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Conditional probability and interferences in generalized measurements with or without definite causal order

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In the context of generalized measurement theory, the Gleason-Busch theorem assures the unique form of the associated probability function. Recently, in Flatt et al. Phys. Rev. A **96**, 062125 (2017), the case of subsequent measurements has been treated, with the derivation of the Lüders rule and its generalization (Kraus update rule). Here we investigate the special case of subsequent measurements where an intermediate measurement is a composition of two measurements (a or b) and the case where the causal order is not defined (a and b or b and a). In both cases interference effects can arise. We show that the associated probability cannot be written univocally, and the distributive property on its arguments cannot be taken for granted. The two probability expressions correspond to the Born rule and the classical probability; they are related to the intrinsic possibility of obtaining definite results for the intermediate measurement. For indefinite causal order, a causal inequality is also deduced. The frontier between the two cases is investigated in the framework of generalized measurements with a toy model, a Mach-Zehnder interferometer with a movable beam splitter.

I. INTRODUCTION

In Quantum Mechanics, probabilities are obtained by the squared modulus of complex amplitudes, which give rise to interference phenomena. In the common example of Young's setup composed of a source, two slits and a screen or movable detector as represented in Fig. 1, the probability to detect an emitted particle in a position x on the backstop wall is given by

$$\begin{aligned} Pr(ab) &= |\psi_a + \psi_b|^2 = \\ &= Pr(a) + Pr(b) + 2\sqrt{Pr(a)Pr(b)} \cos(\arg(\psi_a\psi_b^*)), \end{aligned} \quad (1)$$

where ψ_a, ψ_b are the complex probability amplitudes associated with each slit and $Pr(a) = |\psi_a|^2$, $Pr(b) = |\psi_b|^2$ are the probabilities associated to the opening of the single slits. The above expression is substantially different from the classical probability sum rule

$$Pr^C(ab) = Pr(a) + Pr(b), \quad (2)$$

where interference terms are not present.

The probability function for the quantum case is strictly connected to the Hilbert space structure, where systems are described with respect to a defined basis and where the complex numbers mentioned above correspond to coordinates. With some minimal requirements on the probability function Pr , in 1957 Gleason [2] demonstrated that Pr is univocally defined in a Hilbert space by the trace rule

$$Pr(i) = tr(\rho P_i), \quad (3)$$

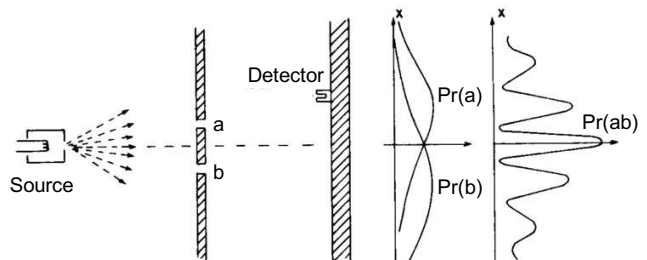


FIG. 1. Scheme of Young's slits experiment. Adapted from Ref. [1].

where $\rho = |\varphi\rangle\langle\varphi|$ is the density matrix of the prepared system and $P_i = |i\rangle\langle i|$ is the projector on the state of interest. In the case of an initial pure state $|\varphi\rangle = |s\rangle$, Eq. (3) corresponds to the Born rule with $Pr(i) = |\langle i|s\rangle|^2$. Gleason's theorem has some limitations; it is valid only for Hilbert spaces with a dimension larger than two and for projective von Neumann measurements [3].

In the framework of the general measurement formalism of positive-operator-valued measures (POVM, also called probability operator-valued measures), in 2003, Busch [4] extended Gleason's theorem for any dimension and for imperfect measurements described by positive operators, effects E_i , instead of projectors. Recently (2017), in the same context of POVM, Flatt, Barnett and Croke applied the Gleason-Busch theorem to subsequent measurements [5]. Considering the operators E_i and F_j associated with the measurements i and j , with i before j , Flatt and coworkers proved that $Pr(i, j)$ takes the general form

$$Pr(i, j) = tr \left(F_j \sum_k K_{ik} \rho K_{ik}^\dagger \right), \quad (4)$$

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where the operators K_{ik} are related to the effects by $E_i = \sum_k K_{ik}^\dagger K_{ik}$. From the above equation and the corresponding one for the conditional probability $Pr(i|j)$, the Kraus update rule [6, 7]

$$\rho \rightarrow \rho'_i = \frac{\sum_k K_{ik} \rho K_{ik}^\dagger}{\text{tr} \left(\sum_k K_{ik}^\dagger K_{ik} \right)} \quad (5)$$

for the state update of the system ρ'_i after the measurement i is recovered. It is worth noting that the Kraus update rule, and its particular case of the Lüders rule [8] valid for ideal measurement and determined by the von Neumann projection postulate, are derived from first principles. There is no need of other postulates than the description of states via the Hilbert space and a few basic requirements for the probability function.

In this article, we apply the formalism of subsequent generalized measurements to the case with two possible and mutually exclusive intermediate measurements (a two-way interferometer) with indefinite measurement order. For both cases, interferences can occur. With the introduction of a new notation of the probability function arguments with respect to Flatt et al., we will show that two possible expressions of the final probability Pr can be derived from Eq. (3). These two expressions correspond to Eqs. (1) and (2) in the example of Young's slits, and are related to the possibility of distinguishing or not the intermediate measurement. The difference between the two forms is the order of the arguments of the probability expressions, where the distributive property can not be taken for granted. The violation of the distributive property in Quantum Mechanics is not new and it has been pointed out since the early years of its formulation [9, 10] and extensively discussed in Quantum Logic. Its connection to the extension of classical probability to quantum probability is well discussed in the literature in the case of perfect projective measurement [11–15]. For imperfect general measurements, when positive operators are considered instead of projectors, some work has been performed by Busch and collaborators [16, 17]. Here we present a general discussion about the probability function for the distinguishable and indistinguishable path cases (the particle-like and wave-like behaviors) in the case of imperfect (unsharp) measurements.

For Young's slits, the frontier between the different cases and the domain of validity of Eqs. (1) and (2), has been extensively discussed in the past. Experimentally, it has been explored in the last decades through investigations of interference effects with molecules with larger and larger masses. Diffraction of large inorganic and organic molecules with masses beyond 25000 atomic mass units has been obtained [18–20]. Here, we discuss this frontier in the context of generalized measurements considering a Mach-Zehnder interferometer with movable beam splitter. This toy model, introduced in the past by Haroche et al. [21, 22], has the interesting feature of allowing to pass from one case to the other continuously, simply considering a variation of the mass of the movable

beam splitter.

