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A new approximation of the Height process of a CSBP

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RÉSUMÉ. Nous codons les arbres de Galton-Walton par un processus de hauteur continu, afin de donner un sens précis à la convergence des forêts d'arbres. Ceci nous permet d'établir la convergence de la forêt d'arbres généalogiques du processus de branchement d'une grande population vers les arbres généalogiques d'un processus de branchement à espace d'état continu (PBEC) limite. L'approximation considérée ici est nouvelle, par rapport à celle qui a été étudiée dans [5].

ABSTRACT. We code Galton-Walton trees by a continuous height process, in order to give a precise meaning to the convergence of forests of trees. This allows us to establish the convergence of the forest of genealogical trees of the branching process of a large population towards the genealogical trees of the limiting continuous state branching process (CSBP). The approximation considered here is new, compared to that which has been studied in [5].

MOTS-CLÉS : Processus de Branchement à Espace d'État Continu; processus de Galton-Watson, processus de Lévy; processus de hauteur

KEYWORDS : Continuous-State Branching Processes; Galton-Watson Processes; Lévy Processes; Height Process



1. Introduction

Continuous state branching processes (or CSBP in short) are the analogues of Galton-Watson (G-W) processes in continuous time and continuous state space. Such classes of processes have been introduced by Jirina [9] and studied by many authors included Grey [8], Lamperti [10], to name but a few. These processes are the only possible weak limits that can be obtained from sequences of rescaled G-W processes, see Lamperti [11].

While rescaled discrete-time G-W processes converge to a CSBP, it has been shown in Duquesne and Le Gall [6] that the genealogical structure of the G-W processes converges too. More precisely, the corresponding rescaled sequences of discrete height process, converges to the height process in continuous time that has been introduced by Le Gall and Le Jan in [12]. For the approximation by continuous time generalized G-W processes we refer to our recent paper [4].

Some work has been also devoted recently to the description of the genealogy of generalized CSBPs, see Dramé and Pardoux [5] and Dramé et al. in [3] for the case of continuous such processes and Li, Pardoux and Wakolbinger [13] for the general case. In [5] Dramé and Pardoux give an approximation of the Height process of a continuous state branching process in terms of a stochastic integral equation with jumps, which is well suited for the case of generalized CSBPs. The present paper studies a new approximation of the genealogy of a continuous time GW process to that of a generalized possibly discontinuous CSBP, under the same assumptions as [5]. Note that the two approximations have the same limit.

We start by looking at the convergence of the renormalized population process, which is both a necessary condition for the convergence of the genealogy, and a first step in the proof of that convergence. Note that the genealogical forest of trees contains more information than the population process, which is why we want to prove the convergence of the genealogy. Figure 1 shows a trajectory of a continuous time population branching process, and two distinct compatible genealogical trees.

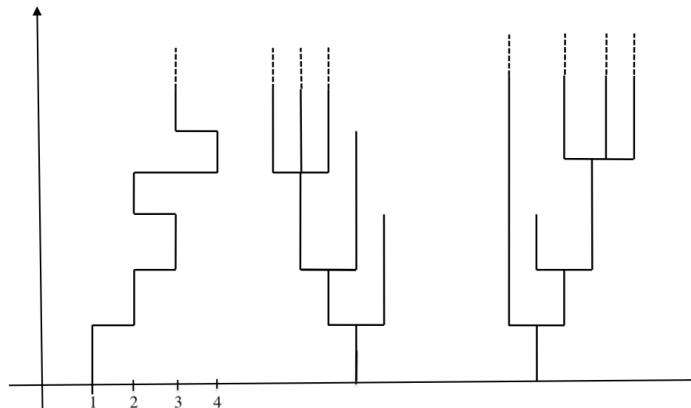


Figure 1. Population process with two distinct compatible genealogical trees.

We have reproduced in Figure 2 a picture from [3], which shows the height process associated to a particular tree. Note that Theorem 3.3 in [3] establishes a correspondence between the law of the height process and the law of the associated genealogical tree, which will be implicitly exploited below.

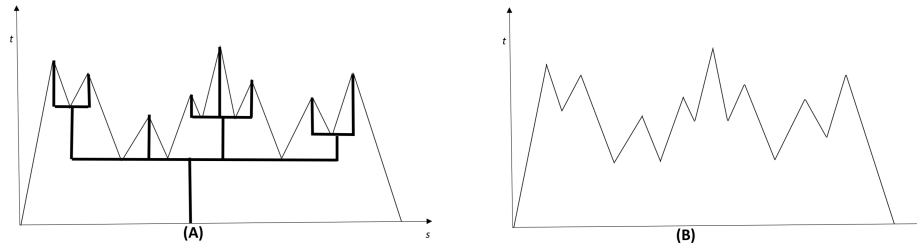


Figure 2. (A) The tree and its associated height process. (B) The height process. The t -axis is real time as well as exploration height, the s -axis is exploration time.

However, it's impossible to draw the genealogical tree of a CSBP, but we can study the associated height process, see [13].

The organization of the paper is as follows : In Section 2 we recall some basic definitions and notions concerning branching processes. Section 3 is devoted to the description of the discrete approximation of both the population process and the height process of its genealogical forest of trees. We prove the convergence of the height process. Finally, for the convenience of the reader, we collect in an Appendix, at the end of this paper, detailed proofs of some propositions. We shall assume that all random variables in the paper are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We shall use the following notations $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. For $x \in \mathbb{R}_+$, $[x]$ denotes the integer part of x .

