Geodesics and Visual boundary of Horospherical Products

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Abstract

Horospherical products of two hyperbolic spaces unify the construction of metric spaces such as the Diestel-Leader graphs, the SOL geometry or the treebolic spaces. Given two proper, geodesically complete, Gromov hyperbolic, Busemann spaces H_p and H_q, we study the geometry of their horospherical product $\mathcal{H} := H_p \bowtie H_q$ through a description of its geodesics.
Specifically we introduce a large family of distances on $H_p \bowtie H_q$. We show that all these distances produce the same large scale geometry. This description allows us to depict the shape of geodesic segments and geodesic lines. The understanding of the geodesics' behaviour leads us to the characterization of the visual boundary of the horospherical products. Our results are based on metric estimates on paths avoiding horospheres in a Gromov hyperbolic space.

Contents

1 Introduction

A horospherical product is a metric space constructed from two Gromov hyperbolic spaces H_p and H_q. It is included in their cartesian product $H_p \times H_q$ and can be seen as a diagonal in it. The definition of this horospherical product makes use of the so-called Busemann functions. Let us assume that there exists a unique geodesic ray k in H_p starting at a given base point $w_p \in H_p$ and leading in the direction of a given point on the boundary $a_p \in \partial H_p$. Then the Busemann function with respect to a_p and w_p

associates to a point $x_p \in H_p$ the delay it has in a race towards a_p against k. Given a base point and a point on the boundary on H_p and H_q we have two respective Busemann functions. We define the height functions of these spaces H_p and H_q to be the opposite of the Busemann functions. Hence the level-lines of the Busemann functions, which are called horospheres, are also the level-lines of the height functions. Then the horospherical product $\mathcal{H} = H_p \bowtie H_q$ is built by gluing H_p with an upside down copy of H_q along their respective horospheres. More precisely with a given height h_p on H_p (Definition 2.1) and a given height h_q on H_q, the horospherical product $\mathcal{H} = H_p \bowtie H_q$ is defined as follows.

$$\mathcal{H} := \{(x_p, x_q) \in H_p \times H_q \ / \ h_p(x_p) + h_q(x_q) = 0\}.$$

Since we are considering only couples of points with opposite heights in this set, we define the height on the horospherical product $\mathcal{H}$ as the height on the first component H_p. This notion of horospherical product generalizes the description of the Diestel-Leader graphs, the SOL geometry and the Cayley 2-complexes of Baumslag-Solitar groups $BS(1,n)$. In the second chapter of [14, Woess], the last three exemples are presented as horocyclic products of metric spaces. We choose the name horospherical product instead of horo*cyclic* product since in higher dimension, level-lines according to a Busemann function are not horocycles but horospheres. As Woess suggested in [14, W], we explore here a generalization for horospherical products.
The Diestel-Leader graphs are horospherical products of two regular trees. If the two trees' degree are equal, their horospherical product is the Cayley graph of a lamplighter group, see [13, Woess] for further details. A motivation to study this construction are the results from [4, Eskin, Fisher, Whyte], [5, E,F,W] and [6, E,F,W]. They state that the Diestel-Leader graphs constructed from two regular trees with no common divisor in their degree are vertex-transitive graphs which are not quasi-isometric to any Cayley Graphs. The existence of such a graph was a long open problem.
The SOL geometry, one of the eight Thurston geometries, is presented in [14, Woess] as the horospherical product of two hyperbolic planes. In [6, Eskin, Fisher, Whyte], they also prove rigidity results on lattices of the SOL geometry. A third example is related to the family of Baumslag-Solitar groups $BS(1,n)$, their Cayley 2-complex are described in [1, Bendikov, Saloff-Coste, Salvatori, Woess] as the horospherical product of a hyperbolic plane and a homogeneous tree. Similar rigidity results as in [4, Eskin, Fisher, Whyte] are presented in [7, Farb, Mosher] and [8, F,M] for Baumslag-Solitar groups.

For our generalization, we require that our components H_p and H_q are two proper geodesically complete Gromov hyperbolic Busemann spaces. A Busemann space is a metric space where the distance between any two geodesics is convex, and metric space X is geodesically complete if and only if a geodesic segment $\alpha : I \to X$ can be prolonged into a geodesic line $\hat{\alpha} : \mathbb{R} \to X$. The Busemann hypothesis suits with the definition of horospherical product since we require that the opposite heights are exactly equal. Furthermore, adding the hypothesis that H_p and H_q are geodesically complete allows us to prove that the horospherical product $\mathcal{H} = H_p \bowtie H_q$ is connected.

There are many possible choices for the distance on $H_p \bowtie H_q$. In this paper we work with a family of length path metrics induced by distances on $H_p \times H_q$ (see precise definition 4.2). We require that the distance on $H_p \bowtie H_q$ comes from a norm N on $\mathbb{R}^2$ that is greater than the normalized ℓ_1 norm. Such norms are called admissible norms. A description of the distances on horospherical products is given by Corollary 4.17. This corollary shows that any distance we described earlier provides the same geodesic shapes, up to an additive constant depending only on H_p, H_q and on the norm N. To do so we introduce a notion of vertical geodesics, which are geodesics heuristically "normal" to horospheres (see precise definition 4.7). The shapes of geodesic segments are described in the following theorem.

Theorem 1.1. *Let $\delta \geq 1$ and let N be an admissible norm. Let H_p and H_q be two proper, geodesically complete, δ-hyperbolic, Busemann spaces. Let $x = (x_p, x_q)$ and $y = (y_p, y_q)$ be two points of $\mathcal{H} = H_p \bowtie H_q$ and let α be a geodesic segment of $(\mathcal{H}, d_{\mathcal{H},N})$ linking x to y. There exists a constant κ depending only on δ and N, and there exist two vertical geodesics $V_1 = (V_{1,p}, V_{1,q})$ and $V_2 = (V_{2,p}, V_{2,q})$ such that:*

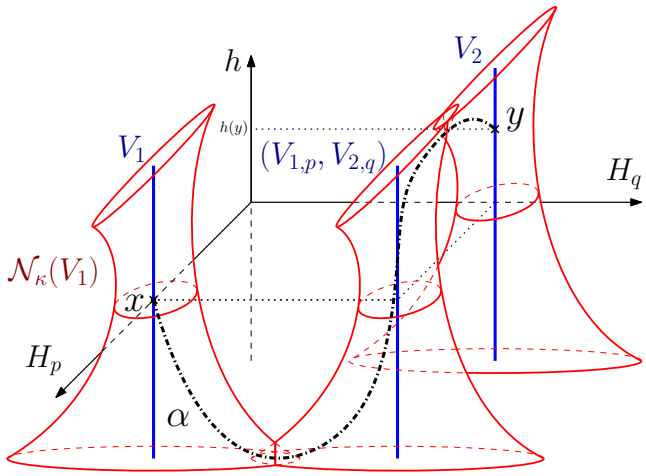

Figure 1: Shape of geodesic segments when $h(x) \leq h(y) - \kappa$ in $\mathcal{H} = H_p \bowtie H_q$. The neighbourhoods' shape are distorted since when going upward, distances are contracted in the "direction" H_p and expanded in the "direction" H_q.

1. *If* $h(x) \leq h(y) - \kappa$ *then* α *is in the* κ*-neighbourhood of* $V_1 \cup (V_{1,p}, V_{2,q}) \cup V_2$
2. *If* $h(x) \geq h(y) + \kappa$ *then* α *is in the* κ*-neighbourhood of* $V_1 \cup (V_{2,p}, V_{1,q}) \cup V_2$
3. *If* $|h(x) - h(y)| \leq \kappa$ *then at least one of the conclusions of* 1. *or* 2. *holds.*

Specifically, V_1 *and* V_2 *can be chosen such that* x *is close to* V_1 *and* y *is close to* V_2*.*

This behaviour is illustrated on Figure 1 for $h(x) \leq h(y) - \kappa$. This result is similar to the hyperbolic case, where a geodesic segment is in the constant neighbourhood of two vertical geodesics. The heuristic comprehension of Theorem 1.1 is, say in the case $h(x) \leq h(y) - \kappa$, that a geodesic segment travels first along a copy of the component H_q (which is upside down) as a geodesic in it, and last travels along a copy of the component H_p as a geodesic in it.
To prove Theorem 1.1 we need to control the lengths of the geodesics' projections on H_p and H_q. This work is done in section 3. The relative distance is defined as the distance minus the difference of height, it can be understood as the distance on horospheres. We first exhibit that in a hyperbolic space the maximal height of a geodesic segment and the relative distance between the end points of that geodesic segment are tightly related. We also have a lower bound on the length of paths avoiding horoballs as in Proposition 1.6 p400 of [2, Bridson, Haefliger]. Then we refine this last result into a control on the length of paths which avoid horoballs and which reach a given point. Since the projections on H_p and H_q of geodesics in $\mathcal{H}$ are such paths, Theorem 1.1 follows.
This result leads us to show the existence of unextendabled geodesics, which are called dead-ends. This was well known for lamplighter groups. This description of geodesic segments also allows us to prove that for any geodesic ray k of $\mathcal{H} = H_p \bowtie H_q$, there exists a vertical geodesic ray at finite Hausdorff distance. Therefore we classify all possible shapes for geodesic lines and then give a description of the visual boundary of $\mathcal{H}$. The notion of H_p-type and H_q-type geodesics at scale κ are described in Definition 5.10 and illustrated on Figure 2. They are essentially geodesics of $\mathcal{H}$ in a constant κ-neighbourhood of geodesics in a copy of H_p or in a copy of H_q in $\mathcal{H}$. We show that the geodesic lines of $H_p \bowtie H_q$ are either H_p-type, H_q-type or both.

Theorem 1.2. *Let* $\delta \geq 1$ *and let* N *be an admissible norm. Let* H_p *and* H_q *be two proper, geodesically complete,* δ*-hyperbolic, Busemann spaces. Let* $\mathcal{H} = H_p \bowtie H_q$ *be the horospherical product of* H_p *and* H_q*. Then there exists* $\kappa \geq 0$ *depending only on* δ *and* N *such that for all geodesic* $\alpha : \mathbb{R} \to \mathcal{H}$ *of* $(\mathcal{H}, d_{\mathcal{H},N})$ *at least one of the two following statements holds.*

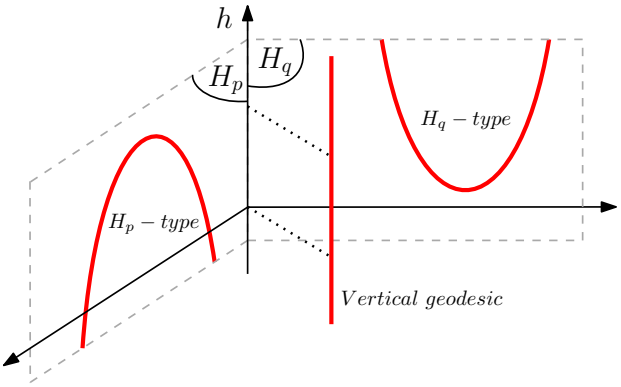

Figure 2: Different type of geodesics in $\mathcal{H} = H_p \bowtie H_q$.

1. *α is a H_p-type geodesic at scale κ of $\mathcal{H}$*
2. *α is a H_q-type geodesic at scale κ of $\mathcal{H}$*

If a geodesic is both H_p-type and H_q-type at scale κ, it is in the κ-neighbourhood of a vertical geodesic of $H_p \bowtie H_q$.

The notion of visual boundary of $\mathbb{H}^2 \bowtie \mathbb{H}^2$ is presented in the work of Troyanov in [12, Troyanov] through several definitions of horizons. We expand the definition and the description of the visual boundary in the general case of horospherical products as follows. Two geodesics are called asymptotic if they are at finite Hausdorff distance from each other. Let $o \in \mathcal{H}$, the visual boundary of $\mathcal{H}$ is then denoted by $\partial_o(\mathcal{H})$ and stands for the set of families of asymptotic geodesic rays starting at o. We have:

Theorem 1.3. *Let $\delta \geq 1$ and let H_p and H_q be two proper, geodesically complete, δ-hyperbolic, Busemann spaces. We fix base points and directions on H_p and H_q as follows, $(w_p, a_p) \in H_p \times \partial H_p$, $(w_q, a_q) \in H_q \times \partial H_q$. Let $\mathcal{H} = H_p \bowtie H_q$ be the horospherical product with respect to (w_p, a_p) and (w_q, a_q). Then the visual boundary of $\mathcal{H}$ with respect to a given point $o = (o_p, o_q)$ is:*

$$\begin{aligned}\partial_o\mathcal{H} =& \Big(\big(\partial H_p \setminus \{a_p\}\big) \times \{a_q\}\Big) \bigcup \Big(\{a_p\} \times \big(\partial H_q \setminus \{a_q\}\big)\Big) \\ =& \Big(\big(\partial H_p \times \{a_q\}\big) \bigcup \big(\{a_p\} \times \partial H_q\big)\Big) \setminus \{(a_p, a_q)\}\end{aligned}$$

This last result is similar to the Proposition 6.4 of [12, Troyanov]. However, unlike Troyanov in his work, we are focusing on minimal geodesics and not on local ones. One can see that this visual boundary neither depends on the chosen admissible norm N nor the base point o.

The figures of this paper depict lemmas and theorems when the two components H_p and H_q are hyperbolic planes $\mathbb{H}^2$, hence when $\mathcal{H} = H_p \bowtie H_q$ is the SOL geometry. In the 2 dimensional figures, we picture the vertical geodesics as getting closer when going upward since the distance contracts in this direction. In the 3 dimensional case we picture the vertical geodesics as straight lines in order to match with their shapes in the SOL geometry.

The paper is organised with first Section 2 which presents Gromov hyperbolic spaces, the notion of vertical geodesics in them and the impact of the Busemann hypothesis on the vertical geodesics. Then Section 3 provides us with an estimate on the length of paths avoiding horoballs in hyperbolic spaces, namely Lemma 3.8, which will be central in our control of the distances in $\mathcal{H}$. In Section 4 we define the horospherical products and give an estimate of their distance through Corollary 4.17. We hence discuss the fact that an entire family of distances are close to each others in $\mathcal{H}$. Last, in Section 5, we prove our three main results. Theorem 1.1 follows from the estimates of Corollary 4.17 on the length of geodesic segments. The description of geodesic lines of Theorem 1.2 follows from Theorem 1.1 and gives us the tools to prove Theorem 1.3.

2 Context

2.1 Gromov hyperbolic spaces

The goal of this section is to recall what is a Gromov hyperbolic space and what are vertical geodesics in such a space. Let H be a proper geodesic metric space, and d be a distance on H. A geodesic line of H is the isometric image of an Euclidean line in H. A geodesic ray of a metric space H is the isometric image of a half Euclidean line in H. A geodesic segment of a metric space H is the isometric image of an Euclidean interval in H. By slight abuse, we will call geodesic, geodesic ray or geodesic segment, a map $\alpha : I \to H$ which parametrises our given geodesic by arclength.

Let $\delta \geq 0$ be a non-negative number. Let x, y and z be three points of H. The geodesic triangle $[x, y] \cup [y, z] \cup [z, x]$ is called δ-slim if any of its sides is included in the δ-neighbourhood of the remaining two. The metric space H is called δ-hyperbolic if every geodesic triangle is δ-slim. A metric space H is called Gromov hyperbolic if there exists $\delta \geq 0$ such that H is a δ-hyperbolic space.

An important property of Gromov hyperbolic spaces is that they admit a nice compactification. Indeed the Gromov boundary allows that. We call two geodesic rays of H equivalent if their images are at finite Hausdorff distance. Let $o \in H$ be a base point. We define $\partial_o H$ the Gromov boundary of H as the set of families of equivalent rays starting from o. In fact, the boundary $\partial_o H$ does not depend on the base point o, hence we will simply denote it by ∂H. For more details, see [10, Ghys, De La Harpe] or chap.III H p.399 of [2, Bridson, Haefliger].

2.2 Vertical geodesics with respect to a boundary point

In this section we fix $\delta \geq 0$, H a proper geodesic δ-hyperbolic space, $w \in H$ a base point and $a \in \partial H$ a point on the boundary of H. We recall the definition of Busemann function firstly presented in the introduction.

$$\forall\ x \in H,\ \beta_a(x, w) = \sup\{\limsup_{t \to +\infty}(d(x, k(t)) - t) \mid k \in a,\ \text{starting from } w\}.$$

We want a notion of height on our hyperbolic spaces, a number tending to $+\infty$ when following a selected direction. It is the reason why we define the height on H as the opposite of the Busemann function.

Definition 2.1 (height with respect to $a \in \partial H$ and $w \in H$)**.** *Let $a \in \partial H$ be a direction in H and let $w \in H$ be a base point. Then we define:*

$$\forall x \in H, \quad h_{(a,w)}(x) = -\beta_a(x, w).$$

Let us write Proposition 2 chap.8 p.136 of [10, Ghys, De La Harpe] with our notations.

Proposition 2.2 ([10], chap.8 p.136)**.** *Let H be a hyperbolic proper geodesic metric space. Let $a \in \partial H$ and $w \in H$, then:*

1. $\lim_{x \to a} h_{(a,w)}(x) = +\infty$
2. $\lim_{x \to b} h_{(a,w)}(x) = -\infty$, $\forall b \in \partial H \setminus \{a\}$
3. $\forall x, y, z \in H, |\beta_a(x, y) + \beta_a(y, z) - \beta_a(x, z)| \leq 200\delta$.

Furthermore, a geodesic ray is in $a \in \partial H$ if and only if its height tends to $+\infty$.

Corollary 2.3. *Let H be a hyperbolic proper geodesic metric space. Let $a \in \partial H$ and $w \in H$, and let $\alpha : [0, +\infty[\to H$ be a geodesic ray. The two following properties are equivalent:*

1. $\lim_{t\to+\infty} h_{(a,w)}(\alpha(t)) = +\infty$

2. $\alpha([0,+\infty[) \in a$.

Proof. As for any geodesic ray $\alpha : [0,+\infty[\to H$ there exists $b \in \partial H$ such that $\alpha([0,+\infty[) \in b$, this proposition is a particular case of Proposition 2.2. □

We will picture our hyperbolic spaces in a way similar to the Log model for the hyperbolic plane. We send $a \in \partial H$ upward to infinity and $\partial H \setminus \{a\}$ downward to infinity. We then call vertical the geodesic rays that are in the equivalence class a.

Definition 2.4 (Vertical geodesics with repsect to $a \in \partial H$)**.** *A geodesic of H which satisfies one of the properties of Corollary 2.3 is called a vertical geodesic relatively to the point a.*

An important property of the height is to be Lipschitz.

Proposition 2.5. *Let $a \in \partial H$ and $w \in H$. The height function $h_a := -\beta_a(\cdot, w)$ is Lipschitz:*

$$\forall x, y \in H, |h_{(a,w)}(x) - h_{(a,w)}(y)| \leq d(x,y).$$

Proof. By using the triangular inequality we have for all $x, y \in H$:

$$\begin{aligned} -h_{(a,w)}(x) = \beta_a(x,w) &= \sup\{\limsup_{t\to+\infty}(d(x,k(t)) - t) \mid \text{k vertical rays starting at } w\} \\ &\leq d(x,y) + \sup\{\limsup_{t\to+\infty}(d(y,k(t)) - t) \mid \text{k vertical rays starting at } w\} \\ &\leq d(x,y) + \beta_a(y,w) \leq d(x,y) - h_{(a,w)}(y). \end{aligned}$$

The result follows by exchanging the roles of x and y. □

From now on, we fix a given $a \in \partial H$ and a given $w \in H$. Therefore we simply denote the height by h instead of $h_{(a,w)}$.

Proposition 2.6. *Let α be a vertical geodesic of H. We have the following control on the height along α:*

$$\forall t_1, t_2 \in \mathbb{R},\ t_2 - t_1 - 200\delta \leq h\big(\alpha(t_2)\big) - h\big(\alpha(t_1)\big) \leq t_2 - t_1 + 200\delta.$$

Proof. Let $t_1, t_2 \in \mathbb{R}$, then:

$$\begin{aligned} h(\alpha(t_2)) - h(\alpha(t_1)) &= \beta\big(\alpha(t_1), w\big) - \beta\big(\alpha(t_2), w\big) \\ &= \beta\big(\alpha(t_1), \alpha(t_2)\big) - \Big(\beta\big(\alpha(t_2), w\big) - \beta\big(\alpha(t_1), w\big) + \beta\big(\alpha(t_1), \alpha(t_2)\big)\Big). \end{aligned}$$

The third point of Proposition 2.2 applied to the last bracket gives:

$$\beta\big(\alpha(t_1), \alpha(t_2)\big) - 200\delta \leq h(\alpha(t_2)) - h(\alpha(t_1)) \leq \beta\big(\alpha(t_1), \alpha(t_2)\big) + 200\delta. \tag{1}$$

Since $t \mapsto \alpha(t + t_2)$ is a vertical geodesic starting at $\alpha(t_2)$ we have:

$$\begin{aligned} \beta\big(\alpha(t_1), \alpha(t_2)\big) &= \sup\Big\{ \limsup_{t\to+\infty} \big(d(\alpha(t_1), k(t)) - t\big) \Big| \text{k vertical rays starting at } \alpha(t_2)\Big\} \\ &\geq \limsup_{t\to+\infty} \Big(d\big(\alpha(t_1), \alpha(t + t_2)\big) - t\Big) \\ &\geq \limsup_{t\to+\infty} \big(|t + t_2 - t_1| - t\big) \geq t_2 - t_1, \text{ for t large enough.} \end{aligned}$$

Using this last inequality in inequality (1) we get $t_2 - t_1 - 200\delta \leq h(\alpha(t_2)) - h(\alpha(t_1))$. The result follows by exchanging the roles of t_1 and t_2. □

Using Proposition 2.6 with $t_1 = 0$ and $t_2 = t$, the next corollary holds.

