# Conditional probability and interferences in generalized measurements

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(Dated: July 24, 2020)

In the context of generalized measurement theory, the Gleason-Busch theorem assures the unique form of the associated probability function. Recently, in Flatt et al. Phys. Rev. A **96**, 062125 (2017), the case of subsequent measurements has been treated, with the derivation of the Lüders rule and its generalization (Krauss update rule). Here we investigate the special case of subsequent measurements where an intermediate measurement is a composition of two measurements ($a$ or $b$) with possible interference effects. In this case, the associated probability cannot be written univocally, and the distributive property on its arguments cannot be taken for granted. Different probability expressions are related to the intrinsic possibility of obtaining definite results for the intermediate measurement. The frontier between the two cases is investigated in the framework of generalized measurements with a toy model, a Mach-Zehnder interferometer with movable beam splitter.

## I. INTRODUCTION

In Quantum Mechanics, probabilities are obtained by the squared modulus of complex amplitudes, which give rise to interference phenomena. In the common example of Young's slits composed of a source, two slits and a screen or movable detector as represented in Fig. 1, the probability to detect an emitted particle in a position $x$ on the backstop wall is given by

$$\begin{aligned} Pr(ab) &= |\psi_a + \psi_b|^2 = \\ &= Pr(a) + Pr(b) + 2\sqrt{Pr(a)Pr(b)}\cos\left(\arg(\psi_a \psi_b^*)\right), \end{aligned} \tag{1}$$

where $\psi_a$, $\psi_b$ are the complex probability amplitudes associated with each slit and $Pr(a) = |\psi_a|^2$, $Pr(b) = |\psi_b|^2$ are the probabilities associated to the opening of the single slits. The above expression is substantially different from the classical probability sum rule

$$Pr^C(ab) = Pr(a) + Pr(b), \tag{2}$$

where interference terms are not present.

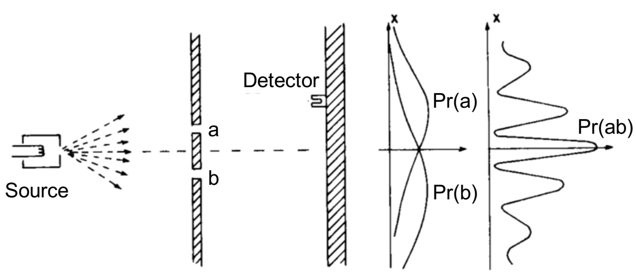


FIG. 1. Scheme of Young's slits experiment. Adapted from Ref. [1].

The probability function for the quantum case is strictly connected to the Hilbert space structure, where systems are described with respect to a defined basis and where the complex numbers mentioned above correspond to coordinates. With some minimal requirements on the probability function $Pr$, in 1957 Gleason [2] demonstrated that $Pr$ is univocally defined in a Hilbert space by the trace rule

$$Pr(i) = tr(\rho P_i), \tag{3}$$

where $\rho = |\varphi\rangle\langle\varphi|$ is the density matrix of the prepared system and $P_i = |i\rangle\langle i|$ is the projector on the state of interest. In the case of an initial pure state $|\varphi\rangle = |s\rangle$, Eq. (3) corresponds to the Born rule with $Pr(i) = |\langle i|s\rangle|^2$. Gleason's theorem has some limitations; it is valid only for Hilbert spaces with a dimension larger than two and for projective von Neumann measurements [3].

In the framework of the general measurement formalism of positive-operator-valued measures (POVM, also called probability operator measures), in 2003, Busch [4] extended Gleason's theorem for any dimension and for imperfect measurements described by positive operators, effects $E_i$, instead of projectors. Recently (2017), in the same context of POVM, Flatt, Barnett and Croke applied the Gleason-Busch theorem to subsequent measurements [5]. Considering the operators $E_i$ and $F_j$ associated with the measurements $i$ and $j$, with $i$ before $j$, Flatt and coworkers proved that $Pr(i, j)$ takes the general form

$$Pr(i, j) = tr\left(F_j \sum_k K_{ik} \rho K_{ik}^\dagger\right), \tag{4}$$

where the operators $K_{ik}$ are related to the effects by $E_i = \sum_k K_{ik}^\dagger K_{ik}$. From the above equation and the corresponding one for the conditional probability $Pr(i|j)$, the Kraus update rule [6, 7]

$$\rho \to \rho_i' = \frac{\sum_k K_{ik} \rho K_{ik}^\dagger}{tr\left(\sum_k K_{ik}^\dagger K_{ik}\right)} \tag{5}$$

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for the state update of the system $\rho_i'$ after the measurement $i$ is recovered. It is worth noting that the Kraus update rule, and its particular case of the Lüders rule [8] valid for ideal measurement and determined by the von Neumann projection postulate, are derived from first principles. There is no need of other postulates than the description of states via the Hilbert space and a few basic requirements for the probability function.

In this article, we apply the formalism of subsequent generalized measurements to the special case with two possible and mutually exclusive intermediate measurements, with possible emerging interference effects. With a change of notations with respect to Flatt et al., we will show that two possible expressions of the final probability $Pr$ can be derived from Eq. (3). They correspond to a wave-like or particle-like behavior, i.e. Eqs. (1) and (2) in the example of Young's slits, and correspond to the possibility of distinguishing or not the intermediate measurement. The difference between the two forms is the order of the arguments of the probability expressions, where the distributive property can not be taken for granted. The violation of the distributive property in Quantum Mechanics is not new and it has been pointed out since the early years of its formulation [9, 10] and extensively discussed in Quantum Logic. Its connection to the extension of classical probability to quantum probability is well discussed in the literature in the case of perfect projective measurement [11–15]. For imperfect general measurements, when positive operators are considered instead of projectors, some work has been performed by Busch and collaborators [16, 17]. Here we present a general discussion about the probability function for the distinguishable and indistinguishable path cases (the particle-like and wave-like behaviors) in the case of imperfect (unsharp) measurements.

