BERRY -ESSEEN BOUND AND CRAMÉR MODERATE DEVIATION EXPANSION FOR A SUPERCRITICAL BRANCHING RANDOM WALK

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Abstract. We consider a supercritical branching random walk where each particle gives birth to a random number of particles of the next generation, which move on the real line, according to a fixed law. Let Z_n be the counting measure which counts the number of particles of nth generation situated in a given region. Under suitable conditions, we establish a Berry-Esseen bound and a Cramér type moderate deviation expansion for Z_n with suitable norming.

1. Introduction

A branching random walk is a system of particles, in which each particle gives birth to new particles of the next generation, whose children move on $\mathbb{R}$. The particles behave independently; the number of children and their displacements are governed by the same probability law for all particles. Important research topics on the model include the study of the asymptotic properties of the counting measure Z_n which counts the number of particles of generation n situated in a Borel set (see e.g. [2, 3, 8, 9, 10, 17, 19, 20, 18]), the study of the fundamental martingale, the norming problem, and the properties of the limit variable (see e.g. [7, 13, 30, 32, 31, 1, 26, 29]), and the positions of the extreme particles (which constitute the boundary of the support of the counting measure Z_n (see e.g. [24, 23, 5, 16]), etc. The study of this model is very interesting especially due to a large number of applications and its close relation with other important models in applied

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probability settings, such as multiplicative cascades, fractals, perpetuities, branching Brownian motion, the quick sort algorithm and infinite particle systems. For close relations to Mandelbrot's cascades, see e.g. [27, 31, 6, 14, 33]; for relations to other important models, see e.g. the recent books [36, 15, 25] and many references therein. In this paper, we consider the asymptotic properties of the counting measure Z_n as $n \to \infty$, by establishing the Berry - Esseen bound and Cramér's moderate deviation expansion for a suitable norming of Z_n. The study of asymptotic properties of Z_n is interesting because it gives a good description of the configuration of the system at time n.

The branching random walk on the real line can be defined precisely as follows. The process begins with one initial particle denoted by the null sequence $\emptyset$, situated at the origin $S_\emptyset = 0$. It gives birth to N children denoted by $\emptyset i = i$, with displacements L_i, $i = 1, \cdots, N$. In general, each particle of generation n, denoted by a sequence $u = u_1 \cdots u_n$ of length n, situated at $S_u \in \mathbb{R}$, gives birth to N_u particles of the next generation, denoted by ui, which move on the real line with displacements L_{ui} so that their positions are $S_{ui} = S_u + L_{ui}, i = 1, \cdots, N_u$. All the random variables $(N_u, L_{u1}, L_{u2}, \cdots)$, indexed by all finite sequences $u \in \mathbb{U} := \cup_{n=0}^{\infty} (\mathbb{N}^*)^n$ (by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$), are independent and identically distributed, defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with values in $\mathbb{N} \times \mathbb{R} \times \mathbb{R} \times \cdots$.

For $n \geqslant 0$, let $\mathbb{T}_n$ be the set of particles of n-th generation. Consider the counting measure

$$Z_n(A) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{S_u \in A\}}, \quad A \subset \mathbb{R},$$

which counts the number of particles of n-th generation situated in A.

Throughout this paper we assume that

$$m := \mathbb{E} N = \mathbb{E}[Z_1(\mathbb{R})] \in (1, \infty),$$

so that the Galton-Watson process formed by the generation sizes survives with positive probability, and

$$F(A) = \mathbb{E}[Z_1(A)], \quad A \subset \mathbb{R},$$

is a finite measure on $\mathbb{R}$ with mass m. Let $\overline{F}$ be the probability mesure on $\mathbb{R}$ defined by

$$\overline{F}(A) = \frac{F(A)}{m}, \quad A \subset \mathbb{R}.$$

Denote its mean and variance by

$$m_0 = \int x\overline{F}(dx) \quad \text{and} \quad \sigma_0^2 = \int (x-m_0)^2\overline{F}(dx). \tag{1.1}$$

We will assume that $\mathbb{E}(\sum_{i=1}^N L_i^2) < \infty$, so that m_0 and σ_0^2 are finite, with

$$m_0 = \frac{1}{m}\mathbb{E}\Big[\sum_{i=1}^N L_i\Big] \quad \text{and} \quad \sigma_0^2 = \frac{1}{m}\mathbb{E}\Big[\sum_{i=1}^N L_i^2\Big] - m_0^2.$$

A central limit theorem for the special case where $(N_u)_{u\in\mathbb{U}}$ and $(L_u)_{u\in\mathbb{U}}$ are two independent families of independent and identically distributed (i.i.d.) random variables was conjectured by Harris [22]. His conjecture states that under suitable conditions we have, for any $x \in \mathbb{R}$,

$$\frac{1}{m^n}Z_n\Big((-\infty, x\sigma_0\sqrt{n}+nm_0]\Big) \overset{n\to\infty}{\longrightarrow} W\Phi(x) \tag{1.2}$$

in probability, where $\Phi(x)$ is the normal distribution function and W is the a.s. limit of the fundamental martingale $\left(\frac{Z_n(\mathbb{R})}{m^n}\right)$ of the Galton-Watson process $(Z_n(\mathbb{R}))$. This conjecture has first been solved by Stam [37], then improved by Asmussen and Kaplan [2, 3] to L^2-convergence and almost sure (a.s.) convergence. The general case has been considered by Klebaner [28] and Biggins [10].

In this paper we will study the Berry -Esseen bound about the rate of convergence in (1.2), and the associated Cramér's moderate deviation expansion.

