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On subgroups of Clifford groups defined by Jordan pairs of rectangular matrices

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Abstract

Some embeddings of general linear groups into hyperbolic Clifford groups are constructed generically by using Jordan pairs of rectangular and alternating matrices over a ring. In low rank cases through exceptional isomorphisms, their direct description and relation to some automorphisms of Clifford groups are given. Generic norms are calculated in details, and equivariant embeddings of representation spaces are constructed also.

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Introduction

This paper studies some embeddings of general linear groups into Clifford groups, which are in general explicit only on certain open dense parts and even in low rank cases made explicit rather complexly, but in fact quite natural once interpreted Jordan theoretically. So-called *Jordan pairs of rectangular matrices* are used in the process, embedded naturally into those of *alternating* matrices in a global setting. Examples in low rank cases suggest that our embeddings are involved with certain *automorphisms* of Clifford groups.

We shall work over an arbitrary commutative base ring k of scalars and consider by the data a couple N, P of finitely generated projective modules. Two direct sums $M := N \oplus P, Q := N \oplus P^*$ (where $P^* := \text{Hom}(P, k)$ is dual to P) are to be considered and moreover from M via the *hyperbolic quadratic module* $\mathbf{H}(M)$ we construct its *special Clifford group* $\mathbf{S}\Gamma(\mathbf{H}(M))$. It is then proved that the general linear group $\mathbf{G}\mathbf{L}(Q)$ is embedded into $\mathbf{S}\Gamma(\mathbf{H}(M))$ by a unique homomorphism λ characterized on an open dense subscheme (Theorem

2.4). This is a non-trivial example worked out for O. Loos's theory [13] of Tits-Kantor-Koecher constructions at the *group level*, and the assurance of existence largely overlaps with some rudiments for Jordan pairs, especially that $(\text{Hom}(P^*, N), \text{Hom}(P, N^*))$ of “rectangular matrices”. They are actually well reformulated by an embedding into that $(\bigwedge^2 M, \bigwedge^2 M^*)$ of “alternating matrices” (Sections one; Proposition 1.4). The generic description of λ is partly remedied by direct ones in some cases of low rank (Section three), one of which involves certain complexity caused by succession of exceptional isomorphisms (Proposition 3.3). These examples suggest relation of λ to some involutive automorphism of $\mathbf{S}\Gamma(\mathbf{H}(M))$ (Subsection 3.1 and Proposition 3.7).

Besides general construction and examples of λ , this paper contains other topics of independent interest which are: calculation of the “generic norm” δ_M of $(\bigwedge^2 M, \bigwedge^2 M^*)$ in terms of components relative to the decomposition $M = N \oplus P$ (Proposition 1.6), and construction of an equivariant embedding ζ of representation spaces relative to λ (Theorem 2.8). The former offers utilities for further studies of spin representations [9], [10]. The latter is reminiscent of the classical fact observed by J. Igusa [4] that *some fundamental representations of symplectic groups occur in spin-representations*; note that the symplectic group of $N \oplus N^*$ may well be embedded into $\mathbf{S}\Gamma(\mathbf{H}(M))$ not only through $\mathbf{G}\mathbf{L}(M)$ by taking $P = N^*$ as Igusa did, but also through $\mathbf{G}\mathbf{L}(Q)$ by taking $P = N$.

Acknowledgment I thank the referee who read an earlier version of this paper and gave a sharp question about triviality or non-triviality of λ . Leading me to new problems together with a conjecture, this enriched the last section of the present paper with more interesting topics affirming that conjecture for low rank cases.

1 Jordan pair of rectangular matrices

1.1. Exterior powers of modules. We begin by fixing notation for M , supposed here an arbitrary finitely generated projective module. We denote by M^* the k -module $\text{Hom}(M, k)$ dual to M , and by $\langle x, f \rangle$ the natural pairing $M \times M^* \rightarrow k$ of evaluations. For each integer $\nu \geq 0$, the ν -th exterior power $(\bigwedge^\nu M) \times (\bigwedge^\nu M^*) \rightarrow k$ is to be considered under the sign convention

$$\langle x_1 \wedge \cdots \wedge x_\nu, f_1 \wedge \cdots \wedge f_\nu \rangle := \varepsilon_\nu \det(\langle x_i, f_j \rangle) \text{ with } \varepsilon_\nu := (-1)^{\nu(\nu-1)/2}, \quad (1.1.1)$$

and moreover putting $\langle z, z^* \rangle := 0$ if $z \in \bigwedge^\nu M, z^* \in \bigwedge^\mu M^*$ with $\nu \neq \mu$ we extend it to the whole $\bigwedge M \times \bigwedge M^*$. Besides the ordinary wedge product $z \wedge z' =: l_z \cdot z'$, we use the (left) *interior product* $z^* \lrcorner z' = d_{z^*} \cdot z'$ systematically, with which $\bigwedge M$ becomes a *left $\bigwedge M^*$ -module* such that any element $f \in M^*$

of degree one operates as the unique anti-derivation d_f extending the map $f : \bigwedge^1 M \rightarrow \bigwedge^0 M$. The left $\bigwedge M$ -module $\bigwedge M^*$ being considered just similarly, there arise natural identifications

$$\begin{aligned} \bigwedge^2 M &\subset \text{Hom}(M^*, M) & \bigwedge^2 M^* &\subset \text{Hom}(M, M^*) \\ A(f) &:= -f \lrcorner A & B(x) &:= -x \lrcorner B, \end{aligned} \quad (1.1.2)$$

under which the pair $(\bigwedge^2 M, \bigwedge^2 M^*)$ comes equipped with the quadratic product $(A, B) \mapsto (-ABA, -BAB)$; this is a *Jordan pair*, such that (A, B) is quasi-invertible if and only if the endomorphism $1 + AB$ of M , dual to that $1 + BA$ of M^* , is invertible in which case the quasi-inverses A^B, B^A are $A^B = (1 + AB)^{-1}A = A(1 + BA)^{-1}$, etcetera [5, Subsection 1.5]. Particular interest is to be laid on the polynomial function

$$\delta_M(A, B) := \langle \exp(A), \exp(B) \rangle \quad (1.1.3)$$

(where \exp denotes *Chevalley's exponential map* [2, Section IV-2]), which we have called the “generic norm” in the introduction, and now labeled M as it will be varied later; being actually a square root of the polynomial $\det(1 + AB)$ [5, Theorem 2.6], this δ_M enjoys “cocycle relations”

$$\delta_M(A, B)\delta_M(A^B, B') = \delta_M(A, B + B'), \quad (1.1.4)$$

$$\delta_M(A', B^A)\delta_M(A, B) = \delta_M(A' + A, B) \quad (1.1.5)$$

for any quasi-invertible (A, B) and for any (A', B') [5, Subsection 3.11]. Exponential maps play a role of relating the Jordan pair $(\bigwedge^2 M, \bigwedge^2 M^*)$ to the *hyperbolic (quadratic) module* $\mathbf{H}(M)$; recall that the latter is the direct sum $M \oplus M^*$ equipped with the quadratic form defined by the pairing $\langle \cdot, \cdot \rangle$, and that the exterior power $\bigwedge M = \bigwedge^+ M \oplus \bigwedge^- M$, decomposed by the parities of degrees, plays the role of “space of spinors” by identifying $\text{End}(\bigwedge M)$ with the Clifford algebra; it has $M \oplus M^*$ embedded by $(x, f) \mapsto l_x + d_f$, and is graded by the “checker-board grading” (see [11, IV, (2.1)] for details).

