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**A NOTE ON SOME MICROLOCAL ESTIMATES USED TO PROVE  
THE CONVERGENCE OF SPLITTING METHODS RELYING ON  
PSEUDO-SPECTRAL DISCRETIZATIONS.**

JOACKIM BERNIER, FERNANDO CASAS, AND NICOLAS CROUSEILLES

ABSTRACT. In [BCC20], we used some classical microlocal estimates to prove the convergence of our splitting methods (for example page A671). In this note, through Corollary 2 and Remark 1, we provide a detailed proof of these estimates. All the proofs rely on results presented in [NR10].

We consider the classes of symbols  $S(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$ ,  $s_1, s_2 \in \mathbb{R}$ . By definition (see Definition 1.1.1 page 19 in [NR10]), it contains the symbols  $a(x, \xi)$  such that

$$\forall \gamma, \delta \in \mathbb{N}, |\partial_x^\gamma \partial_\xi^\delta a(x, \xi)| \lesssim_{\gamma, \delta} (\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}) \langle x \rangle^{-\gamma} \langle \xi \rangle^{-\delta}.$$

We are going to prove the following proposition.

**Proposition 1.** *If  $s_1, s_2 \geq 0$  and  $b \in S(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$  then*

$$\forall u \in \mathcal{S}(\mathbb{R}^d), |(u, b^w u)_{L^2}| \lesssim_{b, s_1, s_2} (u, (\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2})^w u)_{L^2}.$$

This proposition is useful to get the following corollaries.

**Corollary 1.** *If  $s \geq 0$  and  $a \in S(\langle x \rangle^s + \langle \xi \rangle^s; \langle \xi \rangle, \langle x \rangle)$  then*

$$\forall u \in \mathcal{S}(\mathbb{R}^d), \|a^w u\|_{L^2} \lesssim_{a, s} \|u\|_{X^s}$$

where

$$\|u\|_{X^s}^2 := \int_{\mathbb{R}^d} \langle x \rangle^{2s} |u(x)|^2 dx + \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi.$$

*Proof of Corollary 1.* Since we have

$$(\langle x \rangle^s + \langle \xi \rangle^s)^2 \leq 2(\langle x \rangle^{2s} + \langle \xi \rangle^{2s}),$$

applying the Proposition 1.2.9 page 29 and the Theorem 1.2.16 page 31 in [NR10], we get a symbol

$$c \in S((\langle x \rangle^s + \langle \xi \rangle^s)^2; \langle \xi \rangle, \langle x \rangle) \subset S(\langle x \rangle^{2s} + \langle \xi \rangle^{2s}; \langle \xi \rangle, \langle x \rangle),$$

such that

$$c^w = (a^w)^* a^w.$$

Consequently, applying Proposition 1, for  $u \in \mathcal{S}(\mathbb{R}^d)$ , we have

$$\|a^w u\|_{L^2}^2 = (u, c^w u)_{L^2} \leq c_{a, s} (u, (\langle x \rangle^{2s} + \langle \xi \rangle^{2s})^w u)_{L^2} = c_{a, s} \|u\|_{X^s}^2$$

where  $c_{a, s}$  is a constant depending only on  $a$  and  $s$ . □

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**Corollary 2.** *If  $s \geq 0$  and  $\alpha, \beta > 0$  are such that  $\alpha + \beta \leq s$  then*

$$\forall u \in \mathcal{S}(\mathbb{R}^d), \quad \|\langle x \rangle^\alpha \langle \nabla \rangle^\beta u\|_{L^2} \lesssim_{a,s} \|u\|_{X^s}.$$

*Proof of Corollary 1.* Since, by Young, we have

$$\langle x \rangle^\alpha \langle \xi \rangle^\beta \leq \frac{\alpha}{\alpha + \beta} \langle x \rangle^{\alpha + \beta} + \frac{\beta}{\alpha + \beta} \langle \xi \rangle^{\alpha + \beta} \leq \langle x \rangle^s + \langle \xi \rangle^s,$$

applying the Theorem 1.2.16 page 31 in [NR10], we get a symbol

$$a \in S(\langle x \rangle^\alpha \langle \xi \rangle^\beta; \langle \xi \rangle, \langle x \rangle) \subset S(\langle x \rangle^s + \langle \xi \rangle^s; \langle \xi \rangle, \langle x \rangle),$$

such that

$$a^w = \langle x \rangle^\alpha \langle \nabla \rangle^\beta.$$

Consequently, we conclude by applying the Corollary 1.  $\square$

**Remark 1.** This corollary could be easily extended to control the terms we had in [BCC20]. For exemple, provided that  $\alpha + \beta + \gamma \leq s$ , we could also control things like  $\|\langle \nabla \rangle^\gamma \langle x \rangle^\alpha \langle \nabla \rangle^\beta u\|_{L^2(\mathbb{R}^d)}$  or  $\|\langle \partial_{x_1} \rangle^\gamma \langle x_2 \rangle^\alpha \langle \partial_{x_2} \rangle^\beta u\|_{L^2(\mathbb{R}^d)}$ .

