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Hole Locations and a Filling Method for Coprime Planar Arrays for DOA Estimation

Xiao Yang, Yide Wang, and Pascal Chargé

Abstract—Coprime linear arrays (CLAs) have drawn lots of attention. Efforts have been made to reveal the hole locations in the difference coarrays of CLAs, based on which many methods have been proposed to fill the holes, lengthening the consecutive difference coarray part and increasing the effective degrees of freedom (DOFs). Compared with CLAs, two dimensional (2D) coprime planar arrays (CPAs) are more relevant to real applications. However, no closed-form expressions for the hole locations in the difference coarray of a CPA have been found in the open literature, and the advantage in terms of DOFs of CPAs has not been well explored. In this letter, the structure of the difference coarrays of CPAs is investigated and the exact expressions of all hole positions are provided. Then, a holes-filling method is proposed, by which the most critical holes in the difference coarray can be filled such that a difference coarray with more consecutive lags as well as higher effective DOFs can be obtained.

Index Terms—coprime planar arrays, difference coarray, DOA estimation, DOFs, holes-filling

I. INTRODUCTION

Coprime linear arrays (CLAs) have been found to be attractive in direction of arrival (DOA) estimation thanks to their high degrees of freedom (DOFs) [1], [2]. Generally, by exploiting the unique lags in the difference coarray obtained from the covariance matrix of received signals, $O(MN)$ DOFs can be achieved by a CLA with $O(M+N)$ sensor elements. However, since there are holes existing in the difference coarray and only the consecutive segments of the difference coarray can be directly used by subspace based DOA estimation algorithms like MUSIC [3] and ESPRIT [4], the effective DOFs are not as high as expected.

The strategic point in filling the holes in the difference coarray and increasing the effective DOFs is to find the exact expressions of the hole locations. Some efforts have been made to address this issue recently. [5] studies the characteristics of the difference coarrays of CLAs, and provides the expression of the hole positions in the negative part of one cross difference coarray. In [6], the closed-form expressions of the final hole locations in the difference coarrays of CLAs are derived, and by adding a complementary subarray, a hole-free difference coarray can be obtained.

Compared with linear arrays, two dimensional (2D) planar arrays are more relevant to real applications. There are many research studies developed for 2D coprime planar arrays (CPAs). In [7] and [8], a CPA is treated as two uniform planar subarrays, which simplifies the system model, but leads to a significant loss of DOFs. In [9], the CPA geometry is

generalized, resulting in higher DOFs than [7] and [8] with same number of sensor elements. However, it also deals with the subarrays and the significant advantage in terms of DOFs is sacrificed. In order to exploit the high DOFs offered by the coprime geometry, the difference coarrays should be applied instead of the physical arrays. However, due to the existence of holes, the consecutiveness of the difference coarrays of CPAs is highly limited, which significantly reduces the number of effective DOFs. Unfortunately, no closed-form expressions for the hole locations in the difference coarrays of CPAs have been found in the open literature, which rises the major challenge in holes-filling and DOFs-increasing for CPAs.

Compared with other existing 2D sparse array configurations [10]–[12], CPAs are more attractive because of their limited mutual coupling effect property. To offer a better understanding of CPAs and facilitate the future research in this field, in this letter, CPAs are investigated from the perspective of difference coarrays. Closed-form expressions of the exact hole locations are derived, based on which an efficient method is proposed to fill the most critical holes, such that a difference coarray with more consecutive lags can be generated and higher effective DOFs can be obtained. Simulation results are provided to support our proposition.

II. PRELIMINARY

A conventional CPA consists of two uniform square subarrays. The first subarray has $N \times N$ antenna elements with inter-element spacing Md , and the second subarray has $M \times M$ antenna elements with inter-element spacing Nd , where $d = \frac{\lambda}{2}$ with λ the wavelength of incoming signals, and M, N two coprime integers. The locations of the antenna elements of the coprime planar array can be expressed as:

$$\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2 \quad (1)$$

with

$$\mathbb{L}_1 = \{(p_1Md, p_2Md) \mid 0 \leq p_1, p_2 \leq N - 1\} \quad (2)$$

$$\mathbb{L}_2 = \{(q_1Nd, q_2Nd) \mid 0 \leq q_1, q_2 \leq M - 1\} \quad (3)$$

denoting the antenna element locations of the two subarrays respectively. Without loss of generality, we assume that $M < N$. Fig. 1 shows the case where $N = 5$ and $M = 3$.

