



# Notch-Induced Domain Wall Pining in Ferromagnetic Nanowires

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## Chapter 1

### Notch-Induced Domain Wall pinning in ferromagnetic Nanowires

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#### 1. Introduction

In promising new magnetic-memory devices, the data are encoded by the positions of domain walls (DW) in ferromagnetic nanowires (see Ref. 1). In order to ensure the reliability of the storage, DW positions must be fixed, since an uncontrolled drift of a DW induces an undesired modification of the data. In these devices, DW pinning is ensured by patterning notches along the wire. This pinning effect is described experimentally and numerically in several papers (see Refs. 2,3). The aim of this contribution is to rigorously establish a one-dimensional model of notched nanowire and to observe DW pinning in this model.

The paper is organized as follows. In the next section, starting from the three-dimensional classical model of ferromagnetic materials, we derive by asymptotic process an equivalent one-dimensional model of nanowire. In Section 3, we construct a static solution modeling one DW for the 1d model and we prove that this solution is asymptotically stable modulo rotations in Section 4. With this result, we establish rigorously that a notch pins DW. In the last section, we study the effects of an applied magnetic field on the pinned DW. We prove that the DW remains in the notch for small applied field and moves out of the notch if the applied field is large enough.

## 2. One-dimensional model of ferromagnetic wire

### 2.1. Landau-Lifschitz model in 3d

We consider a straight cylindrical ferromagnetic nanowire of length  $2L$ , presenting a radial constriction of length  $2l$  centered in the middle of the wire, with  $0 < l < L$ . This sample is described by  $\Omega_\eta$ :

$$\Omega_\eta = \left\{ (x, y, z), x \in ]-L, L[, \quad (y^2 + z^2)^{\frac{1}{2}} < r(x)\eta \right\},$$

where

$$r(x) = \begin{cases} r_0 & \text{if } |x| > l, \\ r_1 & \text{if } |x| \leq l, \end{cases} \quad (1)$$

with  $0 < r_1 < r_0$ . We denote by  $\mathbf{a}(x)$  the area of the cross section for  $\Omega_1$ :

$$\mathbf{a}(x) = \pi(r(x))^2. \quad (2)$$

The magnetic moment  $\mathbf{m}(t, \mathbf{x})$  is defined for time  $t \geq 0$  and position  $\mathbf{x} = (x, y, z) \in \Omega_\eta$ , takes its values in  $\mathbb{R}^3$ , and satisfies the saturation constraint:

$$|\mathbf{m}(t, \mathbf{x})| = 1 \text{ for } (t, \mathbf{x}) \in \mathbb{R}^+ \times \Omega_\eta, \quad (3)$$

where  $|\cdot|$  is the euclidean norm on  $\mathbb{R}^3$ . The magnetic moment satisfies the Landau-Lifschitz Equation:<sup>4,5</sup>

$$\begin{cases} \partial_t \mathbf{m} = -\mathbf{m} \times \mathbf{h}(\mathbf{m}) - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{h}(\mathbf{m})) & \text{in } \mathbb{R}^+ \times \Omega_\eta, \\ \mathbf{h}(\mathbf{m}) = \ell^2 \Delta \mathbf{m} + \mathbf{h}_d(\mathbf{m}) + \mathbf{h}_a, \\ \frac{\partial \mathbf{m}}{\partial \nu} = 0 & \text{on } \mathbb{R}^+ \times \partial \Omega_\eta, \end{cases} \quad (4)$$

where  $\alpha$  is the gyromagnetic ratio,  $\times$  is the cross product in  $\mathbb{R}^3$ ,  $\nu$  is the outward unit normal on  $\partial \Omega$ ,  $\ell$  is the exchange length,  $\mathbf{h}_a$  is the applied magnetic field. The stray field generated by the magnetization, denoted by  $\mathbf{h}_d(\mathbf{m})$ , satisfies:

$$\text{curl } \mathbf{h}_d(\mathbf{m}) = 0 \quad \text{and} \quad \text{div}(\mathbf{h}_d(\mathbf{m}) + \overline{\mathbf{m}}) = 0, \quad (5)$$

where  $\overline{\mathbf{m}}(t, \mathbf{x}) = \mathbf{m}(t, \mathbf{x})$  for  $\mathbf{x} \in \Omega_\eta$  and zero for  $\mathbf{x} \notin \Omega_\eta$ .

