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Renormalization group equations and the recurrence pole relations in pure quantum gravity

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Abstract

In the framework of dimensional regularization, we propose a generalization of the renormalization group equations in the case of the perturbative quantum gravity that involves renormalization of the metric and of the higher order Riemann curvature couplings. The case of zero cosmological constant is considered. Solving the renormalization group (RG) equations we compute the respective beta functions and derive the recurrence relations, valid at any order in the Newton constant, that relate the higher pole terms $1/(d-4)^n$ to a single pole $1/(d-4)$ in the quantum effective action. Using the recurrence relations we find the exact form for the higher pole counter-terms that appear in 2, 3 and 4 loops and we make certain statements about the general structure of the higher pole counter-terms in any loop. We show that the complete set of the UV divergent terms can be consistently (at any order in the Newton constant) hidden in the bare gravitational action, that includes the terms of higher order in the Riemann tensor, provided the metric and the higher curvature couplings are renormalized according to the RG equations.

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1 Introduction

The ultra-violet (UV) divergences in a quantum field theory are handled by adding the respective counter-terms. The structure of the UV divergences is most conveniently analyzed in the framework of the dimensional regularization that preserves all symmetries present in the quantum system. The divergences then appear as a set of poles when the space-time dimension $d \rightarrow 4$. The respective terms in the quantum effective action are not independent: the higher pole terms $1/(d-4)^n$, $n > 1$ are completely determined by the terms in a single pole $1/(d-4)$. These relations, known in the literature as the pole equations, come out as a consequence of the Renormalization Group equations as was demonstrated by 't Hooft [1]. Earlier the pole equations were derived and used, quite effectively, in the case of 2d sigma-models in which one has to renormalize the metric in the target space [2], [3].

In the case of the perturbative quantum gravity the computation of the loop diagrams is notoriously difficult. The one- and two-loop results are the only available in the literature [4], [5], [6] although the possible structure of the counter-terms that may in principle appear in the higher loops can be analyzed by means of the covariance principle, at least at some lower orders in the curvature. Their number is rapidly increasing with number of loops and in all orders in the Newton constant one deals with an infinite number of possible structures.

It was suggested in [7] (see also [8] and for some later developments [9]) that the Renormalization Group methods should be equally applicable to the non-renormalizable theories. This program if successful would make the non-renormalizable theories to look quite similar to the renormalizable ones. The available in the literature prescriptions, however, are rather implicit. We do not directly rely here on this previous work although it played an important inspirational role for our study.

In the present paper we develop a systematic approach to perturbative quantum gravity that uses the renormalization group equations. The primary goal of the paper is to derive the recurrence pole relations in the case of gravity. Our prescriptions are precise and unambiguous. It should be noted that over the last several decades there have been suggested a number of approaches to quantum gravity that refer to certain versions of the Renormalization Group, in most cases of the Wilsonian type. In order to avoid any possible confusion we would like to stress from the very beginning that none of these approaches will be used here. The closest analogue of the approach that we develop in the present paper is that of the renormalization group equations of 't Hooft [1]. The key important point in our construction, that, to the best of our knowledge, was missing in the earlier approaches to quantum gravity, is the necessity to consider a renormalization of the metric, much in the same way as in a renormalizable QFT one introduces a renormalization of the quantum fields. The peculiarity of this renormalization procedure for the metric is that it is not multiplicative but of a rather general, although still

local, type.

The other important remark is that throughout the paper only the case of pure gravity with zero cosmological constant will be considered. The case of non-zero cosmological constant will be treated in a subsequent work.

The paper is organized as follows. In section 2 we briefly review the method of 't Hooft in the case of a renormalizable quantum field theory (QFT). In section 3 we derive the renormalization group equations in the case of pure quantum gravity. In section 3.1 we consider the renormalization of the metric and in section 3.2 the renormalization of the quantum effective action. In section 4 we solve some of the RG equations and determine the exact form of the higher loop metric beta function and the beta functions for the higher order curvature coupling constants. In section 5 we derive the pole recurrence relations and solve these relations to determine the exact form of the higher pole counter-terms in 2, 3 and 4 loops. In section 6 we focus on the General Relativity (GR) counter-terms and make some statements on their general form. In section 7 we demonstrate that, similarly to the renormalizable theories, the complete set of the UV divergences can be hidden in the bare gravitational action provided the bare metric and the bare higher curvature couplings are expressed in terms of the renormalized quantities. We conclude in section 8.

2 RG equations in renormalizable QFT

Before we start our analysis of quantum gravity we briefly review the derivation of the renormalization group equations in 't Hooft's method in the case of a 4d renormalizable theory [1], see also [3], [7] for a similar review. Consider a dimensionless coupling constant λ . In $d = 4 - \epsilon$ space-time dimensions the bare coupling λ_B has dimension $[\mu^\epsilon]$, where μ is mass scale, as for instance in the case of ϕ^4 theory. In dimensional regularization one develops a series of counter-terms of the classical action such that the bare coupling constant is expressed as a function of a dimensionless renormalized coupling λ_R ,

$$\lambda_B = \mu^\epsilon \left(\lambda_R + \sum_{k=1} \frac{a_k(\lambda_R)}{\epsilon^k} \right). \quad (2.1)$$

The renormalized coupling λ_R is a function of scale μ such that an equation

$$\mu \partial_\mu \lambda_R = -\epsilon \lambda_R + \beta(\lambda_R) \quad (2.2)$$

holds. The bare coupling is supposed to be independent of μ so that $\mu \partial_\mu \lambda_B = 0$. Differentiating equation (2.1) with respect to μ one obtains the following equation

$$\epsilon \sum_{k=1} \frac{a_k}{\epsilon^k} + \beta(\lambda_R) + (-\epsilon \lambda_R + \beta(\lambda_R)) \sum_{k=1} \frac{a'_k(\lambda_R)}{\epsilon^k} = 0, \quad (2.3)$$

where $a'_k(\lambda_R) \equiv \partial_{\lambda_R} a_k(\lambda_R)$ and the terms linear in ϵ cancel out.

The constant term ϵ^0 in (2.3) gives us equation that allows to express the beta function in terms of a single pole a_1 ,

$$\beta(\lambda_R) = a_1(\lambda_R) - \lambda_R a'_1(\lambda_R). \quad (2.4)$$

The vanishing condition for a pole $1/\epsilon^k$, $k \geq 1$ in eq. (2.3) produces a recurrence relation

$$a_{k+1}(\lambda_R) - \lambda_R a'_{k+1}(\lambda_R) = \beta(\lambda_R) a'_k(\lambda_R), \quad k \geq 1 \quad (2.5)$$

This relation together with the beta function (2.4) form a set of recurrence relations that uniquely determine the higher pole residues a_k in (2.1) provided a single pole residue a_1 is given. Similar equations can be written for the renormalization of masses and the quantum fields [1]. The renormalization of fields should not be necessarily multiplicative in general. Besides other things, the pole equations play the role of the consistency conditions to be satisfied in the higher loop calculations. Below we generalize these equations in the case of pure quantum gravity without a cosmological constant.

3 Renormalization group equations in pure quantum gravity

Our starting point is the theory of gravitational field described by the action

$$L_{gr} = -\frac{1}{G_N} \int R \sqrt{g} d^4x = \frac{L_0}{G_N}, \quad (3.1)$$

where G is the Newton constant (notice that we absorb the usual factor 16π in the definition of G_N). This action does not include the cosmological constant that is assumed to vanish. As we have mentioned this above the case of non-vanishing cosmological constant deserves a separate study and will be reported later. Thus, there is only one dimensionful parameter, the Newton constant G_N .

3.1 Renormalization of the metric

In the dimensional regularization one considers the space-time dimension d slightly different from 4. In doing so the otherwise dimensionless quantities acquire some dimension that can be compensated by introducing a new scale μ . In what follows we prefer to keep the dimensionality of the Newton constant to be the same as in $d = 4$, i.e. $[G_N] = 2$. Instead, the bare metric gets some dimensionality $[g_{B,ij}] = -(d - 4)$ and it is the only quantity present in the classical

action (3.1) that has to be renormalized. This renormalization is in fact a field renormalization that takes a general local form,

$$g_{B,ij} = \mu^{-\epsilon}(g_{R,ij} + \sum_{k=1} \epsilon^{-k} h_{k,ij}(g_R)), \quad (3.2)$$

where $\epsilon = 4 - d$ and $g_{R,ij}$ is a dimensionless renormalized metric, $h_{k,ij}(g_R)$ are local covariant functions of the renormalized metric. One has that

$$\mu \partial_\mu g_{R,ij} = \epsilon g_{R,ij} + \beta_{ij}(g_R), \quad (3.3)$$

where the beta function $\beta_{ij}(g_R)$ is a local function of g_R . Eqs.(3.2)-(3.3) are quite similar to the renormalization of the target metric in the $d = 2$ sigma-models [2], [3]. We should, however, stress the obvious differences: the target metric in a sigma model represents an infinite set of couplings while here we deal with a field renormalization.

The bare metric $g_{B,ij}$ is independent of the scale μ so that differentiating the both sides of equation (3.2) and using (3.3) we arrive at the equation

$$-\epsilon \sum_{k=1} \frac{h_k}{\epsilon^k} + \beta(g_R) + \sum_{k=1} \frac{1}{\epsilon^k} h'_k(g_R) \times (\epsilon g_R + \beta(g_R)) = 0, \quad (3.4)$$

where we skip the space-time indices and the terms linear in ϵ already canceled out. Let us explain our notations used in the above expression: for a tensor $h_{ij}(g)$, a local covariant function of the metric, and a tensor $f_{ij}(x)$ we define

$$(h' \times f)_{ij} \equiv \frac{d}{dt} h_{ij}(g + t f)|_{t=0}. \quad (3.5)$$

Matching the coefficients for a constant term in (3.4) one finds the expression for the beta function in terms of a single pole term,

$$\beta(g_R) = h_1(g_R) - h'_1(g_R) \times g_R. \quad (3.6)$$

For the coefficients at a higher pole $1/\epsilon^k$, $k \geq 1$ one finds

$$h_{k+1} - h'_{k+1} \times g_R = h'_k \times \beta(g_R), \quad k \geq 1. \quad (3.7)$$

This is a recurrence relation for the higher pole residues in terms of the lower pole residues. As always in the case of the RG equations, the complete information about the renormalization is contained in a single pole h_1 . In the perturbative expansion with respect to the Newton constant each term can be expressed as a power series in G_N ,

$$h_k = \sum_{l=k} G_N^l h_{k,l}, \quad \beta_{ij} = \sum_{l=1} G_N^l \beta_{l,ij}, \quad (3.8)$$

where $h_{k,l}$ is a local polynomial in curvature of degree l (we count two covariant derivatives acting on a curvature to have same degree 1 as the curvature itself). Notice that $h'_g(g)g$ is a transformation of $h(g)$ under the rescaling of the metric, $g \rightarrow \lambda g$. So that one finds that $h'_{k,l}(g) \times g = (1-l)h_{k,l}$. Applying this to equations (3.6) and (3.7) one finds for the terms in the expansion (3.8),

$$\beta_l = l h_{1,l}, \quad h_{k+1,l} = \frac{1}{l} \sum_{p=1}^{l-1} h'_{k,l-p} \times \beta_p = \frac{1}{l} \sum_{p=1}^{l-1} p h'_{k,l-p} \times h_{1,p}, \quad k \geq 1. \quad (3.9)$$

This is first set of the RG equations that we will deal with. The other one arises when the renormalization of the effective gravitational action is considered. It will be discussed in the next section. We finish this section by saying that the solution to the RG equations for the metric that we have just derived is completely determined by either specifying the single pole terms $\{h_{1,l}\}$ or the beta function terms $\{\beta_l\}$ in the decomposition (3.8). In the two-dimensional sigma-models the primary element is a single pole term that determines the beta function. In the present case it is rather the beta functions $\{\beta_l\}$ that we have to specify first and then determine all terms in the expansion (3.2), (3.8). However, in order to determine the beta function for the metric we have to look at the renormalization of the action. The metric beta function is then completely specified (up to the gauge coordinate transformations that we will discuss) by a single pole in the effective action. These issues are discussed in the next section.

