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A comparison of harmonic modeling methods with application to the interconnection and the control of switched systems

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Abstract

Many harmonic modeling approaches have been introduced in the literature, such as generalized state-space averaging, dynamic phasors, extended harmonic domain, and harmonic state-space. They are capable of capturing both the transient evolution and the steady-state of harmonics. They model the frequency coupling nature of a system and can highlight the frequency couplings within interconnected components. By these modeling techniques, a linear time-periodic model can be converted into a linear time-invariant model, which allows the use of traditional analysis and control methods. This paper presents a state of the art of harmonic modeling approaches. Its contribution is to clearly establish strong links between the different literature approaches and to develop a general harmonic methodology that unifies them. A rigorous theoretical framework for the modeling, analysis and control of linear time-varying systems with an arbitrary number of harmonics is developed. In particular, a harmonic model is proposed from which all the literature models can be easily derived. This paper also deals with the harmonic modeling of switched affine systems. It shows the advantages of harmonic modeling to analyze the frequency couplings within associated switched systems and to the control with active filtering.

Keywords: harmonic modeling, harmonic analysis, frequency couplings, switched systems, interconnected systems, harmonic-based control

1. Introduction

Electricity has transformed the way we communicate, produce, travel and relate to the world. Electrical power is an essential element of modern society which provides for increasingly sophisticated needs. To meet the new needs in transmission and conversion of electrical power, the power systems are more and more complex with non-linear components and complicated frequential behaviors. That leads to several new challenges. The non-linear components introduce couplings

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between frequencies. In particular, high-frequency converters inherently generate numerous harmonics and can interact with each other. These interactions may cause unpredicted instability problems involving many different frequencies, like in the Swiss locomotives case [15]. Historically, in power electronics, the early models only represent the average signal behavior through the State-Space Averaging (SSA) method which involves averaging on a sliding window [14]. These models have been widely used to study small signals. However, they only allow the harmonic analysis of the steady-state system and they have revealed limitations in modeling large signals [5]. This has motivated the researches for an analysis approach capable of characterizing the harmonic dynamics and the frequency couplings.

A review of the literature in this area shows that various frequency domain based approaches have been introduced. Generalized State-Space Averaging (GSSA) [19] and Dynamic Phasors (DP) [13] are generalized averaging approaches to model the harmonic content of general linear systems. They employ Fourier series of time-varying coefficients and have been widely used in power system modeling and control, e.g. [10], [13], [20]. Extended Harmonic Domain (EHD) [12] and Dynamic Harmonic Domain (DHD) [4] have been introduced to analyze the frequency behavior of Linear Time-Periodic (LTP) systems through Linear Time-Invariant (LTI) models in the harmonic domain. These approaches have also been widely used, e.g. [11], [18]. Meanwhile, Harmonic State Space (HSS) has been introduced for the harmonic analysis and the control analysis of LTP systems [21]. The HSS approach makes it possible to represent a LTP system as a LTI state space of infinite dimension and to define a harmonic transfer function for these LTP systems.

The main contribution of this paper is to clearly establish the links between the various harmonic modeling techniques and to develop a general harmonic methodology in a rigorous theoretical framework that unifies them. As a result, the literature approaches have different scopes of application and specificities but some strong links are stated in Section 2 and a new harmonic model is proposed from which all the literature models can easily be derived. This has been addressed in [3], of which this paper is an extended and updated version. Also, the decomposition of non-periodic signals in generalized Fourier series with time-varying coefficients is specified with strong proof and slightly different from the literature. One more contribution of the paper is to show the new possibilities that are given by harmonic modeling techniques for the interconnection and the control of systems, in particular switched systems. These are illustrated through two practical examples. In Section 3 we take advantage of the harmonic modeling to analyze the interconnection of a Boost converter and a Flyback converter. In Section 4 we present a control of a Boost converter that uses the harmonic techniques to act directly on the signals harmonics and to achieve active filtering by rejecting periodic perturbations.

2. Harmonic modeling

In this section, we discuss the various harmonic modeling approaches that have been introduced in the literature. Their links are clearly identified and a general harmonic methodology in a rigorous theoretical framework that unifies them is developed. It results in a harmonic model presented in Proposition 4 from which they can all be derived. We also propose a strong proof for the use of time-varying Fourier coefficients through general Fourier decomposition which is slightly different from the literature. In this context, let us consider a linear time-varying system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{1}$$

with $x(t) \in \mathbb{R}^{n_x}$, $y(t) \in \mathbb{R}^{n_y}$, $u(t) \in \mathbb{R}^{n_u}$, and where $A(t), B(t), C(t), D(t)$ are some real-valued matrices of appropriate dimension. If the matrices $A(t), B(t), C(t), D(t)$ are periodic of same period, the system is a LTP system. If they are constant, the system is a LTI system.

2.1. Generalized averaging

In this section, the various tools of generalized averaging techniques, namely GSSA, DP, EHD and DHD are presented. We describe their scopes of application and their specificities and we identify strong links between the methods. Based on the generality of each approach, a unifying rigorous theoretical framework for harmonic modeling is proposed. It allows to deal with linear time-varying systems and to take into account an arbitrary number of harmonics, possibly infinite. This section also highlights the general Fourier decomposition which is slightly different from the literature by providing a strong proof.

The conventional SSA models only consider the averaging behavior of signals but it is known that higher order harmonics are needed to describe the harmonic behavior accurately [5]. By extension, the SSA approach has resulted in the GSSA. The GSSA approach is a general Fourier series development with time-varying coefficients, first introduced by Sanders [19]. The DP approach has been introduced by Mattavelli, Verghese, and Stankovic [13] to emphasize the time-varying nature of the Fourier coefficients, but it essentially shares the same formulation.

Proposition 1. *Any piecewise C^0 signal $x(\cdot)$ with bounded variations can be represented using a general Fourier series of arbitrary period T by*

$$x(t) = 2 \sum_{k=-\infty}^{\infty} X_k(t) e^{jk\omega t} - x(t-T) \quad a.e. \quad (2)$$

and for a real number $\delta \in]0, T[$, by the non-causal relation

$$x(t-\delta) = \sum_{k=-\infty}^{\infty} X_k(t) e^{jk\omega(t-\delta)} \quad a.e. \quad (3)$$

with time-varying coefficients

$$X_k(t) = \frac{1}{T} \int_{t-T}^t x(p) e^{-jk\omega p} dp \quad (4)$$

where $\omega = \frac{2\pi}{T}$ and "a.e." means almost everywhere.

Proof. Let $x(\cdot)$ be a piecewise continuous signal with bounded variations and let us introduce the signal

$$\tilde{x}_T(t, \tau) = \sum_{k=-\infty}^{\infty} X_k(t) e^{jk\omega\tau} \quad (5)$$

with time-varying coefficients (4). This signal depends on two time scales expressed by the two variables t and τ and is T -periodic with respect to τ . Actually, it is a general Fourier series with respect to the variable τ , plotted for two different values of t in Figure 1. By application of Dirichlet's theorem on Fourier series convergence: for any real number t and for any $\tau \in]t-T, t[$,

$$\tilde{x}_T(t, \tau) = \frac{x(\tau_-) + x(\tau_+)}{2} \quad (6)$$

where $x(\tau_-)$ and $x(\tau_+)$ stands for the left and right limits of $x(\cdot)$ in τ . For a real number $\delta \in]0, T[$, by writing (6) for $\tau = t - \delta$, $\tilde{x}_T(t, t - \delta) = \frac{x((t-\delta)_-) + x((t-\delta)_+)}{2}$ so that (3) holds almost everywhere, except at the discontinuities of $x(\cdot)$. Otherwise, for $\tau = t$, $\tilde{x}_T(t, t) = \frac{x(t_-) + x((t-T)_+)}{2}$ so that (2) holds almost everywhere, except at the discontinuities of $x(\cdot)$. \square

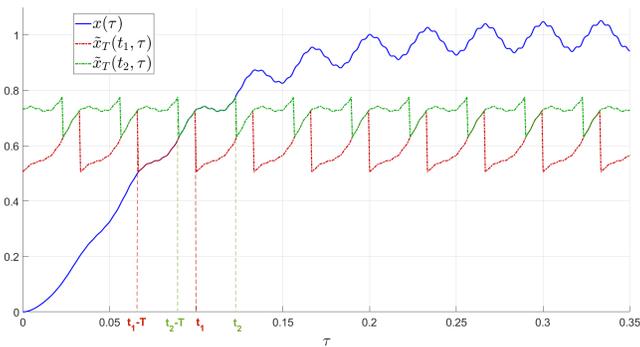


Figure 1: The signal $x(\tau)$ is plotted in blue and the signal $\tilde{x}_T(t, \tau)$ is plotted for $t = t_1$ in red and for $t = t_2$ in green, versus its second time scale τ .