2. The Height process of a continuous state branching process

2.1. Continuous state branching process

A CSBP is a \mathbb{R}_+ -valued strong Markov process with the property that \mathbb{P}_x denoting the law of the process when starts from x at time $t = 0$, $\mathbb{P}_{x+y} = \mathbb{P}_x * \mathbb{P}_y$. More precisely, a CSBP $X^x = (X_t^x, t \geq 0)$ (with initial condition $X_0^x = x$) is a Markov process taking values in $[0, \infty]$, where 0 and ∞ are two absorbing states, and satisfying the branching property ; that is to say, its Laplace transform satisfies $\mathbb{E}[\exp(-\lambda X_t^x)] = \exp\{-xu_t(\lambda)\}$, for $\lambda \geq 0$, for some non negative function $u_t(\lambda)$. According to Silverstein [14], the function u_t is the unique nonnegative solution of the integral equation : $u_t(\lambda) = \lambda - \int_0^t \psi(u_r(\lambda)) dr$, where ψ is called the branching mechanism associated with X^x and is defined by

$$\psi(\lambda) = b\lambda + c\lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z \leq 1\}}) \mu(dz),$$

with $b \in \mathbb{R}$, $c \geq 0$ and μ is a σ -finite measure which satisfies $\int_0^\infty (1 \wedge z^2)\mu(dz) < \infty$. We shall in fact assume in this paper that

$$(\mathbf{H}) : \int_0^\infty (z \wedge z^2)\mu(dz) < \infty \quad \text{and} \quad c > 0.$$

The first assumption implies in particular that the process X^x does not explode and it allows us to write the last integral in the above equation in the following form

$$\psi(\lambda) = b\lambda + c\lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z)\mu(dz). \quad (1)$$

Let us recall that b represents a drift term, c is a diffusion coefficient and μ describes the jumps of the CSBP. The CSBP is then characterized by the triplet (b, c, μ) and can also be defined as the unique non-negative strong solution of a stochastic differential equation. More precisely, from Fu and Li [7] (see also the results in Dawson-Li [2]), we have

$$X_t^x = x - b \int_0^t X_s^x ds + \sqrt{2c} \int_0^t \int_0^{X_s^x} W(ds, du) + \int_0^t \int_0^\infty \int_0^{X_s^x} z \bar{M}(ds, dz, du), \quad (2)$$

where $W(ds, du)$ is a space-time white noise on $(0, \infty)^2$, $M(ds, dz, du)$ is a Poisson random measure on $(0, \infty)^3$, with intensity $ds\mu(dz)du$, and \bar{M} is the compensated measure of M .

2.2. The height process

We shall also interpret below the function ψ defined by (1) as the Laplace exponent of a spectrally positive Lévy process Y . Lamperti [10] observed that CSBPs are connected to Lévy processes with no negative jumps by a simple time-change. More precisely, define

$$A_s^x = \int_0^s X_t^x dt, \quad \tau_s = \inf\{t > 0, A_t^x > s\} \quad \text{and} \quad Y(s) = X_{\tau_s}^x.$$

Then, until its first hitting time of 0, $Y(s)$ is a Lévy process of the form

$$Y(s) = -bs + \sqrt{2c}B(s) + \int_0^s \int_0^\infty z \bar{\Pi}(dr, dz), \quad (3)$$

where B is a standard Brownian motion and $\bar{\Pi}(ds, dz) = \Pi(ds, dz) - ds\mu(dz)$, Π being a Poisson random measure on \mathbb{R}_+^2 independent of B with mean measure $ds\mu(dz)$. We refer the reader to [10] for a proof of that result. To code the genealogy of the CSBP, Le Gall and Le Jan [12] introduced the so-called height process, which is a functional of a Lévy process with Laplace exponent ψ ; see also Duquesne and Le Gall [6]. In this paper, we will use the new definition of the height process H given by Li et al in [13]. Indeed, if the Lévy process Y has the form (3), then the associated height process is given by

$$cH(s) = Y(s) - \inf_{0 \leq r \leq s} Y(r) - \int_0^s \int_0^\infty \left(z + \inf_{r \leq u \leq s} Y(u) - Y(r) \right)^+ \Pi(dr, dz), \quad (4)$$

and it has a continuous modification. Note that the height process H is the one defined in Chapter 1 of [6]. i.e $cH(s) = |\{\bar{Y}^s(r); 0 \leq r \leq s\}|$, where $\bar{Y}^s(r) := \inf_{r \leq u \leq s} Y(u)$ and $|A|$ denotes the Lebesgue measure of the set A . A graphical interpretation of (4) is as shown on Figure 3. Suppose that Y has a unique jump of size z at time s , and let $s' := \inf\{r > s, Y_r = Y_{s-}\}$. On the interval $[s, s']$, H_r equals $Y_r - z$, reflected above $Y_{s-} = Y_s - z$, while for $r \notin [s, s']$, $H_r = Y_r$.