The effective field  $\mathbf{h}(\mathbf{m})$  is derived from the micromagnetism energy  $\mathcal{E}_\eta(\mathbf{m})$  by the formula:

$$\mathbf{h}(\mathbf{m}) = -\partial_{\mathbf{m}} \mathcal{E}_\eta,$$

with:

$$\mathcal{E}_\eta(\mathbf{m}) = \frac{\ell^2}{2} \int_{\Omega_\eta} |\nabla \mathbf{m}|^2 d\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{h}_d(\mathbf{m})|^2 d\mathbf{x} - \int_{\Omega_\eta} \mathbf{m} \cdot \mathbf{h}_a. \quad (6)$$

For smooth solutions, the Landau-Lifschitz equation (4) is equivalent to the Landau-Lifschitz-Gilbert equation:

$$\partial_t \mathbf{m} - \alpha \mathbf{m} \times \partial_t \mathbf{m} = -(1 + \alpha^2) \mathbf{m} \times \mathbf{h}(\mathbf{m}) \text{ in } \mathbb{R}^+ \times \Omega_\eta. \quad (7)$$

Existence of global weak solutions for (7) is established in several papers (see Refs. 6–8). For an initial data  $\mathbf{m}_0 \in H^1(\Omega_\eta; \mathbb{R}^3)$  satisfying the saturation constraint  $|\mathbf{m}_0| = 1$  a.e., there exists  $\mathbf{m}^\eta : \mathbb{R}^+ \times \Omega_\eta \rightarrow \mathbb{R}^3$  such that:

- $\mathbf{m} \in L^\infty(\mathbb{R}^+; H^1(\Omega_\eta)) \cap C^0(\mathbb{R}^+; L^2(\Omega_\eta))$ ,
- $\partial_t \mathbf{m}^\eta \in L^2(\mathbb{R}^+ \times \Omega_\eta)$ ,
- $|\mathbf{m}^\eta| = 1$  a.e.,
- $\mathbf{m}^\eta(0, \cdot) = \mathbf{m}_0$ ,
- For all  $T \geq 0$ , for all  $\Phi \in L^\infty([0, T]; H^1(\Omega_\eta; \mathbb{R}^3))$ ,

$$\begin{aligned} & \int_{[0, T] \times \Omega_\eta} \left( \frac{\partial \mathbf{m}^\eta}{\partial t} - \alpha \mathbf{m}^\eta \times \frac{\partial \mathbf{m}^\eta}{\partial t} \right) \cdot \Phi dt d\mathbf{x} = \\ & (1 + \alpha^2) \int_{[0, T] \times \Omega_\eta} \sum_{i=1}^3 \ell^2 \left( \mathbf{m}^\eta \times \frac{\partial \mathbf{m}^\eta}{\partial \mathbf{x}_i} \right) \cdot \frac{\partial \Phi}{\partial \mathbf{x}_i} dt d\mathbf{x} \quad (8) \end{aligned}$$

$$-(1 + \alpha^2) \int_{[0, T] \times \Omega_\eta} \mathbf{m}^\eta \times (\mathbf{h}_d(\mathbf{m}^\eta) + \mathbf{h}_a) \cdot \Phi dt d\mathbf{x},$$

- for all  $t \geq 0$ ,

$$\mathcal{E}_\eta(\mathbf{m}^\eta(t)) + \frac{\alpha}{1 + \alpha^2} \int_0^t \int_{\Omega_\eta} \left| \frac{\partial \mathbf{m}^\eta}{\partial t}(\tau, \mathbf{x}) \right|^2 d\tau d\mathbf{x} \leq \mathcal{E}_\eta(\mathbf{m}_0), \quad (9)$$

where  $\mathcal{E}_\eta$  is given by (6).

## 2.2. Statement of the asymptotic result

As in Refs. 9,10, by rescaling in the transverse variable  $(y, z)$ , we aim to obtain a one-dimensional asymptotic model of ferromagnetic notched wire:

**Proposition 1.** *Let  $m_0 \in H^1([-L, L]; \mathbb{R}^3)$  satisfy  $|m_0(x)| = 1$  for all  $x$ . For  $(x, y, z) \in \Omega_\eta$ , we set  $\mathbf{m}_0(x, y, z) = m_0(x)$ , and we consider  $\mathbf{m}^\eta$  the weak solution for (7) with initial data  $\mathbf{m}_0$ . We define  $m^\eta : \mathbb{R}^+ \times \Omega_1 \rightarrow \mathbb{R}^3$  by  $m^\eta(t, x, y, z) = \mathbf{m}^\eta(t, x, \eta y, \eta z)$ .*

Then, when  $\eta$  tends to zero, we can extract a subsequence still denoted  $(m^\eta)_\eta$  such that  $m^\eta$  tends to  $m$  in  $L^\infty(0, T; H^1(\Omega_1))$  weak \*. In addition,  $m$  does not depend on  $y$  and  $z$  and satisfies:

- $m \in L^\infty(\mathbb{R}^+; H^1([-L, L])) \cap \mathcal{C}^0(\mathbb{R}^+; L^2([-L, L]))$ ,
- $\partial_t m \in L^2(\mathbb{R}^+ \times [-L, L])$ ,
- $|m| = 1$  a.e.,
- $m(0, \cdot) = m_0$ ,
- For all  $\Phi \in \mathcal{C}_c^\infty(\mathbb{R}^+; H^1([-L, L]; \mathbb{R}^3))$ ,

$$\begin{aligned} & \int_{\mathbb{R}^+ \times [-L, L]} \mathbf{a}(x) \left( \frac{\partial m}{\partial t} - \alpha m \times \frac{\partial m}{\partial t} \right) \cdot \Phi dt dx = \\ & (1 + \alpha^2) \int_{\mathbb{R}^+ \times [-L, L]} \mathbf{a}(x) \ell^2 m \times \frac{\partial m}{\partial x} \cdot \frac{\partial \Phi}{\partial x} dt dx \\ & - (1 + \alpha^2) \int_{\mathbb{R}^+ \times [-L, L]} \mathbf{a}(x) m \times \left( -\frac{1}{2} (m_2 e_2 + m_3 e_3) + \mathbf{h}_a \right) \cdot \Phi dt dx, \end{aligned} \quad (10)$$

where  $m_1, m_2, m_3$  are the coordinates of  $m$  in the canonical basis  $(e_1, e_2, e_3)$  and  $\mathbf{a}(x)$  is defined by (2),

