6) holds if and only if $\mathbb{E} \left(\frac{Z_1}{m_0} \right)^2 < +\infty$ and $\mathbb{E} m_0^{-1} < 1$ by the criterion (1.3) of Guivarc'h and Liu [14, Theorem 1.3]. A quantitative condition (which ensures $\mathbb{E} m_0^{-1} < 1$ for $d = 1$) is missing in the claim of Cohn [8]. This shows that the annealed L^p convergence is rather delicate even for $p = 2$. We mention that Jones [18], Biggins, Cohn and Nerman [5] have studied respectively the L^2 and L^p convergence of multi-type branching processes in varying environment. Their results give sufficient conditions for quenched L^p convergence for multi-type branching processes in random environments. In this paper, we deal with the annealed L^p convergence, which is in general more delicate because there is an additional integral operation. Since we will always deal with the annealed L^p convergence, for simplicity we will just say L^p convergence in the following.

More precisely, we will find a necessary and sufficient condition for the L^p convergence which extends the criterion (1.3) to the multi-type case, and establish the exponential convergence rate. Let $p > 1$ be such that $\mathbb{E} M_0(i, j)^{1-p} < +\infty$ for all $1 \leq i, j \leq d$, and define

$$\kappa(1-p) = \lim_{n \rightarrow +\infty} (\mathbb{E} \|M_{0,n-1}\|^{1-p})^{1/n}.$$

It will be shown that the limit exists and is finite. Under the Furstenberg-Kesten condition **H1**, we will prove that $\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \rightarrow W^i$ in L^p for any $1 \leq i, j \leq d$ if and only if

$$\max_{1 \leq i, j \leq d} \mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i,j)} \right)^p < +\infty \quad \text{and} \quad \kappa(1-p) < 1. \quad (1.7)$$

(cf. Theorem 2.1); moreover, if (1.7) holds, then there exists $\delta \in (0, 1)$ such that

$$\lim_{n \rightarrow +\infty} \delta^{-n} \left(\mathbb{E} \left| \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} - W^i \right|^p \right)^{1/p} = 0 \quad (1.8)$$

(cf. Theorem 2.2). For a single type branching process in random environment, we have $\kappa(1-p) = \mathbb{E}m_0^{1-p}$, so (1.7) coincides with (1.3), and (1.8) corresponds to (1.4) but with less information on the exact exponential rate.

The proof of (1.7) and (1.8) is based on the corresponding results for the associated fundamental martingale (W_n^i) introduced in [11]. Let us recall briefly its construction. For any $n, k \geq 0$, let $\rho_{n,n+k}$ be the spectral radius of $M_{n,n+k}$. Applying the famous Perron-Frobenius theorem (see e.g. [3]), $\rho_{n,n+k}$ is a positive eigenvalue of $M_{n,n+k}$, for which there exist positive right and left eigenvectors $U_{n,n+k}$ and $V_{n,n+k}$ with the normalizations $\|U_{n,n+k}\| = 1$ and $\langle V_{n,n+k}, U_{n,n+k} \rangle = 1$, where $\|x\|$ denotes the L^1 norm of the vector x , and $\langle x, y \rangle$ the scalar product of the vectors x, y . Then, under certain conditions, by the results of Hennion [15, Lemma 3.3 and Theorem 1] the limit

$$U_{n,\infty} := \lim_{k \rightarrow \infty} U_{n,n+k} \quad (1.9)$$

exists a.s., with $U_{n,\infty} > 0$ a.s. and $\|U_{n,\infty}\| = 1$; moreover, there exist random scalars $\lambda_n > 0$ a.s. called the pseudo-spectral radii of the random matrices (M_n) , which satisfy a.s. the relation

$$M_n U_{n+1,\infty} = \lambda_n U_{n,\infty}. \quad (1.10)$$

The relation (1.10) can be iterated to obtain

$$M_{n,n+k} U_{n+k+1,\infty} = \lambda_{n,n+k} U_{n,\infty}, \quad (1.11)$$

where $\lambda_{n,n+k} = \prod_{r=n}^{n+k} \lambda_r$ for $n, k \geq 0$. Then, the non-negative martingale (W_n^i) is defined as follows [11]:

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, U_{n,\infty} \rangle}{\lambda_{0,n-1} U_{0,\infty}(i)}, \quad n \geq 1. \quad (1.12)$$

Assume for simplicity that the Furstenberg-Kesten condition **H1** is satisfied. Assume also that $p > 1$ is such that $\mathbb{E}M_0(i,j)^{1-p} < +\infty$ for all $1 \leq i, j \leq d$. Then we show that W_n^i converges in L^p to the random variable W^i for any $1 \leq i \leq d$ if and only if (1.7) holds (cf. Theorem 2.3); moreover, if (1.7) is satisfied, then

$$\lim_{n \rightarrow +\infty} \delta^{-n} (\mathbb{E}|W_n^i - W^i|^p)^{1/p} = 0 \quad \forall \delta > \delta_c(p), \quad (1.13)$$

with

$$\delta_c(p) = \begin{cases} \kappa(1-p)^{1/p} & \text{if } p \in (1, 2), \\ \max \left\{ \kappa(1-p)^{1/p}, \kappa(-p/2)^{1/p} \right\} & \text{if } p \geq 2 \end{cases} \quad (1.14)$$

(cf. Theorem 2.4). In the case of the single type branching process, the martingale (W_n) coincides with the normalized population size $(Z_n/m_{0,n-1})$, so the relations (1.13) and (1.14) coincide exactly with (1.4) and (1.5). It is known that when $d = 1$, the critical value $\delta_c(p)$ is the best possible for (1.13) to hold (see Huang and Liu [17]).

For the proof, we develop the approach in [17] where the case $d = 1$ was considered. In addition to the complexity related to the products of random matrices, the main difficulty for the multi-dimensional case resides in the fact that W_n^i depends on the whole environment sequence $\xi = (\xi_0, \xi_1, \dots)$, not just on the environment sequence until the present $(\xi_0, \dots, \xi_{n-1})$, contrary to the one-dimensional case. Let us give a short description of the approach. For $p \in (1, 2]$, we first control the quenched L^p norm of the martingale difference $W_{n+1}^i - W_n^i$, using the branching property and the Marcinkiewicz-Zygmund inequality on the L^p norm of sums of independent random variables. This permits us to obtain a bound of $\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p$ in terms of $(\lambda_{0,n-1} U_{0,\infty}(i))^{1-p}$. To overcome the difficulty related to the dependence on the whole environment sequence, we condition on the future $T^n \xi = (\xi_n, \xi_{n+1}, \dots)$ to obtain $\mathbb{E}_{T^n \xi} (\lambda_{0,n-1} U_{0,\infty}(i))^{1-p} \leq C \kappa(1-p)^n$, which gives the correct convergence rate in L^p for the martingale (W_n^i) . For $p > 2$, we use an argument by induction. To get the convergence rate of the normalized population size $Z_n^i(j)/M_{0,n-1}(i, j)$, we prove that the difference $Z_n^i(j)/M_{0,n-1}(i, j) - W_n^i$ goes to 0 in L^p exponentially fast, using the exponential convergence of the products of stochastic matrices due to Seneta [28]. For the necessity, we first establish some spectral properties of the important transfer operator P_s for $s \leq 0$ (see Section 3).

The main results will be presented in Section 2. In Section 3 we establish the spectral properties of the transfer operator P_s that we will need. In Section 4 we prove the criterion for the convergence in L^p of the martingales (W_n^i) , as well as their exponential convergence rate. Similar results for the normalized population size $Z_n^i(j)/M_{0,n-1}(i, j)$ are proved in Section 5.

2. Notation and main results

Let $\mathbb{N} = \{0, 1, \dots\}$ be the set of non-negative integers. The indicator of an event A is denoted by $\mathbb{1}_A$. The symbol $\xrightarrow{\mathbb{P}}$ denotes the convergence in probability with respect to the annealed law \mathbb{P} . For an integer $d \geq 1$, let \mathbb{R}^d be the d -dimensional space of vectors with real coordinates, equipped with the scalar product and the L^1 -norm respectively defined by

$$\langle x, y \rangle := \sum_{i=1}^d x(i) y(i) \quad \text{and} \quad \|x\| := \sum_{i=1}^d |x(i)|, \quad x, y \in \mathbb{R}^d.$$

Let e_i be the d -dimensional vector with 1 in the i -th place and 0 elsewhere. Define also $\mathcal{M}_d(\mathbb{R})$ the set of $d \times d$ matrices with entries in \mathbb{R} , and the operator norm on $\mathcal{M}_d(\mathbb{R})$:

$$\|M\| := \sup_{\|x\|=1} \|Mx\|, \quad M \in \mathcal{M}_d(\mathbb{R}).$$

For a matrix or a vector X , we write $X > 0$ to mean that each entry of X is strictly positive.

Let us define precisely the multi-type branching process in random environment (MBPRE). Let $\xi = (\xi_n)_{n \geq 0}$ be the random environment, which is an independent and identically distributed (i.i.d.) sequence with values in an abstract space \mathbb{X} . To each realization of ξ_n , we associate d probability generating functions : for $1 \leq r \leq d$,

$$f_n^r(s) = \sum_{k_1, \dots, k_d=0}^{\infty} p_{k_1, \dots, k_d}^r(\xi_n) s_1^{k_1} \cdots s_d^{k_d}, \quad s = (s_1, \dots, s_d) \in [0, 1]^d.$$

A MBPRE (Z_n) in the random environment ξ is a process with values in \mathbb{N}^d such that for all $n \geq 0$,

$$Z_{n+1} = \sum_{r=1}^d \sum_{l=1}^{Z_n(r)} N_{l,n}^r, \quad (2.1)$$

where $Z_0 \in \mathbb{N}^d$ is fixed, $Z_n(j)$ represents the number of particles of type j of some population in generation n , and $N_{l,n}^r(j)$ is the offspring of type j at time $n+1$ of the l -th particle of type r in generation n . The random vectors $N_{l,n}^r = (N_{l,n}^r(1), \dots, N_{l,n}^r(d))$, indexed by $l \geq 1$, $n \geq 0$, $r \in \{1, \dots, d\}$, are conditionally independent and have the same probability generating function f_n^r , given the environment ξ . Set $f_n := (f_n^1, \dots, f_n^d)$. When the process starts with one initial particle of type i , that is, when $Z_0 = e_i$, we write Z_n^i instead of Z_n .

Denote by \mathbb{P}_ξ the underline probability when the environment ξ is given; it is called quenched law. Let τ be the law of the environment ξ . Then, the total probability \mathbb{P} , called annealed law, is defined by $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx) \tau(d\xi)$. The expectation with respect to \mathbb{P}_ξ and \mathbb{P} are denoted respectively by \mathbb{E}_ξ and \mathbb{E} . By our notation the quenched probability generating function of $N_{l,n}^r$ is

$$f_n^r(s) = \mathbb{E}_\xi \left(\prod_{j=1}^d s_j^{N_{l,n}^r(j)} \right), \quad s = (s_1, \dots, s_d) \in [0, 1]^d.$$

We introduce the random mean matrices $M_n \in \mathcal{M}_d(\mathbb{R})$ whose entries are defined by

$$M_n(i, j) := \frac{\partial f_n^i}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi [Z_{n+1}(j) | Z_n = e_i], \quad 1 \leq i, j \leq d, \quad n \geq 0,$$

where $\frac{\partial f_n^i}{\partial s_j}(\mathbf{1})$ is the left derivative at $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$ of the d -dimensional probability generating function f_n^i with respect to s_j . For each $1 \leq i, j \leq d$, $M_n(i, j)$

represents the conditioned mean of the number of children of type j produced by a particle of type i at time n . The sequence of random matrices (M_n) is i.i.d. (because the sequence (ξ_n) is i.i.d.). We define the products of these matrices by

$$M_{k,n} := M_k \cdots M_n, \quad 0 \leq k \leq n.$$

Notice that we have

$$\mathbb{E}_\xi Z_{n+1}^i(j) = M_{0,n}(i, j), \quad n \geq 0, \quad 1 \leq i, j \leq d. \quad (2.2)$$

For $n, k \geq 0$, denote by $\rho_{n,n+k}$ the spectral radius of $M_{n,n+k}$. By the Perron-Frobenius theorem (see e.g. [3]), $\rho_{n,n+k}$ is an eigenvalue of $M_{n,n+k}$. Let $U_{n,n+k}$ and $V_{n,n+k}$ be respectively the positive right and left eigenvectors associated to the eigenvalue $\rho_{n,n+k}$, with the normalizations $\|U_{n,n+k}\| = 1$ and $\langle V_{n,n+k}, U_{n,n+k} \rangle = 1$.