The frontier between the different cases, and then the domain of validity of Eqs. (1) and (2), has been extensively discussed in the past. Experimentally, it has been explored in the last decades through investigations of interference effects with molecules with larger and larger masses. Diffraction of large inorganic and organic molecules with masses beyond 25000 atomic mass units has been obtained [18–20]. Here, we discuss this frontier in the context of generalized measurements considering a Mach-Zehnder interferometer with movable beam splitter. This toy model, introduced in the past by Haroche et al. [21, 22], has the interesting feature of allowing to pass from one case to the other continuously, simply considering a variation of the mass of the movable beam splitter.

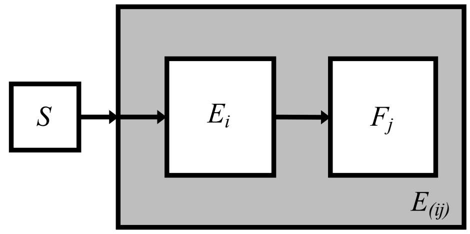


FIG. 2. Scheme of two subsequent measurements. Adapted from Ref. [5].

## II. PROBABILITY FOR SUBSEQUENT MEASUREMENTS

### A. Introduction of new notations

Taking inspiration from the Quantum Logic approach [11–15, 23, 24] and the propositional definition of probability [25–27], we introduce a new notation with the logical operators "$\wedge$" and "$\vee$" to unambiguously discuss the joint probability of series of subsequent measurements. The *conjunction* operator "$\wedge$" is equivalent to "AND" in normal language and to the comma in the previously introduced notation $Pr(i,j)$. The *disjunction* operator "$\vee$" is equivalent to "OR" also indicated with the "+" operator (in Refs. [4, 5] as example). Particular attention has to be payed for measurements $i, j$ that are incompatible. In this case, the logical operator "$\wedge$" is not well defined [12–14, 23], except if the order of subsequent measurements is defined. As already pointed out in the *consistent histories* interpretation of Quantum Mechanics [28–30], differently from standard logic, the operator "$\wedge$" is not symmetric with respect to $i, j$ with $i \wedge j \neq j \wedge i$. With this notation the joint probability defined above for a measurement $j$ obtained after a measurement $i$ can be written as

$$\wp(j \wedge i|s) \equiv Pr(i,j), \tag{6}$$

where we explicitly indicate the system preparation $s$, which is in fact connected to the possible measurement outcomes. We also invert the order of $i, j$ to clearly indicate the sequential order of the measurement or preparation from right to left (preparation $s$, first measurement $i$ and second measurement $j$).

### B. Rewriting probabilities

Before treating in details the Young's slits problem with new introduced notation, we shall rewrite the properties and assumptions of the probability function used by Flatt et al. [5] that lead to Eq. (5). We consider a set of positive-semidefinite operators (effects) $E_i$ of the same POVM with $\sum_i E_i = I$. The requirement properties of the probability function $\nu(E_i) = Pr(i)$ for the

Gleason-Busch theorem are

(P1) $0 \le \nu(E_i) \le 1$.
(P2) $\nu(I) = 1$.
(P3) $\nu(E_i + E_j \ldots) = \nu(E_i) + \nu(E_j) + \ldots$

The function $\nu(E_i)$ is in fact a map from the full set of effects $\mathcal{E}(\mathcal{H})$ acting on the Hilbert space $\mathcal{H}$: $E \to \nu(E)$ with $\nu(E) \in [0, 1]$.

With our notation, the previous propositions become

(P1*) $0 \le \wp(i|s) \le 1$.
(P2*) $\wp(\mathcal{I}|s) = 1$.
(P3*) $\wp(i \vee j \vee \ldots |s) = \wp(i|s) + \wp(j|s) + \ldots$

where $i, j$ are the measurements that correspond to the effects $E_i, E_j$ and $\mathcal{I} = \bigvee_i i$ measurement correspond to the identity operator $I$.

When two subsequent measurements are considered together, Flatt et al. introduced the new function

$$\mu^i_\nu(F_j) = \nu(E_{(ij)}) = Pr(i, j) \tag{7}$$

for the action of the effect $F_j$ after the action of $E_i$ and $E_{(ij)}$ indicating the cumulative effect (see Fig. 2). In addition, the following assumptions are considered by Flatt et al.

(A1) $0 \le \mu^i_\nu(F_j) \le \nu(E_i) < 1$.
(A2) $\mu^i_\nu(I) = \nu(E_i)$.
(A3) $\mu^i_\nu(F_j + F_k + \ldots) = \mu^i_\nu(F_j) + \mu^i_\nu(F_k) + \ldots$.

With our notation, we consider on the same level the measurements $j$ and $i$ and the operator "$\wedge$" indicates the measurement order. The assumptions (A1–2) can simply be rewritten as

(A1*) $0 \le \wp(j \wedge i|s) \le \wp(i|s) \le 1$.
(A2*) $\wp(\mathcal{I} \wedge i|s) = \wp(i|s)$.