The difference coarray of a CPA is defined as

$$\mathbb{D} = \{\mathbf{n}_1 - \mathbf{n}_2 \mid \mathbf{n}_1, \mathbf{n}_2 \in \mathbb{L}\} \quad (4)$$

of which the elements, generated by all pairs of sensors, are known as lags. The difference coarray of the CPA in Fig. 1 is shown in Fig. 2. Assume a far-field and narrowband signal $s(t)$ impinging on a CPA from direction (θ, φ) , where θ and

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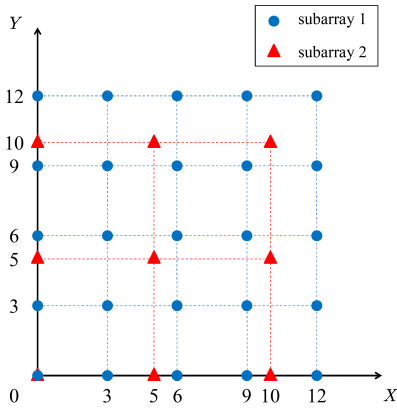


Fig. 1. Conventional coprime planar array with $N = 5$ and $M = 3$.

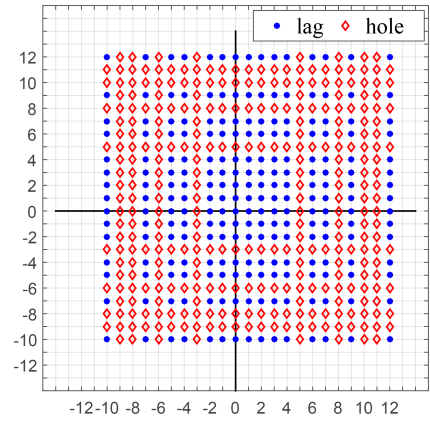


Fig. 3. Cross difference coarray \mathbb{D}_c^{1-2} of a CPA with $N = 5$ and $M = 3$.

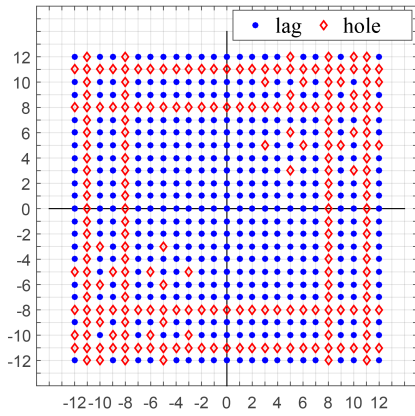


Fig. 2. Difference coarray \mathbb{D} of a CPA with $N = 5$ and $M = 3$.

φ denote the elevation and azimuth angles respectively, then the signal received at the sensor located at \mathbf{n}_1 is given by

$$x_1(t) = s(t) e^{j \frac{2\pi}{\lambda} \mathbf{n}_1(\sin \theta \cos \varphi, \sin \theta \sin \varphi)^T} \quad (5)$$

Let σ_s^2 denote the power of $s(t)$, and the correlation between the signals received by the sensors at \mathbf{n}_1 and \mathbf{n}_2 is given by

$$E \{x_1(t) x_2^*(t)\} = \sigma_s^2 e^{j \frac{2\pi}{\lambda} (\mathbf{n}_1 - \mathbf{n}_2)(\sin \theta \cos \varphi, \sin \theta \sin \varphi)^T} \quad (6)$$

The difference coarray elements $\mathbf{n}_1 - \mathbf{n}_2$ appear in the exponents of the correlation terms, which behave like signals received by virtual sensors (lags) in the difference coarray. Containing much more distinct lags than the number of physical sensors, the difference coarray is usually applied instead of the physical array to exploit its higher number of DOFs [2]. However, the difference coarray is not consecutive, and the effective DOFs are highly limited by the existence of holes.