Let \mathcal{G}_+^0 be the subset of the matrices of $\mathcal{M}_d(\mathbb{R})$ with strictly positive entries. According to the results of Hennion [15, Lemma 3.3 and Theorem 1], if M_0 is allowable in the sense that every row and column contains a strictly positive element, and if the positivity condition

$$\mathbb{P}\left(\bigcup_{n \geq 0} \{M_{0,n} \in \mathcal{G}_+^0\}\right) > 0 \quad (2.3)$$

holds, then the random vectors $U_{n,\infty}$ and the random scalars λ_n are well defined by (1.9) and (1.10), and satisfy (1.11). Note that the sequences $(U_{n,\infty})$ and (λ_n) are stationary ergodic. It is proved in [11, Theorem 1] that the sequence $(W_n^i)_{n \geq 0}$ defined by (1.12) is a non-negative martingale under \mathbb{P}_ξ and \mathbb{P} , with respect to the filtration

$$\mathcal{F}_0 = \sigma(\xi), \quad \mathcal{F}_n = \sigma(\xi, N_{l,k}^r(j), 0 \leq k \leq n-1, 1 \leq r, j \leq d, l \geq 1) \quad \text{for } n \geq 1.$$

Thus \mathbb{P} -a.s. for all $1 \leq i \leq d$, the limit

$$W^i := \lim_{n \rightarrow +\infty} W_n^i \quad (2.4)$$

exists and $\mathbb{E}_\xi W^i \leq 1$ by Fatou's lemma.

Now we introduce a classification of MBPRE's. Under the following moment condition

$$\mathbb{E} \log^+ \|M_0\| < +\infty, \quad (2.5)$$

by an argument of sub-additivity, the limite

$$\gamma := \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\| = \inf_{n \geq 1} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\|,$$

exists; it is called Lyapunov exponent of the sequence $(M_n)_{n \geq 0}$. Moreover, Furstenberg and Kesten established in [9] a strong law of large numbers for $\log \|M_{0,n-1}\|$:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|M_{0,n-1}\| = \gamma \quad \mathbb{P}\text{-a.s.}$$

We say that a MBPRE is supercritical if $\gamma > 0$, critical if $\gamma = 0$, and subcritical if $\gamma < 0$. In this paper, the process (Z_n) will always be supercritical, i.e. $\gamma > 0$, under the conditions that we will assume.

The non-degeneracy of the limit variables W^i has been studied in [11]. In particular, when $\gamma > 0$, it has been proved in [11, Theorem 2.6] that the $X \log X$ condition

$$\mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)} \right) < +\infty \quad \forall 1 \leq i, j \leq d \quad (2.6)$$

is sufficient for the non-degeneracy of each W^i in the sense that $\mathbb{P}(W^i > 0) > 0$, and that this condition is also necessary under the additional condition **H1** that we will introduce below. Moreover, when W^i are non-degenerate, then

$$\mathbb{P}_\xi(W^i > 0) > 0 \quad \text{and} \quad \mathbb{E}_\xi W^i = 1 \quad \text{a.s.}, \quad \text{and} \quad W_n^i \rightarrow W^i \text{ in } L^1. \quad (2.7)$$

In this paper, for a given $p > 1$, we study the convergence in L^p of the fundamental martingale $(W_n^i)_{n \geq 0}$ and the normalized population size $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$, for all $i, j = 1, \dots, d$.

We first consider the martingale $(W_n^i)_{n \geq 0}$, $1 \leq i \leq d$. To formulate our results, we need to introduce some notation and condition. Set

$$I = \{s \leq 0 : \mathbb{E} M_0(i, j)^s < +\infty \quad \forall i, j = 1, \dots, d\}.$$

Obviously, by Hölder's inequality, I is an interval, and if there exists $s \in I$ with $s < 0$, then $M_0 > 0$ \mathbb{P} -a.s., so that condition (2.3) is satisfied. It will be seen in Proposition 3.1 that for $s \in I$ the limit

$$\kappa(s) := \lim_{n \rightarrow +\infty} (\mathbb{E} \|M_{0, n-1}\|^s)^{1/n} \quad (2.8)$$

exists, with $\kappa(s) < +\infty$. Notice that κ is a log-convex function on I . We will need the following condition of Furstenberg and Kesten [9]:

H1. *There exists a constant $D > 1$ such that*

$$1 \leq \frac{\max_{1 \leq i, j \leq d} M_0(i, j)}{\min_{1 \leq i, j \leq d} M_0(i, j)} \leq D.$$

Note that condition **H1** implies condition (2.3).

Our first theorem gives sufficient and necessary conditions for the L^p convergence of the martingales (W_n^i) , $1 \leq i \leq d$.

Theorem 2.1. *Let $p > 1$ be such that $1 - p \in I$. If*

$$\max_{1 \leq i, j \leq d} \mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i, j)} \right)^p < +\infty \quad \text{and} \quad \kappa(1 - p) < 1, \quad (2.9)$$

*then $W_n^i \xrightarrow[n \rightarrow +\infty]{} W^i$ in L^p for any $1 \leq i \leq d$. The converse is also valid when the Furstenberg-Kesten condition **H1** holds.*

It is clear that condition (2.9) implies (2.6). Moreover, (2.9) also implies the supercritical condition $\gamma > 0$ when condition (2.5) holds, since by Jensen's inequality we have $\log \kappa(1-p) \geq (1-p)\gamma$.

Our second theorem shows that the L^p convergence of W_n^i has an exponential rate.

Theorem 2.2. *Let $p > 1$ be such that $1-p \in I$. Assume (2.9).*

1. *If $1 < p \leq 2$, then denoting $\delta_c(p) = \kappa(1-p)^{1/p}$ we have*

$$\limsup_{n \rightarrow +\infty} \delta_c(p)^{-n} (\mathbb{E}|W_n^i - W^i|^p)^{1/p} < +\infty. \quad (2.10)$$

2. *If $p > 2$, then $\delta_c(p) := \max \{ \kappa(1-p)^{1/p}, \kappa(-p/2)^{1/p} \} < 1$, and*

$$\lim_{n \rightarrow +\infty} \delta^{-n} (\mathbb{E}|W_n^i - W^i|^p)^{1/p} = 0 \quad \forall \delta > \delta_c(p). \quad (2.11)$$

In the proof we shall see that in Part 1 the moment condition $\mathbb{E}(\frac{Z_n^i(j)}{M_0(i,j)})^p < +\infty$ for all $1 \leq i, j \leq d$ can be relaxed to $\mathbb{E}(W_1^i)^p < +\infty$ for all $1 \leq i \leq d$.

Note that for $p \geq 2$, by applying Hölder's inequality to $\mathbb{E}\|M_{0,n-1}\|^{-p/2}$ and then letting $n \rightarrow +\infty$, it is easy to see that $\kappa(-p/2)^{2/p} \leq \kappa(1-p)^{1/(p-1)}$. Thus $\kappa(1-p) < 1$ implies $\kappa(-p/2) < 1$, so that $\delta_c(p) < 1$.

Now we investigate the convergence in L^p of the normalized population size $\frac{Z_n^i(j)}{\mathbb{E} Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)}$. Recall that under condition (2.5), **H1** and the supercriticality condition $\gamma > 0$, by the Kesten-Stigum type theorem for a supercritical MBPRE [11, Theorem 2.11], for all $1 \leq i, j \leq d$,

$$\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} W^i, \quad (2.12)$$

and the convergence holds a.s. if additionally $\mathbb{E}(Z_1^i(j)/M_0(i,j))^p < +\infty$ and $\mathbb{E}\|M_0\|^{1-p} < +\infty$ for some $p > 1$ and all $1 \leq i, j \leq d$ (see [11, Theorem 2.13]). By [11, Theorem 2.11] and Sheffé's theorem, under the supercritical condition $\gamma > 0$ and the Furstengerg-Kesten condition **H1**, $\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \rightarrow W^i$ in L^1 if and only if (2.6) holds. From Theorem 2.1 and under condition **H1**, we obtain a criterion for the convergence in L^p of $\frac{Z_n^i(j)}{M_{0,n-1}(i,j)}$.

Theorem 2.3. *Assume condition **H1**. Let $p > 1$ be such that $1-p \in I$. Then*

$\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \xrightarrow[n \rightarrow +\infty]{} W^i$ in L^p for any $1 \leq i, j \leq d$ if and only if (2.9) holds.

Finally, from Theorem 2.2, we deduce an exponential rate of the convergence in L^p of $\frac{Z_n^i(j)}{M_{0,n-1}(i,j)}$.

Theorem 2.4. *Assume condition **H1**. Let $p > 1$ be such that $1-p \in I$ and that (2.9) holds. Then there exists $\delta \in (0, 1)$ such that*

$$\lim_{n \rightarrow +\infty} \delta^{-n} \left(\mathbb{E} \left| \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} - W^i \right|^p \right)^{1/p} = 0. \quad (2.13)$$

3. Spectral properties of the transfer operator P_s

We start this section by giving some notation. Denote by $\mathcal{S} = \{x \in \mathbb{R}^d : x \geq 0, \|x\| = 1\}$. For $x \in \mathcal{S}$ and $M \in \mathcal{G}_+^0$ (the set of matrices with strictly positive entries), define the projective action of M on \mathcal{S} by $M \cdot x := \frac{Mx}{\|Mx\|}$. Let $\mathcal{C}(\mathcal{S})$ be the space of continuous functions on \mathcal{S} with real values. For any $\varphi \in \mathcal{C}(\mathcal{S})$, set

$$\|\varphi\|_\infty = \sup_{x \in \mathcal{S}} \|\varphi x\|.$$

For $s \in I$, define the transfer operator P_s as follows : for all $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s \varphi(x) := \mathbb{E}[\|M_0 x\|^s \varphi(M_0 \cdot x)], \quad x \in \mathcal{S}. \quad (3.1)$$

Define also the conjugate operator P_s^* , such that for $s \in I$ and $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s^* \varphi(x) := \mathbb{E}[\|M_0^T x\|^s \varphi(M_0^T \cdot x)], \quad x \in \mathcal{S}. \quad (3.2)$$

In this section, we investigate the spectral properties of the transfer operator P_s and its conjugate P_s^* for $s \leq 0$. These results extend some properties known in the case $s \geq 0$ (see [25, 6]) to the case $s < 0$. We also give some properties of $\kappa(s)$. The main result is given by the following proposition. We use the notation $\mu(\psi) = \int \psi d\mu$ to denote the integral of ψ with respect the measure μ .

Proposition 3.1. *Assume that $s \in I$. Then $\kappa(s) < +\infty$, the spectral radius of P_s is equal to $\kappa(s)$, and there exists a probability measure ν_s on \mathcal{S} and a strictly positive function $r_s \in \mathcal{C}(\mathcal{S})$ such that*

$$\nu_s P_s = \kappa(s) \nu_s \quad \text{and} \quad P_s r_s = \kappa(s) r_s,$$

where $\nu_s P_s$ denotes the measure on \mathcal{S} such that $(\nu_s P_s)(\psi) = \nu_s(P_s \psi)$ for all $\psi \in \mathcal{C}(\mathcal{S})$. Moreover, $\kappa(s)$ is also the spectral radius of P_s^* , and there exists a probability measure ν_s^* on \mathcal{S} and a strictly positive function $r_s^* \in \mathcal{C}(\mathcal{S})$ such that

$$\nu_s^* P_s^* = \kappa(s) \nu_s^* \quad \text{and} \quad P_s^* r_s^* = \kappa(s) r_s^*.$$

To prove the above proposition, we will use the following Lemma about the properties of $\kappa(s)$, $s \leq 0$.