The rate of convergence in (1.2) has been studied in several papers. Révész [35] considered the special case where the displacements follow the same Gaussian law and conjectured the exact convergence rate; his conjecture was solved by Chen [17]. Gao and Liu [19] improved and extended Chen's result to the general non-lattice case while the lattice case has been considered by Grübel and Kabluchko [21]. All the above mentioned results are about the point-wise convergence without uniformity in x. In this paper, our first objective is to find a uniform bound for the rate of convergence in (1.2) of type Berry-Esseen: we will prove that, under suitable conditions, a.s. for $n \geqslant 1$,

$$\sup_{x\in\mathbb{R}}\left|\frac{1}{m^n}Z_n\Big((-\infty, x\sigma_0\sqrt{n}+nm_0]\Big) - W\Phi(x)\right| \leqslant \frac{M}{\sqrt{n}}, \tag{1.3}$$

where M is a positive and finite random variable (see Theorem 2.1).

The problem of large deviations for the counting measure $Z_n(\cdot)$ has been considered by Biggins: he established in [8] a large deviation principle, which was subsequently improved in [9] to a Bahadur-Rao large deviation asymptotic. Our second objective in this paper is to establish a Cramér's type moderate deviation expansion for Z_n (see Theorem 2.2): we will prove that a.s. for $n \to \infty$ and $x \in [0, o(\sqrt{n})]$,

$$\frac{Z_n\Big((x\sigma_0\sqrt{n} + nm_0, +\infty) \Big)}{m^n W[1-\Phi(x)]} = e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})}\Big[1 + O\Big(\frac{x+1}{\sqrt{n}}\Big)\Big], \tag{1.4}$$

where $t \mapsto \mathscr{L}(t)$ is the Cramér series (see (2.7)). Here we use the usual notation $b_n = O(a_n)$ to mean that the sequence (b_n/a_n) is bounded. (We mention that as (1.4) holds a.s., the bound in $O\Big(\frac{x+1}{\sqrt{n}}\Big)$ may be random.)

Let us explain briefly the key ideas in the proofs. To prove the Berry-Esseen bound (1.3), we use Esseen's smoothing inequality ([34, Theorem V.2.2.]). The key point in this proof is the formula of the characteristic function of $\frac{1}{m^n} Z_n\Big((-\infty, x\sigma_0\sqrt{n} + nm_0]\Big)$, which can be interpreted as $W_n(\frac{it}{\sigma_0\sqrt{n}})f_n(t)$, $t \in \mathbb{R}$, where $(W_n(\lambda))$ is Biggins' martingale with complexed valued parameter λ for the branching random walk (see [11, 12]), and $f_n(t)$ is the characteristic function of the n-fold convolution of $\overline{F}$. Using the results of Biggins [11, 12], Grübel and Kabluchko [21] about the uniform convergence of $W_n(\lambda)$, together with the approach of Petrov [34] for the proof of the Berry-Esseen bound for sums of i.i.d. random variables, we are able to establish (1.3). The Berry-Esseen bound (1.3) is then extended to the changed measure of type Cramér, $Z_n^\theta(A) = \int_A e^{\theta t} Z_n(dt), A \subset \mathbb{R}, \theta \in \mathbb{R}$. This is an important step in establishing the moderate deviation expansion (1.4). Our approach in proving (1.4) is very different from the method of Biggins [9] on the Bahadur-Rao large deviation asymptotic; instead, it is inspired by the ideas of the proof of Cramér's moderate deviation expansion on sums of i.i.d. random variables (see [34]), and the arguments in [12] for the proof of the local limit theorem with large deviations for Z_n.

The main results, Theorems 2.1-2.3, are presented in Section 2. Theorems 2.1 and 2.3 about the Berry-Esseen bound are proved in Section 3, while Theorem 2.2 about the moderate deviation is established in Section 4.

2. Notation and results

We will use the following standard assumptions.

C1. *$N > 0$ a.s. with $m = \mathbb{E}N \in (1, \infty)$, and $\mathbb{E}\Big[\sum_{i=1}^N L_i^2\Big] < \infty$.*

C2. *F is non-degenerate, i.e. it is not concentrated on a single point.*

The first condition in **C1** implies that the underlying Galton -Watson process is supercritical; the second condition in **C1**, together with condition **C2**, implies that the mean m_0 and the variance σ_0^2 defined by (1.1) are finite with $\sigma_0 > 0$.

The Laplace transform of F will be denoted by

$$m(\lambda) = \int_{\mathbb{R}} e^{\lambda t} F(dt) = \mathbb{E}\Big[\sum_{i=1}^N e^{\lambda L_i}\Big], \quad \lambda \in \mathbb{C}. \tag{2.1}$$

Denote by $\text{int}(A)$ the interior of the set A. Set

$$\mathcal{D} = \text{int}\{\theta \in \mathbb{R} : m(\theta) < \infty\}. \tag{2.2}$$

Throughout, we assume that

C3. *$\mathcal{D}$ is non-empty.*

Denote by $Re(\lambda)$ the real part of $\lambda \in \mathbb{C}$. A important role in the proof of Berry-Esseen bound and moderate deviation expansion is played by the martingale of Biggins with complex parameter:

$$W_n(\lambda) = \frac{1}{m(\lambda)^n} \int_{\mathbb{R}} e^{\lambda t} Z_n(dt) = \sum_{u \in \mathbb{T}_n} \frac{e^{\lambda S_u}}{m(\lambda)^n}, \quad n \geqslant 0, Re(\lambda) \in \mathcal{D}.$$