The coefficients $X_k(t)$ are called *generalized Fourier coefficients* in GSSA but also referred to as *phasors* in DP. They are functions of time since the interval under consideration slides as a function of time. $X_k(t)$ is the k^{th} coefficient (or k -phasor) at time t that represents the k^{th} harmonic of the signal at frequency $\frac{k}{T}$. If the signal $x(t)$ is T -periodic after a certain time t_r , then its general Fourier coefficients are constant for $t \geq t_r + T$, so the steady state is defined by constant harmonics, as it can be seen in Figure 2. In other words, in periodic steady state, (2) becomes the well-known Fourier formula $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega t}$. This is why a good suggestion to analyze the harmonic content of DC-DC switching converters is to choose the period T equal to the switching period, so that the harmonics of the signals are constant for a fixed duty cycle. However, it is important to highlight that the proposition holds for any period.

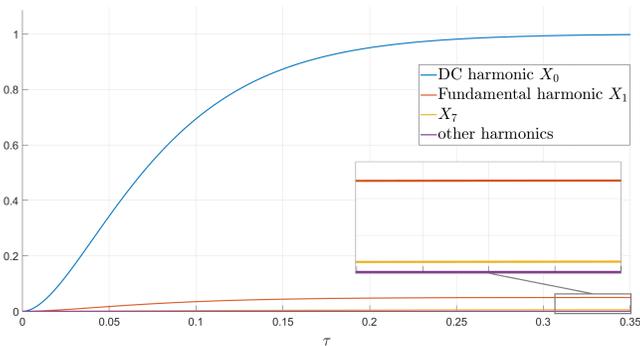


Figure 2: The generalized Fourier coefficients $X_k(\tau)$ of signal $x(\tau)$ (blue signal in Figure 1) are plotted against time. The periodic steady state is clearly defined by constant harmonics.

The Proposition 1 is different from the general Fourier formulas of literature, e.g. [19], [13], at time t (for $\delta = 0$) because the discontinuity of the periodic function extrapolated from one signal period is considered. It results that the Fourier series can not exactly match this signal in general but only if the function extrapolated is continuous, that is to say if the original signal is periodic.

Proposition 2. *The harmonic dynamics of a C^0 , piecewise C^1 signal $x(\cdot)$ of derivative of bounded variations can be described as*

$$\dot{X}_k(t) = \langle \dot{x} \rangle_k(t) - jk\omega X_k(t) \quad a.e. \quad (7)$$

where $\dot{X}_k(\cdot)$ stands for the derivative of the k^{th} general Fourier coefficient of $x(\cdot)$, $\langle \dot{x} \rangle_k(\cdot)$ stands for the k^{th} general Fourier coefficient of the derivative of $x(\cdot)$ and with the term ' $j k \omega$ ' which is the derivative of the k^{th} complex exponential function $e^{jk\omega t}$.

Proof. This proposition can be easily verified by differentiating (4). □

Proposition 3. *The harmonics of a product of two piecewise C^0 signals $y(\cdot)$ and $x(\cdot)$ can be expressed using the following convolution-based procedure*

$$\langle yx \rangle_k(t) = \sum_{i=-\infty}^{\infty} Y_{k-i}(t)X_i(t) \quad a.e. \quad (8)$$

where $Y_k(t), X_k(t)$ are the k^{th} general Fourier coefficients of $y(t), x(t)$.

Proof. This proposition can be easily verified using (3) for both signals. □

In the literature, GSSA has been applied by taking few harmonics into account, although it was acknowledged that higher order harmonics must be incorporated for a more accurate model of a large-signal system. This is because GSSA equations do not allow to add harmonics easily. That has led to EHD approach that systematically describes a LTP system by its state space in the harmonic domain with an arbitrary number of harmonics, possibly infinite [12], [18]. The term DHD has also been introduced in [4] but it shares an identical formulation. EHD/DHD approach is more general than GSSA/DP since it allows to take into account an arbitrarily large number of harmonics. Nevertheless, GSSA/DP is more general since it allows the study of non-periodic systems. It is proposed on the following to take into account an arbitrarily large number of harmonics and not to be limited to LTP systems. Basically, GSSA equations are used to generate the state-space of a general linear system in a chosen harmonic basis. This requires the use of Toeplitz form [21] and generalized here to time-varying Fourier coefficients.

Proposition 4. *A linear time-varying system (1) with piecewise C^0 input signal and matrices can be represented in the harmonic basis of arbitrary period T as a state space model of the form*

$$\begin{aligned} \dot{X}(t) &= (\mathcal{A}(t) - \mathcal{N})X(t) + \mathcal{B}(t)U(t) \\ Y(t) &= \mathcal{C}(t)X(t) + \mathcal{D}(t)U(t) \end{aligned} \quad (9)$$

with

$$\begin{aligned} X(t) &= [\dots X_{-k}(t) \dots X_0(t) \dots X_k(t) \dots]^T \\ Y(t) &= [\dots Y_{-k}(t) \dots Y_0(t) \dots Y_k(t) \dots]^T \\ U(t) &= [\dots U_{-k}(t) \dots U_0(t) \dots U_k(t) \dots]^T \\ \mathcal{N} &= \text{diag}(\dots -jk\omega \dots 0 \dots jk\omega \dots -jk\omega \dots 0 \dots jk\omega \dots) \end{aligned} \quad (10)$$

where the matrix of differentiation \mathcal{N} contains the vector of all the derivative terms of the complex exponential functions $e^{jk\omega t}$ of the harmonic basis replicated as many times as there are state variables in (1), where $X_k(t), Y_k(t), U_k(t)$ are the generalized Fourier coefficients of $x(t), y(t), u(t)$, $\mathcal{A}(t), \mathcal{B}(t), \mathcal{C}(t), \mathcal{D}(t)$ are the Toeplitz form of $A(t), B(t), C(t), D(t)$, and with $\omega = \frac{2\pi}{T}$.