Lemma 3.2. *Assume that $s \in I$. Then*

$$\kappa(s) = \lim_{n \rightarrow +\infty} (\mathbb{E}\|M_{0,n-1}\|^s)^{1/n} = \sup_{n \geq 1} (\mathbb{E}\|M_{0,n-1}\|^s)^{1/n} < +\infty,$$

and there exists $C_s > 0$ such that for all $x, y \in \mathcal{S}$ and $n \geq 1$,

$$\mathbb{E}\|M_{0,n-1}\|^s \leq \mathbb{E}\|M_{0,n-1}x\|^s \leq \mathbb{E}\langle M_{0,n-1}x, y \rangle^s \leq C_s \mathbb{E}\|M_{0,n-1}\|^s.$$

Proof. Notice that the sequence $(\mathbb{E}\|M_{0,n-1}\|^s)_{n \geq 1}$ is super-multiplicative for $s \in I$, so the limit $\kappa(s) = \lim_{n \rightarrow \infty} (\mathbb{E}\|M_{0,n-1}\|^s)^{1/n}$ exists, and

$$\kappa(s) = \sup_{n \geq 1} (\mathbb{E}\|M_{0,n-1}\|^s)^{1/n} \in \mathbb{R}_+ \cup \{+\infty\}.$$

Clearly, for all $x, y \in \mathcal{S}$ and $n \geq 1$, we have the inequalities

$$\mathbb{E}\|M_{0,n-1}\|^s \leq \mathbb{E}\|M_{0,n-1}x\|^s \leq \mathbb{E}\langle M_{0,n-1}x, y \rangle^s \leq \mathbb{E}\left(\max_{1 \leq i, j \leq d} M_{0,n-1}(i, j)^s\right). \quad (3.3)$$

Moreover, since the sequence of matrices (M_n) is i.i.d, for all $n, k \geq 1$ we have

$$\begin{aligned} & \mathbb{E} \max_{1 \leq i, j \leq d} M_{0,n+k-1}(i, j)^s \\ & \leq \mathbb{E} \max_{1 \leq i, j, l \leq d} (M_{0,n-1}(i, l)^s M_{n,n+k-1}(l, j)^s) \\ & \leq \mathbb{E} \max_{1 \leq i, l \leq d} M_{0,n-1}(i, l)^s \mathbb{E} \max_{1 \leq l, j \leq d} M_{0,k-1}(l, j)^s. \end{aligned}$$

Hence $(\mathbb{E} \max_{1 \leq i, j \leq d} M_{0,n-1}(i, j)^s)_{n \geq 1}$ is sub-multiplicative, so that

$$\lim_{n \rightarrow +\infty} (\mathbb{E} \max_{1 \leq i, j \leq d} M_{0,n-1}(i, j)^s)^{1/n} = \inf_{n \geq 1} (\mathbb{E} \max_{1 \leq i, j \leq d} M_{0,n-1}(i, j)^s)^{1/n}.$$

Combining this with (3.3), and letting $n \rightarrow +\infty$, we obtain

$$\kappa(s) \leq \lim_{n \rightarrow +\infty} (\mathbb{E} \max_{1 \leq i, j \leq d} M_{0,n-1}(i, j)^s)^{1/n} \leq \mathbb{E} \max_{1 \leq i, j \leq d} M_0(i, j)^s < +\infty.$$

Furthermore by simple calculations, for all $n \geq 3$ it holds that

$$\begin{aligned} & \mathbb{E} \max_{1 \leq i, j \leq d} M_{0,n-1}(i, j)^s \\ & = \mathbb{E} \max_{1 \leq i, j \leq d} \left(\sum_{1 \leq l_1, l_2 \leq d} M_0(i, l_1) M_{1,n-2}(l_1, l_2) M_{n-1}(l_2, j) \right)^s \\ & \leq \mathbb{E} \left(\min_{1 \leq i, l_1 \leq d} M_0(i, l_1) \min_{1 \leq l_2, j \leq d} M_{n-1}(l_2, j) \sum_{1 \leq l_1, l_2 \leq d} M_{1,n-2}(l_1, l_2) \right)^s \\ & \leq (\mathbb{E} \max_{1 \leq i, j \leq d} M_0(i, j)^s)^2 \mathbb{E}\|M_{0,n-3}\|^s. \end{aligned}$$

It follows that for all $n \geq 3$,

$$\begin{aligned} & \mathbb{E} \max_{1 \leq i, j \leq d} M_{0,n-1}(i, j)^s \\ & \leq (\mathbb{E} \max_{1 \leq i, j \leq d} M_0(i, j)^s)^2 \frac{\mathbb{E}(\|M_{0,n-3}\| \|M_{n-2}\| \|M_{n-1}\|)^s}{(\mathbb{E}\|M_0\|^s)^2} \\ & \leq \left(\frac{\mathbb{E} \max_{1 \leq i, j \leq d} M_0(i, j)^s}{\mathbb{E}\|M_0\|^s} \right)^2 \mathbb{E}\|M_{0,n-1}\|^s. \end{aligned}$$

This, together with (3.3), proves the inequalities of Lemma 3.2 for $n \geq 3$ with

$$C_s = \left(\frac{\mathbb{E} \max_{1 \leq i, j \leq d} M_0(i, j)^s}{\mathbb{E}\|M_0\|^s} \right)^2,$$

which is finite since $s \in I$. It is clear that the inequalities remain valid for $1 \leq n \leq 2$ by modifying slightly the value of C_s (choosing it large enough). This concludes the proof of Lemma 3.2. \square

Proof of Proposition 3.1. We shall use an argument similar to that in the proof of [6, Proposition 4.4] where the case $s \geq 0$ is considered. Let $M^1(\mathcal{S})$ be the set of all probability measures on \mathcal{S} , and $M_b^1(\mathbb{R})$ the set of all finite signed measures on \mathbb{R} equipped with the total variation norm. Since $M^1(\mathcal{S})$ is a compact convex subset of the Banach space $M_b^1(\mathbb{R})$, by the Schauder-Tychonoff theorem applied to the continuous map $\nu \mapsto \nu P_s / \nu P_s(\mathcal{S})$, there exists an invariant probability measure $\nu_s \in M^1(\mathcal{S})$ of this map. Consequently, ν_s is an eigenmeasure of P_s :

$$\nu_s P_s = [\nu_s P_s(\mathcal{S})] \nu_s. \quad (3.4)$$

In the same way there exists an probability eigenmeasure ν_s^* of the operator P_s^* , associated to the eigenvalue $k(s) = \nu_s^* P_s^*(\mathcal{S})$:

$$\nu_s^* P_s^* = k(s) \nu_s^*. \quad (3.5)$$

Set

$$r_s(x) := \frac{1}{k(s)} \int_{\mathcal{S}} \mathbb{E} \langle M_0 x, y \rangle^s \nu_s^*(dy), \quad x \in \mathcal{S}.$$

Since $s \in I$, it is clear that for all $x \in \mathcal{S}$,

$$0 < \frac{1}{k(s)} \mathbb{E} \min_{1 \leq i, j \leq d} M_0(i, j)^s \leq r_s(x) \leq \frac{1}{k(s)} \mathbb{E} \max_{1 \leq i, j \leq d} M_0(i, j)^s < +\infty,$$

and that $r_s \in \mathcal{C}(\mathcal{S})$. Moreover, for all $x \in \mathcal{S}$ we have

$$\begin{aligned} r_s(x) &= \frac{1}{k(s)} \int_{\mathcal{S}} \mathbb{E} \langle x, M_0^T y \rangle^s \nu_s^*(dy) \\ &= \frac{1}{k(s)} \int_{\mathcal{S}} \mathbb{E} [\|M_0^T y\|^s \langle x, M_0^T \cdot y \rangle^s] \nu_s^*(dy) \\ &= \frac{1}{k(s)} \int_{\mathcal{S}} \langle x, y \rangle^s (\nu_s^* P_s^*)(dy) \\ &= \int_{\mathcal{S}} \langle x, y \rangle^s \nu_s^*(dy). \end{aligned} \quad (3.6)$$

Using Fubini's theorem, it follows from the definition of r_s and (3.6) that for all $x \in \mathcal{S}$,

$$\begin{aligned} r_s(x) &= \frac{1}{k(s)} \mathbb{E} [\|M_0 x\|^s \int_{\mathcal{S}} \langle M_0 \cdot x, y \rangle^s \nu_s^*(dy)] \\ &= \frac{1}{k(s)} \mathbb{E} [\|M_0 x\|^s r_s(M_0 \cdot x)] \\ &= \frac{1}{k(s)} P_s r_s(x). \end{aligned} \quad (3.7)$$

So we have proved that $P_s r_s = k(s) r_s$. Now we show that

$$k(s) = \rho(P_s) = \kappa(s),$$

where $\rho(P_s)$ is the spectral radius of P_s .

First we have $k(s) \leq \rho(P_s)$, since $k(s)$ is a positive eigenvalue of P_s .

Next, we prove that $\rho(P_s) \leq \kappa(s)$. By iteration of the operator P_s (using the fact that $M_{0,n-1}$ has the same law as $M_{n-1} \cdots M_0$) and Lemma 3.2, it holds that for all $\varphi \in \mathcal{C}(\mathcal{S})$ and $x \in \mathcal{S}$,

$$P_s^n \varphi(x) = \mathbb{E}[\|M_{0,n-1}x\|^s \varphi(M_{0,n-1} \cdot x)] \leq C_s \|\varphi\|_\infty \mathbb{E}\|M_{0,n-1}\|^s.$$

This implies that

$$\rho(P_s) = \lim_{n \rightarrow \infty} \sup\{\|P_s^n \varphi\|_\infty^{1/n} : \|\varphi\|_\infty = 1\} \leq \lim_{n \rightarrow +\infty} (\mathbb{E}\|M_{0,n-1}\|^s)^{1/n} = \kappa(s).$$

We then prove that $\kappa(s) \leq k(s)$. Iterating the relation $\nu_s^* P_s^* = k(s) \nu_s^*$, we obtain $\nu_s^* P_s^{*n} = k(s)^n \nu_s^*$, so that

$$k(s)^n = \nu_s^* P_s^{*n}(\mathcal{S}) = \int_{\mathcal{S}} \mathbb{E}\|M_{0,n-1}^T y\|^s \nu_s^*(dy) \geq \mathbb{E}\|M_{0,n-1}^T\|^s.$$

This implies $\kappa(s) \leq k(s)$, since

$$\kappa(s) = \lim_{n \rightarrow +\infty} (\mathbb{E}\|M_{0,n-1}\|^s)^{1/n} = \lim_{n \rightarrow +\infty} (\mathbb{E}\|M_{0,n-1}^T\|^s)^{1/n}.$$

So we have proved the equalities $k(s) = \rho(P_s) = \kappa(s)$. This together with (3.5) and (3.7) implies

$$\nu_s^* P_s^* = \kappa(s) \nu_s^* \quad \text{and} \quad P_s r_s = \kappa(s) r_s.$$

Changing the roles of P_s and P_s^* , by the same arguments we can prove that

$$\nu_s P_s = \kappa(s) \nu_s \quad \text{and} \quad P_s^* r_s^* = \kappa(s) r_s^*$$

for some strictly positive function $r_s^* \in \mathcal{C}(\mathcal{S})$, and that $\kappa(s)$ is also the spectral radius of P_s^* . This concludes the proof of Proposition 3.1. \square

4. Convergence in L^p of the martingale W_n^i

In this section, we prove Theorems 2.1 and 2.2 giving sufficient and necessary conditions for the L^p convergence of W_n^i , $1 \leq i \leq d$, with an exponential speed. First we formulate the following result.

Theorem 4.1. *Let $p > 1$ be such that $1 - p \in I$. Consider the assertions:*

$$W_n^i \xrightarrow[n \rightarrow +\infty]{} W^i \quad \text{in } L^p \quad \forall i = 1, \dots, d \quad (4.1)$$

$$\mathbb{E}(W_1^i)^p < +\infty \quad \forall i = 1, \dots, d \quad \text{and} \quad \kappa(1-p) < 1. \quad (4.2)$$

$$\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)^p < +\infty \quad \forall i, j = 1, \dots, d \quad \text{and} \quad \kappa(1-p) < 1. \quad (4.3)$$

If $1 < p \leq 2$, then we have the implications: (4.3) \Rightarrow (4.1) \Leftrightarrow (4.2). If $p > 2$, then we have: (4.3) \Rightarrow (4.1) \Rightarrow (4.2). When the Furstenberg-Kesten condition **H1** holds, then for each $p > 1$, (4.3) \Leftrightarrow (4.1) \Leftrightarrow (4.2).