Corollary 2.7. *Let α be a vertical geodesic parametrised by arclength and such that $h(\alpha(0)) = 0$. We have:*

$$\forall t \in \mathbb{R},\ |h(\alpha(t)) - t| \leq 200\delta.$$

In the sequel we want to apply the slim triangles property on ideal triangles, hence we need the following result of [3, Coornaert, Delzant, Papadopoulos].

Property 2.8 (Proposition 2.2 page 19 of [3])**.** *Let a, b and c be three points of $X \cup \partial X$. Let α, β, γ be three geodesics of X linking respectively b to c, c to a, and a to b. Then every point of α is at distance less than 24δ from the union $\beta \cup \gamma$.*

2.3 Busemann spaces

We recall here some material from Chap.8 and Chap.12 of [11, Papadopoulos] about Busemann spaces. Busemann spaces are metric spaces where the distance between geodesics are convex functions. To make it more precise, a metric space X is called Busemann if it is geodesic, and if for every pair of geodesics parametrized by arclength $\gamma : [a, b] \to X$ and $\gamma' : [a', b'] \to X$, the following function is convex:

$$\begin{aligned} D_{\gamma,\gamma'} : [a, b] \times [a', b'] &\to X \\ (t, t') &\mapsto d_X(\gamma(t), \gamma'(t')). \end{aligned}$$

As an example, all $CAT(0)$ spaces are Busemann spaces. However, being $CAT(0)$ is stronger than being Busemann convex by Theorem 1.3 of [9, Foertsch, Lytchak, Schroeder]. As an example, strictly convex Banach spaces are all Busemann spaces, but they are $CAT(0)$ if and only if they are Hilbert spaces. Something interesting in Busemann spaces is that two points are always linked by a unique geodesic (see 8.1.4 p.203 of Papadopoulos [11, Papadopoulos] for further details). The next proposition gives us informations on the height functions.

Property 2.9 (Prop. 12.1.5 in p.263 of Papadopoulos [11])**.** *Let $\delta \geq 0$ be a non negative number. Let H be a proper δ-hyperbolic, Busemann space. For every geodesic α, the function $t \mapsto -h(\alpha(t))$ is convex.*

From now on, H will be a proper, Gromov hyperbolic, Busemann space. The Busemann hypothesis implies that the height along geodesic behaves nicely. This means that we can drop the constant 200δ from Corollary 2.7. It is the main reason why we require our spaces to be Busemann spaces.

Proposition 2.10. *Let H be a δ-hyperbolic and Busemann space and let V be a path of H. Then V is a vertical geodesic if and only if $\exists c \in \mathbb{R}$ such that $\forall t \in \mathbb{R},\ h(V(t)) = t + c$.*

Proof. Let V be a vertical geodesic in H. By Property 2.9 we have that $t \mapsto -h(V(t))$ is convex. Furthermore, from Corollary 2.7, we get $|h(V(t)) - t| \leq 200\delta$. Thereby the bounded convex function $t \mapsto t - h(V(t))$ is constant. Then there exists a real number c such that $\forall t \in \mathbb{R},\ h(V(t)) = t + c$.
We now assume that there exists a real number c such that $\forall t \in \mathbb{R},\ h(V(t)) = t + c$. Therefore, for all real numbers t_1 and t_2 we have $d\big(V(t_1), V(t_2)\big) \geq \Delta h\big(V(t_1), V(t_2)\big) = |t_1 - t_2|$. By definition V is a connected path, hence $|t_1 - t_2| \geq d\big(V(t_1), V(t_2)\big)$ which implies with the previous sentence that $|t_1 - t_2| = d\big(V(t_1), V(t_2)\big)$, then V is a geodesic. Furthermore $\lim\limits_{t \to +\infty} h(V(t)) = +\infty$, which implies by definition that V is a vertical geodesic. □

A metric space is called geodesically complete if all its geodesic segments can be prolonged into geodesic lines. By adding the hypothesis of geodesically completeness on a hyperbolic Busemann space H we get that any point of H is included in a vertical geodesic line.

Property 2.11. *Let H be a δ-hyperbolic Busemann geodesically complete space. Then for all $x \in H$ there exists a vertical geodesic $V_x : \mathbb{R} \to H$ such that V_x contains x*

Proof. Let us consider in this proof $w \in H$ and $a \in \partial H$, from which we constructed the height h of our space H. Then by definition we have $h_{(a,w)} = h$. Proposition 12.2.4 of [11, Papadopoulos] ensures the existence of a geodesic ray $R_x \in a$ starting at x. Furthermore as H is geodesically complete R_x can be prolonged into a geodesic $V_x : \mathbb{R} \to H$ such that $V_x([0; +\infty[) \in a$. Hence V_x is a vertical geodesic from Definition 2.4. □

In this section we defined all the objects we will use in hyperbolic spaces. We will now focus on proving length estimates on specific paths. They will appear in Section 4 as the projection of geodesics in a horospherical product.

3 Metric estimates in Gromov hyperbolic Busemann spaces

3.1 Metric description of geodesics

This section focuses on length estimates in Gromov hyperbolic Busemann spaces. The central result is Lemma 3.8, which present a lower bound on the length of a path staying between two horospheres. Before moving to the technical results of this section, let us introduce some notations.

Notation 3.1. *Unless otherwise specified, H will be a Gromov hyperbolic Busemann geodesically complete proper space. Let $\gamma : I \to H$ be a connected path. Let us denote the maximal height and the minimal height of this path as follows:*

$$h^+(\gamma) = \sup_{t \in I} \{h(\gamma(t))\},$$
$$h^-(\gamma) = \inf_{t \in I} \{h(\gamma(t))\}.$$

Let x and y be two points of H, we denote the height difference between them by:

$$\Delta h(x, y) = |h(x) - h(y)|.$$

We define the relative distance between two points x and y of H as:

$$d_r(x, y) = d(x, y) - \Delta h(x, y).$$

Let us denote V_x a vertical geodesic containing x, we will consider it to be parametrised by arclength. Thanks to Proposition 2.10 we choose a parametrisation by arclength such that $\forall t \in \mathbb{R},\ h(V_x(t)) = t + 0$.

The relative distance between two points quantifies how far a point is from the nearest vertical geodesic containing the other point. Next lemma tells us that in order to connect two points a geodesic needs to go sufficiently high. This height is controlled by the relative distance between those two points.

Lemma 3.2. *Let H be a δ-hyperbolic and Busemann metric space, let x and y be two elements of H such that $h(x) \leq h(y)$, and let α be a geodesic linking x to y. Let us denote $z = \alpha\left(\Delta h(x, y) + \frac{1}{2} d_r(x, y)\right)$, $x_1 := V_x\left(h(y) + \frac{1}{2} d_r(x, y)\right)$ the point of V_x at height $h(y) + \frac{1}{2} d_r(x, y)$ and $y_1 := V_y\left(h(y) + \frac{1}{2} d_r(x, y)\right)$ the point of V_y at the same height $h(y) + \frac{1}{2} d_r(x, y)$. Then we have:*

1. $h^+(\alpha) \geq h(y) + \frac{1}{2} d_r(x, y) - 96\delta$
2. $d(z, x_1) \leq 144\delta$
3. $d(z, y_1) \leq 144\delta$
4. $d(x_1, y_1) \leq 288\delta$.

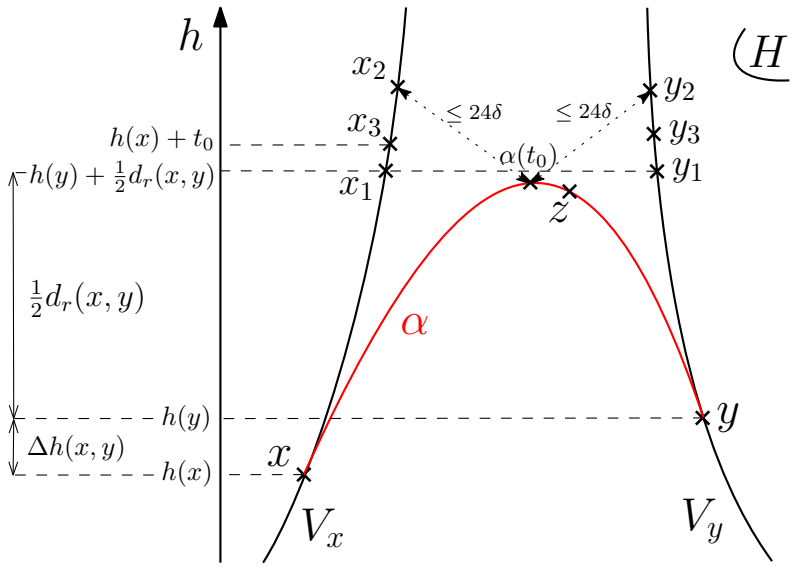

Figure 3: Proof of Lemma 3.2

Proof. The lemma and its proof are illustrated on Figure 3. Following Property 2.8, the triple of geodesics α, V_x and V_y is a 24δ-slim triangle. Since the sets $\{t \in [0, d(x,y)] | d(\alpha(t), V_x) \leq 24\delta\}$ and $\{t \in [0, d(x,y)] | d(\alpha(t), V_y) \leq 24\delta\}$ are closed sets covering $[0, d(x,y)]$, their intersection is non empty. Hence there exists $t_0 \in [0, d(x,y)]$, $x_2 \in V_x$ and $y_2 \in V_y$ such that $d(\alpha(t_0), x_2) \leq 24\delta$ and $d(\alpha(t_0), y_2) \leq 24\delta$. Let us first prove that t_0 is close to $\Delta h(x,y) + \frac{1}{2}d_r(x,y)$. By the triangular inequality we have that:

$$|t_0 - d(x, x_2)| = |d(x, \alpha(t_0)) - d(x, x_2)| \leq d(x_2, \alpha(t_0)) \leq 24\delta.$$

Let us denote $x_3 := V_x(h(x) + t_0)$ the point of V_x at height $h(x) + t_0$, and $y_3 = V_y(h(y) + d(x,y) - t_0)$ the point of V_y at height $h(y) + d(x,y) - t_0$. Then by the triangular inequality:

$$\begin{aligned} d(\alpha(t_0), x_3) &\leq d(\alpha(t_0), x_2) + d(x_2, x_3) = d(\alpha(t_0), x_2) + |d(x, x_2) - d(x, x_3)| \\ &\leq d(\alpha(t_0), x_2) + |d(x, x_2) - t_0| \leq 48\delta. \end{aligned} \tag{2}$$

In the last inequality we used that $d(x, x_3) = t_0$, which holds by the definition of x_3. We show in the same way that $d(\alpha(t_0), y_3) \leq 48\delta$. By the triangular inequality we have $d(x_3, y_3) \leq 96\delta$. As the height function is Lipschitz we have $\Delta h(x_3, y_3) \leq d(x_3, y_3) \leq 96\delta$, which provides us with:

$$\begin{aligned} \left|\frac{1}{2}d_r(x,y) + \Delta h(x,y) - t_0\right| &= \frac{1}{2}\left|d_r(x,y) + \Delta h(x,y) + h(y) - h(x) - 2t_0\right| \\ &= \frac{1}{2}|h(y) + d(x,y) - t_0 - (h(x) + t_0)| = \frac{1}{2}\Delta h(x_3, y_3) \leq \frac{96\delta}{2} \leq 48\delta. \end{aligned} \tag{3}$$

In particular it gives us that $d(z, \alpha(t_0)) \leq 48\delta$. We are now ready to prove the first point using inequalities (2) and (3):

$$\begin{aligned} h^+(\alpha) &\geq h(\alpha(t_0)) \geq h(x_3) - \Delta h(\alpha(t_0), x_3) \geq h(x) + t_0 - 48\delta \\ &\geq h(x) + \frac{1}{2}d_r(x,y) + \Delta h(x,y) - 96\delta \geq h(y) + \frac{1}{2}d_r(x,y) - 96\delta, \text{ as we have } h(x) \leq h(y). \end{aligned}$$

The second point of our lemma is proved by the sequel:

$$\begin{aligned} d(z, x_1) &\leq d(z, \alpha(t_0)) + d(\alpha(t_0), x_1) \leq 48\delta + d(\alpha(t_0), x_3) + d(x_3, x_1) \\ &\leq 96\delta + \left|t_0 + h(x) - \left(\frac{1}{2}d_r(x,y) + h(y)\right)\right| = 96\delta + \left|t_0 - \left(\Delta h(x,y) + \frac{1}{2}d_r(x,y)\right)\right| \leq 144\delta. \end{aligned}$$

The proof of 3. is similar, and 4. is obtained from 2. and 3. by the triangular inequality. □

The next lemma shows that in the case where $h(x) \leq h(y)$ a geodesic linking x to y is almost vertical until it reaches the height $h(y)$.

Lemma 3.3. *Let H be a δ-hyperbolic and Busemann space. Let x and y be two points of H such that $h(x) \leq h(y)$. We define $x' := V_x(h(y))$ to be the point of the vertical geodesic V_x at the same height as y. Then:*

$$|d_r(x,y) - d(x',y)| \leq 54\delta. \tag{4}$$

Proof. Since H is δ-hyperbolic, the geodesic triangle $[x,y] \cup [y,x'] \cup [x',x]$ is δ-slim. Then there exists $p_1 \in [x,x']$, $p_2 \in [x',y]$ and $m \in [x,y]$ such that $d(p_1,m) \leq \delta$ and $d(p_2,m) \leq \delta$. Hence, $h^-([x',y]) - \delta \leq h(m) \leq h^+([x,x']) + \delta$. Let $R_{x'}$ and R_y be two vertical geodesic rays respectively contained in V_x and V_y and respectively starting at x' and y. Then Property 2.8 used on the ideal triangle $R_x \cup R_y \cup [x',y]$ implies that $h^-([x',y]) \geq h(y) - 24\delta$, therefore we have $h^+([x,x']) = h(y)$. Then $h(y) - 25\delta \leq h(m) \leq h(y) + \delta$ holds. It follows that m and x' are close to each other:

$$\begin{aligned} d(m,x') &\leq d(m,p_1) + d(p_1,x') \leq \delta + \Delta h(p_1,x') \leq \delta + \Delta h(p_1,m) + \Delta h(m,y) + \Delta h(y,x') \\ &\leq \delta + d(p_1,m) + 25\delta + 0 \leq 27\delta. \end{aligned} \tag{5}$$

Then we give an estimate on the distance between x and m:

$$|d(x,m) - \Delta h(x,y)| = |d(x,m) - d(x,x')| \leq d(m,x') \leq 27\delta. \tag{6}$$

However $d_r(x,y) = d(x,y) - \Delta h(x,y)$ and $d(x,y) = d(x,m) + d(m,y)$, therefore:

$$d_r(x,y) = d(x,m) + d(m,y) - \Delta h(x,y). \tag{7}$$

Combining inequalities (6) and (7) we have $|d_r(x,y) - d(m,y)| \leq 27\delta$. Then:

$$|d_r(x,y) - d(x',y)| \leq 27\delta + d(x',m) \leq 54\delta.$$

□

The lemmas of this last section allow us to prove the estimate lemmas of the next one.

3.2 Length estimate of paths avoiding horospheres

Consider a path γ and a geodesic α that links the two same points of a proper, Gromov hyperbolic, Busemann space. We prove in this section that if the height of γ does not reach the maximal height of the geodesic α, then γ is much longer than α. Furthermore, its length increases exponentially on the difference of maximal height between γ and α. To do so we need Proposition 1.6 p400 of [2, Bridson, Haefliger]. We denote by $l(c)$ the length of a path c.

Proposition 3.4 ([2])**.** *Let X be a δ-hyperbolic geodesic space. Let c be a continuous path in X. If $[p,q]$ is a geodesic segment connecting the endpoints of c, then for every $x \in [p,q]$:*

$$d(x, im(c)) \leq \delta |\log_2 l(c)| + 1.$$

This result implies that a path of H between x and y which avoids the ball centred in the middle of a geodesic $[x,y]$ has length greater than an exponential in the distance $d(x,y)$. From now on we will add as convention that $\delta \geq 1$. For all $\delta_1 \leq \delta_2$ a δ_1-slim triangle is also δ_2-slim, hence all δ_1-hyperbolic spaces are δ_2-hyperbolic spaces. That is why we can assume that all Gromov hyperbolic spaces are δ-hyperbolic with $\delta \geq 1$. It allows us to consider $\frac{1}{\delta}$ as a well defined term, we hence avoid different cases in the proof of the following lemma. We also use this assumption to simplify constants appearing in the proof. The next result is a similar control on the length of path as Proposition 3.4, but we consider that the path is avoiding a horosphere instead of avoiding a ball in H.

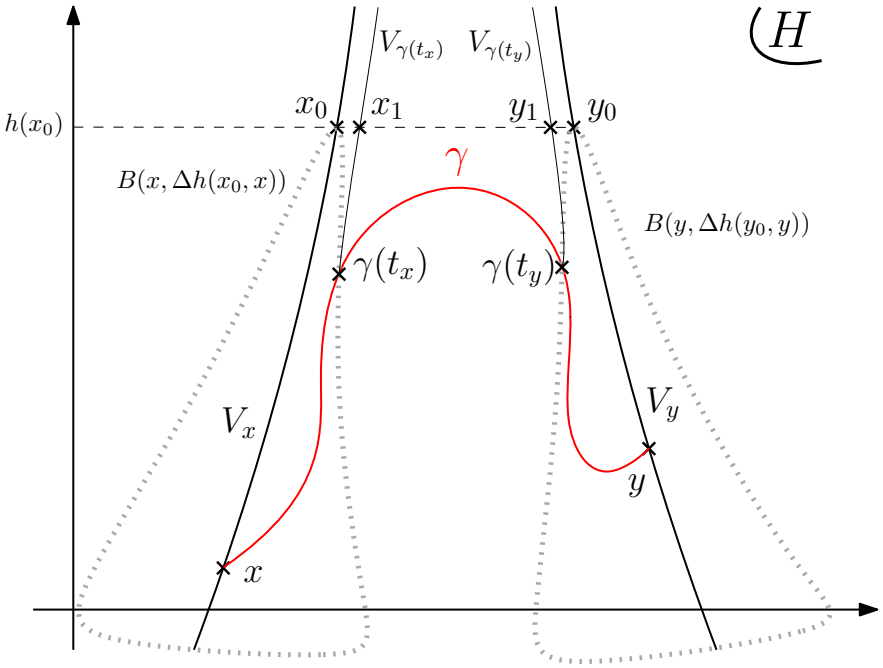

Figure 4: Proof of Lemma 3.5

Lemma 3.5. *Let $\delta \geq 1$ and H be a proper, δ-hyperbolic, Busemann space. Let x and $y \in H$ and let V_x (respt. V_y) be a vertical geodesic containing x (respt. y). Let us consider $t_0 \geq \max(h(x), h(y))$ and let us denote $x_0 := V_x(t_0)$ and $y_0 := V_y(t_0)$. Assume that $d(x_0, y_0) > 768\delta$.*
Then for all connected path $\gamma : [0, T] \to H$ such that $\gamma(0) = x$, $\gamma(T) = y$ and $h^+(\gamma) \leq h(x_0)$ we have:

$$l(\gamma) \geq \Delta h(x, x_0) + \Delta h(y, y_0) + 2^{-386} 2^{\frac{1}{2\delta} d(x_0, y_0)} - 24\delta. \tag{8}$$

For trees when $\delta = 0$ this Lemma still makes sense. Indeed, if δ tends to 0 then the length of the path described in this Lemma tends to infinity, which is consistent with the fact that such a path does not exist in trees. The proof would use the fact that in Proposition 3.4 we have $d(x, im(c)) = 0$ when $\delta = 0$ since 0-hyperbolic spaces are real trees.