3.2 Renormalization of the action

The renormalized gravitational action with all counter-terms added is a function of the renormalized metric g_R . The counter-terms are divided on two classes: those that vanish on-shell, we call them L_k , and those that do not, we call them V_k . V_k are invariants constructed from the Riemann tensor and its covariant derivatives that remain non-trivial provided the Bianchi identities are used. It is not a goal of the present paper to give a classification of all possible non-equivalent curvature invariants of a given order. We, however, assume that this classification can be done and, possibly, already exists in the mathematical literature although we are not aware of any relevant publications. In the classical part of the action we have to add terms that are due to the Riemann tensor only, we call them W , with the appropriate coupling constants that have to be renormalized in order to absorb the UV divergences due to V_k . It is convenient to choose a basis of integral invariants of degree $(l+1)$ constructed from the Riemann tensor and its covariant derivatives, $\{P_l, l \geq 2\}$, and expand W and V_k with respect to this basis

$$W = \sum_{l=2} W_l, \quad V_k = \sum_{l \geq k} G_N^{l-1} V_{k,l}, \quad W_l = G_N^{l-1} \lambda_l P_l, \quad V_{k,l} = v_{k,l} P_l, \quad (3.10)$$

where $\{\lambda_l\}$ is a set of dimensionless coupling constants and, since for each l there will be a certain number of independent invariants, both λ_l and $v_{k,l}$ are supposed to have an extra index

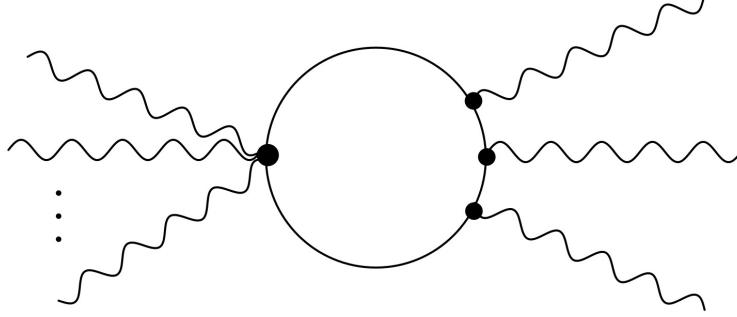


Figure 1: One-loop graviton diagram with three GR cubic vertices and one l -point vertex due to a higher curvature term.

to enumerate the different invariants of same degree $l + 1$. In $d = 4$ the invariant quadratic in the Riemann tensor can be expressed in terms of invariants quadratic in the Ricci tensor and Ricci scalar and the Euler topological invariant. Therefore, in first order ($l = 1$) the UV divergent terms vanish on-shell and one has that $v_{1,1} = 0$. In the second order (two loops), $l = 2$, there is only one independent invariant that can be constructed from the Riemann tensor, $P_2 = \int d^4x \sqrt{g} R_{ab}{}^{cd} R_{cdmn} R^{mnab}$. The numerical value of $v_{1,2}$ was first computed by Goroff and Sagnotti [5] and later by van de Venn [6].

Similarly to (3.10) we expand the counter-terms L_k in powers of the Newton constant G ,

$$L_k = \sum_{l \geq k} G_N^{l-1} L_{k,l}, \quad (3.11)$$

where $L_{k,l}$ contains the curvature invariants of degree $l + 1$. The terms $L_{k,l}$ vanish on-shell and, hence, have to contain at least one power of the Ricci tensor or Ricci scalar.

3.2.1 Higher order Riemann curvature terms in gravitational action

It should be noted that by adding W to the action L_0 one does not change the graviton propagator. Indeed, expanding metric over Minkowski spacetime, $g_{ij} = \eta_{ij} + \sqrt{G} \phi_{ij}$, where ϕ_{ij} is a perturbation, one finds that the terms

$$W_l \sim \Lambda_l (\sqrt{G})^{l+1} \int d^4x (\partial \partial \phi)^{l+1}, \quad l \geq 2 \quad (3.12)$$

where $\Lambda_l = G^{l-1} \lambda_l$, start with a term of $(l + 1)$ -th order in perturbation and, since $l \geq 2$, do not contribute to the graviton propagator. These new terms lead to an additional $(l + 1)$ -point vertex with a new coupling in the graviton Feynman diagrams. Each leg in the vertex has two spacetime derivatives. The corresponding coupling constant is $\Lambda_l (\sqrt{G})^{l+1}$. For $l = 2$ this is a 3-point vertex with two space-time derivatives at each leg. We remind that a (usual) 3-point vertex in General Relativity has effectively a coupling constant \sqrt{G} and at maximum it has one

leg with two derivatives. Effectively, the expansion will go with respect to all available coupling constants: $G, \Lambda_2, \Lambda_3, \dots$. We do not give here a detailed analysis of the corresponding UV divergences. It is, however, instructive to observe certain rules by looking at some simple examples and by using the dimensionality arguments. We first remark that by the dimensionality the UV divergent term that is linear in Λ_p and that contains $(l+1)$ -th power of the curvature has the following form,

$$\sum_{p=2}^{l-1} \Lambda_p G^{l-p} \int d^4x \mathcal{R}^{l+1}, \quad (3.13)$$

\mathcal{R} stands here for any curvature, the Ricci tensor or Riemann tensor. A one-loop diagram that may produce a UV divergent term of this type for $p = l - 1$ is shown in Fig.1. It contains one Λ_p vertex and three GR 3-point vertices. For other values of p the diagram would include a graviton vertex of V_p and $r = (l - 1) - p$ internal GR graviton lines. It is clear that in the renormalization of the coupling Λ_l there may appear only the couplings Λ_p with $p \leq l - 1$. So that the upper limit in the sum in (3.13). Same restriction comes from the condition that the number of the internal graviton lines $r \geq 0$. A similar analysis shows that a Feynman diagram with $\Lambda_{p_1}, \Lambda_{p_2}, \dots, \Lambda_{p_n}$ vertices, r internal GR lines and m GR vertices such that $p_1 + \dots + p_n - n + m = l + 1$ produces a UV divergent term of $(l + 1)$ -th order in curvature,

$$\Lambda_{p_1} \dots \Lambda_{p_n} G^{(n+r)} \int d^4x \mathcal{R}^{l+1}, \quad p_1 + \dots + p_n = (l - 1) - r. \quad (3.14)$$

The counter-term has to have dimension zero, so that one gets a condition for p_i as above. Since $r \geq 0$ one has a condition on the values of p_i :

$$p_1 + \dots + p_n \leq l - 1. \quad (3.15)$$

This discussion is not a rigorous proof of validity of this bound in general. So that its status is conjectural. Although we will not need it in the most of our consideration below this bound will help to avoid certain ambiguities in the beta function equations that will be discussed in section 7.2. Thus, the lowest parameter $v_{1,2}$ (or the counter-term $V_{1,2}$) is independent of any λ . On the other hand, the higher order counter-terms, $V_{k,l}$ and $L_{k,l}$, $l \geq 3$ can be polynomial functions of $\lambda_p, p \leq l - 1$ provided the condition (3.15) is satisfied. We see that, when the higher curvature terms are present, the lower loops may give contributions to the curvature terms that appear at the GR loop order l . For convenience, we will still refer to l as a loop order.

3.2.2 RG equations for λ couplings

In d space-time dimensions the bare coupling λ_l^B has dimension $\mu^{l(d-4)}$. Expressing the bare couplings in terms of dimensionless renormalized quantities λ_l^R one has

$$\lambda_l^B = \mu^{-l\epsilon} \left(\lambda_l^R + \sum_{k=1} \epsilon^{-k} a_{k,l}(\lambda^R) \right), \quad l \geq 2 \quad (3.16)$$

The renormalized couplings λ_l^R satisfy equation

$$\mu \partial_\mu \lambda_l^R = \epsilon l \lambda_l^R + \hat{\beta}_l, \quad (3.17)$$

where $\hat{\beta}_l$ is the beta function for coupling λ_l . The renormalization group equations for the couplings λ_l , $l \geq 2$ read

$$\begin{aligned} \hat{\beta}_l &= l a_{1,l} - \sum_{p=2}^{l-1} p \lambda_p \partial_{\lambda_p} a_{1,l}, \\ \left(1 - \frac{1}{l} \sum_{p=2}^{l-1} p \lambda_p \partial_{\lambda_p} \right) a_{k+1,l} &= \sum_{p=2}^{l-1} \hat{\beta}_p \partial_{\lambda_p} a_{k,l}. \end{aligned} \quad (3.18)$$

Here we take into account that in the renormalization of a coupling λ_l there may be involved only the couplings λ_p with $p \leq l-1$.

3.2.3 Modified RG equations for metric

In general, the bare metric can be a function of the renormalized couplings λ^R . So that the terms $h_{(k)}$ in the expansion (3.2) are, in general, functions of both the renormalized metric g_R and the renormalized couplings $\{\lambda_l^R\}$. The condition $\mu \partial_\mu g_B = 0$ then leads to the modified RG equations,

$$\begin{aligned} \beta_1 &= h_{1,1}, \quad \beta_l = l h_{1,l} - \sum_{p=2}^{l-1} p \lambda_p \partial_{\lambda_p} h_{1,l}, \quad l \geq 2 \\ \left(1 - \frac{1}{l} \sum_{p=2}^{l-1} p \lambda_p \partial_{\lambda_p} \right) h_{k+1,l} &= \frac{1}{l} \sum_{p=1}^{l-1} h'_{k,l-p} \times \beta_p + \frac{1}{l} \sum_{p=2}^{l-1} \hat{\beta}_p \partial_{\lambda_p} h_{k,l}, \quad k \leq l-1 \end{aligned} \quad (3.19)$$

These equations present a modification of the metric RG equations considered in section 3.1.