Proof. Let us consider (1) with piecewise C^0 input signal and matrices. The k^{th} coefficient of $\dot{x}(\cdot)$ and $y(\cdot)$ are

$$\begin{aligned} \langle \dot{x} \rangle_k(t) &= \langle Ax \rangle_k(t) + \langle Bu \rangle_k(t) \\ Y_k(t) &= \langle Cx \rangle_k(t) + \langle Du \rangle_k(t) \end{aligned} \quad (11)$$

By the convolution-based procedure (8),

$$\begin{aligned} \langle \dot{x} \rangle_k(t) &= \sum_{i=-\infty}^{\infty} A_{k-i}(t)X_i(t) + \sum_{i=-\infty}^{\infty} B_{k-i}(t)U_i(t) \\ Y_k(t) &= \sum_{i=-\infty}^{\infty} C_{k-i}(t)X_i(t) + \sum_{i=-\infty}^{\infty} D_{k-i}(t)U_i(t) \end{aligned} \quad (12)$$

The infinite vectors of the harmonics of $\dot{x}(t)$ and $y(t)$ can be written with the Toeplitz form of matrices as

$$\begin{aligned} \langle \dot{x} \rangle(t) &= \mathcal{A}(t)X(t) + \mathcal{B}(t)U(t) \\ Y(t) &= \mathcal{C}(t)X(t) + \mathcal{D}(t)U(t) \end{aligned} \quad (13)$$

By (7), we also write

$$\langle \dot{x} \rangle(t) = \dot{X}(t) - \mathcal{N}X(t) \quad (14)$$

so that the derivative of the infinite vector $X(t)$ of the harmonics of $x(t)$ can be expressed as $\dot{X}(t) = (\mathcal{A}(t) - \mathcal{N})X(t) + \mathcal{B}(t)U(t)$. \square

The state-space formulation (9) is an extension of GSSA approach to an infinite number of harmonics, and an extension of EHD approach to the study of non-periodic systems. It provides the state space of a general linear system in a harmonic basis of chosen period. If the investigated system is LTP of period T , then by choosing T or a multiple of T as the harmonic basis period, the Fourier coefficients of the matrices appear to be time-invariant, so are their Toeplitz form. In other words, a LTP model of period T is converted into a LTI model of the form (9) with $\mathcal{A}(t) = \mathcal{A}$, $\mathcal{B}(t) = \mathcal{B}$, $\mathcal{C}(t) = \mathcal{C}$, $\mathcal{D}(t) = \mathcal{D}$. Similar results are presented in EHD and DHD approaches for which the harmonic basis period is imposed equal to the system matrices period, but the proposed modeling is more general since it can also be applied to non-periodic systems, opening new possibilities as to model AC-DC and DC-AC converters.

2.2. Harmonic state-space

In this section, the tools of HSS approach are presented and we identify the strong links with the generalized averaging methods and with the theoretical framework developed in section 2.1. The fundamental notion behind the development of the traditional analysis tools like Bode diagram for LTI systems is that a sinusoidal input at a given frequency is mapped by the LTI transfer function into a sinusoidal output of the same frequency, but with possibly different amplitude and phase. In contrast, if a sinusoid is input to a LTP system, several sinusoids may appear in the output signal. The corresponding output signal may oscillates at the frequency of the input signal plus integer multiples of the system fundamental frequency [22]. As a result, the LTI analysis Laplace domain cannot be used as it stands in LTP context. This has motivated the researches for a solution to

LTP systems analysis. The Floquet solution is a geometrically periodic signal whose transient state has a complex exponential behavior [6]. EMP signals have been introduced by Hill [8] through periodic extension to the kernel function of Laplace to make the connection with the assumed Floquet solution. More details about Floquet theory resulting in EMP signals can be found in [2].

Definition 1. An exponentially modulated periodic (EMP) signal $x(\cdot)$ can be expressed as the complex Fourier series of a periodic signal of fundamental period $T = \frac{2\pi}{\omega}$ modulated by a complex exponential

$$x(t) = e^{st} \sum_{k=-\infty}^{\infty} X_k e^{jk\omega t} \quad (15)$$

where $s \in \mathbb{C}$. It can be written on compact form

$$x(t) = e^{st} \Gamma(t) X \quad (16)$$

with

$$\begin{aligned} \Gamma(t) &= [\dots e^{-jk\omega t} \dots 1 \dots e^{jk\omega t} \dots] \\ X &= [\dots X_{-k} \dots X_0 \dots X_k \dots]^T \end{aligned} \quad (17)$$

Proposition 5. EMP signal spaces defined for a period T are fundamental signal spaces for LTP systems of period T .

Proof. A LTP system of period T maps an EMP signal of period T to an EMP signal of same fundamental frequency but with different amplitudes and phases, see [21]. \square

This frequency separation property reveals the strong analogy that EMP signals are to LTP systems as complex exponentials are to LTI systems, as illustrated in Figure 3.

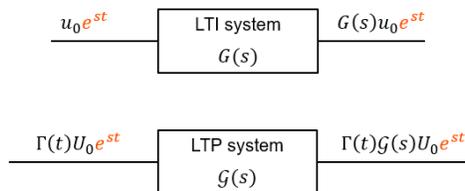


Figure 3: EMP signals have frequency separation property through LTP systems (down) as complex exponential signals have through LTI systems (up).

Proposition 6. A LTP system (1) of matrices $A(t), B(t), C(t), D(t)$ of period T can be written in the harmonic basis of period T as an infinite LTI system of the form

$$\begin{aligned} sX &= (\mathcal{A} - \mathcal{N})X + \mathcal{B}U \\ Y &= \mathcal{C}X + \mathcal{D}U \end{aligned} \quad (18)$$

with

$$\begin{aligned} X &= [\dots X_{-k} \dots X_0 \dots X_k \dots]^T \\ Y &= [\dots Y_{-k} \dots Y_0 \dots Y_k \dots]^T \\ U &= [\dots U_{-k} \dots U_0 \dots U_k \dots]^T \\ \mathcal{N} &= \text{diag}(\dots -jk\omega \dots 0 \dots jk\omega \dots -jk\omega \dots 0 \dots jk\omega \dots) \end{aligned} \quad (19)$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are the Toeplitz matrices of $A(t), B(t), C(t), D(t)$ and $s \in \mathbb{C}$.

Proof. The proof is based on the harmonic balance approach using EMP fundamental signals [21]. \square

The state-space form (18) with the frequency separation property of EMP signals through LTP systems allows to define a transfer function for LTP systems in the harmonic domain. The *harmonic transfer function* of a LTP system (18) is given by the infinite matrix $\mathcal{G}(s) = \mathcal{C}((s\mathcal{U}_I - (\mathcal{A} - \mathcal{N}))^{-1}\mathcal{B} + \mathcal{D})$ where \mathcal{U}_I stands for the identity matrix. This time-invariant matrix contains the transfer functions between each input harmonic and each output harmonic. It is a linear operator of the harmonic domain that allows to analyze harmonic dynamics and couplings using traditional LTI analysis techniques, as pole-zero map [16] or Nyquist plot [15]. Another strong analogy is given in this paper: HSS domain is to LTP systems what Laplace domain is for LTI systems. As opposed to the Laplace domain of unique reference frame e^{st} , HSS domain contains an infinite range of reference frames $(e^{(s+jk\omega)t})_{k \in \mathbb{Z}}$. The complex exponential e^{st} is related to the harmonic transient evolution. Its derivative is se^{st} so that the multiplication by s denotes differentiation over time on direction s . The product sX stands for the dynamic of the state harmonics.

2.3. Discussion

In this section, we synthesize the links between the literature approaches and we define the bonds with the unified model of Proposition (4). It is easy to derive all the literature harmonic models from the one proposed in this paper so that it is possible to combine their advantages. We also discuss the choice of the harmonic basis period. Many publications have focused on harmonic content, resulting in several methods. The literature has not described precisely the links between these approaches because of the differences in their writing and goals [9]. DP and GSSA are essentially identical, as they are based on the same generalized Fourier representation (3) with time-varying coefficients (4) and generate the same equations (7) and (8). Equation (2) has been introduced in this paper to manage some issues that may occur when moving back to the temporal domain because the relation (3) is not causal. In the literature, these approaches have only been developed for few harmonics. Mostly, only the first harmonic at fundamental frequency is incorporated [19], [20], or some known harmonics of the system are selected to design a controller [7]. What is not clearly stated in the literature is that EHD and DHD approaches are a systematical writing of the LTI state space model that one can get by applying GSSA equations to a LTP system with an arbitrary number of harmonics. Moreover, HSS is a frequential analysis domain for LTP systems that is analogous to the Laplace domain for LTI system analysis. It is a purely frequential release of EHD that can be derived from EHD state space by Laplace transform. The links between the different literature harmonic modeling approaches which have been clearly identified in this paper are summarized in Figure 4.