Proof. One can follow the idea of the proof on Figure 4. We will consider γ to be parametrised by arclength. Let $B(x, \Delta h(x_0, x)) \subset H$ be the ball of radius $h(x_0) - h(x)$ centred on x, and let $m \in B(x, \Delta h(x_0, x))$ be a point in this ball. Then:

$$d_r(m, x) = d(m, x) - \Delta h(m, x) \leq \Delta h(x, x_0) - \Delta h(m, x) \leq \Delta h(x_0, m).$$

Let us first assume that $h(m) \geq h(x)$, then:

$$h(m) + \frac{d_r(m, x)}{2} \leq h(m) + \frac{\Delta h(x_0, m)}{2} \leq h(m) + \frac{h(x_0) - h(m)}{2} = \frac{h(x_0)}{2} + \frac{h(m)}{2} \leq h(x_0). \tag{9}$$

By Lemma 3.2 we have:

$$d\left(V_x\left(h(m) + \frac{d_r(m, x)}{2}\right), V_m\left(h(m) + \frac{d_r(m, x)}{2}\right)\right) \leq 288\delta.$$

We now assume that $h(m) \leq h(x)$, then:

$$h(x) + \frac{d_r(x, m)}{2} \leq h(x) + \frac{d(x, m)}{2} \leq h(x) + \frac{\Delta h(x, x_0)}{2} \leq h(x_0).$$

Then Lemma 3.2 provides:

$$d\left(V_x\left(h(x)+\frac{d_r(m,x)}{2}\right),V_m\left(h(x)+\frac{d_r(m,x)}{2}\right)\right)\leq 288\delta.$$

Since H is a Busemann space, the function $t\to d(V_x(t),V_m(t))$ is convex. Furthermore $t\to d(V_x(t),V_m(t))$ is bounded on $[0;+\infty[$ as H is Gromov hyperbolic, hence $t\to d(V_x(t),V_m(t))$ is a non increasing function. Therefore both cases $h(m)\leq h(x)$ and $h(x)\leq h(m)$ give us that:

$$d\Big(x_0,V_m\left(h(x_0)\right)\Big)=d\Big(V_x\left(h(x_0)\right),V_m(h(x_0))\Big)\leq 288\delta. \tag{10}$$

In other words, all points of $B(x,\Delta h(x_0,x))$ belong to a vertical geodesic passing nearby x_0. By the same reasoning we have $\forall n\in B(y,\Delta h(y_0,y))$:

$$d\Big(y_0,V_n\left(h(y_0)\right)\Big)\leq 288\delta. \tag{11}$$

Then by the triangular inequality:

$$\begin{aligned}d\Big(V_m(h(x_0)),V_n(h(y_0))\Big)&\geq -d\Big(x_0,V_m\left(h(x_0)\right)\Big)+d(x_0,y_0)-d\Big(y_0,V_n\left(h(y_0)\right)\Big)\\&\geq 768\delta-288\delta-288\delta\geq 192\delta.\end{aligned} \tag{12}$$

Specifically $d(V_m(h(x_0)),V_n(h(y_0)))=d(V_m(h(x_0)),V_n(h(x_0)))>0$ which implies that $m\neq n$. Then $B(x,\Delta h(x_0,x))\cap B(y,\Delta h(y_0,y))=\varnothing$. By continuity of γ we deduce the existence of the two following times $t_x\leq t_y$ such that:

$$\begin{aligned}t_x&=\inf\{t\in[0,T]\mid d(\gamma(t),x)=\Delta h(x,x_0)\},\\t_y&=\sup\{t\in[0,T]\mid d(\gamma(t),y)=\Delta h(y,y_0)\}.\end{aligned}$$

In order to have a lower bound on the length of γ we will need to split this path into three parts:

$$\gamma=\gamma_{|[0,t_x]}\cup\gamma_{|[t_x,t_y]}\cup\gamma_{|[t_y,T]}.$$

As γ is parametrised by arclength and $d(\gamma(0),\gamma(t_x))=\Delta h(x,x_0)$ we have that:

$$l\left(\gamma_{|[0,t_x]}\right)\geq\Delta h(x,x_0). \tag{13}$$

For similar reasons we also have:

$$l\left(\gamma_{|[t_y,T]}\right)\geq\Delta h(y,y_0). \tag{14}$$

We will now focus on proving a lower bound for the length of $\gamma_{|[t_x,t_y]}$.

We want to construct a path γ' joining $x_1=V_{\gamma(t_x)}(h(x_0))$ to $y_1=V_{\gamma(t_y)}(h(x_0))$, that stays below $h(x_0)$ and such that $\gamma_{|[t_x,t_y]}$ is contained in γ'. Let $x_1:=V_{\gamma(t_x)}(h(x_0))$ and $y_1:=V_{\gamma(t_y)}(h(x_0))$. We construct γ' by gluing paths together:

$$\gamma'=\begin{cases}V_{\gamma(t_x)} & \text{from } x_1 \text{ to } \gamma(t_x)\\ \gamma & \text{from } \gamma(t_x) \text{ to } \gamma(t_y)\\ V_{\gamma(t_y)} & \text{from } \gamma(t_y) \text{ to } y_1\end{cases}$$

Applying inequalities (10) and (11) used on $\gamma(t_x)$ and $\gamma(t_y)$ we get:

$$d(x_0,x_1)\leq 288\delta, \tag{15}$$

$$d(y_0,y_1)\leq 288\delta. \tag{16}$$

In order to apply Proposition 3.4 to γ' we need to check that there exists a point A of the geodesic segment $[x_1, y_1]$ such that $h(A) \geq h(x_0)$. Applying Lemma 3.2 to $[x_1, y_1]$ and since $h(x_1) = h(y_1)$ we get:

$$h^+([x_1, y_1]) \geq \frac{d_r(x_1, y_1)}{2} + h(x_0) - 96\delta = \frac{d(x_1, y_1)}{2} + h(x_0) - 96\delta.$$

Thanks to the triangular inequality and inequalities (15) and (16):

$$h^+([x_1, y_1]) \geq \frac{d(y_0, x_0) - d(x_0, x_1) - d(y_0, y_1)}{2} + h(x_0) - 96\delta \geq \frac{d(x_0, y_0)}{2} + h(x_0) - 384\delta.$$

Since by hypothesis $d(x_0, y_0) > 768\delta$, there exists a point A of $[x_1, y_1]$ exactly at the height:

$$h(A) = \frac{d(x_0, y_0)}{2} + h(x_0) - 384\delta.$$

We can then apply Proposition 3.4 to get:

$$\begin{aligned}\delta|\log_2(l(\gamma'))| + 1 \geq d(A, \gamma') \geq \Delta h(A, x_0) &\geq \frac{d(x_0, y_0)}{2} + h(x_0) - 384\delta - h(x_0) \\ &\geq \frac{d(x_0, y_0)}{2} - 384\delta.\end{aligned}$$

Since $\delta \geq 1$, last inequality implies that $l(\gamma') \geq 2^{-385} 2^{\frac{1}{2\delta} d(x_0, y_0)}$. Now we use this inequality to have a lower bound on the length of $\gamma_{|[t_x, t_y]}$:

$$\begin{aligned}l(\gamma_{|[t_x, t_y]}) &\geq l(\gamma') - \Delta h(\gamma(t_x), x_0) - \Delta h(\gamma(t_y), y_0) \\ &\geq 2^{-385} 2^{\frac{1}{2\delta} d(x_0, y_0)} - \Delta h(\gamma(t_x), x_0) - \Delta h(\gamma(t_y), y_0). \qquad (17)\end{aligned}$$

We claim that $l\left(\gamma_{|[t_x, t_y]}\right) \geq \Delta h(\gamma(t_x), x_0) + \Delta h(\gamma(t_y), y_0) - 48\delta$, hence:

$$l\left(\gamma_{|[t_x, t_y]}\right) \geq 2^{-386} 2^{\frac{1}{2\delta} d(x_0, y_0)} - 24\delta, \qquad (18)$$

which ends the proof by combining inequality (18) with inequalities (13) and (14).

Proof of the claim. Inequality (12) with $m = \gamma(t_x)$ and $n = \gamma(t_y)$ gives $d(x_1, y_1) \geq 192\delta$. We want to prove that $h^+([\gamma(t_x), \gamma(t_y)]) \geq h(x_1) - 24\delta$. First, by Lemma 2.8 we have that $[\gamma(t_x), \gamma(t_y)] \cup V_{\gamma(t_x)} \cup V_{\gamma(t_y)}$ is a 24δ-slim triangle. Then there exist three times t_0, t_1 and t_2 such that $d\left(V_{\gamma(t_x)}(t_1), \gamma(t_0)\right) \leq 24\delta$ and such that $d\left(V_{\gamma(t_y)}(t_2), \gamma(t_0)\right) \leq 24\delta$. Then:

$$\begin{aligned}|t_1 - t_2| = \Delta h\left(V_{\gamma(t_x)}(t_1), V_{\gamma(t_y)}(t_2)\right) &\leq d\left(V_{\gamma(t_x)}(t_1), V_{\gamma(t_y)}(t_2)\right) \\ &\leq d\left(V_{\gamma(t_x)}(t_1), \gamma(t_0)\right) + d\left(\gamma(t_0), V_{\gamma(t_y)}(t_2)\right) \leq 48\delta. \qquad (19)\end{aligned}$$

We will show by contradiction that either $t_1 = h(V_{\gamma(t_x)}(t_1)) \geq h(x_0)$ or $t_2 = h(V_{\gamma(t_y)}(t_2)) \geq h(x_0)$. Assume that $t_1 < h(x_0)$ and $t_2 < h(x_0)$. Then by the triangular inequality:

$$\begin{aligned}d\big(V_{\gamma(t_x)}(t_1), V_{\gamma(t_y)}(t_2)\big) &\geq d\big(V_{\gamma(t_y)}(t_2), V_{\gamma(t_x)}(t_2)\big) - d\big(V_{\gamma(t_x)}(t_2), V_{\gamma(t_x)}(t_1)\big) \\ &\geq d\big(V_{\gamma(t_y)}(t_2), V_{\gamma(t_x)}(t_2)\big) - 48\delta, \text{ since } |t_1 - t_2| \leq 48\delta \text{ by equation (19).}\end{aligned}$$

As H is a Busemann space, the function $t \mapsto d\big(V_{\gamma(t_x)}(t), V_{\gamma(t_y)}(t)\big)$ is non increasing. Furthermore, $h(x_0) \geq t_2$ hence:

$$\begin{aligned}48\delta &\geq d\big(V_{\gamma(t_x)}(t_1), V_{\gamma(t_x)}(t_2)\big) \geq d\big(V_{\gamma(t_x)}(t_2), V_{\gamma(t_y)}(t_2)\big) - 48\delta \\ &\geq d\big(V_{\gamma(t_x)}(h(x_0)), V_{\gamma(t_y)}(h(x_0))\big) - 48\delta \geq d(x_1, y_1) - 48\delta \\ &\geq d(x_0, y_0) - d(x_0, x_1) - d(y_0, y_1) - 48\delta \geq d(x_0, y_0) - 624\delta, \text{ by inequalities (15) and (16),} \\ &\geq 49\delta, \text{ since } d(x_0, y_0) \geq 768\delta \text{ by hypothesis,}\end{aligned}$$

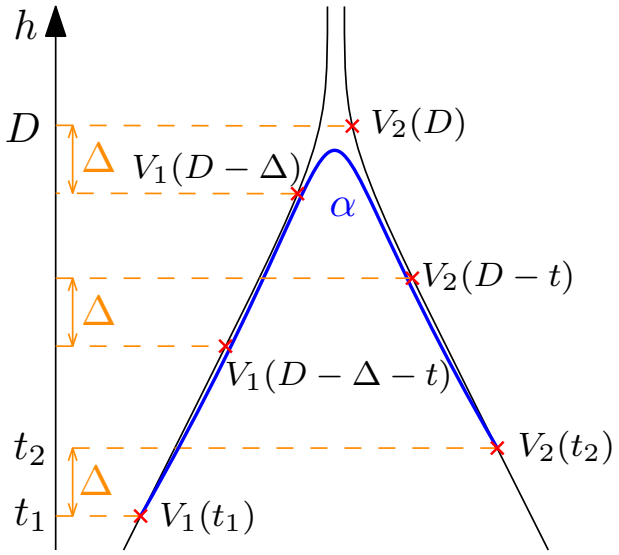

Figure 5: Proof of Lemma 3.6

which is impossible. Therefore $t_1 \geq h(x_0)$ or $t_2 \geq h(x_0)$. We assume without loss of generality that $t_1 \geq h(x_0)$, then:

$$\Delta h\big(\gamma(t_0), V_{\gamma(t_x)}(t_1)\big) \leq d\big(\gamma(t_0), V_{\gamma(t_x)}(t_1)\big) \leq 24\delta,$$

which implies:

$$h^+([\gamma(t_x), \gamma(t_y)]) \geq h(\gamma(t_0)) \geq h\left(V_{\gamma(t_x)}(t_1)\right) - \Delta h\big(\gamma(t_0), V_{\gamma(t_x)}(t_1)\big) \geq h(x_0) - 24\delta,$$

and gives us:

$$\begin{aligned} l\left(\gamma_{|[t_x,t_y]}\right) &\geq h^+([\gamma(t_x), \gamma(t_y)]) - h(\gamma(t_x)) + h^+([\gamma(t_x), \gamma(t_y)]) - h(\gamma(t_y)) \\ &\geq h(x_0) - 24\delta - h(\gamma(t_x)) + h(x_0) - 24\delta - h(\gamma(t_y)) \\ &\geq \Delta h(\gamma(t_x), x_0) + \Delta h(\gamma(t_y), y_0) - 48\delta. \end{aligned} \tag{20}$$

□

Next lemma shows that we are able to control the relative distance of a couple of points travelling along two vertical geodesics.

Lemma 3.6 (Backwards control). *Let $\delta \geq 0$ and H be a proper, δ-hyperbolic, Busemann space. Let V_1 and V_2 be two vertical geodesics of H. Then for all couple of times (t_1, t_2) and for all $t \in \left[0, \frac{1}{2} d_r(V_1(t_1), V_2(t_2))\right]$:*

$$\left| d_r\left(V_1\left(t_1 + \frac{1}{2} d_r(V_1(t_1), V_2(t_2)) - t\right), V_2\left(t_2 + \frac{1}{2} d_r(V_1(t_1), V_2(t_2)) - t\right)\right) - 2t \right| \leq 288\delta.$$

Proof. To simplify the computations, we use the following notation, $D := t_2 + \frac{1}{2} d_r(V_1(t_1), V_2(t_2))$ and $\Delta = |t_1 - t_2|$. The term Δ is the difference of height between $V_1(t_1)$ and $V_2(t_2)$ since vertical geodesics are parametrised by their height. Then we have to prove that $\forall t \in \left[0, \frac{1}{2} d_r(V_1(t_1), V_2(t_2))\right]$, $|d_r(V_1(D-\Delta-t), V_2(D-t)) - 2t| \leq 288\delta$. We can assume without loss of generality that $t_1 \leq t_2$. Lemma 3.2 applied with $x = V_1(t_1)$ and with $y = V_2(t_2)$ gives us $d(V_1(D), V_2(D)) \leq 288\delta$. Furthermore, the relative distance is smaller than the distance, hence $d_r(V_1(D), V_2(D)) \leq 288\delta$. Now if we move the

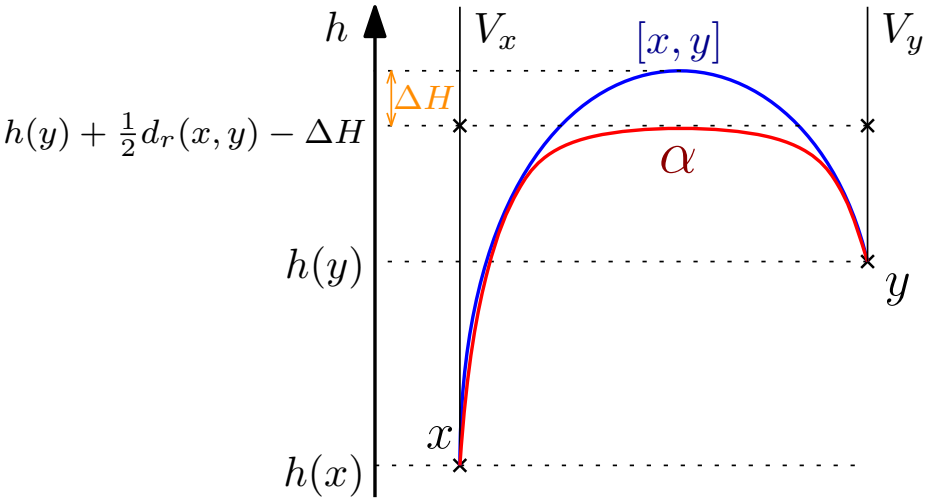

Figure 6: Proof of Lemma 3.7

two points backward from $V_1(D-\Delta)$ and $V_2(D)$ along V_1 and V_2, we have for $t \in [0, D]$:

$$\begin{aligned} d_r(V_1(D-\Delta-t), V_2(D-t)) =& d(V_1(D-\Delta-t), V_2(D-t)) - \Delta \qquad (21)\\ \leq& d(V_1(D-\Delta-t), V_1(D-\Delta)) + d(V_1(D-\Delta), V_2(D))\\ &+ d(V_2(D), V_2(D-t)) - \Delta,\\ &\text{furthermore } V_1 \text{ and } V_2 \text{ are geodesics, then:}\\ \leq& t + d(V_1(D-\Delta), V_1(D)) + d(V_1(D), V_2(D)) + t - \Delta\\ \leq& t + \Delta + 288\delta + t - \Delta \leq 2t + 288\delta. \qquad (22) \end{aligned}$$

Let us consider a geodesic α between $V_1(t_1)$ and $V_2(t_2)$. Since H is a Busemann space, and thanks to Lemma 3.2 we have $d\left(V_1(D-\Delta-t), \alpha(D-\Delta-t_1-t)\right) \leq 144\delta$ and $d\left(V_2(D-t), \alpha(D-t_1+t)\right) \leq 144\delta$. Then the second part of our inequality follows:

$$\begin{aligned} d_r(V_1(D-\Delta-t), V_2(D-t)) =& d(V_1(D-\Delta-t), V_2(D-t)) - \Delta\\ \geq& d(\alpha(D-\Delta-t_1-t), \alpha(D-t_1+t))\\ &- d(V_1(D-\Delta-t), \alpha(D-\Delta-t_1-t))\\ &- d(V_2(D-t), \alpha(D-t_1+t)) - \Delta\\ \geq& d(\alpha(D-\Delta-t_1-t), \alpha(D-t_1+t)) - 288\delta - \Delta\\ \geq& 2t + \Delta - 288\delta - \Delta \geq 2t - 288\delta. \qquad (23) \end{aligned}$$

□

The next lemma is a slight generalisation of Lemma 3.5. The difference is we control the length of a path with its maximal height instead of the distance between the projection of its extremities on a horosphere.

Lemma 3.7. *Let $\delta \geq 1$ and H be a proper, δ-hyperbolic, Busemann space. Let $x, y \in H$ such that $h(x) \leq h(y)$. Let α be a path connecting x to y with $h^+(\alpha) \leq h(y) + \frac{1}{2}d_r(x,y) - \Delta H$ and where ΔH is a positive number such that $\Delta H > 555\delta$. Then:*

$$l(\alpha) \geq d(x,y) + 2^{-530} 2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 24\delta.$$

Proof. This proof is illustrated on Figure 6. Since $h^+(\alpha) \geq h(y)$ we have that $\frac{1}{2}d_r(x,y) \geq \Delta H$. Applying Lemma 3.6 with $V_1 = V_x$, $V_2 = V_y$, $t_1 = h(x)$, $t_2 = h(y)$ and $t = \Delta H$ we have:

$$\left| d_r\left(V_x\left(h(x) + \frac{1}{2}d_r(x,y) - \Delta H\right), V_y\left(h(y) + \frac{1}{2}d_r(x,y) - \Delta H\right)\right) - 2\Delta H \right| \leq 288\delta.$$

Then we have:

$$d_r\left(V_x\left(h(x) + \frac{1}{2}d_r(x,y) - \Delta H\right), V_y\left(h(y) + \frac{1}{2}d_r(x,y) - \Delta H\right)\right) \geq 2\Delta H - 288\delta.$$

Furthermore, Lemma 3.3 applied on $V_x\left(h(x) + \frac{1}{2}d_r(x,y) - \Delta H\right)$ and $V_y\left(h(y) + \frac{1}{2}d_r(x,y) - \Delta H\right)$ gives (notice that the only difference between the two sides of the following inequality is the height in the vertical geodesic V_x):

$$\begin{aligned} &d_r\left(V_x\left(h(x) + \frac{1}{2}d_r(x,y) - \Delta H\right), V_y\left(h(y) + \frac{1}{2}d_r(x,y) - \Delta H\right)\right) \\ &\leq d\left(V_x\left(h(y) + \frac{1}{2}d_r(x,y) - \Delta H\right), V_y\left(h(y) + \frac{1}{2}d_r(x,y) - \Delta H\right)\right) + 54\delta. \end{aligned}$$

Then:

$$d\left(V_x\left(h(y) + \frac{1}{2}d_r(x,y) - \Delta H\right), V_y\left(h(y) + \frac{1}{2}d_r(x,y) - \Delta H\right)\right) \geq 2\Delta H - 342\delta > 768\delta. \tag{24}$$

Let us denote $t_0 = h(y) + \frac{1}{2}d_r(x,y) - \Delta H$. Thanks to inequality (24) the hypothesis of Lemma 3.5 holds with $x_0 = V_x\left(h(y) + \frac{1}{2}d_r(x,y) - \Delta H\right)$ and $y_0 = V_y\left(h(y) + \frac{1}{2}d_r(x,y) - \Delta H\right)$. Applying this lemma on α provides:

$$\begin{aligned} l(\alpha) &\geq \Delta h(x,x_0) + \Delta h(y,y_0) + 2^{-386}2^{\frac{1}{2\delta}d(x_0,y_0)} - 24\delta \\ &\geq h(y) + \frac{1}{2}d_r(x,y) - \Delta H - h(x) + h(y) + \frac{1}{2}d_r(x,y) - \Delta H - h(y) + 2^{-386}2^{\frac{1}{2\delta}d(x_0,y_0)} - 24\delta \\ &\geq \Delta h(y,x) + d_r(y,x) - 2\Delta H + 2^{-386}2^{\frac{1}{2\delta}d(x_0,y_0)} - 24\delta \\ &\geq d(x,y) - 2\Delta H + 2^{-386}2^{\frac{1}{2\delta}(2\Delta H - 288\delta)} - 24\delta, \text{ by equation (24).} \\ &\geq d(x,y) + 2^{-530}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 24\delta. \end{aligned}$$

□

This previous lemma tells us that a path needs to reach a sufficient height for its length not to increase to much. We give now a generalization of Lemma 3.7, where the path reaches a given low height before going to its end point. This lemma will be the central result for the understanding of the geodesic shapes in a horospherical product.