The proof of the Proposition 1 relies on the following technical lemma whose proof is given in the subsection 1.1 of the Appendix.

**Lemma 1.** *If  $a \in S(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$ , there exists  $r_1 \in S(\langle x \rangle^{s_1-1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$ ,  $r_2 \in S(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2-1}; \langle \xi \rangle, \langle x \rangle)$  such that*

$$A_a = a^w + r_1^w + r_2^w$$

where  $A_a$  is the Anti-Wick operator with symbol  $a$  (see Definition 1.7.3 page 53 in [NR10]).

Finally, we focus on the proof of the Proposition 1.

*Proof of Proposition 1.* Let  $s_1, s_2 \geq 0$  and  $b \in S(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$ . We aim at proving that

$$(1) \quad \forall u \in \mathcal{S}(\mathbb{R}^d), \quad |(u, b^w u)_{L^2}| \lesssim_{b, s_1, s_2} (u, (\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2})^w u)_{L^2}.$$

Since  $b \in S(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$ , there exists a constant  $c_b > 0$  such that

$$\forall x, \xi \in \mathbb{R}^d, \quad a(x, \xi) := c_b(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}) \pm b(x, \xi) \geq 0.$$

Since it is clear that  $\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2} \in S(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$ , we have  $a \in S(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$  and so, by applying the Lemma 1, we get  $r_1 \in S(\langle x \rangle^{s_1-1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$  and  $r_2 \in S(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2-1}; \langle \xi \rangle, \langle x \rangle)$  such that

$$A_a = a^w + r_1^w + r_2^w.$$

Since the symbol  $a$  is nonnegative, by applying the Proposition 1.7.6 page 53 in [NR10], we know that  $A_a$  is a nonnegative operator and so

$$(2) \quad (u, A_a u)_{L^2} = c_b(u, (\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2})^w u)_{L^2} \pm (u, b^w u)_{L^2} + (u, r_1^w u)_{L^2} + (u, r_2^w u)_{L^2} \geq 0.$$

Finally, we have to control  $(u, r_1^w u)_{L^2}$  and  $(u, r_2^w u)_{L^2}$ . By symmetry, we only focus on  $(u, r_1^w u)_{L^2}$ .

We proceed by induction.

- *Case  $s_1 > 1$ .* By the induction assumption, we know that

$$(u, r_1^w u)_{L^2} \lesssim_{b, s_1, s_2} (u, (\langle x \rangle^{s_1-1} + \langle \xi \rangle^{s_2})^w u)_{L^2},$$

and so since  $\langle x \rangle^{s_1-1} \leq \langle x \rangle^{s_1}$  we have

$$(u, r_1^w u)_{L^2} \lesssim_{b, s_1, s_2} (u, (\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2})^w u)_{L^2}.$$

- *Case  $s_1 \leq 1$ .* Applying Theorem 1.2.16 page 31 in [NR10], we get a symbol  $f \in S(\langle \xi \rangle^{-s_2} \langle x \rangle^{s_1-1} + 1; \langle \xi \rangle, \langle x \rangle) \subset S(1; 1, 1)$  such that

$$f^w = \langle \nabla \rangle^{-s_2/2} r_1^w \langle \nabla \rangle^{-s_2/2}.$$

Consequently, applying the Calderón-Vaillancourt theorem (Theorem 1.7.14 page 58 in [NR10]), this operator is bounded in  $L^2$  and so

$$\forall u \in \mathcal{S}(\mathbb{R}^d), (r_1^w \langle \nabla \rangle^{-s_2/2} u, \langle \nabla \rangle^{-s_2/2} u)_{L^2} \lesssim_{r_1^w, s_2} \|u\|_{L^2}^2.$$

As a consequence, the change of function  $u \leftarrow \langle \nabla \rangle^{-s_2/2} u$  provides the estimate

$$\forall u \in \mathcal{S}(\mathbb{R}^d), (r_1^w u, u)_{L^2} \lesssim_{r_1, s_2} (u, \langle \nabla \rangle^{s_2} u)_{L^2} \lesssim_{r_1, s_2} (u, (\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2})^w u)_{L^2}.$$

In any case, we have

$$(r_1^w u, u)_{L^2} + (r_2^w u, u)_{L^2} \lesssim_{b, s_1, s_2} (u, (\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2})^w u)_{L^2}.$$

Plugging this estimate in (2) provides naturally the estimate (1) we aimed at proving.  $\square$