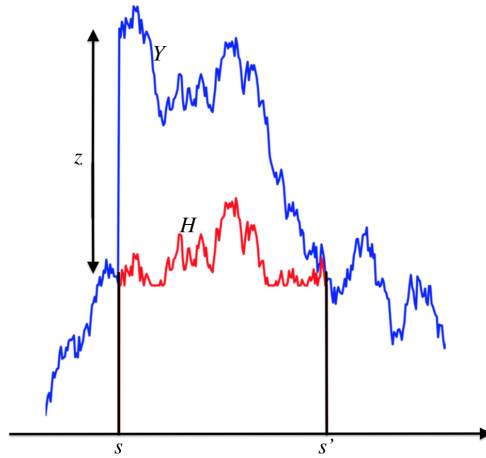


Figure 3. Trajectories of Y and H

3. Approximation of the Height process

In the following, we consider a specific forest of Bellman-Harris trees, obtained by Poissonian sampling of the height process H . In other words, let $\alpha > 0$ and we consider a standard Poisson process with intensity α . We denote by $\tau_1^\alpha \leq \tau_2^\alpha \leq \dots$ the jump times of this Poisson process. If H is seen as the contour process of a continuous tree, consider the forest of the smaller trees carried by the vector $H(\tau_1^\alpha), H(\tau_2^\alpha), \dots$. We have

Proposition 3.1 (Theorem 3.2.1 in [6]) *The trees in this forest are trees which are distributed as the family tree of a continuous-time Galton-Watson process starting with one individual at time 0 and such that :*

- * Lifetimes of individuals have exponential distributions with parameter $\psi'(\psi^{-1}(\alpha))$;
- * The offspring distribution is the law of the variable η with generating function :

$$h(s) = \mathbb{E}(s^\eta) = s + \frac{\psi((1-s)\psi^{-1}(\alpha))}{\psi^{-1}(\alpha)\psi'(\psi^{-1}(\alpha))}.$$

Duquesne and Le Gall obtain an embedding of (the height processes of) a family of continuous-time Galton Watson processes by a Poissonian sampling from the arguments of H (corresponding to the leaves of the forest).

Now we consider a population evolving in continuous time with m ancestors at time $t = 0$, in which each individual lives for an exponential time with parameter $\psi'(\psi^{-1}(\alpha))$, and is replaced by η a random number of children according to the probability generating function h .

We will first renormalize this model, then we will present the results of convergence of the population process, and finally we will prove the convergence of the height process of its genealogical tree.

Let $N \geq 1$ be an integer which will eventually go to infinity. In the next two sections, we choose a sequence $\delta_N \downarrow 0$ such that, as $N \rightarrow \infty$,

$$\frac{1}{N} \int_{\delta_N}^{+\infty} \mu(dz) \rightarrow 0. \tag{5}$$

Because of assumption **(H)** this implies in particular that

$$\frac{1}{N} \int_{\delta_N}^{+\infty} z\mu(dz) \rightarrow 0.$$

Moreover, we will need to consider the truncated branching mechanism

$$\psi_{\delta_N}(\lambda) = c\lambda^2 + \int_{\delta_N}^{\infty} (e^{-\lambda z} - 1 + \lambda z)\mu(dz). \tag{6}$$

To get a reasonable approximation, we will now set $\alpha = \psi_{\delta_N}(N)$ in the limit of large populations, where α was defined in above.

3.1. A discrete mass approximation

In this subsection, we obtain a CSBP as a scaling limit of continuous time Galton–Watson branching processes. In other words, the aim of this subsection is to set up a "discrete mass - continuous time" approximation of (2). To this end, we set

$$h_N(s) = s + \frac{\psi_{\delta_N}((1-s)N)}{N\psi'_{\delta_N}(N)}, \quad |s| \leq 1.$$

It is easy to see that $s \rightarrow h_N(s)$ is an analytic function in $(-1, 1)$ satisfying $h_N(1) = 1$ and

$$\frac{d^n}{ds^n} h_N(0) \geq 0, \quad n \geq 0.$$

Therefore h_N is a probability generating function. and we have

$$h_N(s) = \sum_{\ell \geq 0} v_N(\ell) s^\ell, \quad |s| \leq 1,$$

where v_N is probability measure on \mathbb{Z}_+ . Given an arbitrary $x > 0$, the approximation of (2) will be given by the total mass $X^{N,x}$ of a population of individuals, each of which has mass $1/N$. The initial mass is $X_0^{N,x} = [Nx]/N$, and $X^{N,x}$ follows a Markovian jump dynamics : from its current state k/N ,

$$X^{N,x} \text{ jumps to } \begin{cases} \frac{k+\ell-1}{N} & \text{at rate } \psi'_{\delta_N}(N)v_N(\ell)k, \text{ for all } \ell \geq 2; \\ \frac{k-1}{N} & \text{at rate } \psi'_{\delta_N}(N)v_N(0)k. \end{cases}$$

In this process, each individual dies without descendant at rate

$$\frac{\psi_{\delta_N}(N)}{N} = cN + \int_{\delta_N}^{\infty} z\mu(dz) - \frac{1}{N} \int_{\delta_N}^{\infty} (1 - e^{-Nz})\mu(dz),$$

it dies and leaves two descendants at rate

$$cN + \frac{1}{N} \int_{\delta_N}^{\infty} \frac{(Nz)^2}{2} e^{-Nz} \mu(dz),$$

and it dies and leaves k descendants ($k \geq 3$) at rate

$$\frac{1}{N} \int_{\delta_N}^{\infty} \frac{(Nz)^k}{k!} e^{-Nz} \mu(dz).$$

We note that $NX_t^{N,x}$ is a continuous time branching process with $m = [Nx]$ ancestors. More precisely, each individual lives for an exponential time with parameter $\psi'_{\delta_N}(N)$, and is replaced by a random number of children according to the probability generating function h_N .