Lemma 3.8. *Let $\delta \geq 1$ and H be a proper, δ-hyperbolic, Busemann space. Let $x, y, m \in H$ such that $h(m) \leq h(x) \leq h(y)$ and let $\alpha : [0,T] \to H$ be a path connecting x to y such that $h^-(\alpha) = h(m)$. With the notation $\Delta H = h(y) + \frac{1}{2}d_r(x,y) - h^+(\alpha)$ we have:*

$$l(\alpha) \geq 2\Delta h(x,m) + d(x,y) + 2^{-850}2^{\frac{1}{\delta}\Delta H} - 1 - \max(0, 2\Delta H) - 1700\delta.$$

Proof. This proof is illustrated on Figure 7. We first assume that $\Delta H > 850\delta$, we postpone the other cases to the end of this proof. Let V_x and V_m be vertical geodesics respectively containing x and m. We call $x_1 = V_x(h(y))$ and $m_1 = V_m(h(y))$ the points of V_x and V_m at height $h(y)$. First, Lemma 3.3 provides $|d(x_1,y) - d_r(x,y)| \leq 54\delta$. Then we consider a geodesic triangle between the three points x_1, m_1 and y. Lemma 3.2 tells us that $h^+([x_1,y]) \geq h(y) + \frac{1}{2}d_r(x_1,y) - 96\delta \geq h(y) + \frac{1}{2}d_r(x,y) - 123\delta$.

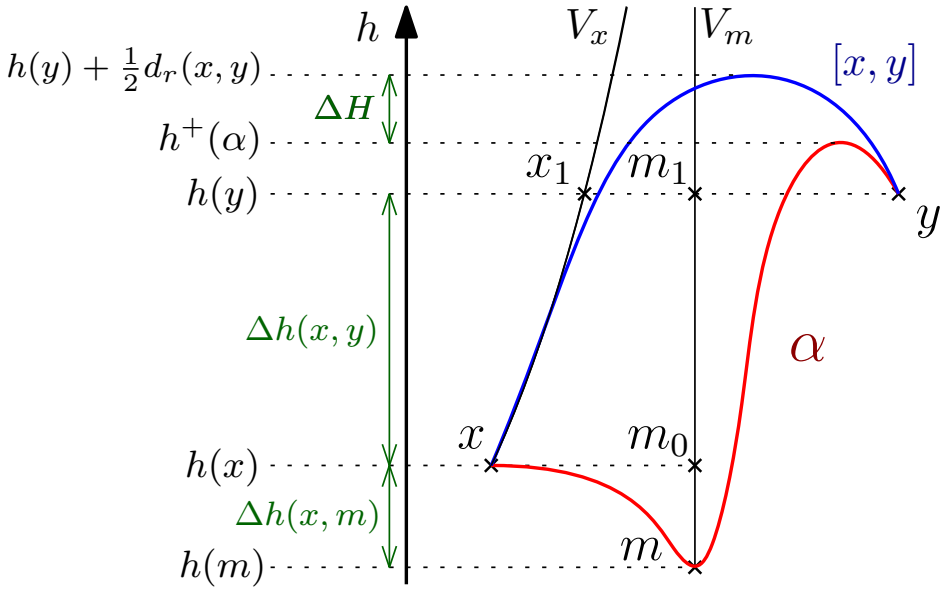

Figure 7: Proof of Lemma 3.8

Since $[x_1, y]$ is included in the δ-neighbourhood of the two other sides of the geodesic triangle, one of the two following inequalities holds:

$$1)\ h^+([x_1,m_1]) \geq h(y) + \frac{1}{2}d_r(x,y) - 124\delta$$
$$2)\ h^+([m_1,y]) \geq h(y) + \frac{1}{2}d_r(x,y) - 124\delta.$$

We first assume 1) that $h^+([x_1,m_1]) \geq h(y) + \frac{1}{2}d_r(x,y) - 124\delta$, hence:

$$d(x_1,m_1) \geq d_r(x,y) - 248\delta. \tag{25}$$

Let us denote $m_0 = V_m(h(x))$ the point of V_m at height $h(x)$. By considering the 2δ-slim quadrilateral between the points x, x_1, m_0, m_1 we have that $[x_1,m_1]$ is in the 2δ- neighbourhood of $[x_1,x] \cup [x,m_0] \cup [m_0,m]$. Furthermore $d_r(x,y) \geq 2(h^+(\alpha) - h(y)) + 2\Delta H \geq 2\Delta H \geq 1700\delta$ by assumption, then $h^+([x_1,m_1]) \geq h(y) + \frac{1}{2}d_r(x,y) - 124\delta \geq h(y) + 726\delta$. Since $h^+([x_1,x]) = h^+([m_0,m_1]) = h(y)$ we have that $h^+([x,m_0]) \geq h^+([x_1,m_1]) - 2\delta \geq h(y) + 724\delta$. Moreover:

$$d_r(x,m_0) = d(x,m_0) \geq h^+([x,m_0]) - h(x) \geq h(y) - h(x) + 724\delta \geq \Delta h(x,y) + 724\delta,$$

which allows us to use Lemma 3.6 on V_x and V_m with $t = \frac{1}{2}d_r(x,m_0) - \Delta h(x,y) \geq 0$ and $t_1 = t_2 = h(x)$. It gives:

$$\left| d_r\Big(V_x\big(h(x) + \Delta h(x,y)\big), V_m\big(h(x) + \Delta h(x,y)\big)\Big) - d_r(x,m_0) + 2\Delta h(x,y) \right| \leq 288\delta,$$

which implies in particular:

$$d_r\Big(V_x\big(h(y)\big), V_m\big(h(y)\big)\Big) + 2\Delta h(x,y) - 288\delta \leq d_r(x,m_0). \tag{26}$$

Combining inequalities (25) and (26) we have $d(x,m_0) = d_r(x,m_0) \geq d_r(x,y) + 2\Delta h(x,y) - 536\delta$. Lemma 3.3 used on x and m then gives:

$$d_r(x,m) \geq d(x,m_0) - 54\delta \geq d_r(x,y) + 2\Delta h(x,y) - 590\delta. \tag{27}$$

Let us denote α_1 the part of α linking x to m and α_2 the part of α linking m to y. We have:

$$\begin{aligned}
h^+(\alpha_1) \leq & h^+(\alpha) \leq h(y) + \frac{1}{2}d_r(x,y) - \Delta H \leq h(x) + \Delta h(x,y) + \frac{1}{2}d_r(x,y) - \Delta H \\
\leq & h(x) + \frac{1}{2}\left(2\Delta h(x,y) + d_r(x,y)\right) - \Delta H \leq h(x) + \frac{1}{2}\left(d_r(x,m) + 590\delta\right) - \Delta H, \text{ by inequality (27).} \\
\leq & h(x) + \frac{1}{2}d_r(x,m) + 295\delta - \Delta H \leq h(x) + \frac{1}{2}d_r(x,m) - \Delta H',
\end{aligned}$$

with $\Delta H' = \Delta H - 295\delta$. By assumption $\Delta H > 850\delta$, hence $\Delta H' > 555\delta$ which allows us to apply Lemma 3.7 on α_1. It follows:

$$\begin{aligned}
l(\alpha_1) \geq & d(x,m) + 2^{-530}2^{\frac{1}{\delta}\Delta H'} - 2\Delta H' - 24\delta \\
\geq & \Delta h(x,m) + d_r(x,m) + 2^{-825}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 614\delta, \text{ since } \Delta H' = \Delta H - 295\delta. \\
\geq & \Delta h(x,m) + d_r(x,y) - 590\delta + 2^{-825}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 614\delta, \text{ by inequality (27)} \\
\geq & \Delta h(x,m) + d_r(x,y) + 2^{-825}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 1204\delta.
\end{aligned}$$

We use in the following inequalities that $l(\alpha_2) \geq d(m,y) \geq \Delta h(m,y)$, we have:

$$\begin{aligned}
l(\alpha) \geq & l(\alpha_1) + l(\alpha_2) \geq \Delta h(x,m) + d_r(x,y) + 2^{-825}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 1204\delta + \Delta h(m,y) \\
\geq & 2\Delta h(x,m) + \Delta h(x,y) + d_r(x,y) + 2^{-825}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 1204\delta \\
\geq & 2\Delta h(x,m) + d(x,y) + 2^{-825}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 1204\delta \\
\geq & 2\Delta h(x,m) + d(x,y) + 2^{-850}2^{\frac{1}{\delta}\Delta H} - 1 - 2\Delta H - 1700\delta, \\
\geq & 2\Delta h(x,m) + d(x,y) + 2^{-850}2^{\frac{1}{\delta}\Delta H} - 1 - \max(0, 2\Delta H) - 1700\delta, \text{ since } \Delta H > 850\delta \geq 0,
\end{aligned}$$

which ends the proof for case 1).

Now assume that 2) holds, which is $h^+([m_1,y]) \geq h(y) + \frac{1}{2}d_r(x,y) - 124\delta$. It implies $d(m_1,y) \geq d_r(x,y) - 248\delta$, then:

$$\begin{aligned}
h^+(\alpha_2) \leq & h^+(\alpha) \leq h(y) + \frac{1}{2}d_r(x,y) - \Delta H \leq h(y) + \frac{1}{2}d_r(m_1,y) + 124\delta - \Delta H \\
& \leq h(y) + \frac{1}{2}d_r(m_1,y) - \Delta H'',
\end{aligned}$$

with $\Delta H'' = \Delta H - 124\delta$. Lemma 3.3 provides us with:

$$d_r(m,y) \geq d(m_1,y) - 54\delta \geq d_r(x,y) - 302\delta. \tag{28}$$

Since $\Delta H > 850\delta$, we have $\Delta H'' > 726\delta$ which allows us to apply Lemma 3.7 on α_2. It follows that:

$$\begin{aligned}
l(\alpha_2) \geq & d(y,m) + 2^{-530}2^{\frac{1}{\delta}\Delta H''} - 2\Delta H'' - 24\delta \\
\geq & \Delta h(y,m) + d_r(y,m) + 2^{-654}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 272\delta, \text{ since } \Delta H'' = \Delta H - 124\delta. \\
\geq & \Delta h(y,m) + d_r(x,y) + 2^{-654}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 574\delta, \text{ by inequality (26).}
\end{aligned}$$

Hence:

$$\begin{aligned}
l(\alpha) \geq & l(\alpha_1) + l(\alpha_2) \geq \Delta h(x,m) + \Delta h(y,m) + d_r(x,y) + 2^{-654}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 574\delta \\
\geq & 2\Delta h(x,m) + \Delta h(y,x) + d_r(x,y) + 2^{-654}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 574\delta \\
\geq & 2\Delta h(x,m) + d(x,y) + 2^{-654}2^{\frac{1}{\delta}\Delta H} - 2\Delta H - 574\delta \\
\geq & 2\Delta h(x,m) + d(x,y) + 2^{-850}2^{\frac{1}{\delta}\Delta H} - 1 - \max(0, 2\Delta H) - 1700\delta.
\end{aligned}$$

There remains to treat the case when $\Delta H \leq 850\delta$, where $\Delta H = h(y) + \frac{1}{2}d_r(x,y) - h^+(\alpha)$. Let n denote a point of α such that $h(n) = h^+(\alpha)$. If m comes before n, we have $l(\alpha) \geq d(x,m) + d(m,n) + d(n,y)$. Otherwise n comes before m and we have $l(\alpha) \geq d(x,n) + d(n,m) + d(m,y)$. Since $h(m) \leq h(x) \leq h(y) \leq h(n)$ we always have:

$$\begin{aligned} l(\alpha) &\geq \Delta h(x,m) + \Delta h(m,n) + \Delta h(n,y) \\ &\geq \Delta h(x,m) + \Delta h(m,x) + \Delta h(x,y) + \Delta h(y,n) + \Delta h(y,n) \geq 2\Delta h(x,m) + \Delta h(x,y) + 2(h^+(\alpha) - h(y)) \\ &\geq 2\Delta h(x,m) + \Delta h(x,y) + d_r(x,y) - 2\Delta H \geq 2\Delta h(m,x) + d(x,y) - 1700\delta. \end{aligned}$$

Furthermore $\Delta H \leq 850\delta$, then $2^{-850}2^{\frac{1}{\delta}\Delta H} \leq 1$. Therefore:

$$l(\alpha) \geq 2\Delta h(m,x) + d(x,y) + 2^{-850}2^{\frac{1}{\delta}\Delta H} - 1 - \max(0, 2\Delta H) - 1700\delta,$$

which ends the proof for the remaining case. □

4 Horospherical products

4.1 Definitions

In this part we generalize the definition of horospherical product, as seen in [5, Eskin, Fisher, Whyte] for two trees or two hyperbolic planes, to any pair of proper, geodesically complete, Gromov hyperbolic, Busemann spaces. We recall that given a proper, δ-hyperbolic space H with distinguished $a \in \partial H$ and $w \in H$, we defined the height function on H in Definition 2.1 from the Busemann functions with respect to a and w.

Definition 4.1 (Horospherical product)**.** *Let H_p and H_q be two $\delta-$hyperbolic spaces. We fix the base points $w_p \in H_p$, $w_q \in H_q$ and the directions in the boundaries $a_p \in \partial H_p$, $a_q \in \partial H_q$. We consider their heights functions h_p and h_q respectively on H_p and H_q. We define the horospherical product of H_p and H_q, denoted $H_p \bowtie H_q = \mathcal{H}$, by:*

$$\mathcal{H} := \left\{(x_p, x_q) \in H_p \times H_q \;/\; h_p(x_p) + h_q(x_q) = 0\right\}.$$

From now on, with slight abuse, we omit the base points and fixed points on the boundary in the construction of the horospherical product. The metric space $\mathcal{H}$ refers to a horospherical product of two Gromov hyperbolic Busemann spaces. We choose to denote H_p and H_q the two components in order to identify easily which objects are in which component.
One of our goals is to understand the shape of geodesics in $\mathcal{H}$ according to a given distance on it. In a cartesian product the chosen distance changes the behaviour of geodesics. However we show that in a horopsherical product the shape of geodesics does not change for a large family of distances, up to an additive constant.

We will define the distances on $H_p \bowtie H_q = \mathcal{H}$ as length path metrics induced by distances on $H_p \times H_q$. A lot of natural distances on the cartesian product $H_p \times H_q$ come from norms on the vector space $\mathbb{R}^2$. Let N be such a norm and let us denote $d_N := N(d_{H_p}, d_{H_q})$, the length $l_N(\gamma)$ of a path $\gamma = (\gamma_p, \gamma_q)$ in the metric space $\left(H_p \times H_q, d_N\right)$ is defined by:

$$l_N(\gamma) = \sup_{\theta \in \Theta([t_1,t_2])} \left(\sum_{i=1}^{n_\theta - 1} d_N(\gamma(\theta_i), \gamma(\theta_{i+1})) \right).$$

Where $\Theta([t_1,t_2])$ is the set of subdivisions of $[t_1,t_2]$. Then the N-path metrics on $\mathcal{H}$ is:

Definition 4.2 (The N-path metrics on $\mathcal{H}$)**.** *Let N be a norm on the vector space $\mathbb{R}^2$. The N-path metric on $\mathcal{H} := H_p \bowtie H_q$, denoted by $d_{\mathcal{H},N}$, is the length path metric induced by the distance $N(d_{H_p}, d_{H_q})$ on $H_p \times H_q$. For all x and y in $\mathcal{H}$ we have:*

$$d_{\mathcal{H},N}(x,y) = \inf\{l_N(\gamma) | \gamma \text{ path in } \mathcal{H} \text{ linking } x \text{ to } y\}. \tag{29}$$

Any norm N on $\mathbb{R}^2$ can be normalized such that $N(1,1) = 1$. We call admissible any such norm which satisfies an additional condition.

Definition 4.3 (Admissible norm)**.** *Let N be a norm on the vector space $\mathbb{R}^2$ such that $N(1,1) = 1$. The norm N is called admissible if and only if for all real a and b we have:*

$$N(a,b) \geq \frac{a+b}{2}. \tag{30}$$

Since all norms are equivalent in $\mathbb{R}^2$, there exists a constant $C_N \geq 1$ such that:

$$N(a,b) \leq C_N \frac{a+b}{2}. \tag{31}$$

As an example, any l_p norm with $p \geq 1$ is admissible.

Property 4.4. *Let N be an admissible norm on the vector space $\mathbb{R}^2$. Let $\gamma := (\gamma_p, \gamma_q) \subset H_p \times H_q$ be a connected path. Then we have:*

$$\frac{l_{H_p}(\gamma_p) + l_{H_q}(\gamma_q)}{2} \leq l_N(\gamma) \leq C_N \frac{l_{H_p}(\gamma_p) + l_{H_q}(\gamma_q)}{2}.$$

Proof. Let $\gamma := (\gamma_p, \gamma_q) : [t_1, t_2] \to H_p \times H_q$ be a connected path and θ a subdivision of $[t_1, t_2]$, then by the definition of the length:

$$\begin{aligned}
l_N(\gamma) &\geq \sum_{i=1}^{n_\theta - 1} d_N(\gamma(\theta_i), \gamma(\theta_{i+1})) = \sum_{i=1}^{n_\theta - 1} N\Big(d_{H_p}\big(\gamma_p(\theta_i), \gamma_p(\theta_{i+1})\big), d_{H_q}\big(\gamma_q(\theta_i), \gamma_q(\theta_{i+1})\big)\Big) \\
&\geq \sum_{i=1}^{n_\theta - 1} \frac{1}{2}\Big(d_{H_p}\big(\gamma_p(\theta_i), \gamma_p(\theta_{i+1})\big) + d_{H_q}\big(\gamma_q(\theta_i), \gamma_q(\theta_{i+1})\big)\Big), \text{ since } N \text{ is admissible.} \\
&\geq \frac{1}{2}\left(\sum_{i=1}^{n_\theta - 1} d_{H_p}\big(\gamma_p(\theta_i), \gamma_p(\theta_{i+1})\big) + \sum_{i=1}^{n_\theta - 1} d_{H_q}\big(\gamma_q(\theta_i), \gamma_q(\theta_{i+1})\big)\right).
\end{aligned}$$

Any couple of subdivision θ_1 and θ_2 can be merge into a subdivision θ that contains θ_1 and θ_2. Furthermore the last inequality holds for any subdivision θ, hence by taking the supremum on all the subdivisions we have:

$$l_N(\gamma) \geq \frac{l_{H_p}(\gamma_p) + l_{H_q}(\gamma_q)}{2}.$$

Furthermore, we have that $\forall a, b \in \mathbb{R}$, $N(a,b) \leq C_N \frac{a+b}{2}$, hence:

$$\begin{aligned}
\sum_{i=1}^{n_\theta - 1} d_N(\gamma(\theta_i), \gamma(\theta_{i+1})) &\leq \frac{C_N}{2}\left(\sum_{i=1}^{n_\theta - 1} d_{H_p}(\gamma_p(\theta_i), \gamma(\theta_{i+1})) + \sum_{i=1}^{n_\theta - 1} d_{H_q}(\gamma_q(\theta_i), \gamma_q(\theta_{i+1}))\right) \\
&\leq C_N \frac{l_{H_p}(\gamma_p) + l_{H_p}(\gamma_p)}{2}
\end{aligned}$$

Since last inequality holds for any subdivision θ, we have that $l_N(\gamma) \leq C_N \frac{l_{H_p}(\gamma_p) + l_{H_p}(\gamma_p)}{2}$.

□

The definition of height on H_p and H_q is used to construct a height function on $H_p \bowtie H_q$.

Definition 4.5 (Height on $\mathcal{H}$)**.** *The height $h(x)$ of a point $x = (x_p, x_q) \in H_p \bowtie H_q$ is defined as $h(x) = h_p(x_p) = -h_q(x_q)$.*

On Gromov hyperbolic spaces we have that de distance between two points is greater than their height difference. The same occurs on horospherical products given with an admissible norm. Let x and y be two points of $\mathcal{H}$, and let us denote $\Delta h(x,y) := |h(x) - h(y)|$ their height difference.

Lemma 4.6. *Let N be a admissible norm, and let $d_{\mathcal{H},N}$ the distance on $\mathcal{H} = H_p \bowtie H_q$ induced by N. Let $x = (x_p, x_q)$ and $y = (y_p, y_q)$ be two points of $\mathcal{H}$, we have:*

$$\forall x, y \in \mathcal{H}, \quad d_{\mathcal{H},N}(x, y) \geq \Delta h(x, y). \tag{32}$$

Proof. Since N is admissible we have:

$$\begin{aligned} d_{\mathcal{H},N}(x, y) &\geq \frac{d_{H_p}(x_p, y_p) + d_{H_q}(y_p, y_q)}{2} \geq \frac{\Delta h(x_p, y_p) + \Delta h(x_q, y_q)}{2} \\ &\geq \Delta h(x_p, y_p) = \Delta h(x, y). \end{aligned}$$

□

Following Proposition 2.10, we define a notion of vertical paths in a horospherical product.