It is clear that the assertions of Theorem 2.1 follow from Theorem 4.1. Theorem 4.1 is slightly stronger in the sense that for $1 < p \leq 2$, it gives a sufficient and necessary condition without assuming the Furstenberg-Kesten condition H1.

The following Lemma will be useful to investigate the convergence in L^p , and it is a direct consequence of the Marcinkiewicz-Zygmund inequality in [7, Theorem 1.5], as stated in [27, Lemma 1.4].

Lemma 4.2. *Let $(X_k)_{k \geq 1}$ be a sequence of i.i.d. random centered variables. Then for all $n \geq 1$ and $p > 1$:*

$$\mathbb{E} \left| \sum_{k=1}^n X_k \right|^p \leq \begin{cases} (B_p)^p \mathbb{E} |X_k|^p n, & \text{if } 1 < p \leq 2, \\ (B_p)^p \mathbb{E} |X_k|^p n^{\frac{p}{2}}, & \text{if } p > 2, \end{cases}$$

where $B_p = 2 \min \{k^{1/2} : k \in \mathbb{N}, k \geq \frac{p}{2}\}$.

In a last article, we proved the following result [11, Lemma 7.1]. It gives some properties on the products of random matrices $M_{n,n+k}$ under the Furstenberg-Kesten condition H1.

Lemma 4.3. *Assume condition H1. Then:*

1. for all $n, k \geq 0$ and $1 \leq i, j, r \leq d$, \mathbb{P} -a.s.,

$$\frac{1}{D} \leq \frac{M_{n,n+k}(i, j)}{M_{n,n+k}(i, r)} \leq D \quad \text{and} \quad \frac{1}{D} \leq \frac{M_{n,n+k}(i, j)}{M_{n,n+k}(r, j)} \leq D; \quad (4.4)$$

2. for all $n, k \geq 0$ and $1 \leq i, j \leq d$, \mathbb{P} -a.s.,

$$\frac{1}{dD^2} \leq \frac{M_{n,n+k}(i, j) U_{n+k+1, \infty}(j)}{\lambda_{n,n+k} U_{n, \infty}(i)} \leq 1. \quad (4.5)$$

Proof of Theorems 4.1 and 2.2. By iterating (2.1), it is easy to see that the process $(Z_n)_{n \geq 0}$ satisfies the relation

$$Z_{n+k} = \sum_{j=1}^d \sum_{l=1}^{Z_n(j)} Z_{l,n,k}^j, \quad n \geq 0, k \geq 1, \quad (4.6)$$

where $Z_{l,n,k}^j(r)$ denotes the number of the offspring of type r at time $n+k$ of the l -th particle of type j in the generation n ; conditional on the environment ξ , the random vectors $Z_{l,n,k}^j = (Z_{l,n,k}^j(1), \dots, Z_{l,n,k}^j(d))$ indexed by $l \in \mathbb{N}^*$ and $j \in \{1, \dots, d\}$ (for fixed n and k) are independent, each has the probability generating function $f_n^j \circ f_{n+1} \circ \dots \circ f_{n+k-1}$. Combining (4.6), (1.12) and (1.11), we have, for all $n, k \geq 0$

and $1 \leq i \leq d$,

$$\begin{aligned}
W_{n+k}^i - W_n^i &= \sum_{r=1}^d \frac{U_{n+k,\infty}(r)}{\lambda_{0,n+k-1}U_{0,\infty}(i)} \sum_{j=1}^d \sum_{l=1}^{Z_n^i(j)} Z_{l,n,k}^j(r) - W_n^i \\
&= \sum_{j=1}^d \frac{U_{n,\infty}(j)}{\lambda_{0,n-1}U_{0,\infty}(i)} \sum_{l=1}^{Z_n^i(j)} \sum_{r=1}^d \frac{U_{n+k,\infty}(r)Z_{l,n,k}^j(r)}{\lambda_{n,n+k-1}U_{n,\infty}(j)} - W_n^i \\
&= \sum_{j=1}^d \frac{U_{n,\infty}(j)}{\lambda_{0,n-1}U_{0,\infty}(i)} \sum_{l=1}^{Z_n^i(j)} (W_{l,n,k}^j - 1), \tag{4.7}
\end{aligned}$$

where

$$W_{l,n,k}^j := \frac{\langle Z_{l,n,k}^j, U_{n+k,\infty} \rangle}{\lambda_{n,n+k-1}U_{n,\infty}(j)}.$$

Let T be the shift operator of the environment sequence:

$$T\xi = (\xi_1, \xi_2, \dots) \quad \text{if} \quad \xi = (\xi_0, \xi_1, \dots),$$

and let T^n be its n -fold iteration. It is clear that, given the environment ξ , the random variables $W_{l,n,k}^j$, $l \geq 1$, are i.i.d.; they are independent of ξ_0, \dots, ξ_{n-1} and Z_n^i , and have the same distribution as $W_{n,k}^j$, where $(W_{n,k}^j)_{k \geq 0}$ is the martingale associated to a MBPRE starting with one individual of type j , in the shift random environment $T^n\xi$.

We divide the proof into 5 steps.

Step 1. We first prove that for $1 < p \leq 2$, we have the implications (4.2) \Rightarrow (4.1) of Theorem 4.1, and (4.2) \Rightarrow (2.10) of Theorem 2.2. We assume that $1 < p \leq 2$ and (4.2). Applying (4.7), the convexity of the function $x \mapsto x^p$ (together with the fact that $\sum_{j=1}^d U_{n,\infty}(j) = 1$), Lemma 4.2 and (1.11), for all $n \geq 0$, $k \geq 1$ and $1 \leq i \leq d$, \mathbb{P} -a.s., we have (for $1 < p \leq 2$),

$$\begin{aligned}
\mathbb{E}_\xi |W_{n+k}^i - W_n^i|^p &\leq \mathbb{E}_\xi \left(\sum_{j=1}^d \frac{U_{n,\infty}(j)}{\lambda_{0,n-1}U_{0,\infty}(i)} \left| \sum_{l=1}^{Z_n^i(j)} (W_{l,n,k}^j - 1) \right| \right)^p \\
&\leq \sum_{j=1}^d \frac{U_{n,\infty}(j)}{(\lambda_{0,n-1}U_{0,\infty}(i))^p} \mathbb{E}_\xi \left(\left| \sum_{l=1}^{Z_n^i(j)} (W_{l,n,k}^j - 1) \right| \right)^p \\
&\leq B_p^p \sum_{j=1}^d \frac{U_{n,\infty}(j)}{(\lambda_{0,n-1}U_{0,\infty}(i))^p} \mathbb{E}_\xi Z_n^i(j) \mathbb{E}_\xi |W_{n,k}^j - 1|^p \\
&\leq B_p^p \sigma_{n,k}(p) \sum_{j=1}^d \frac{M_{0,n-1}(i,j)U_{n,\infty}(j)}{(\lambda_{0,n-1}U_{0,\infty}(i))^p} \\
&= B_p^p \sigma_{n,k}(p) (\lambda_{0,n-1}U_{0,\infty}(i))^{1-p}, \tag{4.8}
\end{aligned}$$

with

$$\sigma_{n,k}(p) = \max_{1 \leq j \leq d} \mathbb{E}_\xi |W_{n,k}^j - 1|^p. \quad (4.9)$$

Using again (1.11) and Lemma 3.2 together with the fact that $M_{0,n-1}$ is independent of $T^n \xi$, we get that for all $s \in I$, $n \geq 1$ and $1 \leq i \leq d$, \mathbb{P} -a.s.

$$\begin{cases} \mathbb{E}_{T^n \xi} \lambda_{0,n-1}^s = \mathbb{E}_{T^n \xi} \|M_{0,n-1} U_{n,\infty}\|^s \leq C_s \kappa(s)^n; \\ \mathbb{E}_{T^n \xi} (\lambda_{0,n-1} U_{0,\infty}(i))^s = \mathbb{E}_{T^n \xi} \langle M_{0,n-1} U_{n,\infty}, e_i \rangle^s \leq C_s \kappa(s)^n. \end{cases} \quad (4.10)$$

Taking expectation in (4.8), by (4.10) we get that for all $n \geq 0$, $k \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned} \mathbb{E} |W_{n+k}^i - W_n^i|^p &\leq B_p^p \mathbb{E} [\sigma_{n,k}(p) \mathbb{E}_{T^n \xi} (\lambda_{0,n-1} U_{0,\infty}(i))^{1-p}] \\ &\leq B_p^p C_{1-p} \mathbb{E} \sigma_{0,k}(p) \kappa(1-p)^n. \end{aligned} \quad (4.11)$$

From condition (4.2) we have $\mathbb{E} \sigma_{0,1}(p) < +\infty$ and $\kappa(1-p) < 1$. So by the triangular inequality of L^p , it follows from (4.11) that for all $1 \leq i \leq d$, with $C = B_p [C_{1-p} \mathbb{E} \sigma_{0,1}(p)]^{1/p}$,

$$\sup_{n \geq 0} (\mathbb{E} (W_n^i)^p)^{1/p} \leq 1 + C \sum_{n=0}^{+\infty} \kappa(1-p)^{n/p} < +\infty. \quad (4.12)$$

Therefore for all $1 \leq i \leq d$, (W_n^i) is a martingale bounded in L^p , so that it converges in L^p . This proves the implication (4.2) \Rightarrow (4.1) of Theorem 4.1. Furthermore (4.12) implies that $\sup_{k \geq 0} \mathbb{E} \sigma_{0,k}(p) < +\infty$. So, by letting $k \rightarrow +\infty$ in (4.11) we get (2.10) of Theorem 2.2.

Step 2. We next prove the implication (4.3) \Rightarrow (4.2) of Theorem 4.1 for any $p > 1$, which, in particular, will conclude the proof of Theorem 4.1 for $1 < p \leq 2$. By (1.10) we have $0 \leq \frac{M_0(i,j), U_{1,\infty}(j)}{\lambda_0 U_{0,\infty}(i)} \leq 1$ a.s. for all $1 \leq i, j \leq d$. So by the triangular inequality of L^p , it follows that for $p > 1$ and $1 \leq i \leq d$,

$$(\mathbb{E} (W_1^i)^p)^{1/p} = \left(\mathbb{E} \left(\frac{\langle Z_1^i, U_{1,\infty} \rangle}{\lambda_0 U_{0,\infty}(i)} \right)^p \right)^{1/p} \quad (4.13)$$

$$= \left(\mathbb{E} \left(\sum_{j=1}^d \frac{M_0(i,j), U_{1,\infty}(j)}{\lambda_0 U_{0,\infty}(i)} \frac{Z_1^i(j)}{M_0(i,j)} \right)^p \right)^{1/p} \quad (4.14)$$

$$\leq \sum_{j=1}^d \left(\mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i,j)} \right)^p \right)^{1/p}. \quad (4.15)$$

Therefore the implication (4.3) \Rightarrow (4.2) of Theorem 4.1 holds.