III. HOLES LOCATIONS

According to the definition of the difference coarray (4), it comes that

$$\mathbb{D} = \mathbb{D}_c^{1-2} \cup \mathbb{D}_c^{2-1} \cup \mathbb{D}_s^1 \cup \mathbb{D}_s^2 \quad (7)$$

where \mathbb{D}_c^{1-2} and \mathbb{D}_c^{2-1} denote the cross difference coarrays of the two subarrays with

$$\mathbb{D}_c^{1-2} = \{\mathbf{n}_1 - \mathbf{n}_2 \mid \mathbf{n}_1 \in \mathbb{L}_1, \mathbf{n}_2 \in \mathbb{L}_2\} \quad (8)$$

$$\mathbb{D}_c^{2-1} = \{\mathbf{n}_2 - \mathbf{n}_1 \mid \mathbf{n}_1 \in \mathbb{L}_1, \mathbf{n}_2 \in \mathbb{L}_2\} \quad (9)$$

and \mathbb{D}_s^1 and \mathbb{D}_s^2 denote the self difference coarrays of the two subarrays with

$$\mathbb{D}_s^1 = \{\mathbf{n}_1 - \mathbf{n}_2 \mid \mathbf{n}_1, \mathbf{n}_2 \in \mathbb{L}_1\} \quad (10)$$

$$\mathbb{D}_s^2 = \{\mathbf{n}_1 - \mathbf{n}_2 \mid \mathbf{n}_1, \mathbf{n}_2 \in \mathbb{L}_2\} \quad (11)$$

Being different from CLAs, the self difference coarrays of CPAs do not form a subset of the cross difference coarrays. Therefore the self difference coarrays and cross difference coarrays of CPAs should be both analyzed. We first discuss the characteristics of \mathbb{D}_c^{1-2} (8). Taking d as unit, \mathbb{D}_c^{1-2} can be expressed as

$$\mathbb{D}_c^{1-2} = \{(p_1M - q_1N, p_2M - q_2N) \mid 0 \leq p_1, p_2 \leq N-1, 0 \leq q_1, q_2 \leq M-1\} \quad (12)$$

According to the values of p_1, p_2, q_1 and q_2 , it is easy to see that \mathbb{D}_c^{1-2} is located in $\{(x, y) \mid -(M-1)N \leq x, y \leq (N-1)M\}$. The set \mathbb{D}_c^{1-2} associated to the example CPA in Fig. 1 is shown in Fig. 3.

In the following, the locations of the holes in \mathbb{D}_c^{1-2} will be provided with detailed proof. Four general rules can be summarized as:

- 1) In $\{(x, y) \mid 0 \leq x \leq (N-1)M, -(M-1)N \leq y \leq (N-1)M\}$, the position (x, y) would be a hole if $x = aM + bN$, with $a \geq 0, b \geq 1$;
- 2) In $\{(x, y) \mid -(M-1)N \leq x \leq 0, -(M-1)N \leq y \leq (N-1)M\}$, the position (x, y) would be a hole if $x = aM + bN$, with $a \leq -1, b \leq 0$;
- 3) In $\{(x, y) \mid -(M-1)N \leq x \leq (N-1)M, 0 \leq y \leq (N-1)M\}$, the position (x, y) would be a hole if $y = aM + bN$, with $a \geq 0, b \geq 1$;
- 4) In $\{(x, y) \mid -(M-1)N \leq x \leq (N-1)M, -(M-1)N \leq y \leq 0\}$, the position (x, y) would be a hole if $y = aM + bN$, with $a \leq -1, b \leq 0$.

To prove rule 1), it is sufficient and necessary to prove that for position (x, y) , if $x = aM + bN$ with $a \geq 0, b \geq 1$, it is a hole, and if $x \neq aM + bN$ with $a \geq 0, b \geq 1$, it is not a hole. The proof is given in the following.