Definition 4.7 (Vertical paths in $\mathcal{H}$)**.** *Let $V : \mathbb{R} \to \mathcal{H}$ be a connected path. We say that V is vertical if and only if there exists a parametrisation by arclength of V such that $h(V(t)) = t$ for all t.*

Actually, a vertical path of a horospherical product is a geodesic.

Lemma 4.8. *Let N be an admissible norm. Let $V : \mathbb{R} \to \mathcal{H}$ be a vertical path. Then V is a geodesic of $(\mathcal{H}, d_{\mathcal{H},N})$.*

Proof. Let $t_1, t_2 \in \mathbb{R}$. The path V is vertical therefore $\Delta h\big(V(t_1), V(t_2)\big) = |t_1 - t_2|$. Since V is connected and parametrised by arclength, we have that:

$$\begin{aligned} |t_1 - t_2| = l_N\left(V_{|[t_1,t_2]}\right) &\geq d_{\mathcal{H},N}\big(V(t_1), V(t_2)\big) \\ &\geq \Delta h\big(V(t_1), V(t_2)\big) = |t_1 - t_2|. \end{aligned}$$

Then $d_{\mathcal{H},N}\big(V(t_1), V(t_2)\big) = |t_1 - t_2|$, which ends the proof. □

Such geodesics are called vertical geodesics. Next proposition tells us that vertical geodesics of $H_p \bowtie H_q$ are exactly couples of vertical geodesics of H_p and H_q.

Proposition 4.9. *Let N be an admissible norm and let $V = (V_p, V_q) : \mathbb{R} \to \mathcal{H}$ be a geodesic of $(\mathcal{H}, d_{\mathcal{H},N})$. The two following properties are equivalent:*

1. *V is a vertical geodesic of $(\mathcal{H}, d_{\mathcal{H},N})$*
2. *V_p and V_q are respectively vertical geodesics of H_p and H_q.*

Proof. Let us first assume that V be a vertical geodesic, we have for all real t that $h(V_p(t)) = h(V(t)) = t$, hence $\forall t_1, t_2 \in \mathbb{R}$:

$$d_{H_p}\big(V_p(t_1), V_p(t_2)\big) \geq \Delta h\big(V_p(t_1), V_p(t_2)\big) = |t_1 - t_2|. \tag{33}$$

Similarly we have that $d_{H_q}\big(V_q(t_1), V_q(t_2)\big) \geq |t_1 - t_2|$. Using that N is admissible and that V is a geodesic we have:

$$\begin{aligned} d_{H_p}\big(V_p(t_1), V_p(t_2)\big) &= 2\frac{d_{H_p}\big(V_p(t_1), V_p(t_2)\big) + d_{H_q}\big(V_q(t_1), V_q(t_2)\big)}{2} - d_{H_q}\big(V_q(t_1), V_q(t_2)\big) \\ &\leq 2d_{\mathcal{H},N}\big(V(t_1), V(t_2)\big) - |t_1 - t_2| = |t_1 - t_2|. \end{aligned}$$

Combine with inequality (33) we have that $d_{H_p}\big(V_p(t_1), V_p(t_2)\big) = |t_1 - t_2|$, hence V_p is a vertical geodesic of H_p. Similarly, V_q is a vertical geodesic H_q.
Let us assume that V_p and V_q are vertical geodesics of H_p and H_q. Let $t_1, t_2 \in \mathbb{R}$, we have:

$$\begin{aligned}
d_{\mathcal{H},N}(V(t_1), V(t_2)) &= \sup_{\theta \in \Theta([t_1,t_2])} \left(\sum_{i=1}^{n_\theta - 1} d_N(V(\theta_i), V(\theta_{i+1})) \right) \\
&= \sup_{\theta \in \Theta([t_1,t_2])} \left(\sum_{i=1}^{n_\theta - 1} N\Big(d_{H_p}\big(V_p(\theta_i), V_p(\theta_{i+1})\big), d_{H_q}\big(V_q(\theta_i), V_q(\theta_{i+1})\big)\Big) \right) \\
&= \sup_{\theta \in \Theta([t_1,t_2])} \left(\sum_{i=1}^{n_\theta - 1} N\Big(\Delta h\big(V_p(\theta_i), V_p(\theta_{i+1})\big), \Delta h\big(V_q(\theta_i), V_q(\theta_{i+1})\big)\Big) \right) \\
&= \sup_{\theta \in \Theta([t_1,t_2])} \left(N(1,1) \sum_{i=1}^{n_\theta - 1} \Delta h\big(V_p(\theta_i), V_p(\theta_{i+1})\big) \right) \\
&= N(1,1)\Delta h\big(V_p(t_1), V_p(t_2)\big) = |t_1 - t_2|, \text{ since } N(1,1) = 1.
\end{aligned}$$

Where $\Theta([t_1, t_2])$ is the set of subdivision of $[t_1, t_2]$. Hence the proposition is proved. □

This previous result is the main reason why we are working with distances which came from admissible norms.

Definition 4.10. *A geodesic ray of $\mathcal{H} = H_p \bowtie H_q$ is called vertical if it is a subset of a vertical geodesic.*

A metric space is called geodesically complete if all its geodesic segments can be prolonged into geodesic lines. If H_p and H_q are proper hyperbolic geodesically complete Busemann spaces, their horospherical product $\mathcal{H}$ is connected.

Property 4.11. *Let H_p and H_q be two proper, geodesically complete, δ-hyperbolic, Busemann spaces. Let $\mathcal{H} = H_p \bowtie H_q$ be their horospherical product. Then $\mathcal{H}$ is connected, furthermore $\frac{1}{2}(d_{H_p} + d_{H_q}) \leq d_{\mathcal{H}} \leq 2C_N(d_{H_p} + d_{H_q})$.*

Proof. Let $x = (x_p, x_q)$ and $y = (y_p, y_q)$ be two points of $\mathcal{H}$. From Property 2.11, there exists a vertical geodesic V_{x_q} such that x_q is in the image of V_{x_q}, and there exists a vertical geodesic V_{y_p} such that y_p is in the image of V_{y_p}. Let y_q' be the point of V_{x_q} at height $h(y_q)$. Let α_p be a geodesic of H_p linking x_p to y_p and let α_q' be a geodesic of H_q linking y_q' to y_q. We will connect x to y with a path composed with pieces of α_p, α_q', V_{x_q} and V_{y_p}.
We first link (x_p, x_q) to (y_p, y_q') with α_p and V_{x_q}. It is possible since V_{x_q} is parametrised by its height. More precisely we construct the following path c_1:

$$\forall t \in [0, d(x_p, y_p)],\ c_1(t) = \Big(\alpha_p(t), V_{x_q}\big(-h(\alpha_p(t))\big)\Big).$$

Since V_{x_q} is parametrised by its height, we have $h\left(V_{x_q}\big(-h(\alpha_p(t))\big)\right) = -h(\alpha_p(t))$ which implies $c_1(t) \in \mathcal{H}$. Furthermore, using the fact that the height is 1-Lipschitz, we have $\forall t_1, t_2 \in [0, d(x_p, y_p)]$:

$$d_{H_q}\Big(V_{x_q}\big(-h(\alpha_p(t_1))\big), V_{x_q}\big(-h(\alpha_p(t_2))\big)\Big) = |h(\alpha_p(t_1)) - h(\alpha_p(t_2))| \leq d_{H_p}(\alpha_p(t_1), \alpha_p(t_2)).$$

Hence $c_{1,q} : t \mapsto V_{x_q}\big(-h(\alpha_p(t))\big)$ is a connected path such that $l(c_{1,q}) \leq l(\alpha_p) \leq d_{H_p}(x_p, y_p)$. Hence c_1 is a connected path linking (x_p, x_q) to (y_p, y_q'). Using Property 4.4 on c_1 provides us with:

$$\begin{aligned}
l_N(c_1) &\leq \frac{C_N}{2}(l(c_{1,q}) + l(\alpha_p)) \leq C_N l(\alpha_p) \\
&\leq C_N d_{H_p}(x_p, y_p)
\end{aligned}$$

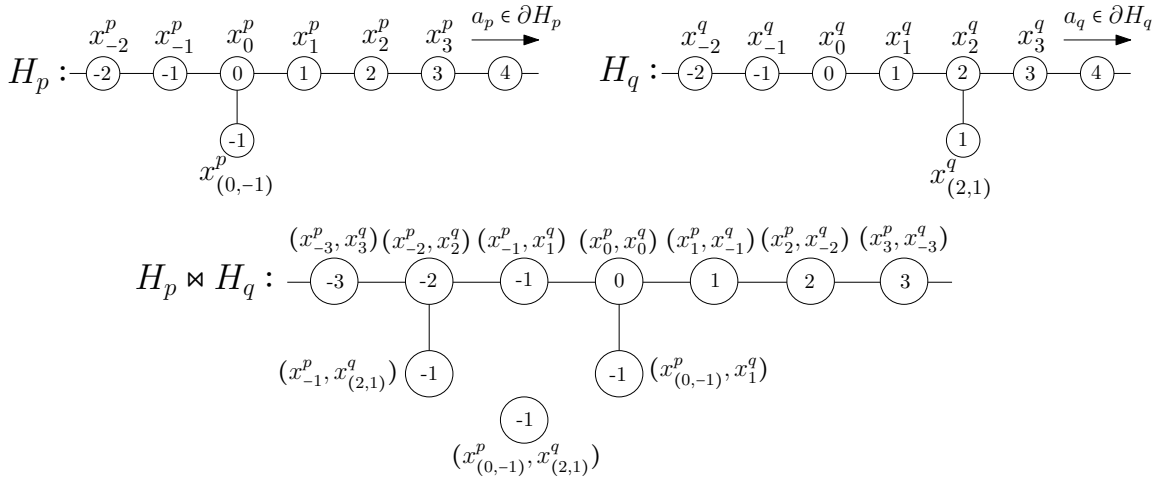

Figure 8: Example of horospherical product which is not connected. The number in a vertex is the height of that vertex.

We recall that by definition $y'_q = V_{x_q}(h(y_q))$. We show similarly that $c_2 : t \mapsto \Big(V_{y_p}\big(-h(\alpha'_q(t))\big), \alpha'_q(t)\Big)$ is a connected path linking (y_p, y'_q) to (y_p, y_q) such that:

$$\begin{aligned} l(c_2) &\leq C_N d_{H_q}(y'_q, y_q) \leq C_N\big(d_{H_q}(y'_q, x_q) + d_{H_q}(x_q, y_q)\big) \\ &= C_N\big(\Delta h(x_q, y_q) + d_{H_q}(x_q, y_q)\big), \text{ since } y'_q = V_{x_q}(h(y_q)) \\ &\leq 2C_N d_{H_q}(x_q, y_q). \end{aligned}$$

Hence, there exists a connected path $c = c_1 \cup c_2$ linking x to y such that:

$$l(c) \leq C_N d_{H_p}(x_p, y_p) + 2C_N d_{H_q}(x_q, y_q) \leq 2C_N\big(d_{H_p}(x_p, y_p) + d_{H_q}(x_q, y_q)\big). \tag{34}$$

□

However if the two components H_p and H_q are not geodesically complete, $\mathcal{H}$ may not be connected.

Example 4.12. *Let H_p and H_q be two graphs, constructed from an infinite line $\mathbb{Z}$ (indexed by $\mathbb{Z}$) with an additional vertex glued on the 0 for H_p and on the -2 for H_q. Their construction are illustrated on figure 8. They are two 0-hyperbolic Busemann spaces which are not geodesically complete. Let $w_p \in H_p$ be the vertex indexed by 0 in H_p, and let $w_q \in H_q$ be the vertex indexed by -2 in H_q. We choose them to be the base points of H_p and H_q. Since ∂H_p and ∂H_q contain two points each, we fix in both cases the point of the boundary a_p or a_q to be the one that contains the geodesic ray indexed by $\mathbb{N}$. On figure 8, we denoted the height of a vertex inside this one. Then the horospherical product $\mathcal{H} = H_p \bowtie H_q$ taken with the ℓ_1 path metric is not connected. Since some vertices of H_p and H_q are not contained in a vertical geodesic, one may not be able to adapt its height correctly while constructing a path joining $\Big(x^p_{-1}, x^q_{(2,1)}\Big)$ to $\Big(x^p_{(0,-1)}, x^q_{(2,1)}\Big)$.*

It is not clear that a horospherical product is still connected without the hypothesis that H_p and H_q are Busemann spaces. In that case we would need a "coarse" definition of horospherical product. Indeed, the height along geodesics would not be smooth as in Proposition 2.10, therefore the condition requiring to have two exact opposite heights would not suits.

4.2 Examples

A first example of horospherical product is the family of Diestel-Leader graphs. They are by construction horospherical products of two trees.

Definition 4.13 (Diestel-Leader graph $DL(p,q)$)**.** *Let $p \geq 2$ and $q \geq 2$ be two integers. Let T_p be the p-homogeneous tree and T_q be the q-homogeneous tree. The two graphs T_p and T_q are 0-hyperbolic proper geodesically complete Busemann spaces. The Diestel-Leader graph $DL(p,q)$ is defined by $DL(p,q) = T_p \bowtie T_q$.*

We see T_p and T_q as connected metric spaces with the usual distance on them. By choosing half of the ℓ_1 path metric on $DL(p,q)$, this horospherical product becomes a graph with the usual distance on it. Indeed, the set of vertices of $DL(p,q)$ is then defined by the subset of couples of vertices of $T_p \times T_q$ included in $DL(p,q)$. In this horospherical product, two points (x_p, x_q) and (y_p, y_q) of $DL(p,q)$ are connected by an edge if and only if x_p and y_p are connected by an edge in T_p and if x_q and y_q are connected by an edge in T_q. Furthermore, when $p = q$, there is a one-to-one correspondance between $DL(q,q)$ and the Cayley graph of the lamplighter group $\mathbb{Z}_q \wr \mathbb{Z}$, see [13, Woess] for further details.

The SOL geometry is the Riemannian manifold with coordinates $(x,y,z) \in \mathbb{R}^3$ and with the Riemannian metric $ds^2 = dz^2 + e^{2z}dx^2 + e^{-2z}dy^2$. It is the horospherical product of two hyperbolic planes, it is described in [14, Woess]. Let us consider $\mathbb{H}^2$ the Log model of the hyperbolic plane, defined as the Riemannian manifold with coordinates $(x,z) \in \mathbb{R}^2$ and with the Riemannian metric $ds^2 = dz^2 + e^{-2z}dx^2$. We fix $w = (0,0)$ as the base point of $\mathbb{H}$ and the "upward" direction a as the point on the boundary. In that case the height function in regards to (a,w) taken on a point $(x,z) \in \mathbb{H}$ is $h_{(a,w)}(x,z) = z$. We now look at the horospherical product $\mathbb{H}^2 \bowtie \mathbb{H}^2 := \{(x_1,z_1,x_2,z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 | z_1 = -z_2\}$ taken with the ℓ_2 path metric. Since the second and the fourth variable are exactly opposite, we merge them into one. Hence we have that $\mathbb{H}^2 \bowtie \mathbb{H}^2$ is isometric to the space $\{(x_1,x_2,z_1) \in \mathbb{R}^3\}$ with the metric

$$ds^2 = dz_1^2 + e^{-2z_1}dx_1^2 + dz_1^2 + e^{2z_1}dx_2^2 = 2dz_1^2 + e^{-2z_1}dx_1^2 + e^{2z_1}dx_2^2.$$

Changing the coordinates by dividing x_1 and x_2 by two tells us that this space is isometric to SOL.

Depending on the case, we either used the ℓ_1 path metric or the ℓ_2 path metric. Proposition 4.18 tells us that it does not matter, up to an additive uniform constant. Quasi-isometric rigidity results have been proved in the Diestel-Leader graphs and the SOL geometry with the same techniques in [5, Eskin, Fisher, Whyte] and [6, E,F,W].

The horospherical product of a hyperbolic plane and a regular tree has been studied as the 2-complex of Baumslag-Solitar groups in [1, Bendikov, Saloff-Coste, Salvatori, Woess]. They are called the treebolic spaces. The distance they choose on the treebolic spaces is similar to ours. In fact our Proposition 4.17 and their Proposition 2.8 page 9 (in [1]) tell us they are equal up to an additive constant. Rigidity results on the treebolic spaces were brought up in [7, Farb, Mosher] and [8, F,M].

The previous examples were already known, however our construction still works for many other spaces. As an example, a geodesically complete manifold with a curvature lower than a negative constant could be used as the component H_p or H_q in the horospherical product.

4.3 Length of geodesic segments in $\mathcal{H}$

From now on, unless otherwise specified, H_p and H_q will always be two proper, geodesically complete, δ-hyperbolic, Busemann spaces with $\delta \geq 1$, and N will always be an admissible norm. Let x and y be two points of $\mathcal{H} := H_p \bowtie H_q$, and let α be a geodesic of $\mathcal{H}$ connecting them. We first prove an upper bound on the length of α by computing the length of a path $\gamma \subset \mathcal{H}$ linking x to y

Lemma 4.14. *Let x and y be points of the horospherical product $\mathcal{H} = H_p \bowtie H_q$. There exists a path γ connecting $x = (x_p, x_q)$ to $y = (y_p, y_q)$ such that:*

$$l_N(\gamma) \leq d_r(x_q, y_q) + d_r(x_p, y_p) + \Delta h(x,y) + 1152\delta C_N.$$

Proof. Without loss of generality, we assume $h(x) \leq h(y)$. One can follow the idea of the proof on Figure 9. We consider V_{x_p} and V_{y_p} two vertical geodesics of H_p containing x_p and y_p respectively. Similarly let V_{x_q} and V_{y_q} be two vertical geodesics of H_q containing x_q and y_q respectively. We will use them to construct γ. Let A_1 be the point of the vertical geodesic $(V_{x_p}, V_{x_q}) \subset \mathcal{H}$ at height $h(x) - \frac{1}{2}d_r(x_q, y_q)$ and A_2 be the point of the vertical geodesic $(V_{x_p}, V_{y_q}) \subset \mathcal{H}$ at the same height $h(x) - \frac{1}{2}d_r(x_q, y_q)$. Let A_3 be the point of the vertical geodesic (V_{x_p}, V_{y_q}) at height $h(y) + \frac{1}{2}d_r(x_p, y_p)$ and A_4 be the point of the vertical geodesic (V_{y_p}, V_{y_q}) at the same height $h(y) + \frac{1}{2}d_r(x_p, y_p)$. Then $\gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_5$ is constructed as follows:

- γ_1 is the part of (V_{x_p}, V_{x_q}) linking x to A_1.
- γ_2 is a geodesic linking A_1 to A_2. Such a geodesic exists by Property 4.11.
- γ_3 is the part of (V_{x_p}, V_{y_q}) linking A_2 to A_3.
- γ_4 is a geodesic linking A_3 to A_4. Such a geodesic exists by Property 4.11.
- γ_5 is the part of (V_{y_p}, V_{y_q}) linking A_4 to y.

In fact A_1 and A_2 are close to each other. Indeed, the two points $A_1 = (A_{1,p}, A_{1,q})$ and $A_2 = (A_{2,p}, A_{2,q})$ are characterised by the two geodesics (V_{x_p}, V_{x_q}) and (V_{x_p}, V_{y_q}). Then, because $-h(y) = h_q(y_q) \leq h_q(x_q)$, Lemma 3.2 applied on x_q and y_q in H_q gives us $d_{H_q}(A_{1,q}, A_{2,q}) \leq 288\delta$. Furthermore Property 4.11 provides us with $d_{\mathcal{H},N} \leq 2C_N(d_{H_p} + d_{H_q})$, however we have that $A_{1,p} = A_{2,p}$ hence:

$$d_{\mathcal{H},N}(A_1, A_2) \leq 576\delta C_N. \tag{35}$$

Lemma 3.2 applied on x_p and y_p provides similarly:

$$d_{\mathcal{H},N}(A_3, A_4) \leq 576\delta C_N, \tag{36}$$

which gives us:

$$\begin{aligned}
l_N(\gamma) =& l_N(\gamma_1) + l_N(\gamma_2) + l_N(\gamma_3) + l_N(\gamma_4) + l_N(\gamma_5) \\
=& d_{\mathcal{H},N}(x, A_1) + d_{\mathcal{H},N}(A_1, A_2) + d_{\mathcal{H},N}(A_2, A_3) + d_{\mathcal{H},N}(A_3, A_4) + d_{\mathcal{H},N}(A_4, y) \\
& \text{Since } \gamma_1,\ \gamma_3 \text{ and } \gamma_5 \text{ are vertical geodesics, we have:} \\
=& \Delta h(x, A_1) + d_{\mathcal{H},N}(A_1, A_2) + \Delta h(A_2, A_3) + d_{\mathcal{H},N}(A_3, A_4) + \Delta h(A_4, y) \\
=& \frac{1}{2}d_r(x_q, y_q) + d_{\mathcal{H},N}(A_1, A_2) + \frac{1}{2}d_r(x_q, y_q) + \frac{1}{2}d_r(x_p, y_p) + \Delta h(x, y) + d_{\mathcal{H},N}(A_3, A_4) + \frac{1}{2}d_r(x_p, y_p) \\
\leq& d_r(x_q, y_q) + d_r(x_p, y_p) + \Delta h(x, y) + 1152\delta C_N, \text{ by inequalities (35) and (36).}
\end{aligned}$$

□

We are aiming to use Lemma 3.8 on the two components $\alpha_p \subset H_p$ and $\alpha_q \subset H_q$ of α to obtain lower bounds on their lengths. We hence need the following lemma to ensure us that when α is a geodesic, the exponential term in the inequality of Lemma 3.8 will be small.