The model developed in this article in Proposition 4 is a generalization of all the harmonic modeling models as we can not only choose the number of harmonics to take into account but also consider time-varying systems. From this model can be derived all the averaging models by truncation or selection of harmonics and HSS model by Laplace transform. It can be seen as a bridge model whose benefit is to combine easily the advantages of all the harmonic techniques. The bonds with the bridge model are summarized in Figure 5.

In practice, the Fourier series have to be truncated so that they can be computed. Then, the accuracy of the truncated model highly relies on the choice of the harmonic basis period and the number of harmonics. The impact of the basis period has been weakly developed so far. Actually, it is a compromise between model complexity and accuracy. A rigorous theoretical basis

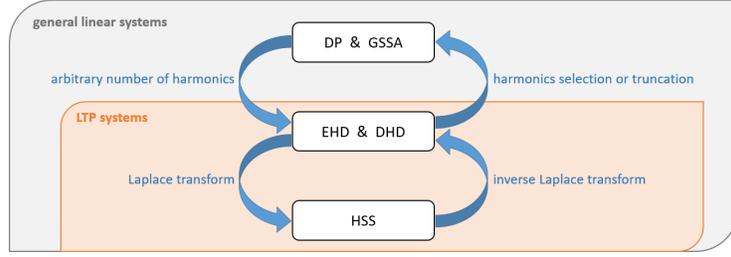


Figure 4: Links between the different harmonic modeling approaches.

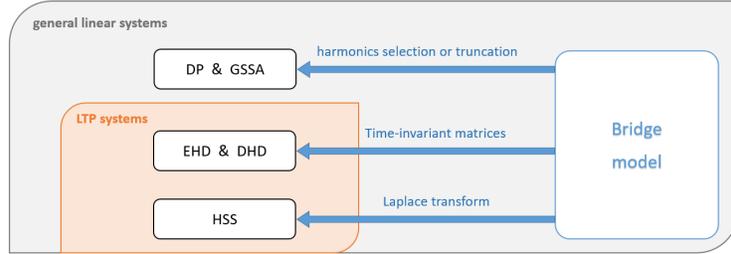


Figure 5: Links between the bridge model unifying the different harmonic modeling approaches. All harmonic models can be derived from the state space of the bridge model in Proposition 4.

for determining the truncation order is difficult to develop as it depends on numerous parameters. For example, high-order harmonics can be excited with respect to the transient dynamics. It is still an open issue but we provide on the following some serious directions to assess the model accuracy with respect to the choice of the harmonic basis period and the maximal order of harmonics. The main idea is that the harmonic basis period T specifies the frequency spectrum precision as the frequencies represented in the harmonic basis are separated from $\frac{1}{T}$ and the maximal order of harmonics h specifies the frequency bandwidth as the frequencies represented are between $\frac{-h}{T}$ and $\frac{h}{T}$. The model order is then $(2h + 1)n_x$, with n_x the number of physical variables and $(2h + 1)$ the number of harmonic variables for each physical variable (the dc component, h positive harmonics and h negative harmonics).

Practically speaking, we should choose the period large enough to capture the low frequencies and include enough harmonics to capture the high frequencies. Since the general harmonic model has the great advantage to allow to choose an arbitrary number of harmonics and to change this number easily, it is possible to add more and more harmonics until the accuracy suits our goal, or on the contrary to remove more and more harmonics until the model order is low enough for our needs. This compromise depends on the application and specifications. Moreover, by choosing wisely the harmonic basis period (if possible as a multiple of all relevant signal periods), we can both limit the model order and get to a LTI model with very little accuracy loss. As an example, let's consider the waveform $x(t) = a_1 \sin(\omega_0 t) + a_3 \sin(3\omega_0 t)$. By choosing $T = \frac{2\pi}{\omega_0}$, only three harmonics at ω_0 , $2\omega_0$, and $3\omega_0$ would be needed to model perfectly the signal. More generally, by choosing $T = \frac{2\pi N}{\omega_0}$, $N \in \mathbb{N}^*$, we need $3N$ harmonics. However, by choosing T arbitrarily, we may need much more harmonics and they have no reason to be constant. Similarly, the better choice to model a

DC-DC switched system seems to be a harmonic basis of period equal to the switching period, so that the state-space model is LTI in steady state and few harmonics are needed for accuracy.

Harmonic modeling has proved to be an interesting tool to analyze the harmonic content of a system. It can also be used to make good choices when associating components by characterizing their harmonic couplings. This point has not been much explored and a striking example is developed in Section 3, where the models of two different converters of type Boost and Flyback are simply interconnected in a common harmonic basis. The choice of the harmonic basis period also allows to target some specific harmonics for example to reject oscillations, which is illustrated with the active filtering of a Boost converter output voltage in Section 4.

3. Harmonic analysis of associated switched systems

The harmonic modeling can be employed to improve the analysis of interconnected switched systems. Not only does it allow to represent switched systems by continuous models with good precision, but it also contributes to the analysis of their interconnection by capturing their harmonic couplings. We develop on the following a harmonic model of a switched affine system and show the resulting models for a Boost converter and a Flyback converter. Afterthat, we build the harmonic model of the interconnection of these two converters, from which we can analyze the harmonic couplings between them and the resulting global harmonic content. We neglect in this section the discontinuous mode of the converters. A particular attention will be paid to the choice of a common harmonic basis to these converters made possible by the discussion on the harmonic basis period.

3.1. Harmonic modeling of switched affine systems

A switched affine system can be described by a state-space model of the form

$$\dot{x}(t) = A(d(t))x(t) + B(d(t))u(t) \quad (20)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the input vector and $d(t)$ is the switching signal. The time-varying matrices can be decomposed as

$$\begin{aligned} A(d(t)) &= A_{ind} + d(t)A_{dep} \\ B(d(t)) &= B_{ind} + d(t)B_{dep} \end{aligned} \quad (21)$$

where $d(t)$ is scalar and A_{dep}, B_{dep} are some constant matrices. It is possible to control the behavior of a switched affine system by the switching signal $d(t)$. It is a discontinuous signal that contains many harmonics at multiples of the switching frequency f_{sw} . For a fixed duty cycle, the switching signal is periodic of period T_{sw} . Thus, an interesting idea to describe a switched affine system by an almost LTI state-space model is to choose this period to build the harmonic basis. That is to say, we choose the basis of complex exponential functions $(e^{jk\omega t})_{k \in \mathbb{Z}}$, with $\omega = 2\pi f_{sw}$. By Proposition 4, the corresponding state space is

$$\dot{X}(t) = \mathcal{A}(\mathcal{D}(t))X(t) + \mathcal{B}(\mathcal{D}(t))U(t) \quad (22)$$

with

$$\begin{aligned} X(t) &= [X_{\{1\}}(t) \dots X_{\{n_x\}}(t)]^T \\ U(t) &= [U_{\{1\}}(t) \dots U_{\{n_u\}}(t)]^T \\ \mathcal{A}(\mathcal{D}(t)) &= A_{ind} \otimes \mathcal{U}_I - \tilde{\mathcal{N}} + A_{dep} \otimes \mathcal{D}(t) \\ \mathcal{B}(\mathcal{D}(t)) &= B_{ind} \otimes \mathcal{U}_I + B_{dep} \otimes \mathcal{D}(t) \\ \tilde{\mathcal{N}} &= \text{diag}(\mathcal{N}, \dots, \mathcal{N}) \end{aligned} \quad (23)$$

where $X_{\{j\}}(t)$ is the vector of harmonics of the j^{th} state variable, $U_{\{l\}}(t)$ is the vector of harmonics of the l^{th} input variable, $\mathcal{D}(t)$ is the Toeplitz matrix of the switching signal $d(t)$, $\mathcal{N} = \text{diag}(\dots -jk\omega \dots 0 \dots jk\omega \dots)$ is the matrix of differentiation, \mathcal{U}_I is the identity matrix, and where \otimes stands for the Kronecker product. n_x is the number of states and n_u is the number of inputs. This state-space model is LTV, but a great advantage of it is that the Toeplitz matrices reveal the couplings between input, control and output frequencies. Moreover, each physical variable is described by many harmonic variables that vary much less over time, which reduces the simulation step time. Typically, for a constant duty cycle, the switching signal is periodic of the same period than the harmonic basis, so the Toeplitz matrix \mathcal{D} is constant, and for periodic inputs of period T_{sw} , the vector U is constant and the model is LTI.