Based on [6], for any integer I in the range of $[0, (N-1)M]$, we can always find two integers a_0 and b_0 , such that

$$I = a_0M + b_0N \quad (13)$$

Let $b' = b_0 \bmod M$, then $b_0 = xM + b', b' \in [0, M-1]$. Then we can get

$$I = a'M + b'N \quad (14)$$

with $a' = a_0 + xN$, $b' = b_0 - xM$, and $\lceil -b' \frac{N}{M} \rceil \leq a' \leq N - 1 - \lceil b' \frac{N}{M} \rceil$.

Then, according to (12), we need to find out if there exist p_1 and q_1 , with $0 \leq p_1 \leq N - 1$, $0 \leq q_1 \leq M - 1$, satisfying

$$p_1M - q_1N = I = a'M + b'N \quad (15)$$

The problem can be split into three cases:

- i) $0 \leq a' \leq N - 1$, $b' = 0$;
- ii) $\lceil \frac{N}{M} \rceil - N \leq a' < 0$, $0 < b' \leq M - 1$;
- iii) $0 \leq a' \leq N - 1$, $0 < b' \leq M - 1$.

For i), (15) can be written as

$$(p_1 - a')M = q_1N \quad (16)$$

Since

$$\begin{cases} 0 \leq a' \leq N - 1, & b' = 0 \\ 0 \leq p_1 \leq N - 1, & 0 \leq q_1 \leq M - 1 \end{cases} \quad (17)$$

for any a' in this case, there exist $p_1 = a'$ and $q_1 = 0$ satisfying (16). Therefore, the positions corresponding to such a' and b' are not holes.

For ii), noticing that $q_1 + b' \geq 1$, (15) can be written as

$$\frac{N}{M} = \frac{p_1 - a'}{q_1 + b'} \quad (18)$$

Since

$$\begin{cases} \lceil \frac{N}{M} \rceil - N \leq a' < 0, & 0 < b' \leq M - 1 \\ 0 \leq p_1 \leq N - 1, & 0 \leq q_1 \leq M - 1 \end{cases} \quad (19)$$

for any a' , b' in this case, there exist p_1 , q_1 with $p_1 - a' = N$ and $q_1 + b' = M$, satisfying (18). Therefore, the positions corresponding to such a' and b' are not holes.

For iii), similar with ii), (15) can be written as (18). Considering that M , N are two coprime integers, $(p_1 - a')$ and $(q_1 + b')$ should exactly equal to N and M respectively. Since

$$\begin{cases} 0 \leq a' \leq N - 1 \\ 0 \leq p_1 \leq N - 1 \end{cases} \quad (20)$$

for any a' in this case, we cannot find p_1 such that $p_1 - a' = N$. Therefore, the positions corresponding to such a' and b' are holes.

The proof of rule 1) ends, and the proof of the other three is similar and omitted here.

Then, we focus on the characteristics of \mathbb{D}_c^{2-1} (9), which can be expressed as

$$\mathbb{D}_c^{2-1} = \{(q_1N - p_1M, q_2N - p_2M) \mid 0 \leq p_1, p_2 \leq N - 1, 0 \leq q_1, q_2 \leq M - 1\} \quad (21)$$

and it is easy to see that \mathbb{D}_c^{2-1} is located in $\{(x, y) \mid -(N - 1)M \leq x, y \leq (M - 1)N\}$. By the same method, we can get four similar general rules of the holes in \mathbb{D}_c^{2-1} :

- 5) In $\{(x, y) \mid 0 \leq x \leq (M - 1)N, -(N - 1)M \leq y \leq (M - 1)N\}$, the position (x, y) would be a hole if $x = aM + bN$, with $a \geq 1$, $b \geq 0$;
- 6) In $\{(x, y) \mid -(N - 1)M \leq x \leq 0, -(N - 1)M \leq y \leq (M - 1)N\}$, the position (x, y) would be a hole if $x = aM + bN$, with $a \leq 0$, $b \leq -1$;
- 7) In $\{(x, y) \mid -(N - 1)M \leq x \leq (M - 1)N, 0 \leq y \leq (M - 1)N\}$, the position (x, y) would be a hole if $y = aM + bN$, with $a \geq 1$, $b \geq 0$;

- 8) In $\{(x, y) \mid -(N - 1)M \leq x \leq (M - 1)N, -(N - 1)M \leq y \leq 0\}$, the position (x, y) would be a hole if $y = aM + bN$, with $a \leq 0$, $b \leq -1$.