(A2*) is now a tautology. For (A3), the rewriting is ambiguous. $\mu^i_\nu(F_j + F_k + \ldots)$ can be written in fact in two different forms:

$$\wp((j \wedge i) \vee (k \wedge i)|s) \tag{8}$$

or

$$\wp((j \vee k) \wedge i|s). \tag{9}$$

The discussion on this ambiguity is the key point of the present work. If the distributive property is considered valid, the two expressions are equivalent. But the validity of distributivity cannot be taken for granted. As anticipated in the introduction, the violation of the distributive law in quantum phenomena is well known since the early years of Quantum Mechanics [9, 31]. In particular in Quantum Logic [10, 11, 14, 24, 32–34] this is related to the properties of orthomodular lattices, associated to sets of yes/no experiments, where the distributivity on their elements is not always valid. In the next sections we will in particular discuss this violation in the context of general measurements.

## III. INTERFERENCES IN THE POVM FORMALISM

### A. General considerations

To investigate the difference between Eqs. (8) and (9), we come back the specific example of Young's slits where we consider the possibility to measure or flag the passage through each slit. Before that, a short introduction to generalized measurements is mandatory. In the framework of POVM, the single measurements are described by the positive-valued operators $E_\ell = K^\dagger_\ell K_\ell$, where $K_\ell$ operators are determined by the unitary interaction between the system we want to study and the detector, both considered as quantum systems. The general expression for $K_\ell$ is given by [35, 36]

$$K_\ell = \sum_{i,j} \alpha_{ij} \langle \ell^{det} | \Phi^{det}_i \rangle |j\rangle \langle i| , \tag{10}$$

where $\alpha_{ij}$ depend on the action of the unitary matrix $U_{int}$ describing the interaction between the system and the detector. The initial state is described by $|\Psi^0_i\rangle = |i\rangle |\Phi_0\rangle$, where $|i\rangle$ and $|\Phi_0\rangle$ describe the initial state of the system and the detector, respectively. After their mutual interaction, the system and detector states are described by $|\Psi^0_i\rangle \to |\Psi_i\rangle = |\varphi_i\rangle |\Phi_i\rangle = \sum_j \alpha_{ij} |j\rangle |\Phi_i\rangle$. $|\Phi^{det}_i\rangle$ describes the detector state after the interaction with the system in an initial state $|i\rangle$. Finally, $|\ell^{det}\rangle$ represents the detector state corresponding to the macroscopic outcome of the measurement device. In the case of a non-destructive measurement, the above formula is simplified to

$$K_\ell = \sum_i \langle \ell^{det} | \Phi^{det}_i \rangle |i\rangle \langle i| . \tag{11}$$

### B. Distinguishable case

In the case of Young's slits, we consider that the detection of the path taken by the particle is possible and is non-destructive. The formula corresponding to Eq. (8) becomes $\wp((d \wedge a) \vee (d \wedge b)|s)$ and depends on the operators $K_a, K_b$ and $K_d$. $K_a, K_b$ are related to the detection of the path $a$ or $b$, and the corresponding effects are

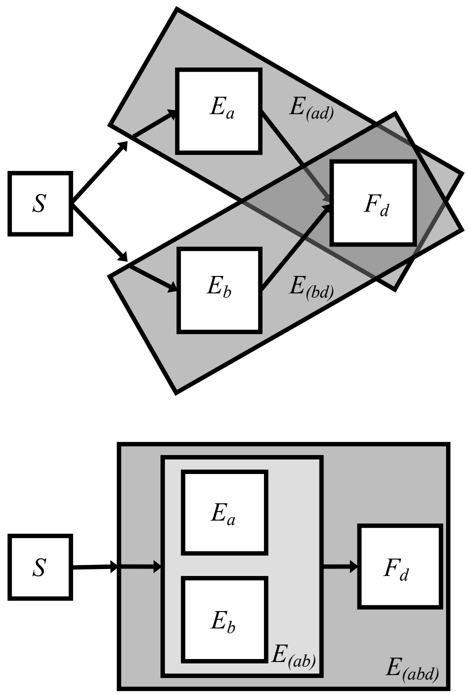


FIG. 3. Schemes of subsequential measurements corresponding to the Young's slits experiment for the case where the path of the particle can be detected (top) or not (bottom).

$E_a = K_a^\dagger K_a$, $E_b = K_b^\dagger K_b$. $K_d$ is related to the detection $d$ on the screen with $F_d = K_d^\dagger K_d$ The combination of $E_a$ and $E_b$ with $E_d$ can be assimilated to the effects $E_{(ad)}$ and $E_{(bd)}$ as in Eq. (7), and for which the property (P3)/(P3*) can be applied. In this case we have

$$\wp((d \wedge a) \vee (d \wedge b)|s) = \wp(d \wedge a|s) + \wp(d \wedge b|s) = \\ = tr(F_d K_a \rho K_a^\dagger) + tr(F_d K_b \rho K_b^\dagger). \quad (12)$$

The above equation corresponds to the classic probability sum rule, i.e. the particle-like probability in Eq. (2). The fact that we can decompose the measurement in two separate operators $E_{(ad)}$ and $E_{(bd)}$ (Fig. 3, top)) implicitly means that the different paths can be distinguished and we have just a duplicated version of the basic subsequent measurement represented in Fig. 2. This case can be easily treated with the formalism introduced by Flatt et al. with the introduction of the probability functions $\mu_\nu^a(F_d)$ and $\mu_\nu^b(F_d)$.