In the case of measurements with indefinite order, extensively discussed in the literature in the last years [23–29], we will show that they can be treated with the same approach as for the two-path interferometer. Here too, the presence of interference or not is related in this case to two possible, but not equivalent, expressions of the probability function.

II. PROBABILITY FOR SUBSEQUENT MEASUREMENTS

A. Introduction of new notations

Taking inspiration from the Quantum Logic approach [11–15, 30, 31] and the propositional definition of probability [32–34], we introduce a new notation with the logical operators “ \wedge ” and “ \vee ” to unambiguously discuss the joint probability of series of subsequent measurements. The *conjunction* operator “ \wedge ” is equivalent to “AND” in normal language and to the comma in the previously introduced notation $Pr(i, j)$. The *disjunction* operator “ \vee ” is equivalent to “OR” also indicated with the “+” operator (in Refs. [4, 5] as example). Particular attention has to be paid for measurements i, j that are incompatible. In this case, the logical operator “ \wedge ” is not well defined [12–14, 30], except if the order of subsequent measurements is defined. As already pointed out in the *consistent histories* interpretation of Quantum Mechanics [35–37], differently from standard logic, the operator “ \wedge ” is not symmetric with respect to i, j with $i \wedge j \neq j \wedge i$. With this notation, the joint probability defined above for a measurement j obtained after a measurement i can be written as

$$\wp(j \wedge i | s) \equiv Pr(i, j), \quad (6)$$

where we explicitly indicate the system preparation s , which is in fact connected to the possible measurement outcomes. We also invert the order of i, j to clearly indicate the sequential order of the measurement or preparation from right to left (preparation s , first measurement i and second measurement j).

B. Rewriting probabilities

Before treating in detail the Young's slits problem with the new introduced notation, we shall rewrite the properties and assumptions of the probability function used by Flatt et al. [5] that lead to Eq. (5). We consider a set of positive-semidefinite operators (effects) E_i of the same POVM with $\sum_i E_i = I$. The requirement properties of the probability function $\nu(E_i) = Pr(i)$ for the

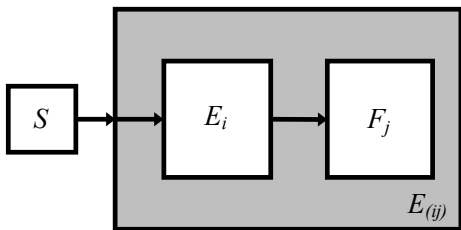


FIG. 2. Scheme of two subsequent measurements. Adapted from Ref. [5].

Gleason-Busch theorem are

- (P1) $0 \leq \nu(E_i) \leq 1$.
- (P2) $\nu(I) = 1$.
- (P3) $\nu(E_i + E_j \dots) = \nu(E_i) + \nu(E_j) + \dots$

The function $\nu(E_i)$ is in fact a map from the full set of effects $\mathcal{E}(\mathcal{H})$ acting on the Hilbert space \mathcal{H} : $E \rightarrow \nu(E)$ with $\nu(E) \in [0, 1]$.

With our notation, the previous propositions become

- (P1*) $0 \leq \wp(i|s) \leq 1$.
- (P2*) $\wp(\mathcal{I}|s) = 1$.
- (P3*) $\wp(i \vee j \vee \dots |s) = \wp(i|s) + \wp(j|s) + \dots$

where i, j are the measurements that correspond to the effects E_i, E_j and $\mathcal{I} = \bigvee_i i$ measurement correspond to the identity operator I .

When two subsequent measurements are considered together, Flatt et al. introduced the new function

$$\mu_\nu^i(F_j) = \nu(E_{(ij)}) = Pr(i, j) \quad (7)$$

for the action of the effect F_j after the action of E_i and $E_{(ij)}$ indicating the cumulative effect (see Fig. 2). In addition, the following assumptions are considered by Flatt et al.

- (A1) $0 \leq \mu_\nu^i(F_j) \leq \nu(E_i) < 1$.
- (A2) $\mu_\nu^i(I) = \nu(E_i)$.
- (A3) $\mu_\nu^i(F_j + F_k + \dots) = \mu_\nu^i(F_j) + \mu_\nu^i(F_k) + \dots$

With our notation, we consider on the same level the measurements j and i and the operator “ \wedge ” indicates the measurement order. The assumptions (A1–2) can simply be rewritten as

- (A1*) $0 \leq \wp(j \wedge i|s) \leq \wp(i|s) \leq 1$.
- (A2*) $\wp(\mathcal{I} \wedge i|s) = \wp(i|s)$.

(A2*) is now a tautology. For (A3), the rewriting is ambiguous. $\mu_\nu^i(F_j + F_k + \dots)$ can be written in fact in two different forms:

$$\wp((j \wedge i) \vee (k \wedge i)|s) \quad (8)$$

or

$$\wp((j \vee k) \wedge i|s). \quad (9)$$

Another situation that cannot be easily treated with the formalism of Flatt et al. is the case of subsequent measurements with indefinite causal order. Equation (7) implies in fact a defined causal order because of the nesting of the two probability functions. The treatment of indefinite causal order phenomena and the associated possible interferences is well defined for elementary processes in the context of quantum field theory (e.g. in the relativistic Compton scattering [38]). When a subsequent interaction with measuring detectors is considered, the situation is more complicated and it has been the center of interest of several works in the last years [23–25, 29]. Considering two measurements represented by two effects E_i and E_j with an indefinite order and a final measurement represented by F_d , similarly to Eqs. (8) and (9), the associated probability can be written in two ways

$$\wp(k \wedge [(i \wedge j) \vee (j \wedge i)]|s) \quad (10)$$

or

$$\wp((k \wedge i \wedge j) \vee (k \wedge j \wedge i)|s). \quad (11)$$

In the following sections, it will be shown that the formulas (8)–(9) and (10)–(11) will lead to different probability expressions. These different expressions, obtained from a unique definition of the probability function, will be the key point of the present work.