- for all  $t \geq 0$ ,

$$\mathcal{E}(m(t)) + \frac{\alpha}{1 + \alpha^2} \int_0^t \int_{[-L, L]} \mathbf{a}(x) \left| \frac{\partial m}{\partial t}(\tau, x) \right|^2 d\tau dx \leq \mathcal{E}(m_0), \quad (11)$$

where

$$\begin{aligned} \mathcal{E}(m(t)) &= \frac{\ell^2}{2} \int_{[-L, L]} \mathbf{a}(x) |\partial_x m(t, x)|^2 dx \\ &+ \frac{1}{4} \int_{[-L, L]} \mathbf{a}(x) ((m_2(t, x))^2 + (m_3(t, x))^2) dx \\ &- \int_{[-L, L]} \mathbf{a}(x) m(t, x) \cdot \mathbf{h}_a dx. \end{aligned}$$

**Remark 1.** As already observed,<sup>9,10</sup> the equivalent demagnetizing field in the limit 1d model is local, *i.e.* its value at the point  $x$  only depends on the value of  $m$  at the point  $x$ .

### 2.3. Proof of Proposition 1

We adapt the proof detailed in Ref. 9 in the present case of notched straight nanowire.

**First step: bound for the rescaled variables.** We define the following rescaled quantities: for  $t \geq 0$  and  $(x, y, z) \in \Omega_1$ ,  $m^\eta(t, x, y, z) = \mathbf{m}^\eta(t, x, \eta y, \eta z)$  and for  $t \geq 0$  and  $(x, y, z) \in \mathbb{R}^3$ ,  $h^\eta(t, x, y, z) = \mathbf{h}(\mathbf{m}^\eta)(t, x, \eta y, \eta z)$ . By using this rescaling in the Energy Estimate (9), we obtain that for all  $t \in \mathbb{R}^+$ ,

$$\tilde{\mathcal{E}}_\eta(m^\eta)(t) + \frac{\alpha}{1 + \alpha^2} \int_0^t \int_{\Omega_1} \left| \frac{\partial m^\eta}{\partial t} \right|^2 dt d\mathbf{x} \leq \tilde{\mathcal{E}}_\eta(m_0), \quad (12)$$

where

$$\begin{aligned} \tilde{\mathcal{E}}_\eta(m^\eta)(t) &:= \frac{1}{\eta^2} \mathcal{E}_\eta(\mathbf{m}^\eta(t)) \\ &= \frac{\ell^2}{2} \int_{\Omega_1} |\partial_x m^\eta(t, \mathbf{x})|^2 d\mathbf{x} + \frac{\ell^2}{2\eta^2} \int_{\Omega_1} (|\partial_y m^\eta(t, \mathbf{x})|^2 + |\partial_z m^\eta(t, \mathbf{x})|^2) d\mathbf{x} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} |h^\eta(t, \mathbf{x})|^2 d\mathbf{x} - \int_{\Omega_1} m^\eta(t, \mathbf{x}) \cdot \mathbf{h}_a. \end{aligned} \quad (13)$$

Since  $-\mathbf{h}_a$  is an orthogonal projection for the  $L^2(\mathbb{R}^3)$ -inner product, we have:

$$\|\mathbf{h}_a(\mathbf{m}^\eta(t, \cdot))\|_{L^2(\mathbb{R}^3)}^2 \leq \|\mathbf{m}^\eta(t, \cdot)\|_{L^2(\Omega_\eta)}^2 \leq \text{meas}(\Omega_\eta)$$

as  $|\mathbf{m}^\eta| = 1$  a.e. By rescaling, we obtain that for almost every  $t \geq 0$ ,

$$\|h^\eta(t, \cdot)\|_{L^2(\mathbb{R}^3)}^2 \leq \text{meas}(\Omega_1). \quad (14)$$

In addition, using also the saturation constraint, for all  $t \geq 0$ ,

$$\left| \int_{\Omega_1} \mathbf{h}_a \cdot m^\eta(t, \mathbf{x}) d\mathbf{x} \right| \leq |\mathbf{h}_a| \text{meas}(\Omega_1). \quad (15)$$

Since  $\mathbf{m}_0$  does not depend on the rescaled variable  $(y, z)$ , using (14) and (15), we obtain that for all  $\eta > 0$ ,

$$\tilde{\mathcal{E}}_\eta(\mathbf{m}_0) \leq K := \frac{\ell^2}{2} \int_{-L}^L \mathbf{a}(x) |\partial_x m_0(x)|^2 dx + \text{meas}(\Omega_1) \left( \frac{1}{2} + 2|\mathbf{h}_a| \right).$$