Let $\mathcal{D}([0, \infty), \mathbb{R}_+)$ denote the space of functions from $[0, \infty)$ into \mathbb{R}_+ which are right continuous and have left limits at any $t > 0$. We shall always equip the space $\mathcal{D}([0, \infty), \mathbb{R}_+)$ with the Skorohod topology. The main limit proposition of this subsection is a consequence of Theorem 4.1 in [4].

Proposition 3.2 *Suppose that Assumptions (H) is satisfied. Then, as $N \rightarrow +\infty$, $\{X_t^{N,x}, t \geq 0\}$ converges to $\{X_t^x, t \geq 0\}$ in distribution on $\mathcal{D}([0, \infty), \mathbb{R}_+)$, where X^x is the unique solution of the SDE (2).*

3.2. The approximate height process

In this subsection, we shall define $\{H^N(s), s \geq 0\}$, the height process associated to the population process $\{X_t^{N,x}, t \geq 0\}$. After we show that the rescaled exploration process of the corresponding Galton-Watson genealogical forest of trees, converges in a functional sense, to the continuous height process associated with the CSBP. We will first need to write precisely the evolution of $\{H^N(s), s \geq 0\}$, the height process of the forest of trees representing the population described in section 3. To this end, to any $\delta > 0$, we define

$$Y_\delta(s) = - \left(b + \int_\delta^\infty z \mu(dz) \right) s + \sqrt{2c} B(s) + \int_0^s \int_\delta^\infty z \Pi(dr, dz).$$

and we associate H_δ the exploration process defined with the Lévy process Y_δ . In other words, we have suppressed the small jumps, smaller than δ , i.e (4) takes the following form

$$cH_\delta(s) = Y_\delta(s) - \inf_{0 \leq r \leq s} Y_\delta(r) - \int_0^s \int_\delta^\infty \left(z + \inf_{r \leq u \leq s} Y_\delta(u) - Y_\delta(r) \right)^+ \Pi(dr, dz). \quad (7)$$

We consider for each $N \geq 1$ a Poisson process $\{P_s^N, s \geq 0\}$ with intensity $\psi_{\delta_N}(N)$ independent from $\{Y(s), s \geq 0\}$. We denote by $\tau_1^N \leq \tau_2^N \leq \dots$ the jump times of this Poisson process. The height process $\{H^N(s), s \geq 0\}$ is simply the piecewise affine function of slope $\pm 2N$ passing through the values

$$0, H_{\delta_N}(\tau_1^N), \min_{s \in [\tau_1^N, \tau_2^N]} H_{\delta_N}(s), H_{\delta_N}(\tau_2^N), \min_{s \in [\tau_2^N, \tau_3^N]} H_{\delta_N}(s), \dots, H_{\delta_N}(\tau_n^N), \min_{s \in [\tau_n^N, \tau_{n+1}^N]} H_{\delta_N}(s), \dots$$

see Duquesne and Le Gall [6]. We are ready to state the main result of this paper. Recall the process H defined in (4).

Theorem 3.3 *For any $s > 0$, $H^N(s) \rightarrow H(s)$ in probability, locally uniformly in s , as $N \rightarrow \infty$.*

To prove this theorem, we will proceed in several steps. So, for any $s > 0$, we define

$$Y^{ref}(s) = Y(s) - \inf_{0 \leq r \leq s} Y(r) \quad \text{and} \quad Y_{\delta_N}^{ref}(s) = Y_{\delta_N}(s) - \inf_{0 \leq r \leq s} Y_{\delta_N}(r).$$

From now on, we do as if Y^{ref} and $Y_{\delta_N}^{ref}$ were deterministic, only P^N (and the τ_k^N 's) are random. For any $N \geq 1$, $s > 0$, we define

$$K_N(s) = \frac{1}{2N} H_{\delta_N}(\tau_1^N) + \frac{1}{2N} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} \left\{ (H_{\delta_N}(\tau_k^N) - \min_{r \in [\tau_k^N, \tau_{k+1}^N]} H_{\delta_N}(r)) + (H_{\delta_N}(\tau_{k+1}^N) - \min_{r \in [\tau_k^N, \tau_{k+1}^N]} H_{\delta_N}(r)) \right\} \quad (8)$$

It is not hard to see that $K^N(s)$ is the time taken by the process H^N to reach the point $H_{\delta_N}(\tau_{[\psi_{\delta_N}(N)s]}^N)$. So we get by our construction that

$$H^N(K_N(s)) = H_{\delta_N}(\tau_{[\psi_{\delta_N}(N)s]}^N). \quad (9)$$

For the proof of Theorem 3.3 we will need the two following Propositions.

Proposition 3.4 For any $s > 0$, $\tau_{[\psi_{\delta_N}(N)s]}^N \rightarrow s$ a.s, as $N \rightarrow \infty$.