C. General considerations for POVM operators

To investigate the difference between Eqs. (8) and (9), we come back the specific example of Young’s slits where we consider the possibility to measure or flag the passage through each slit. Before that, a short introduction to generalized measurements is mandatory. In the framework of POVM, the single measurements are described by the positive-valued operators $E_\ell = K_\ell^\dagger K_\ell$, where K_ℓ operators are determined by the unitary interaction between the system we want to study and the detector, both considered as quantum systems. The general expression for K_ℓ is given by [39, 40]

$$K_\ell = \sum_{i,j} \alpha_{ij} \langle \ell^{det} | \Phi_i^{det} \rangle |j\rangle \langle i|, \quad (12)$$

where α_{ij} depend on the action of the unitary matrix U_{int} describing the interaction between the system and the detector. The initial state is described by $|\Psi_i^0\rangle = |i\rangle |\Phi_0\rangle$,

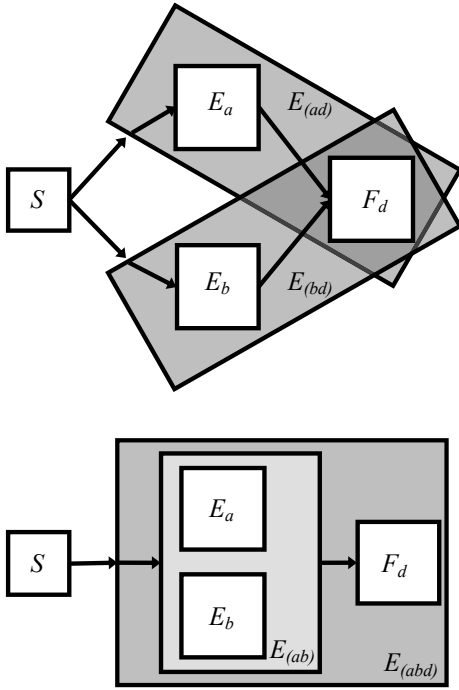


FIG. 3. Schemes of subsequential measurements corresponding to the Young's slits experiment for the case where the path of the particle can be detected (top) or not (bottom).

where $|i\rangle$ and $|\Phi_0\rangle$ describe the initial state of the system and the detector, respectively. After their mutual interaction, the system and detector states are described by $|\Psi_i^0\rangle \rightarrow |\Psi_i\rangle = |\varphi_i\rangle |\Phi_i^{det}\rangle = \sum_j \alpha_{ij} |j\rangle |\Phi_i^{det}\rangle$. $|\Phi_i^{det}\rangle$ describes the detector state after the interaction with the system in an initial state $|i\rangle$. Finally, $|\ell^{det}\rangle$ represents the detector state corresponding to the macroscopic outcome of the measurement device. In the case of a non-destructive measurement, the above formula is simplified to

$$K_\ell = \sum_i \langle \ell^{det} | \Phi_i^{det} \rangle |i\rangle \langle i|. \quad (13)$$

III. INTERFERENCES IN A TWO-PATH INTERFEROMETER

A. Distinguishable paths

In the case of Young's slits, we consider that the detection of the path taken by the particle is possible and is non-destructive. The formula corresponding to Eq. (8) becomes $\wp((d \wedge a) \vee (d \wedge b)|s)$ and depends on the operators K_a, K_b and K_d . K_a, K_b are related to the detection of the path a or b , and the corresponding effects are $E_a = K_a^\dagger K_a$, $E_b = K_b^\dagger K_b$. K_d is related to the detection d on the screen with $F_d = K_d^\dagger K_d$. The combination of E_a and E_b with F_d can be assimilated to the effects

$E_{(ad)}$ and $E_{(bd)}$ as in Eq. (7), and for which the property (P3)/(P3*) can be applied. In this case we have

$$\begin{aligned} \wp((d \wedge a) \vee (d \wedge b)|s) &= \wp(d \wedge a|s) + \wp(d \wedge b|s) = \\ &= \text{tr}(F_d K_a \rho K_a^\dagger) + \text{tr}(F_d K_b \rho K_b^\dagger). \end{aligned} \quad (14)$$

The above equation corresponds to the classic probability sum rule, i.e. the particle-like probability in Eq. (2). The fact that we can decompose the measurement in two separate operators $E_{(ad)}$ and $E_{(bd)}$ (Fig. 3, top) implicitly means that the different paths can be distinguished and we have just a duplicated version of the basic subsequent measurement represented in Fig. 2. This case can be easily treated with the formalism introduced by Flatt et al. with the introduction of the probability functions $\mu_\nu^a(F_d)$ and $\mu_\nu^b(F_d)$.

In the case of ideal projective measurements, we have $E_a = K_a = P_a = |a\rangle \langle a|$ and $E_b = K_b = P_b = |b\rangle \langle b|$ where we used the properties of projectors $P_i^\dagger = P_i$ and $P_i P_i = P_i$. The above equation then becomes [15]

$$\begin{aligned} \wp((d \wedge a) \vee (d \wedge b)|s) &= \\ &= |\langle d|U_{(ad)}|a\rangle \langle a|U_{(sa)}|s\rangle|^2 + |\langle d|U_{(bd)}|b\rangle \langle b|U_{(sb)}|s\rangle|^2, \end{aligned} \quad (15)$$

where the unitary operators U correspond to the evolution of the different parts of the apparatus.

The expression of $\wp((d \wedge a) \vee (d \wedge b)|s)$ can also be directly obtained by the trace reduction of the density matrix ρ with respect to detector base $|a^{det}\rangle$ and $|b^{det}\rangle$. In this case we have

$$\wp((d \wedge a) \vee (d \wedge b)|s) = \text{tr}(F_d \rho_r), \quad (16)$$

with $\rho_r = \text{tr}_{a^{det}, b^{det}}(|\Psi_i\rangle \langle \Psi_i|)$ and where $|\Psi_i\rangle = \sum_j \alpha_{ij} |j\rangle |\Phi_i\rangle$. From the linearity of the trace operator, it is easy to verify that the previous expression is equivalent to Eq. (14). This indicates that the use of the trace over the undetected $|a^{det}\rangle, |b^{det}\rangle$ states implicitly implies an interaction between the system and the which-path detectors, even if they are not directly involved in the measurement.

B. Indistinguishable paths and discussion

In the case we can not distinguish which path is taken by the particle, the $a \vee b$ cannot be decomposed and we have to deal with the expression

$$\wp(d \wedge (a \vee b)|s) = \text{tr}(F_d K_{a \vee b} \rho K_{a \vee b}^\dagger). \quad (17)$$

The operator $K_{a \vee b}$ can be built in three different ways:

1. from path-detectors with the same final state after the interaction with the system ($|a^{det}\rangle = |b^{det}\rangle$),
2. from a complementary measurement c (e.g. a series of detectors on the slit walls),

3. via a detector state $|q^{det}\rangle$ belonging to the span generated by the vectors $|a^{det}\rangle$ and $|b^{det}\rangle$.

As we will see, a genuine $a \vee b$ measurement is related to the first two cases only. The third approach is in fact related to the *quantum eraser* case and it is discussed separately in the next section.