In the case of ideal projective measurements, we have $E_a = K_a = P_a = |a\rangle\langle a|$ and $E_b = K_b = P_b = |b\rangle\langle b|$ where we used the properties of projectors $P_i^\dagger = P_i$ and $P_i P_i = P_i$. The above equation then becomes [15]

$$\wp((d \wedge a) \vee (d \wedge b)|s) = \\ = |\langle d|U_{(ad)}|a\rangle\langle a|U_{(sa)}|s\rangle|^2 + |\langle d|U_{(bd)}|b\rangle\langle b|U_{(sb)}|s\rangle|^2, \quad (13)$$

where the unitary operators $U$ correspond to the evolution of the different parts of the apparatus.

The expression of $\wp((d \wedge a) \vee (d \wedge b)|s)$ can also be directly obtained by the trace reduction of the density matrix $\rho$ with respect to detector base $|a^{det}\rangle$ and $|b^{det}\rangle$. In this case we have

$$\wp((d \wedge a) \vee (d \wedge b)|s) = tr(K_d \rho_r K_d^\dagger), \quad (14)$$

with $\rho_r = tr_{a^{det},b^{det}}(|\Psi_i\rangle\langle\Psi_i|)$ and where $|\Psi_i\rangle = \sum_j \alpha_{ij}|j\rangle|\Phi_i\rangle$. From the linearity of the trace operator, it is easy to verify that the previous expression is equivalent to Eq. (12). This indicates that the use of the trace over the undetected $|a^{det}\rangle, |b^{det}\rangle$ states implicitly implies an interaction between the system and the which-path detectors, even if they are not directly involved in the measurement.

### C. Indistinguishable case

In the case we can not distinguish which path is taken by the particle, the $a \vee b$ cannot be decomposed and we have to deal with the expression

$$\wp(d \wedge (a \vee b)|s) = tr(F_d K_{a\vee b} \rho K_{a\vee b}^\dagger). \quad (15)$$

The operator $K_{a\vee b}$ can be defined in two different ways:

1. from a complementary measurement $c$ (e.g. a series of detectors on the slit walls),
2. via a detector state $|q^{det}\rangle$ belonging to the span generated by the vectors $|a^{det}\rangle$ and $|b^{det}\rangle$.

As we will see, a genuine $a \vee b$ measurement is related to a complementary case only. The second approach is in fact related to the *quantum eraser* case and it is discussed separately in the next section.

If we consider a complementary measurement $c$ to both $a$ and $b$ measurements, we have that $c \wedge b = 0, c \wedge b = 0$ and $c = \mathcal{I} - a \vee b$. $E_{a\vee b}$ corresponds to the absence of signal in the measurement $E_c$, then, using the property of the set of effects of the POVM for which $\sum_{i=a,b,c} E_i = I$, we have $E_{a\vee b} = I - E_c = E_a + E_b$. $K_{a\vee b}$ can be written as [17, 35, 37]

$$K_{a\vee b} = U_{a\vee b}\sqrt{E_a + E_b}, \quad (16)$$

where $U_{a\vee b}$ is a unitary matrix that depends on the details of the interaction between $|a^{det}, b^{det}\rangle$ and the propagating particle-wave.

We can see that in this general case, $\wp(d \wedge (a \vee b)|s) \neq \wp((d \wedge a) \vee (d \wedge b)|s)$ and the distributive property on the arguments of $\wp$ is violated. In the case of ideal projective measurements, $K_{a\vee b}$ can be explicitly written. In this case we have that $E_{a\vee b} = E_a + E_b = P_a + P_b$ and Eq. (15) becomes [15] (see also Refs. [11, 13, 17])

$$\wp(d \wedge (a \vee b)|s) = \\ = |\langle d|U_{(ad)}|a\rangle\langle a|U_{(sa)}|s\rangle + \langle d|U_{(bd)}|b\rangle\langle b|U_{(sb)}|s\rangle|^2, \quad (17)$$

which is equivalent to the quantum form of the probability in Eq. (1), i.e. equivalent to the Born rule.

For the indistinguishable case, the measurement $a \vee b$ corresponds to an atomic operator $E_{a\vee b} \equiv E_{(ab)}$ that cannot be decomposed in terms of $E_a, E_b$. The cumulative effect $E_{(abd)}$ depends then on all three measurements $a$, $b$ and $d$ and can be represented by the scheme in the bottom of Fig. 3.

### D. The quantum eraser revisited

In the Quantum Logic context, a measurement representing $a \vee b$ can be built from a vector $|q^{det}\rangle = \alpha\,|a^{det}\rangle + \beta\,|b^{det}\rangle$ [14], with $\alpha, \beta \neq 0$, which belongs to the span generated by the vectors $|a^{det}\rangle$ and $|b^{det}\rangle$. Using Eq. (11) with $\ell = q$, we can then write

$$K_{a\vee b} \equiv K_q = \alpha^* K_a + \beta^* K_b, \tag{18}$$

where $|\alpha|^2 + |\beta|^2 = 1$ for a normalized probability. Once inserted in Eq. (15), the above expression gives rise to mixed $\langle a^{det}|b^{det}\rangle$ terms and then to interference phenomena. This is in fact the case of the *quantum eraser* [16, 38–42], where instead of the direct path detection via $|a^{det}\rangle\,, |b^{det}\rangle$, a combination of them is considered and interference terms appear.