3.2.4 RG equations for quantum action

In the renormalizable theories the renormalization procedure goes in few steps. The quantum action, which is a sum of the classical action and the counter-terms, is supposed to be a function of the renormalized fields and the renormalized couplings and masses. It does not depend on the scale μ . This condition imposes certain equations on the residues of the poles $1/(d-4)^k$ in the quantum action. Then, in the renormalizable theories the quantum action takes the form of the classical (bare) action provided it is expressed in terms of the bare fields, couplings and masses. The latter, on the other hand, are functions of the renormalized quantities. These steps can be repeated in a non-renormalizable theory. The RG equations for the quantum action in a rather general theory were previously considered in [7]. However, the construction in [7] was

not accompanied, in a coherent way, by a suitable renormalization of the fields and couplings. Their equations are different from those considered in the present paper.

The quantum gravitational action is a function of all renormalized quantities, the metric and the higher curvature couplings, and it takes a general form

$$L_Q(g_R(\mu), \lambda_l^R(\mu)) = \mu^{-\epsilon} \left(\frac{1}{G} L_0(g_R) + W(g_R, \lambda_R) + \sum_{k=1} \frac{1}{\epsilon^k} (L_k(g_R, \lambda_R) + V_k(g_R, \lambda_R)) \right). \quad (3.20)$$

The power of μ is uniquely determined by the requirement that the action to have dimension zero. The quantum action does not depend on the scale μ , the differential equation $\mu \partial_\mu L_Q = 0$ leads to equation

$$\begin{aligned} & -\epsilon \left(\frac{1}{G_N} L_0 + \sum_{k=1} \frac{1}{\epsilon^k} (L_k + V_k) \right) + \frac{1}{G_N} L'_0 \cdot (\epsilon g + \beta) + \sum_{k=1} \frac{1}{\epsilon^k} (L'_k + V'_k) \cdot (\epsilon g + \beta) \\ & + \sum_{l=2} G_N^{l-1} [-\epsilon \lambda_l P_l + \lambda_l P'_l \cdot (\epsilon g + \beta) + P_l(\epsilon l \lambda_l + \hat{\beta}_l)] \\ & + \sum_{k=1} \frac{1}{\epsilon^k} \sum_{p=2} \partial_{\lambda_p} (L_k + V_k) (\epsilon p \lambda_p + \hat{\beta}_p) = 0, \end{aligned} \quad (3.21)$$

where in order to simplify the expression we drop the subscript R in the renormalized metric g_R and in the renormalized higher curvature couplings λ_l^R , the second line in (3.21) is due to differentiation of W . In order to simplify further the formulas, throughout the paper, we will maximally use the index-free notations. Let us explain our notations used in the above expression. Each term in the action (3.20) is a functional of the renormalized metric g_R . L' stands for the metric variation so that it is a local tensor $(L')^{ij} = \frac{\delta L}{\delta g_{ij}(x)}$. In particular, $(L'_0)^{ij} = (R^{ij} - \frac{1}{2} g^{ij} R) = G^{ij}$ is the Einstein tensor. Then we define

$$L' \cdot \beta \equiv \frac{d}{dt} L[g + t\beta]|_{t=0} = \int d^4x \sqrt{g} (L')^{ij} \beta_{ij}. \quad (3.22)$$

Each term in the action (3.20) is an integral over spacetime. Therefore, we will systematically neglect any total derivatives that may appear under the integral.

We notice that the equation (3.21) is invariant under a redefinition of the metric beta function,

$$\beta_{ij} \rightarrow \beta_{ij} + \nabla_i \xi_j + \nabla_j \xi_i. \quad (3.23)$$

This is a usual ambiguity for the metric beta function. We stress that the metric beta function in equation (3.21) is the one that we studied in Section 3.1, see (3.3) and (3.20).

Consider a rescaled metric λg_{ij} . Then with our notations one has that for any functional of the metric $\frac{dW}{d\lambda}|_{\lambda=1} = W' \cdot g$. In particular, this gives us a relation

$$T'_l \cdot g = -(l-1)T_l \quad (3.24)$$

for any curvature polynomial of degree $l + 1$ (any two covariant derivatives are counted as one curvature degree). In particular, this relation holds for $T_l = P_l$, polynomials of the Riemann tensor and its covariant derivatives. In the first line of (3.21) the linear in ϵ terms cancel due to relation $-L_0 + L'_0 \cdot g = 0$ that is an extension of (3.24) for $l = 0$.

4 Beta functions

The vanishing of a constant, ϵ^0 , term in (3.21) will give us the relations to determine the beta functions β_{ij} and $\hat{\beta}_l$. One has that

$$L_1 + V_1 - \frac{1}{G} L'_0 \cdot \beta - (L'_1 + V'_1) \cdot g - \sum_{l=2} G^{l-1} (\lambda_l P'_l \cdot \beta + P_l \hat{\beta}_l) - \sum_{p=2} p \lambda_p \partial_{\lambda_p} (L_1 + V_1) = 0 \quad (4.1)$$

In order to proceed further we use the expansion of the counter-terms in series with respect to the Newton constant G , (3.10), (3.11). One finds that

$$l L_{1,l} - L'_0 \cdot \beta_l - P_l (\hat{\beta}_l - l v_{1,l}) - \sum_{m=2}^{l-1} \lambda_m P'_m \cdot \beta_{l-m} - \sum_{p=2}^{l-1} p \lambda_p \partial_{\lambda_p} (L_{1,l} + V_{1,l}) = 0, \quad (4.2)$$

where $l \geq 1$ and we used (3.24). Notice that the 3rd term in (4.2) is non-trivial for $l \geq 2$ and the 4th and 5th terms are non-zero for $l \geq 3$.

We will use the following general representation for terms $L_{1,l}$

$$L_{1,l} = L'_0 \cdot X^{(l)} = \int d^4 x \sqrt{g} G^{ij} X_{ij}^{(l)}, \quad (4.3)$$

where $X_{ij}^{(l)}$ is a curvature polynomial of degree l (any two covariant derivatives are counted as one curvature degree). We note that, provided that $L_{1,l}$ is given, the term $X_{ij}^{(l)}$ is determined up to a redefinition

$$X_{ij}^{(l)} \rightarrow X_{ij}^{(l)} + \nabla_i \xi_j + \nabla_j \xi_i. \quad (4.4)$$

Below we consider some particular values of l .

$l = 1$

Equation (4.2) in this case reduces to only first two terms,

$$-L_{1,1} + L'_0 \cdot \beta_1 = 0. \quad (4.5)$$

In one loop, the quantum effective action contains terms quadratic in the Ricci scalar and in the Ricci tensor (the square of the Riemann tensor reduces to these two invariants plus a topological Euler number that we neglect in our study). We represent

$$L_{1,1} = \int d^4 x \sqrt{g} (a G_{ij}^2 + b G^2) = \int d^4 x \sqrt{g} G^{ij} X_{ij}^{(1)}, \quad X_{ij}^{(1)} = a G_{ij} + b g_{ij} G, \quad (4.6)$$

where $G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R$, $G = g^{ij}G_{ij} = -R$. Values of a and b are available in the literature and are known to depend on the gauge. The equation for the beta function β_1 is

$$L'_0 \cdot X^{(1)} = L'_0 \cdot \beta_1. \quad (4.7)$$

A solution of this equation is

$$(\beta_1)_{ij} = X_{ij}^{(1)} = a G_{ij} + b g_{ij}G. \quad (4.8)$$

This solution is not unique. One can add to (4.8) a term of the form $\nabla_i \xi_j + \nabla_j \xi_i$ where ξ_i is an arbitrary vector field. This is a general ambiguity (3.23) for the beta function of the metric. We notice that at this order the beta function (4.8) vanishes on-shell ($G_{ij} = 0$).

$l = 2$

In this order equation (4.2) has more terms,

$$-2L_{1,2} + L'_0 \cdot \beta_2 + P_2 (\hat{\beta}_2 - 2v_{1,2}) = 0. \quad (4.9)$$

We remind that, by definition, $L_{1,2}$ vanishes on-shell and hence can be presented in the form

$$L_{1,2} = L'_0 \cdot X^{(2)} = \int d^4x \sqrt{g} G^{ij} X_{ij}^{(2)}, \quad (4.10)$$

where $X^{(2)}$ is quadratic in curvature, it does not necessarily vanish on-shell, in particular it may contain a term quadratic in the Riemann tensor (see below). On the other hand, P_2 is invariant cubic in the Riemann tensor. In $d = 4$ there is only one such invariant,

$$P_2 = \int d^4x \sqrt{g} R_{ijkl} R^{klmn} R_{mn}{}^{ij}. \quad (4.11)$$

One sees that in (4.9) the first two terms vanish on-shell and the last term does not vanish. This means that the first two terms and the last term in the equation are independent and should vanish separately. This gives us the two loop beta functions for the metric and for the cubic coupling λ_2 ,

$$(\beta_2)_{ij} = 2X_{ij}^{(2)}, \quad \hat{\beta}_2 = 2v_{1,2}. \quad (4.12)$$

Tensor $X_{ij}^{(2)}$ is a local tensor quadratic in curvature or its covariant derivatives. Its general form is

$$\begin{aligned} X_{ij}^{(2)} &= c_0 g_{ij} R_{nkml} R^{nkml} + G_{kl} Y_{ij}^{kl}, \\ Y_{ij}^{kl} &= c_1 R_i{}^{(kl)}{}_j + c_2 (g^{kl} G_{ij} + \delta_i^{(k} \delta_j^{l)}) G + c_3 G^{kl} g_{ij} + c_4 g^{kl} g_{ij} G + c_5 G_i^{(k} \delta_j^{l)}, \end{aligned} \quad (4.13)$$

where we did not include the term with covariant derivatives of scalar curvature since it has the form of a gauge transform (4.4). We also did not include a product of two Riemann tensors with two free indices since in $d = 4$ it is expressed in terms of other curvature invariants,

$$R_i^{nkl} R_{jnkl} = 2R_{ikjl} R^{kl} - RR_{ij} + 2R_{ik} R_j^k + \frac{1}{4} g_{ij} (R_{nkml}^2 - 4R_{kl}^2 + R^2). \quad (4.14)$$

The simplest way to get this relation is to vary the $d = 4$ Euler density, see for instance [10] and, for an alternative derivation in terms of the Weyl tensor, [11]. When derived (4.13) we also took into account the fact that a certain combination of tensors is orthogonal to the Einstein tensor G_{kl} ,

$$G_{kl} \tilde{Y}_{ij}^{kl} = 0, \quad \tilde{Y}_{ij}^{kl} = g^{kl} G_{ij} - \delta_i^{(k} \delta_j^{l)} G. \quad (4.15)$$

We see that in two loops there may appear a term (proportional to c_0) in the metric beta function that does not vanish on-shell. The term due to c_1 is linear in G_{ij} and the other terms are quadratic in G_{ij} .