Step 3. We now prove that for $p > 2$, we have the implications (4.3) \Rightarrow (4.1) of Theorem 4.1, and (4.3) \Rightarrow (2.11) of Theorem 2.2. Assume $p > 2$ and (4.3). In the following $C > 0$ will be a constant which may depend on p and which may differ from line to line. Applying (4.7), the inequality $(\sum_{j=1}^d x_j)^p \leq d^{p-1} \sum_{j=1}^d x_j^p$, $x_j \geq 0$

for any $1 \leq j \leq d$, and Lemma 4.2, for all $n \geq 0$, $k \geq 1$ and $1 \leq i \leq d$, \mathbb{P} -a.s. we have

$$\begin{aligned}
& \mathbb{E}_\xi |W_{n+k}^i - W_n^i|^p \\
& \leq d^{p-1} B_p^p \sum_{j=1}^d \left(\frac{U_{n,\infty}(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \right)^p \mathbb{E}_\xi (Z_n^i(j))^{p/2} \mathbb{E}_\xi |W_{n,k}^j - 1|^p \\
& \leq C \sigma_{n,k}(p) \sum_{j=1}^d (U_{n,\infty}(j))^{p/2} \mathbb{E}_\xi \left(\frac{U_{n,\infty}(j) Z_n^i(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \right)^{p/2} (\lambda_{0,n-1} U_{0,\infty}(i))^{-p/2} \\
& \leq C \sigma_{n,k}(p) \left(\sum_{j=1}^d U_{n,\infty}(j) \right) \mathbb{E}_\xi (W_n^i)^{p/2} (\lambda_{0,n-1} U_{0,\infty}(i))^{-p/2} \\
& \leq C \sigma_{n,k}(p) \mathbb{E}_\xi (W_n^i)^{p/2} (\lambda_{0,n-1} U_{0,\infty}(i))^{-p/2}, \tag{4.16}
\end{aligned}$$

with $\sigma_{n,k}(p)$ defined as in (4.9) (for $p > 2$). Set $j_p \in \mathbb{N}$ the unique integer such that $1 < \frac{p}{2^{j_p}} \leq 2$. For all $n \geq 0$, $1 \leq i \leq d$ and $1 \leq j \leq j_p$, define

$$a_{n,j}^i(p) := (\lambda_{0,n-1} U_{0,\infty}(i))^{p/2^{j_p} - p} \mathbb{E}_\xi (W_n^i)^{p/2^{j_p}}. \tag{4.17}$$

Taking expectation in (4.16), we obtain that for all $n \geq 0$, $k \geq 1$ and $1 \leq i \leq d$,

$$\mathbb{E} |W_{n+k}^i - W_n^i|^p \leq C \mathbb{E} [\sigma_{n,k}(p) \mathbb{E}_{T^n \xi} a_{n,1}^i(p)]. \tag{4.18}$$

To prove (4.1) of Theorem 4.1, and (2.11) of Theorem 2.2, it is enough to show that there exists a constant $C_1 > 0$ (which may depend on p) such that for all $n \geq 0$, $1 \leq i \leq d$, $1 \leq j \leq j_p$ and $\delta > \delta_c(p)$,

$$\delta^{-n} (\mathbb{E}_{T^n \xi} a_{n,j}^i(p))^{1/p} \leq C_1 \quad \mathbb{P}\text{-a.s.} \tag{4.19}$$

In fact, combining (4.18) and (4.19) for $j = 1$, it follows that for all $1 \leq i \leq d$ and $\delta > \delta_c(p)$,

$$\sup_{n \geq 0} (\mathbb{E} (W_n^i)^p)^{1/p} \leq 1 + C (\mathbb{E} \sigma_{0,1}(p))^{1/p} \sum_{n=0}^{+\infty} \delta^n. \tag{4.20}$$

Condition (4.3) implies that $\mathbb{E} \sigma_{0,1}(p) < +\infty$ and $\delta_c(p) < 1$. Therefore, applying (4.20) with $\delta \in (\delta_c(p), 1)$, we deduce that (W_n^i) is a martingale bounded in L^p , for all $1 \leq i \leq d$. Hence, (W_n^i) , $1 \leq i \leq d$, converge in L^p , and we have $\sup_{k \geq 0} \mathbb{E} \sigma_{0,k}(p) < +\infty$. This proves the implication (4.3) \Rightarrow (4.1). Moreover, combining again (4.18) and (4.19), and letting $k \rightarrow +\infty$, we obtain that for all $n \geq 0$, $1 \leq i \leq d$ and $\delta \in (\delta_c(p), 1)$,

$$(\mathbb{E} |W^i - W_n^i|^p)^{1/p} \leq C \delta^n,$$

which implies (2.11) and ends the proof of Theorem 2.2 in the case $p > 2$.

It remains to prove (4.19). We will prove it by iteration on j . First consider the case $j = j_p$. By definition of j_p we have $1 < p/2^{j_p} \leq 2$. So, by the triangular inequality in $L^{p/2^{j_p}}$ under \mathbb{P}_ξ and (4.8), it follows that for all $1 \leq i \leq d$ and $n \geq 0$, \mathbb{P} -a.s.,

$$\begin{aligned} a_{n,j_p}^i(p)^{2^{j_p}/p} &\leq (\lambda_{0,n-1} U_{0,\infty}(i))^{1-2^{j_p}} \\ &\quad + (\lambda_{0,n-1} U_{0,\infty}(i))^{1-2^{j_p}} \sum_{l=0}^{n-1} (\mathbb{E}_\xi |W_{l+1}^i - W_l^i|^{p/2^{j_p}})^{2^{j_p}/p} \\ &\leq (\lambda_{0,n-1} U_{0,\infty}(i))^{1-2^{j_p}} \\ &\quad + C \sum_{l=0}^{n-1} \left[\sigma_{l,1} \left(\frac{p}{2^{j_p}} \right) \right]^{2^{j_p}/p} (\lambda_{0,l-1} U_{0,\infty}(i))^{2^{j_p}/p - 2^{j_p}} \lambda_{l,n-1}^{1-2^{j_p}}. \end{aligned} \quad (4.21)$$

Taking the $L^{p/2^{j_p}}$ -norm under $\mathbb{P}_{T^n \xi}$ on both sides, and using the triangular inequality in $L^{p/2^{j_p}}$ and inequalities (4.10), we obtain that for all $1 \leq i \leq d$ and $n \geq 0$, \mathbb{P} -a.s.,

$$\begin{aligned} &(\mathbb{E}_{T^n \xi} a_{n,j_p}^i(p))^{2^{j_p}/p} \\ &\leq (\mathbb{E}_{T^n \xi} (\lambda_{0,n-1} U_{0,\infty}(i))^{p/2^{j_p} - p})^{2^{j_p}/p} \\ &\quad + C \sum_{l=0}^{n-1} \left\{ \mathbb{E}_{T^n \xi} \left[\mathbb{E}_{T^l \xi} \left[(\lambda_{0,l-1} U_{0,\infty}(i))^{1-p} \right] \sigma_{l,1} \left(\frac{p}{2^{j_p}} \right) \lambda_{l,n-1}^{p/2^{j_p} - p} \right] \right\}^{2^{j_p}/p} \\ &\leq C \kappa \left(\frac{p}{2^{j_p}} - p \right)^{\frac{n 2^{j_p}}{p}} + C \sum_{l=0}^{n-1} \left\{ \kappa (1-p)^l \mathbb{E}_{T^n \xi} \left[\sigma_{l,1} \left(\frac{p}{2^{j_p}} \right) \lambda_{l,n-1}^{p/2^{j_p} - p} \right] \right\}^{2^{j_p}/p}. \end{aligned} \quad (4.22)$$

Notice that if $1 \leq j \leq j_p$, then we have $1 - p < \frac{p}{2^j} - p < -\frac{p}{2}$. Since κ is log-convex on I , we obtain that

$$\max_{1 \leq j \leq j_p} \left\{ \kappa \left(\frac{p}{2^j} - p \right) \right\} \leq \max \left\{ \kappa(1-p), \kappa \left(-\frac{p}{2} \right) \right\} = \delta_c(p)^p. \quad (4.23)$$

We now deal with the second term in (4.22), by calculating first the conditional expectation $\mathbb{E}_{T^{l+1} \xi}$. By the triangular inequalities of $L^{p/2^j}$ under \mathbb{P}_ξ and $\mathbb{P}_{T^{l+1} \xi}$, and inequalities (4.10), it holds that for all $l \geq 0$ and $1 \leq j \leq j_p$, \mathbb{P} -a.s.,

$$\begin{aligned} &\left\{ \mathbb{E}_{T^{l+1} \xi} \left(\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_l^{p/2^j - p} \right) \right\}^{2^j/p} \\ &= \left\{ \mathbb{E}_{T^{l+1} \xi} \left(\max_{1 \leq r \leq d} \mathbb{E}_\xi |W_{l,k}^r - 1|^{p/2^j} \lambda_l^{p/2^j - p} \right) \right\}^{2^j/p} \\ &\leq \left\{ \mathbb{E}_{T^{l+1} \xi} \left(\max_{1 \leq r \leq d} \mathbb{E}_\xi (W_{l,1}^r)^{p/2^j} \lambda_l^{p/2^j - p} \right) \right\}^{2^j/p} + (\mathbb{E}_{T^{l+1} \xi} \lambda_l^{p/2^j - p})^{2^j/p} \\ &\leq \left\{ \sum_{r=1}^d \mathbb{E}_{T^{l+1} \xi} \left[\mathbb{E}_\xi (W_{l,1}^r)^{p/2^j} \lambda_l^{p/2^j - p} \right] \right\}^{2^j/p} + C \kappa \left(\frac{p}{2^j} - p \right)^{2^j/p}. \end{aligned}$$

Therefore, using inequality (4.23), condition (4.3) and again the triangular inequalities of $L^{p/2^j}$ under \mathbb{P}_ξ and $\mathbb{P}_{T^{l+1}\xi}$, we see that for all $l \geq 0$ and $1 \leq j \leq j_p$, \mathbb{P} -a.s.,

$$\begin{aligned}
& \left(\mathbb{E}_{T^{l+1}\xi} \left(\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_l^{p/2^j - p} \right) \right)^{2^j/p} \\
& \leq d^{2^j/p} \max_{1 \leq r \leq d} \left(\mathbb{E}_{T^{l+1}\xi} \left[\lambda_l^{-p} \mathbb{E}_\xi \left(\lambda_l W_{l,1}^r \right)^{p/2^j} \right] \right)^{2^j/p} + C \\
& \leq C \max_{1 \leq r \leq d} \left(\mathbb{E}_{T^{l+1}\xi} \left[\lambda_l^{-p} \mathbb{E}_\xi \left(\lambda_l W_{l,1}^r \mathbb{1}_{\{\lambda_l W_{l,1}^r \leq 1\}} \right)^{p/2^j} \right] \right)^{2^j/p} \\
& \quad + C \max_{1 \leq r \leq d} \left(\mathbb{E}_{T^{l+1}\xi} \left[\lambda_l^{-p} \mathbb{E}_\xi \left(\lambda_l W_{l,1}^r \mathbb{1}_{\{\lambda_l W_{l,1}^r > 1\}} \right)^{p/2^j} \right] \right)^{2^j/p} + C \\
& \leq C \max_{1 \leq r \leq d} \left(\mathbb{E}_{T^{l+1}\xi} \left[\lambda_l^{1-p} \mathbb{E}_\xi W_{l,1}^r \right] \right)^{2^j/p} + C \max_{1 \leq r \leq d} \left(\mathbb{E}_{T^{l+1}\xi} \left(W_{l,1}^r \right)^p \right)^{2^j/p} + C.
\end{aligned}$$

We know that $(W_{l,k}^r)_{k \geq 0}$ is the martingale associated to a MBPRE starting with one individual of type r , in the shift random environment $T^l \xi$. In particular we have $\mathbb{E}_\xi W_{l,1}^r = 1$ a.s. Therefore, applying again (4.10), (4.23) and condition (4.3), it follows that for all $l \geq 0$ and $1 \leq j \leq j_p$, \mathbb{P} -a.s.,

$$\begin{aligned}
& \left(\mathbb{E}_{T^{l+1}\xi} \left(\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_l^{p/2^j - p} \right) \right)^{2^j/p} \\
& \leq C \max_{1 \leq r \leq d} \left(\mathbb{E}_{T^{l+1}\xi} \lambda_l^{1-p} \right)^{2^j/p} + C \max_{1 \leq r \leq d} \left(\mathbb{E}_{T^{l+1}\xi} \left(W_{l,1}^r \right)^p \right)^{2^j/p} + C \\
& \leq C \kappa (1-p)^{2^j/p} + C \max_{1 \leq r \leq d} \left(\mathbb{E}_{T^{l+1}\xi} \left(W_{l,1}^r \right)^p \right)^{2^j/p} + C \\
& \leq C \left(1 + \max_{1 \leq r \leq d} \left(\mathbb{E}_{T^{l+1}\xi} \left(W_{l,1}^r \right)^p \right)^{2^j/p} \right). \tag{4.24}
\end{aligned}$$