1.2. Exterior powers of linear maps. We shall consider the case where M is given as the direct sum of two arbitrary finitely generated and projective modules:

$$M = N \oplus P. \quad (1.2.1)$$

Applying the convention (1.1.1) to P , we settle $\bigwedge^\nu P$ dual to $\bigwedge^\nu P^*$ and identify $\text{Hom}(\bigwedge^\nu P^*, \bigwedge^\nu N)$ with $(\bigwedge^\nu N) \otimes (\bigwedge^\nu P)$; moreover, embedding the latter into $\bigwedge^{2\nu} M$ by the map $z \otimes w \mapsto z' \wedge w'$ induced from the ν -th exterior powers $z \mapsto z', w \mapsto w'$ of natural inclusions of $N \rightarrow M, P \rightarrow M$ together with the wedge product in $\bigwedge M$, we construct a linear map

$$\lambda_{\pm}^\nu : \text{Hom}(\bigwedge^\nu P^*, \bigwedge^\nu N) \longrightarrow \bigwedge^{2\nu} M. \quad (1.2.2)$$

This is in fact an isomorphism onto one of those direct factors as we have in $\bigwedge^{2\nu} M$ decomposed to the sum of the $(\bigwedge^{2\nu-i} N) \otimes (\bigwedge^i P)$, $0 \leq i \leq 2\nu$; in particular, we have a decomposition

$$\bigwedge^2 M \xrightarrow{\sim} (\bigwedge^2 N) \oplus \text{Hom}(P^*, N) \oplus (\bigwedge^2 P). \quad (1.2.3)$$

On the external two factors of the decomposition (1.2.3), the exponential map $\exp : \bigwedge^2 M \rightarrow \bigwedge^+ M$ considered for M induces those for N, P by naturality; as for the central factor, say $u \in \text{Hom}(P^*, N)$ any element of it and $\exp(\lambda_+^1(u)) =: \sum_{\nu \geq 0} \exp(\lambda_+^1(u))_{2\nu}$ its exponential decomposed to homogeneous components, we claim that

$$\exp(\lambda_+^1(u))_{2\nu} = \lambda_+^\nu(\bigwedge^\nu u) \in \bigwedge^{2\nu} M. \quad (1.2.4)$$

To see this, choose finite n -tuples $(x_1, \dots, x_n) \in N^n$, $(y_1, \dots, y_n) \in P^n$ such that $u = \sum_i x_i \otimes y_i$ and use abbreviations such as $x_I := x_\alpha \wedge x_\beta \wedge \dots$, for any set $I =: \{\alpha < \beta < \dots\}$ of finite indexes; it is then immediate that

$$\bigwedge^\nu u = \varepsilon_\nu \sum_I x_I \otimes y_I, \quad (1.2.5)$$

where the summation ranges over all the sets I of ν -indexes. On the other hand, since $\lambda_+^1(u)$ equals $\sum_i (x_i, 0) \wedge (0, y_i)$, its exponential is the product of the mutually commuting n -elements $1 + (x_i, 0) \wedge (0, y_i)$; thus its 2ν -component $\exp(\lambda_+^1(u))_{2\nu}$ equals the sum of all partial products $p_I := \prod_{i \in I} (x_i, 0) \wedge (0, y_i)$, for I ranging over just as in (1.2.5). Then the proof completes, by soon encountering the signature ε_ν in the relation $p_I = \varepsilon_\nu \lambda_+^\nu(x_I \otimes y_I)$.

1.3. The Jordan pair $(\mathfrak{V}^+, \mathfrak{V}^-)$. The previous λ_+^ν is in fact to be coupled with the analogous map

$$\lambda_-^\nu : \text{Hom}(\bigwedge^\nu P, \bigwedge^\nu N^*) \longrightarrow \bigwedge^{2\nu} M^* \quad (1.3.1)$$

constructed just similarly from (N^*, P^*) by making obvious identifications $N^* \oplus P^* \cong M^*$, $P^{**} \cong P$; furthermore, as in [5, Subsection 1.6], synonyms such as $M^+ := M$, $M^- := M^*$, together with a mod 2-suffix $\sigma \in \{+, -\}$, which is to equilibrate the two roles indicated by \pm , will be occasionally employed. With these being said, we now consider the pair

$$\mathfrak{V}^+ := \text{Hom}(P^*, N), \quad \mathfrak{V}^- := \text{Hom}(P, N^*) \quad (1.3.2)$$

of k -modules equipped with the quadratic product $(u, v) \mapsto (uv^*u, vu^*v)$. This is needless to say a Jordan pair, locally isomorphic to that of *rectangular matrices*, but as for its fundamental properties about quasi-inverses it is possible to have a global proof, not resorting to calculation in local coordinates. The Bergmann operators $B_+(u, v) \in \text{End}(\mathfrak{V}^+)$ and $B_-(v, u) \in \text{End}(\mathfrak{V}^-)$ being by

definition of the forms $u_0 \mapsto (1 - uv^*)u_0(1 - v^*u)$ and $v_0 \mapsto (1 - vu^*)v_0(1 - u^*v)$, respectively, it is obvious that (u, v) is quasi-invertible if all the endomorphisms

$$\begin{aligned} 1 - uv^* &\in \text{End}(N^*), & 1 - v^*u &\in \text{End}(P^*), \\ 1 - vu^* &\in \text{End}(N), & 1 - u^*v &\in \text{End}(P) \end{aligned} \quad (1.3.3)$$

are invertible, in which case one has the quasi-inverses to be of the forms $u^v = (1 - uv^*)^{-1}u = u(1 - v^*u)^{-1}$, etcetera; as for the converse, which appears likely but cumbersome to be precise in terms of actual matrices, we shall subsume it into the following

1.4. Proposition. (a) *The pair $(\lambda_+^1, \lambda_-^1)$ of linear maps is a homomorphism of Jordan pairs; in particular, if (u, v) is quasi-invertible then so is $(\lambda_+^1(u), \lambda_-^1(v))$.*

(b) *Suppose (u, v) arbitrary. The four endmorphisms listed in (1.3.3) have the same determinant which is equal to $\delta_M(\lambda_+^1(u), \lambda_-^1(v))$; it is invertible if and only if (u, v) is quasi-invertible.*

Proof. (a) Let $w = \sum_i x_i \otimes y_i$ be any element of $\mathfrak{V}^\sigma \cong N^\sigma \otimes P^\sigma$ and regard $\lambda_\sigma^1(w) =: W$ as a linear map $M^{-\sigma} \rightarrow M^\sigma$ by the rule (1.1.2); then the image $W(\xi)$ of any $\xi = (f, g) \in M^{-\sigma} \cong N^{-\sigma} \oplus P^{-\sigma}$ equals the sum of the $-(f, g) \lrcorner ((x_i, 0) \wedge (0, y_i))$, namely $\sum_i (\langle y_i, g \rangle x_i, -\langle x_i, f \rangle y_i)$; therefore $W(f, g) = (w(g), -w^*(f))$ which, interpreted as matrices of left actions and now duplicated for two cases $u \in \mathfrak{V}^+$, $v \in \mathfrak{V}^-$, yields

$$\lambda_+^1(u) = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \quad \lambda_-^1(v) = \begin{pmatrix} 0 & v \\ -v^* & 0 \end{pmatrix}. \quad (1.4.1)$$

It is then immediate to get identities $-\lambda_+^1(u)\lambda_-^1(v)\lambda_+^1(u) = \lambda_+^1(uv^*u)$ and $-\lambda_-^1(v)\lambda_+^1(u)\lambda_-^1(v) = \lambda_-^1(vu^*v)$, showing that $(\lambda_+^1, \lambda_-^1)$ is a homomorphism. The last statement is then a consequence of naturalness of quasi-inverses, see [12, Proposition 3.2].

(b) Since $1 - uv^*$ and $1 - vu^*$ are dual to each other and so are $1 - u^*v$ and $1 - v^*u$, in order to prove the first statement it suffices to prove

$$\det(1 - uv^*) = \delta_M(\lambda_+^1(u), \lambda_-^1(v)) = \det(1 - u^*v); \quad (1.4.2)$$

furthermore, on account of the last statement of the part (a), these identities will make also clear the unsettled part, the if part, of the second statement. We prove (1.4.2). Since the formula (1.2.4) and the analogous one for $\lambda_-^\nu(v)$ convert $\delta_M(\lambda_+^1(u), \lambda_-^1(v))$ to the sum of the $\langle \lambda_+^\nu(\bigwedge^\nu u), \lambda_-^\nu(\bigwedge^\nu v) \rangle$ for $\nu \geq 0$, by a well-known formula for determinants of type $\det(1 + h)$ (see for example the formula (19) of [1, Section III-8.5]), it suffices to prove, for any X in $\text{Hom}(\bigwedge^\nu P^*, \bigwedge^\nu N)$ and any Y in $\text{Hom}(\bigwedge^\nu P, \bigwedge^\nu N^*)$,

$$(-1)^\nu \text{trace}(XY^*) = \langle \lambda_+^\nu(X), \lambda_-^\nu(Y) \rangle = (-1)^\nu \text{trace}(X^*Y). \quad (1.4.3)$$

The problem being now linear in each arguments X, Y , we may suppose them to be of the forms $z \otimes w, z^* \otimes w^*$, where z, w, z^*, w^* are any elements of ν -th exterior powers of N, P, N^*, P^* . In this case, both $XY^* = \langle w, w^* \rangle z \otimes z^*$ and $X^*Y = \langle z, z^* \rangle w \otimes w^*$ have indeed the same trace $\langle z, z^* \rangle \langle w, w^* \rangle$, and since $\lambda_+^\nu(X) = z \wedge w, \lambda_-^\nu(Y) = z^* \wedge w^*$ under the identifications we have made and indicated by dashes in settling (1.2.2), their pairing $\langle \lambda_+^\nu(X), \lambda_-^\nu(Y) \rangle$ is now equal to $\langle w^* \lrcorner (z \wedge w), z^* \rangle$ by the duality between left interior- and right wedge-products; since $w^* \lrcorner (z \wedge w) = (-1)^\nu \langle w, w^* \rangle z$, this completes the proof.