$l = 3$

In the cubic order (three loops) all terms in equation (4.1) contribute,

$$-3L_{1,3} + L'_0 \cdot \beta_3 + P_3 (\hat{\beta}_3 - 3v_{1,3}) + \lambda_2 P'_2 \cdot \beta_1 + 2\lambda_2 \partial_{\lambda_2} (L_{1,3} + v_{1,3} P_3) = 0. \quad (4.16)$$

In this equation the terms containing P_3 do not vanish on-shell. Hence the sum of these terms has to vanish separately from the other terms. This gives us a relation for the beta function for the coupling λ_3 in front of the quartic power of the Riemann tensor,

$$\hat{\beta}_3 = 3v_{1,3} - 2\lambda_2 \partial_{\lambda_2} v_{1,3}. \quad (4.17)$$

We remind that $v_{1,3}$ is linear function of λ_2 . The rest of equation (4.16) can be resolved for a three-loop beta function for the metric,

$$(\beta_3)_{ij} = 3X_{ij}^{(3)} - 2\lambda_2 \partial_{\lambda_2} X_{ij}^{(3)} - \lambda_2 (a (P'_2)_{ij} + b g_{ij} g^{kl} (P'_2)_{kl}), \quad (4.18)$$

where a and b are those that appeared in the one-loop equation (4.6) and $(P'_2)_{ij}$ is a metric variation of invariant P_2 , see for instance [12],

$$(P'_2)_{ij} = -3R_i^{kln} R_{ln}{}^{mp} R_{mpjk} + \frac{1}{2} g_{ij} R_{klmn} R^{mn}{}_{ab} R^{abkl} - 6\nabla^k \nabla^l (R_{ikmn} R_{jl}{}^{mn}), \quad (4.19)$$

$$g^{kl} (P'_2)_{kl} = -R_{klmn} R^{mn}{}_{ab} R^{abkl} - \frac{3}{2} \square (R_{abkl} R^{abkl}) - 12\nabla^k \nabla^l (R_{kilj} G^{ij}) + O(G^2), \quad (4.20)$$

where in (4.20) we used (4.14) and $O(G^2)$ stands for terms quadratic in G_{ij} .

$l = 4$

In the quartic order (four loops) one finds

$$\begin{aligned}
& -4L_{1,4} + L'_0 \cdot \beta_4 + P_4(\hat{\beta}_4 - 4v_{1,4}) + \lambda_2 P'_2 \cdot \beta_2 + \lambda_3 P'_3 \cdot \beta_1 \\
& + (2\lambda_2 \partial_{\lambda_2} + 3\lambda_3 \partial_{\lambda_3})(L_{1,4} + v_{1,4} P_4) = 0.
\end{aligned} \tag{4.21}$$

As before, this equation splits on two parts: the first part contain terms that have at least one power of G_{ij} and the other part contain terms that are due to the Riemann tensor only. The first part can be used to determine the metric beta function β_4 while the second part is used to determine the beta function for the higher curvature coupling λ_4 . In this equation the 3rd term is due to the Riemann tensor only and, thus, it does not vanish on-shell whilst the 1st, 2nd and 5th terms contain at least one power of the Einstein tensor G_{ij} . The 4th term has a part due to the Riemann tensor only and the other part that has at least one power of G_{ij} ,

$$\begin{aligned}
P'_2 \cdot \beta_2 &= \int 2c_0 \left(-\text{Tr}(R^2) \text{Tr}(R^3) - \frac{3}{2} \text{Tr}(R^2) \square \text{Tr}(R^2) \right) \\
& - \int 24c_0 G^{ij} R_{ikjl} \nabla^k \nabla^l \text{Tr}(R^2) + \int 2c_1 G^{ij} R_{ikjl} (P'_2)^{kl} + O(G^2),
\end{aligned} \tag{4.22}$$

where $(P'_2)_{ij}$ is given by eq.(4.19) and $O(G^2)$ stands for terms quadratic in the Einstein tensor and $\text{Tr}(R^n)$ is the trace of a product of n copies of the Riemann tensor, $\text{Tr}(R^n) = R^{i_1 i_2}_{i_3 i_4} R^{i_3 i_4}_{i_5 i_6} \dots R^{i_{2n-1} i_{2n}}_{i_1 i_2}$. The second line in (4.22) contains at least one power of the Einstein tensor and, thus, should be taken into account in (4.21) when one determines the four loop beta function for the metric,

$$\begin{aligned}
(\beta_4)_{ij} &= 4X_{ij}^{(4)} - (2\lambda_2 \partial_{\lambda_2} + 3\lambda_3 \partial_{\lambda_3}) X_{ij}^{(4)} + 24c_0 \lambda_2 R_{ikjl} \nabla^k \nabla^l (\text{Tr} R^2) \\
& + 6c_1 \lambda_2 R_{ikjl} R^{kabc} R_{bc}{}^{mn} R^l{}_{amn} - 12c_1 \lambda_2 R_{ikjl} \nabla^a \nabla^b (R^k{}_{amn} R_b{}^{lmn}) \\
& - \lambda_3 (a(P'_3)_{ij} + b g^{kl} (P'_3)_{kl}) + O(G),
\end{aligned} \tag{4.23}$$

where $O(G)$ are terms linear in G_{ij} .

There is more than one invariant of 5th order that can be constructed from the Riemann tensor. Here we list some of such invariants,

$$P_4^{(1)} = \int \text{Tr}(R^2) \text{Tr}(R^3), \quad P_4^{(2)} = \int \text{Tr} R^2 \square \text{Tr} R^2, \quad P_4^{(3)} = \int \text{Tr}(R^5), \quad \dots \tag{4.24}$$

Respectively, the coupling constant has an extra index $\lambda_4^{(a)}$, $a = 1, 2, \dots$. From eq.(4.21) we find the respective beta functions

$$\begin{aligned}
\hat{\beta}_4^{(1)} &= 4v_{1,4}^{(1)} - (2\lambda_2 \partial_{\lambda_2} + 3\lambda_3 \partial_{\lambda_3}) v_{1,4}^{(1)} + 2\lambda_2 c_0, \\
\hat{\beta}_4^{(2)} &= 4v_{1,4}^{(2)} - (2\lambda_2 \partial_{\lambda_2} + 3\lambda_3 \partial_{\lambda_3}) v_{1,4}^{(2)} + 3\lambda_2 c_0, \\
\hat{\beta}_4^{(a)} &= 4v_{1,4}^{(a)} - (2\lambda_2 \partial_{\lambda_2} + 3\lambda_3 \partial_{\lambda_3}) v_{1,4}^{(a)}, \quad a = 3, \dots
\end{aligned} \tag{4.25}$$

5 The higher pole counterterms

Now it is time to look at the higher poles in the RG equation (3.21). The vanishing condition for the coefficient in front of the pole $1/\epsilon^k$, $k \geq 1$ gives us equation,

$$\begin{aligned} & -L_{k+1} - V_{k+1} + L'_{k+1} \cdot g + V'_{k+1} \cdot g + \sum_{p=2} p \lambda_p \partial_{\lambda_p} (L_{k+1} + V_{k+1}) \\ & + L'_k \cdot \beta + V'_k \cdot \beta + \sum_{p=2} \hat{\beta}_p \partial_{\lambda_p} (L_k + V_k) = 0, \quad k \geq 1 \end{aligned} \quad (5.1)$$

This is a recurrence relation that can be used to determine the counter-terms L_{k+1} and V_{k+1} provided the lower pole counter-terms L_p , V_p , $p = k, k-1, \dots$ are given. In order to start the recurrence procedure one has to know the single pole terms L_1 and V_1 .

Expanding in powers of the Newton constant as in (3.10) and (3.11) we find the recurrence relation for terms $L_{k,l}$ and $V_{k,l}$ that appear in this expansion,

$$\begin{aligned} & \left(1 - \frac{1}{l} \sum_{p=2}^{l-1} p \lambda_p \partial_{\lambda_p}\right) (L_{k+1,l} + V_{k+1,l}) \\ & = \frac{1}{l} \sum_{p=1}^{l-k} (L'_{k,l-p} + V'_{k,l-p}) \cdot \beta_p + \frac{1}{l} \sum_{p=2}^{l-1} \hat{\beta}_p \partial_{\lambda_p} (L_{k,l} + V_{k,l}), \quad l \geq k+1 \end{aligned} \quad (5.2)$$

where β_p are the coefficients in the power series (with respect to G_N) of the metric beta function and $\hat{\beta}_p$ is the beta function for a higher curvature coupling λ_p .

In the effective action the expansion goes in two directions: the powers series in $1/\epsilon$ and the powers of the Newton constant G_N . Interchanging the order of these two expansions one finds

$$\sum_{k=1} \frac{1}{\epsilon^k} \sum_{l=k} G_N^{l-1} (L_{k,l} + V_{k,l}) = \sum_{l=1} G_N^{l-1} \sum_{k=1}^l \frac{1}{\epsilon^k} (L_{k,l} + V_{k,l}) \quad (5.3)$$

The expansion on the right hand side is the loop expansion that indicates that in l -th loop the UV divergences run from a single pole $k = 1$ to the highest pole $k = l$. We stress once again that the basic information is always contained in the single pole. The higher poles are expressed in terms of the single pole using equation (5.2).

Below in this section we analyze the solutions of Eq.(5.2) for certain values of l and k .

5.1 $l = 2$

In this case there is no dependance on the couplings λ_p and the RG equation (5.2) takes a rather simple form

$$L_{2,2} + V_{2,2} = \frac{1}{2} L'_{1,1} \cdot \beta_1, \quad (5.4)$$

where we take into account that $V_{1,1} = 0$. In this equation the right hand side vanishes on-shell since β_1 is linear in the Einstein tensor. On the left hand side of this equation $L_{2,2}$ also vanishes since by assumption $L_{k,l}$ contain at least one power of G_{ij} . Since $V_{2,2}$ is the only term in (5.4) that does not contain the Ricci tensor or the Ricci scalar it has to vanish identically,

$$V_{2,2} = 0 \quad (v_{2,2} = 0). \quad (5.5)$$

We remark that the two loop result (5.5) was first obtained, using methods different from ours, by Chase in 1982 [13] (see also discussion in [14]). The counter-term $L_{1,1}$ takes the form (4.6). Using the beta function (4.8) we find

$$\begin{aligned} L_{2,2} = & \int (a^2 R_{ijnl} G^{ml} G^{ij} + \frac{a^2}{2} G_{ij} \square G^{ij} - \frac{1}{2} (a^2 + 6b^2 + 4ab) G \square G \\ & - (\frac{3}{4} a^2 + ab) G G_{ij}^2 + \frac{1}{4} (ab + a^2) G^3). \end{aligned} \quad (5.6)$$

We see that $L_{2,2}$ is at least quadratic in G_{ij} .