3.2. Choice of a common harmonic basis

In this part, we raise concern about the choice of the harmonic basis, in particular its period. It has been presented in the last part that an interesting choice to model a switched system is to choose the period of the harmonic basis equal to the switching period, so that the model is almost LTI (LTI in steady state) and precisely describes the system behavior with a limited number of harmonics. However, by choosing any multiple of the switching period, similar results can be obtained. The drawback is that we have to increase the number of harmonics taken into account for the same precision. That is the reason why there has been little mention of it in the literature. However, it could be the key point to find a common harmonic basis for two switched systems in order to associate them.

We are interested in a common harmonic basis for two switched systems of different switching period. A harmonic basis can be said to be wisely chosen with respect to a system if all the significant frequencies of the system are contained in the basis. For a switched system, any complex exponential basis of period equal to a multiple of the switching period is a good choice, even more the switching period itself. Furthermore, for two switched systems with different switching period, a wise choice is a complex exponential basis of period equal to a common multiple of both switching periods. This choice must lead to an almost LTI state-space representation of the harmonic behavior of the interconnection. Unfortunately, the case where the frequencies do not have any common multiple is an issue. An idea could be to approximate a common multiple, but the sensitivity has to be studied since the harmonics would not be constant and the accuracy would be degraded. Another issue is that the number of harmonics has to be increased. Greater the common multiple is (with respect to the switching frequencies), more the number of harmonics need to be increased for accuracy. Taking the smallest common multiple is a wise choice to limit the number of harmonics.

3.3. Application to the interconnection of a Boost converter and a Flyback converter

The harmonic techniques can be applied to a Boost converter and a Flyback converter. The resulting harmonic models are shown on the following. Then, a common harmonic basis is chosen to model their interconnection and capture the couplings between their frequencies.

The Boost converter is a high-frequency switched system shown in Figure 6. Its instantaneous

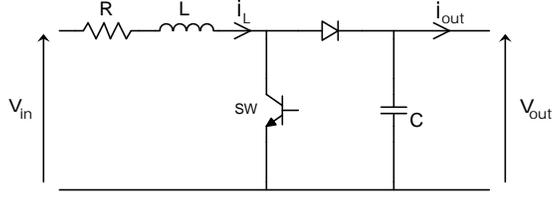


Figure 6: Circuit diagram of a Boost converter.

temporal model is expressed as a switched affine representation (20) with

$$\begin{aligned}
 x(t) &= \begin{bmatrix} i_L(t) \\ v_{out}(t) \end{bmatrix} \\
 u(t) &= \begin{bmatrix} v_{in}(t) \\ i_{out}(t) \end{bmatrix} \\
 A(d(t)) &= \begin{bmatrix} -\frac{R}{L} & -\frac{1-d(t)}{L} \\ \frac{1-d(t)}{C} & 0 \end{bmatrix} \\
 B(d(t)) &= \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & -\frac{1}{C} \end{bmatrix}
 \end{aligned} \tag{24}$$

where $i_L(t)$ is the current through inductance, $v_{out}(t)$ is the output voltage, $v_{in}(t)$ is the input voltage, i_{out} is the output current that is required from the load ($i_{out} = \frac{v_{out}}{R_{load}}$ in case of constant resistive load), and $d(t)$ is the discontinuous switching signal, usually coming from pulse width modulation (pwm). For a constant duty cycle, this last signal is periodic at switching frequency f_{boost} .

The harmonic model of a Boost converter in a complex exponential basis $(e^{jk\omega t})_{k \in \mathbb{Z}}$ of given frequency f is expressed as in (22) with

$$\begin{aligned}
 X(t) &= \begin{bmatrix} I_L(t) \\ V_{out}(t) \end{bmatrix} \\
 U(t) &= \begin{bmatrix} V_{in}(t) \\ I_{out}(t) \end{bmatrix} \\
 A(\mathcal{D}(t)) &= \begin{bmatrix} \frac{-R}{L}\mathcal{U}_I - \mathcal{N} & \frac{-1}{L}(\mathcal{U}_I - \mathcal{D}(t)) \\ \frac{1}{C}(\mathcal{U}_I - \mathcal{D}(t)) & -\mathcal{N} \end{bmatrix} \\
 B(\mathcal{D}(t)) &= \begin{bmatrix} \frac{1}{L}\mathcal{U}_I & \mathcal{Z} \\ \mathcal{Z} & \frac{-1}{C}\mathcal{U}_I \end{bmatrix}
 \end{aligned} \tag{25}$$

with $\omega = 2\pi f$, where $I_L(t)$, $V_{out}(t)$, $V_{in}(t)$, $I_{out}(t)$ are the harmonic vectors of the inductance current, the output voltage, the input voltage and the output current, $\mathcal{D}(t)$ is the Toeplitz form of the switching law $d(t)$, \mathcal{U}_I is the identity matrix and \mathcal{Z} is the zero matrix.

This model of infinite dimension is relevant for any given frequency of the harmonic basis. However, in practice, the model has to be truncated, which can affect its accuracy. The harmonic model of the Boost converter gives very accurate results by choosing the complex exponential

basis of frequency f_{boost} and by taking into account three harmonics at least to approximate well the switching effects. Same results can be obtained by choosing the complex exponential basis of frequency $f = \frac{f_{boost}}{N}$ and by taking into account $h = 3N$ harmonics at least.

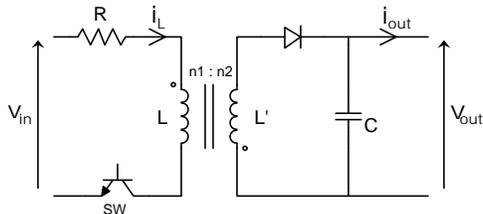


Figure 7: Circuit diagram of a Flyback converter.

The Flyback converter is a high-frequency switched system shown in Figure 7. Let us define the transformer rate $n = \frac{n_1}{n_2}$. Its instantaneous temporal model is expressed as a switched affine representation (20) with

$$\begin{aligned}
 x(t) &= \begin{bmatrix} i_L(t) \\ v_{out}(t) \end{bmatrix} \\
 u(t) &= \begin{bmatrix} v_{in}(t) \\ i_{out}(t) \end{bmatrix} \\
 A(d(t)) &= \begin{bmatrix} -\frac{R}{L} & -\frac{n(1-d(t))}{L} \\ \frac{n(1-d(t))}{C} & 0 \end{bmatrix} \\
 B(d(t)) &= \begin{bmatrix} \frac{d(t)}{L} & 0 \\ 0 & -\frac{1}{C} \end{bmatrix}
 \end{aligned} \tag{26}$$

where $i_L(t)$ is the current through inductance, $v_{out}(t)$ is the output voltage, $v_{in}(t)$ is the input voltage, i_{out} is the output current that is required from the load, and $d(t)$ is the switching signal. For a constant duty cycle, this last signal is periodic at switching frequency $f_{flyback}$.