In the case where the measurement of the passage of a particle in one path or the other induces the same detector state $|ab^{det}\rangle \equiv |a^{det}\rangle = |b^{det}\rangle$, the operator corresponding to $a \vee b$ can be written as

$$K_{a \vee b} = \sum_{i=a,b} \langle \ell^{det} | ab^{det} \rangle |i\rangle \langle i| = \sum_{i=a,b} K_i, \quad (18)$$

where K_a and K_b are associated to the effects E_a and E_b represented in the bottom of Fig. 3 and $|\ell^{det}\rangle$ is a generic state corresponding to the $a \vee b$ measure ($|\ell^{det}\rangle = |ab^{det}\rangle$ for an ideal measurement). Note that this is not the case for Kraus operators K_{ik} as in Eq. (5) where different detector states Φ_k^{det} correspond to the same system state $|i\rangle$. Here in opposite, different system states $|a\rangle$ and $|b\rangle$ correspond to the same detector state $|ab^{det}\rangle$, and the trace operator in Eqs. (5) and (17) cannot be separated into different terms.

If we consider a complementary measurement c to both a and b measurements, we have that $c \wedge a = 0$, $c \wedge b = 0$ and $c = \mathcal{I} - a \vee b$. $E_{a \vee b}$ corresponds to the absence of signal in the measurement E_c , then, using the property of the set of effects of the POVM for which $\sum_{i=a,b,c} E_i = I$, we have $E_{a \vee b} = I - E_c = E_a + E_b$. $K_{a \vee b}$ can be written as [17, 39, 41]

$$K_{a \vee b} = U_{a \vee b} \sqrt{E_a + E_b}, \quad (19)$$

where $U_{a \vee b}$ is a unitary matrix that depends on the details of the interaction between $|a^{det}\rangle, |b^{det}\rangle$ and the propagating particle-wave.

In the case of ideal projective measurements for both situations discussed above, $K_{a \vee b}$ can be explicitly written. In this case we have that $E_{a \vee b} = E_a + E_b = P_a + P_b$ and Eq. (17) becomes [15] (see also Refs. [11, 13, 17])

$$\begin{aligned} \wp(d \wedge (a \vee b) | s) &= \\ &= |\langle d | U_{(ad)} | a \rangle \langle a | U_{(sa)} | s \rangle + \langle d | U_{(bd)} | b \rangle \langle b | U_{(sb)} | s \rangle|^2, \end{aligned} \quad (20)$$

which is equivalent to the quantum form of the probability in Eq. (1), i.e. equivalent to the Born rule.

In the general case represented in Eqs. (14) and (17) (and in the particular case in Eqs. (15) and (20)),

$$\wp(d \wedge (a \vee b) | s) \neq \wp((d \wedge a) \vee (d \wedge b) | s) \quad (21)$$

and the distributive property on the arguments of \wp is violated. Equations (14) and (15) reproduce the sum rule valid for the classical probability Pr^C (Eq. (2)). However, Eqs. (17) and (20) present additional interference terms and are compatible with the Born rule. If the distributive property is considered valid, the two expressions

should be equivalent. But the validity of distributivity cannot in fact be taken for granted. As anticipated in the introduction, the violation of the distributive law in quantum phenomena is well known since the early years of Quantum Mechanics [9, 42]. In particular in Quantum Logic [10, 11, 14, 31, 43–45] this is related to the properties of orthomodular lattices, associated to sets of yes/no experiments, where the distributivity on their elements is not always valid.

For the indistinguishable case, the measurement $a \vee b$ corresponds to an atomic operator $E_{a \vee b} \equiv E_{(ab)}$ that cannot be decomposed in terms of E_a, E_b . The cumulative effect $E_{(abd)}$ depends then on all three measurements a, b and d and can be represented by the scheme in the bottom of Fig. 3.

C. The quantum eraser revisited

In the Quantum Logic context, a measurement representing $a \vee b$ can be built from a vector $|q^{det}\rangle = \alpha |a^{det}\rangle + \beta |b^{det}\rangle$ [14], with $\alpha, \beta \neq 0$, which belongs to the span generated by the vectors $|a^{det}\rangle$ and $|b^{det}\rangle$. Using Eq. (13) with $\ell = q$, we can then write

$$K_{a \vee b} \equiv K_q = \alpha^* K_a + \beta^* K_b, \quad (22)$$

where $|\alpha|^2 + |\beta|^2 = 1$ for a normalized probability. Once inserted in Eq. (17), the above expression gives rise to mixed $\langle a^{det} | b^{det} \rangle$ terms and then to interference phenomena. This is in fact the case of the *quantum eraser* [16, 46–50], where instead of the direct path detection via $|a^{det}\rangle, |b^{det}\rangle$, a combination of them is considered and interference terms appear.

This is a situation not equivalent to the case with a complementary measurement $c = \mathcal{I} - a \vee b$. Even if we recover the presence of interferences with the use of $|q^{det}\rangle$ instead of $|a^{det}\rangle$ or $|b^{det}\rangle$, we are dealing with a single measurement q that corresponds to the probability $\wp(d \wedge q | s)$, and not $\wp(d \wedge (a \vee b) | s)$. Similarly to a, b measurements, we could consider the alternative measurement given by the vector $|r^{det}\rangle = -e^{i\phi} \beta |a^{det}\rangle + e^{i\phi} \alpha |b^{det}\rangle$ orthogonal to $|q^{det}\rangle$. When both possible measurements q and r are considered, we can write down the probabilities $\wp((d \wedge q) \vee (d \wedge r) | s)$ and $\wp(d \wedge (q \vee r) | s)$. $|r^{det}\rangle, |q^{det}\rangle$ and $|a^{det}\rangle, |b^{det}\rangle$ are two different bases describing the detection and they are related by a unitary transformation. Because of the property of the unitary transformation, it can be demonstrated (see App. A for the detailed calculations) that the combination of the two measurements r and q and the which-path a and b are completely equivalent and

$$\wp((d \wedge q) \vee (d \wedge r) | s) = \wp((d \wedge a) \vee (d \wedge b) | s). \quad (23)$$

The interference terms present in the separate terms $\wp(d \wedge q | s)$ and $\wp(d \wedge r | s)$, completely compensate in $\wp((d \wedge q) \vee (d \wedge r) | s) = \wp(d \wedge q | s) + \wp(d \wedge r | s)$ like in the well known results on the quantum eraser.

For the case of $\wp(d \wedge (q \vee r)|s)$ probability, the situation is more complicated because it depends on the values of α, β, ϕ but also on the choice of α', β' for building $K_{q \vee r} = \alpha' K_q + \beta' K_r$. With this last consideration, we can conclude that in fact the construction of $K_{a \vee b}$ via Eq. (22) is not equivalent to a genuine which-path ignorance, but it is a special case where a different detector state basis is considered.