Then, by a similar calculation as in (4.13), for all $1 \leq r \leq d$ and $l \geq 0$, \mathbb{P} -a.s., we have

$$\begin{aligned}
\left(\mathbb{E}_{T^{l+1}\xi} \left(W_{l,1}^r \right)^p \right)^{1/p} &= \left(\mathbb{E}_{T^{l+1}\xi} \left(\frac{\langle N_l^r, U_{l+1,\infty} \rangle}{\lambda_l U_{l,\infty}(r)} \right)^p \right)^{1/p} \\
&= \left(\mathbb{E}_{T^{l+1}\xi} \left(\sum_{j=1}^d \frac{M_l(r,j) U_{l+1,\infty}(j)}{\lambda_l U_{l,\infty}(r)} \frac{N_l^r}{M_l(r,j)} \right)^p \right)^{1/p} \\
&\leq \sum_{j=1}^d \left(\mathbb{E}_{T^{l+1}\xi} \left(\frac{N_l^r(j)}{M_l(r,j)} \right)^p \right)^{1/p} \\
&= \sum_{j=1}^d \left(\mathbb{E} \left(\frac{Z_1^r(j)}{M_0(r,j)} \right)^p \right)^{1/p} < +\infty. \tag{4.25}
\end{aligned}$$

Putting together (4.24) and (4.25), we get that for all $l \geq 0$ and $1 \leq j \leq j_p$, \mathbb{P} -a.s.,

$$\mathbb{E}_{T^{l+1}\xi} \left(\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_l^{p/2^j - p} \right) \leq C. \quad (4.26)$$

Therefore, for all $n \geq 0$ and $0 \leq l \leq n-1$, \mathbb{P} -a.s., (the value of the constant C may change from line to line),

$$\begin{aligned} \mathbb{E}_{T^n\xi} \left[\sigma_{l,1} \left(\frac{p}{2^{j_p}} \right) \lambda_{l,n-1}^{p/2^{j_p} - p} \right] &= \mathbb{E}_{T^n\xi} \left[\mathbb{E}_{T^{l+1}\xi} \left(\sigma_{l,1} \left(\frac{p}{2^{j_p}} \right) \lambda_l^{p/2^{j_p} - p} \right) \lambda_{l+1,n-1}^{p/2^{j_p} - p} \right] \\ &\leq C \mathbb{E}_{T^n\xi} \lambda_{l+1,n-1}^{p/2^{j_p} - p} \\ &\leq C \left[\kappa \left(\frac{p}{2^{j_p}} - p \right) \right]^{n-1-l} \\ &\leq C \delta_c(p)^{(n-1-l)p}, \end{aligned}$$

where the last two inequalities hold by (4.10) and (4.23). Combining this with (4.22) and (4.23), we obtain that for all $1 \leq i \leq d$ and $n \geq 0$, \mathbb{P} -a.s.,

$$\begin{aligned} \left(\mathbb{E}_{T^n\xi} a_{n,j_p}^i(p) \right)^{2^{j_p}/p} &\leq C \delta_c(p)^{n2^{j_p}} \\ &\leq C \delta_c(p)^{n2^{j_p}} + C \sum_{l=0}^{n-1} \left(\delta_c(p)^{lp} \delta_c(p)^{(n-1-l)p} \right)^{2^{j_p}/p} \\ &\leq C(1 + \delta_c(p)^{-2^{j_p}} n) \delta_c(p)^{n2^{j_p}}. \end{aligned}$$

So (4.19) holds for $j = j_p$.

Now suppose that (4.19) holds for $j+1 \leq j_p$ with $j \geq 1$. We will prove that it still holds for j . By recurrence this will prove that (4.19) holds for all $j = 1, \dots, j_p$. Since $j+1$ satisfies (4.19), for all $n \geq 0$, $1 \leq i \leq d$, and $\delta > \delta_c(p)$,

$$\delta^{-n} \left(\mathbb{E}_{T^n\xi} a_{n,j+1}^i(p) \right)^{1/p} \leq C \quad \mathbb{P}\text{-a.s.} \quad (4.27)$$

By definition of j_p we have $p/2^j > 2$. Corresponding to (4.22), with the same argument as in its proof but applying (4.16) instead of (4.8), we obtain that for all $1 \leq i \leq d$ and $n \geq 0$, \mathbb{P} -a.s.,

$$\begin{aligned} &\left(\mathbb{E}_{T^n\xi} a_{n,j}^i(p) \right)^{2^j/p} \\ &\leq C \kappa \left(\frac{p}{2^j} - p \right)^{n2^j/p} + C \times \\ &\quad \sum_{l=0}^{n-1} \left\{ \mathbb{E}_{T^n\xi} \left[\sigma_{l,1} \left(\frac{p}{2^j} \right) \mathbb{E}_\xi \left(W_l^i \right)^{\frac{p}{2^{j+1}}} \left(\lambda_{0,l-1} U_{0,\infty}(i) \right)^{\frac{p}{2^{j+1}} - p} \lambda_{l,n-1}^{p/2^j - p} \right] \right\}^{2^j/p} \\ &= C \kappa \left(\frac{p}{2^j} - p \right)^{n2^j/p} + C \sum_{l=0}^{n-1} \left(\mathbb{E}_{T^n\xi} \left[\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_{l,n-1}^{p/2^j - p} \mathbb{E}_{T^l\xi} a_{l,j+1}^i(p) \right] \right)^{2^j/p}. \end{aligned} \quad (4.28)$$

This enables us to obtain the desired bound of $\mathbb{E}_{T^n\xi} a_{n,j}^i(p)$ from that of $\mathbb{E}_{T^n\xi} a_{n,j+1}^i(p)$. In fact, combining this with the recurrence hypothesis (4.27), together with (4.23), (4.26) and (4.10), we obtain that for all $n \geq 0$, $1 \leq i \leq d$, and $\delta > \delta_c(p)$,

$$\begin{aligned}
(\mathbb{E}_{T^n\xi} a_{n,j}^i(p))^{2^j/p} &\leq C \delta_c(p)^{n2^j} + C \times \\
&\quad \sum_{l=0}^{n-1} \delta^{l2^j} \left\{ \mathbb{E}_{T^n\xi} \left[\mathbb{E}_{T^{l+1}\xi} \left[\sigma_{l,1} \left(\frac{p}{2^j} \right) \lambda_l^{p/2^j-p} \right] \lambda_{l+1,n-1}^{p/2^j-p} \right] \right\}^{2^j/p} \\
&\leq C \delta_c(p)^{n2^j} + C \sum_{l=0}^{n-1} \delta^{l2^j} \kappa \left(\frac{p}{2^j} - p \right)^{(n-1-l)2^j/p} \\
&\leq C (\delta_c(p)^{n2^j} + n \delta^{(n-1)2^j}) \\
&\leq C (1 + \delta^{-2^j} n) \delta^{n2^j}. \tag{4.29}
\end{aligned}$$

So (4.19) also holds for j . Therefore, by recurrence, we have proved that (4.19) holds for all $j = 1, \dots, j_p$.

Step 4. For any $p > 1$, we prove the implication (4.1) \Rightarrow (4.2) of Theorem 4.1. Assume $p > 1$ and condition (4.1), that is, the martingale $(W_n^i)_{n \geq 0}$ converges in L^p , for all $1 \leq i \leq d$. In particular we have $\mathbb{E}(W_1^i)^p < +\infty$, $\mathbb{E}(W^i)^p < +\infty$, and $\mathbb{E}(W^i) = 1$ for all $1 \leq i \leq d$. It was observed in [11, Theorem 2.6] that $\mathbb{E}_\xi(W^i) = 1$ a.s. when W^i are non-degenerate. In fact $\mathbb{E}_\xi(W^i) = 1$ a.s. whenever $\mathbb{E}(W^i) = 1$, because $\mathbb{E}_\xi(W^i) \leq 1$ by Fatou's lemma.

Notice that for all $1 \leq i \leq d$, the limit variables W^i satisfy

$$W^i = \sum_{j=1}^d \frac{U_{1,\infty}(j)}{\lambda_0 U_{0,\infty}(i)} \sum_{l=1}^{Z_1^j} W^j(l, T\xi) \quad \mathbb{P}\text{-a.s.},$$

where for all $1 \leq j \leq d$, under the conditional law \mathbb{P}_ξ , $(W^j(l, T\xi))_{l \geq 1}$ is a sequence of i.i.d. random variables, also independent of Z_1^j , with common distribution $\mathbb{P}_\xi(W^j(l, T\xi) \in \cdot) = \mathbb{P}_{T\xi}(W^j \in \cdot)$. So, by the strict sub-additivity of the function $x \mapsto x^p$ on \mathbb{R}_+^* , we get that for all $1 \leq i \leq d$,

$$\mathbb{E}_\xi(W^i)^p > \sum_{j=1}^d \frac{M_0(i, j) U_{1,\infty}(j)^p}{\lambda_0^p U_{0,\infty}(i)^p} \mathbb{E}_{T\xi}(W^j)^p \quad \mathbb{P}\text{-a.s.},$$

using the fact that $\mathbb{E}_\xi(W^i) = 1$ a.s. This, together with (1.10), implies that for all $1 \leq i \leq d$, \mathbb{P} -a.s.,

$$\begin{aligned}
\mathbb{E}_\xi(W^i)^p U_{0,\infty}(i)^{p-1} &> \lambda_0^{1-p} \sum_{j=1}^d \frac{M_0(i, j) U_{1,\infty}(j)}{\lambda_0 U_{0,\infty}(i)} \mathbb{E}_{T\xi}(W^j)^p U_{1,\infty}(j)^{p-1} \\
&> \lambda_0^{1-p} \min_{1 \leq j \leq d} (\mathbb{E}_{T\xi}(W^j)^p U_{1,\infty}(j)^{p-1}).
\end{aligned}$$

Therefore we obtain

$$\min_{1 \leq i \leq d} (\mathbb{E}_\xi(W^i)^p U_{0,\infty}(i)^{p-1}) > \lambda_0^{1-p} \min_{1 \leq i \leq d} (\mathbb{E}_{T\xi}(W^i)^p U_{1,\infty}(i)^{p-1}) \quad \mathbb{P}\text{-a.s.} \quad (4.30)$$

On the other hand, by (3.1) and (1.10), the transfer operator P_{1-p} satisfies the following property: for all $\varphi \in \mathcal{C}(\mathcal{S})$, \mathbb{P} -a.s.,

$$P_{1-p}\varphi(U_{1,\infty}) = \mathbb{E}_{T\xi}[\|M_0 U_{1,\infty}\|^{1-p} \varphi(M_0 \cdot U_{1,\infty})] = \mathbb{E}_{T\xi}[\lambda_0^{1-p} \varphi(U_{0,\infty})]. \quad (4.31)$$

Using (4.31) with $\varphi = r_{1-p}$, and combining this with Proposition 3.1, we get

$$\mathbb{E}_{T\xi}[\lambda_0^{1-p} r_{1-p}(U_{0,\infty})] = \kappa(1-p) r_{1-p}(U_{1,\infty}) \quad \mathbb{P}\text{-a.s.} \quad (4.32)$$

Moreover, by Proposition 3.1 we know that r_{1-p} is a strictly positive continuous function on \mathcal{S} . This implies that

$$\begin{aligned} 0 < \mathbb{E} \left[\min_{1 \leq i \leq d} (\mathbb{E}_\xi(W^i)^p U_{0,\infty}(i)^{p-1}) r_{1-p}(U_{0,\infty}) \right] \\ \leq \|r_{1-p}\|_\infty \min_{1 \leq i \leq d} \mathbb{E}(W^i)^p < +\infty. \end{aligned} \quad (4.33)$$

Therefore, putting together (4.30), (4.32) and (4.33), we obtain

$$\begin{aligned} & \mathbb{E} \left[\min_{1 \leq i \leq d} (\mathbb{E}_\xi(W^i)^p U_{0,\infty}(i)^{p-1}) r_{1-p}(U_{0,\infty}) \right] \\ & > \mathbb{E} \left[\mathbb{E}_{T\xi}[\lambda_0^{1-p} r_{1-p}(U_{0,\infty})] \min_{1 \leq i \leq d} (\mathbb{E}_{T\xi}(W^i)^p U_{1,\infty}(i)^{p-1}) \right] \\ & = \kappa(1-p) \mathbb{E} \left[\min_{1 \leq i \leq d} (\mathbb{E}_{T\xi}(W^i)^p U_{1,\infty}(i)^{p-1}) r_{1-p}(U_{1,\infty}) \right] \\ & = \kappa(1-p) \mathbb{E} \left[\min_{1 \leq i \leq d} (\mathbb{E}_\xi(W^i)^p U_{0,\infty}(i)^{p-1}) r_{1-p}(U_{0,\infty}) \right], \end{aligned}$$

so $\kappa(1-p) < 1$. This ends the proof of (4.1) \Rightarrow (4.2).