So, by the rescaled energy estimate (12), we obtain that for all  $\eta > 0$  and all  $t \geq 0$ ,

$$\begin{aligned} \frac{\ell^2}{2} \|\partial_x m^\eta(t, \cdot)\|_{L^2(\Omega_1)}^2 + \frac{\ell^2}{2\eta^2} \left( \|\partial_y m^\eta(t, \cdot)\|_{L^2(\Omega_1)}^2 + \|\partial_z m^\eta(t, \cdot)\|_{L^2(\Omega_1)}^2 \right) \\ + \frac{1}{2} \|h^\eta(t, \cdot)\|_{L^2(\Omega_1)}^2 + \int_0^t \|\partial_t m^\eta\|_{L^2(\Omega_1)}^2 \leq K. \end{aligned} \quad (16)$$

This yields uniform bounds for  $\partial_x m^\eta$ ,  $\frac{1}{\eta} \partial_y m^\eta$ ,  $\frac{1}{\eta} \partial_z m^\eta$  and  $h^\eta$  in  $L^\infty(\mathbb{R}^+; L^2(\Omega_1))$ , and for  $\partial_t m^\eta$  in  $L^2(\mathbb{R}^+ \times \Omega_1)$ . So, using also the saturation constraint  $|m^\eta| = 1$  *a.e.*, we can extract a subsequence still denoted by  $m^\eta$  and  $h^\eta$  such that

$$m^\eta \rightharpoonup m \text{ in } L^\infty(\mathbb{R}^+; H^1(\Omega_1)) \text{ weak*},$$

$$\partial_y m^\eta \text{ and } \partial_z m^\eta \longrightarrow 0 \text{ in } L^\infty(\mathbb{R}^+; L^2(\Omega_1)),$$

$$\partial_t m^\eta \rightharpoonup \partial_t m \text{ in } L^2(\mathbb{R}^+; L^2(\Omega_1)) \text{ weak},$$

$$h^\eta \rightharpoonup h \text{ in } L^\infty(\mathbb{R}^+; L^2(\Omega_1)) \text{ weak*}.$$

By Aubin-Lions-Simon lemma,<sup>11</sup> we can assume that  $m^\eta$  tends strongly to  $m$  in  $\mathcal{C}^0(0, T; L^2(\Omega_1)) \cup L^\infty(0, T; L^p(\Omega_1))$  and almost everywhere, so that  $|m| = 1$  *a.e.* in  $\mathbb{R}^+ \times \Omega_1$ . In particular, we remark that the limit  $m$  does not depend on the transverse variable  $(y, z)$ .

**Second step: limit equation.** Since for all  $\eta > 0$ ,  $m^\eta(0, \cdot) = m_0$ , since  $m^\eta$  tends to  $m$  in  $\mathcal{C}^0(\mathbb{R}^+; L^2(\Omega_1))$ , we have

$$m(0, \cdot) = m_0.$$

We take in (8) a test function  $\Phi$  such that  $\partial_y \Phi = \partial_z \Phi = 0$ . We obtain that for all  $\eta$ ,

$$\begin{aligned} & \int_{[0, T] \times \Omega_1} \left( \frac{\partial m^\eta}{\partial t} - \alpha m^\eta \times \frac{\partial m^\eta}{\partial t} \right) \cdot \Phi dt d\mathbf{x} = \\ & (1 + \alpha^2) \int_{[0, T] \times \Omega_1} \ell^2 \left( m^\eta \times \frac{\partial m^\eta}{\partial x} \right) \cdot \frac{\partial \Phi}{\partial x} dt d\mathbf{x} \\ & -(1 + \alpha^2) \int_{[0, T] \times \Omega_1} m^\eta \times (h^\eta + \mathbf{h}_a) \cdot \Phi dt d\mathbf{x}. \end{aligned}$$

Since  $m^\eta \longrightarrow m$  strongly in  $L^\infty(0, T; L^2(\Omega_1))$ , since  $\partial_t m^\eta$ ,  $\partial_x m^\eta$  and  $h^\eta$  tend respectively to  $\partial_t m$ ,  $\partial_x m$  and  $h$  weakly in  $L^2(0, T \times \Omega_1)$ , we obtain that:

$$\begin{aligned}
& \int_{[0,T] \times \Omega_1} \left( \frac{\partial m}{\partial t} - \alpha m \times \frac{\partial m}{\partial t} \right) \cdot \Phi dt d\mathbf{x} = \\
& (1 + \alpha^2) \int_{[0,T] \times \Omega_1} \ell^2 \left( m \times \frac{\partial m}{\partial x} \right) \cdot \frac{\partial \Phi}{\partial x} dt d\mathbf{x} \\
& -(1 + \alpha^2) \int_{[0,T] \times \Omega_1} m \times (h + \mathbf{h}_a) \cdot \Phi dt d\mathbf{x}.
\end{aligned}$$