D. A toy model with a Mach-Zehnder interferometer

The fundamental difference between distinguishable and indistinguishable cases, i.e., the use of Eq. (16) or Eq. (19) for the probability function, is the coupling between the considered system and the possible which-path detector(s) and/or the environment, but also the information that can be extracted from the detector(s) outputs. Such a coupling has been extensively studied in the context of decoherence theory [51, 52]. In this section we consider a very simple case to investigate the limits of Eqs. (16) and (19) in terms of effects thanks to a toy system where we can continuously tune the detectability of the taken path.

We consider a Mach-Zehnder interferometer with a movable beam splitter (represented in Fig. 4), an example discussed in the literature and realized experimentally with atoms in resonant cavities [21, 22]. Here we treat the problem in terms of effects in a POVM framework. A discussion of the Mach-Zehnder interferometer in terms of unsharp detection has been already discussed by Busch and Shilladay [16]. In this past work, the unsharpness of the detection was studied in terms of measurement mixing between the two paths, like in the quantum eraser case discussed in Sec. III C. The cases of distinguishability or indistinguishability of the paths were also treated, but not the frontier between them, which on the contrary is the main subject of the following paragraphs. A more general treatment of the continuous passage between distinguishable and indistinguishable cases has been done in the past by B.-G. Englert [53]. The visibility of interferences and of the *distinguishability* quantity (in terms of the amount of which-way information can be extracted) is discussed, but not the relation to the probability function in the context of POVM.

The system considered here is composed by a single-photon source emitting monochromatic photons $|1_{in}\rangle$ interacting with: a movable beam splitter $BS1$, two mirrors M, M' , a phase shifter PS that induces a phase ϕ , a second (fixed) beam splitter $BS2$ and two detectors d_1 and d_2 , following the scheme represented in Fig. 4. The movable beam splitter, with a mass m , can move with respect to a pivot and is connected to a fixed part by a spring that corresponds to a resonant angular frequency ω . The beam splitter-spring system is described by a harmonic oscillator with energy spectrum $\mathcal{E}_n = \omega \hbar (n + \frac{1}{2})$. When the photon is reflected from the first beam splitter, a momentum kick $\Delta P = \sqrt{2}p$, with $p = 2\pi\hbar/\lambda$ the impulse

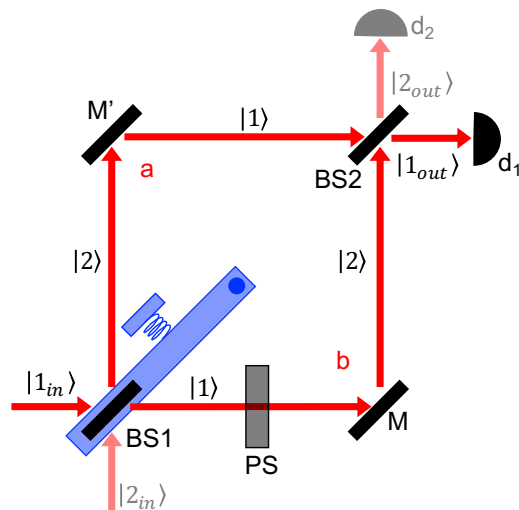


FIG. 4. Scheme of the Mach-Zehnder interferometer with a movable beam splitter $BS1$, a phase shifter with ϕ with two possible incoming beams $|1_{in}\rangle$ and $|2_{in}\rangle$ and outputs $|1_{out}\rangle$ and $|2_{out}\rangle$ measured by the detectors d_1 and d_2 . Photons parallel to the incoming photon (horizontal propagation in the figure) are indicated by the states $|1\rangle$ and with $|2\rangle$ otherwise (vertical propagation).

of the photon, is transferred to $BS1$ with a translation from its ground state $|0\rangle_{BS}$ to the coherent state $|\alpha_{BS}\rangle$ with $\alpha = ip/\sqrt{m\omega\hbar}$ [21, 22].

In analogy to the Young's slits, we can consider the interferometer arm with the reflection from the movable beam splitter as the path a , and path b otherwise (see Fig. 4).

In the case of a fixed beam splitter, the state of the beam splitter itself does not change after the passage of the photon and the state corresponding to the photon is

$$|1_{in}\rangle \rightarrow |\phi\rangle = \frac{1}{2}|1_{out}\rangle - \frac{i}{2}|2_{out}\rangle - \frac{e^{i\phi}}{2}|1_{out}\rangle + \frac{ie^{i\phi}}{2}|2_{out}\rangle \quad (24)$$

The probability of detecting something on the detector d_1 depends on the operator $K_{d_1} = |0_{out}\rangle\langle 1_{out}|$ and the corresponding effect $F_{d_1} = K_{d_1}^\dagger K_{d_1} = |1_{out}\rangle\langle 1_{out}|$. Because of the impossibility of determining the path taken by the photon, the corresponding probability is $\wp(d_1 \wedge (a \vee b)|s)$. The complementary detection c representing $K_{a \vee b}$ could be constituted by a series of detectors around the beam splitter $BS1$, like the wall detection in the case of the Young's slits, to insure the interaction (reflection or transmission) of the incoming photon $|1_{in}\rangle$ with $BS1$. The probability is then given by

$$\wp(d_1 \wedge (a \vee b)|s) = \text{tr}(F_{d_1} \rho') = \frac{1}{2} [1 + \cos(\phi)], \quad (25)$$

with $\rho' = |\phi\rangle\langle\phi|$ and $|\phi\rangle$ given by Eq. (24). We recover the standard formula of the Mach-Zehnder interferometer [16, 41].

We consider now that the beam splitter $BS1$ can move and that its state after the recoil is described by the coherent state $|\alpha_{BS}\rangle$. Considering the initial state $|1_{in}\rangle|0_{BS}\rangle$ describing the photon-beam splitter system, after the interaction between the incoming photon with the first beamsplitter $BS1$ (and the mirrors M and M' and the second beam splitter $BS2$), the photon/mirror state $|\phi\rangle$ is described by

$$\begin{aligned} |1_{in}\rangle|0_{BS}\rangle &\rightarrow |\phi\rangle|\Phi_{BS}\rangle = \\ &= -\frac{1}{2}|1_{out}\rangle|\alpha_{BS}\rangle - \frac{i}{2}|2_{out}\rangle|\alpha_{BS}\rangle + \\ &\quad -\frac{e^{i\phi}}{2}|1_{out}\rangle|0_{BS}\rangle + \frac{ie^{i\phi}}{2}|2_{out}\rangle|0_{BS}\rangle, \end{aligned} \quad (26)$$

where $|\phi\rangle$ is state of the photon at the exit of the interferometer and $|\Phi_{BS}\rangle$ is the state of the movable beam splitter after the passage of the photon.