Step 5. To conclude the proof, it remains to show that under the Furstenberg-Kesten condition **H1** we have (4.2) \Rightarrow (4.3) for all $p > 1$. By (4.5) in Lemma 4.3, we know that, under **H1**, we have for all $n \geq 0$ and $1 \leq i, j \leq d$

$$\frac{1}{dD^2} \leq \frac{M_{0,n-1}(i,j)U_{n,\infty}(j)}{\lambda_{0,n-1}U_{0,\infty}(i)} \leq 1 \quad \mathbb{P}\text{-a.s.}$$

Therefore we obtain that for all $n \geq 0$ and $1 \leq i, j \leq d$,

$$\begin{aligned} \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} & \leq dD^2 \frac{M_{0,n-1}(i,j)U_{n,\infty}(j)}{\lambda_{0,n-1}U_{0,\infty}(i)} \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \\ & = dD^2 \frac{Z_n^i(j)U_{n,\infty}(j)}{\lambda_{0,n-1}U_{0,\infty}(i)} \leq dD^2 W_n^i. \end{aligned} \quad (4.34)$$

The implication (4.2) \Rightarrow (4.3) follows from (4.34) with $n = 1$.

This ends the proof of Theorems 4.1 and 2.2. \square

5. Convergence in L^p of the normalized population size $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$

In this section we give proof of Theorems 2.3 and 2.4 about the convergence in L^p of $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$ and its exponential convergence rate, under the Furstenberg-Kesten condition **H1**.

5.1. Auxiliary results

We need some preliminary results concerning the products of random matrices $M_{n,n+k}$. The following proposition was established by Hennion in [15, Theorem 1], which provides an analog of the Perron-Frobenius theorem for products of random matrices.

Proposition 5.1. *Assume that $M_0 > 0$ \mathbb{P} -a.s. Then for all $n \geq 0$ and $1 \leq i, j \leq d$, as $k \rightarrow +\infty$, \mathbb{P} -a.s.,*

$$M_{n,n+k}(i, j) \underset{k \rightarrow +\infty}{\sim} \rho_{n,n+k} U_{n,n+k}(i) V_{n,n+k}(j).$$

For $1 \leq i \leq d$, let $(\Pi_n^i)_{n \geq 0}$ be the sequence of random matrices in $\mathcal{M}_d(\mathbb{R})$ such that for all $1 \leq j, r \leq d$,

$$\Pi_0^i(j, r) := \delta_{i,r}, \quad \Pi_n^i(j, r) := \frac{M_{0,n-1}(i, r) M_n(r, j)}{M_{0,n}(i, j)}, \quad n \geq 1.$$

By definition all the entries of the i -th column of Π_0^i are equal to 1, the others are 0; each Π_n^i is a stochastic matrix. For $n, k \geq 0$ let

$$\Pi_{n+k,n}^i := \Pi_{n+k}^i \cdots \Pi_n^i$$

be the products of the matrices Π_n^i . Clearly each $\Pi_{n+k,n}^i$ is a non-negative stochastic random matrix. The following lemma concerns the convergence of the products $\Pi_{n+k,n}^i$ of random matrices and its exponential rate as $k \rightarrow +\infty$, which will be very useful for the study of the L^p convergence of the normalized population size $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$.

Lemma 5.2. *Assume the Furstenberg-Kesten condition **H1**. Then for all $n \geq 0$ and $1 \leq i \leq d$, as $k \rightarrow +\infty$, the sequence $(\Pi_{n+k,n}^i)_{k \geq 0}$ converges \mathbb{P} -a.s. to the random matrix $\Pi_{\infty,n}^i$ such that:*

(1) For all $1 \leq j, r \leq d$,

$$\Pi_{\infty,0}^i(j, r) := \Pi_0^i(j, r), \quad \Pi_{\infty,n}^i(j, r) := \frac{M_{0,n-1}(i, r) U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)}, \quad n \geq 1.$$

(2) There exist $C > 0$ and $\delta \in (0, 1)$ such that for all $k \geq 0$ and $1 \leq i \leq d$,

$$\sup_{n \geq 0} \|\Pi_{n+k,n}^i - \Pi_{\infty,n}^i\| \leq C \delta^k \quad \mathbb{P}\text{-a.s.} \quad (5.1)$$

For the proof of Lemma 5.2, we will use the following result established by Seneta [28, Theorem 4.19], which gives the convergence of products of stochastic matrices with an exponential rate.

Lemma 5.3 ([28]). *Assume that $(P_n)_{n \geq 0}$ is a sequence of stochastic matrices such that for some $\varepsilon \in (0, 1)$, all $n \geq 0$ and all $1 \leq i, j \leq d$,*

$$P_n(i, j) \geq \varepsilon.$$

Then, for all $n \geq 0$ the product $P_{n+k, n} := P_{n+k} \cdots P_n$ converges as $k \rightarrow +\infty$ to some matrix $P_{\infty, n}$, and there exist two constants $C > 0$ and $\delta \in (0, 1)$ depending only on ε such that

$$\sup_{n \geq 0} \|P_{n+k, n} - P_{\infty, n}\| \leq C\delta^k.$$

Proof of Lemma 5.2. (1) First we show by induction on k that for all $n \geq 1$, $k \geq 0$ and $1 \leq i, j, r \leq d$,

$$\Pi_{n+k, n}^i(j, r) = \frac{M_{0, n-1}(i, r)M_{n, n+k}(r, j)}{M_{0, n+k}(i, j)}. \quad (5.2)$$

Obviously, by definition of Π_n^i , (5.2) holds for $k = 0$. Assume that (5.2) holds for some $k \geq 0$. Then, for all $n \geq 1$ and $1 \leq i, j, r \leq d$ we have

$$\begin{aligned} \Pi_{n+k+1, n}^i(j, r) &= \sum_{l=1}^d \Pi_{n+k+1}^i(j, l) \Pi_{n+k, n}^i(l, r) \\ &= \sum_{l=1}^d \frac{M_{0, n+k}(i, l)M_{n+k+1}(l, j)}{M_{0, n+k+1}(i, j)} \frac{M_{0, n-1}(i, r)M_{n, n+k}(r, l)}{M_{0, n+k}(i, l)} \\ &= \frac{M_{0, n-1}(i, r)}{M_{0, n+k+1}(i, j)} \sum_{l=1}^d M_{n, n+k}(r, l)M_{n+k+1}(l, j) \\ &= \frac{M_{0, n-1}(i, r)M_{n, n+k+1}(r, j)}{M_{0, n+k+1}(i, j)}. \end{aligned}$$

So (5.2) holds for $k + 1$. Therefore by reduction (5.2) holds for all $k \geq 0$.

Combining (5.2) with Proposition 5.1, we deduce that for all $n \geq 1$ and $1 \leq i, j, r \leq d$, \mathbb{P} -a.s. as $k \rightarrow +\infty$,

$$\begin{aligned} \Pi_{n+k, n}^i(j, r) &\sim \frac{M_{0, n-1}(i, r)\rho_{n, n+k}U_{n, n+k}(r)V_{n, n+k}(j)}{\rho_{0, n+k}U_{0, n+k}(i)V_{0, n+k}(j)} \\ &\sim \frac{\rho_{n, n+k}V_{n, n+k}(j)}{\rho_{0, n+k}V_{0, n+k}(j)} \frac{M_{0, n-1}(i, r)U_{n, \infty}(r)}{U_{0, \infty}(i)} \\ &= \left(\prod_{l=0}^{n-1} \frac{\rho_{l+1, n+k}V_{l+1, n+k}(j)}{\rho_{l, n+k}V_{l, n+k}(j)} \right) \frac{M_{0, n-1}(i, r)U_{n, \infty}(r)}{U_{0, \infty}(i)}. \quad (5.3) \end{aligned}$$

By [11, Proposition 2.2] we know that for all $l, n \geq 0$ and $1 \leq j \leq d$,

$$\lambda_l = \lim_{k \rightarrow +\infty} \frac{\rho_{l,n+k} V_{l,n+k}(j)}{\rho_{l+1,n+k} V_{l+1,n+k}(j)} \quad \mathbb{P}\text{-a.s.}$$

This, together with (5.3), implies that for $n \geq 1$ and $1 \leq j, r \leq d$, as $k \rightarrow +\infty$,

$$\Pi_{n+k,n}^i(j, r) \rightarrow \Pi_{\infty,n}^i(j, r) = \frac{M_{0,n-1}(i, r) U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \quad \mathbb{P}\text{-a.s.}$$

Hence, as $k \rightarrow \infty$, \mathbb{P} -a.s., $\Pi_{k,0}^i = \Pi_{k,1}^i \Pi_0^i \rightarrow \Pi_{\infty,1}^i \Pi_0^i = \Pi_{\infty,0}^i$, where

$$\Pi_{\infty,0}^i(j, r) = \sum_{l=1}^d \Pi_{\infty,1}^i(j, l) \Pi_0^i(l, r) = \sum_{l=1}^d \Pi_{\infty,1}^i(j, l) \delta_{i,r} = \delta_{i,r}, \quad 1 \leq j, r \leq d.$$

(2) By (4.4) in Lemma 4.3, we have, for all $n \geq 1$ and $1 \leq i, j, r \leq d$,

$$\frac{1}{\Pi_n^i(j, r)} = \sum_{l=1}^d \frac{M_{0,n-1}(i, l) M_n(l, j)}{M_{0,n-1}(i, r) M_n(r, j)} \leq dD^2 \quad \mathbb{P}\text{-a.s.},$$

or equivalently

$$\Pi_n^i(j, r) \geq \frac{1}{dD^2} \quad \mathbb{P}\text{-a.s.} \quad (5.4)$$

Since $(\Pi_n^i)_{n \geq 0}$ is a sequence of positive stochastic matrices satisfying (5.4), by Lemma 5.3, there exist two constants $C > 0$ and $\delta \in (0, 1)$ such that for all $k \geq 0$ and $1 \leq i \leq d$,

$$\sup_{n \geq 0} \|\Pi_{n+k,n}^i - \Pi_{\infty,n}^i\| \leq C\delta^k, \quad \mathbb{P}\text{-a.s.}$$

This concludes the proof of Lemma 5.2. \square

5.2. Proof of Theorems 2.3 and 2.4

For all $n \geq 0$ and $1 \leq i, j \leq d$, set

$$\bar{Z}_n^i(j) := \frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i, j)}.$$

First we show that (2.9) is a necessary condition for the convergence in L^p of the normalized population size $\bar{Z}_n^i(j)$, $1 \leq i, j \leq d$. Assume that $(\bar{Z}_n^i(j))_{n \geq 0}$ converges in L^p for all $1 \leq i, j \leq d$. For $n \geq 0$ and $1 \leq i \leq d$ we have, by the definition of W_n^i and (1.11),

$$W_n^i = \sum_{j=1}^d \frac{M_{0,n-1}(i, j) U_{n,\infty}(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \bar{Z}_n^i(j) \leq \max_{1 \leq j \leq d} \bar{Z}_n^i(j).$$

This implies that the martingale $(W_n^i)_{n \geq 0}$, $1 \leq i \leq d$, is bounded in L^p , hence converges in L^p . So by Theorem 2.1, condition (2.9) holds.