Proof. It is easy to see that $\tau_{[\psi_{\delta_N}(N)s]}^N = \frac{1}{\psi_{\delta_N}(N)} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} \xi_k$, where $(\xi_k)_{k \geq 1}$ is an sequence of independent and identically distributed (i.i.d) $\sim \text{Exp}(1)$. The desired result follows easily from the law of large numbers. ■

Proposition 3.5 For any $s > 0$, $K_N(s) \rightarrow s$ in probability, as $N \rightarrow \infty$.

Proof. Let us rewrite (8) in the form

$$K_N(s) = \frac{1}{N} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} (H_{\delta_N}(\tau_k^N) - \min_{r \in [\tau_k^N, \tau_{k+1}^N]} H_{\delta_N}(r)) + \frac{1}{2N} \left(H_{\delta_N}(\tau_1^N) + \sum_{k=1}^{[\psi_{\delta_N}(N)s]} (H_{\delta_N}(\tau_{k+1}^N) - H_{\delta_N}(\tau_k^N)) \right) = K_1^N(s) + K_2^N(s) + K_3^N(s), \text{ with}$$

$$K_1^N(s) = \frac{1}{cN} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} \left\{ (cH_{\delta_N}(\tau_k^N) - \min_{r \in [\tau_k^N, \tau_{k+1}^N]} cH_{\delta_N}(r)) - (Y_{\delta_N}^{ref}(\tau_k^N) - \min_{r \in [\tau_k^N, \tau_{k+1}^N]} Y_{\delta_N}^{ref}(r)) \right\}$$

$$K_2^N(s) = \frac{1}{cN} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} (Y_{\delta_N}^{ref}(\tau_k^N) - \min_{r \in [\tau_k^N, \tau_{k+1}^N]} Y_{\delta_N}^{ref}(r)) \quad \text{and} \quad K_3^N(s) = \frac{1}{2N} H_{\delta_N}(\tau_{[\psi_{\delta_N}(N)s]}^N).$$

A standard argument combined with Proposition 3.4 yields $K_3^N(s) \rightarrow 0$ a.s, as $N \rightarrow \infty$, for any $s > 0$. The Proposition is now a consequence of the two next Propositions. ■

We now state two Propositions whose proofs will be given in the Appendix.

Proposition 3.6 For any $s > 0$, $K_1^N(s) \rightarrow 0$ in probability, as $N \rightarrow \infty$.

Proposition 3.7 For any $s > 0$, $K_2^N(s) \rightarrow s$ in probability, as $N \rightarrow \infty$.

For the proof of the last Proposition, we need a basic result on Lévy processes. To this end, let us define $\Gamma(s) = \max_{0 \leq r \leq s} (-Y_{\delta_N}(r))$, where Y_{δ_N} is again a Lévy process with characteristic exponent ψ_{δ_N} . The following result is Corollary 2, chapter VII in [1].

Lemma 3.8 Since τ_1^N is an exponential random variable with parameter $\psi_{\delta_N}(N)$, independent of Y_{δ_N} , $\Gamma(\tau_1^N)$ has an exponential distribution with parameter N .

Now, let us define

$$w_\delta^N(a, b) = \sup_{a \leq r, s \leq b, |s-r| \leq \delta} |H^N(s) - H^N(r)|,$$

$$\text{and } w_{N, \delta}(a, b) = \sup_{a \leq r, s \leq b, |s-r| \leq \delta} |H_{\delta_N}(s) - H_{\delta_N}(r)|.$$

We shall also need below the

Proposition 3.9 For all $\varepsilon > 0$, $\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}(w_\delta^N(a, b) \geq \varepsilon) = 0$.

Proof. We have

$$\{w_\delta^N(a, b) > \varepsilon\} \subset \left\{ \sup_{a < s < b} |\mathcal{H}_N(s) - s| + \sup_{k_a \leq k \leq k_b} (\tau_{k+1}^N - \tau_k^N) > \delta \right\}$$

$$\cup \{w_{N, 3\delta}(a - \delta, b + \delta) > \varepsilon\},$$

where $\mathcal{H}_N(s) = \tau_{\lfloor \frac{a}{\psi_{\delta_N}(N) K_N^{-1}(s)} \rfloor}^N$, $k_a = \lfloor \frac{a}{\psi_{\delta_N}(N)} \rfloor$ and $k_b = \lfloor \frac{b}{\psi_{\delta_N}(N)} \rfloor - 1$.

So the result follows from both the two following facts : for each $\delta > 0$,

$$\forall \delta > 0, \mathbb{P} \left(\sup_{a < r < b} |\mathcal{H}_N(r) - r| > \delta \right) \rightarrow 0, \text{ as } N \rightarrow \infty, \quad (10)$$

$$\forall \varepsilon > 0, \lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}(w_{N, 3\delta}(a - \delta, b - \delta) > \varepsilon) = 0. \quad (11)$$