1.5. More on δ_M . Let us examine the polynomial $\delta_M(A, B)$ in more details than the previous case $(A, B) = (\lambda_+^1(u), \lambda_-^1(v))$. Coupling the decomposition (1.2.3) with the analogous one for $\bigwedge^2 M^*$, we shall note here the following addendum to (1.4.1): as maps intertwining between $M = N \oplus P$ and $M^* = N^* \oplus P^*$ according to the rule (1.1.2), any elements $A = a \oplus u \oplus c \in \bigwedge^2 M, B = b \oplus v \oplus d \in \bigwedge^2 M^*$ with components

$$\begin{aligned} a &\in \bigwedge^2 N, & u &\in \text{Hom}(P^*, N), & c &\in \bigwedge^2 P, \\ b &\in \bigwedge^2 N^*, & v &\in \text{Hom}(P, N^*), & d &\in \bigwedge^2 P^* \end{aligned}$$

are represented as alternating matrices of type

$$a \oplus u \oplus c = \begin{pmatrix} a & u \\ -u^* & c \end{pmatrix}, \quad b \oplus v \oplus d = \begin{pmatrix} b & v \\ -v^* & d \end{pmatrix}. \quad (1.5.1)$$

We propose to give a method of calculating $\delta_M(A, B)$ in terms of these components “generically” (Preposition 1.6). We begin by noting that, from naturalness of exponential maps, $\delta_M(A, B)$ reduces to either $\delta_N(a, b)$ or $\delta_P(c, d)$ according to those cases where (A, B) is either $(a \oplus 0 \oplus 0, b \oplus 0 \oplus 0)$ or $(0 \oplus 0 \oplus c, 0 \oplus 0 \oplus d)$; both $\delta_M(a \oplus 0 \oplus 0, 0 \oplus 0 \oplus d)$ and $\delta_M(0 \oplus 0 \oplus c, b \oplus 0 \oplus 0)$ are 1, since except for constant terms all pairings coming from $(\bigwedge N^\pm) \times (\bigwedge P^\mp)$ vanish; here and in the following, we shall tacitly forget the order $\langle z, z^* \rangle =: \langle z^*, z \rangle$ of arguments in the pairing $\langle \cdot, \cdot \rangle$, which will thus appear “symmetric”. Along the same line, combined also with (1.4.2), we get the following generalization of what we have said:

$$\begin{aligned} \delta_M(0 \oplus u \oplus c, b \oplus 0 \oplus 0) &= \delta_M(a \oplus u \oplus 0, 0 \oplus 0 \oplus d) = \\ &= \delta_M(a \oplus 0 \oplus 0, 0 \oplus v \oplus d) = \delta_M(0 \oplus 0 \oplus c, b \oplus v \oplus 0) = 1, \end{aligned} \quad (1.5.2)$$

$$\begin{aligned} \delta_M(a \oplus u \oplus c, 0 \oplus 0 \oplus d) &= \delta_M(0 \oplus 0 \oplus c, b \oplus v \oplus d) = \delta_P(c, d), \\ \delta_M(a \oplus 0 \oplus 0, b \oplus v \oplus d) &= \delta_M(a \oplus u \oplus c, b \oplus 0 \oplus 0) = \delta_N(a, b), \end{aligned} \quad (1.5.3)$$

$$\begin{aligned} \delta_M(a \oplus u \oplus c, 0 \oplus v \oplus 0) &= \delta_M(0 \oplus u \oplus 0, b \oplus v \oplus d) = \\ &= \det(1 - uv^*) = \det(1 - u^*v). \end{aligned} \quad (1.5.4)$$

Note, however, that similar values such as $\delta_M(a \oplus 0 \oplus c, 0 \oplus v \oplus 0)$ are no longer trivial, since, for example, $\exp(a \oplus 0 \oplus c)$ has components coming from the $\text{Hom}(\bigwedge^\nu(P^*), \bigwedge^\nu(N))$; as for these, we shall prove

$$\begin{aligned} \delta_M(a \oplus 0 \oplus c, 0 \oplus v \oplus 0) &= \delta_N(a, (\bigwedge^2 v)c) = \delta_P(c, (\bigwedge^2 v^*)a), \\ \delta_M(0 \oplus u \oplus 0, b \oplus 0 \oplus d) &= \delta_N((\bigwedge^2 u)d, b) = \delta_P((\bigwedge^2 u^*)b, d). \end{aligned} \quad (1.5.5)$$

Indeed, let $\delta^{(1)}$ denote the left-hand side $\delta_M(a \oplus 0 \oplus c, 0 \oplus v \oplus 0)$ of the first line, and $\delta^{(2)}$ that of the second line. We begin by regarding $\delta^{(1)}$ as $\delta_M(A + A', B)$, with $A := a \oplus 0 \oplus 0$, $A' := 0 \oplus 0 \oplus c$, $B := 0 \oplus v \oplus 0$. In this case, $\delta_M(A, B) = 1$ by (1.5.2) and the quasi-inverse $B^A = B(1 + AB)^{-1}$, calculated in terms of matrices of type (1.5.1), has components $0 \oplus v \oplus (\bigwedge^2 v^*)a$; thus, the cocycle relation (1.1.5) converts $\delta^{(1)}$ to $\delta_P(c, (\bigwedge^2 v^*)a)$, on account of (1.5.3). Similarly, starting from $A := 0 \oplus 0 \oplus c$, $A' := a \oplus 0 \oplus 0$, in which case $B^A = (\bigwedge^2 v)c \oplus v \oplus 0$, we get $\delta^{(1)} = \delta_N(a, (\bigwedge^2 v)c)$. Furthermore, proof of the second line for $\delta^{(2)}$ is also similar, by applying now the relation (1.1.4) to $A := 0 \oplus u \oplus 0$ and $\{B, B'\} := \{b \oplus 0 \oplus 0, 0 \oplus 0 \oplus d\}$. We shall proceed along the same line of thus proving (1.5.5), to prove the following

1.6. Proposition. (a) *If $(c, d) \in (\bigwedge^2 P) \times (\bigwedge^2 P^*)$ is quasi-invertible then*

$$\delta_M(a \oplus u \oplus c, b \oplus v \oplus d) = \delta_M(a_1 \oplus u_1 \oplus 0, b_1 \oplus v \oplus 0) \delta_P(c, d) \quad (1.6.1)$$

with $a_1 := a + (\bigwedge^2 u)d^c$, $u_1 := u(1 + dc)^{-1}$, $b_1 := b + (\bigwedge^2 v)c^d$; in particular,

$$\delta_M(a \oplus u \oplus c, b \oplus v \oplus 0) = \delta_M(a \oplus u \oplus 0, (b + (\bigwedge^2 v)c) \oplus v \oplus 0). \quad (1.6.2)$$

(b) *If $(u, v) \in \mathfrak{B}^+ \times \mathfrak{B}^-$ is quasi-invertible then*

$$\delta_M(a \oplus u \oplus 0, b \oplus v \oplus 0) = (\det h) \delta_N(a', b) \quad (1.6.3)$$

with $h := 1 - uv^*$, $a' := (\bigwedge^2 h)^{-1}a$.

Proof. (a) We begin with the special case (1.6.2), which is proved similarly to (1.5.5) by applying (1.1.5) to $A' := a \oplus u \oplus 0$, $A := 0 \oplus 0 \oplus c$, $B := b \oplus v \oplus 0$. In order to prove (1.6.1), we let now $A := a \oplus u \oplus c$, $B := 0 \oplus 0 \oplus d$, $B' := b \oplus v \oplus 0$, in which case $\delta_M(A, B) = \delta_P(c, d)$ (1.5.3) is by supposition invertible, and use (1.1.4) to decompose the left-hand side of (1.6.1) to $\delta_M(A^B, B') \delta_P(c, d)$. Calculating

$$1 + AB = \begin{pmatrix} 1 & ud \\ 0 & 1 + cd \end{pmatrix} = \begin{pmatrix} 1 & -ud(1 + cd)^{-1} \\ 0 & (1 + cd)^{-1} \end{pmatrix}^{-1}$$

and noting that besides two expressions $d(1 + cd)^{-1} = (1 + dc)^{-1}d$ of d^c the expression $1 - d(1 + cd)^{-1}c$ equals $(1 + dc)^{-1}$, we see that the quasi-inverse $A^B =$

$(1 + AB)^{-1}A$ has components $a_1 \oplus u_1 \oplus c^d$ with our a_1, u_1 ; thus, on account of (1.6.2), we get the desired relation $\delta_M(A^B, B') = \delta_M(a_1 \oplus u_1 \oplus 0, b_1 \oplus v \oplus 0)$.