## 1. APPENDIX

**1.1. Proof of Lemma 1.** We aim at proving that if  $a \in S(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$  then there exists  $r_1 \in S(\langle x \rangle^{s_1-1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$ ,  $r_2 \in S(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2-1}; \langle \xi \rangle, \langle x \rangle)$  such that

$$A_a = a^w + r_1^w + r_2^w.$$

Applying the Proposition 1.7.9 page 55, we know that

$$A_a = b^w$$

where

$$(3) \quad b(x, \xi) := 2^d \int_{\mathbb{R}^d} a(y, \eta) e^{-|x-y|^2 - |\xi-\eta|^2} \frac{dy}{(2\pi)^{d/2}} \frac{d\eta}{(2\pi)^{d/2}}.$$

Consequently, we just have to decompose  $b$  in the good classes.

Naturally, the Taylor expansion at the order 1 of  $a$  in  $(x, \xi)$  is

$$\begin{aligned} a(y, \eta) &= a(x, \xi) + \int_0^1 \partial_x a(x + t(y-x), \xi + t(\eta - \xi)) dt (y-x) \\ &\quad + \int_0^1 \partial_\xi a(x + t(y-x), \xi + t(\eta - \xi)) dt (\eta - \xi). \end{aligned}$$

Plugging this expansion in (3) and realizing the change of coordinate  $(y, \eta) \leftarrow (y-x, \eta - \xi)$ , we are naturally led to set

$$r_1(x, \xi) = 2^d \int_{\mathbb{R}^d} \int_0^1 \partial_x a(x + ty, \xi + t\eta) y e^{-|y|^2 - |\eta|^2} dt \frac{dy}{(2\pi)^{d/2}} \frac{d\eta}{(2\pi)^{d/2}}$$

and

$$r_2(x, \xi) = 2^d \int_{\mathbb{R}^d} \int_0^1 \partial_\xi a(x + ty, \xi + t\eta) \eta e^{-|y|^2 - |\eta|^2} dt \frac{dy}{(2\pi)^{d/2}} \frac{d\eta}{(2\pi)^{d/2}}.$$

By symmetry, we just check that  $r_1 \in S(\langle x \rangle^{s_1-1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$ .

By definition, we just have to prove that

$$(4) \quad |\partial_x^\alpha \partial_\xi^\beta r_1(x, \xi)| \lesssim_{\alpha, \beta} (\langle x \rangle^{s_1-1} + \langle \xi \rangle^{s_2}) \langle x \rangle^{-\alpha} \langle \xi \rangle^{-\beta}.$$

Since, by assumption, we know that

$$|\partial_x^\gamma \partial_\xi^\delta a(x, \xi)| \lesssim_{\gamma, \delta} (\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}) \langle x \rangle^{-\gamma} \langle \xi \rangle^{-\delta},$$

we deduce that

$$|\partial_x^{\alpha+1} \partial_\xi^\beta a(x + ty, \xi + t\eta)| \lesssim_{\alpha, \beta} (\langle x + ty \rangle^{s_1} + \langle \xi + t\eta \rangle^{s_2}) \langle x + ty \rangle^{-\alpha-1} \langle \xi + t\eta \rangle^{-\beta}.$$

Recalling the Peetre's inequality (0.1.2) page 19 :

$$\langle x + y \rangle^s \lesssim_s \langle x \rangle^s \langle y \rangle^{|s|}, \quad \forall x, y \in \mathbb{R}^d, s \in \mathbb{R},$$

we get

$$(5) \quad \begin{aligned} |\partial_x^{\alpha+1} \partial_\xi^\beta a(x + ty, \xi + t\eta)| &\lesssim_{\alpha, \beta, s_1, s_2} (\langle x \rangle^{s_1} \langle y \rangle^{s_1} + \langle \xi \rangle^{s_2} \langle \eta \rangle^{s_2}) \langle x \rangle^{-\alpha-1} \langle \xi \rangle^{-\beta} \langle y \rangle^{\alpha+1} \langle \eta \rangle^\beta \\ &\lesssim_{\alpha, \beta, s_1, s_2} (\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}) \langle x \rangle^{-\alpha-1} \langle \xi \rangle^{-\beta} \langle y \rangle^{\alpha+1+s_1} \langle \eta \rangle^{\beta+s_2}. \end{aligned}$$

Finally, observing that

$$(\langle x \rangle^{s_1} + \langle \xi \rangle^{s_2}) \langle x \rangle^{-1} \leq \langle x \rangle^{s_1-1} + \langle \xi \rangle^{s_2},$$

plugging (5) in the definition of  $r_1$  yields to (4), i.e.  $r_1 \in S(\langle x \rangle^{s_1-1} + \langle \xi \rangle^{s_2}; \langle \xi \rangle, \langle x \rangle)$ .

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