5.2 $l = 3$

Two values of k are possible: $k = 1$ and $k = 2$. For $k = 1$ the RG equation (5.2) is

$$(1 - \frac{2}{3} \lambda_2 \partial_{\lambda_2})(L_{2,3} + V_{2,3}) = \frac{1}{3} (L'_{1,2} + V'_{1,2}) \cdot \beta_1 + \frac{1}{3} L'_{1,1} \cdot \beta_2 + \frac{1}{3} \hat{\beta}_2 \partial_{\lambda_2} (L_{1,3} + V_{1,3}) \quad (5.7)$$

while for $k = 2$ the RG equation is

$$(1 - \frac{2}{3} \lambda_2 \partial_{\lambda_2})(L_{3,3} + V_{3,3}) = \frac{1}{3} L'_{2,2} \cdot \beta_1 + \frac{1}{3} \hat{\beta}_2 \partial_{\lambda_2} (L_{2,3} + V_{2,3}), \quad (5.8)$$

where, as we have shown earlier, $\hat{\beta}_2 = 2v_{1,2}$. In this order the counter-terms can be at most linear in λ_2 so that $V_{1,3} = V_{1,3}^{(0)} + V_{1,3}^{(2)} \lambda_2$ and the same for the counter-terms $L_{1,3}$. Here and below we use the notations for a linear function of λ : $f(\lambda) = f^{(0)} + \sum_{a=2} f^{(a)} \lambda_a$. Separating in each equation the terms vanishing on-shell and the terms non-vanishing on-shell one solves these two equations and obtains

$$\begin{aligned} V_{2,3} &= \frac{2}{3} v_{1,2} V_{1,3}^{(2)}, \\ L_{2,3} &= \frac{2}{3} v_{1,2} L_{1,3}^{(2)} + \frac{1}{3} (L'_{1,2} + V'_{1,2}) \cdot \beta_1 + \frac{1}{3} L'_{1,1} \cdot \beta_2, \\ V_{3,3} &= 0 \quad (v_{3,3} = 0), \quad L_{3,3} = \frac{1}{3} L'_{2,2} \cdot \beta_1. \end{aligned} \quad (5.9)$$

Few things we should notice. First of all, none of the counter-terms $V_{k,3}$, $k = 2, 3$ and $L_{k,3}$, $k = 2, 3$ depends on λ_2 . Next, $V_{3,3}$ vanishes identically in exact parallel with the vanishing of $V_{2,2}$ and $V_{1,1}$. Finally, looking a bit more careful at $L_{3,3}$ we can see that it is at least quadratic in the Einstein tensor, similarly to $L_{2,2}$ (5.6) and $L_{1,1}$ (4.6). We will see whether some of these observations persist to a higher loop order.

5.3 $l = 4$

In this case three values of k are possible: $k = 1$, $k = 2$ and $k = 3$. For the beta function $\hat{\beta}$ one has that $\hat{\beta}_2 = 2v_{1,2}$ is independent of λ and that $\hat{\beta}_3 = 3v_{1,3}^{(0)} + v_{1,3}^{(2)}\lambda_2$ is linear function of λ_2 . Here $v_{1,3} = v_{1,3}^{(0)} + v_{1,3}^{(2)}\lambda_2$. The metric beta functions β_1 and β_2 do not depend on λ while $\beta_3 = \beta_3^{(0)} + \beta_3^{(2)}\lambda_2$ is linear in λ_2 (see (4.18)). The analysis of the RG equations (5.2) goes along same line as for $l = 3$. We skip the details of the analysis and below summarize the results.

$k = 1$

$$\begin{aligned}
V_{2,4}^{(0)} &= \widehat{V}_{2,4} + \frac{v_{1,2}}{2}V_{1,4}^{(2)} + \frac{3}{4}v_{1,3}^{(0)}V_{1,4}^{(3)}, \quad \widehat{V}_{2,4} = \frac{1}{4}(L'_{1,2} + V'_{1,2}) \cdot \beta_2|_{G_{ij}=0}, \\
L_{2,4}^{(0)} &= \frac{1}{4}(L_{1,3}^{(0)} + V_{1,3}^{(0)})' \cdot \beta_1 + \widehat{L}_{2,4} + \frac{1}{4}(L_{1,1})' \cdot \beta_3^{(0)} + \frac{1}{2}v_{1,2}L_{1,4}^{(2)} + \frac{3}{4}v_{1,3}^{(0)}L_{1,4}^{(3)}, \\
\widehat{L}_{2,4} &= \frac{1}{4}(L'_{1,2} + V'_{1,2}) \cdot \beta_2 - \widehat{V}_{2,4}, \quad V_{2,4}^{(2)} = \frac{1}{2}v_{1,3}^{(2)}V_{1,4}^{(3)}, \\
L_{2,4}^{(2)} &= \frac{1}{2}(L_{1,3}^{(2)} + V_{1,3}^{(2)})' \cdot \beta_1 + \frac{1}{2}L'_{1,1} \cdot \beta_3^{(2)} + \frac{1}{2}v_{1,3}^{(2)}L_{1,4}^{(3)}, \\
V_{2,4}^{(3)} &= L_{2,4}^{(3)} = 0.
\end{aligned} \tag{5.10}$$

$k = 2$

$$\begin{aligned}
V_{3,4}^{(0)} &= \frac{1}{2}v_{1,2}V_{2,4}^{(2)}, \\
L_{3,4}^{(0)} &= \frac{1}{4}(L'_{2,3} + V'_{2,3}) \cdot \beta_1 + \frac{1}{4}L'_{2,2} \cdot \beta_2 + \frac{1}{2}v_{1,2}L_{2,4}^{(2)}, \\
V_{3,4}^{(2)} &= L_{3,4}^{(2)} = 0, \quad V_{3,4}^{(3)} = L_{3,4}^{(3)} = 0.
\end{aligned} \tag{5.11}$$

$k = 3$

$$V_{4,4} = 0, \quad L_{4,4} = \frac{1}{4}L'_{3,3} \cdot \beta_1. \tag{5.12}$$

The counter-terms (5.11) and (5.12) do not depend on λ_2 or λ_3 . Counter-term $V_{4,4}$ vanishes identically similarly to $V_{3,3}$ and $V_{2,2}$. A careful analysis (which we perform in the next section) demonstrates that $L_{4,4}$ is a polynomial in G_{ij} that starts with a quadratic term. This is similar to what we have found for $L_{2,2}$ and $L_{3,3}$ and what was known for $L_{1,1}$ (4.6).

6 The GR counter-terms

As we have seen above, the counter-terms have different origins. Some of them originate from the Feynman diagrams with only the usual GR vertices, that come from the General Relativity

action, and the other counter-terms come from the diagrams where additional vertices due to the higher curvature couplings are present. In this section our goal is to isolate those counter-terms that are due to the GR vertices only. We call them the GR counter-terms. They can be obtained from the total counter-terms $V_{k,l}$ and $L_{k,l}$ by taking the limit of vanishing couplings $\{\lambda_p\}$ and neglecting derivatives of the total counter-terms with respect to $\lambda_p, p \geq 2$. We will denote the GR counter-terms vanishing on-shell as $\mathcal{L}_{k,l}$ and the counter-terms non-vanishing on-shell as $\mathcal{V}_{k,l}$. The recurrence relations for the GR counter-terms are obtained from (5.2) in the limiting procedure just described,

$$\mathcal{L}_{k+1,l} + \mathcal{V}_{k+1,l} = \frac{1}{l} \sum_{p=1}^{l-k} (\mathcal{L}'_{k,l-p} + \mathcal{V}'_{k,l-p}) \cdot \beta_p^{(0)}, \quad l \geq k+1 \quad (6.1)$$

where $\beta_p^{(0)}$ is the metric beta function in the limit of vanishing λ . For $p=1$ and $p=2$ it is the same as β_1 and β_2 . As one can see from our analysis in the previous section $V_{k,k} = \mathcal{V}_{k,k}$ and $L_{k,k} = \mathcal{L}_{k,k}$ for $k=1, 2, 3, 4$.

6.1 Some general properties of GR counter-terms in the highest pole $k=l$

The GR counter-terms in the highest pole satisfy equation

$$\mathcal{L}_{k+1,k+1} + \mathcal{V}_{k+1,k+1} = \frac{1}{k+1} (\mathcal{L}'_{k,k} \cdot \beta_1 + \mathcal{V}'_{k,k} \cdot \beta_1), \quad k \geq 1 \quad (6.2)$$

where β_1 is the metric beta function in one loop (4.8). The right hand side of this equation necessarily contains at least one power of G_{ij} (due to β_1) so does $\mathcal{L}_{k+1,k+1}$ in the left hand side of this equation. The only term that does not contain G_{ij} is $\mathcal{V}_{k+1,k+1}$ and, hence, it has to vanish,

$$\mathcal{V}_{k+1,k+1} = 0, \quad k \geq 1. \quad (6.3)$$

So that one has the following

Statement 1. The GR counter-terms in the highest pole $k=l$ at any given loop order l vanish on-shell.

Taking into account (6.3) the RG equation (6.2) can be written in a simpler form

$$\mathcal{L}_{k+1,k+1} = \frac{1}{k+1} \mathcal{L}'_{k,k} \cdot \beta_1. \quad (6.4)$$

We have seen that $L_{2,2}$ and $L_{3,3}$ vanish quadratically in G_{ij} . This property can be extended for the GR counter-terms $\mathcal{L}_{k,k}$ for any $k \geq 2$. This can be done using the induction. The counter-term $L_{2,2}$ is at least quadratic in G_{ij} . Let us assume that $\mathcal{L}_{k,k}$ $k > 2$ is at least quadratic in G_{ij} . So that it can be represented in the form

$$\mathcal{L}_{k,k} = \int G^{ij} Y_{ij}^{(k)}, \quad (6.5)$$

where $Y_{(k)}^{ij}$ has a term linear in the Einstein tensor G . Then, varying (6.5) with respect to metric and neglecting the terms quadratic and of a higher order in G_{ij} one has

$$(\mathcal{L}'_{k,k})_{ij} = -2R_i^a{}^b{}_j Y_{ab}^{(k)} + \nabla_i \nabla^a Y_{aj}^{(k)} + \nabla_j \nabla^a Y_{ai}^{(k)} - g_{ij} \nabla^a \nabla^b Y_{ab}^{(k)} - \square Y_{ab}^{(k)} - \nabla_i \nabla_j Y^{(k)} + g_{ij} \square Y^{(k)},$$

where $Y^{(k)} = g^{ij} Y_{ij}^{(k)}$. Substituting this into eq.(6.4) one finds that $\mathcal{L}_{k+1,k+1}$ can be represented in a form similar to (6.5) with

$$Y_{ij}^{(k+1)} = \frac{1}{k+1} (-2a R_{iajb} Y_{(k)}^{ab} + a g_{ij} \square Y^{(k)} - a \square Y_{ij}^{(k)} - (a+2b) g_{ij} (\nabla^a \nabla^b Y_{ab}^{(k)} - \square Y^{(k)})) + O(G^2)$$

that demonstrates that $Y_{ij}^{(k+1)}$ as well has necessarily a term linear in the Einstein tensor. Hence, one concludes that $\mathcal{L}_{k+1,k+1}$ is at least quadratic in G_{ij} , i.e.

$$\mathcal{L}_{k+1,k+1} = O(G^2), \quad k \geq 1 \quad (6.6)$$

Thus, one has the following

Statement 2. *At any loop order l the GR counter-terms in the highest pole $k = l$ vanish on-shell quadratically.*

We remark here that both a and b that appear in the one-loop counter-term $L_{1,1}$ (4.6) depend on the gauge conditions, see [15] and [16]. There may exist a certain gauge for which both $a = 0$ and $b = 0$, see [15] where some of such gauge conditions were found. In this case all the GR counter-terms in the highest poles vanish identically, $\mathcal{L}_{k,k} = 0$, $k \geq 1$ (as well as the one-loop beta function $\beta_{1,ij}$). This is so up to a topological Euler term which we ignore here. On the other hand, in an alternative approach which makes use of the so-called unique quantum effective action of Vilkovisky one ends up with the certain non-vanishing values of a and b which are claimed to be unique in quantum gravity [17], [18]. In any case, it makes sense to keep the consideration general and consider the arbitrary non-vanishing a and b . Then our Statement 2 is non-trivial since it restricts the possible dependence of the highest pole counter-terms on the Einstein tensor. For instance, a possibility that $\mathcal{L}_{k,k}$ may vanish by a power law $O(G^k)$ depending on the value of k is ruled out by Statement 2. Among other things, our results may also serve as a consistency condition to be used as a tool to check the higher loop calculations.