Lemma 4.15. *Let $C = 2853\delta C_N + 2^{851}$ and let $e : \mathbb{R} \to \mathbb{R}$ be a map defined by $\forall t \in \mathbb{R}$, $e(t) = \frac{1}{C}2^{C^{-1}t} - 2\max(0, t)$. Then $\forall t \in \mathbb{R}$:*

1. $e(t) \geq -7C^2$
2. $(\ e(t) \leq 2853\delta C_N\ ) \Rightarrow (\ t \leq 3C^2\ )$.

Proof. For all time t, we have that $e(t) = \frac{1}{C}2^{C^{-1}t} - 2\max(0, t) \leq \frac{1}{C}2^{C^{-1}t} - 2t =: e_1(t)$. The derivative of e_1 is $e_1'(t) = \frac{\log(2)}{C^2}2^{C^{-1}t} - 2$, which is non negative $\forall t \geq C\log_2\left(\frac{2}{\log(2)}C^2\right)$ and non positive otherwise. Then $\forall t \in \mathbb{R}$:

$$\begin{aligned}
e_1(t) &\geq e_1\left(\log_2\left(\frac{2}{\log(2)}C^2\right)\right) \geq \frac{2C}{\log(2)} - 2C\log_2\left(\frac{2}{\log(2)}C^2\right) \geq \frac{2C}{\log(2)} - 4C\log_2\left(\sqrt{\frac{2}{\log(2)}}C\right) \\
&\geq \frac{2C}{\log(2)} - 4\sqrt{\frac{2}{\log(2)}}C^2 \geq -4\sqrt{\frac{2}{\log(2)}}C^2 \geq -7C^2.
\end{aligned}$$

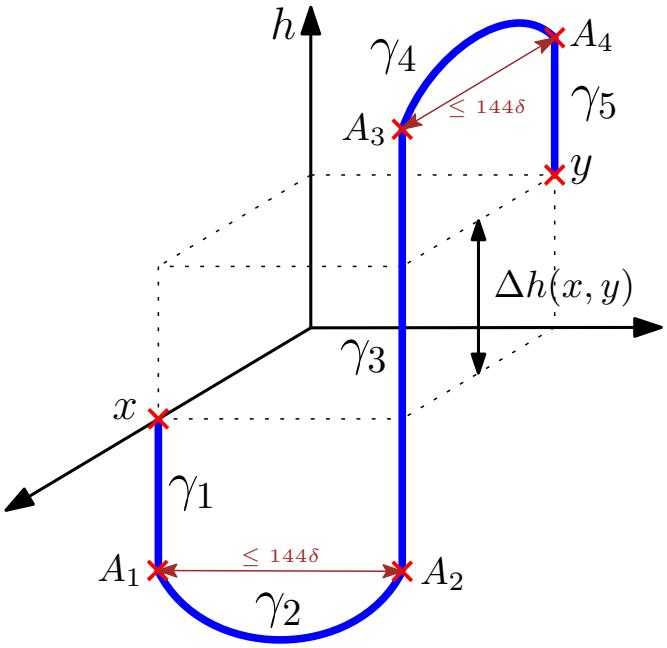

Figure 9: Construction of the path γ when $h(x) \leq h(y)$ for Lemma 4.14.

Since $C \geq \frac{2}{\log(2)}$ we have $3C^2 \geq C\log_2(C^3) \geq C\log_2\left(\frac{2}{\log(2)}C^2\right)$, then e_1 is non decreasing on $[C\log_2(C^3); +\infty[$. We show that $e_1(3C^2) \geq 2853\delta C_N$:

$$e_1(3C^2) \geq e_1(C\log_2(C^3)) = \frac{1}{C}2^{\frac{C\log_2(C^3)}{C}} - 2C\log_2(C^3) = C(C - 6\log_2(C)).$$

Since $C \geq 2^{851}$ we have $C - 6\log_2(C) \geq 1$ and since $C \geq 2853\delta C_N$ we have that $e_1(3C^2) \geq C \times 1 \geq 2853\delta C_N$ which provides $\forall t \in [3C^2; +\infty[$ we have $e_1(t) \geq 2853\delta C_N$. Furthermore $\forall t \in \mathbb{R}^+$, $e_1(t) = e(t)$, hence $\forall t \in [3C^2; +\infty[$ we have $e(t) \geq 2853\delta C_N$ which implies point 2. of this lemma. □

The following lemma provides us with a lower bound matching Lemma 4.14, and a first control on the heights a geodesic segment must reach.

Lemma 4.16. *Let $x = (x_p, x_q)$ and $y = (y_p, y_q)$ be two points of $\mathcal{H} = H_p \bowtie H_q$ such that $h(x) \leq h(y)$. Let $\alpha = (\alpha_p, \alpha_q)$ be a geodesic segment of $\mathcal{H}$ linking x to y. Let $C_0 = (2853\delta C_N + 2^{851})^2$, we have:*

1. $l(\alpha) \geq \Delta h(x,y) + d_r(x_q, y_q) + d_r(x_p, y_p) - 15C_0$
2. $h^+(\alpha) \geq h(y) + \frac{1}{2}d_r(x_p, y_p) - 3C_0$
3. $h^-(\alpha) \leq h(x) - \frac{1}{2}d_r(x_q, y_q) + 3C_0$.

Proof. Let us denote $\Delta H^+ = h(y) + \frac{1}{2}d_r(x_p, y_p) - h^+(\alpha)$ and $\Delta H^- = h^-(\alpha) - \left(h(x) - \frac{1}{2}d_r(x_q, y_q)\right)$. Let m be a point of α at height $h^-(\alpha) = h(x) - \frac{1}{2}d_r(x_q, y_q) + \Delta H^-$, and n be a point of α at height $h^+(\alpha) = h(y) + \frac{1}{2}d_r(x_p, y_p) - \Delta H^+$. Then Lemma 3.8 used on α_p gives us:

$$\begin{aligned}
l(\alpha_p) \geq& 2\Delta h(x_p, m_p) + d(x_p, y_p) + 2^{-850}2^{\frac{1}{\delta}\Delta H^+} - 1 - 2\max(0, \Delta H^+) - 1700\delta \\
\geq& 2h(x_p) - 2\left(h(x_p) - \frac{1}{2}d_r(x_q, y_q) + \Delta H^-\right) + d(x_p, y_p) + 2^{-850}2^{\frac{1}{\delta}\Delta H^+} - 1 \\
& - 2\max(0, \Delta H^+) - 1700\delta \\
\geq& d_r(x_q, y_q) + d_r(x_p, y_p) + \Delta h(x,y) + 2^{-850}2^{\frac{1}{\delta}\Delta H^+} - 1 - 2\max(0, \Delta H^+) - 2\Delta H^- - 1700\delta.
\end{aligned}$$

Since $h(x_q) \geq h(y_q)$ and $h(n_q) = h(y_q) - \frac{1}{2}d_r(x_p, y_p) + \Delta H^+$, Lemma 3.8 used on α_q provides similarly:

$$l(\alpha_q) \geq d_r(x_p, y_p) + d_r(x_q, y_q) + \Delta h(x, y) + 2^{-850}2^{\frac{1}{\delta}\Delta H^-} - 1 - 2\max(0, \Delta H^-) - 2\Delta H^+ - 1700\delta.$$

Hence by Property 4.4:

$$\begin{aligned} l_N(\alpha) \geq \frac{1}{2}(l(\alpha_p) + l(\alpha_q)) \geq & d_r(x_p, y_p) + d_r(x_q, y_q) + \Delta h(x, y) - 1700\delta + 2^{-851}2^{\frac{1}{\delta}\Delta H^-} \\ & + 2^{-851}2^{\frac{1}{\delta}\Delta H^+} - 2\max(0, \Delta H^-) - 2\max(0, \Delta H^+) - 1. \end{aligned} \tag{37}$$

Furthermore, we know by Lemma 4.14 that $l_N(\alpha) \leq \Delta h(x, y) + d_r(x_p, y_p) + d_r(x_q, y_q) + 1152\delta C_N$. Since $C_N \geq 1$ we have:

$$2852\delta C_N \geq 2^{-851}2^{\frac{1}{\delta}\Delta H^-} - 2\max(0, \Delta H^-) + 2^{-851}2^{\frac{1}{\delta}\Delta H^+} - 2\max(0, \Delta H^+) - 1.$$

Let us denote $S := \max\{\Delta H^-, \Delta H^+\}$. Therefore we have $2^{-851}2^{\frac{1}{\delta}S} - 2\max(0, S) - 1 \leq 2852\delta C_N$. By assumption $\delta \geq 1$ hence $2^{-851}2^{\frac{1}{\delta}S} - 2\max(0, S) \leq 2853\delta C_N$. Furthermore, for $C = 2853\delta C_N + 2^{851}$, we have both $2^{-851} \geq \frac{1}{C}$ and $\frac{1}{\delta} \geq \frac{1}{C}$. Then we have $\frac{1}{C}2^{\frac{S}{C}} - 2\max(0, S) \leq 2853\delta C_N$. Lemma 4.15 provides $S \leq 3C^2 = 3C_0$ which implies points 2. and 3. of our lemma. Lemma 4.15 also provides us with:

$$-14C_0 \leq 2^{-851}2^{\frac{1}{\delta}\Delta H^-} - 2\max(0, \Delta H^-) + 2^{-851}2^{\frac{1}{\delta}\Delta H^+} - 2\max(0, \Delta H^+).$$

Last inequality is a lower bound of the term we want to remove in inequality (37). The first point of our lemma hence follows since $1700\delta + 1 \leq C_0$. □

Combining Lemma 4.14 and 4.16 we get the following corollary.

Corollary 4.17. *Let N be an admissible norm and let $C_0 = (2853\delta C_N + 2^{851})^2$. The length of a geodesic segment α connecting x to y in $(\mathcal{H}, d_{\mathcal{H},N})$ is controlled as follows:*

$$\left|l_N(\alpha) - \left(\Delta h(x, y) + d_r(x_p, y_p) + d_r(x_q, y_q)\right)\right| \leq 15C_0,$$

which gives us a control on the N-path metric, for all points x and y in $\mathcal{H}$ we have:

$$\left|d_{\mathcal{H},N}(x, y) - \left(\Delta h(x, y) + d_r(x_p, y_p) + d_r(x_q, y_q)\right)\right| \leq 15C_0.$$

This result is central as it shows that the shape of geodesics does not depend on the N-path metric chosen for the distance on the horospherical product.

Corollary 4.18. *Let $r \geq 1$. For all x and y in $\mathcal{H} = H_p \bowtie H_q$ we have:*

$$\left|d_{\mathcal{H},\ell_r}(x, y) - d_{\mathcal{H},\ell_1}(x, y)\right| \leq 30(5706\delta + 2^{851})^2.$$

Proof. The ℓ_r norm inequalities provide us with:

$$\sqrt[r]{{d_{H_p}}^r + {d_{H_q}}^r} \leq d_{H_p} + d_{H_q} \leq 2^{\frac{r-1}{r}}\sqrt[r]{{d_{H_p}}^r + {d_{H_q}}^r}.$$

Hence we have $\frac{\sqrt[r]{2}}{2}\left(d_{H_p} + d_{H_q}\right) \leq \sqrt[r]{{d_{H_p}}^r + {d_{H_q}}^r} \leq d_{H_p} + d_{H_q}$. Then the ℓ_r norms are admissible norms with $C_{\ell_r} \leq 2$, which ends the proof. □

The next corollary tells us that changing this distance does not change the large scale geometry of $\mathcal{H}$.

Corollary 4.19. *Let N_1 and N_2 be two admissible norms. Then the metric spaces $(\mathcal{H}, d_{\mathcal{H},N_1})$ and $(\mathcal{H}, d_{\mathcal{H},N_2})$ are quasi-isometric.*

The control on the distances of Lemma 4.17 will help us understand the shape of geodesic segments and geodesic lines in a horospherical product.

5 Shapes of geodesics and visual boundary of $\mathcal{H}$

5.1 Shapes of geodesic segments

In this section we focus on the shape of geodesics. We recall that in all the following H_p and H_q are assumed to be two proper, geodesically complete, δ-hyperbolic, Busemann spaces with $\delta \geq 1$, and N is assumed to be an admissible norm.

The next lemma gives a control on the maximal and minimal height of a geodesic segment in a horospherical product. It is similar to a traveller problem, who needs to walk from x to y passing by m and n. This result follows from the inequalities on maximal and minimal heights of Lemma 4.16 combined with Lemma 4.14.

Lemma 5.1. *Let $x = (x_p, x_q)$ and $y = (y_p, y_q)$ be two points of $\mathcal{H} = H_p \bowtie H_q$ such that $h(x) \leq h(y)$. Let N be an admissible norm and let $\alpha = (\alpha_p, \alpha_q)$ be a geodesic of $(\mathcal{H}, d_{\mathcal{H},N})$ linking x to y. Let $C_0 = (2853\delta C_N + 2^{851})^2$, we have:*

1. $\left|h^-(\alpha) - \left(h(x) - \frac{1}{2}d_r(x_q, y_q)\right)\right| \leq 4C_0$
2. $\left|h^+(\alpha) - \left(h(y) + \frac{1}{2}d_r(x_p, y_p)\right)\right| \leq 4C_0$.

Proof. Let us consider a point m of α such that $h(m) = h^-(\alpha)$ and a point n of α such that $h(n) = h^+(\alpha)$. Then m comes before n or n comes before m. In both cases, since $h(m) \leq h(x) \leq h(y) \leq h(n)$ and by Lemma 4.6 we have:

$$\begin{aligned} l_N(\alpha) &\geq \Delta h(x,y) + 2(h(x) - h^-(\alpha)) + 2(h^+(\alpha) - h(y)) \\ &\geq \Delta h(x,y) + 2(h(x) - h^-(\alpha)) + d_r(x_p, y_p) - 6C_0, \text{ by Lemma 4.16.} \end{aligned}$$

Furthermore Lemma 4.14 provides $l_N(\alpha) \leq \Delta h(x,y) + d_r(x_p, y_p) + d_r(x_q, y_q) + C_0$, hence:

$$\Delta h(x,y) + d_r(x_p, y_p) + d_r(x_q, y_q) + C_0 \geq \Delta h(x,y) + 2(h(x) - h^-(\alpha)) + d_r(x_p, y_p) - 6C_0,$$

which implies $\left(h(x) - \frac{1}{2}d_r(x_q, y_q)\right) - h^-(\alpha) \leq 4C_0$. In combination with the third point of Lemma 4.16 it proves the first point of our Lemma 5.1. The second point is proved similarly. □

Lemma 5.2. *Let N be an admissible norm and let $C_0 = (2853\delta C_N + 2^{851})^2$. Let$x = (x_p, x_q)$ and $y = (y_p, y_q)$ be two points of $\mathcal{H} = H_p \bowtie H_q$. Let $\alpha = (\alpha_p, \alpha_q)$ be a geodesic of $(\mathcal{H}, d_{\mathcal{H},N})$ linking x to y. Then there exist two points $a = (a_p, a_q)$, $b = (b_p, b_q)$ of α such that $h(a) = h(x)$, $h(b) = h(y)$ with the following properties:*

1. *If $h(x) \leq h(y) - 7C_0$ then:*
   - *(a) $h^-(\alpha) = h^-([x,a])$ and $h^+(\alpha) = h^+([b,y])$*
   - *(b) $|d_r(x_q, a_q) - d_r(x_q, y_q)| \leq 16C_0$ and $d_r(x_p, a_p) \leq 22C_0$*
   - *(c) $|d_r(y_p, b_p) - d_r(x_p, y_p)| \leq 16C_0$ and $d_r(y_q, b_q) \leq 22C_0$*
   - *(d) $|d_{\mathcal{H},N}(a,b) - \Delta h(a,b)| \leq 13C_0$.*
2. *If $h(y) \leq h(x) - 7C_0$ then (a), (b), (c) and (d) hold by switching the roles of x and y and switching the roles of a and b.*
3. *If $|h(x) - h(y)| \leq 7C_0$ at least one of the two previous conclusions is satisfied.*

Lemma 5.2 is illustrated on Figure 10. Its notations will be used in all section 5.

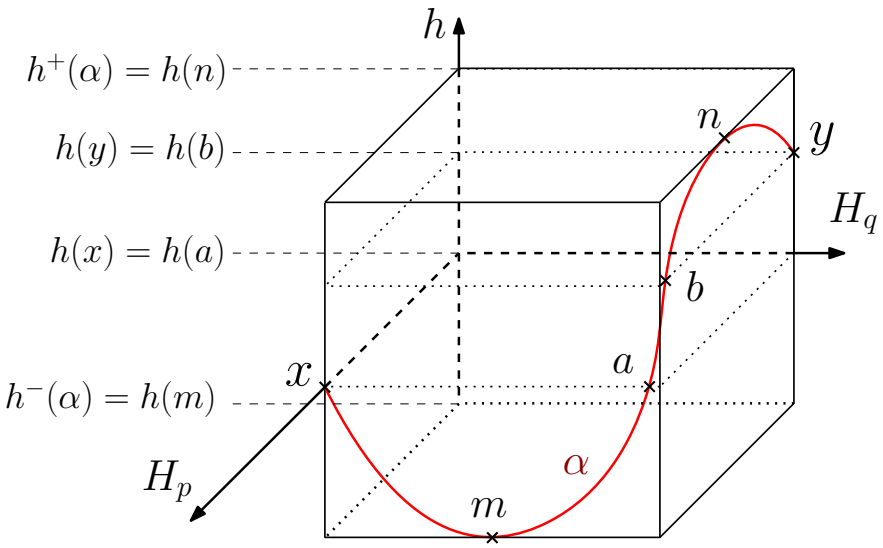

Figure 10: Notations of Lemma 5.2.

Proof. Let us consider a point m of α such that $h(m) = h^-(\alpha)$ and a point n of α such that $h(n) = h^+(\alpha)$. We first assume that m comes before n in α oriented from x to y. Let us call a the first point between m and n at height $h(x)$ and b the last point between m and n at height $h(y)$. Property (a) of our Lemma is then satisfied. Let us denote α_1 the part of α linking x to a, α_2 the part of α linking a to b and α_3 the part of α linking b to y. We have that m is a point of α_1 and that n is a point of α_3. Inequalities 2. and 3. of Lemma 4.16 used on α_1 provide $l_N(\alpha_1) \geq d(x,m) + d(m,a) \geq 2\Delta h(x,m) \geq d_r(x_q,y_q) - 6C_0$ and similarly $l_N(\alpha_3) \geq d_r(x_p,y_p) - 6C_0$. Furthermore we have $l_N(\alpha_2) \geq \Delta h(x,y)$. Combining $l_N(\alpha_1) = l_N(\alpha) - l_N(\alpha_2) - l_N(\alpha_3)$ and Lemma 4.14 we have:

$$\begin{aligned} l_N(\alpha_1) &\leq \Delta h(x,y) + d_r(x_p,y_p) + d_r(x_q,y_q) + C_0 - \Delta h(x,y) - d_r(x_p,y_p) + 6C_0 \\ &\leq d_r(x_q,y_q) + 7C_0. \end{aligned} \tag{38}$$

We have similarly that $l_N(\alpha_3) \leq d_r(x_p,y_p) + 7C_0$ and that $d_{\mathcal{H},N}(a,b) = l_N(\alpha_2) \leq \Delta h(x,y) + 13C_0$. It gives us $|d_{\mathcal{H},N}(a,b) - \Delta h(x,y)| \leq 13C_0$, point (d) of our lemma. Furthermore, using Lemma 5.1 on α and α_1 provides:

$$\begin{aligned} \left| h^-(\alpha) - \left(h(x) - \frac{1}{2} d_r(x_q,y_q) \right) \right| &\leq 4C_0, \\ \left| h^-(\alpha_1) - \left(h(x) - \frac{1}{2} d_r(x_q,a_q) \right) \right| &\leq 4C_0. \end{aligned}$$

Since $h^-(\alpha) = h^-(\alpha_1)$ we have:

$$|d_r(x_q,a_q) - d_r(x_q,y_q)| \leq 16C_0, \tag{39}$$

which is the first inequality of (b). Using the first point of Lemma 4.16 on α_1 in combination with inequality (38) gives us:

$$\begin{aligned} d_r(x_q,y_q) + 7C_0 \geq & l_N(\alpha_1) \geq \Delta h(x,a) + d_r(x_p,a_p) + d_r(x_q,a_q) - 15C_0 \\ \geq & d_r(x_p,a_p) + d_r(x_q,a_q) - 15C_0 \\ \geq & d_r(x_p,a_p) + d_r(x_q,y_q) - 31C_0, \text{ by inequality (39).} \end{aligned}$$

Then $d_r(x_p,y_p) \leq 38C_0$ the second inequality of point (b) holds. We prove similarly the inequality (c) of this lemma. This ends the proof when m comes before n. If n comes before m, the proof is still

working by orienting α from y to x hence switching the roles between x and y.