The harmonic model of a Flyback converter (with $n = \frac{n_1}{n_2}$) in the complex exponential basis $(e^{jk\omega t})_{k \in \mathbb{Z}}$ of given frequency f is expressed as in (22) with

$$\begin{aligned}
 X(t) &= \begin{bmatrix} I_L(t) \\ V_{out}(t) \end{bmatrix} \\
 U(t) &= \begin{bmatrix} V_{in}(t) \\ I_{out}(t) \end{bmatrix} \\
 A(\mathcal{D}(t)) &= \begin{bmatrix} -\frac{R}{L}\mathcal{U}_I - \mathcal{N} & -\frac{n}{L}(\mathcal{U}_I - \mathcal{D}(t)) \\ \frac{n}{C}(\mathcal{U}_I - \mathcal{D}(t)) & -\mathcal{N} \end{bmatrix} \\
 B(\mathcal{D}(t)) &= \begin{bmatrix} \frac{1}{L}\mathcal{D}(t) & \mathcal{Z} \\ \mathcal{Z} & -\frac{1}{C}\mathcal{U}_I \end{bmatrix}
 \end{aligned} \tag{27}$$

with $\omega = 2\pi f$, where $I_L(t), V_{out}(t), V_{in}(t), I_{out}(t)$ are the harmonic vectors of the inductance current, the output voltage, the input voltage and the output current, $\mathcal{D}(t)$ is the Toeplitz form of the switching law $d(t)$, \mathcal{U}_I is the identity matrix and \mathcal{Z} is the zero matrix.

The harmonic model of the Flyback converter gives very accurate results by choosing the complex exponential basis of frequency $f_{flyback}$ and by taking into account three harmonics at least. Same results can be obtained by choosing the complex exponential basis of frequency $f = \frac{f_{flyback}}{N}$ and by taking into account $h = 3N$ harmonics at least.

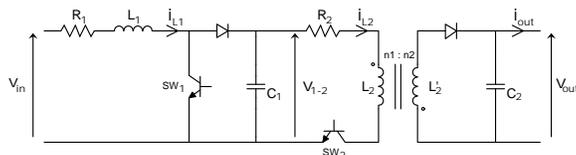


Figure 8: Circuit diagram of the interconnection of a Boost and a Flyback converter.

Let us consider a Boost converter of switching frequency $f_{boost} = 100kHz$ interconnected with a Flyback converter of switching frequency $f_{flyback} = 75kHz$, as shown in Figure 8. The output voltage of the Boost converter imposes the input voltage of the Flyback converter as $v_{1-2} = v_{out}^{boost} = v_{in}^{flyback}$. The output current of the Boost converter is imposed by the inductance current and the switching law of the Flyback converter as $i_{out}^{boost} = d^{flyback} i_L^{flyback}$. Through a common harmonic basis, we have simple relations as $V_{1-2} = V_{out}^{boost} = V_{in}^{flyback}$ and $I_{out}^{boost} = \mathcal{D}^{flyback} I_L^{flyback}$.

The harmonic model of the interconnection of a Boost converter and a Flyback converter with a constant resistive load in the complex exponential basis $(e^{jk\omega t})_{k \in \mathbb{Z}}$ of given frequency f is expressed as in (22) with

$$\begin{aligned}
 X(t) &= \begin{bmatrix} I_{L1}(t) \\ V_{1-2}(t) \\ I_{L2}(t) \\ V_{out}(t) \end{bmatrix} \\
 U(t) &= V_{in}(t) \\
 A(\mathcal{D}(t)) &= \begin{bmatrix} \frac{-R_1}{L_1} \mathcal{U}_I - \mathcal{N} & \frac{-1}{L_1} (\mathcal{U}_I - \mathcal{D}_1) & \mathcal{Z} & \mathcal{Z} \\ \frac{1}{C_1} (\mathcal{U}_I - \mathcal{D}_1) & -\mathcal{N} & \frac{-1}{C_1} \mathcal{D}_2 & \mathcal{Z} \\ \mathcal{Z} & \frac{1}{L_2} \mathcal{D}_2 & \frac{-R_2}{L_2} \mathcal{U}_I - \mathcal{N} & \frac{-n}{L_2} (\mathcal{U}_I - \mathcal{D}_2) \\ \mathcal{Z} & \mathcal{Z} & \frac{n}{C_2} (\mathcal{U}_I - \mathcal{D}_2) & \frac{-1}{R_{load} C_2} \mathcal{U}_I - \mathcal{N} \end{bmatrix} \\
 B(\mathcal{D}(t)) &= \begin{bmatrix} \frac{1}{L_1} \mathcal{U}_I \\ \mathcal{Z} \\ \mathcal{Z} \\ \mathcal{Z} \end{bmatrix}
 \end{aligned} \tag{28}$$

with $\omega = 2\pi f$, where index 1 refers to the Boost converter and index 2 refers to the Flyback converter, where V_{in} is the harmonic vector of the input voltage of the Boost converter (viewed as a perturbation), V_{out} is the harmonic vector of the output voltage of the Flyback converter, V_{1-2} is the harmonic vector of the intermediate voltage between them, $\mathcal{D}_1(t), \mathcal{D}_2(t)$ are the Toeplitz forms of the switching signals of the Boost and the Flyback (time indices have been left for clarity), \mathcal{U}_I is the identity matrix and \mathcal{Z} is the zero matrix.

A wise choice of common harmonic basis for this association is the complex exponential basis of frequency $f = GCD(f_{boost}, f_{flyback})$ equal to the greatest common divisor of both switching

frequency. Here, $f = 25kHz$. As $f = \frac{f_{boost}}{4}$, we need $h = 12$ harmonics at least for the Boost commutation modeling accuracy. As $f = \frac{f_{flyback}}{3}$, we need $h = 9$ harmonics at least for the Flyback commutation modeling accuracy. Thus, the complex exponential basis of frequency $f = 25kHz$ may be a wise choice and requires only $h = 12$ harmonics for accuracy. The output voltage of the interconnection of a Boost converter and a Flyback converter is plotted in Figure 9 by open-loop simulation of the continuous harmonic model (28) with $f = 25kHz$ and truncated to 12 harmonics. It is compared to the output voltage from the interconnected discontinuous temporal models to demonstrate its accuracy. Constant duty cycles are applied to both converters. The converters parameters can be found in Figure 10. We have considered a constant resistive load $R_{load} = 10\Omega$.

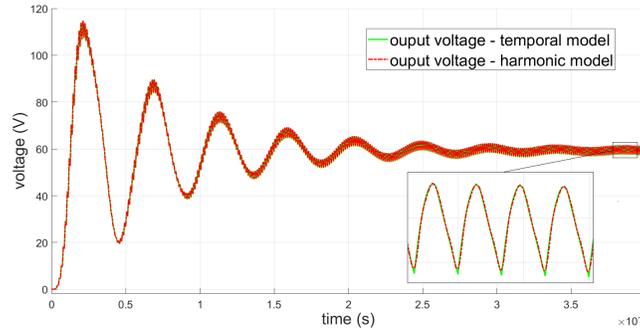


Figure 9: Comparison of the output voltage from temporal model and harmonic model in the exponential basis of frequency $f = 25kHz$ and with 12 harmonics.

parameters	Boost	Flyback
R	$1m\Omega$	$1m\Omega$
L	$0.1mH$	$0.1mH$
C	$10\mu H$	$10\mu H$
n	×	2
f_{sw}	$100kHz$	$75kHz$

Figure 10: Parameters of the Boost converter and the Flyback converter used for simulation.

The harmonic model (25) has several advantages. First, it is a continuous model that reveals the high-frequency behavior of both converters. It is possible, by simulation of this model, to analyze the influences of commutations with larger time steps than for temporal discontinuous models but still very accurate results. In addition, it highlights the couplings between the harmonics generated by the commutations in both converters. Finally, it becomes a LTI model in a steady state given by constant duty cycles, so it allows to use traditional control laws. It has to be reminded that these advantages may disappear if the common harmonic basis is not wisely chosen, for example if we do not know the switching periods of the converters or if they do not have any common multiple.

In this section, we raised concern about the choice of the harmonic basis, in particular its period. We have taken advantage of this new parameter to interconnect two switched systems in a common harmonic basis, so that we can analyze their interconnection with an almost LTI continuous model.

Actually, by making another choice for the period of the harmonic basis, other possibilities can be opened. In disturbed environment, by choosing a harmonic basis at a perturbation frequency, it could permit to directly control the harmonics and reject this frequency. This point has not been much explored yet and it is considered in the next part.