The operator $K_b = \langle 0_{BS}|\Phi_{BS}\rangle|\phi\rangle\langle 1_{in}|$ can be associated to the branch b where there is no momentum transfer to $BS1$, which remains in the $|0_{BS}\rangle$ state. For the branch a , we cannot directly use $\langle \alpha_{BS}|\Phi_{BS}\rangle|\phi\rangle\langle 1_{in}|$ as K_a operator. Due to the non-orthogonality of $|\alpha_{BS}\rangle$ and $|0_{BS}\rangle$, this leads to the possibility of having $E_a + E_b > 1$, violating the basic POVM properties. Considering that we can identify a coherent state only if its corresponding signal is above the quantum shot noise of the system, instead of $|\alpha_{BS}\rangle$ we can consider its Gram-Schmidt orthogonalization $|\alpha'_{BS}\rangle$ with respect to $|0_{BS}\rangle$

$$|\alpha'_{BS}\rangle = \frac{|\alpha_{BS}\rangle - \langle 0_{BS}|\alpha_{BS}\rangle|0_{BS}\rangle}{\sqrt{1 - |\langle 0_{BS}|\alpha_{BS}\rangle|^2}}. \quad (27)$$

The corresponding which-path operators are then

$$\begin{aligned} K_a &= \langle \alpha'_{BS}|\phi\rangle\langle 1_{in}| = \frac{1}{2}\sqrt{1 - |\langle 0_{BS}|\alpha_{BS}\rangle|^2}|1_{out}\rangle\langle 1_{in}| + \\ &\quad - \frac{i}{2}\sqrt{1 - |\langle 0_{BS}|\alpha_{BS}\rangle|^2}\langle \alpha'_{BS}|\alpha_{BS}\rangle|2_{out}\rangle\langle 1_{in}| \end{aligned} \quad (28)$$

and

$$\begin{aligned} K_b &= \langle 0_{BS}|\Psi\rangle\langle 1_{in}| = \\ &= -\frac{1}{2}\langle 0_{BS}|\alpha_{BS}\rangle|1_{out}\rangle\langle 1_{in}| - \langle 0_{BS}|\alpha_{BS}\rangle\frac{i}{2}|2_{out}\rangle\langle 1_{in}| + \\ &\quad - \frac{e^{i\phi}}{2}|1_{out}\rangle\langle 1_{in}| + \frac{ie^{i\phi}}{2}|2_{out}\rangle\langle 1_{in}|. \end{aligned} \quad (29)$$

The corresponding probabilities of the single paths become

$$\wp(d_1 \wedge a|s) = \text{tr}(F_{d_1} K_a \rho K_a^\dagger) = \frac{1}{4}(1 - |\langle 0_{BS}|\alpha_{BS}\rangle|^2) \quad (30)$$

$$\wp(d_1 \wedge b|s) = \text{tr}(F_{d_1} K_b \rho K_b^\dagger) = \frac{1}{4}|1 + e^{i\phi}\langle 0_{BS}|\alpha_{BS}\rangle|^2 = \quad (31)$$

$$= \frac{1}{4}[1 + |\langle 0_{BS}|\alpha_{BS}\rangle|^2 + 2\Re(e^{i\phi}\langle 0_{BS}|\alpha_{BS}\rangle)]$$

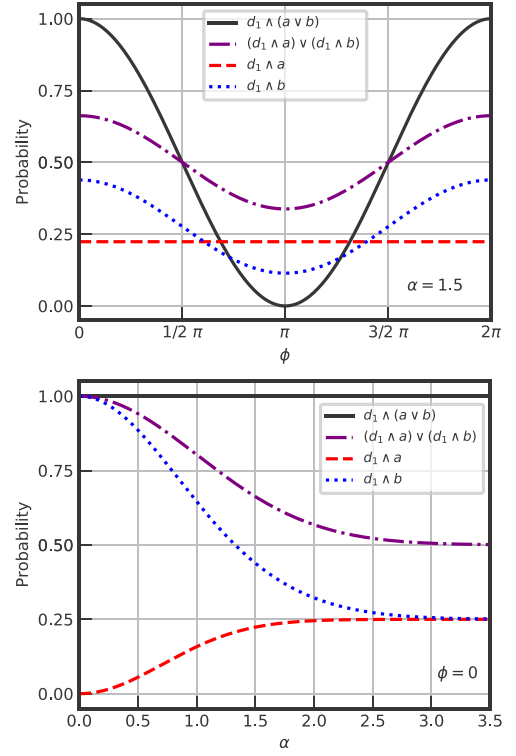


FIG. 5. Dependency of the different probability expressions on ϕ and α with the fixed value $\alpha = 1.5$ (top) and with the fixed value $\phi = 0$ (bottom).

Finally, we have then

$$\wp((d_1 \wedge a) \vee (d_1 \wedge b)|s) = \frac{1}{2} \left[1 + e^{-\frac{|\alpha|^2}{2}} \cos(\phi) \right], \quad (32)$$

where we used $\langle \alpha_{BS}|0_{BS}\rangle = e^{-\frac{|\alpha|^2}{2}}$.

As we can see in Fig. 5, for each probability relative to a specific path, an interference term is always present and is proportional to the overlap between the $|0\rangle_{BS}$ and $|\alpha_{BS}\rangle$ states. Only the probability corresponding to the path b is sensitive to the phase of the PS .

In the limit $\langle 0_{BS}|\alpha_{BS}\rangle \rightarrow 0$ (corresponding to $\alpha \rightarrow \infty$ and $m \rightarrow 0$, see Fig. 5 bottom), we have a pure particle-like behavior with $\wp(d_1 \wedge a|s) = \wp(d_1 \wedge b|s) = 1/4$ and $\wp((d_1 \wedge a) \vee (d_1 \wedge b)|s) = 1/2$.

In the limit $\langle 0_{BS}|\alpha_{BS}\rangle \rightarrow 1$ (corresponding to $\alpha \rightarrow 0$ and $m \rightarrow \infty$, see Fig. 5 bottom), we have instead $\wp(d_1 \wedge a|s) = 0$ and $\wp(d_1 \wedge b|s) = 1$. From the detection of $|0\rangle_{BS}$, no information on the taken path can be extracted. This is similar to the case 3 discussed in Sec. III B, where both path detectors provide the same output. In this limit case $\wp(d_1 \wedge b|s)$ (and then $\wp((d_1 \wedge a) \vee (d_1 \wedge b)|s)$) is *de facto* equivalent to $\wp(d_1 \wedge (a \vee b)|s)$ treated in the previous section. The behavior of the different formulas as function of ϕ and α is shown in Fig. 5.