We will now prove that if $h(x) \leq h(y) - 7C_0$ then m comes before n on α oriented from x to y. Let us assume that $h(x) \leq h(y) - 7C_0$. We will proceed by contradiction, let us assume that n comes before m, using $h(m) \leq h(x) \leq h(y) \leq h(n)$ it implies:

$$\begin{aligned} l_N(\alpha) \geq & d_{\mathcal{H},N}(x,n) + d_{\mathcal{H},N}(n,m) + d_{\mathcal{H},N}(m,y) \geq \Delta h(x,n) + \Delta h(n,m) + \Delta h(m,y) \\ \geq & \Delta h(x,y) + \Delta h(y,n) + \Delta h(m,x) + \Delta h(x,y) + \Delta h(y,n) + \Delta h(m,x) + \Delta h(x,y) \\ \geq & 2\Delta h(x,y) + \Delta h(x,y) + 2\Delta h(m,x) + 2\Delta(y,n) \\ \geq & 14C_0 + \Delta h(x,y) + 2(h(x) - h^-(\alpha)) + 2(h^+(\alpha) - h(y)). \end{aligned}$$

However Lemma 4.16 applied on α provides $h^+(\alpha) \geq h(y) + \frac{1}{2}d_r(x_p, y_p) - 3C_0$ and $h^-(\alpha) \leq h(x) - \frac{1}{2}d_r(x_q, y_q) + 3C_0$. Then:

$$\begin{aligned} l_N(\alpha) \geq & 14C_0 + \Delta h(x,y) + d_r(x_p,y_p) + d_r(x_q,y_q) - 12C_0 \\ \geq & \Delta h(x,y) + d_r(x_p,y_p) + d_r(x_q,y_q) + 2C_0, \end{aligned}$$

which contradict Lemma 4.14. Hence, if $h(x) \leq h(y) - 7C_0$, the point m comes before the point n and by the first part of the proof, 1. holds. Similarly, if $h(y) \leq h(x) - 7C_0$ then n comes before m and then 2. holds. Otherwise when $|h(x) - h(y)| \leq 7C_0$ both cases could happened, then 1. or 2. hold. □

This previous lemma essentially means that if x is sufficiently below y, the geodesic α first travels in a copy of H_q in order to "loose" the relative distance between x_q and y_q, then it travels upward using a vertical geodesic from a to b until it can "lose" the relative distance between x_p and y_p by travelling in a copy of H_p. It looks like three successive geodesics of hyperbolic spaces, glued together. The idea is that the geodesic follows a shape similar to the path γ we constructed in Lemma 4.14. We formalize this in the following theorem, which tells us that a geodesic segment is in the constant neighbourhood of three vertical geodesics. It can be understood as an extension of the fact that in a hyperbolic space, a geodesic segment is in a constant neighbourhood of two vertical geodesics.

Theorem 5.3. *Let N be an admissible norm. Let $x = (x_p, x_q)$ and $y = (y_p, y_q)$ be two points of $\mathcal{H} = H_p \bowtie H_q$ and let α be a geodesic segment of $(\mathcal{H}, d_{\mathcal{H},N})$ linking x to y. Let $C_0 = (2853\delta C_N + 2^{851})^2$, there exist two vertical geodesics $V_1 = (V_{1,p}, V_{1,q})$ and $V_2 = (V_{2,p}, V_{2,q})$ such that:*

1. *If $h(x) \leq h(y) - 7C_0$ then α is in the $196C_0C_N$-neighbourhood of $V_1 \cup (V_{1,p}, V_{2,q}) \cup V_2$*
2. *If $h(x) \geq h(y) + 7C_0$ then α is in the $196C_0C_N$-neighbourhood of $V_1 \cup (V_{2,p}, V_{1,q}) \cup V_2$*
3. *If $|h(x) - h(y)| \leq 7C_0$ then at least one of the conclusions of 1. or 2. holds.*

Specifically V_1 and V_2 can be chosen such that x is close to V_1 and y is close to V_2.

Figure 11 pictures the $196C_0C_N$-neighbourhood of such vertical geodesics when $h(x) \leq h(y) - 7C_0$. When $|h(x) - h(y)| \leq 7C_0$, there are two possible shapes for a geodesic segment. In some cases, two points can be linked by two different geodesics, one of type 1 and one of type 2.

Proof. Let $m = (m_p, m_q)$ be a point of α such that $h(m) = h^-(\alpha)$, and $n = (n_p, n_q)$ be a point of α such that $h(n) = h^+(\alpha)$. Then by Lemma 5.1 we have:

$$\left|\Delta h(x,m) - \frac{1}{2}d_r(x_q,y_q)\right| \leq 4C_0. \tag{40}$$

We show similarly that:

$$\left|\Delta h(y,n) - \frac{1}{2}d_r(x_p,y_p)\right| \leq 4C_0. \tag{41}$$

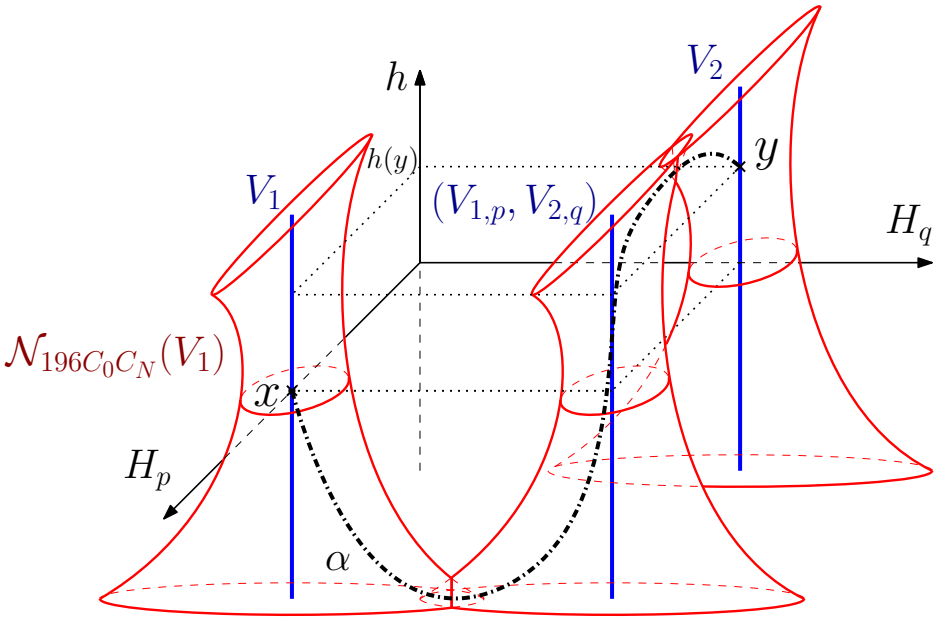

Figure 11: Theorem 5.3. The neighbourhood's shapes are distorted since when going upward, distances are contracted in the "direction" H_p and expanded in the "direction" H_q.

In the first case we assume that $h(x) \leq h(y) - 7C_0$. With notations as in Lemma 5.2, and by inequality (38), we have that $l_N([x,a]) \leq d_r(x_q, y_q) + 7C_0$, hence:

$$\begin{aligned} l_N([x,m]) =& l_N([x,a]) - l_N([a,m]) \leq d_r(x_q,y_q) + 7C_0 - \Delta h(a,m) \\ \leq& \frac{1}{2} d_r(x_q,y_q) + 11C_0, \text{ since } \Delta h(x,m) = \Delta h(a,m). \end{aligned} \tag{42}$$

It follows from this inequality that:

$$\begin{aligned} d_{H_p}(x_p,m_p) =& 2d_{H_p\times H_q}(x,m) - d_{H_q}(x_q,m_q) \leq 2d_{\mathcal{H},N}(x,m) - d_{H_q}(x_q,m_q) \\ \leq& 2l_N([x,m]) - d_{H_q}(x_q,m_q) \leq d_r(x_q,y_q) + 22C_0 - \Delta h(x,m) \leq \frac{1}{2} d_r(x_q,y_q) + 26C_0. \end{aligned}$$

Then:

$$\begin{aligned} d_r(x_p,m_p) =& d_{H_p}(x_p,m_p) - \Delta h(x,m) \leq \frac{1}{2} d_r(x_q,y_q) + 26C_0 - \Delta h(x,m) \\ \leq& 30C_0, \text{ by inequality } (40). \end{aligned}$$

Similarly $d_r(x_q, m_q) \leq 30C_0$. Let us consider the vertical geodesic V_{m_p} of H_p containing m_p, and the vertical geodesic V_{x_q} of H_q containing x_q. Let us denote x'_p the point of V_{m_p} at the height $h(x)$. Since $d_r(x_p, m_p) \leq 30C_0$, Lemma 3.3 applied on x_p and m_p provides $d_{H_p}(x_p, x'_p) \leq 31C_0$. We will then consider two paths of H_p. The first one is $\alpha_{1,p} = [x_p, m_p]$, the part of α_p linking x_p to m_p. The second one is $[m_p, x'_p]$ a piece of vertical geodesic linking m_p to x'_p. We show that these two paths have close length. Using Property 4.4 with inequalities (40) and (42) provides us with:

$$\begin{aligned} l_{H_p}([x_p,m_p]) &\leq 2l_N([x,m]) - l_{H_q}([x_q,m_q]) \leq 2\left(\frac{1}{2} d_r(x_q,y_q) + 11C_0\right) - \Delta h(x,m) \\ &\leq \Delta h(x,m) + 30C_0 \end{aligned}$$

Furthermore $l_{H_p}([x_p, m_p]) \geq \Delta h(x,m)$ and we know that $l_{H_p}([m_p, x'_p]) = \Delta h(x,m)$, hence:

$$\left| l_{H_p}([x_p,m_p]) - l_{H_p}([m_p,x'_p]) \right| \leq 30C_0$$

We already proved that their end points are also close to each other $d(x_p, x'_p) \leq 31C_0$. Since $\delta \leq C_0$, the property of hyperbolicity of H_p gives us that $\alpha_{1,p}$ is in the $(31 + 30 + 1)C_0 = 62C_0$-neighbourhood of $[m_p, x'_p]$, a part of the vertical geodesic V_{m_p}. We show similarly that $\alpha_{1,q}$ is in the $62C_0$-neighbourhood of V_{x_q}. Since N is an admissible norm, Property 4.11 gives us that α_1 is in the $124C_0C_N$-neighbourhood of (V_{m_p}, V_{x_q}). We show similarly that α_3, the portion of α linking n to y, is in the $124C_0C_N$-neighbourhood of (V_{y_p}, V_{n_q}). We now focus on α_2, the portion of α linking m to n. Let us denote $[m_p, n_p]$ the path $\alpha_{2,p}$ and $[m_q, n_q]$ the path $\alpha_{2,q}$. Then Lemma 5.1 provides us with:

$$\left|\Delta h(m,n) - \left(\Delta h(x,y) + \frac{1}{2}d_r(x_q,y_q) + \frac{1}{2}d_r(x_p,y_p)\right)\right| \leq 8C_0. \tag{43}$$

However from Lemma 4.14 and since $1152\delta C_N \leq C_0$:

$$\begin{aligned} l_N(\alpha_2) =& l_N(\alpha) - l_N(\alpha_1) - l_N(\alpha_3) \\ \leq& \Delta h(x,y) + d_r(x_p,y_p) + d_r(x_q,y_q) + C_0 - \Delta h(x,m) - \Delta h(n,y) \\ \leq& \Delta h(x,y) + \frac{1}{2}d_r(x_p,y_p) + \frac{1}{2}d_r(x_q,y_q) + 9C_0, \text{ by inequalities } (40) \text{ and } (41). \end{aligned}$$

It follows from this inequality and the fact that N is admissible that:

$$\begin{aligned} d_{H_p}(m_p,n_p) &\leq 2l_N(\alpha_2) - d_{H_q}(m_q,n_q) \leq 2\Delta h(x,y) + d_r(x_p,y_p) + d_r(x_q,y_q) + 18C_0 - \Delta h(m,n) \\ &\leq \Delta h(m,n) + 34C_0, \text{ by inequality } (43). \end{aligned}$$

Thus:

$$d_r(m_p,n_p) = d_{H_p}(m_p,n_p) - \Delta h(m,n) \leq 34C_0.$$

In the same way we have $d_r(m_q,n_q) \leq 34C_0$. Let us denote n'_p the point of V_{m_p} at the height $h(n_p)$. Since $d_r(x_p,m_p) \leq 34C_0$, Lemma 3.3 applied on m_p and n_p provides:

$$d_{H_p}(m_p,n'_p) \leq 35C_0 \tag{44}$$

Hence we have proved that $\alpha_{2,p}$ and $[m_p,n'_p]$ have their end points close to each other. Let us now prove that these paths have close lengths. We have that $l_{H_p}([m_p,n'_p]) = \Delta h(m,n)$, and from inequalities (40) and (41) we have:

$$\begin{aligned} l_{H_p}([m_p,n_p]) &\leq 2l_N(\alpha_{2,p}) - l_{H_q}([m_q,n_q]) = 2\Big(l_N(\alpha) - l_N(\alpha_1) - l_N(\alpha_3)\Big) - \Delta h(m,n) \\ &\leq 2\Big(15C_0 + \Delta h(x,y) + d_r(x_p,y_p) + d_r(x_q,y_q) - \Delta h(x,m) - \Delta h(n,y)\Big) - \Delta h(m,n) \\ &\leq 2\Big(\Delta h(x,y) + d_r(x_p,y_p) + d_r(x_q,y_q) - \Delta h(x,m) - \Delta h(n,y)\Big) - \Delta h(m,n) \\ &\leq 2\Big(\Delta h(x,y) + \Delta h(x,m) + \Delta h(n,y) + 16C_0\Big) - \Delta h(m,n) + 30C_0 \leq \Delta h(m,n) + 62C_0 \end{aligned}$$

As $l_{H_p}([m_p,n_p]) \geq \Delta h(m,n)$ we obtain:

$$|l_{H_p}([m_p,n_p]) - l_{H_p}([m_p,n'_p])| \leq 62C_0 \tag{45}$$

Then by similar arguments as for the path $\alpha_{1,p}$, inequalities (44) and (45) show that $\alpha_{2,p}$ is in the $(35 + 62 + 1)C_0 = 98C_0$ neighbourhood of V_{m_p}. Similarly we prove that $\alpha_{2,q}$ is in the $98C_0$ neighbourhood of V_{n_q}. Since N is an admissible norm, Property 4.11 gives us that α_2 is in the $196C_0C_N$-neighbourhood of (V_{m_p}, V_{n_q}).

In the second case, we assume that $h(y) \leq h(x) - 7C_0$. Then by switching the role of x and y, Lemma 5.2 gives us the result identically.

In the third case, we assume that $|h(x) - h(y)| \leq 7C_0$. Then Lemma 5.2 tells us that on of the two previous situations prevails, which proves the result. □

5.2 Coarse monotonicity

The fact that a geodesic is following a vertical geodesic is related to the next definition.

Definition 5.4. *Let C be a non negative number. A geodesic $\alpha : I \to \mathcal{H}$ of $\mathcal{H} = H_p \rtimes H_q$ is called C-coarsely increasing if $\forall t_1, t_2 \in I$:*

$$\big(t_2 > t_1 + C \big) \Rightarrow \big(h(\alpha(t_2)) > h(\alpha(t_2)) \big).$$

The geodesic α is called C-coarsely decreasing if $\forall t_1, t_2 \in I$:

$$\big(t_2 > t_1 + C \big) \Rightarrow \big(h(\alpha(t_2)) < h(\alpha(t_2)) \big).$$

The next lemma links the coarse monotonicity and the fact that a geodesic segment is close to vertical geodesics.

Lemma 5.5. *Let N be an admissible norm and let $C_0 = (2853\delta C_N + 2^{851})^2$. Let $x = (x_p, x_q)$ and $y = (y_p, y_q)$ be two points of $\mathcal{H} = H_p \rtimes H_q$ and let α be a geodesic segment of $(\mathcal{H}, d_{\mathcal{H},N})$ linking x to y. Let $m \in \alpha$ and $n \in \alpha$ be two points in $\mathcal{H}$ such that $h^-(\alpha) = h(m)$ and $h^+(\alpha) = h(n)$. We have:*

1. *If $h(x) \leq h(y) - 7C_0$, then α is $17C_0$-coarsely decreasing on $[x, m]$ and $17C_0$-coarsely increasing on $[m, n]$ and $17C_0$-coarsely decreasing on $[n, y]$.*
2. *If $h(x) \geq h(y) + 7C_0$, then α is $17C_0$-coarsely increasing on $[x, n]$ and $17C_0$-coarsely decreasing on $[n, m]$ and $17C_0$-coarsely increasing on $[m, n]$.*
3. *If $|h(x) - h(y)| \leq 7C_0$ then the conclusions of 1. or 2. holds.*

Proof. Assume that $h(x) \leq h(y) - 7C_0$. Then from inequality (42) in the proof of Theorem 5.3, $l_N([x, m]) \leq \frac{1}{2} d_r(x_q, y_q) + 11C_0$. Furthermore Lemma 5.1 gives us that $\left|\Delta h(x, m) - \frac{1}{2} d_r(x_q, y_q)\right| \leq 4C_0$. Then:

$$l_N([x, m]) \leq \Delta h(x, m) + 15C_0. \tag{46}$$

We will proceed by contradiction, assume that $[x, m]$ is not $15C_0$-coarsely decreasing, then there exists $i_1 \in \alpha$, $i_2 \in \alpha$ such that $h(i_1) = h(i_2)$ and $l([i_1, i_2]) > 15C_0$. Hence:

$$\begin{aligned} l_N([x, m]) &\geq l_N([x, i_1]) + l_N([i_1, i_2]) + l_N([i_2, m]) \geq \Delta h(x, i_1) + l_N([i_1, i_2]) + \Delta h(i_2, m) \\ &> \Delta h(x, m) + 15C_0, \end{aligned}$$

which contradicts inequality (46). Then $[x, m]$ is $15C_0$-coarsely decreasing. We show in a similar way that $[m, n]$ is $17C_0$-coarsely increasing and that $[n, y]$ is $15C_0$-coarsely decreasing. This proves the first point of our lemma. The second point is proved by switching the roles of x and y. We now assume $|h(x) - h(y)| \leq 7C_0$, as in the proof of Theorem 5.3 the inequality (42) or a corresponding inequality holds, which ends the proof. □

5.3 Shapes of geodesic rays and geodesic lines

In this section we are focusing on using the previous results to get informations on the shapes of geodesic rays and geodesic lines. We first link the coarse monotonicity of a geodesic ray to the fact that it is close to a vertical geodesic. Let $\lambda \geq 1$ and $c \geq 0$, a (λ, c)-quasigeodesic of the metric space $(\mathcal{H}, d_{\mathcal{H},N})$ is the image of a function $\phi : \mathbb{R} \to \mathcal{H}$ verifying that $\forall t_1, t_2 \in \mathbb{R}$:

$$\frac{|t_1 - t_2|}{\lambda} - c \leq d_{\mathcal{H},N}\big(\phi(t_1), \phi(t_2)\big) \leq \lambda |t_1 - t_2| + c \tag{47}$$

Lemma 5.6. *Let N be an admissible norm and let $C_0 = (2853\delta C_N + 2^{851})^2$. Let $\alpha = (\alpha_p, \alpha_q)$ be a geodesic ray of $(\mathcal{H}, d_{\mathcal{H},N})$ and let K be a positive number such that α is K-coarsely monotone. Then α_p and α_q are $(1, 26C_0 + 8K)$-quasigeodesics.*

Proof. Let t_1 and t_2 be two times. Let us denote $x = (x_p, x_q) = \alpha(t_1)$ and $y = (y_p, y_q) = \alpha(t_2)$. We apply Lemma 5.2 on the part of α linking x to y denoted by $[x, y]$. By K-coarse monotonicity of α we have that $d(x, a)_{\mathcal{H},N} \leq K$ and $d_{\mathcal{H},N}(b, y) \leq K$. Hence using d) of Lemma 5.2:

$$\begin{aligned}\Delta h(x,y) \leq d_{\mathcal{H},N}(x,y) &\leq d_{\mathcal{H},N}(x,a) + d_{\mathcal{H},N}(a,b) + d_{\mathcal{H},N}(b,y) \leq K + \Delta h(a,b) + 13C_0 + K \\ &\leq \Delta h(x,y) + \Delta h(x,a) + \Delta h(b,y) + 13C_0 + 2K \leq \Delta h(x,y) + 13C_0 + 4K.\end{aligned}$$

Furthermore, $d_{H_p}(x_p, y_p) \geq \Delta h(x_p, y_p) = \Delta h(x, y)$ and $d_{H_q}(x_q, y_q) \geq \Delta h(x, y)$. Since N is an admissible norm we have:

$$\begin{aligned}\Delta h(x,y) \leq d_{H_p}(x_p,y_p) &= 2d_{H_p \times H_q}(x,y) - d_{H_q}(x_q,y_q) \leq 2d_{\mathcal{H},N}(x,y) - d_{H_q}(x_q,y_q) \\ &\leq 2\Delta h(x,y) + 13C_0 + 4K - \Delta h(x,y) \leq \Delta h(x,y) + 13C_0 + 4K.\end{aligned}$$

Hence:

$$d_{\mathcal{H},N}(x,y) - 26C_0 - 8K \leq d_{H_p}(x_p,y_p) \leq d_{\mathcal{H},N}(x,y) + 26C_0 + 8K,$$

By definition we have $x_p = \alpha_p(t_1)$, $y_p = \alpha_p(t_2)$ and $d_{\mathcal{H},N}(x, y) = |t_1 - t_2|$. Then α_p is a $(1, 26C_0 + 8K)$-quasigeodesic ray. We prove similarly that α_q is a $(1, 26C_0 + 8K)$-quasigeodesic ray. □

We will now make use of the rigidity property of quasi-geodesics in Gromov hyperbolic spaces, presented in Theorem 3.1 p.41 of [3, Coornaert, Delzant, Papadopoulos].