When $\lambda = 0$, $W_n := W_n(0) = \frac{Z_n(\mathbb{R})}{m^n}$ is the fundamental martingale of the Galton -Watson process $(Z_n(\mathbb{R}))$, whose a.s. limit is denoted by W. The famous Kesten-Stigum theorem states that W is non degenerate if and only if $\mathbb{E}N \log_+ N < \infty$ (see [4]), where $\log_+ x = \max\{0, \log x\}$ denotes the positive part of $\log x$. By the martingale convergence theorem for non-negative martingales, we have for all $\theta \in \mathcal{D}$,

$$W_n(\theta) \overset{n\to\infty}{\to} W(\theta), \quad \text{a.s.}$$

Notice that when $N > 0$ a.s. we have $W_n(\theta) > 0$ a.s. for all $n \geqslant 0$ and $\theta \in \mathcal{D}$. Biggins [7, Theorem A] gave a necessary and sufficient condition for the non-degeneracy of $W(\theta)$: $\mathbb{E}W(\theta) > 0$ if and only if

$$\mathbb{E}[W_1(\theta) \log_+ W_1(\theta)] < \infty \quad \text{and} \quad \theta \in (\theta_-, \theta_+), \tag{2.3}$$

where $(\theta_-, \theta_+) \subset \mathcal{D}$ denotes by the open interval on which $\frac{\theta m'(\theta)}{m(\theta)} < \log m(\theta)$, i.e.

$$\theta_- = \inf \Big\{ \theta \in \mathcal{D} : \frac{\theta m'(\theta)}{m(\theta)} < \log m(\theta) \Big\},$$
$$\theta_+ = \sup \Big\{ \theta \in \mathcal{D} : \frac{\theta m'(\theta)}{m(\theta)} < \log m(\theta) \Big\}.$$

Moreover, when **C1** and (2.3) hold,

$$W(\theta) > 0 \text{ a.s. and } \quad \mathbb{E}W(\theta) = 1. \tag{2.4}$$

We see that $0 \in (\theta_-, \theta_+)$, so that this interval is non-empty. The endpoints of the interval $\mathcal{D}$ and the quantities θ_-, θ_+ are allowed to be infinite. We will need the following moment condition which is slightly stronger than (2.3).

C4. *There are $\gamma > 1$ and $K_0 > 0$ with $(-K_0, K_0) \subset (\theta_-, \theta_+)$ such that*

$$\mathbb{E}W_1^{\gamma}(\theta) < \infty \quad \forall \theta \in (-K_0, K_0).$$

By the argument of the proof of [12, Theorem 2], we know that under hypothesis **C4**, for every compact subset C of $V := \{\lambda = \theta + i\eta : \theta \in (-K_0, K_0), \eta \in \mathbb{R}\}$, a.s.

$$\sup_{\lambda \in C} |W_n(\lambda) - W(\lambda)| \overset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad W(\lambda) \text{ is analytic in } C. \tag{2.5}$$

Our first result gives the Berry-Esseen bound for Z_n:

Theorem 2.1. *Assume conditions **C1** -**C4**. Then, a.s. for all $n \geqslant 1$,*

$$\sup_{x \in \mathbb{R}} \left| \frac{Z_n\Big((-\infty, x\sigma_0\sqrt{n} + nm_0]\Big)}{m^n} - W\Phi(x) \right| \leqslant \frac{M}{\sqrt{n}},$$

where M is a positive and finite random variable.

To state the result corresponding to the Cramér type moderate deviation expansion for Z_n, we need more notation. Consider the measure

$$F_\theta(dx) = \frac{e^{\theta x}}{m(\theta)} F(dx), \quad \theta \in \mathcal{D}. \tag{2.6}$$

We see that F_θ is a distribution function with finite mean m_θ and variance σ_θ^2, given by

$$m_\theta = \frac{m'(\theta)}{m(\theta)}, \quad \sigma_\theta^2 = \frac{m''(\theta)}{m(\theta)} - \Big(\frac{m'(\theta)}{m(\theta)}\Big)^2;$$

moreover, $\sigma_\theta > 0$ when F is non-degenerate. Consider the change of measure of type Cramer for Z_n: for $\theta \in \mathcal{D}$,

$$Z_n^\theta(dx) = e^{\theta x} Z_n(dx),$$

namely,

$$Z_n^\theta(A) = \sum_{u \in \mathbb{T}_n} e^{\theta S_u} \mathbb{1}_{\{S_u \in A\}}, \quad A \subset \mathbb{R}.$$

Let X be a random variable with distribution $\overline{F} := \frac{F}{m}$, and

$$\Lambda(\theta) := \log \mathbb{E} e^{\theta X} = \log m(\theta) - \log m$$

be its cumulant generating function. Then $\Lambda(\theta)$ is analytic on $\mathcal{D}$, with $\Lambda'(\theta) = m_\theta$ and $\Lambda''(\theta) = \sigma_\theta^2$. Denote by $\gamma_k := \Lambda^{(k)}(0)$ the cumulant of order k of the random variable X. We shall use the Cramér series (see [34, Theorem VIII.2.2]):

$$\mathscr{L}(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3} t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}} t^2 + \dots \tag{2.7}$$

which converges for $|t|$ small enough.

Theorem 2.2. *Assume conditions **C1** -**C4**. Then we have, for $0 \leqslant x = o(\sqrt{n})$, as $n \to \infty$, a.s.*

$$\frac{Z_n\Big((x\sigma_0\sqrt{n} + nm_0, +\infty) \Big)}{m^n W[1-\Phi(x)]} = e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})} \Big[1 + O\Big(\frac{x+1}{\sqrt{n}}\Big)\Big], \tag{2.8}$$

and

$$\frac{Z_n\Big((-\infty, -x\sigma_0\sqrt{n} + nm_0) \Big)}{m^n W\Phi(-x)} = e^{-\frac{x^3}{\sqrt{n}}\mathscr{L}(-\frac{x}{\sqrt{n}})} \Big[1 + O\Big(\frac{x+1}{\sqrt{n}}\Big)\Big]. \tag{2.9}$$

As a by-product in the proof of Theorem 2.2, we obtain a Berry - Esseen bound for the changed measure Z_n^θ with uniformity in θ.

Theorem 2.3. *Assume conditions **C1** -**C4**. Then, there exists a constant $0 < K < K_0$ such that a.s. for all $n \geqslant 1$,*

$$\sup_{\theta \in [-K,K]} \sup_{x \in \mathbb{R}} \Big| \frac{Z_n^\theta\Big((-\infty, x\sigma_\theta\sqrt{n} + nm_\theta]\Big)}{m(\theta)^n} - W(\theta)\Phi(x) \Big| \leqslant \frac{M}{\sqrt{n}},$$

where M is a positive and finite random variable.