This is a situation not equivalent to the case with a complementary measurement $c = \mathcal{I} - a \vee b$. Even if we recover the presence of interferences with the use of $|q^{det}\rangle$ instead of $|a^{det}\rangle$ or $|b^{det}\rangle$, we are dealing with a single measurement $q$ that corresponds to the probability $\wp(d \wedge q|s)$, and not $\wp(d \wedge (a \vee b)|s)$. Similarly to $a, b$ measurements, we could consider the alternative measurement given by the vector $|r^{det}\rangle = -e^{i\phi}\beta\,|a^{det}\rangle + e^{i\phi}\alpha\,|b^{det}\rangle$ orthogonal to $|q^{det}\rangle$. When both possible measurements $q$ and $r$ are considered, we can write down the probabilities $\wp((d\wedge q)\vee(d\wedge r))|s)$ and $\wp(d\wedge(q\vee r)|s)$. $|r^{det}\rangle\,, |q^{det}\rangle$ and $|a^{det}\rangle\,, |b^{det}\rangle$ are two different bases describing the detection and they are related by a unitary transformation. Because of the property of the unitary transformation, it can be demonstrated (see App. A for the detailed calculations) that the combination of the two measurements $r$ and $q$ and the which-path $a$ and $b$ are completely equivalent and

$$\wp((d \wedge q) \vee (d \wedge r))|s) = \wp((d \wedge a) \vee (d \wedge b))|s). \tag{19}$$

The interference terms present in the separate terms $\wp(d \wedge q|s)$ and $\wp(d \wedge r|s)$, completely compensate in $\wp((d\wedge q)\vee(d\wedge r))|s) = \wp(d\wedge q|s) + \wp(d\wedge r|s)$ like in the well known results on the quantum eraser.

For the case of $\wp(d \wedge (q \vee r))|s)$ probability, the situation is more complicated because it depends on the values of $\alpha, \beta, \phi$ but also on the choice of $\alpha', \beta'$ for building $K_{q\vee r} = \alpha' K_q + \beta' K_r$. With this last consideration, we can conclude that in fact the construction of $K_{a\vee b}$ via Eq. (18) is not equivalent to a genuine which-path ignorance, but it is a special case where a different detector state basis is considered.

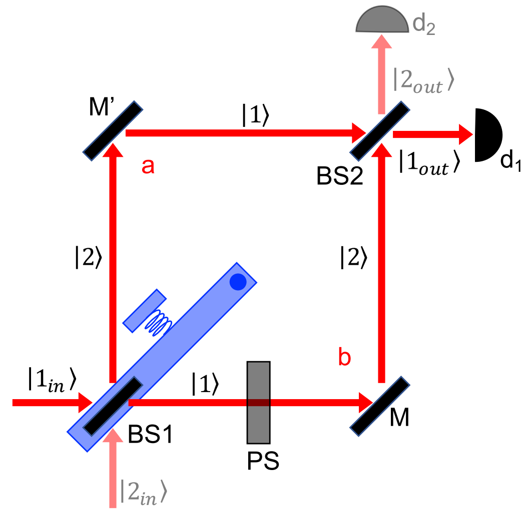


FIG. 4. Scheme of the Mach-Zehnder interferometer with a movable beam splitter BS1, a phase shifter with $\phi$ with two possible incoming beams $|1_{in}\rangle$ and $|2_{in}\rangle$ and outputs $|1_{out}\rangle$ and $|2_{out}\rangle$ measured by the detectors $d_1$ and $d_2$. Photons parallel to the incoming photon (horizontal propagation in the figure) are indicated by the states $|1\rangle$ and with $|2\rangle$ otherwise (vertical propagation).

## IV. DISCUSSIONS AND A TOY MODEL

### A. Distinguishing between distinguishable and indistinguishable cases

The fundamental difference between distinguishable and indistinguishable cases, i.e., the use of Eq. (14) or Eq. (16) for the probability function, is the coupling between the considered system and the possible which-path detector(s) and/or the environment. Such a coupling has been extensively studied in the context of the decoherence theory [43, 44]. In this section we consider a very simple case to consider the limits of Eqs. (14) and (16) in terms of effects thanks to a toy system where we can continuously tune the detectability of the taken path.

We consider a Mach-Zehnder interferometer with a movable beam splitter (represented in Fig. 4), an example discussed in the literature and realized experimentally with atoms in resonant cavities [21, 22]. Here we treat the problem in terms of effects in a POVM framework. A discussion of the Mach-Zehnder interferometer in terms of unsharp detection has been already discussed by Busch and Shilladay [16]. In this past work, the unsharpness of the detection is studied in terms of measurement mixing between the two paths, like in the quantum eraser case discussed in Sec. III D. The cases of distinguishability or indistinguishability of the paths is also treated, but not the frontier between them, which on the contrary is the main subject of the following paragraphs.

### B. A Mach-Zehnder interferometer with a movable beam splitter

The system considered here is composed by a single-photon source emitting monochromatic photons $|1_{in}\rangle$ interacting with: a movable beam splitter $BS1$, two mirrors $M, M'$, a phase shifter $PS$ that induces a phase $\phi$, a second (fixed) beam splitter $BS2$ and two detectors $d_1$ and $d_2$, following the scheme represented in Fig. 4. The movable beam splitter, with a mass $m$, can move with respect to a pivot and is connected to a fixed part by a spring that corresponds to a resonant angular frequency $\omega$. The beam splitter-spring system is described by a harmonic oscillator with energy spectrum $\mathcal{E}_n = \omega\hbar(n + \frac{1}{2})$. When the photon is reflected from the first beam splitter, a momentum kick $\Delta P = \sqrt{2}p$, with $p = 2\pi\hbar/\lambda$ the impulse of the photon, is transferred to $BS1$ with a translation from its ground state $|0\rangle_{BS}$ to the coherent state $|\alpha_{BS}\rangle$ with $\alpha = ip/\sqrt{m\omega\hbar}$ [21, 22].