Theorem 5.7 ([3])**.** *Let H be a δ-hyperbolic geodesic space. If $f : \mathbb{R} \to H$ is a (λ, k)-quasi geodesic, then there exists a constant $\kappa > 0$ depending only on δ, λ and k such that the image of f is in the κ-neighbourhood of a geodesic in H.*

Lemma 5.8. *Let N be an admissible norm and let T_1 and T_2 be two real numbers. Let $\alpha = (\alpha_p, \alpha_q) : [T_1, +\infty[\to \mathcal{H}$ be a geodesic ray of $(\mathcal{H}, d_{\mathcal{H},N})$. Let K be a positive number such that α is K-coarsely monotone. Then there exists a constant $\kappa > 0$ depending only on K, δ and N such that α is in the κ-neighbourhood of a vertical geodesic ray $V : [T_2; +\infty[\to \mathcal{H}$ and such that $d_{\mathcal{H},N}\big(\alpha(T_1), V(T_2)\big) \leq \kappa$.*

Proof. We assume without loss of generality that $\lim\limits_{t \to +\infty} h(\alpha(t)) = +\infty$. Let $C_0 = (2853\delta C_N + 2^{851})^2$, by Lemma 5.6, α_p is a $(1, 26C_0 + 8K)$-quasi geodesic ray. Then Theorem 5.7 says there exists $\kappa_p > 0$ depending only on $26C_0 + 8K$ and δ such that α_p is in the κ_p-neighbourhood of a geodesic V_p. Since C_0 depends only on δ and N, κ_p depends only on K, δ and N. Then $\lim\limits_{t \to +\infty} h(\alpha(t)) = +\infty$ gives us $\lim\limits_{t \to +\infty} h(V_p(t)) = +\infty$ which implies that V_p is a vertical geodesic of H_p. We will now build the vertical geodesic we want in H_q. We have $\lim\limits_{t \to +\infty} h(\alpha_q(t)) = -\infty$ and by Lemma 5.6:

$$\Delta h(\alpha_q(t_1), \alpha_q(t_2)) - 26C_0 - 8K \leq d_{H_q}(\alpha_q(t_1), \alpha_q(t_2)) \leq \Delta h(\alpha_q(t_1), \alpha_q(t_2)) + 26C_0 + 8K.$$

Since H_q is Busemann, there exists a vertical geodesic ray β starting at $\alpha_q(T_1)$. Since β is parametrised by its height, $\alpha_q \cup \beta$ is also a $(1, 26C_0 + 8K)$-quasi geodesic, hence there exists κ_q and V_q depending only on K, δ and N such that $\alpha_q \cup \beta$ is in the κ_q-neighbourhood of V_q. Since $\lim\limits_{t \to -\infty} h(V_q(t)) = +\infty$, V_q is a vertical geodesic of H_q. Furthermore, by Property 4.11, $d_{\mathcal{H},N} \leq 2C_N(d_{H_p} + d_{H_q})$, hence there exists κ depending only on K, δ and N such that α is in the κ-neighbourhood (for $d_{\mathcal{H},N}$) of (V_p, V_q), a vertical geodesic of $(\mathcal{H}, d_{\mathcal{H},N})$. Since $h(\alpha(t)) \geq h(\alpha(T_1)) - 26C_0 - 8K =: M$, α is in the κ-neighbourhood of $\Big(V_p\big([M - \kappa; +\infty[\big), V_q\big(]-\infty; -M + \kappa]\big)\Big)$ which is a vertical geodesic ray.
We will now show that the starting points of α and V are close to each other. Let us denote T_1' a time

such that $d_{\mathcal{H},N}\big(\alpha(T_1), V(T_1')\big) \leq \kappa$, then $\Delta h\big(\alpha(T_1), V(T_1')\big) \leq \kappa$, hence $|T_1' - M| \leq 26C_0 + 8K + \kappa$. Then by the triangular inequality:

$$\begin{aligned} d_{\mathcal{H},N}\Big(\alpha(T_1), V(M-\kappa)\Big) \leq & d_{\mathcal{H},N}\Big(\alpha(T_1), V(T_1')\Big) + d_{\mathcal{H},N}\Big(V(T_1'), V(M-\kappa)\Big) \\ \leq & \kappa + 26C_0 + 8K + \kappa + \kappa = 26C_0 + 8K + 3\kappa \end{aligned}$$

Let us denote $\kappa' := 26C_0 + 8K + 3\kappa \geq \kappa$ and $T_2 := M - \kappa$. Hence $\alpha : [T_1; +\infty[\to \mathcal{H}$ is in the κ'-neighbourhood of a vertical geodesic ray $V : [T_2 : +\infty[\to \mathcal{H}$, we have $d_{\mathcal{H},N}\big(\alpha(T_1), V(T_2)\big) \leq \kappa'$ and κ' depends only on δ and K. □

Lemma 5.9. *Let N be an admissible norm and let $\alpha : \mathbb{R}^+ \to \mathcal{H}$ be a geodesic ray of $(\mathcal{H}, d_{\mathcal{H},N})$. Then α changes its $17C_0$-coarse monotonicity at most once.*

Proof. Let $\alpha : \mathbb{R}^+ \to \mathcal{H}$ be a geodesic ray. Thanks to Lemma 5.5 α changes at most twice of $17C_0$-coarse monotonicity. Indeed, assume it changes three times, applying Lemma 5.5 on the geodesic segment which includes these three times provides a contradiction. We will show in the following that it actually only changes once.

Assume α changes twice of $17C_0$-coarse monotonicity. Then α must be first $17C_0$-coarsely increasing or $17C_0$-coarsely decreasing. We assume without loss of generality that α is first $17C_0$-coarsely decreasing. Then there exist $t_1, t_2, t_3 \in \mathbb{R}$ such that α is $17C_0$-coarsely decreasing on $[\alpha(t_1), \alpha(t_2)]$ then $17C_0$-coarsely increasing on $[\alpha(t_2), \alpha(t_3)]$ then $17C_0$-coarsely decreasing on $[\alpha(t_3), \alpha(+\infty)[$. Hence Lemma 5.8 applied on $[\alpha(t_3), \alpha(+\infty)[$ implies that there exists $\kappa > 0$ depending only on δ (since the constant of coarse monotonicity depends only on δ) and a vertical geodesic ray $V = (V_p, V_q)$ such that $[\alpha(t_3), \alpha(+\infty)[$ is in the κ-neighbourhood of V. Since $h^+([\alpha(t_3), \alpha(+\infty)[) < +\infty$, we have that $\lim_{t \to +\infty} h(\alpha(t)) = -\infty$, hence there exists $t_4 \geq t_3$ such that $h(\alpha(t_4)) \leq h(\alpha(t_1)) - 7C_0$. Then Lemma 5.5 tells us that α is first $17C_0$-coarsely increasing, which contradicts what we assumed. □

We have classified the possible shapes of geodesic rays. Since geodesics lines are two geodesic rays glued together, we will be able to classify their shapes too.

Definition 5.10. *Let N be an admissible norm and let $\alpha = (\alpha_p, \alpha_q) : \mathbb{R} \to \mathcal{H}$ be a path of $(\mathcal{H}, d_{\mathcal{H},N})$. Let $\kappa \geq 0$.*

1. *α is called H_p-type at scale κ if and only if:*
   - *(a) α_p is in a κ-neighbourhood of a geodesic of H_p*
   - *(b) α_q is in a κ-neighbourhood of a vertical geodesic of H_q.*
2. *α is called H_q-type at scale κ if and only if:*
   - *(a) α_q is in a κ-neighbourhood of a geodesic of H_q*
   - *(b) α_p is in a κ-neighbourhood of a vertical geodesic of H_p.*

The H_p-type paths follow geodesics of H_p, meaning that they are close to a geodesic in a copy of H_p inside $\mathcal{H}$. The H_q-type paths follow geodesics of H_q.

Remark 5.11. *In a horospherical product, being close to a vertical geodesic is equivalent to be both H_p-type and H_q-type.*

Theorem 5.12. *Let N be an admissible norm. There exists $\kappa \geq 0$ depending only on δ and N such that for any $\alpha : \mathbb{R} \to \mathcal{H}$ geodesic of $(\mathcal{H}, d_{\mathcal{H},N})$ at least one of the two following statements holds.*

1. *α is a H_p-type geodesic at scale κ of $(\mathcal{H}, d_{\mathcal{H},N})$*
2. *α is a H_q-type geodesic at scale κ of $(\mathcal{H}, d_{\mathcal{H},N})$*

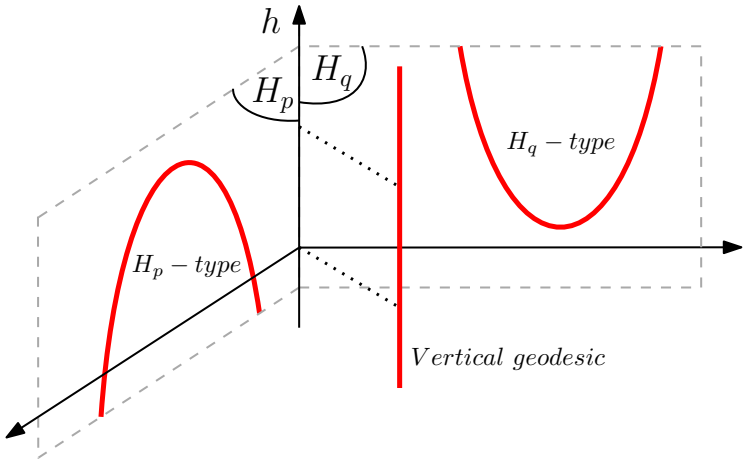

Figure 12: Different type of geodesics in $\mathcal{H} = H_p \bowtie H_q$.

Proof. It follows from Lemma 5.9 that α changes its coarse monotonicity at most once. Otherwise there would exist a geodesic ray included in α that changes at least two times of coarse monotonicity. We cut α in two coarsely monotone geodesic rays $\alpha_1 : [0, +\infty[\to \mathcal{H}$ and $\alpha_2 : [0, +\infty[\to \mathcal{H}$ such that up to a parametrization $\alpha_1(0) = \alpha_2(0)$ and $\alpha_1 \cup \alpha_2 = \alpha$. By Lemma 5.8 there exists κ_1 and κ_2 depending only on δ such that α_1 is in the κ_1-neighbourhood of a vertical geodesic ray $V_1 = (V_{1,p}, V_{1,q}) : [0; +\infty[\to \mathcal{H}$ and such that α_2 is in the κ_2-neighbourhood of a vertical geodesic ray $V_2 = (V_{2,p}, V_{2,q}) : [0; +\infty[\to \mathcal{H}$. This lemma also gives us $d_{\mathcal{H},N}\big(\alpha_1(0), V_1(0)\big) \leq \kappa_1$ and $d_{\mathcal{H},N}\big(\alpha_2(0), V_2(0)\big) \leq \kappa_2$.
Assume that $\lim\limits_{t\to+\infty} h(V_{1,p}(t)) = \lim\limits_{t\to+\infty} h(V_{2,p}(t)) = +\infty$, then they are both vertical rays hence are close to a common vertical geodesic ray. Furthermore $\lim\limits_{t\to+\infty} h(V_{1,q}(t)) = \lim\limits_{t\to+\infty} h(V_{2,q}(t)) = -\infty$ in that case. Let W_q be the non continuous path of H_q defined as follows.

$$W_q(t) = \begin{cases} V_{1,q}(-t) & \forall t \in]-\infty; 0] \\ V_{2,q}(t) & \forall t \in]0; +\infty[\end{cases}$$

We now prove that $W_q : \mathbb{R} \to H_q$ is a quasigeodesic of H_q. Let t_1 and t_2 be two real numbers. Since $V_{1,q}$ and $V_{2,q}$ are geodesics, $d_{H_q}(W_q(t_1), W_q(t_2)) = |t_1 - t_2|$ if t_1 and t_2 are both non positive or both positive. Thereby we can assume without loss of generality that t_1 is non positive and that t_2 is positive. We also assume without loss of generality that $|t_1| \geq |t_2|$. The quasi-isometric upper bound is given by:

$$\begin{aligned} d_{H_q}\big(W_q(t_1), W_q(t_2)\big) &= d_{H_q}\big(V_{1,q}(-t_1), V_{2,q}(t_2)\big) \\ &\leq d_{H_q}\big(V_{1,q}(-t_1), V_{1,q}(0)\big) + d_{H_q}\big(V_{1,q}(0), V_{2,q}(0)\big) + d_{H_q}\big(V_{2,q}(0), V_{2,q}(t_2)\big) \\ &\leq |t_1| + \kappa_1 + \kappa_2 + |t_2| \\ &\leq |t_1 - t_2| + \kappa_1 + \kappa_2, \text{ since } t_1 \text{ and } t_2 \text{ have different signs.} \end{aligned}$$

It remains to prove the lower bound of the quasi-geodesic definition on W_q.

$$\begin{aligned} d_{H_q}\big(W_q(t_1), W_q(t_2)\big) &= d_{H_q}\big(V_{1,q}(-t_1), V_{2,q}(t_2)\big) \\ &\geq \frac{1}{2C_N} d_{\mathcal{H},N}\big(V_1(-t_1), V_2(t_2)\big) - d_{H_p}\big(V_{1,p}(-t_1), V_{2,p}(t_2)\big) \\ &\geq \frac{1}{2C_N} d_{\mathcal{H},N}\big(\alpha(t_1), \alpha(t_2)\big) - \frac{\kappa_1 + \kappa_2}{C_N} - d_{H_p}\big(V_{1,p}(-t_1), V_{2,p}(t_2)\big). \end{aligned} \tag{48}$$

The Busemann assumption on H_p provides us with:

$$d_{H_p}\big(V_{1,p}(-t_1), V_{2,p}(-t_1)\big) \leq d_{H_p}\big(V_{1,p}(0), V_{2,p}(0)\big) \leq \kappa_1 + \kappa_2.$$

Since α is a geodesic and by using the triangular inequality on (48) we have:

$$\begin{aligned} d_{H_q}\big(W_q(t_1), W_q(t_2)\big) &\geq \frac{|t_1 - t_2|}{2C_N} - d_{H_p}\big(V_{1,p}(-t_1), V_{2,p}(-t_1)\big) - d_{H_p}\big(V_{2,p}(-t_1), V_{2,p}(t_2)\big) - \frac{\kappa_1 + \kappa_2}{C_N} \\ &\geq \frac{|t_1 - t_2|}{2C_N} - \Delta h\big(V_{2,q}(-t_1), V_{2,q}(t_2\big) - \left(\frac{1}{C_N} + 1\right)(\kappa_1 + \kappa_2). \end{aligned}$$

Assume that $\Delta h\big(V_{2,q}(-t_1), V_{2,q}(t_2)\big) \leq \frac{|t_1 - t_2|}{4C_N}$, then:

$$d_{H_q}\big(W_q(t_1), W_q(t_2)\big) \geq \frac{|t_1 - t_2|}{4C_N} - \left(\frac{1}{C_N} + 1\right)(\kappa_1 + \kappa_2).$$

Hence W_q is a $\left(\frac{1}{4C_N}, \left(\frac{1}{C_N} + 1\right)(\kappa_1 + \kappa_2)\right)$ quasi-geodesic, which was the remaining case. Since κ_1 and κ_2 depend only on δ and N, there exists a constant κ' depending only on δ and N such that $V_{1,q} \cup V_{2,q}$ is in the κ'-neighbourhood of a geodesic of H_q. The geodesic α is a H_q-type geodesic in this case. Assume $\lim\limits_{t \to +\infty} h(V_{1,p}(t)) = \lim\limits_{t \to +\infty} h(V_{2,p}(t)) = -\infty$, we prove similarly that α is a H_p-type geodesic. □

If a geodesic is both H_p-type at scale κ and H_q-type at scale κ, then it is in a κ-neighbourhood of a vertical geodesic of $\mathcal{H}$.

5.4 Visual boundary of $\mathcal{H}$

We will now look at the visual boundary of our horospherical products. This notion is described for the SOL geometry in the work of Troyanov [12, Troyanov] through the objects called geodesic horizons. We extend one of the definitions presented in page 4 of [12, Troyanov] for horospherical products.

Definition 5.13. *Two geodesic of a metric space X are called asymptotically equivalent if they are at finite Hausdorff distance from each other.*

Definition 5.14. *Let X be a metric space and let o be a base point of X. The visual boundary of X is the set of asymptotic equivalence classes of geodesic rays $\alpha : \mathbb{R}^+ \to$ such that $\alpha(0) = o$. It is denoted $\partial_o X$.*

We will use a result of [11, Papadopoulos] to describe the visual boundary of horospherical products.

Property 5.15 (Property 10.1.7 p.234 of [11])**.** *Let X be a proper Busemann space, let q be a point in X and let $r : [0, +\infty[\to X$ be a geodesic ray. Then, there exists a unique geodesic ray r' starting at q that is asymptotic to r.*

Theorem 5.16. *Let N be an admissible norm. We fix base points and directions $(w_p, a_p) \in H_p \times \partial H_p$, $(w_q, a_q) \in H_q \times \partial H_q$. Let $\mathcal{H} = H_p \bowtie H_q$ be the horospherical product with respect to (w_p, a_p) and (w_q, a_q). Then the visual boundary of $(\mathcal{H}, d_{\mathcal{H},N})$ with respect to a base point $o = (o_p, o_q)$ is given by:*

$$\begin{aligned} \partial_o \mathcal{H} =& \Big(\big(\partial H_p \setminus \{a_p\}\big) \times \{a_q\}\Big) \bigcup \Big(\{a_p\} \times \big(\partial H_q \setminus \{a_q\}\big)\Big) \\ =& \Big(\big(\partial H_p \times \{a_q\}\big) \bigcup \big(\{a_p\} \times \partial H_q\big)\Big) \setminus \{(a_p, a_q)\} \end{aligned}$$

The fact that (a_p, a_q) is not allowed as a direction in $\mathcal{H}$ is understandable since both heights in H_p and H_q would tend to $+\infty$, which is impossible by the definition of $\mathcal{H}$.

Proof. Let α be a geodesic ray. Lemma 5.9 implies that there exists $t_0 \in \mathbb{R}$ such that α is coarsely monotone on $[t_0, +\infty[$. Then Lemma 5.8 tells us that $\alpha([t_0, +\infty[)$ is at finite Hausdorff distance from a vertical geodesic ray $V = (V_p, V_q)$, hence α is also at finite Hausdorff distance from V. Since H_p is Busemann and proper, Property 5.15 ensure us there exists V_p' a vertical geodesic ray such that V_p and V_p' are at finite Hausdorff distance with $V_p'(0) = o_p$. Similarly, there exists V_q' a vertical geodesic ray of H_q with $V_q'(0) = o_q$ such that V_q and V_q' are at finite Hausdorff distance. Since there is at least

one vertical geodesic ray $V' = (V'_q, V'_p)$ in every asymptotic equivalence class of geodesic rays, $\partial_o\mathcal{H}$ is the set of asymptotic equivalence classes of vertical geodesic rays starting at o. Hence an asymptotic equivalence class can be identified by the couple of directions of a vertical geodesic ray. Then $\partial_o\mathcal{H}$ can be identified to:

$$\Big(\big(\partial H_p \smallsetminus \{a_p\}\big) \times \{a_q\}\Big) \bigcup \Big(\{a_p\} \times \big(\partial H_q \smallsetminus \{a_q\}\big)\Big).$$

the union between downward directions and upward directions, which proves the theorem. □

Example 5.17. *In the case of SOL, H_p and H_q are hyperbolic planes $\mathbb{H}_2$, hence their boundaries are $\partial H_p = \partial\mathbb{H}_2 = S^1$ and $\partial H_q = S^1$. Then $\partial_o SOL$ can be identified to the following set:*

$$\big(S^1 \smallsetminus \{a_p\}\big) \times \{a_q\} \bigcup \{a_p\} \times \big(S^1 \smallsetminus \{a_q\}\big). \tag{49}$$

It can be seen as two lines at infinity, one upward $\{a_p\} \times \big(S^1 \smallsetminus \{a_q\}\big)$ and the other one downward $\big(S^1 \smallsetminus \{a_p\}\big) \times \{a_q\}$.

It is similar to Proposition 6.4 of [12, Troyanov], however it is not the same result.

A possible way of generalising Theorems 5.3, 5.12 and 5.16 is looking at what happens when the Busemann hypothesis of our components H_p and H_q is removed. However in that case it is already unclear how to make a relevant definition for the horospherical product.

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