3. Proof of Theorems 2.1 and 2.3

We first recall some known results in the form of two lemmas which will be used for the proof of Theorems 2.1 and 2.3.

The first lemma concerns the Cramér change of measure (2.6), see [34, Theorem VIII.2.2, inequalities (2.31) and (2.32)]).

Lemma 3.1. *Let X be a real random variable with distribution G. Suppose that $Var(X) > 0$ and that there exist strictly positive constants H, c such that*

$$|\log \mathbb{E}e^{\theta X}| \leqslant c \ \ \text{for all} \ \ \theta \in (-H, H).$$

Let X_θ be a real random variable with distribution G_θ defined by

$$G_\theta(dx) = \frac{e^{\theta x} G(dx)}{\mathbb{E}e^{\theta X}}, \quad \theta \in (-H, H).$$

Then there exist strictly positive constants H_1, c_1, c_2 with $H_1 < H$, such that for all $\theta \in (-H_1, H_1)$,

$$Var(X_\theta) \geqslant c_1 \ \ \text{and} \ \ \mathbb{E}|X_\theta - \mathbb{E}X_\theta|^3 \leqslant c_2.$$

We see that under **C2** and **C3**, the distribution $G = \overline{F}$ satisfies the conditions of this lemma. Indeed, if X is a random variable with distribution $\overline{F}$, then by condition **C2** about the non-degeneracy of F, we have $Var(X) > 0$. By condition **C3**, the set $\mathcal{D}$ defined by (2.2) is an open interval containing 0. Notice that $\log \mathbb{E}e^{\theta X} = \log \frac{m(\theta)}{m} < \infty$ for all $\theta \in \mathcal{D}$. Hence there exist constants $H, c > 0$ such that $|\log \mathbb{E}e^{\theta X}| \leqslant c$ for all $\theta \in (-H, H)$.

The second lemma is about the exponential convergence rate of $W_n(\theta)$, see [21, Lemma 3.3]. In fact in [21, Lemma 3.3] the result is only given for the lattice case, but the proof therein remains valid for the non-lattice case.

Lemma 3.2. *Assume condition **C1**-**C3**. There exist two constants $0 < K < K_0$ and $c \in (0,1)$ such that a.s. for all $n \geqslant 0$,*

$$\sup_{\theta \in [-K, K]} |W_n(\theta) - W(\theta)| \leqslant M_1 c^n,$$

where M_1 is a positive and finite random variable.

Notice that Theorem 2.1 follows from Theorem 2.3 with $\theta = 0$, by the fact that $m(0) = m$ and $W(0) = W$. So we only proceed to prove Theorem 2.3.

Proof of Theorem 2.3. From Lemma 3.2, to prove Theorem 2.3, it is enough to show that there is a constant $0 < K < K_0$ such that

$$\sup_{\theta \in [-K,K]} \sup_{x \in \mathbb{R}} \left| \frac{Z_n^\theta\Big((-\infty, x\sigma_\theta\sqrt{n} + nm_\theta]\Big)}{m(\theta)^n} - W_n(\theta)\Phi(x) \right| \leqslant \frac{M}{\sqrt{n}},$$

where M is a positive and finite random variable. Consider the random measure

$$\nu_n^\theta(A) = \frac{Z_n^\theta\Big(\sigma_\theta\sqrt{n}A + nm_\theta\Big)}{m(\theta)^n}, \quad A \subset \mathbb{R},$$

with the usual notation $aA + b = \{ax + b : x \in A\}$. Its distribution function is

$$\nu_n^\theta(x) = \frac{Z_n^\theta\Big((-\infty, x\sigma_\theta\sqrt{n} + nm_\theta]\Big)}{m(\theta)^n}, \quad x \in \mathbb{R}.$$

The characteristic function of the random measure ν_n^θ is

$$\begin{aligned}\psi_n^\theta(t) = \int_{\mathbb{R}} e^{itx}\nu_n^\theta(dx) &= \frac{1}{m(\theta)^n}\sum_{u \in \mathbb{T}_n} \exp\Big\{\Big(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\Big)S_u - \frac{it\,nm_\theta}{\sigma_\theta\sqrt{n}}\Big\} \\ &= W_n\Big(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\Big)f_n^\theta(t), \quad t \in \mathbb{R}, \end{aligned} \tag{3.1}$$

where $f_n^\theta(t) = \frac{1}{m(\theta)^n} m\Big(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\Big)^n e^{-\frac{it\,nm_\theta}{\sigma_\theta\sqrt{n}}}$. Denote by F_θ^{*n} the n-fold convolution of F_θ. It is not difficult to see that

$$f_n^\theta(t) = \int_{\mathbb{R}} e^{\frac{it(x - nm_\theta)}{\sigma_\theta\sqrt{n}}} F_\theta^{*n}(dx),$$

which is the characteristic function of $\frac{\overline{S}_n - nm_\theta}{\sigma_\theta\sqrt{n}}$, where $\overline{S}_n$ is the sum of independent random variables $\{\overline{X}_i\}_{i=1}^n$ with the same law F_θ.