In analogy to the Young's slits, we can consider the interferometer arm with the reflection from the movable beam splitter as the path $a$, and path $b$ otherwise (see Fig. 4).

### C. No-path detection case

In the case of a fixed beam splitter, the state of the beam splitter itself does not change after the passage of the photon and the state corresponding to the photon is

$$|1_{in}\rangle \to |\phi\rangle = - \frac{1}{2}|1_{out}\rangle - \frac{i}{2}|2_{out}\rangle - \frac{e^{i\phi}}{2}|1_{out}\rangle + \frac{ie^{i\phi}}{2}|2_{out}\rangle \quad (20)$$

The probability of detecting something on the detector $d_1$ depends on the operator $K_{d_1} = |0_{out}\rangle\langle 1_{out}|$ and the corresponding effect $F_{d_1} = K^{\dagger}_{d_1}K_{d_1} = |1_{out}\rangle\langle 1_{out}|$. Because of the impossibility of determining the path taken by the photon, the corresponding probability is $\wp(d_1 \wedge (a \vee b)|s)$. The complementary detection $c$ representing $K_{a\vee b}$ could be constitued by a series of detectors around the beam splitter $BS1$, like the wall detection in the case of the Young's slits, to insure the interaction (reflection or transmission) of the incoming photon $|1_{in}\rangle$ with $BS1$. The probability is then given by

$$\wp(d_1 \wedge (a \vee b)|s) = tr(F_{d_1}\rho') = \frac{1}{2}\left[1 + \cos(\phi)\right], \quad (21)$$

with $\rho' = |\phi\rangle\langle\phi|$ and $|\phi\rangle$ given by Eq. (20). We recover the standard formula of the Mach-Zehnder interferometer [16, 37].

### D. Which-path detection and probabilities

We consider now that the beam splitter $BS1$ can move and that its state after the recoil is described by the coherent state $|\alpha_{BS}\rangle$. Considering the initial state $|1_{in}\rangle|0_{BS}\rangle$ describing the photon-beam splitter system, after the interaction between the incoming photon with the first beamsplitter $BS1$ (and the mirrors $M$ and $M'$ and the second beam splitter $BS2$), the photon/mirror state $|\phi\rangle$ is described by

$$\begin{aligned} |1_{in}\rangle|0_{BS}\rangle \to= |\phi\rangle|\Phi_{BS}\rangle = \\ = -\frac{1}{2}|1_{out}\rangle|\alpha_{BS}\rangle - \frac{i}{2}|2_{out}\rangle|\alpha_{BS}\rangle + \\ - \frac{e^{i\phi}}{2}|1_{out}\rangle|0_{BS}\rangle + \frac{ie^{i\phi}}{2}|2_{out}\rangle|0_{BS}\rangle, \end{aligned} \quad (22)$$

where $|\phi\rangle$ is state of the photon at the exit of the interferometer and $|\Phi_{BS}\rangle$ is the state of the movable beam splitter after the passage of the photon.

The operator $K_b = \langle 0_{BS}|\Phi_{BS}\rangle|\phi\rangle\langle 1_{in}|$ can be associated to the branch $b$ where there is no momentum transfer to $BS1$, which remains in the $|0_{BS}\rangle$ state. For the branch $a$, we cannot directly use $\langle\alpha_{BS}|\Phi_{BS}\rangle|\phi\rangle\langle 1_{in}|$ as $K_a$ operator. Due to the non-orthogonality of $|\alpha_{BS}\rangle$ and $|0_{BS}\rangle$, this leads to the possibility of having $E_a + E_b > 1$, violating the basic POVM properties. Considering that we can identify a coherent state only if its corresponding signal is above the quantum shot noise of the system, instead of $|\alpha_{BS}\rangle$ we can consider its Gram-Schmith orthogonalization $|\alpha'_{BS}\rangle$ with respect to $|0_{BS}\rangle$

$$|\alpha'_{BS}\rangle = \frac{|\alpha_{BS}\rangle - \langle 0_{BS}|\alpha_{BS}\rangle|0_{BS}\rangle}{\sqrt{1 - |\langle 0_{BS}|\alpha_{BS}\rangle|^2}}. \quad (23)$$

The corresponding which-path operators are then

$$\begin{aligned} K_a = \langle\alpha'_{BS}|\phi\rangle\langle 1_{in}| = \frac{1}{2}\sqrt{1 - |\langle 0_{BS}|\alpha_{BS}\rangle|^2}\,|1_{out}\rangle\langle 1_{in}| + \\ - \frac{i}{2}\sqrt{1 - |\langle 0_{BS}|\alpha_{BS}\rangle|^2}\,\langle\alpha'_{BS}|\alpha_{BS}\rangle|2_{out}\rangle\langle 1_{in}| \end{aligned} \quad (24)$$

and

$$\begin{aligned} K_b = \langle 0_{BS}|\Psi\rangle\langle 1_{in}| = \\ = -\frac{1}{2}\langle 0_{BS}|\alpha_{BS}\rangle|1_{out}\rangle\langle 1_{in}| - \langle 0_{BS}|\alpha_{BS}\rangle\frac{i}{2}|2_{out}\rangle\langle 1_{in}| + \\ - \frac{e^{i\phi}}{2}|1_{out}\rangle\langle 1_{in}| + \frac{ie^{i\phi}}{2}|2_{out}\rangle\langle 1_{in}|. \end{aligned} \quad (25)$$