Since  $m$  and  $\Phi$  do not depend on  $y$  and  $z$ , denoting by  $\mathbf{h}$  the average of  $h$  in the cross section:

$$\mathbf{h}(t, x) = \frac{1}{\mathbf{a}(x)} \int_{(y,z) \in B(0,r(x))} h(t, x, y, z) dy dz$$

(we recall that  $\mathbf{a}(x)$  (resp.  $r(x)$ ) is the area (resp. the radius) of the cross section of  $\Omega_1$  at the point  $x$ ), we obtain that

$$\begin{aligned}
& \int_{[0,T] \times [-L,L]} \mathbf{a}(x) \left( \frac{\partial m}{\partial t} - \alpha m \times \frac{\partial m}{\partial t} \right) \cdot \Phi dt dx = \\
& (1 + \alpha^2) \int_{[0,T] \times [-L,L]} \ell^2 \mathbf{a}(x) m \times \frac{\partial m}{\partial x} \cdot \frac{\partial \Phi}{\partial x} dt dx \\
& -(1 + \alpha^2) \int_{[0,T] \times [-L,L]} \mathbf{a}(x) m \times (\mathbf{h} + \mathbf{h}_a) \cdot \Phi dt dx.
\end{aligned}$$

**Third step: limit for the demagnetizing field.** We aim to characterize the weak limit  $h$  of  $h^\eta$ . We proceed as in Ref. 10. By rescaling (5) in the variables  $y$  and  $z$ , the following relations hold in  $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^3)$ :

$$\partial_y h^{\eta,3} - \partial_z h^{\eta,2} = 0, \quad \partial_z h^{\eta,1} - \eta \partial_x h^{\eta,3} = 0, \quad \partial_y h^{\eta,1} - \eta \partial_x h^{\eta,2} = 0,$$

$$\eta \partial_x (h^{\eta,1} + \overline{m^{\eta,1}}) + \partial_y (h^{\eta,2} + \overline{m^{\eta,2}}) + \partial_z (h^{\eta,3} + \overline{m^{\eta,3}}) = 0. \quad (17)$$

Since  $h^\eta \rightharpoonup h$  and  $m^\eta \rightharpoonup m$  in  $\mathcal{D}'(\mathbb{R}^+ \times \Omega_1)$ , by taking the limit in  $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^3)$  when  $\eta$  tends to zero, we obtain that  $\partial_y h^1 = \partial_z h^1 = 0$ , so  $h^1 = 0$  since  $h^1 \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3))$ , and

$$\partial_y h^3 - \partial_z h^2 = 0 \quad \text{and} \quad \partial_y (h^2 + \overline{m^2}) + \partial_z (h^3 + \overline{m^3}) = 0. \quad (18)$$

Therefore,  $(h^2, h^3)$  is the 2-dimensional demagnetizing field generated by  $(m^2, m^3)$  in the plane  $(t, x) \times \mathbb{R}^2$ .



Let us prove now that for all  $t \geq 0$ ,  $h^\eta(t, \cdot) \rightharpoonup h(t, \cdot)$  weakly in  $L^2(\mathbb{R}^3)$ . We fix  $t \in \mathbb{R}^+$ . By (16), the sequence  $h^\eta(t, \cdot)$  is bounded in  $L^2(\mathbb{R}^3)$ , so we can extract a subsequence such that  $(h^{\eta_n}(t, \cdot))_n$  tends to a limit  $u$  in  $L^2(\mathbb{R}^3)$  weak. Now, System (17) also holds for a fixed  $t$  in  $\mathcal{D}'(\mathbb{R}^3)$ . So by taking the limit when  $\eta_n$  tends to zero, since  $h^{\eta_n}(t, \cdot) \rightharpoonup u$  and  $m^{\eta_n}(t, \cdot) \rightharpoonup m(t, \cdot)$  in  $\mathcal{D}'(\mathbb{R}^3)$ , we obtain that  $u^1 = 0$  and  $(u^2, u^3)$  is the 2-dimensional demagnetizing field generated by  $(m^2(t, x, \cdot), m^3(t, x, \cdot))$ . Therefore  $u = h(t, \cdot)$ . In particular, the weak limit does not depend on the extraction, so the whole sequence  $h^\eta(t, \cdot)$  tends weakly to  $h(t, \cdot)$ .

Let us characterize  $h^\perp = (h^2, h^3)$ . We denote  $m^\perp = (m^2, m^3)$ . We fix  $(t, x) \in \mathbb{R}^+ \times [-L, L]$ . By the first equation in (18), there exists  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $h_\perp(t, x) = \begin{pmatrix} \partial_y \varphi \\ \partial_z \varphi \end{pmatrix}$ , and  $\varphi$  satisfies

$$(\partial_y^2 + \partial_z^2)\varphi = -\partial_y \overline{m^2} - \partial_z \overline{m^3}.$$

Using that the laplacian kernel in  $\mathbb{R}^2$  is  $\frac{1}{2\pi} \log |Y|$ , with  $Y = (y, z)$ , we obtain that:

$$h^\perp(t, x, Y) = \frac{1}{2\pi} \int_{Z \in \partial B_2(0, r(x))} \frac{Y - Z}{|Y - Z|^2} m^\perp(t, x) \cdot \nu(Z) d\sigma(Z).$$