By Esseen's smoothing inequality (see [34, Theorem V.2.2]), we get for all $T > 0$, a.s.

$$\begin{aligned} &\sup_{x \in \mathbb{R}} \Big|\nu_n^\theta(x) - W_n(\theta)\Phi(x)\Big| \\ &\leqslant \frac{1}{\pi}\int_{-T}^{T} \left| \frac{W_n\Big(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\Big)f_n^\theta(t) - W_n(\theta)e^{-t^2/2}}{t} \right| dt + W_n(\theta)\frac{c}{T}, \end{aligned} \tag{3.2}$$

where c is a deterministic positive constant. From Lemma 3.1, there exist strictly positive constants K, c_1, c_2 with $K < \min\{H_1, K_0\}$ such that for all $|\theta| \leqslant K$

$$\sigma_\theta^2 \geqslant c_1 \quad \text{and} \quad \mathbb{E}|\overline{X} - m_\theta|^3 \leqslant c_2. \tag{3.3}$$

Take $T = a\sigma_\theta\sqrt{n}$ with $a = \inf_{\theta\in[-K,K]} \frac{\sigma_\theta^2}{4\mathbb{E}|\overline{X}-m_\theta|^3} \geqslant \frac{c_1}{c_2} > 0$. For $0 < \varepsilon < a$, we split the integral on the right-hand side of (3.2) into two parts $|t| < \varepsilon\sigma_\theta\sqrt{n}$ and $\varepsilon\sigma_\theta\sqrt{n} \leqslant |t| \leqslant a\sigma_\theta\sqrt{n}$ to get

$$\sup_{\theta\in[-K,K]} \sup_{x\in\mathbb{R}} \left|\nu_n^\theta(x) - W_n(\theta)\Phi(x)\right| \leqslant \frac{1}{\pi}(I_1 + I_2) + \frac{c}{a\sqrt{n}} \sup_{\theta\in[-K,K]} \frac{W_n(\theta)}{\sigma_\theta},$$

where

$$I_1 = \sup_{\theta\in[-K,K]} \int_{|t|<\varepsilon\sigma_\theta\sqrt{n}} \left|\frac{W_n\left(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\right) f_n^\theta(t) - W_n(\theta)e^{-t^2/2}}{t}\right| dt,$$

$$I_2 = \sup_{\theta\in[-K,K]} \int_{\varepsilon\sigma_\theta\sqrt{n}\leqslant|t|\leqslant a\sigma_\theta\sqrt{n}} \left|\frac{W_n\left(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\right) f_n^\theta(t) - W_n(\theta)e^{-t^2/2}}{t}\right| dt.$$

In the following, M_i denotes a positive and finite random variable. By Lemma 3.2 and the lower bound (3.3) of σ_θ, $\sup_{\theta\in[-K,K]} \frac{W_n(\theta)}{\sigma_\theta} \leqslant M_2$ a.s. Hence, it remains to show that a.s., $I_1 \leqslant \frac{M_3}{\sqrt{n}}$ and $I_2 \leqslant \frac{M_4}{\sqrt{n}}$.

For I_1, we see that

$$\begin{aligned} I_1 \leqslant & \sup_{\theta\in[-K,K]} \sup_{\frac{|t|}{\sigma_\theta\sqrt{n}}\leqslant\varepsilon} \left|W_n\Big(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\Big)\right| \int_{|t|<\varepsilon\sigma_\theta\sqrt{n}} \frac{|f_n^\theta(t) - e^{-t^2/2}|}{|t|} dt \\ & + \sup_{\theta\in[-K,K]} \int_{|t|<\varepsilon\sigma_\theta\sqrt{n}} \frac{\left|W_n\left(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\right) - W_n(\theta)\right|}{|t|} e^{-t^2/2} dt. \end{aligned} \tag{3.4}$$

By the uniform convergence (2.5) of $W_n(\cdot)$, we have

$$\sup_{\theta\in[-K,K]} \sup_{\frac{|t|}{\sigma_\theta\sqrt{n}}\leqslant\varepsilon} \left|W_n\Big(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\Big)\right| \leqslant M_5. \tag{3.5}$$

Recall that $t \mapsto f_n^\theta(t)$ is the characteristic function of $\frac{\overline{S}_n - nm_\theta}{\sigma_\theta\sqrt{n}}$. Then by [34, Lemma V.2.1], for $|t| \leqslant \dfrac{\sigma_\theta^3\sqrt{n}}{4\mathbb{E}|\overline{X} - m_\theta|^3}$, we have

$$\frac{|f_n^\theta(t) - e^{-t^2/2}|}{|t|} \leqslant \frac{\mathbb{E}|\overline{X} - m_\theta|^3}{\sigma_\theta^3\sqrt{n}} t^2 e^{-t^2/3} \leqslant \frac{c_2}{c_1\sqrt{n}} t^2 e^{-t^2/3}. \tag{3.6}$$

Therefore (3.6) holds for $|t| \leqslant \varepsilon\sigma_\theta\sqrt{n}$ since $\varepsilon\sigma_\theta\sqrt{n} \leqslant \dfrac{\sigma_\theta^3\sqrt{n}}{4\mathbb{E}|\overline{X} - m_\theta|^3}$. From (3.5), (3.6) and the fact that $\int_{\mathbb{R}} |t|^2 e^{-t^2/3}dt < \infty$, we see that the first term in (3.4) is bounded by $\frac{M_6}{\sqrt{n}}$.

Now we consider the second term in (3.4). Since $\int_{\mathbb{R}} e^{-t^2/2}dt = \sqrt{2\pi}$, we need only to show that

$$\sup_{\theta\in[-K,K]} \sup_{\frac{|t|}{\sigma_\theta\sqrt{n}}\leqslant\varepsilon} \frac{1}{|t|}\Big|W_n\Big(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\Big) - W_n(\theta)\Big| \leqslant \frac{M_7}{\sqrt{n}}. \tag{3.7}$$

Notice that $W_n(\lambda)$ is a.s. analytic in the strip $Re(\lambda) \in (-K_0, K_0)$. Let $0 < K_1 < K_0$. By the mean value theorem, when $\theta \in [-K_1, K_1]$ and $\frac{|t|}{\sigma_\theta\sqrt{n}} \leqslant \varepsilon$, we have

$$\Big|W_n\Big(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\Big) - W_n(\theta)\Big| \leqslant \frac{|t|}{\sigma_\theta\sqrt{n}} \max_{\eta\in[-\varepsilon,\varepsilon]} |W_n'(\theta + i\eta)| \tag{3.8}$$