Except to the limit case with $\langle 0_{BS}|\alpha_{BS}\rangle \rightarrow 0$ ($m \rightarrow 0$), the two equations (25) and (32) lead to different forms of the probability function. Once more, the expressions

$d_1 \wedge (a \vee b)$ and $(d_1 \wedge a) \vee (d_1 \wedge b)$ cannot be considered equivalent with the violation of the distributivity property.

IV. INTERFERENCES IN SUBSEQUENT MEASUREMENTS WITH INDEFINITE CAUSAL ORDER

A. General considerations for indefinite causal order measurements

Another two-way system where interference effects occur is the case of subsequent measurements with an indefinite causal order. To treat such class of subsequent measurements, the new formalism of the process matrices has been developed in the last years [23, 24, 29], and the presence of interference effects in measurements with indefinite order has been proven experimentally [54]. In particular it has been demonstrated that, for two subsequent measurements a and b , the probability relation [23–25]

$$Pr(a, b) = \lambda Pr^{a \preceq b}(a, b) + (1 - \lambda) Pr^{b \preceq a}(a, b), \quad (33)$$

valid for a probabilistic composition between two measurements with defined causal order (where the measurement order is indicated by the operator “ \preceq ” and with $0 \leq \lambda \leq 1$) is violated for indefinite causal phenomena.

With the probability definition and notation introduced in the previous sections, both cases of two-way interferences and two causal orders are simply discussed on the same footing and without introducing new notations in the probability function like $Pr^{a \preceq b}$.

In the case a single causal sequence of emissions from a source s , with a measurement a followed by a measurement b and final detection d , the corresponding probability is

$$\wp(d \wedge b \wedge a | s) = \text{tr}(F_d K_b K_a \rho K_a^\dagger K_b^\dagger), \quad (34)$$

where K_a and K_b are the Kraus operators associated to the effects E_a and E_b corresponding to the intermediate measurements, respectively, and F_d is the effect corresponding to the final detection. Note that due to the implicit prior of the order “ a after b ” in the notation $Pr^{a \preceq b}$, in our notation $Pr^{a \preceq b}(a, b)$ corresponds in fact to $\wp(d | b \wedge a \wedge s) = \wp(d \wedge b \wedge a | s) / \wp(b \wedge a | s)$.

Similarly to the which-path case, if we consider the same subsequent measurements but where the order between a and b is unknown, the probabilities $\wp((d \wedge b \wedge a) \vee (d \wedge a \wedge b) | s)$ and $\wp(d \wedge [(b \wedge a) \vee (a \wedge b)] | s)$ lead to two different final expressions.

B. Distinguishable causal order

This case corresponds to the situation where we can implicitly determine the measurement order, similarly to the case where each path in the Young’s slits can

be determined, and the associated probability writing is $\wp((d \wedge b \wedge a) \vee (d \wedge a \wedge b) | s)$. The different sequences of measurements can be associated to the effects $E_{(abd)}$ and $E_{(bad)}$ and each corresponding probability can be constructed from a nested version of Eq. (7). Moreover, in this case the property (P3)/(P3*) can be applied:

$$\begin{aligned} \wp((d \wedge b \wedge a) \vee (d \wedge a \wedge b) | s) &= \wp(d \wedge a \wedge b | s) + \wp(d \wedge b \wedge a | s) = \\ &= \text{tr}(F_d K_b K_a \rho K_a^\dagger K_b^\dagger) + \text{tr}(F_d K_a K_b \rho K_b^\dagger K_a^\dagger) = \\ &= \wp(a \wedge b | s) \wp(d | a \wedge b \wedge s) + \wp(b \wedge a | s) \wp(d | b \wedge a \wedge s). \end{aligned} \quad (35)$$

The last equality of the above equation allows to appreciate the equivalence to Eq. (33), which is manifestly not violated where $\wp(a \wedge b | s) = \lambda$.

C. Indistinguishable causal order and discussion

If the order of the measurement is intrinsically indefinite, $\wp(d \wedge [(b \wedge a) \vee (a \wedge b)] | s)$ has to be considered with

$$\wp(d \wedge [(b \wedge a) \vee (a \wedge b)] | s) = \text{tr}(F_d K_{a \overline{\wedge} b} \rho K_{a \overline{\wedge} b}^\dagger), \quad (36)$$

where the Kraus operator $K_{a \overline{\wedge} b}$ corresponds to the indefinite ordered measurement $(b \wedge a) \vee (a \wedge b)$. Compared to Eq. (35), additional interference terms can be present and we have

$$\wp(d \wedge [(b \wedge a) \vee (a \wedge b)] | s) \neq \wp((d \wedge b \wedge a) \vee (d \wedge a \wedge b) | s). \quad (37)$$

In order to appreciate the differences between the two expressions, we can consider ideal measurements for a and b with the assumption that they can be described by non-commutative and non-orthogonal projectors P_a and P_b acting in the large Hilbert space composed by the inputs and outputs of the two measurements [5, 23]. With these assumptions, Eq. (35) can be written as

$$\begin{aligned} \wp((d \wedge b \wedge a) \vee (d \wedge a \wedge b) | s) &= \\ &= \text{tr}(P_d P_b P_a \rho P_a P_b) + \text{tr}(P_d P_a P_b \rho P_b P_a). \end{aligned} \quad (38)$$

On the other hand, Eq. (36) becomes

$$\begin{aligned} \wp(d \wedge [(b \wedge a) \vee (a \wedge b)] | s) &= \\ &= \text{tr}(P_d P_b P_a \rho P_a P_b) + \text{tr}(P_d P_a P_b \rho P_b P_a) + \\ &+ \text{tr}(P_d P_b P_a \rho P_b P_a) + \text{tr}(P_d P_b P_a \rho P_b P_a) + \dots \end{aligned} \quad (39)$$

where the cross terms $\text{tr}(P_d P_b P_a \rho P_b P_a)$ and $\text{tr}(P_d P_b P_a \rho P_b P_a)$ are present (plus additional terms related to the non-commutativity between P_a and P_b). The above equation violates Eq. (33), as obtained in the work of Brukner and coworkers with the process matrices formalism [23–25].