6.2 Some general properties of GR counter-terms in a sub-leading pole $k = l - 1$

Consider now the first sub-leading pole at a given loop order. From the recurrence relation (5.2) we find

$$\mathcal{L}_{k+1,k+2} + \mathcal{V}_{k+1,k+2} = \frac{1}{k+2} \left((\mathcal{L}'_{k,k+1} + \mathcal{V}'_{k,k+1}) \cdot \beta_1 + \mathcal{L}'_{k,k} \cdot \beta_2 \right), \quad (6.7)$$

where we already took into account (6.3). Beta function β_2 is given by Eqs.(4.12)-(4.13). It does not necessarily vanish on-shell. However, by our Statement 2, $\mathcal{L}'_{k,k}$ vanishes linearly in G_{ij} . Therefore, all terms on the r.h.s of (6.7) vanish on-shell. On the l.h.s. of (6.7) $\mathcal{L}_{k+1,k+2}$ vanishes at least linearly in G_{ij} . Thus, we see that in eq. (6.7) there is only one term, $\mathcal{V}_{k+1,k+2}$, that does not vanish on-shell. And hence it has to be zero,

$$\mathcal{V}_{k+1,k+2} = 0, \quad k \geq 1 \quad (6.8)$$

Statement 3. At any loop order l the GR counter-terms in the first sub-leading pole $k = l - 1$ do not contain any terms that are due to the Riemann tensor only.

We note that (6.8) is valid starting with $k = 1$ and, thus, is not true for $k = 0$. Indeed, $V_{1,2} = v_{1,2}P_2$ is a single pole that appears in two loops. It is proportional to cubic invariant P_2 (4.11). Let us consider (6.7) for $k = 1$,

$$\mathcal{L}_{2,3} = \frac{1}{3} (L'_{1,2} + V'_{1,2}) \cdot \beta_1 + L'_{1,1} \cdot \beta_2. \quad (6.9)$$

It is an expression for a second order pole that appears in three loops. We can not claim that $L_{2,3}$ is quadratic in G_{ij} . Indeed, on the r.h.s. $V'_{1,2} = v_{1,2}P'_2$ is given by (4.19) that is cubic in the Riemann tensor. Taking that β_1 is linear in G_{ij} we conclude that $L_{2,3}$ necessarily contains a term of the form $G_{ij}(\mathcal{R}^3)^{ij}$, where \mathcal{R} stands for the Riemann tensor. Similar reasonings are valid for the 3rd term in the r.h.s. of (6.9). The direct calculation gives us the following expression,

$$\begin{aligned} \mathcal{L}_{2,3} = & \int \left(-av_{1,2}G_{ij}R_i{}^{kln}R_{ln}{}^{mp}R_{mpjk} + v_{1,2}\left(\frac{a}{6} - \frac{b}{3}\right)G \text{Tr} R^3 \right. \\ & \left. - 2av_{1,2}G_{ij}\nabla^k\nabla^n(R_{ikmn}R_{jl}{}^{mn}) - \left(\frac{b}{2}v_{1,2} + c_0(3b+a)\right)G\Box \text{Tr} R^2 + O(G^2) \right). \end{aligned} \quad (6.10)$$

For a larger value of $k \geq 3$ equation (6.7), provided one uses (6.8), reduces to

$$\mathcal{L}_{k+1,k+2} = \frac{1}{k+2} \left(\mathcal{L}'_{k,k+1} \cdot \beta_1 + \mathcal{L}'_{k,k} \cdot \beta_2 \right). \quad (6.11)$$

Clearly, in the r.h.s. of this equation one always has a term that vanishes linearly in G_{ij} . This, in particular, rules out the possibility for $\mathcal{L}_{k,k+1}$ to vanish by a power law with the power growing with k .

6.3 Some general properties of GR counter-terms in a sub-leading pole $k = l - 2$

The recurrence relation (5.2) in this order leads to equation

$$\mathcal{L}_{k+1,k+3} + \mathcal{V}_{k+1,k+3} = \frac{1}{k+3} ((\mathcal{L}'_{k,k+2} + \mathcal{V}'_{k,k+2}) \cdot \beta_1 + (\mathcal{L}'_{k,k+1} + \mathcal{V}'_{k,k+1}) \cdot \beta_2 + \mathcal{L}'_{k,k} \cdot \beta_3^{(0)}), \quad (6.12)$$

where we already took into account that $\mathcal{V}_{k,k} = 0$, $k \geq 1$. We have shown earlier that $\mathcal{V}_{k,k+1} = 0$, $k \geq 2$. However, for $k = 1$ it is non-zero since $\mathcal{V}_{1,2}$ is a non-trivial single pole. Looking at equation (6.12) we note that β_1 is linear in G_{ij} , $\mathcal{L}_{k,k}$ is quadratic in G_{ij} and, hence, $\mathcal{L}'_{k,k}$ is at least linear in G_{ij} . On the other hand, $\mathcal{L}_{k,k+1}$ is linear in G_{ij} and hence $\mathcal{L}'_{k,k+1}$ may be non-vanishing if $G_{ij} = 0$. Putting $G_{ij} = 0$ on both sides of (6.12) we find that

$$\mathcal{V}_{k+1,k+3} = \frac{1}{k+3} \mathcal{L}'_{k,k+1} \cdot \beta_2|_{G_{ij}=0}, \quad k \geq 2. \quad (6.13)$$

This indicates that the second sub-leading pole may not vanish on-shell. Considering the decreasing order of the pole at a fixed loop order l , this is the first time when a higher pole may not vanish on-shell. The value $k = 1$ is a special case. One has more non-vanishing terms in equation (6.12) in this case,

$$\mathcal{L}_{2,4} + \mathcal{V}_{2,4} = \frac{1}{4} ((\mathcal{L}'_{1,3} + \mathcal{V}'_{1,3}) \cdot \beta_1 + (L'_{1,2} + V'_{1,2}) \cdot \beta_2 + L'_{1,1} \cdot \beta_3^{(0)}). \quad (6.14)$$

The terms in the r.h.s. of this equation are due to a single pole ($k = 1$). We recall that $L_{1,1}$ is quadratic in G_{ij} and hence $L'_{1,1}$ is linear in G_{ij} . Putting G_{ij} on both sides of (6.14) we find

$$\mathcal{V}_{2,4} = \frac{1}{4} (L'_{1,2} + V'_{1,2}) \cdot \beta_2|_{G_{ij}=0}. \quad (6.15)$$

This expression is what we called $\widehat{V}_{2,4}$ in (5.10). A general form for the beta function $\beta_{2,ij}$ is given by (4.12), (4.13). Imposing $G_{ij} = 0$ only the term with c_0 survives in (4.13). This term is proportional to g_{ij} so that (6.15) is simplified in the limit $G_{ij} = 0$,

$$\mathcal{V}_{2,4} = \frac{1}{2} c_0 (L'_{1,2} + V'_{1,2}) \cdot g \text{Tr} R^2|_{G_{ij}=0}. \quad (6.16)$$

$L_{1,2}$ has the form (4.10), (4.13) while $V_{1,2} = v_{1,2} P_2$, whose metric variation is given by (4.19), (4.20). Putting everything together we find

$$\mathcal{V}_{2,4} = \frac{3}{2} c_0 (c_0 - \frac{1}{2} v_{1,2}) \int \text{Tr} R^2 \square \text{Tr} R^2 - \frac{1}{2} c_0 v_{1,2} \int \text{Tr} R^2 \text{Tr} R^3. \quad (6.17)$$

A similar analysis can be done for $k \geq 2$, see equation (6.13). Since $\mathcal{L}_{k,k+1}$ is linear in G_{ij} it can be written as follows

$$\mathcal{L}_{k,k+1} = \int G_{ij} Z_{(k)}^{ij} + O(G^2), \quad (6.18)$$

where tensor $Z_{(k)}^{ij}$ has a curvature order k , it is constructed from the Riemann tensor so that it does not vanish on-shell. Then one finds that

$$(\mathcal{L}'_{k,k+1})_{ij}g^{ij} = -\nabla_\alpha \nabla_\beta Z_{(k)}^{\alpha\beta} + \square Z_{(k)}^{\alpha\beta} g_{\alpha\beta} + O(G). \quad (6.19)$$

Using this equation and (4.12), (4.13) for the beta function β_2 one finds

$$\mathcal{V}_{k+1,k+3} = \frac{2c_0}{k+3} \int (-\nabla_\alpha \nabla_\beta Z_{(k)}^{\alpha\beta} + \square Z_{(k)}^{\alpha\beta} g_{\alpha\beta}) \text{Tr} R^2. \quad (6.20)$$

Our analysis in this subsection can be summarized in the following

Statement 4. *At any loop order $l \geq 4$ the GR counter-terms in a sub-leading pole $k = l - 2$ do not necessarily vanish on-shell: there may appear some terms that are due to the Riemann tensor only.*

The appearance of the non-vanishing on-shell terms is possible in other sub-leading poles, $k = l - 3, l - 4, \dots$. This can be easily analyzed using our equation (6.1). We, however, do not consider this question in the present paper.