By Cauchy's formula, when $|\lambda| < K_1$,

$$W_n'(\lambda) = \frac{1}{2\pi i}\int_{|z|=K_1} \frac{W_n(z)}{(z-\lambda)^2}dz.$$

By (2.5), a.s. for all $n \geqslant 1$ and all $z \in \mathbb{C}$ with $|z| \leqslant K_1$, $|W_n(z)| \leqslant M_8$. When $|\lambda| \leqslant K_1/2$ and $|z| = K_1$, $|z - \lambda| \geqslant K_1 - K_1/2 = K_1/2$, so that $|\frac{W_n(z)}{(z-\lambda)^2}| \leqslant \frac{4M_8}{K_1^2}$. Therefore for all $n \geqslant 1$, a.s.

$$\max_{|\lambda|\leqslant K_1/2} |W_n'(\lambda)| \leqslant \frac{4M_8}{K_1}.$$

Therefore from (3.8) and (3.3), we see that (3.7) holds when $K < K_1/4$ and $\varepsilon < K_1/4$. This concludes that the second term in (3.4) is bounded by $\frac{M_9}{\sqrt{n}}$. Therefore from (3.4) we get $I_1 \leqslant \frac{M_{10}}{\sqrt{n}}$.

For I_2, using the constraint in the integral of I_2, we have $\frac{1}{|t|} \leqslant \frac{1}{\varepsilon\sigma_\theta\sqrt{n}}$, so that

$$I_2 \leqslant \sup_{\theta\in[-K,K]} \frac{1}{\varepsilon\sigma_\theta\sqrt{n}} \int\limits_{\varepsilon\leqslant\frac{|t|}{\sigma_\theta\sqrt{n}}\leqslant a} \Big|W_n\Big(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\Big) f_n^\theta(t)\Big|dt$$
$$+ \sup_{\theta\in[-K,K]} \frac{W_n(\theta)}{\varepsilon\sigma_\theta\sqrt{n}} \int\limits_{\varepsilon\leqslant\frac{|t|}{\sigma_\theta\sqrt{n}}\leqslant a} e^{-t^2/2}dt.$$

It is shown in the proof of [12, Lemma 5] that as $n \to \infty$,

$$\sup_{\theta\in[-K,K]} \sqrt{n} \int\limits_{\varepsilon\leqslant\eta\leqslant a} \Big| W_n\Big(\theta + i\eta\Big) f_n^{\theta}(\sigma_\theta \sqrt{n}\eta)\Big| d\eta \to 0 \quad \text{a.s.,}$$

which can be rewritten as

$$\sup_{\theta\in[-K,K]} \frac{1}{\sigma_\theta} \int\limits_{\varepsilon\leqslant\frac{|t|}{\sigma_\theta\sqrt{n}}\leqslant a} \Big| W_n\Big(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\Big) f_n^{\theta}(t)\Big| dt \to 0 \quad \text{a.s.}$$

Therefore,

$$\sup_{\theta\in[-K,K]} \frac{1}{\varepsilon\sigma_\theta\sqrt{n}} \int\limits_{\varepsilon\leqslant\frac{|t|}{\sigma_\theta\sqrt{n}}\leqslant a} \Big| W_n\Big(\theta + \frac{it}{\sigma_\theta\sqrt{n}}\Big) f_n^{\theta}(t)\Big| dt \leqslant \frac{M_{11}}{\sqrt{n}} \quad \text{a.s.}$$

This, together with $\sup_{\theta\in[-K,K]} \frac{W_n(\theta)}{\sigma_\theta} \leqslant M_{12}$, implies that $I_2 \leqslant \dfrac{M_{13}}{\sqrt{n}}$. Thus the proof of Theorem 2.3 is completed. □

strictly positive constant

4. Proof of Theorem 2.2

In this section we prove Theorem 2.2, the Cramér type moderate deviation expansion for Z_n.

Proof of Theorem 2.2. We will only prove (2.8), as the proof of (2.9) is similar.

For $x \in [0,1]$, Theorem 2.8 is a direct consequence of Theorem 2.1, as we will see in the following. For $n \geqslant 1$,

$$\begin{aligned}
&\left| \frac{Z_n\Big((x\sigma_0\sqrt{n} + nm_0, +\infty) \Big)}{m^n W[1-\Phi(x)] e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})}} - 1 \right| \\
&= \frac{1}{W[1-\Phi(x)] e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})}} \left| \frac{Z_n(\mathbb{R})}{m^n} - \frac{Z_n\Big((-\infty, x\sigma_0\sqrt{n} + nm_0) \Big)}{m^n} \right. \\
&\qquad\qquad \left. - W(1-\Phi(x)) e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})} \right|. \qquad (4.1)
\end{aligned}$$

Since $\sup_{x\in[0,1]} |\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})| \to 0$, there exists n_0 large enough such that for all $x \in [0,1]$ and $n \geqslant n_0$, $e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})} \geqslant 1/2$. Using this and the fact that $1-\Phi(x) \geqslant c := 1-\Phi(1)$ for all $x \in [0,1]$, from (4.1) we get for all $n \geqslant n_0$,

$$\begin{aligned}
&\Big|\frac{Z_n\Big((x\sigma_0\sqrt{n}+nm_0,+\infty)\Big)}{m^n W[1-\Phi(x)]e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})}}-1\Big| \\
&\leqslant \frac{2}{cW}\Big|\frac{Z_n(\mathbb{R})}{m^n}-W\Big| + \frac{2}{cW}\Big|-\frac{Z_n\Big((-\infty,x\sigma_0\sqrt{n}+nm_0)\Big)}{m^n}+W\Phi(x)\Big| \\
&\quad + \frac{2}{cW}\Big|W(1-\Phi(x))\Big(1-e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})}\Big)\Big|.
\end{aligned} \tag{4.2}$$