PROOF OF (10). It follows from a combination of Propositions 3.4 and 3.5 that $\mathcal{H}_N(s) \rightarrow s$ in probability, for any $s > 0$. Moreover for any $N, s \rightarrow \mathcal{H}_N(s)$ is increasing. Let $M \geq 1$ and $a = s_0 < s_1 < \dots < s_M = b$ be such that $\sup_{0 \leq i \leq M-1} (s_{i+1} - s_i) \leq \delta/2$. For any $\varepsilon > 0$, we can choose N_ε large enough such that for all $N \geq N_\varepsilon$, $\mathbb{P}(\bigcap_{i=0}^M \{|\mathcal{H}_N(s_i) - s_i| \leq \delta/2\}) \geq 1 - \varepsilon$. But for any $s_I \leq s \leq s_{i+1}$, on the event $\bigcap_{i=0}^M \{|\mathcal{H}_N(s_i) - s_i| \leq \delta/2\}$,

$$s - \delta \leq s_i - \delta/2 \leq \mathcal{H}_N(s_i) \leq \mathcal{H}_N(s) \leq \mathcal{H}_N(s_{i+1}) \leq s_{i+1} + \delta/2 \leq s + \delta,$$

hence we have shown that for $N \geq N_\varepsilon$, the following property equivalent to (10)

$$\mathbb{P} \left(\sup_{a < s < b} |\mathcal{H}_N(s) - s| \leq \delta \right) \geq 1 - \varepsilon.$$

PROOF OF (11). Since $H_{\delta_N}(s) \rightarrow H(s)$ uniformly in $[(a-1) \vee 0, b+1]$ in probability, hence it converges in law in $\mathcal{C}([(a-1) \vee 0, b+1])$, hence the sequence $\{H_{\delta_N}, N \geq 1\}$ is tight in $\mathcal{C}([(a-1) \vee 0, b+1])$, from which (11) follows. \blacksquare

Proof of Theorem 3.3 : From (9), we have

$$\begin{aligned} |H^N(s) - H(s)| \leq & |H^N(s) - H^N(K_N(s))| + \left| H_{\delta_N} \left(\tau_{[\psi_{\delta_N}(N)s]}^N \right) - H \left(\tau_{[\psi_{\delta_N}(N)s]}^N \right) \right| \\ & + \left| H \left(\tau_{[\psi_{\delta_N}(N)s]}^N \right) - H(s) \right|. \end{aligned}$$

A combination of Propositions 3.5 and 3.9 implies that the first term on the right tends to 0 in probability, as $N \rightarrow +\infty$. Since $H_{\delta_N} \rightarrow H$ *a.s.* locally uniformly in s , and from Proposition 3.4, $\tau_{[\psi_{\delta_N}(N)s]} \rightarrow s$ *a.s.*, the second term tends to 0 *a.s.* Finally the last term tends to 0 *a.s.* thanks again to Proposition 3.4 and the continuity of H .

We have just proved that for each $s > 0$, $H^N(s) \rightarrow H(s)$ in probability, as $N \rightarrow +\infty$. Since from Proposition 3.9, H is tight in $\mathcal{C}([0, s])$ for all $s > 0$, the convergence is locally uniform in s . ■

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5. Appendix

In this section we detail the proofs of Propositions 3.6 and 3.7.

Proof of Proposition 3.6 : In this proof, we will use the following notations

$$\min_{\tau_k^N \leq r \leq \tau_{k+1}^N} Y_{\delta_N}^{ref}(r) = Y_{\delta_N}^{ref}(r_Y^{k,N}) \quad \text{and} \quad \min_{\tau_k^N \leq r \leq \tau_{k+1}^N} H_{\delta_N}(r) = H_{\delta_N}(r_H^{k,N}).$$

Let us define

$$U_{\delta_N}(s) = \int_0^s \int_{\delta_N}^{\infty} \left(z + \inf_{r \leq u \leq s} Y_{\delta_N}(u) - Y_{\delta_N}(r) \right)^+ \Pi(dr, dz).$$

We first note that $V_{U_{\delta_N}}[0, s]$, the total variation of U_{δ_N} on the interval $[0, s]$, satisfies

$$\int_0^s \int_{\delta_N}^{\infty} z \Pi(dr, dz) \leq V_{U_{\delta_N}}[0, s] \leq 2 \int_0^s \int_{\delta_N}^{\infty} z \Pi(dr, dz). \quad (12)$$

However, we can rewrite (7) indexed by δ_N in the following form $cH_{\delta_N}(s) = Y_{\delta_N}^{ref}(s) - U_{\delta_N}(s)$. It is not hard to obtain the following inequality

$$\begin{aligned} U_{\delta_N}(r_Y^{k,N}) &= Y_{\delta_N}^{ref}(r_Y^{k,N}) - cH_{\delta_N}(r_Y^{k,N}) \leq \min_{\tau_k^N \leq r \leq \tau_{k+1}^N} Y_{\delta_N}^{ref}(r) - \min_{\tau_k^N \leq r \leq \tau_{k+1}^N} H_{\delta_N}(r) \\ &\leq Y_{\delta_N}^{ref}(r_H^{k,N}) - cH_{\delta_N}(r_H^{k,N}) = U_{\delta_N}(r_H^{k,N}) \end{aligned} \quad (13)$$