The corresponding probabilities of the single paths become

$$\wp(d_1 \wedge a|s) = tr(F_{d_1}K_a\rho K^{\dagger}_a) = \frac{1}{4}\left(1 - |\langle 0_{BS}|\alpha_{BS}\rangle|^2\right) \quad (26)$$

$$\wp(d_1 \wedge b|s) = tr(F_{d_1}K_b\rho K^{\dagger}_b) = \frac{1}{4}\left|1 + e^{i\phi}\langle 0_{BS}|\alpha_{BS}\rangle\right|^2 = \quad (27)$$

$$= \frac{1}{4}\left[1 + |\langle 0_{BS}|\alpha_{BS}\rangle|^2 + 2\Re e(e^{i\phi}\langle 0_{BS}|\alpha_{BS}\rangle)\right]$$

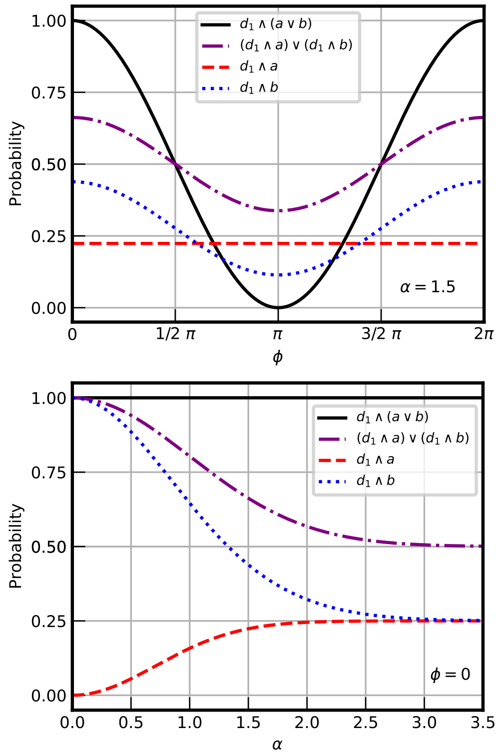


FIG. 5. Dependency of the different probability expressions as function of $\phi$ (with the fixed value $\alpha = 1.5$, top) and as function of $\alpha$ (with the fixed value $\phi = 0$, bottom).

Finally, we have then

$$\wp((d_1 \wedge a) \vee (d_1 \wedge b)|s) = \frac{1}{2}\left[1 + e^{-\frac{|\alpha|^2}{2}} \cos(\phi)\right], \quad (28)$$

where we used $\langle \alpha_{BS}|0_{BS}\rangle = e^{-\frac{|\alpha|^2}{2}}$.

As we can see in Fig. 5, for each probability relative to a specific path, an interference term is always present and is proportional to the overlap between the $|0\rangle_{BS}$ and $|\alpha_{BS}\rangle$ states. Only the probability corresponding to the path $b$ is sensitive to the phase of the $PS$.

In the limit $\langle 0_{BS}|\alpha_{BS}\rangle \to 0$ (corresponding to $\alpha \to \infty$ and $m \to 0$, see Fig. 5 bottom), we have a pure particle-like behavior with $\wp(d_1 \wedge a|s) = \wp(d_1 \wedge b|s) = 1/4$ and $\wp((d_1 \wedge a) \vee (d_1 \wedge b)|s) = 1/2$.

In the limit $\langle 0_{BS}|\alpha_{BS}\rangle \to 1$ (corresponding to $\alpha \to 0$ and $m \to \infty$, see Fig. 5 bottom), we have instead $\wp(d_1 \wedge a|s) = 0$ and $\wp(d_1 \wedge b|s) = 1$. From the detection of $|0\rangle_{BS}$, no information on the taken path can be extracted. In this limit case $\wp(d_1 \wedge b|s)$ (and then $\wp((d_1 \wedge a) \vee (d_1 \wedge b)|s)$) is *de facto* equivalent to $\wp(d_1 \wedge (a \vee b)|s)$ treated in the previous section. The behavior of the different formulas as function of $\phi$ and $\alpha$ is shown in Fig. 5.

Except to the limit case with $\langle 0_{BS}|\alpha_{BS}\rangle \to 0$ ($m \to 0$), the two equations (21) and (28) lead to different forms of the probability function. Once more, the expressions $d_1 \wedge (a \vee b)$ and $(d_1 \wedge a) \vee (d_1 \wedge b)$ cannot be considered equivalent with the violation of the distributivity property.

## V. CONCLUSION

In conclusion, we present a formulation of the probability function in the context of generalized measurements for subsequent detections with several possible paths. From the assumption of the Hilbert space structure for the description of systems, Gleason-Busch theorem assures that the trace of the density operator univocally defines the form of the probability function. Flatt and coworkers demonstrate that from this result, when subsequent measurements are considered, the Kraus updating rule is reconstructed. Here we apply the same methodology to a two-path case with a renewed notation. Two different expressions of the probability are found, $\wp((d \wedge a) \vee (d \wedge b)|s)$ and $\wp(d \wedge (a \vee b)|s)$, which are related to the possibility of distinguishing or not the trajectory in the measurement system. In fact, the distributive property of the probability function arguments cannot be taken for granted. From the first expression, the classical law or probability $Pr^C(a \vee b) = Pr^C(a) + Pr^C(b)$ is recovered. The use of the reduced trace over the undetected states of the path-detectors leads to this same expression.