(b) Put $A := a \oplus u \oplus 0$, $B := 0 \oplus v \oplus 0$, $B' := b \oplus 0 \oplus 0$, in which case $\delta_M(A, B) = \det h$ (1.5.4) is by supposition invertible, and use (1.1.5) to reduce the problem to proving $\delta_M(A^B, B') = \delta_N(a', b)$. Again we calculate $1 + AB$ which, on account of $v(1 - u^*v)^{-1} = (1 - vu^*)^{-1}v = v^u$, is now of the form

$$\begin{pmatrix} 1 - uv^* & av \\ 0 & 1 - u^*v \end{pmatrix} = \begin{pmatrix} (1 - uv^*)^{-1} & -(1 - uv^*)^{-1}av^u \\ 0 & (1 - u^*v)^{-1} \end{pmatrix}^{-1}.$$

Therefore, by $A^B = (1 + AB)^{-1}A$ together with $1 + v^u u^* = (1 + vu^*)^{-1}$, we get $A^B = a' \oplus u^v \oplus 0$. On account of (1.5.3), this proves the desired relation.

2 Embeddings

2.1. The homomorphisms X_{\pm}, X_0 . We turn attention to group schemes and adapt the functorial point of view [3]. Put

$$Q := N \oplus P^* \tag{2.1.1}$$

and express elements of $\text{End}(Q)$ as matrices with entries in array

$$\begin{pmatrix} \text{End}(N) & \text{Hom}(P^*, N) \\ \text{Hom}(N, P^*) & \text{End}(P^*) \end{pmatrix} \tag{2.1.2}$$

acting from the left. Similar matrix notation being repeated for all scalar extensions too, we have then homomorphisms

$$\mathbf{W}(\mathfrak{A}^+) \xrightarrow{X_+} \mathbf{GL}(Q) \xleftarrow{X_-} \mathbf{W}(\mathfrak{A}^-) \\ X_+(u) := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad X_-(v) := \begin{pmatrix} 1 & 0 \\ -v^* & 1 \end{pmatrix} \tag{2.1.3}$$

of k -group schemes, where \mathbf{W} denotes the covariant functor constructing vector bundles from finitely generated projective modules: $\mathbf{W}(\mathfrak{A}^+)(k') := \mathfrak{A}^+ \otimes k'$ for each scalar extension k'/k , etcetera; and when describing scheme-morphisms, letters like u, v in (2.1.3) are to be understood ranging over all scalar extensions too. In an analogous fashion we introduce a homomorphism

$$X_0 : \mathbf{GL}(N) \times \mathbf{GL}(P) \longrightarrow \mathbf{GL}(Q) \\ X_0(g, h) := \begin{pmatrix} g & 0 \\ 0 & h^{*-1} \end{pmatrix}. \tag{2.1.4}$$

Actual computation of the product $X_-(v)X_0(g, h)X_+(u) =: \xi(v, (g, h), u)$ shows at once that ξ is an *open embedding* with image the principal open subscheme

$\Omega \subset \mathbf{GL}(Q)$ defined by the condition that the $(1, 1)$ -entry in the sense of matrix (2.1.2) is invertible; since the k -group scheme $\mathbf{GL}(Q)$ is smooth with connected fibers, this proves that Ω is scheme-theoretically dense in $\mathbf{GL}(Q)$.

2.2. Relation to the Jordan pair $(\mathfrak{A}^+, \mathfrak{A}^-)$. We need to know about the induced group germ structure on Ω , for which it is convenient to be conformity with [13]. The decomposition (2.1.1) is now used to endow $\mathbf{GL}(Q)$ with the action ψ of the multiplicative group $\mathbf{G}_{\mathbf{m}k}$ such that ψ_t , for any invertible scalar t , multiplies t^{j-i} to each (i, j) -entry; this amounts to setting $\psi_t := \text{Int}(X_0(t, 1))$ and makes $\mathbf{W}(\mathfrak{A}^\pm)$ to be of weights ± 1 , while $\mathbf{GL}(N) \times \mathbf{GL}(P)$ remains invariant. There exists an action $\rho := (\rho_+, \rho_-)$ of the k -group $\mathbf{GL}(N) \times \mathbf{GL}(P)$ on the pair $(\mathfrak{A}^+, \mathfrak{A}^-)$ by

$$\rho_+(g, h) \cdot u := guh^*, \quad \rho_-(g, h) \cdot v := h^{*-1}vg^{-1} \quad (2.2.1)$$

which is also recovered by conjugations in $\mathbf{GL}(Q)$ and compatible with the Jordan product $(u, v) \mapsto (uv^*u, vu^*v)$. Moreover, from (1.4) it follows that, on the open subscheme $\mathcal{W} \subset \mathbf{W}(\mathfrak{A}^+) \times \mathbf{W}(\mathfrak{A}^-) =: \mathbf{X}$ of quasi-invertible pairs, there exists a well-defined morphism

$$\begin{aligned} b : \mathcal{W} &\longrightarrow \mathbf{GL}(N) \times \mathbf{GL}(P) \\ b(u, v) &:= (1 - uv^*, 1 - u^*v). \end{aligned} \quad (2.2.2)$$

For the morphism $p : \mathbf{X} \rightarrow \mathbf{GL}(Q)$ defined by $p(u, v) := X_+(u)X_-(v)$, easy computation shows \mathcal{W} to be the inverse image $p^{-1}(\Omega)$ on which is induced the morphism $(u, v) \mapsto X_-(v^u)X_0(b(u, v))X_+(u^v)$ by p . In the terminology of [13], the datum $(\mathbf{GL}(Q), \psi)$ is thus an “elementary system” with $(\mathfrak{A}^+, \mathfrak{A}^-)$ the associated Jordan pair, which becomes a “Jordan system” after equipped with the action ρ and the morphism b (to be called the “*Bergmann morphism*”). There is known a theorem of generators and relations [13, 4.14], which we shall apply for construction an embedding λ of $\mathbf{GL}(Q)$.

2.3. Partial description of λ . Provisionally supposing M any finitely generated projective module and considering the hyperbolic module $\mathbf{H}(M)$, we recall the similar description [5] as above for the *special Clifford group* $\mathbf{S}\Gamma(\mathbf{H}(M))$, the normalizer taken in $\mathbf{GL}(\bigwedge^+ M) \times \mathbf{GL}(\bigwedge^- M)$ of the embedded $M \oplus M^* \subset \text{End}(\bigwedge M)$: namely, we have two homomorphisms

$$\begin{aligned} \mathbf{W}(\bigwedge^2 M) &\xrightarrow{Y_+} \mathbf{GL}(\bigwedge M) \xleftarrow{Y_-} \mathbf{W}(\bigwedge^2 M^*) \\ Y_+(A) &:= l_{\exp(A)}, \quad Y_-(B) := d_{\exp(B)}, \end{aligned} \quad (2.3.1)$$

which are in fact factoring through the *spin group*, the kernel of the *norm character* $\mu : \mathbf{S}\Gamma(\mathbf{H}(M)) \rightarrow \mathbf{G}_{\mathbf{m}k}$, and the homomorphism

$$\begin{aligned} Y_0 : \mathbf{G}_{\mathbf{m}k} \times \mathbf{GL}(M) &\longrightarrow \mathbf{GL}(\bigwedge M) \\ Y_0(t, T) &:= t(\det T)^{-1} \bigwedge T, \end{aligned} \quad (2.3.2)$$

which yields the character $(t, T) \mapsto t^2(\det T)^{-1}$ after composed with μ ; Y_0 normalizes Y_\pm with $(t, T) \mapsto (\wedge^2 T, \wedge^2 T^{*-1})$ the induced actions on the Jordan pair $(\wedge^2 M, \wedge^2 M^*)$ and with $(A, B) \mapsto (\delta_M(A, B), 1 + AB)$ the Bergmann morphism. In the case where $M = N \oplus P$ as in (1.2), we have homomorphisms λ_\pm^1 which may well be understood with sources $\mathbf{W}(\mathfrak{Q}^\pm)$ and targets $\mathbf{W}(\wedge^2(M^\pm))$; analogously, we introduce the homomorphism

$$\begin{aligned} \lambda_0 : \mathbf{GL}(N) \times \mathbf{GL}(P) &\longrightarrow \mathbf{G}_{\mathfrak{m}^k} \times \mathbf{GL}(M) \\ \lambda_0(g, h) &:= (\det g, g \oplus h). \end{aligned} \quad (2.3.3)$$

2.4. Theorem. *Let N and P be any finitely generated projective modules and put $M := N \oplus P$, $Q := N \oplus P^*$. There exists one and only one homomorphism*

$$\lambda : \mathbf{GL}(Q) \longrightarrow \mathbf{S}\Gamma(\mathbf{H}(M)) \quad (2.4.1)$$

of group schemes such that $\lambda \circ X_\pm = Y_\pm \circ \lambda_\pm^1$ and $\lambda \circ X_0 = Y_0 \circ \lambda_0$. This is monomorphic, converting the norm μ to the determinant: $\det = \mu \circ \lambda$.