Now, we have

$$\begin{aligned} K_1^N(s) &= \frac{1}{cN} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} \left\{ (cH_{\delta_N}(\tau_k^N) - \min_{r \in [\tau_k^N, \tau_{k+1}^N]} cH_{\delta_N}(r)) - (Y_{\delta_N}^{ref}(\tau_k^N) - \min_{r \in [\tau_k^N, \tau_{k+1}^N]} Y_{\delta_N}^{ref}(r)) \right\} \\ &= \frac{1}{cN} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} \left\{ -U_{\delta_N}(\tau_k^N) + \min_{r \in [\tau_k^N, \tau_{k+1}^N]} Y_{\delta_N}^{ref}(r) - \min_{r \in [\tau_k^N, \tau_{k+1}^N]} cH_{\delta_N}(r) \right\} \\ &= \frac{1}{cN} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} \gamma_N(k), \end{aligned}$$

and (13) implies that $|\gamma_N(k)| \leq \sup_{r_Y^{k,N} \leq r \leq r_H^{k,N}} |U_{\delta_N}(\tau_k^N) - U_{\delta_N}(r)|$. Now from (12)

$$\left| \sum_{k=1}^{[\psi_{\delta_N}(N)s]} \gamma_N(k) \right| \leq V_{U_{\delta_N}}[0, s] \leq 2 \int_0^s \int_{\delta_N}^{\infty} z \Pi(dr, dz),$$

which implies that $|K_1^N(s)| \leq \frac{2}{cN} \int_0^s \int_{\delta_N}^\infty z \Pi(dr, dz)$.

The result follows easily from this estimate combined with (5). ■

Proof of Proposition 3.7 : We have

$$K_2^N(s) = \frac{\psi_{\delta_N}(N)}{cN^2} \times \frac{1}{\psi_{\delta_N}(N)} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} N(Y_{\delta_N}^{ref}(\tau_k^N) - \min_{r \in [\tau_k^N, \tau_{k+1}^N]} Y_{\delta_N}^{ref}(r)).$$

We first notice that $0 \leq e^{-\lambda} - 1 + \lambda \leq \lambda$, for all $\lambda \geq 0$, this implies $\frac{\psi_{\delta_N}(N)}{cN^2} \rightarrow 1$, as $N \rightarrow \infty$. Let Γ' and Y'_{δ_N} be independent copies of Γ and Y_{δ_N} respectively. We notice that

$$\begin{aligned} Y_{\delta_N}^{ref}(\tau_k^N) - \min_{r \in [\tau_k^N, \tau_{k+1}^N]} Y_{\delta_N}^{ref}(r) &\stackrel{(d)}{=} \left(\max_{r \in [\tau_k^N, \tau_{k+1}^N]} (-Y'_{\delta_N}(r)) \right) \wedge Y_{\delta_N}^{ref}(\tau_k^N) \\ &:= \Gamma'(\tau_{k+1}^N - \tau_k^N) \wedge Y_{\delta_N}^{ref}(\tau_k^N). \end{aligned}$$

Let $(\Xi_k)_{k \geq 1}$ be an sequence of i.i.d random variables whose common law is that of $N\Gamma(\tau_1^N)$, such that in addition for any $k \geq 1$, Ξ_k and $\{Y_{\delta_N}^{ref}(r), r \leq \tau_k^N\}$ are independent. We notice from Lemma 3.8 that Ξ_1 has a standard exponential distribution. The Proposition is now a consequence the next lemma. ■

Lemma 5.1 For any $s > 0$,

$$\frac{1}{\psi_{\delta_N}(N)} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} \Xi_k \wedge NY_{\delta_N}^{ref}(\tau_k^N) \rightarrow s \text{ in probability, as } N \rightarrow \infty.$$

Proof. Let $\varepsilon > 0$, which will eventually go to zero. Let $g_\varepsilon, h_\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}$ be two functions defined by

$$g_\varepsilon(z) = \begin{cases} 1, & \text{if } z \leq \varepsilon, \\ -\varepsilon^{-1}z + 2, & \text{if } \varepsilon < z \leq 2\varepsilon, \\ 0, & \text{if } z > 2\varepsilon, \end{cases} \quad \text{and,} \quad h_\varepsilon(z) = \begin{cases} 0, & \text{if } z \leq \varepsilon, \\ -\varepsilon^{-1}z - 1, & \text{if } \varepsilon < z \leq 2\varepsilon, \\ 1, & \text{if } z > 2\varepsilon. \end{cases}$$

It is not hard to see that

$$I_1^N(s, \varepsilon) + I_2^N(s, \varepsilon) \leq \frac{1}{\psi_{\delta_N}(N)} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} \Xi_k \wedge NY_{\delta_N}^{ref}(\tau_k^N) \leq J_1^N(s) + J_2^N(s, \varepsilon) + J_3^N(s, \varepsilon),$$

where

$$I_1^N(s, \varepsilon) = \frac{1 - e^{-N\varepsilon}}{\psi_{\delta_N}(N)} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} h_\varepsilon(Y_{\delta_N}^{ref}(\tau_k^N)) \quad [\text{where using the identity } \mathbb{E}(\Xi_k \wedge N\varepsilon) = 1 - e^{-N\varepsilon}],$$

$$I_2^N(s, \varepsilon) = \frac{1}{\psi_{\delta_N}(N)} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} (\Xi_k \wedge N\varepsilon - \mathbb{E}(\Xi_k \wedge N\varepsilon)) \mathbf{1}_{\{Y_{\delta_N}^{ref}(\tau_k^N) > \varepsilon\}}, \quad J_1^N(s) = \frac{1}{\psi_{\delta_N}(N)} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} \Xi_k,$$

$$J_2^N(s, \varepsilon) = \frac{1}{\psi_{\delta_N}(N)} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} g_\varepsilon(Y_{\delta_N}^{ref}(\tau_k^N)), \quad \text{and } J_3^N(s, \varepsilon) = \frac{1}{\psi_{\delta_N}(N)} \sum_{k=1}^{[\psi_{\delta_N}(N)s]} (\Xi_k - 1) \mathbf{1}_{\{Y_{\delta_N}^{ref}(\tau_k^N) \leq \varepsilon\}}.$$