Considering that the two self difference coarrays \mathbb{D}_s^1 (10) and \mathbb{D}_s^2 (11), expressed as

$$\mathbb{D}_s^1 = \{(aM, bM) \mid -(N - 1) \leq a, b \leq (N - 1)\} \quad (22)$$

$$\mathbb{D}_s^2 = \{(aN, bN) \mid -(M - 1) \leq a, b \leq (M - 1)\} \quad (23)$$

can fill some holes in \mathbb{D}_c^{1-2} and \mathbb{D}_c^{2-1} (7); and with rules 1) – 8), we can get the precise expressions of the hole locations in \mathbb{D} in the 1st quadrant \mathbb{H}_1 and 2nd quadrant \mathbb{H}_2 :

$$\mathbb{H}_1 = \mathbb{H}_{11} \cup \mathbb{H}_{12} \cup \mathbb{H}_{13} \cup \mathbb{H}_{14} \quad (24)$$

$$\mathbb{H}_2 = \mathbb{H}_{21} \cup \mathbb{H}_{22} \quad (25)$$

with

$$\mathbb{H}_{11} = \{(x, y) \mid x = aM + bN, a \geq 1, b \geq 1, 0 \leq x, y \leq (N - 1)M\} \quad (26)$$

$$\mathbb{H}_{12} = \{(x, y) \mid x = aN, y = bM, a \geq 1, b \geq 1, 0 \leq x, y \leq (N - 1)M\} \quad (27)$$

$$\mathbb{H}_{13} = \{(x, y) \mid x = aM, y = bN, a \geq 1, b \geq 1, 0 \leq x, y \leq (N - 1)M\} \quad (28)$$

$$\mathbb{H}_{14} = \{(x, y) \mid y = aM + bN, a \geq 1, b \geq 1, 0 \leq x, y \leq (N - 1)M\} \quad (29)$$

and

$$\mathbb{H}_{21} = \{(x, y) \mid x = aM + bN, a \leq -1, b \leq -1, -(N - 1)M \leq x \leq 0, 0 \leq y \leq (N - 1)M\} \quad (30)$$

$$\mathbb{H}_{22} = \{(x, y) \mid y = aM + bN, a \geq 1, b \geq 1, -(N - 1)M \leq x \leq 0, 0 \leq y \leq (N - 1)M\} \quad (31)$$

According to the definition of \mathbb{D} , the hole locations in the 3rd and 4th quadrants are centrosymmetric to those in the 1st and 2nd quadrants. Fig. 2 illustrates our proposition.

IV. PROPOSED HOLES-FILLING METHOD

Based on the above analysis, it can be seen that there exist some critical holes in the 1st and 3rd quadrants, which sparsely locate inside the range of $\{(x, y) \mid -(M + N) < x, y < (M + N)\}$ and break the consecutiveness of the difference coarray in such range. For the critical holes in the 1st quadrant, which belong to \mathbb{H}_{12} and \mathbb{H}_{13} , we propose a holes-filling method, by which such holes can be filled by adding few additional sensor elements, and because of the centrosymmetry of the difference coarrays, the critical holes in the 3rd quadrant will also be filled as long as the critical holes in the 1st quadrant are filled.