Proof. The uniqueness and two properties monomorphic and $\det = \mu \circ \lambda$ are all evident by the harmless restriction to the dense subscheme $\Omega \subset \mathbf{GL}(Q)$; notice that by [13, 3.8] the homomorphism f stated in [13, 4.14] is monomorphic whenever so are f_0 and f_\pm . Furthermore, since everything has been interpreted to the category of Jordan systems, in order to prove the existence of λ it suffices to verify that *the pair $(\lambda_0, (\lambda_+^1, \lambda_-^1))$ is a homomorphism of Jordan systems*, in other words ([13, 5.1 (6), (7)]) that

$$\lambda_+^1(\rho_+(g, h) \cdot u) = (\wedge^2 T) \cdot A, \quad \lambda_-^1(\rho_-(g, h) \cdot v) = (\wedge^2 T^{*-1}) \cdot B \quad (2.4.2)$$

$$\lambda_0(b(u, v)) = (\delta_M(A, B), 1 + AB) \quad (2.4.3)$$

for any $u, v, (g, h)$ with $A := \lambda_+^1(u)$, $B := \lambda_-^1(v)$, $T := g \oplus h$. Now, from (2.2.1) with computation in matrices of types (1.4.1) and (2.1.2) follows (2.4.2), and similarly $1 + AB = g \oplus h$ once (g, h) is taken to be $b(u, v)$, see (2.2.2); then $\det(g) = \delta_M(A, B)$ also by (1.4), whence (2.4.3), on account of (2.3.3).

Remark. λ seems something like \wedge ; more precisely, letting i denote the isomorphism $\mathbf{S}\Gamma(\mathbf{H}(M)) \xrightarrow{\sim} \mathbf{S}\Gamma(\mathbf{H}(Q))$ induced from the natural isometry $\mathbf{H}(M) \xrightarrow{\sim} \mathbf{H}(Q)$ of switching P and P^* , one might well ask what is the composite $i \circ \lambda$. We conjecture that this is $i \circ \kappa \circ Y_0(\det(\cdot), \cdot)$ with κ an *isomorphism* $\mathbf{S}\Gamma(\mathbf{H}(Q)) \xrightarrow{\sim} \mathbf{S}\Gamma(\mathbf{H}(M))$, namely that there exists an isomorphism κ such that $\lambda(\gamma) = \kappa(\wedge \gamma)$ for all point γ of $\mathbf{GL}(Q)$. In the case where M has rank two or three, we shall give an explicit description of κ (Section three).

2.5. Close-up of $\text{End}(\wedge Q)$. We proceed to give a method to calculate the exterior powers of X_\pm in terms of λ_\pm^1 , to be used in proving the next theorem

(Theorem 2.8). We begin by rewriting (2.1.3) as

$$X_+(u) = 1 + \tilde{u}, \quad X_-(v) = 1 + \tilde{v}, \quad (2.5.1)$$

where $w \mapsto \tilde{w}$ denotes the map from $\mathfrak{V}^\sigma \cong N^\sigma \oplus P^\sigma$ to $\text{End}(Q) \cong Q \otimes Q^*$ such that $(n \otimes p)^\sim = (n \oplus 0) \otimes (0 \oplus p)$ for $\sigma = +$ and $(n^* \otimes p^*)^\sim = -(0 \oplus p^*) \otimes (n^* \oplus 0)$ for $\sigma = -$; here and in the following, elements of Q , Q^* are to be written as $n \oplus p^*$, $n^* \oplus p$ and those of M , M^* as (n, p) , (n^*, p^*) . Introducing the direct sum k -module

$$E := N \oplus P^* \oplus N^* \oplus P, \quad (2.5.2)$$

we have an injection $j : \text{End}(Q) \rightarrow \bigwedge^2 E$ just as in (1.2); furthermore, since M^\pm are direct factors in E , we have induced injections

$$\bigwedge M \xrightarrow{\iota^+} \bigwedge E \xleftarrow{\iota^-} \bigwedge M^* \quad (2.5.3)$$

also; it is then easy to see that, for $w \in \mathfrak{V}^\sigma$, $\sigma = \pm$,

$$j(\tilde{w}) = \iota_\sigma(\lambda_\sigma^1(w)). \quad (2.5.4)$$

On the other hand, since E is the module underlying $\mathbf{H}(Q)$ and since the embedded subspaces $M^\pm \subset E$ are totally singular, both of them generate subalgebras isomorphic to exterior algebras; explicitly, we have injective homomorphisms

$$\bigwedge M \xrightarrow{\varphi^+} \text{End}(\bigwedge Q) \xleftarrow{\varphi^-} \bigwedge M^* \quad (2.5.5)$$

of k -algebras, which are described on the degree-one-parts as

$$\varphi_+(\xi) = l_{n \oplus 0} + d_{0 \oplus p}, \quad \varphi_-(\xi^*) = l_{0 \oplus p^*} + d_{n^* \oplus 0} \quad (2.5.6)$$

for $\xi = (n, p)$ in M and $\xi^* = (n^*, p^*)$ in M^* . As yet another description, we recall [7, 1.6] that there exists a module-isomorphism

$$\varphi : \bigwedge E \xrightarrow{\sim} \text{End}(\bigwedge Q) \quad (2.5.7)$$

characterized by the unital property $\varphi(1) = 1$ and the identity

$$\varphi((n, p^*, n^*, p) \wedge Z) = (l_{n \oplus p^*} + d_{n^* \oplus p}) \circ \varphi(Z) - \varphi(F \lrcorner Z) \quad (2.5.8)$$

for (n, p^*, n^*, p) in E and Z in $\bigwedge E$, with F denotes the linear form $E \rightarrow k$ taking value $\langle n_0, n^* \rangle + \langle p, p_0^* \rangle$ at (n_0, p_0^*, n_0^*, p_0) . Since such an F annihilates the embedded M (M^*) if $n^* = 0$ (respectively, $p = 0$), it follows in particular that φ becomes algebra homomorphisms after restricted to either $\bigwedge M$ or $\bigwedge M^*$; in fact, comparing (2.5.6) with (2.5.8), we find that $\varphi_\pm = \varphi \circ \iota_\pm$.

2.6. Proposition. *For $u \in \mathfrak{V}^+$ and $v \in \mathfrak{V}^-$ one has*

$$\bigwedge X_+(u) = \varphi_+(\exp(\lambda_+^1(u))), \quad \bigwedge X_-(v) = \varphi_-(\exp(\lambda_-^1(v))). \quad (2.6.1)$$

Proof. From $\varphi_{\pm} = \varphi \circ \iota_{\pm}$ follows $\varphi_{\pm} \circ \exp = \text{Exp} \circ \iota_{\pm}$, where Exp denotes the map $\bigwedge^2 E \rightarrow \text{End}(\bigwedge Q)$ introduced in [7, Section 2]. This, combined with (2.5.4), proves $\varphi_{\sigma}(\exp(\lambda_{\sigma}^1(w))) = \text{Exp}(j(\tilde{w}))$ (for w in \mathfrak{B}^{σ} , $\sigma = \pm$), with the second member being $\bigwedge(1 + \tilde{w})$ by [7, 2.2.1]; in view of (2.5.1), we are done.

2.7. The embedding ζ . Let L denote the invertible module $\bigwedge^d P$, with d the rank of P . Besides the natural decomposition of $\bigwedge^d Q$ to the direct sum of $(\bigwedge^{\nu} N) \otimes (\bigwedge^{d-\nu} P^*)$ for $0 \leq \nu \leq d$, there exists an isomorphism

$$\begin{aligned} (\bigwedge^{d-\nu} P^*) \otimes L &\xrightarrow{\sim} \bigwedge^{\nu} P \\ z^* \otimes \omega &\mapsto z^* \lrcorner \omega \end{aligned} \quad (2.7.1)$$

for each ν ; thus, tensoring L (from the right) to $\bigwedge^d Q$ and $\bigwedge^{\nu} N$ (from the left) to (2.7.1), after which the target $(\bigwedge^{\nu} N) \otimes (\bigwedge^{\nu} P)$ becomes embedded into $\bigwedge^{2\nu} M \subset \bigwedge^+ M$ as in (1.2), we obtain a linear map

$$\zeta : (\bigwedge^d Q) \otimes L \longrightarrow \bigwedge^+ M \quad (2.7.2)$$

which is by construction an isomorphism onto a direct factor. Furthermore, on the source $(\bigwedge^d Q) \otimes L$ the k -group $\mathbf{GL}(Q)$ acts by the d -th exterior power of the standard representation tensored with Id_L , while on the target acts $\mathbf{S}\Gamma(\mathbf{H}(M))$ by the even spin representation.