Writing in complex notations  $\mathfrak{y} = y + jz$ ,  $\mathfrak{m} = m^2(t, x) + jm^3(t, x)$  and  $\mathfrak{h} = h^2(t, x, Y) + jh^3(t, x, Y)$ , we have:

$$\begin{aligned} \mathfrak{h} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathfrak{y} - r(x)e^{j\theta}}{|\mathfrak{y} - r(x)e^{j\theta}|^2} \Re(\mathfrak{m}e^{-j\theta}) r(x) d\theta, \\ &= \frac{1}{2j\pi} r(x) \int_{\mathfrak{z} \in \mathcal{C}(0,1)} \frac{\mathfrak{m} + \mathfrak{z}^2 \bar{\mathfrak{m}}}{2\mathfrak{z}(\mathfrak{z}\bar{\mathfrak{y}} - r(x))} d\mathfrak{z}. \end{aligned}$$

For  $Y \in B(0, r(x))$ , the only pole of  $\mathfrak{z} \mapsto \frac{\mathfrak{m} + \mathfrak{z}^2 \bar{\mathfrak{m}}}{2\mathfrak{z}(\mathfrak{z}\bar{\mathfrak{y}} - r(x))}$  is zero and by the residue formula we obtain that  $\mathfrak{h} = -\frac{1}{2}\mathfrak{m}$ , that is

$$h(t, x, Y) = -\frac{1}{2} \begin{pmatrix} 0 \\ m^2(t, x) \\ m^3(t, x) \end{pmatrix}.$$

In particular,  $h(t, x, Y)$  does not depend on  $Y \in B(0, r(x))$ , so the average satisfies:

$$\mathbf{h}(t, x) = -\frac{1}{2} \begin{pmatrix} 0 \\ m^2(t, x) \\ m^3(t, x) \end{pmatrix}.$$

**Last step: limit energy estimate.** We have proved that for all  $t \geq 0$ ,  $h^\eta(t, \cdot)$  tends to  $h(t, \cdot)$  weakly in  $L^2(\mathbb{R}^3)$ . In addition, for all  $t$ , we have

$$\int_{\mathbb{R}^3} |h^\eta(t, \mathbf{x})|^2 d\mathbf{x} = - \int_{\Omega_1} m^\eta(t, \mathbf{x}) \cdot h^\eta(t, \mathbf{x}) d\mathbf{x}.$$

Since  $m^\eta(t, \cdot)$  tends to  $m(t, \cdot)$  in  $L^2(\Omega)$  strongly, since  $h^\eta(t, \cdot)$  tends to  $h(t, \cdot)$  weakly in  $L^2(\mathbb{R}^3)$ , we have

$$\int_{\mathbb{R}^3} |h^\eta(t, \mathbf{x})|^2 d\mathbf{x} \longrightarrow - \int_{\Omega_1} m(t, \mathbf{x}) \cdot h(t, \mathbf{x}) d\mathbf{x} = \int_{\Omega_1} \frac{1}{2} (|m_2(t, \mathbf{x})|^2 + |m_3(t, \mathbf{x})|^2) d\mathbf{x}.$$

Since  $m^\eta(t, \cdot)$  tends to  $m(t, \cdot)$  strongly in  $L^2(\Omega_1)$ , we have:

$$- \int_{\Omega_1} m^\eta(t, \mathbf{x}) \cdot \mathbf{h}_a d\mathbf{x} \longrightarrow - \int_{\Omega_1} m(t, \mathbf{x}) \cdot \mathbf{h}_a d\mathbf{x}.$$

In particular, for  $t = 0$ , using that  $m_0$  does not depend on  $y$  and  $z$ , we obtain that

$$\frac{1}{\eta^2} \mathcal{E}_\eta(\mathbf{m}_0) \longrightarrow \mathcal{E}(m_0).$$

Let us prove that for all  $t \geq 0$ ,  $m^\eta(t, \cdot) \rightharpoonup m(t, \cdot)$  weakly in  $H^1(\Omega_1)$ .

We fix  $t \geq 0$ . First, we know that  $m^\eta \rightarrow m$  in  $\mathcal{C}^0(0, T; L^2(\Omega_1))$  so  $m^\eta(t, \cdot) \rightarrow m(t, \cdot)$  strongly in  $L^2(\Omega_1)$ . In addition, by the rescaled energy estimate (16),  $\nabla m^\eta(t, \cdot)$  is bounded in  $L^2(\Omega_1)$ . Thus for all subsequence  $\nabla m^{\eta_n}$ , we can extract a sub sequence which tends weakly in  $L^2(\Omega)$  to a limit  $w$ . By uniqueness of the limit in  $\mathcal{D}'(\Omega_1)$ ,  $w = \nabla m(t, \cdot)$ . This limit does not depend on the subsequence, so all the sequence tends to this limit.