In the last display, by Theorem 2.1, when $n\to\infty$, the two first terms are $O\Big(\frac{1}{\sqrt{n}}\Big)$. We will show below that the third term is also $O\Big(\frac{1}{\sqrt{n}}\Big)$. In fact, using the inequality $|1-e^t| \leqslant |t|e^t$ for $t\in\mathbb{R}$ and the fact that $\sup_{x\in[0,1]}|\mathscr{L}(\frac{x}{\sqrt{n}})|$ is bounded for $n\geqslant n_0$, we obtain for $x\in[0,1]$, as $n\to\infty$,

$$\Big|1-e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})}\Big| \leqslant \Big|\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})\Big|e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})} = O\Big(\frac{1}{\sqrt{n}}\Big).$$

This implies that the third term in (4.2) is $O\Big(\frac{1}{\sqrt{n}}\Big)$. From (4.2) and the above estimations, we see that for $x\in[0,1]$, as $n\to\infty$,

$$\Big|\frac{Z_n\Big((x\sigma_0\sqrt{n}+nm_0,+\infty)\Big)}{m^n W[1-\Phi(x)]e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})}}-1\Big| = O\Big(\frac{1}{\sqrt{n}}\Big),$$

which implies

$$\frac{Z_n\Big((x\sigma_0\sqrt{n}+nm_0,+\infty)\Big)}{m^n W[1-\Phi(x)]} = e^{\frac{x^3}{\sqrt{n}}\mathscr{L}(\frac{x}{\sqrt{n}})}\Big[1+O\Big(\frac{1}{\sqrt{n}}\Big)\Big].$$

This ends the proof of (2.8) in the case where $x\in[0,1]$.

We now deal with the case $1<x=o(\sqrt{n})$. For $u\in(\mathbb{N}^*)^n$, set

$$V_u = \frac{S_u - nm_\theta}{\sigma_\theta\sqrt{n}}.$$

Recalling that $\Lambda(\theta) = \log \mathbb{E} e^{\theta X} = \log \frac{m(\theta)}{m}$ and $\Lambda'(\theta) = m_\theta$, we have

$$\begin{aligned} I := \frac{1}{m^n} Z_n\Big((x\sigma_0\sqrt{n} + nm_0, +\infty)\Big) &= \frac{1}{m^n} \sum_{u\in\mathbb{T}_n} \mathbb{1}_{\left\{S_u > x\sigma_0\sqrt{n}+nm_0\right\}} \\ &= e^{-n[\theta\Lambda'(\theta)-\Lambda(\theta)]} \sum_{u\in\mathbb{T}_n} e^{-\theta\sigma_\theta\sqrt{n}V_u} \cdot \frac{e^{\theta S_u}}{m(\theta)^n} \mathbb{1}_{\left\{V_u > \frac{\sigma_0 x}{\sigma_\theta} + \frac{(m_0-m_\theta)\sqrt{n}}{\sigma_\theta}\right\}}. \end{aligned} \tag{4.3}$$

Because $\Lambda(\theta)$ is analytic on $\mathcal{D}$ with $\Lambda(0) = 0$, it has the Taylor expansion

$$\Lambda(\theta) = \sum_{k=1}^{\infty} \frac{\gamma_k}{k!}\theta^k, \qquad \text{where } \gamma_k = \Lambda^{(k)}(0), \quad \theta \in \mathcal{D}, \tag{4.4}$$

which implies that

$$\Lambda'(\theta) - \Lambda'(0) = \sum_{k=2}^{\infty} \frac{\gamma_k}{(k-1)!}\theta^{k-1}. \tag{4.5}$$

Consider the equation

$$\sqrt{n}(m_\theta - m_0) = \sigma_0 x, \qquad \text{namely} \quad \Lambda'(\theta) - \Lambda'(0) = \frac{\sigma_0 x}{\sqrt{n}}. \tag{4.6}$$

Set $t = \frac{x}{\sqrt{n}}$, from (4.5) and (4.6), we get

$$\sigma_0 t = \sum_{k=2}^{\infty} \frac{\gamma_k}{(k-1)!}\theta^{k-1}. \tag{4.7}$$

Since $\gamma_2 = \sigma_0^2 > 0$, the equation (4.7) has the unique solution given by

$$\theta = \frac{t}{\gamma_2{}^{1/2}} - \frac{\gamma_3}{2\gamma_2^2}t^2 - \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{6\gamma_2^{7/2}}t^3 + \dots. \tag{4.8}$$

Observe that from (4.4) and (4.5), for any $\theta \in \mathcal{D}$,

$$\theta\Lambda'(\theta) - \Lambda(\theta) = \sum_{k=1}^{\infty} \frac{\gamma_k}{(k-1)!}\theta^k - \sum_{k=1}^{\infty} \frac{\gamma_k}{k!}\theta^k = \sum_{k=2}^{\infty} \frac{k-1}{k!}\gamma_k\theta^k.$$

Choosing θ to be the unique real root of the equation (4.7), which is given by (4.8), we obtain (see [34, Theorem VIII.2.2] for details)

$$\theta\Lambda'(\theta) - \Lambda(\theta) = \frac{t^2}{2} - t^3\mathscr{L}(t) = \frac{x^2}{2n} - \frac{x^3}{n^{3/2}}\mathscr{L}\Big(\frac{x}{\sqrt{n}}\Big), \tag{4.9}$$

where $\mathscr{L}(t)$ is the Cramér series defined in (2.7), which converges for $|t|$ small enough. Substituting (4.6) into (4.3) and using (4.9), we get

$$\begin{aligned} I &= e^{-\frac{x^2}{2}+\frac{x^3}{\sqrt{n}}\mathscr{L}\left(\frac{x}{\sqrt{n}}\right)} \sum_{u\in\mathbb{T}_n} e^{-\theta\sigma_\theta\sqrt{n}V_u} \frac{e^{\theta S_u}}{m(\theta)^n}\mathbb{1}_{\{V_u>0\}} \\ &= e^{-\frac{x^2}{2}+\frac{x^3}{\sqrt{n}}\mathscr{L}\left(\frac{x}{\sqrt{n}}\right)} \int_0^\infty e^{-\theta\sigma_\theta\sqrt{n}y}\overline{Z}_n^\theta(dy), \end{aligned} \tag{4.10}$$