With regards to the $\wp(d \wedge (a \vee b)|s)$, the associated operator $K_{a\vee b}$ to the $a \vee b$ measurement, can be interpreted ambiguously. $K_{a\vee b}$ can be built from a complementary measurement of $a$ and $b$ ($c = NOT(a \vee b)$) leading to a final expression corresponding to the standard Born rule for the case of perfect projective measurements. If $K_{a\vee b}$ is constructed by a mixing of path-detector states, we recover the situation of the *quantum eraser*. We are in fact considering the probability $\wp(d \wedge q|s)$ associated with the state $|q^{det}\rangle = \alpha\,|a^{det}\rangle + \beta\,|b^{det}\rangle$, which depends on the choice of constants $\alpha$ and $\beta$, i.e. a special case of which-path ignorance.

The frontier between the intrinsic possibility to distinguish a path or not is related to the coupling of the studied system with the path-detectors and/or the environment. This topic is widely studied in the literature, and in particular in the context of decoherence theory. Here we consider the very simple case of a Mach-Zehnder interferometer with a movable beam splitter, which is also well known in the literature but treated here in the context of generalized measurements. We demonstrate here that varying the mass of the beam splitter, we can continuously pass from the distinguishable path case, where $\wp((d \wedge a) \vee (d \wedge b)|s)$ is valid, to the indistinguishable path case, where $\wp(d \wedge (a \vee b)|s)$ should be used instead. This toy model reveals once more the complementarity of nature, but also underlines once more the advantages of generalized measurement theory with respect to ideal projective measurements, where unsharp detections revealing particle-like and wave-like behavior at the same time can unambiguously be treated.

## ACKNOWLEDGMENTS

I would like to thank very much M. Romanelli and M. Walschaers for their constructive critics to the previous version of the manuscript, but also C. Fabre, V. Parigi and N. Paul for the stimulating discussions and support. I would like also to thank A. Caticha and N. Carrara for the encouragement and suggestions after a first talk on a primordial version of the presented work.


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### Appendix A: Quantum eraser probabilities

We consider two path detector bases $|p_1\rangle, |p_2\rangle$, e.g. corresponding to $|a^{det}\rangle, |b^{det}\rangle$ states of the Young's slit case, and a final detection $d$. We consider a perfect which-path measurement where $|p_j\rangle$ are associated to the system state $|j\rangle$. The associated operators $K_j = \sum_i \langle p_j|p_i\rangle |i\rangle \langle i|$ is then equivalent to the projectors $P_j = |j\rangle \langle j|$. We consider two different orthogonal states $|q_1\rangle, |q_2\rangle$ related by the unitary transformation $|q_j\rangle = V_{ji} |p_i\rangle$. For each measurement $q_j$, the associated operator

$$K'_j = \sum_i \langle q_j|p_i\rangle |i\rangle \langle i| = \sum_i V_{ji} |i\rangle \langle i| . \tag{A1}$$

For each single measurement $q_j$, we have

$$\begin{aligned}
\wp(d \wedge q_j|s) &= tr(K_d K'_j \rho K'^{\dagger}_j K^{\dagger}_d) = \\
&= tr(K_d(V_{j1}P_1 + V_{j2}P_2)\rho_S(V^*_{j1}P_1 + V^*_{j2}P_2)P_d) = \\
&\quad tr((\sum_i V_{ji}P_i)\rho(\sum_{i'} V^*_{ji'}P_{i'})P_d) = \\
&= \sum_{i,i'} V_{ji}V^*_{ji'}\ tr(P_i\rho_S P_{i'} P_d) = \\
&= \sum_i |V_{ji}|^2\ tr(P_i\rho_S P_i P_d) + \sum_{i,i'\neq i} V_{ji}V^*_{ji'}\ tr(P_i\rho_S P_{i'} P_d).
\end{aligned} \tag{A2}$$

When we consider the probability relative to the measurement $(d \wedge q_1) \vee (d \wedge q_2)$, we have

$$\begin{aligned}
&\wp((d \wedge q_1) \vee (d \wedge q_2)|s) = \\
&= \sum_{j,i} |V_{ji}|^2\ tr(P_i\rho_S P_i P_d) + \sum_{ij,,i'\neq i} V_{ji}V^*_{ji'}\ tr(P_i\rho_S P_{i'} P_d) = \\
&= \sum_i tr(P_i\rho_S P_i P_d) + \sum_{i,i'\neq i,j} V_{ji}V^*_{ji'}\ tr(P_i\rho_S P_{i'} P_d)
\end{aligned} \tag{A3}$$

where we used the unitary matrix property $\sum_j |V_{ji}|^2 = 1$. The second term of the expression is in fact equal to zero because of other property of unitarity of $V$ matrices $\sum_j V_{ji}V^*_{ji'} = \delta_{i,i'}$ in a sum over $i, i' \neq i$. Finally we have

$$\begin{aligned}
\wp((d \wedge q_1) \vee (d \wedge q_2)|s) &= \sum_i tr(P_i\rho P_i P_d) = \\
&= \wp(d \wedge p_1) \vee (d \wedge p_2)|s).
\end{aligned} \tag{A4}$$

Independently of the choice of the orthogonal and complete base, probability is always the same and equal to a particle-like behavior, even if the single $d \wedge q_j$ measurement can provoke interference effects.