2.8. Theorem. ζ is equivariant relative to the group homomorphism λ .

Proof. This amounts to saying that one has

$$\zeta((\bigwedge^d \gamma) \cdot Z) \otimes \omega = \lambda(\gamma) \cdot \zeta(Z \otimes \omega) \quad (2.8.1)$$

for all points (γ, Z, ω) of the k -scheme $\mathbf{GL}(Q) \times \mathbf{W}(\bigwedge^d Q) \times \mathbf{W}(L)$, which are harmless to be supposed k -points in the proof. Furthermore, since the problem is homomorphic in γ and linear in Z , we may suppose γ to be of those forms

$$X_+(n \otimes p), \quad X_-(n^* \otimes p^*), \quad X_0(g, h) \quad (2.8.2)$$

as in the succeeding explanation of (2.5.1) or in (2.1.4), and Z to be

$$Z = (z_{\nu} \oplus 0) \wedge (0 \oplus z_{d-\nu}^*), \quad (2.8.3)$$

whence, by (2.7.1),

$$\zeta(Z \otimes \omega) = (z_{\nu}, 0) \wedge (0, z_{d-\nu}^* \lrcorner \omega), \quad (2.8.4)$$

with z_{ν} in $\bigwedge^{\nu} N$ and $z_{d-\nu}^*$ in $\bigwedge^{d-\nu} P$, $0 \leq \nu \leq d$; here and in the following, previous symbols like $z \oplus 0$, $(z, 0)$, etc., are to be abused also for *exterior tensors* over Q , M , etc.

In order to prove (2.8.1) for the first two cases in (2.8.2), we rewrite $\gamma = X_\sigma(w)$ as in (2.5.4) for short and notice that $\exp(\lambda_\sigma^1(w)) = 1 + \lambda_\sigma^1(w)$; thus, $\bigwedge \gamma$ equals $\varphi_\sigma(1 + \lambda_\sigma^1(w))$ by (2.6.1), while since $\lambda_\sigma^1(w)$ is now either $(n, 0) \wedge (0, p)$ or $(n^*, 0) \wedge (0, p^*)$, combining (2.5.6) with $\lambda \circ X_\sigma = Y_\sigma \circ \lambda_\sigma^1$ we find the couple $(\bigwedge \gamma, \lambda(\gamma))$ to be either $(1 + l_{n \oplus 0} \circ d_{0 \oplus p}, 1 + l_{(n,0) \wedge (0,p)})$ or $(1 + d_{n^* \oplus 0} \circ l_{0 \oplus p^*}, 1 + d_{(n^*,0) \wedge (0,p^*)})$; so the problem (2.8.1) now takes the forms

$$\zeta(Z' \otimes \omega) = (n, 0) \wedge (0, p) \wedge \zeta(Z \otimes \omega), \quad (2.8.5)$$

$$\zeta(Z'' \otimes \omega) = (n^*, 0) \lrcorner ((0, p^*) \lrcorner \zeta(Z \otimes \omega)), \quad (2.8.6)$$

where $Z' := (n \oplus 0) \wedge ((0 \oplus p) \lrcorner Z)$ and $Z'' := (n^* \oplus 0) \lrcorner ((0 \oplus p^*) \wedge Z)$. To prove (2.8.5), we infer $Z' = (-1)^\nu ((n \wedge z_\nu) \oplus 0) \wedge (0 \oplus (p \lrcorner z_{d-\nu}^*))$ from (2.8.3), so that

$$\zeta(Z' \otimes \omega) = (-1)^\nu (n \wedge z_\nu, 0) \wedge (0, (p \lrcorner z_{d-\nu}^*) \lrcorner \omega) \quad (2.8.7)$$

by (2.7.1); however $(p \lrcorner z_{d-\nu}^*) \lrcorner \omega =: z_{\nu+1}^{(1)}$ equals $p \wedge (z_{d-\nu}^* \lrcorner \omega) =: z_{\nu+1}^{(2)}$, since for any $(\nu+1)$ -tensor $z_{\nu+1}^*$ over M^* the obvious nullity of $p \lrcorner (z_{\nu+1}^* \wedge z_{d-\nu}^*)$ equates $(p \lrcorner z_{\nu+1}^*) \wedge z_{d-\nu}^*$ to $(-1)^\nu z_{\nu+1}^* \wedge (p \lrcorner z_{d-\nu}^*)$, which acted by the pairing with ω proves $z_{\nu+1}^{(1)}$ and $z_{\nu+1}^{(2)} = (-1)^\nu (z_{d-\nu}^* \lrcorner \omega) \wedge p$ to be indeed taking the *same* value at $z_{\nu+1}^*$. Therefore, from (2.8.7) the following calculation proceeds, proving (2.8.5), on account of (2.8.4):

$$\begin{aligned} \zeta(Z' \otimes \omega) &= (-1)^\nu (n \wedge z_\nu, 0) \wedge (0, p \wedge (z_{d-\nu}^* \lrcorner \omega)) \\ &= (-1)^\nu (n, 0) \wedge (z_\nu, 0) \wedge (0, p) \wedge (0, z_{d-\nu}^* \lrcorner \omega) \\ &= (n, 0) \wedge (0, p) \wedge (z_\nu, 0) \wedge (0, z_{d-\nu}^* \lrcorner \omega). \end{aligned}$$

As to (2.8.6), for which we have $Z'' = (-1)^\nu ((n^* \lrcorner z_\nu) \oplus 0) \wedge (0 \oplus (p^* \wedge z_{d-\nu}^*))$ by (2.8.3) and moreover $(0, p^*) \lrcorner \zeta(Z \otimes \omega) = (-1)^\nu (z_\nu, 0) \wedge (0, p^* \lrcorner (z_{d-\nu}^* \lrcorner \omega))$ by (2.8.4), it is actually easier to have both members equated, which are $(-1)^\nu (n^* \lrcorner z_\nu, 0) \wedge (0, p^* \lrcorner (z_{d-\nu}^* \lrcorner \omega))$.

For the last case in (2.8.2), from (2.1.4) we calculate the left-hand side of (2.8.1) as the product $((\bigwedge^\nu g) \cdot z_\nu, 0) \wedge (0, w_\nu) =: \Pi$, where

$$w_\nu := \left((\bigwedge^{d-\nu} (h^{*-1}) \cdot z_{d-\nu}^*) \lrcorner \omega = (\det h)^{-1} (\bigwedge^\nu h) \cdot (z_{d-\nu}^* \lrcorner \omega) \in \bigwedge^\nu P; \right.$$

thus, on account of (2.8.4), for $\Gamma := (\det h)^{-1} \bigwedge (g \oplus h) \in \text{GL}(\bigwedge M)$ one has $\Pi = \Gamma \cdot \zeta(Z \otimes \omega)$; however, (2.3.3) combined with $\lambda \circ X_0 = Y_0 \circ \lambda_0$ shows Γ to be $Y_0(\lambda_0(g, h)) = \lambda(\gamma)$. This completes our proof.

3 Examples

3.1. The case of two invertible modules. Consider the case where both N and P are of *rank one*. We have $\bigwedge^- M = M$, and since λ_+^1 identifies $\mathfrak{V}^+ = \text{Hom}(P^*, N)$ with $\bigwedge^2 M$, there arises an identification

$$\bigwedge^+ M \cong k \oplus \mathfrak{V}^+. \quad (3.1.1)$$

Settling $\mathfrak{V}^- = \text{Hom}(P, N^*)$ dual to \mathfrak{V}^+ under the pairing $\{u, v\}$ which either $uv^* \in \text{End}(N) \cong k$ or $u^*v \in \text{End}(P) \cong k$ defines, we shall employ for elements of $\text{End}(\bigwedge^+ M)$ matrix notation of type

$$\begin{bmatrix} k & \mathfrak{V}^- \\ \mathfrak{V}^+ & k \end{bmatrix}$$

acting from the left; similar matrix notation being understood for all scalar extensions too. Then $\langle \lambda_+^1(u), \lambda_-^1(v) \rangle = -\{u, v\}$ by (1.4.3), so that on $\bigwedge^+ M$ the action $Y_-(\lambda_-^1(v))$ is $[\alpha_0, u_0] \mapsto [\alpha_0 - \{u_0, v\}, u_0]$; apparently $Y_+(\lambda_+^1(u))$ is $[\alpha_0, u_0] \mapsto [\alpha_0, u_0 + \alpha_0 u]$; and since a point (g, h) of $\mathbf{GL}(N) \times \mathbf{GL}(P)$ is now a pair of invertible scalars, we have $\lambda_0(g, h) = (t, T)$ with $t := g$, $T := \text{diag}(g, h)$, whence $Y_0(\lambda_0(g, h))$ is $[\alpha_0, u_0] \mapsto [h^{-1}\alpha_0, gu_0]$. These compared with (2.1.3), (2.1.4) soon prove, as having been so on the dense subscheme Ω , that the composite $\rho_+ \circ \lambda$ with the even spin representation $\rho_+ : \mathbf{S}\Gamma(\mathbf{H}(M)) \rightarrow \mathbf{GL}(\bigwedge^+ M)$ is the isomorphism

$$\gamma = \begin{pmatrix} \alpha & u \\ v^* & \beta \end{pmatrix} \mapsto \hat{\gamma} := \begin{bmatrix} \beta & v \\ u & \alpha \end{bmatrix} : \mathbf{GL}(Q) \xrightarrow{\sim} \mathbf{GL}(\bigwedge^+ M). \quad (3.1.2)$$