4. Control with active filtering of switched systems

The harmonic modeling can be employed to improve the control of industrial systems. Not only does it allow the analysis of frequency behavior, but it also extends the notion of equilibrium points. Indeed, a T -periodic signal is constant in the harmonic basis of period T , so that a periodic steady state can be viewed as an equilibrium point. It also enables to act separately on harmonics and to analyze their interactions. These advantages have been used to design a LQ controller for a full-bridge rectifier in [7]. The harmonic model has been linearized for a few harmonics and a feedback control synthesis in harmonic domain has given harmonic rejection criteria. We develop on the following a globally asymptotically stable control law for switched affine systems with periodic perturbation rejection. This control takes advantage of harmonic modeling without linearization to eliminate a periodic input perturbation.

4.1. Equilibrium points in the harmonic domain

Any periodic steady state of period T can be viewed as an equilibrium point in the harmonic basis of same period T . Then, the harmonic equilibrium point is defined by a vector of constant harmonics for each original periodic state variables. In instance, as we are dealing with switched systems, an equilibrium is defined both by the state variables and the switching signal (the control signal). The equilibrium points defined in the harmonic domain are called *harmonic equilibrium points* on the following.

Definition 2. The set of harmonic equilibrium points of a switched system of the form (22) is defined for a periodic input signal of period equal to the period of the harmonic basis by

$$\mathcal{E} = \{(X^e, D^e) \mid X^e = -\mathcal{A}(D^e)^{-1}\mathcal{B}(D^e)U\} \quad (29)$$

where X^e contains the harmonics of all state variables in steady state and D^e contains the harmonics of the switching signal in steady state. For control purposes, we are interested in accessible harmonic equilibrium points $\mathcal{E}_a = \{(X^e, D^e) \mid X^e = -\mathcal{A}(D^e)^{-1}\mathcal{B}(D^e)U, \mathcal{A}(D^e) \text{ Hurwitz}\}$

Now that equilibrium points have been defined in the harmonic basis, a control law can be built to reach one of them.

4.2. Embedded control of switched systems in the harmonic domain

The embedded control for switched systems has been proposed in [1]. It has the advantage of been an adaptive switched control design methodology with a global asymptotic stability property. Please note that other more traditional control laws like Proportional Integrator (PI) or Linear Quadratic (LQ) can also be used after linearization in the harmonic domain. The main asset of the harmonic domain is that we can use the same control techniques than in temporal domain meanwhile we control the harmonic behavior. The drawback to keep in mind is the number of variables that increases.

Proposition 7. *Considering the system (22), any accessible equilibrium point $(X^e, D^e) \in \mathcal{E}_a$ is globally asymptotically stable by the control law*

$$D(t) = D^e - \mathcal{K}\mathcal{B}^T(X(t), U)\mathcal{P}(X(t) - X^e) \quad (30)$$

with $\mathcal{K} > 0$, $\mathcal{P} > 0$ solution of the Lyapunov equation $\mathcal{A}(D^e)^T\mathcal{P} + \mathcal{P}\mathcal{A}(D^e) + \mathcal{Q} = 0$, and

$$\mathcal{B}(X(t), U) = \sum_{j=1}^{n_x} (A_{dep\{C_j\}} \otimes X_{\{j\}_T}(t)) + \sum_{l=1}^{n_u} (B_{dep\{C_l\}} \otimes U_{\{l\}_T}) \quad (31)$$

where $A_{dep\{C_j\}}, B_{dep\{C_l\}}$ are respectively the j^{th} column of A_{dep} and the l^{th} column of B_{dep} , and with $X_{\{j\}_T}(t), U_{\{l\}_T}$ the Toeplitz matrices of the harmonics of the j^{th} state variable and the l^{th} input variable.

Proof. This proposition can be verified by a Lyapunov function approach similar to the one in [1] applied in the harmonic domain. \square

\mathcal{Q} is the Lyapunov gain to spread the control effort and \mathcal{K} is the gain to adjust the control significance. This control law can be exploited either to solve a tracking problem or to eliminate some unwanted harmonics, simply by reaching a harmonic equilibrium of chosen harmonics amplitude and phase.

4.3. Application to the control of a Boost converter

In this part, we present some results about a controller using harmonic modeling to eliminate unwanted harmonics. The chosen system to demonstrate is a Boost converter shown in Figure (6), of switching frequency $f_{sw} = 100kHz$ and connected to a constant resistive load. Its temporal model is expressed as in (20) with

$$\begin{aligned} x(t) &= \begin{bmatrix} i_L(t) \\ v_{out}(t) \end{bmatrix} \\ u(t) &= v_{in}(t) \\ A(d(t)) &= \begin{bmatrix} -\frac{R}{L} & -\frac{1-d(t)}{L} \\ \frac{1-d(t)}{C} & -\frac{1}{R_{load}C} \end{bmatrix} \\ B(d(t)) &= \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \end{aligned} \quad (32)$$

where $i_L(t)$ is the current through inductance, $v_{out}(t)$ is the output voltage, $v_{in}(t)$ is the input voltage (viewed as a perturbation), R_{load} is the constant resistive load, and $d(t)$ is the switching signal coming from pulse width modulation (pwm). High-frequency input perturbations are not relevant as they are filtered by the system and low-frequency ones can be overcome by adaptive control techniques, but a specific frequency range remains critical, as shown in Figure 11. We can see in our case that the critical range of frequency is between $200Hz$ and $10kHz$ with the worst case around $5kHz$, quite far from the high switching frequency of $100kHz$. That leads to the assumption that high frequencies due to commutations can be neglected for rejection of such frequencies. This assumption may be checked by using same techniques than in the last section to find a common harmonic basis for switching harmonics and perturbation harmonics.

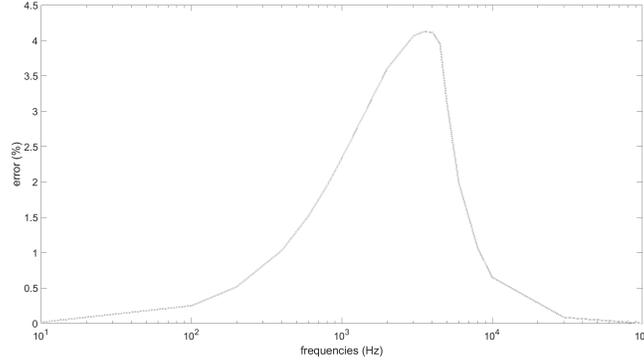


Figure 11: Influence of a periodic input perturbation $v_{in}(t) = v_{in_c} + v_{in_{var}} \sin(2\pi ft)$. A traditional control law is used, improved by adaptive techniques. The maximal amplitude error on the output voltage is plotted against the frequency of the perturbation .

Another assumption is that we know the period T_{in} of the input perturbation, so that by choosing the harmonic basis of period $T = T_{in}$, the periodic perturbation is viewed as some constant harmonics perturbations. This second assumption is a strong one, but in practice the perturbation would come from another associated system whose frequency may be known, measured or estimated with good precision.