V. CONCLUSION

In conclusion, we present a formulation of the probability function in the context of generalized measurements for subsequent detections with several possible paths, and for subsequent detections with indefinite order. From the assumption of the Hilbert space structure for the description of systems, Gleason-Busch theorem assures that the trace of the density operator univocally defines the form of the probability function. Flatt and coworkers demonstrate that from this result, when subsequent measurements are considered, the Kraus updating rule is reconstructed. Here we apply the same methodology to a two-path case with a renewed notation but also to the case of intermediate measurements without define causal order.

For the two-path case, two different expressions of the probability are found, $\wp((d \wedge a) \vee (d \wedge b)|s)$ and $\wp(d \wedge (a \vee b)|s)$, which are related to the possibility of distinguishing or not the trajectory in the measurement system. In fact, the distributive property of the probability function arguments cannot be taken for granted. From the first expression, the classical law or probability $Pr^C(a \vee b) = Pr^C(a) + Pr^C(b)$ is recovered. The use of the reduced trace over the undetected states of the path-detectors leads to this same expression. With regards to the $\wp(d \wedge (a \vee b)|s)$, the associated operator $K_{a \vee b}$ to the $a \vee b$ measurement, can be interpreted ambiguously. $K_{a \vee b}$ can be built from measurements corresponding to the same final detector state independently from the path, or from a complementary measurement of a and b ($c = NOT(a \vee b)$). Both approaches lead to a final expression corresponding to the standard Born rule for the case of perfect projective measurements. If $K_{a \vee b}$ is constructed by a mixing of path-detector states, we recover the situation of the *quantum eraser*. We are in fact considering the probability $\wp(d \wedge q|s)$ associated with the state $|q^{det}\rangle = \alpha |a^{det}\rangle + \beta |b^{det}\rangle$, which depends on the choice of constants α and β , i.e. a special case of which-path ignorance.

The frontier between the intrinsic possibility to distinguish a path or not is related to the coupling of the studied system with the path-detectors and/or the environ-

ment. This topic is widely studied in the literature, and in particular in the context of decoherence theory. Here we consider the very simple case of a Mach-Zehnder interferometer with a movable beam splitter, which is also well known in the literature but treated here in the context of generalized measurements. We demonstrate here that by varying the mass of the beam splitter, we can continuously pass from the distinguishable path case, where $\wp((d \wedge a) \vee (d \wedge b)|s)$ is valid, to the indistinguishable path case, where $\wp(d \wedge (a \vee b)|s)$ should be used instead. This toy model reveals once more the complementarity of nature, but also underlines once more the advantages of generalized measurement theory with respect to ideal projective measurements, where unsharp detections revealing particle-like and wave-like behavior at the same time can be treated unambiguously.

With the same probability notation introduced here, another two-way problem is treated, namely the sequence of detections with an indefinite causal order. For the case of two indefinite ordered measurements a and b , two different formulation of the probability function are allowed, $\wp((d \wedge b \wedge a) \vee (d \wedge a \wedge b)|s)$ and $\wp(d \wedge [(b \wedge a) \vee (a \wedge b)]|s)$. Similarly to the two-path case, the difference between the two expressions is related to the possibility to distinguish or not the causal order of the measurements. The first expression corresponds to a statistical sum of probabilities relative to the two possible measurement sequences with definite order. In this case the ‘‘causal equality’’ (Eq. (33)) is valid. In the second expression, additional interference terms appear with the violation of the causal equality, similarly to the results obtained with the process matrices formalism.

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Appendix A: Quantum eraser probabilities

We consider two path detector bases $|p_1\rangle, |p_2\rangle$, e.g. corresponding to $|a^{det}\rangle, |b^{det}\rangle$ states of the Young’s slit case, and a final detection d . We consider a perfect which-path measurement where $|p_j\rangle$ are associated to the system state $|j\rangle$. The associated operators $K_j = \sum_i \langle p_j | p_i \rangle |i\rangle \langle i|$ is then equivalent to the projectors $P_j = |j\rangle \langle j|$. We consider two different orthogonal states $|q_1\rangle, |q_2\rangle$ related by the unitary transformation $|q_j\rangle = V_{ji} |p_i\rangle$. For each measurement q_j , the associated operator

$$K'_j = \sum_i \langle q_j | p_i \rangle |i\rangle \langle i| = \sum_i V_{ji} |i\rangle \langle i|. \quad (\text{A1})$$

For each single measurement q_j , we have

$$\begin{aligned} \wp(d \wedge q_j | s) &= \text{tr}(K_d K'_j \rho K'^{\dagger}_j K_d) = \\ &= \text{tr}(K_d (V_{j1} P_1 + V_{j2} P_2) \rho_S (V_{j1}^* P_1 + V_{j2}^* P_2) P_d) = \\ &= \text{tr}\left(\left(\sum_i V_{ji} P_i\right) \rho \left(\sum_{i'} V_{ji'}^* P_{i'}\right) P_d\right) = \\ &= \sum_{i, i'} V_{ji} V_{ji'}^* \text{tr}(P_i \rho_S P_{i'} P_d) = \\ &= \sum_i |V_{ji}|^2 \text{tr}(P_i \rho_S P_i P_d) + \sum_{i, i' \neq i} V_{ji} V_{ji'}^* \text{tr}(P_i \rho_S P_{i'} P_d). \end{aligned} \quad (\text{A2})$$

When we consider the probability relative to the measurement $(d \wedge q_1) \vee (d \wedge q_2)$, we have

$$\begin{aligned} \wp((d \wedge q_1) \vee (d \wedge q_2) | s) &= \\ &= \sum_{j, i} |V_{ji}|^2 \text{tr}(P_i \rho_S P_i P_d) + \sum_{i, j, i' \neq i} V_{ji} V_{ji'}^* \text{tr}(P_i \rho_S P_{i'} P_d) = \\ &= \sum_i \text{tr}(P_i \rho_S P_i P_d) + \sum_{i, i' \neq i, j} V_{ji} V_{ji'}^* \text{tr}(P_i \rho_S P_{i'} P_d) \end{aligned} \quad (\text{A3})$$

where we used the unitary matrix property $\sum_j |V_{ji}|^2 = 1$. The second term of the expression is in fact equal to zero because of other property of unitarity of V matrices $\sum_j V_{ji} V_{ji'}^* = \delta_{i, i'}$ in a sum over $i, i' \neq i$. Finally we have

$$\begin{aligned} \wp((d \wedge q_1) \vee (d \wedge q_2) | s) &= \sum_i \text{tr}(P_i \rho P_i P_d) = \\ &= \wp(d \wedge p_1) \vee (d \wedge p_2) | s). \end{aligned} \quad (\text{A4})$$

Independently of the choice of the orthogonal and complete base, probability is always the same and equal to a particle-like behavior, even if the single $d \wedge q_j$ measurement can provoke interference effects.