As for the composite $\rho_- \circ \lambda$, both $Y_+(\lambda_+^1(u))$ and $Y_-(\lambda_-^1(v))$ are trivial and $Y_0(\lambda_0(g, h))$ acts as $\text{diag}(h^{-1}g, 1)$; since $h^{-1}g = \det(X_0(g, h))$, the result is $\gamma \mapsto \text{diag}(\det(\gamma), 1)$. Therefore we arrive at

$$\lambda(\gamma) = (\hat{\gamma}, \text{diag}(\det \gamma, 1)), \quad (3.1.3)$$

matching with the well-known fact that $\mathbf{S}\Gamma(\mathbf{H}(M))$ equals the fiber product of $\mathbf{GL}(\bigwedge^\pm M)$ relative to the determinants. In fact, the construction (3.1.2) may well be adapted by changing roles of Q , M , and this enables us to have an isomorphism $\mathbf{S}\Gamma(\mathbf{H}(Q)) \xrightarrow{\sim} \mathbf{S}\Gamma(\mathbf{H}(M))$ induced from

$$\begin{aligned} \kappa : \mathbf{GL}(\bigwedge^+ Q) \times \mathbf{GL}(\bigwedge^- Q) &\xrightarrow{\sim} \mathbf{GL}(\bigwedge^+ M) \times \mathbf{GL}(\bigwedge^- M) \\ \kappa(\hat{T}, \gamma) &:= (\hat{\gamma}, T). \end{aligned} \quad (3.1.4)$$

A point is that $\lambda(\gamma) = \kappa(\bigwedge(\gamma))$, since $\bigwedge(\gamma) = (\text{diag}(1, \det \gamma), \gamma)$ in the situation of (3.1.4). Incidentally, since $d = 1$, $L = P$ in the notation of (2.7), the embedding ζ is now the natural isomorphism identifying $\bigwedge^d Q \otimes L = (N \oplus$

$P^*) \otimes P$ with $\bigwedge^+ M$, see (3.1.1), and it is easy to work out a direct verification for the fact (2.8) that ζ is equivariant relative to the group isomorphism (3.1.2).

3.2. An embedding ι . Suppose N to be of rank two and P invertible, so that M is of rank three. In this case, $\mathbf{S}\Gamma(\mathbf{H}(M))$ is faithfully represented in $\mathbf{GL}(\bigwedge^- M) \times \mathbf{G}_{\mathbf{m}k}$ by the morphism with components (ρ_-, μ) , as the closed subgroup scheme of those points (S, t) such that $\det S = t^2$ [8, Theorem 1.5]. We know $\mu \circ \lambda = \det$, so $\rho_- \circ \lambda$ remains to be made explicit which should be something quadratic. Let us consider the closed embedding

$$\iota : \mathbf{GL}(\bigwedge^2 Q) \longrightarrow \mathbf{GL}(\bigwedge^- M) \quad (3.2.1)$$

constructed in the following way: identify $\bigwedge^2 Q$ with $(\bigwedge^2 N) \oplus (N \otimes P^*)$, to which tensoring P yields an isomorphism

$$(\bigwedge^2 Q) \otimes P \xrightarrow{\sim} (L \otimes P) \oplus N \quad (\text{where } L := \bigwedge^2 N); \quad (3.2.2)$$

its target, say R , occupies partial factors of $(\bigwedge^3 M) \oplus M$, so that $\mathbf{GL}(R)$ is in natural way a closed subgroup of $\mathbf{GL}(\bigwedge^- M)$; thus, (3.2.2) acted by \mathbf{GL} takes the wanted form (3.2.1). We notice by passing that only the invertibility of P has been used so far.

3.3. Proposition. *If N is of rank two and P invertible, then $\rho_- \circ \lambda$ factors through ι by the natural homomorphism $\bigwedge^2 : \mathbf{GL}(Q) \rightarrow \mathbf{GL}(\bigwedge^2 Q)$ of taking second exterior powers.*

This, $\rho_- \circ \lambda = \iota \circ \bigwedge^2$, matches with the earlier anticipation. To the proof harmless restriction to k -points of Ω applies again, which combined with Theorem 2.4 reduces the problem to verifying

$$\rho_- (Y_+(\lambda_+^1(u))) = \iota (\bigwedge^2 X_+(u)), \quad (3.3.1)$$

$$\rho_- (Y_-(\lambda_-^1(v))) = \iota (\bigwedge^2 X_-(v)), \quad (3.3.2)$$

$$\rho_- (Y_0(\det(g), g \oplus \eta)) = \iota (\bigwedge^2 X_0(g, \eta)) \quad (3.3.3)$$

for $u \in \mathfrak{V}^+$, $v \in \mathfrak{V}^-$, $g \in \mathbf{GL}(N)$, and $\eta \in k^* \cong \mathbf{GL}(P)$. Since $\bigwedge^2 Q$ decomposes to $L \oplus (N \otimes P^*)$ with L invertible, elements of $\text{End}(\bigwedge^2 Q)$ are expressed as matrices of type

$$\left[\begin{array}{cc} k & \text{Hom}(N \otimes P^*, L) \\ \text{Hom}(L, N \otimes P^*) & k \end{array} \right] \quad (3.3.4)$$

acting from the left, while those of $\text{End}(\bigwedge^- M)$ are of type

$$\left(\begin{array}{ccc} \text{End}(N) & \text{Hom}(P, N) & \text{Hom}(L \otimes P, N) \\ \text{Hom}(N, P) & k & \text{Hom}(L \otimes P, P) \\ \text{Hom}(N, L \otimes P) & \text{Hom}(P, L \otimes P) & k \end{array} \right) \quad (3.3.5)$$

acting on $\bigwedge^- M = N \oplus P \oplus (L \otimes P)$ from the left; and we notice that, the operation tensoring Id_P gives rise to isomorphisms

$$\text{Hom}(N \otimes P^*, L) \xrightarrow{\sim} \text{Hom}(N, L \otimes P), \quad (3.3.6)$$

$$\text{Hom}(L, N \otimes P^*) \xrightarrow{\sim} \text{Hom}(L \otimes P, N), \quad (3.3.7)$$

as well as the obvious $\text{End}(N \otimes P^*) \xrightarrow{\sim} \text{End}(N)$, after these of which being treated as identifications ι becomes mere insertion of the array (3.3.4) into the external four corners of (3.3.5). Therefore, verification of an identity $\rho_-(\lambda(\gamma)) = \iota(\bigwedge^2 \gamma)$ consists of the following at most three steps: describe $\bigwedge^2 \gamma$ as the matrix of type (3.3.4); similarly for $\rho_-(\lambda(\gamma))$ of type (3.3.5); and chase entries along the isomorphisms (3.3.6), (3.3.7), if necessary. The first step is carried out under the identification $\bigwedge^2 Q \subset \text{Hom}(Q^*, Q)$, describing the components $a \in L$, $B \in N \otimes P^* \cong \text{Hom}(P, N)$ of an arbitrary element $\mathfrak{p} \in \bigwedge^2 Q$ in the form of two-by-two alternating matrix with entries a , B , $-B^*$, 0 , on which the action of $\bigwedge^2 \gamma$ is $\mathfrak{p} \mapsto \gamma \mathfrak{p} \gamma^*$ [5, (1.4.6)]; on account of (2.1.3), (2.1.4), we get

$$\bigwedge^2 X_+(u) = \begin{bmatrix} 1 & (B \mapsto Bu^* - uB^*) \\ 0 & 1 \end{bmatrix}, \quad (3.3.8)$$

$$\bigwedge^2 X_-(u) = \begin{bmatrix} 1 & 0 \\ (a \mapsto -av) & 1 \end{bmatrix}, \quad (3.3.9)$$

$$\bigwedge^2 X_0(g, \eta) = \begin{bmatrix} \det g & 0 \\ 0 & g \otimes \eta^{-1} \end{bmatrix}. \quad (3.3.10)$$

In fact, once relations $t(\det T)^{-1} = \eta^{-1}$ and $\bigwedge^- T = \text{diag}(g, \eta, \eta \det g)$ have been noticed for $(t, T) := (\det g, g \oplus \eta)$, it is also apparent from (2.3.2) that

$$\rho_-(Y_0(\det g, g \oplus \eta)) = \begin{pmatrix} \eta^{-1}g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det g \end{pmatrix}; \quad (3.3.11)$$

thus, dispensing with the third step, (3.3.10) compared to (3.3.11) proves (3.3.3). Verification of (3.3.1), (3.3.2) is less apparent, which needs some auxiliary definitions and remarks.