6.4 Remarks on previous works

We finish this section with some remarks concerning the compatibility of our work with the previous results in the literature. The only earlier paper, we are aware of, that actually computed the higher pole counter-terms in quantum gravity is a paper of Goroff and Sagnotti [5]. In equation (3.18) of this paper they presented the result of an off-shell 2-loop calculation. It contains both the single pole terms and the double poles, i.e. in our notations they computed $V_{1,2}, L_{1,2}$ and $V_{2,2}, L_{2,2}$. We are here interested in the double pole counter-terms. Unfortunately, the comparison of these earlier results with our analysis shows certain signs of disagreement. Indeed, in (3.18) of [5] the double pole includes terms cubic in the Ricci tensor, that are absent in our eq.(5.6). Even more, in (3.18) there presents a term which is linear in $G_{ij}, R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\sigma} R^\delta_\sigma$, although in our analysis the counter-term $L_{2,2}$ is necessarily quadratic in G_{ij} . We believe that a possible source of the disagreement is the following. The authors of [5], using a weak field approximation over Minkowski spacetime, compute in two loops the UV divergences for the cubic vertices that contain six derivatives and the result of the calculation then is used to fix the coefficients in front of the possible cubic curvature invariants. They count nine cubic invariants (see their equations (3.12 a-c)): $I_1 = R \square R, I_2 = R^3, I_3 = R_{ij} \square R_{ij}, I_4 = R R_{ij}^2, I_5 = R_{ik} R_{jl} R^{ijkl}, I_6 = R_i^j R_j^k R_k^i, I_7 = R R_{ijkl} R^{ijkl}, I_8 = R_{ij} R^{iklm} R^j_{klm}, I_9 = R_{ij}{}^{kl} R_{kl}{}^{mp} R_{mp}{}^{ij}$. In their calculation the authors of [5] drop the trace of the metric perturbation and its divergences. In this way they apparently can not determine the coefficients in front of invariants that contain the Ricci scalar. They, however, say that they can determine the coefficients for five invariants I_3, I_5, I_6, I_8 and I_9 that do not contain the Ricci scalar.

It appears that the authors of [5] were not aware of the relation (4.14). Using this relation one finds that invariant I_8 is not an independent invariant,

$$I_8 = \frac{1}{4}(I_2 + I_7) + 2(-I_4 + I_5 + I_6). \quad (6.21)$$

So that, in reality, there are only 8 independent, cubic in curvature, invariants and not 9 as was assumed in [5]. Invariant I_8 has to be excluded. This means that, if everything were consistent, one would have been able to fix the coefficients for four (not five!) invariants that did not contain the Ricci scalar. This of course would correct the numerical factors in (3.18) of [5].

The other related earlier work that discusses (but does not compute) the higher pole counter-terms in quantum gravity is [8]. It was assumed in [8] that the GR highest pole, what we call $\mathcal{L}_{l,l}$, has a first order zero, i.e. vanishes on-shell linearly. This was important for that the renormalization scheme suggested in [8] actually worked. As we demonstrate this here the counter-terms $\mathcal{L}_{l,l}$ vanish quadratically so that the scenario suggested in [8] can not be realized in pure quantum gravity (with zero cosmological constant).

7 Quantum action as a renormalized gravitational action

In this paper we have introduced two renormalization group equations: one for the metric (3.2) and the other for the quantum effective action (3.20). These two sets of equations appear to know about each other through the metric beta function β_{ij} : it is determined by the single pole terms in the effective action by means of the equation (4.1), (4.2). In this section we want to show that there is a deeper relation between the two RG equations. This relation can in fact be anticipated taking into account the way the renormalization works in the case of the renormalizable field theories. Indeed, in a renormalizable field theory all UV divergences in the quantum effective action can be hidden in the field renormalization and the renormalization of the coupling constants so that the quantum action takes the original classical (bare) form if expressed in terms of the bare fields and couplings. In the case of gravity the classical (bare) action should include not only the original General Relativity term $G_N^{-1}L_0$ but also the higher curvature terms $W = \sum_{l=2} G_N^{l-1} \lambda_l P_l$ that are needed to renormalize the UV divergent terms $V_1 = \sum_{l=2} v_{1,l} P_l$. The analysis in this section we do in two steps. First, we demonstrate that the renormalization of the bare GR action correctly reproduces a certain class of the UV divergent terms that we specify below. This is true not only for the single pole terms but also for the higher order poles. For the sake of completeness, the latter agreement we check in detail for the double pole. It is of course guaranteed by the RG structure that the higher poles agree provided the single pole is the same. Then, in the second part of this section we make a general statement that all UV divergent terms can be hidden in the renormalization of the total gravitational action, $L_{\text{gr}} = G_N^{-1}L_0 + W$. There we focus only on the single pole terms.

7.1 Renormalization of the GR action

As we have explained above we first concentrate our attention on the terms that are independent of the higher curvature couplings. Therefore, in our analysis in this section we assume that such terms are not present, i.e. consider the case when $v_{1,l} = 0$ and $\lambda_l = 0$. Since the quantum action depends on $v_{1,l}$ and λ_l analytically this limit can always be arranged. Then for the corresponding part in the quantum action (3.20) one finds

$$\bar{L}_Q(g_R) = L_Q(g_R, \lambda_l = 0, v_{1,l} = 0) = \mu^{-\epsilon} \left(\frac{1}{G_N} L_0(g_R) + \sum_{k=1} \frac{1}{\epsilon^k} (L_k(g_R) + V_k(g_R)) \right), \quad (7.1)$$

where $V_1(g_R) = 0$ as we just have explained. Respectively, in the expression for the metric beta function one has to put $v_{1,l} = 0$ and $\lambda_l = 0$.

We now want to show that (7.1) is in fact identical to the classical GR gravitational action expressed in terms of the bare metric according to (3.2). Thus, our claim is that

$$\bar{L}_Q(g_R) = \frac{1}{G_N} L_0(g_B(g_R)), \quad L_0 = - \int d^4x \sqrt{g} R. \quad (7.2)$$

So that all divergences that are present in (7.1) are, effectively, hidden in the classical gravitational action. Provided $g_B(g_R)$ takes the form (3.2) one has that

$$\frac{1}{G_N} L_0(g_B(g_R)) = \frac{\mu^{-\epsilon}}{G_N} L_0(g_R + \sum_{k=1} \epsilon^{-k} h_{(k)}(g_R)). \quad (7.3)$$

The latter expression can be expanded as a formal power series in ϵ^{-1} ,

$$\frac{1}{G_N} L_0(g_B(g_R)) = \frac{\mu^{-\epsilon}}{G_N} (L_0 + L'_0 \cdot h_1 \epsilon^{-1} + \frac{1}{2} ((L''_0 \times h_1) \cdot h_1 + 2L'_0 \cdot h_2) \epsilon^{-2} + \dots), \quad (7.4)$$

where we use the definitions (3.5) and (3.22). The expansion (7.4) is an appropriate generalization of Faà di Bruno's formula for derivatives of a composite function. A simple form of it is expressed in terms of the Bell polynomials. Here we concentrate our attention only on the first few terms in this formula.

Comparing the expressions (7.1) and (7.4) we see that the first term in both expressions is the same, $G_N^{-1} L_0$. So that let us look at the second term in both expressions. In equation (7.4) this term is

$$\frac{1}{G_N} \epsilon^{-1} L'_0 \cdot h_1 = \epsilon^{-1} \sum_{l=1} G_N^{l-1} L'_0 \cdot h_{1,l} = \epsilon^{-1} \sum_{l=1} G_N^{l-1} l^{-1} L'_0 \cdot \beta_l, \quad (7.5)$$

where we used the expansion in powers of G_N , $h_1 = \sum_{l=1} G_N^l h_{1,l}$, and in the last equality we used the relation (3.9) between the metric beta function β_l ($\beta = \sum_{l=1} G_N^l \beta_l$) and $h_{1,l}$:

$h_{1,l} = l^{-1}\beta_l$. The next point is that eq.(4.2), in the limit we consider in this section, reduces to equation

$$L_{1,l} = l^{-1}L'_0 \cdot \beta_l = L'_0 \cdot h_{1,l}. \quad (7.6)$$

So that eq.(7.5) is precisely $\epsilon^{-1}L_1 = \epsilon^{-1}\sum_{l=1} G_N^{l-1} L_{1,l}$. We remind once again that $V_1 = 0$ in the limit we consider here so that only L_1 appears as a single pole. We thus have proved that the single pole terms in both expressions (7.1) and (7.4) are identical. Since the higher poles are determined by the single pole and the metric beta function (that is by itself related to the single pole and is, thus, the same in both cases) this is sufficient for proving that all other terms (higher poles) in (7.1) and (7.4) are the same and the two expressions are identical. Below we check the equality for the double pole ϵ^{-2} in (7.1) and (7.4) in order to see how the equality works in a higher order and to check that there are no hidden underwater stones in this easy proof.

The double pole in eq.(7.1) can be represented as follows

$$\begin{aligned} \epsilon^{-2}(L_2 + V_2) &= \epsilon^{-2} \sum_{l=2} G_N^{l-1} (L_{2,l} + V_{2,l}) = \epsilon^{-2} \sum_{l=2} G_N^{l-1} \sum_{p=1}^{l-1} \frac{p}{l} L'_{1,l-p} \cdot h_{1,p} \\ &= \epsilon^{-2} \sum_{l=2} G_N^{l-1} \sum_{p=1}^{l-1} \frac{p}{l} \left((L''_0 \times h_{1,p}) \cdot h_{1,l-p} + L'_0 \cdot (h'_{1,l-p} \times h_{1,p}) \right), \end{aligned} \quad (7.7)$$

where we used the relation (5.2) for $k = 1$ and the relation $\beta_p = ph_{1,p}$ (3.9). In the second line we used the relation obtained by differentiating w.r.t. metric the equation (7.6): $L'_{1,l-p} \cdot h_{1,p} = (L''_0 \times h_{1,p}) \cdot h_{1,l-p} + L'_0 \cdot (h'_{1,l-p} \times h_{1,p})$.

On the other hand, in equation (7.4) the double pole is

$$\epsilon^{-2} \sum_{l=2} G_N^{l-1} \frac{1}{2} \sum_{p=1}^{l-1} \left((L''_0 \times h_{1,l-p}) \cdot h_{1,p} + 2 \frac{p}{l} L'_0 \cdot (h'_{1,l-p} \times h_{1,p}) \right), \quad (7.8)$$

where we used the second relation in (3.9) to express $h_{2,l}$ in terms of $h_{1,p}$, $p = 1, \dots, l-1$.

Comparing the expressions (7.7) and (7.8) we see that the second term, due to L'_0 , is the same in both expressions. Then we note that L''_0 is symmetric so that

$$(L''_0 \times h_{1,p}) \cdot h_{1,l-p} = (L''_0 \times h_{1,l-p}) \cdot h_{1,p}. \quad (7.9)$$

This can be demonstrated by a direct computation of L''_0 . Indeed, for two symmetric tensors A_{ij} and B_{ij} one finds that

$$\begin{aligned} (L''_0 \times A) \cdot B &= \frac{1}{2} \int (\nabla^a \nabla_i A_{ja} + \nabla^a \nabla_j A_{ia} - \square A_{ij} - \nabla_i \nabla_j A \\ &\quad - g_{ij} (\nabla^a \nabla^b A_{ab} - \square A - A_{ab} R^{ab} + \frac{1}{2} AR) + AR_{ij} - RA_{ij}) B^{ij} \\ &= (L''_0 \times B) \cdot A, \end{aligned} \quad (7.10)$$

where $A = g^{ij}A_{ij}$. Eq.(7.9) can be viewed as a symmetric scalar product $h_{1,l-p} \star h_{1,p}$. For a symmetric product like this one has the identity

$$\sum_{p=1}^{l-1} \frac{p}{l} h_{1,l-p} \star h_{1,p} = \frac{1}{2} \sum_{p=1}^{l-1} h_{1,l-p} \star h_{1,p} \quad (7.11)$$

that is the final ingredient needed for the demonstration of the equality of (7.7) and (7.8).