Therefore,  $\partial_x m^\eta(t, \cdot)$  tends to  $\partial_x m(t, \cdot)$  in  $L^2(\Omega_1)$  weak. So, by classical convexity arguments, for all  $t$ ,

$$\int_{\Omega_1} |\partial_x m(t, \mathbf{x})|^2 d\mathbf{x} \leq \liminf \int_{\Omega_1} |\partial_x m^\eta(t, \mathbf{x})|^2 d\mathbf{x}.$$

In addition, using also that  $\partial_t m^\eta$  tends weakly to  $\partial_t m$  in  $L^2(0, t; L^2(\Omega_1))$ , we have by weak semicontinuity arguments that

$$\int_0^t \int_{\Omega_1} |\partial_t m(t, x)|^2 dx dt \leq \liminf \int_0^t \int_{\Omega_1} |\partial_t m^\eta(t, \mathbf{x})|^2 d\mathbf{x} dt.$$

Thus, since the limit  $m$  does not depend on the transverse variable, we obtain that for all  $t \geq 0$ ,

$$\mathcal{E}(m(t, \cdot)) + \int_0^t \int_{[-L, L]} \mathbf{a}(x) |\partial_t m(t, x)|^2 dx dt \leq \mathcal{E}(m_0).$$

This concludes the proof of Theorem 1.  $\square$

### 2.4. Landau-Lifschitz 1d model

The limit LLG equation obtained in Proposition 1 is the following:

$$\begin{cases} \partial_t m - \alpha m \times \partial_t m = -(1 + \alpha^2)m \times \mathcal{H}, \\ \mathcal{H} = \frac{1}{\mathbf{a}} \ell^2 \partial_x (\mathbf{a} \partial_x m) + \frac{1}{2}(m_2 e_2 + m_3 e_3) + \mathbf{h}_a, \end{cases} \quad (19)$$

with the boundary condition:

$$\partial_x m(t, -L) = \partial_x m(t, L) = 0 \text{ for all } t. \quad (20)$$

This system is formally equivalent to the following 1d Landau-Lifschitz equation:

$$\begin{cases} \partial_t m = -m \times \mathcal{H} - \alpha m \times (m \times \mathcal{H}), \\ \mathcal{H} = \frac{1}{\mathbf{a}} \ell^2 \partial_x (\mathbf{a} \partial_x m) - \frac{1}{2}(m_2 e_2 + m_3 e_3) + \mathbf{h}_a, \end{cases} \quad (21)$$

with the boundary condition (20). Thereafter, we will deal with strong solutions, that is solutions so that for all  $t$ ,  $x \mapsto \mathbf{a} \partial_x m$  is in  $H^1([-L, L])$ . In particular, this fact implies that  $m$  satisfies the jump condition:

$$[m] = [\mathbf{a} \partial_x m] = 0 \text{ at the points } -l \text{ and } l. \quad (22)$$

### 3. Static DW in infinite notched nanowire

We consider in this section the infinite wire case ( $L = +\infty$ ) with vanishing applied field modeled by the system:

$$\begin{cases} \partial_t m - \alpha \partial_t m = -(1 + \alpha^2)m \times \mathcal{H}(m), \\ \mathcal{H}_0(m) = \frac{1}{\mathbf{a}} \ell^2 \partial_x (\mathbf{a} \partial_x m) - \frac{1}{2}(m_2 e_2 + m_3 e_3). \end{cases} \quad (23)$$

In this section, we look for a static solution  $\mathbf{m}_0 : \mathbb{R} \rightarrow S^2$  describing one DW between a left-hand-side  $-e_1$  domain and a right-hand-side  $+e_1$  domain. This stationary solution satisfies:

$$\mathbf{m}_0 \times \mathcal{H}_0(\mathbf{m}_0) = 0,$$

and if we look for  $\mathbf{m}_0$  on the form  $\mathbf{m}_0 = \begin{pmatrix} \sin \theta_0 \cos \phi \\ \cos \theta_0 \cos \phi \\ \sin \phi \end{pmatrix}$ , we obtain that

$(\theta_0, \phi)$  satisfies:

$$\begin{cases} \ell^2 \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x \theta_0) - \ell^2 (\partial_x \phi)^2 \cos \theta_0 \sin \theta_0 + \frac{1}{2} \sin \theta_0 \cos \theta_0 = 0, \\ \ell^2 \partial_x (\mathbf{a} \cos \theta_0 \partial_x \phi) = 0. \end{cases}$$

In order to describe a solution with one DW, we assume that  $\theta_0$  tends to  $-\frac{\pi}{2}$  (resp.  $+\frac{\pi}{2}$ ) when  $x$  tends to  $-\infty$  (resp.  $+\infty$ ). From these limit conditions, the second equation above imply that  $\mathbf{a} \cos \theta \partial_x \phi$  vanishes on  $\mathbb{R}$ , and since  $\theta$  is not constant, we obtain that  $\partial_x \phi = 0$ . So,  $\phi$  is constant on  $\mathbb{R}$  and  $\theta_0$  satisfies:

$$\ell^2 \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x \theta_0) + \frac{1}{2} \sin \theta_0 \cos \theta_0 = 0. \quad (24)$$

We remark that by invariance of the system by rotation around the axis  $e_1$ , we can assume that  $\phi = 0$ . We establish the following theorem.