3.4. The “hat” isomorphisms. Provisionally we let N, P be arbitrary. As has already appeared in in (3.3.8), there exists a bilinear composition coupling $B \in N \otimes P^* \cong \text{Hom}(P, N)$ and $u \in \mathfrak{V}^+ = \text{Hom}(P^*, N)$ to the following tensor which is apparently an alternating map:

$$[B, u] := Bu^* - uB^* \in \bigwedge^2 N \subset \text{Hom}(N^*, N). \quad (3.4.1)$$

This amounts to $[B, n \otimes p] = B(p) \wedge n$, because $(Bu^* - uB^*)(f)$ equals $\langle n, f \rangle B(p) - \langle B(p), f \rangle n = -f \lrcorner (B(p) \wedge n)$ for $u = n \otimes p$ and $f \in N^*$. Furthermore, we have linear maps

$$\begin{aligned} u \mapsto \hat{u} : \mathfrak{Y}^+ = \text{Hom}(P^*, N) &\longrightarrow \text{Hom}(N, \wedge^2(N) \otimes P) \\ \hat{u}(x) &:= (x \wedge n) \otimes p \quad \text{for } u = n \otimes p, \end{aligned} \quad (3.4.2)$$

$$\begin{aligned} v \mapsto \hat{v} : \mathfrak{Y}^- = \text{Hom}(P, N^*) &\longrightarrow \text{Hom}(\wedge^2(N) \otimes P, N) \\ \hat{v}(a \otimes p) &:= v(p) \lrcorner a, \end{aligned} \quad (3.4.3)$$

both identified naturally with tensorizations $r \otimes \text{Id}_P$, $d \otimes \text{Id}_{P^*}$, where r denotes the map $N \rightarrow \text{Hom}(N, \wedge^2(N))$ constructing right-wedge products, namely $r(n)(x) := x \wedge n$, and d the map $N^* \rightarrow \text{Hom}(\wedge^2(N), N)$ constructing left-interior products. In particular, both (3.4.2) and (3.4.3) are *isomorphisms* if N is of *rank two*. If P is *invertible*, on the other hand, we claim that (3.4.1) relates to (3.4.2) under the following relation (where $\omega \in P$ is arbitrary):

$$[B, u] \otimes \omega = \hat{u}(B(\omega)). \quad (3.4.4)$$

Indeed, supposing $u = n \otimes p$ harmlessly yields $[B, u] \otimes \omega = (B(p) \wedge n) \otimes \omega$ by the remark after (3.4.1), while $\hat{u}(B(\omega)) = (B(\omega) \wedge n) \otimes p$ by (3.4.2); thus, the assertion is a consequence of the fact that every bilinear map defined on an invertible module is symmetric. The relation (3.4.4) may be read as an identity $(B \mapsto Bu^* - uB^*) = \hat{u}$, for which the isomorphism (3.3.6) makes sense; furthermore, so does the isomorphism (3.3.7) for $(a \mapsto -av) = \hat{v}$, since (3.4.3) reads $-a(v(\omega)) = \hat{v}(a \otimes \omega)$.

3.5. Verification of (3.3.1), (3.3.2). Returning to the situation of Proposition 3.3, we shall complete the proof. By the remark made above, it suffices to prove

$$\rho_-(Y_+(\lambda_+^1(u))) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hat{u} & 0 & 1 \end{pmatrix}, \quad \rho_-(Y_-(\lambda_-^1(v))) = \begin{pmatrix} 1 & 0 & \hat{v} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By linearity we may suppose again $u = n \otimes p$, so that $\lambda_+^1(u) = (n, 0) \wedge (0, p)$, which acted upon $M = N \oplus P$ by the wedge product annihilates P and sends $(x, 0)$ (where $x \in N$) to the element of $\wedge^3 M \cong L \otimes P$ identified with $(x \wedge n) \otimes p = \hat{u}(x)$, see (3.4.2); since $Y_+(\lambda_+^1(u)) = \text{Id} + l_{(n,0) \wedge (0,p)}$, this proves the first formula. As to the second, v may be supposed to be $n^* \otimes p^* \in N^* \otimes P^*$, in which case the interior product by $\lambda_-^1(v) = (n^*, 0) \wedge (0, p^*)$ converts an element $a \otimes \omega \in \wedge^3 M \cong L \otimes P$ to $\langle \omega, p^* \rangle n^* \lrcorner a = v(\omega) \lrcorner a = \hat{v}(a \otimes \omega)$, see (3.4.3); since $Y_-(\lambda_-^1(v)) = \text{Id} + d_{(n^*,0) \wedge (0,p^*)}$, this completes our proof.

3.6. The isomorphism κ . Since N is of rank two with $L = \bigwedge^2 N$, we have an isomorphism

$$\begin{aligned} \theta : N^* \otimes L &\xrightarrow{\sim} N \\ \theta(f \otimes a) &:= -f \lrcorner a \end{aligned} \tag{3.6.1}$$

given by the interior product, in which the sign minus is employed so that the transposed inverse $\theta^{*-1} : N \otimes L^* \rightarrow N^*$ be the ordinary $x \otimes b \mapsto x \lrcorner b$; the identification $L^* \cong \bigwedge^2 N^*$ being made as in (1.1). The isomorphism θ , together with obvious isomorphisms $P^* \otimes L \xrightarrow{\sim} \bigwedge^3 Q$ and $(L \otimes P)^* \otimes L \xrightarrow{\sim} P^*$, gives rise to an isomorphism

$$(\bigwedge^- M)^* \otimes L \xrightarrow{\sim} \bigwedge^- Q \tag{3.6.2}$$

which we shall use to regard the operation $T \mapsto T^{*-1}$ as an isomorphism $\mathbf{GL}(\bigwedge^- Q) \xrightarrow{\sim} \mathbf{GL}(\bigwedge^- M)$ of group schemes. On account of our faithful representations of rank three special Clifford groups, it is then natural to construct the isomorphism

$$\begin{aligned} \kappa : \mathbf{S}\Gamma(\mathbf{H}(Q)) &\xrightarrow{\sim} \mathbf{S}\Gamma(\mathbf{H}(M)) \\ \kappa(T, t) &:= (tT^{*-1}, t). \end{aligned} \tag{3.6.3}$$

3.7. Proposition. *One has $\lambda(\gamma) = \kappa(\bigwedge \gamma)$ for any point γ of $\mathbf{GL}(Q)$.*

Proof. Since we know both $\lambda(\gamma)$ and $\bigwedge \gamma$ have the same norm $\det \gamma$, it remains to check the identity $\rho_-(\lambda(\gamma)) = (\det \gamma) (\bigwedge^- \gamma)^{*-1}$ in $\mathbf{GL}(\bigwedge^- M)$. Similarly to the proof of Proposition 3.3, we may restrict ourselves to three cases $\gamma = X_0(g, \eta)$, $X_+(u)$, $X_-(v)$ of k -points, for each of which the left-hand side $\rho_-(\lambda(\gamma))$ has been made explicit. For the case $\gamma = X_0(g, \eta)$, transporting structures through θ makes g act on $N^* \otimes L$ as $g^{*-1} \otimes (\det g)$, whence the transposed inverse $(\bigwedge^- \gamma)^{*-1}$ multiplied by $\det \gamma = (\det g)\eta^{-1}$ takes the wanted form (3.3.11). Remaining two cases, in both of which $\det \gamma$ is 1, involve the fact $\theta^{*-1}(x \otimes b) = x \lrcorner b$ we have mentioned after (3.6.1): namely, through θ^{*-1} and through the obvious identification $(L \otimes P) \otimes L^* \cong P$, we have $-u^* = \hat{u} \otimes 1$ and $v = \hat{v} \otimes 1$ easily verified along definitions, which shows that the operation $T \mapsto T^{*-1}$ converts $\bigwedge^- X_+(u)$ and $\bigwedge^- X_-(v)$ to those matrices as displayed in (3.5). This completes our proof.

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