We first focus on the elements of \mathbb{H}_{12} (27). Considering the assumption $M < N$, the critical holes belonging to \mathbb{H}_{12} , which can be described as

$$\mathbb{H}'_{12} = \{(N, kM) \mid 1 \leq k \leq K, K = \lfloor \frac{N}{M} \rfloor + 1\} \quad (32)$$

can be filled by adding an additional antenna element at position (N, KM) . The proof is given in the following by two steps:

- a) the position (N, KM) is not occupied by the original antenna elements, which can be proved by contradiction:

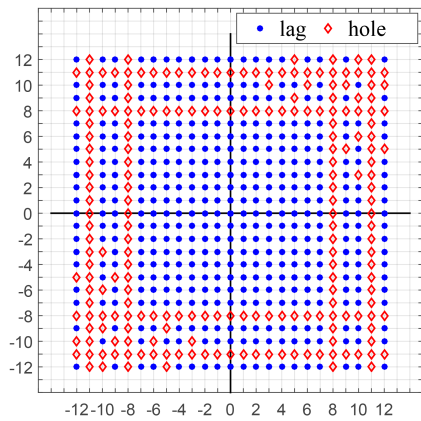


Fig. 4. Holes filled difference coarray of a CPA with $N = 5$ and $M = 3$.

If there exists an original antenna element located at the position of (N, KM) , the lag (N, KM) generated by the pair of antenna elements located at (N, KM) and $(0, 0)$ would be found in the difference coarray, which contradicts our above analysis of the holes locations (27) and the step a) has been proved.

- b) the critical holes of \mathbb{H}'_{12} can be filled by the lags generated by the additional antenna element located at (N, KM) and the original antenna elements located at $(0, tM)$, with $0 \leq t \leq K - 1$.

According to (2), because

$$t \leq K - 1 = \lfloor \frac{N}{M} \rfloor < \frac{N}{M} < N - 1 \quad (33)$$

the antenna elements located at $(0, tM)$ exist in the original coprime planar array. The lags generated by such antenna elements and the added antenna element located at (N, KM) can be expressed as

$$\mathbb{C} = \{(N, gM) \mid g = K - t\} \quad (34)$$

It is clear that

$$\mathbb{C} = \mathbb{H}'_{12} \quad (35)$$

and the step b) has been proved.

From the above two steps, it has been proved that the critical holes of \mathbb{H}'_{12} can be filled by adding an additional antenna element at (N, KM) . Then, for the critical holes belonging to \mathbb{H}_{13} , which can be described as

$$\mathbb{H}'_{13} = \{(kM, N) \mid 1 \leq k \leq K, K = \lfloor \frac{N}{M} \rfloor + 1\} \quad (36)$$

they can be filled by adding another additional antenna element at position (KM, N) . The proof is similar as above and omitted here.

For the example CPA in Fig. 1 with the corresponding difference coarray shown in Fig. 2, the critical holes inside the range of $\{(x, y) \mid -8 < x, y < 8\}$ can be filled by adding two additional sensor elements at positions $(5, 6)$ and $(6, 5)$. The difference coarray of the CPA with the added sensors is shown in Fig. 4. Compared with the difference coarray of the original CPA in Fig. 2, it can be seen that the holes filled difference coarray has a larger consecutive part with much more consecutive lags.

In general, for the a CPA with the proposed holes-filling method, by judiciously adding two additional sensor elements,

we can obtain a difference coarray with a consecutive part located at $\{(x, y) \mid -M - N + 1 \leq x, y \leq M + N - 1\}$, with the total number of consecutive lags being $(2M + 2N - 1)^2$. Compared with the difference coarrays of the conventional CPAs, the critical holes which break the consecutiveness of the difference coarray are filled by the proposed holes-filling method, enlarging the consecutive part of the difference coarrays and increasing the effective DOFs of the CPAs.

V. CONCLUSION

In this letter, CPAs are investigated from the perspective of difference coarrays. Closed-form expressions of the hole locations in the difference coarrays of CPAs are deduced, providing a basis for the future research. Then based on the knowledge of the hole locations, a holes-filling method is proposed. By judiciously adding two additional sensor elements, the most critical holes in the difference coarray can be filled, such that the consecutive part of the difference coarray can be enlarged, which increases the effective DOFs. The filling of the remaining holes in the difference coarray and the compressive sensing based DOA estimation [13] for CPAs would be the future research orientations.

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