The harmonic model of a Boost converter with a periodic input perturbation of period T_{in} in the complex exponential basis $(e^{jk\omega_{in}t})_{k \in \mathbb{Z}}$ is expressed as in (22) with

$$\begin{aligned}
 X(t) &= \begin{bmatrix} I_L(t) \\ V_{out}(t) \end{bmatrix} \\
 U(t) &= V_{in}(t) \\
 \mathcal{A}(\mathcal{D}(t)) &= \begin{bmatrix} \frac{-R}{L} \mathcal{U}_I - \mathcal{N} & \frac{-1}{L} (\mathcal{U}_I - \mathcal{D}(t)) \\ \frac{1}{C} (\mathcal{U}_I - \mathcal{D}(t)) & \frac{-1}{R_{load} C} \mathcal{U}_I - \mathcal{N} \end{bmatrix} \\
 \mathcal{B}(\mathcal{D}(t)) &= \begin{bmatrix} \frac{1}{L} \mathcal{U}_I \\ \mathcal{Z} \end{bmatrix}
 \end{aligned} \tag{33}$$

where $I_L(t), V_{out}(t), V_{in}(t), I_{out}(t)$ are the harmonic vectors of the inductance current, the output voltage, the input voltage and the output current, $\mathcal{D}(t)$ is the Toeplitz form of the switching law $d(t)$, \mathcal{U}_I is the identity matrix and \mathcal{Z} is the zero matrix. The control objective is to regulate the output voltage and avoid its oscillations. The corresponding reference output voltage harmonic vector contains zeros everywhere except at its dc component. Some integrators are used to reject static errors on each harmonics. Figure 12 shows the better rejection of a periodic perturbation $v_{in}(t) = 30 + 3 \sin(2\pi ft)$ of critical frequency $f = 5kHz$ by harmonic-based control. What is remarkable about this result is that the control laws are of the same type and complexity. What is new in the harmonic-based control law is that we control directly the harmonics of the output voltage so we have more parameters and we can adjust them to eliminate some unwanted harmonics and still regulate the dc component (the output voltage average). This is illustrated by Figure 13 in which the harmonics of the regulated output voltage are plotted.

To reject the unwanted harmonics, the controller takes into account the couplings between input, control and output harmonics, so that it generates a control law with some harmonics to

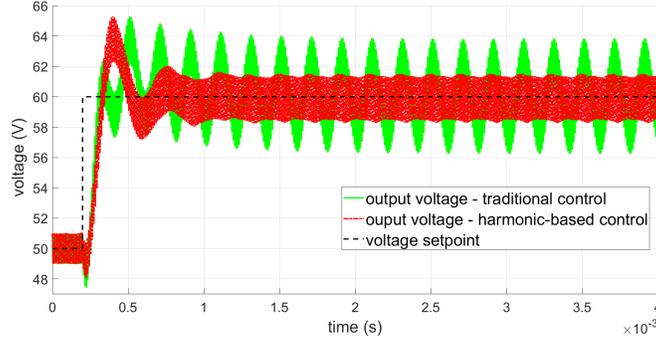


Figure 12: Output regulation with a periodic input perturbation of critical frequency $f = 5kHz$. The resulting output voltages are plotted for traditional (temporal) control in green and harmonic-based control in red.

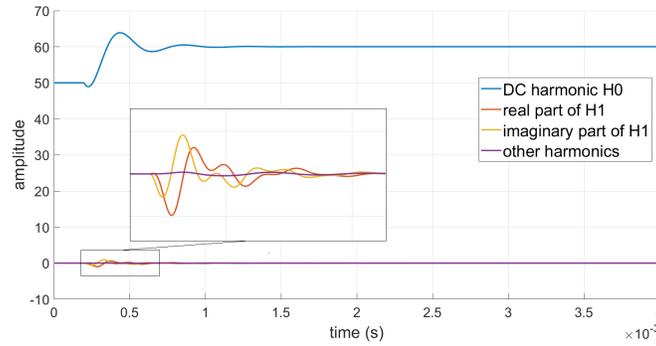


Figure 13: Output regulation with a periodic input perturbation rejection by harmonic-based control. The resulting output voltage harmonics are plotted.

compensate the oscillations on the output voltage. It can be understood as some harmonic injections to compensate oscillations. The control law to reject the input oscillation is shown in Figure 14 and its corresponding harmonics are shown in Figure 15. Obviously, the relevance of this control method depends on the assumption that we know the frequency of the perturbation with sufficient accuracy. The sensitivity of the approach to the perturbation frequency knowledge is an issue where several parameters are involved and should be studied in further works.

5. Conclusion

The harmonic techniques can capture the harmonic content of a system and provides a better understanding of frequency phenomena. Many different harmonic approaches have been developed but this article shows that they are based on the same equations and ideas. The differences in their writing are only due to their different scopes of application, for control design or harmonic analysis, for LTP systems or non-periodic ones. In this paper, a more general state-space model is developed to reveal the links between the different approaches. From this unified modeling methodology can be easily derived all the different methods presented in the literature.

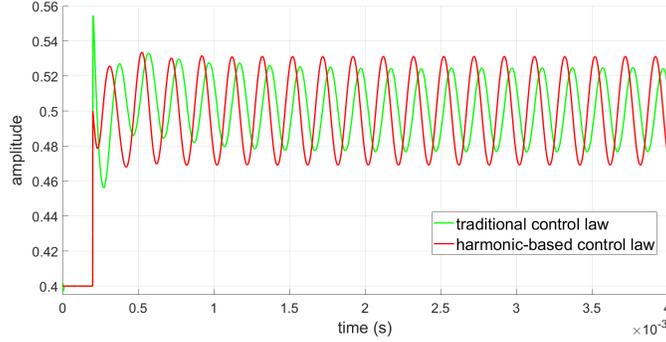


Figure 14: Output regulation with a periodic input perturbation of critical frequency $f = 5kHz$. The resulting duty-cycle commands are plotted for traditional (temporal) control in green and harmonic-based control in red.

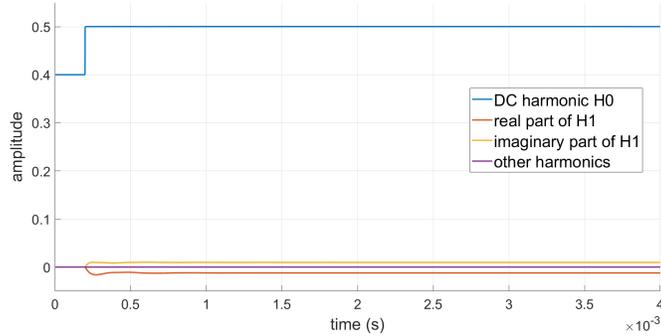


Figure 15: Output regulation with a periodic input perturbation rejection by harmonic-based control. The resulting command law harmonics are plotted.

It has been revealed that the number of harmonics taken into account is of relevance. This article shows that one key point of harmonic modeling techniques is the choice of the harmonic basis period. The period has to be large enough to capture the significant low frequencies and the number of harmonics has to be large enough to capture the significant high frequencies. Moreover, a wise choice of the period as a common multiple of all the significant periods of the signals oscillations may lead to a LTI state-space harmonic model, which allows the use of traditional analysis and control tools with extensive results as for harmonic dynamics modeling, frequency couplings analysis and control with harmonic rejection. The influence of the period on the type of the resulting harmonic model is illustrated in the table in Figure 16.

One advantage of harmonic modeling is the possibility to capture the high-frequency behavior of a switched system through an almost LTI continuous model. However, this result requires the knowledge of the switching frequency and the model has no reason to be LTI if it is not precisely known. It also reveals the couplings between the input, control and output harmonics and, under the assumption that we find a common harmonic basis, it may reveal the frequency couplings within interconnected systems, even associated switched systems. Once more, the results would be simpler to analyze with a wise choice of harmonic basis, which means not only to know precisely their switching frequencies but also to find a common divisor of both frequencies.

TD \ HD	LTI	LTP	LTV
LTI	$\forall T$	\times	\times
LTP (T_0)	T multiple of T_0	T not multiple of T_0	\times
LTV	\times	\times	$\forall T$

Figure 16: Types of model in the temporal domain (TD) and the resulting types of model in the harmonic domain (HD) with respect to the chosen basis period T .

Meanwhile, harmonic modeling techniques can be exploited to control industrial systems. One can define and verify some harmonic criteria that are difficult to express in temporal domain for control design, for example to reject some oscillations. Furthermore, it is possible to control directly the harmonics. By such an harmonic-based control approach, either periodic tracking problems can be solved as several simple regulation problems on harmonic variables or some harmonics can be rejected. This implies to deal with many additional variables and control parameters.

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