Now we prove that (2.9) is sufficient for the convergence in L^p of $\bar{Z}_n^i(j)$, $1 \leq i, j \leq d$, and establish meanwhile Theorem 2.4 about its convergence rate. Assume (2.9). By the definition of the branching process (Z_n^i) (cf. (4.6)), we have the following decomposition: for all $1 \leq i, j \leq d$ and $n, k \geq 1$,

$$\begin{aligned} \bar{Z}_{n+k}^i(j) &= \sum_{r=1}^d \frac{M_{n,n+k-1}(r,j)}{M_{0,n+k-1}(i,j)} \sum_{l=1}^{Z_n^i(r)} \frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} \\ &= \sum_{r=1}^d \frac{M_{n,n+k-1}(r,j)}{M_{0,n+k-1}(i,j)} Z_n^i(r) \\ &\quad + \sum_{r=1}^d \frac{M_{n,n+k-1}(r,j)}{M_{0,n+k-1}(i,j)} \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right). \end{aligned} \quad (5.5)$$

Combining (5.5) and (5.2), we get that for all $1 \leq i, j \leq d$ and $n, k \geq 1$,

$$\begin{aligned} \bar{Z}_{n+k}^i(j) &= \sum_{r=1}^d \Pi_{n+k-1,n}^i(j,r) \bar{Z}_n^i(r) \\ &\quad + \sum_{r=1}^d \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)} \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right) \\ &= \langle \Pi_{n+k-1,n}^i \bar{Z}_n^i, e_j \rangle + R_{n,k}^i(j), \end{aligned} \quad (5.6)$$

with

$$R_{n,k}^i(j) := \sum_{r=1}^d \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)} \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right).$$

Notice that by the definition of W_n^i and that of $\Pi_{\infty,n}^i$ (cf. Lemma 5.2 (1)),

$$W_n^i = \sum_{r=1}^d \Pi_{\infty,n}^i(j,r) \bar{Z}_n^i(r) = \langle \Pi_{\infty,n}^i \bar{Z}_n^i, e_j \rangle \quad (5.7)$$

for any $1 \leq i, j \leq d$. Using (5.6) and (5.7), together with the triangular inequality

in L^p under \mathbb{P} , we obtain that for all $1 \leq i, j \leq d$ and $n, k \geq 1$,

$$\begin{aligned}
& \left(\mathbb{E} |\bar{Z}_{n+k}^i(j) - W^i|^p \right)^{1/p} \\
&= \left(\mathbb{E} |\langle \Pi_{n+k-1,n}^i \bar{Z}_n^i, e_j \rangle - W^i + R_{n,k}^i(j)|^p \right)^{1/p} \\
&= \left(\mathbb{E} |W_n^i - W^i + \langle (\Pi_{n+k-1,n}^i - \Pi_{\infty,n}^i) \bar{Z}_n^i, e_j \rangle + R_{n,k}^i(j)|^p \right)^{1/p} \\
&\leq \left(\mathbb{E} |W_n^i - W^i|^p \right)^{1/p} + \left(\mathbb{E} \|(\Pi_{n+k-1,n}^i - \Pi_{\infty,n}^i) \bar{Z}_n^i\|^p \right)^{1/p} \\
&\quad + \max_{1 \leq j \leq d} \left(\mathbb{E} |R_{n,k}^i(j)|^p \right)^{1/p} \\
&= J_1^i(n) + J_2^i(n, k) + J_3^i(n, k). \tag{5.8}
\end{aligned}$$

In the following $C > 0$ will be a constant which may depend on p and which may differ from line to line.

Control of $J_1^i(n)$. By condition (2.9) and Theorem 2.2 we get that there exists $\delta_1 \in (\delta_c(p), 1)$ such that for all $n \geq 1$ and $1 \leq i \leq d$,

$$J_1^i(n) = \left(\mathbb{E} |W_n^i - W^i|^p \right)^{1/p} \leq C \delta_1^n. \tag{5.9}$$

Control of $J_2^i(n, k)$. Applying the relation (5.1) of Lemma 5.2, we get that there exists $\delta_2 \in (0, 1)$ such that for all $n, k \geq 1$ and $1 \leq i \leq d$,

$$J_2^i(n, k) = \left(\mathbb{E} \|(\Pi_{n+k-1,n}^i - \Pi_{\infty,n}^i) \bar{Z}_n^i\|^p \right)^{1/p} \leq C \left(\mathbb{E} \|\bar{Z}_n^i\|^p \right)^{1/p} \delta_2^k. \tag{5.10}$$

Combining (4.34) and Theorem 2.1, and using condition (2.9), we obtain that, for all $1 \leq i \leq d$,

$$\sup_{n \geq 0} \left(\mathbb{E} \|\bar{Z}_n^i\|^p \right)^{1/p} \leq d^2 D^2 \sup_{n \geq 0} \left(\mathbb{E} (W_n^i)^p \right)^{1/p} < +\infty. \tag{5.11}$$

This, together with (5.10), implies that for all $n, k \geq 1$ and $1 \leq i \leq d$,

$$J_2^i(n, k) \leq C \delta_2^k. \tag{5.12}$$

Control of $J_3^i(n, k)$ for $1 < p \leq 2$. Assume that $1 < p \leq 2$. Using the convexity of the function $x \mapsto x^p$ (together with $\sum_{r=1}^d \Pi_{n+k-1,n}^i(j, r) = 1$) and Lemma 4.2,

for all $n, k \geq 1$ and $1 \leq i, j \leq d$, \mathbb{P} -a.s., we have

$$\begin{aligned}
\mathbb{E}_\xi |R_{n,k}^i(j)|^p &\leq \mathbb{E}_\xi \left(\sum_{r=1}^d \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)} \left| \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right) \right| \right)^p \\
&\leq \mathbb{E}_\xi \left(\sum_{r=1}^d \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)^p} \left| \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right) \right|^p \right) \\
&\leq B_p^p \sum_{r=1}^d \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi Z_n^i(r) \mathbb{E}_\xi \left| \frac{Z_{1,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right|^p \\
&= B_p^p \sum_{r=1}^d \Pi_{n+k-1,n}^i(j,r) M_{0,n-1}(i,r)^{1-p} \mathbb{E}_\xi \left| \frac{Z_{1,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right|^p \\
&\leq B_p^p \bar{\sigma}_{n,k}(p) \max_{1 \leq r \leq d} M_{0,n-1}(i,r)^{1-p},
\end{aligned}$$

where

$$\bar{\sigma}_{n,k}(p) = \max_{1 \leq r, j \leq d} \mathbb{E}_\xi \left| \frac{Z_{1,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right|^p.$$

So, by taking expectation and using the independence between $\bar{\sigma}_{n,k}(p)$ and $M_{0,n-1}$, we get that for all $n, k \geq 1$ and $1 \leq i \leq d$,

$$J_3^i(n, k)^p \leq B_p^p \mathbb{E} \bar{\sigma}_{0,k}(p) \sum_{r=1}^d \mathbb{E} [M_{0,n-1}(i, r)^{1-p}]. \quad (5.13)$$

By (5.11) we have

$$\sup_{k \geq 0} \mathbb{E} \bar{\sigma}_{0,k}(p) \leq d^2 \max_{1 \leq r, j \leq d} \sup_{k \geq 0} \mathbb{E} |\bar{Z}_k^r(j) - 1|^p < +\infty. \quad (5.14)$$

Therefore, putting together the relations (5.13) and (5.14) with Lemma 3.2, we get that for $n, k \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned}
J_3^i(n, k)^p &\leq C \sup_{k \geq 0} \mathbb{E} \bar{\sigma}_{0,k}(p) \sum_{r=1}^d \mathbb{E} [\langle M_{0,n-1} e_r, e_i \rangle^{1-p}] \\
&\leq C \kappa (1-p)^n \\
&\leq C \delta_c(p)^{np}
\end{aligned} \quad (5.15)$$

(recall that the value of C may change from line to line by our convention).

Control of $J_3^i(n, k)$ for $p \geq 2$. Assume that $p \geq 2$. Similar to the preceding case, by the convexity of $x \mapsto x^p$ (together with $\sum_{r=1}^d \Pi_{n+k-1,n}^i(j, r) = 1$) and Lemma

4.2, for all $n, k \geq 1$ and $1 \leq i, j \leq d$, \mathbb{P} -a.s.,

$$\begin{aligned}
\mathbb{E}_\xi |R_{n,k}^i(j)|^p &\leq \mathbb{E}_\xi \left(\sum_{r=1}^d \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)} \left| \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right) \right| \right)^p \\
&\leq B_p^p \sum_{r=1}^d \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi (Z_n^i(r))^{p/2} \mathbb{E}_\xi \left| \frac{Z_{1,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right|^p \\
&\leq B_p^p \bar{\sigma}_{n,k}(p) \sum_{r=1}^d \Pi_{n+k-1,n}^i(j,r) \mathbb{E}_\xi (\bar{Z}_n^i(r))^{p/2} M_{0,n-1}(i,r)^{-p/2} \\
&\leq B_p^p \bar{\sigma}_{n,k}(p) \max_{1 \leq r \leq d} \left\{ \mathbb{E}_\xi (\bar{Z}_n^i(r))^{p/2} M_{0,n-1}(i,r)^{-p/2} \right\}. \tag{5.16}
\end{aligned}$$

Notice that (5.14) still holds when $p \geq 2$. Therefore, taking expectation in (5.16) and using (5.14), we obtain that for all $n, k \geq 1$ and $1 \leq i \leq d$,

$$\begin{aligned}
J_3^i(n, k)^p &\leq B_p^p \mathbb{E} \bar{\sigma}_{0,k}(p) \sum_{1 \leq r \leq d} \mathbb{E} \left[\mathbb{E}_\xi (\bar{Z}_n^i(r))^{p/2} M_{0,n-1}(i,r)^{-p/2} \right] \\
&\leq C \sum_{r=1}^d \mathbb{E} \left[\mathbb{E}_\xi (\bar{Z}_n^i(r))^{p/2} M_{0,n-1}(i,r)^{-p/2} \right]. \tag{5.17}
\end{aligned}$$

Using (4.4) in Lemma 4.3 and (1.11), for all $n \geq 1$ and $1 \leq i, r \leq d$, \mathbb{P} -a.s., we have

$$\begin{aligned}
M_{0,n-1}(i, r) &\geq \frac{1}{dD} \|M_{0,n-1}(i, \cdot)\| \\
&\geq \frac{1}{dD} \langle M_{0,n-1}(i, \cdot), U_{n,\infty} \rangle \\
&= \frac{1}{dD} \lambda_{0,n-1} U_{0,\infty}(i).
\end{aligned}$$

Combining this with (5.17) and (4.34), we get that for $p > 2$, $n, k \geq 1$ and $1 \leq i \leq d$,

$$J_3^i(n, k)^p \leq C \mathbb{E} \left[\mathbb{E}_\xi (W_n^i)^{p/2} (\lambda_{0,n-1} U_{0,\infty}(i))^{-p/2} \right] = C \mathbb{E} a_{n,1}^i(p),$$

where $a_{n,1}^i(p)$ is defined in (4.17) with $j = 1$. This, together with (4.19) (which holds under condition (2.9)), implies that there exists $\delta_3 \in (\delta_c(p), 1)$ such that for all $n, k \geq 1$ and $1 \leq i \leq d$,

$$J_3^i(n, k)^p \leq C \delta_3^{np}. \tag{5.18}$$

Combining (5.8), (5.9), (5.12), (5.15) and (5.18), we obtain that for all $n, k \geq 1$ and $1 \leq i, j \leq d$,

$$\left(\mathbb{E} |\bar{Z}_{n+k}^i(j) - W^i|^p \right)^{1/p} \leq C(\delta_1^n + \delta_2^k + \delta_3^n).$$

Applying this inequality with n replaced by $\lfloor n/2 \rfloor$ (the integral part of $n/2$) and taking $k = n - \lfloor n/2 \rfloor$, we see that for all $n \geq 1$ and $1 \leq i, j \leq d$,

$$\left(\mathbb{E}|\bar{Z}_n^i(j) - W^i|^p\right)^{1/p} \leq C(\delta_1^{n/2} + \delta_2^{n/2} + \delta_3^{n/2}) \leq C\delta^n,$$

with $\delta = \max\{\delta_1^{1/2}, \delta_2^{1/2}, \delta_3^{1/2}\} < 1$. Therefore, for any $1 \leq i, j \leq d$ the normalized population size $\bar{Z}_n^i(j)$ converges in L^p to W^i with an exponential speed, which gives (2.11). This concludes the proof of Theorems 2.3 and 2.4.

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