However, we first have

$$\begin{aligned}\mathbb{E}[I_2^N(s, \varepsilon)]^2 &\leq \frac{1}{(\Psi_{\delta_N}(N))^2} \sum_{k=1}^{[\Psi_{\delta_N}(N)s]} \text{Var}(\Xi_k \wedge N\varepsilon) \\ &= \frac{1}{(\Psi_{\delta_N}(N))^2} \sum_{k=1}^{[\Psi_{\delta_N}(N)s]} (1 - e^{-2N\varepsilon} - 2N\varepsilon e^{-N\varepsilon}) \longrightarrow 0, \text{ as } N \rightarrow +\infty.\end{aligned}$$

We can prove similarly that $\mathbb{E}[J_3^N(s, \varepsilon)]^2 \longrightarrow 0$, as $N \rightarrow +\infty$. Combining Lemma 5.2 below and the fact that $Y^{ref}(r) > 0$ *dr a.s.*, we deduce

$$I_1^N(s, \varepsilon) \xrightarrow[N \rightarrow \infty]{} \int_0^s h_\varepsilon(Y_{\delta_N}^{ref}(r)) dr \xrightarrow[\varepsilon \rightarrow 0]{} s, \quad \text{and} \quad J_2^N(s, \varepsilon) \xrightarrow[N \rightarrow \infty]{} \int_0^s g_\varepsilon(Y_{\delta_N}^{ref}(r)) dr \xrightarrow[\varepsilon \rightarrow 0]{} 0.$$

In addition, we deduce from the law of large numbers that $J_1^N(s) \longrightarrow s$, as $N \rightarrow +\infty$. The desired result follows by combining the above arguments. \blacksquare

We finally establish a last result which we have used in the last proof.

Lemma 5.2 *For any $h \in \mathcal{C}(\mathbb{R}_+; [0, 1])$,*

$$\frac{1}{\Psi_{\delta_N}(N)} \sum_{k=1}^{[\Psi_{\delta_N}(N)s]} h(Y_{\delta_N}^{ref}(\tau_k^N)) \longrightarrow \int_0^s h(Y^{ref}(r)) dr \quad \text{in probability, as } N \rightarrow \infty.$$

Proof. We have

$$\begin{aligned}&\frac{1}{\Psi_{\delta_N}(N)} \sum_{k=1}^{[\Psi_{\delta_N}(N)s]} h(Y_{\delta_N}^{ref}(\tau_k^N)) - \int_0^s h(Y^{ref}(r)) dr \\ &= \frac{1}{\Psi_{\delta_N}(N)} \sum_{k=1}^{[\Psi_{\delta_N}(N)s]} h(Y_{\delta_N}^{ref}(\tau_k^N)) - \frac{1}{\Psi_{\delta_N}(N)} \int_{[0, s]} h(Y_{\delta_N}^{ref}(r)) dP_r^N \\ &+ \frac{1}{\Psi_{\delta_N}(N)} \int_{[0, s]} h(Y_{\delta_N}^{ref}(r)) dP_r^N - \int_0^s h(Y_{\delta_N}^{ref}(r)) dr \\ &+ \int_0^s h(Y_{\delta_N}^{ref}(r)) dr - \int_0^s h(Y^{ref}(r)) dr = A_N(s) + B_N(s) + C_N(s).\end{aligned}$$

First $C_N(s) \rightarrow 0$ follows readily from $\sup_{0 \leq r \leq s} |h(Y^{ref}(r)) - h(Y_{\delta_N}^{ref}(r))| \rightarrow 0$, as $N \rightarrow \infty$, since h is continuous and $\sup_{0 \leq r \leq s} |Y^{ref}(r) - Y_{\delta_N}^{ref}(r)| \rightarrow 0$, as $N \rightarrow \infty$. Next we have $B^N(s) = \frac{1}{\Psi_{\delta_N}(N)} \int_{[0, s]} h(Y_{\delta_N}^{ref}(r)) [dP_r^N - \Psi_{\delta_N}(N) dr]$. We have $\mathbb{E}[B^N(s)] = 0$, while $\text{Var}(B^N(s)) = \frac{1}{\Psi_{\delta_N}(N)} \mathbb{E} \int_0^s h(Y_{\delta_N}^{ref}(r))^2 dr$, which clearly tends to 0 as $N \rightarrow \infty$, since h is bounded and $\Psi_{\delta_N}(N) \rightarrow \infty$. Consequently $B^N(s) \rightarrow 0$ in probability, as $N \rightarrow \infty$. It remains to consider A^N . Since $0 \leq h(y) \leq 1$, $|A^N(s)| \leq \frac{1}{\Psi_{\delta_N}(N)} |P_s^N - \Psi_{\delta_N}(N)s| \rightarrow 0$ *a.s.* from the strong law of large numbers. The result follows. \blacksquare