where $\overline{Z}_n^\theta$ is the finite measure on $\mathbb{R}$ defined by

$$\overline{Z}_n^\theta(A) = \sum_{u\in\mathbb{T}_n} \frac{e^{\theta S_u}}{m(\theta)^n}\mathbb{1}_{\{V_u\in A\}}, \quad A\subset\mathbb{R},$$

whose mass satisfies $\mathbb{E}\overline{Z}_n^\theta(\mathbb{R}) = 1$. From $t = \frac{x}{\sqrt{n}}$ and $x = o(\sqrt{n})$, it follows that $t \to 0$ as $n\to\infty$. By the inverse function theorem for analytic functions, the series on the right-hand side of (4.8) is absolutely convergent for $|t|$ small enough. Moreover, from (4.8), we have $\theta\to 0^+$ as $n\to\infty$. Hence, for sufficiently large n_0 and all $n \geqslant n_0$, we have $|\theta| \leqslant K$, where K is defined as in Theorem 2.3. Therefore, denoting

$$l_{n,\theta}(y) = \overline{Z}_n^\theta((-\infty, y]) - W(\theta)\Phi(y), \quad y\in\mathbb{R},$$

from Theorem 2.3 we get for all $n\geqslant n_0$,

$$\sup_{y\in\mathbb{R}} |l_{n,\theta}(y)| \leqslant \frac{M}{\sqrt{n}}, \tag{4.11}$$

where M is a positive and finite random variable independent of n and θ. Notice that

$$\begin{aligned} \int_0^\infty e^{-\theta\sigma_\theta\sqrt{n}y}\overline{Z}_n^\theta(dy) &= \int_0^\infty e^{-\theta\sigma_\theta\sqrt{n}y} dl_{n,\theta}(y) + \frac{W(\theta)}{\sqrt{2\pi}}\int_0^\infty e^{-\theta\sigma_\theta\sqrt{n}y-\frac{y^2}{2}}dy \\ &=: I_1 + W(\theta)I_2. \end{aligned} \tag{4.12}$$

Estimate of I_1. Using the integration by parts and the bound (4.11), we get that for $n\geqslant n_0$,

$$|I_1| \leqslant |l_{n,\theta}(0)| + \theta\sigma_\theta\sqrt{n}\int_0^\infty e^{-\theta\sigma_\theta\sqrt{n}y}|l_{n,\theta}(y)|dy \leqslant \frac{2M}{\sqrt{n}}. \tag{4.13}$$

Estimate of I_2. The integral I_2 appears in the proof of Cramér's moderate deviation expansion theorem for sums of i.i.d. random variables (see [34, Theorem VIII.2.2]), where the following results have been proved:

(i) there exist some positive constants C_1, C_2 such that for all $\theta \in [-K, K]$ and all n large enough,
$$C_1 \leqslant \theta\sigma_\theta\sqrt{n}I_2 \leqslant C_2;$$

(ii) the integral I_2 admits the following asymptotic expansion :
$$I_2 = e^{\frac{x^2}{2}}[1-\Phi(x)]\Big[1+O\Big(\frac{x}{\sqrt{n}}\Big)\Big]. \tag{4.14}$$

By the definition of σ_θ, the mapping $\theta \mapsto \sigma_\theta$ is strictly positive and continuous on $[-K, K]$. Hence, there exist positive constants C_3, C_4 such that for all $\theta \in [-K, K]$,
$$C_3 \leqslant \theta\sqrt{n}I_2 \leqslant C_4. \tag{4.15}$$

Notice that by (2.4), for all $\theta \in [-K, K]$, $W(\theta) > 0$ a.s. Moreover, $W(\theta)$ is a.s. continuous in θ by the continuity and uniform convergence of $W_n(\theta)$ on $[-K, K]$. Combining this with (4.15), we get

$$M_3 \leqslant \theta\sqrt{n}W(\theta)I_2 \leqslant M_4. \tag{4.16}$$

We now come back to (4.12), and let θ be defined by (4.8). Recall that for $n \geqslant n_0$, $|\theta| \leqslant K$. From (4.12), (4.16) and (4.13), we have, as $n \to \infty$,
$$\int_0^\infty e^{-\theta\sigma_\theta\sqrt{n}y}\overline{Z}_n^\theta(dy) = W(\theta)I_2\Big(1+\frac{\sqrt{n}I_1}{\sqrt{n}W(\theta)I_2}\Big) = W(\theta)I_2\Big(1+O(\theta)\Big). \tag{4.17}$$

According to the analyticity of $W(\theta)$ on $[-K, K]$ and using the mean value theorem one see that $|W(\theta)-W| = |W(\theta)-W(0)| \leqslant M_5\theta$. Since $\theta = O\left(\frac{x}{\sqrt{n}}\right)$ by (4.8), it follows from (4.17) and (4.14) that
$$\begin{aligned}\int_0^\infty e^{-\theta\sigma_\theta\sqrt{n}y}\overline{Z}_n^\theta(dy) &= (W+O(\theta))I_2(1+O(\theta)) \\ &= We^{\frac{x^2}{2}}[1-\Phi(x)]\Big[1+O\Big(\frac{x}{\sqrt{n}}\Big)\Big].\end{aligned} \tag{4.18}$$

Combining this with (4.10) yields
$$I = We^{\frac{x^3}{\sqrt{n}}\mathscr{L}\left(\frac{x}{\sqrt{n}}\right)}[1-\Phi(x)]\Big[1+O(\frac{x}{\sqrt{n}})\Big],$$

which concludes the proof of (2.8).

□

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