Some remarks are in order. The relation (7.2) may have been anticipated. Indeed, many authors have noticed earlier that any counter-term of the form $\int G_{ij}X^{ij}$ that contains at least one power of the Einstein tensor (and, thus, vanishing on-shell) may be absorbed in the original General Relativity action by means of a redefinition of the metric, $g_{ij} \rightarrow g_{ij} + G_N X_{ij}$ (see, for instance, [19] for a relevant discussion). Since the single pole terms $L_{1,l}$ are of this type it is of course natural to expect that by a similar redefinition all of them (and the higher poles related to $L_{1,l}$ by the RG equations) can be consistently hidden inside the classical action L_0 . The metric renormalization (3.2) could be viewed as a consistent way of doing a redefinition of this type. However, the point in this section is still non-trivial. The related higher poles contain $V_{k,l}$, $k \geq 2$ that do not vanish on-shell. Nevertheless, all such terms will go away as soon as metric in the classical action is redefined as (3.2), (3.9). The trick is done by the higher order variations of the classical action L_0 that produce terms that are non-vanishing on-shell. That the entire procedure is self-consistent is guaranteed by the renormalization group equations.

7.2 Renormalization of the total gravitational action

A natural question arises whether one can generalize this property to the complete set of counter-terms, i.e. for a generic case of non-vanishing $\{\lambda_l\}$ and $\{v_{1,l}\}$? A reasonable guess is that the respective generalization of the classical action should include the higher order terms that are due to the Riemann tensor only,

$$L_{\text{gr}}(g, \lambda) = G_N^{-1} L_0 + \sum_{l=2} G_N^{l-1} \lambda_l P_l, \quad (7.12)$$

where, additionally to the renormalization of the metric, one has to include the renormalization of the higher order couplings $\{\lambda_l\}$. Notice that the higher order curvature terms include only the terms with the Riemann tensor. We are now going to show that this is indeed the right form of the bare gravitational action.

Consider (7.12) as a function of the bare metric g_B and the bare coupling constants $\{\lambda_l^B\}$ (3.16) and expand in a formal power series as above. It is sufficient to look at the UV divergent

terms in a single pole (the higher poles are derived from a single pole by the RG equations),

$$L_{gr}(g_B, \lambda_B) = L_{gr}(g_R, \lambda_R) + \epsilon^{-1} Q_1 + \dots, \quad Q_1 = \sum_{l=1} G_N^{l-1} Q_{1,l},$$

$$Q_{1,1} = L'_0 \cdot h_{1,1}, \quad Q_{1,l} = L'_0 \cdot h_{1,l} + a_{1,l} P_l + \sum_{p=2}^{l-1} \lambda_p P'_p \cdot h_{1,l-p}, \quad l \geq 2 \quad (7.13)$$

The last term in $Q_{1,l}$ is non-trivial for $l \geq 3$. We now have to incorporate the dependence of all quantities such as the metric, the beta functions and the terms in the quantum action on the couplings $\{\lambda_p\}$. Therefore, all these quantities are assumed to be decomposed in the power series with respect to λ . For a quantity A_l that appears in a loop order $l \geq 2$ one thus has that

$$A_l = A_l^{(0)} + \sum_{p=2}^{l-1} A_l^{(p)} \lambda_p + \sum_{p_1, p_2=2}^{p_1+p_2=l-1} A_l^{(p_1 p_2)} \lambda_{p_1} \lambda_{p_2} + \dots + \sum_{p_1, \dots, p_n=2}^{p_1+\dots+p_n=l-1} A_l^{(p_1, \dots, p_n)} \lambda_{p_1} \dots \lambda_{p_n} + \dots (7.14)$$

In each sum the condition (3.15) is assumed to be satisfied. For any given l there is a finite number of terms in (7.14). With these definitions we find for $Q_{1,l}$, $l \geq 2$

$$\begin{aligned} Q_{1,l}^{(0)} &= L'_0 \cdot h_{1,l}^{(0)} + a_{1,l}^{(0)} P_l, \\ Q_{1,l}^{(p)} &= L'_0 \cdot h_{1,l}^{(p)} + a_{1,l}^{(p)} + P'_p \cdot h_{1,l-p}^{(0)}, \\ Q_{1,l}^{(p_1 \dots p_n)} &= L'_0 \cdot h_{1,l}^{(p_1 \dots p_n)} + P'_{p_1} \cdot h_{1,l-p_1}^{(p_2 \dots p_n)} + a_{1,l}^{(p_1 \dots p_n)} P_l, \quad n \geq 2 \end{aligned} \quad (7.15)$$

Symmetrization over indices p_1, p_2, \dots, p_n is assumed in the third line of this equation. Assuming the form (7.14) for the quantities that enter the beta function equations (3.18) and (3.20) one finds $\beta_1 = h_{1,1}$ for $l = 1$ and

$$\begin{aligned} \hat{\beta}_l^{(0)} &= l a_{1,l}^{(0)}, \quad \hat{\beta}_l^{(p)} = (l-p) a_{1,l}^{(p)}, \quad \hat{\beta}_l^{(p_1 \dots p_n)} = (l-p_1-\dots-p_n) a_{1,l}^{(p_1 \dots p_n)}, \quad n \geq 2 \\ \beta_l^{(0)} &= l h_{1,l}^{(0)}, \quad \beta_l^{(p)} = (l-p) h_{1,l}^{(p)}, \quad \beta_l^{(p_1 \dots p_n)} = (l-p_1-\dots-p_n) h_{1,l}^{(p_1 \dots p_n)}, \quad n \geq 2 \end{aligned} \quad (7.16)$$

for $l \geq 2$. Notice that if $p_1 + \dots + p_n = l$ then the corresponding beta function vanishes and $a_{1,l}^{(p_1 \dots p_n)}$ and $h_{1,l}^{(p_1 \dots p_n)}$ are not determined by the equations and can be arbitrary. We assume that this ambiguity does not happen. In particular, this is so provided the condition (3.15) holds.

The next step is to expand the equation (4.2) in the series (7.14). First we start with $l = 1$ term, see eq.(4.5), $L_{1,1} - L'_0 \cdot \beta_1 = 0$. Taking into account that $\beta_1 = h_{1,1}$ this equation can be re-written as

$$Q_{1,1} = L_{1,1}. \quad (7.17)$$

Then we consider $l \geq 2$. In zeroth order ($n = 0$) in λ one has

$$l L_{1,l}^{(0)} - L'_0 \cdot \beta_l^{(0)} - P_l(\hat{\beta}_l^{(0)} - l v_{1,l}^{(0)}) = 0 \quad (7.18)$$

that can be re-written, using (7.16), as

$$L'_0 \cdot h_{1,l}^{(0)} + a_{1,l}^{(0)} P_l = L_{1,l}^{(0)} + v_{1,l}^{(0)} P_l. \quad (7.19)$$

The left hand side of this equation is $Q_{1,l}^{(0)}$ (7.15) and, thus, one has that

$$Q_{1,l}^{(0)} = L_{1,l}^{(0)} + V_{1,l}^{(0)}. \quad (7.20)$$

In linear order ($n = 1$) the equation (4.2), after one used (7.16) and dropped the overall factor $(l - p)$, reduces to

$$L_{1,l}^{(p)} - L'_0 \cdot h_{1,l}^{(p)} - a_{1,l}^{(p)} P_l + v_{1,l}^{(p)} P_l - P'_p \cdot h_{1,l-p}^{(0)} = 0. \quad (7.21)$$

It can be re-written in terms of $Q_{1,l}^{(p)}$ (7.15),

$$Q_{1,l}^{(p)} = L_{1,l}^{(p)} + V_{1,l}^{(p)}, \quad (7.22)$$

where $V_{1,l}^{(p)} = v_{1,l}^{(p)} P_l$.

The same algorithm works in n -th order of equation (4.2). Using equations (7.16) and dropping the overall factor $(l - p_1 - \dots - p_n)$ (that is non-zero due to condition (3.15)), one finds

$$L_{1,l}^{(p_1 \dots p_n)} - L'_0 \cdot h_{1,l}^{(p_1 \dots p_n)} - a_{1,l}^{(p_1 \dots p_n)} P_l + v_{1,l}^{(p_1 \dots p_n)} P_l - P'_{p_1} \cdot h_{1,l-p_1}^{(p_2 \dots p_n)} = 0. \quad (7.23)$$

This equation can be re-written as

$$Q_{1,l}^{(p_1 \dots p_n)} = L_{1,l}^{(p_1 \dots p_n)} + V_{1,l}^{(p_1 \dots p_n)}, \quad n \geq 2 \quad (7.24)$$

where $V_{1,l}^{(p_1 \dots p_n)} = v_{1,l}^{(p_1 \dots p_n)} P_l$.

Collecting now equations (7.17), (7.20), (7.22) and (7.24) one finds that

$$Q_{1,l} = L_{1,l} + V_{1,l}, \quad l \geq 1 \quad (7.25)$$

So that a single pole Q_1 in the power series expansion of the bare gravitational action (7.13) is indeed identical to a single pole $L_1 + V_1$ in the quantum effective action. This proves our final statement.

Statement 5. *The complete set of the UV divergent terms in quantum gravity can be consistently hidden in the bare gravitational action (7.12), that includes terms of a higher order in the Riemann tensor, expressed in terms of the bare metric and the bare higher curvature couplings.*

This completes the present analysis.

8 Conclusion

We have formulated a renormalization group approach to the perturbative quantum gravity based on 't Hooft's method developed earlier for the renormalizable theories. Our formulation includes the renormalization of the metric, of the higher Riemann curvature couplings and the renormalization of the quantum action. The equations (3.18), (3.20) and (5.2) form the complete set of the renormalization group recurrence equations that can be solved to determine the higher pole counter-terms in the quantum action. The metric and the higher coupling beta functions are determined by solving eq.(4.2). These equations and, based on them, Statements 1-5 constitute the main result of this paper. The analysis in the present paper has been done in the spacetime dimension $d = 4$. It is of interest to generalize it to other values of d .

We suspect that the approach developed in the present paper can be extended to other, conventionally considered as non-renormalizable, theories such as a scalar field theory with a non-renormalizable potential and the interacting theories of the Horndeski type. We plan to consider these theories elsewhere.

Our approach may have many applications in quantum gravity. The recurrence equations that we derived can be used as a consistency check in a higher loop calculation, that are in the case of quantum gravity are very laborious and time consuming, provided one will be performed in the future. The other possible application is related to the computation of the black hole entropy in the perturbative quantum gravity, along the lines developed in [20]. Finally, it would be interesting to analyze whether one can reconcile the approach developed in this paper with the renormalization ideas suggested in [21]. A related direction is to extend the present approach to the case of gravity with a non-zero cosmological constant. This will be considered in a subsequent work.

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