**Theorem 1.** *For all  $l > 0$ , there exists a continuous, odd, increasing map  $\theta_0 : \mathbb{R} \rightarrow \mathbb{R}$ , tending to  $-\frac{\pi}{2}$  (resp.  $+\frac{\pi}{2}$ ) when  $x$  tends to  $-\infty$  (resp.  $+\infty$ ), and such that  $\mathbf{m}_0 : \mathbb{R} \rightarrow S^2$  given by*

$$\mathbf{m}_0(x) = \begin{pmatrix} \sin \theta_0(x) \\ \cos \theta_0(x) \\ 0 \end{pmatrix}$$

*is a static solution for (23) with the jump conditions (22).*

**Proof.** As already said, if  $x \mapsto \mathbf{m}(x) = \begin{pmatrix} \sin \theta(x) \\ \cos \theta(x) \\ 0 \end{pmatrix}$  is a static solution of (23), we have:

$$\ell^2 \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x \theta) + \frac{1}{2} \sin \theta \cos \theta = 0 \text{ in } \mathcal{D}'(\mathbb{R}). \quad (25)$$

We assume that  $\theta$  is odd, so  $\theta(0) = 0$ . We will prove the existence of  $\theta_0$  by a shooting method, *i.e.* we will find  $p_0$  such that the solution of the Cauchy problem

$$\begin{cases} \ell^2 \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x \theta_0) + \frac{1}{2} \sin \theta \cos \theta = 0, \\ \theta_0(0) = 0, \quad \theta'_0(0) = p_0, \end{cases}$$

satisfies the desired limit conditions:  $\theta_0(x) \rightarrow \frac{\pi}{2}$  when  $x$  tends to  $+\infty$ .

By (25), the quantity  $\mathcal{F}(x) := \ell^2 (\theta'(x))^2 + \frac{1}{2} (\sin^2 \theta(x))$  is constant in each interval  $[0, l]$  and  $]l, +\infty[$ . We denote  $p_0 = \theta'(0)$ , so that  $\theta$  is solution on  $[0, l]$  of the Cauchy problem:

$$\begin{cases} \ell^2 \theta'' + \frac{1}{2} \sin \theta \cos \theta = 0, \\ \theta(0) = 0, \quad \theta'(0) = p_0. \end{cases} \quad (26)$$

From the conservation of  $\mathcal{F}$ , we have:

$$\forall x \in [0, l[, \mathcal{F}(x) = \mathcal{F}(0) = \ell^2(p_0)^2,$$

so, denoting by  $\theta'(l^-)$  the limit of  $\theta'(x)$  when  $x$  tends to  $l$  with  $x < l$ , we have:

$$\theta'(l^-) = \left( (p_0)^2 - \frac{1}{2\ell^2} \sin^2 \theta(l) \right)^{\frac{1}{2}}. \quad (27)$$

From the jump conditions (22),  $[\mathbf{m}] = [\mathbf{a}\partial_x \mathbf{m}] = 0$  at  $l$ , so denoting by  $\theta'(l^+)$  the limit of  $\theta'(x)$  when  $x$  tends to  $l$  with  $x > l$ , we have  $\pi(r_1)^2 \theta'(l^-) = \pi(r_1)^2 \theta'(l^+)$ , that is

$$\gamma \theta'(l^-) = \theta'(l^+),$$

where  $\gamma$  is the ratio of the cross-section areas:

$$\gamma = \frac{(r_1)^2}{(r_0)^2}. \quad (28)$$

In particular, by continuity of  $\theta$  and using (27), we have:

$$\theta'(l^+) = \gamma \left( (p_0)^2 - \frac{1}{2\ell^2} \sin^2 \theta(l) \right)^{\frac{1}{2}}. \quad (29)$$

On  $]l + \infty[, \theta$  satisfies the pendulum equation and tends to  $\frac{\pi}{2}$  when  $x$  tends to  $+\infty$ , so the trajectory  $x \mapsto (\theta(x), \theta'(x))$  is on the separatrix of the phase portrait, *i.e.* the curve  $p = \frac{1}{\ell\sqrt{2}} \cos \theta$ . In particular,

$$\forall x \in ]l, +\infty[, \mathcal{F}(x) = \frac{1}{2},$$

and taking the limit when  $x$  tends to  $l$ ,  $x > l$ , using (29), we obtain that

$$\gamma^2 \left( \ell^2(p_0)^2 - \frac{1}{2} \sin^2 \theta(l) \right) + \frac{1}{2} \sin^2(\theta(l)) = \frac{1}{2}.$$

Therefore,

$$\theta(l) = \arcsin \sqrt{\frac{1 - 2\gamma^2 \ell^2(p_0)^2}{1 - \gamma^2}}.$$

On  $[0, l]$ , we have  $\theta'(x) = \sqrt{\ell^2(p_0)^2 - \frac{1}{2} \sin^2 \theta(x)}$ . Dividing this equality by its second member and integrating in  $x$  between 0 and  $l$ , we obtain that

$$l = \int_0^{\theta(l)} \frac{du}{\sqrt{\ell^2(p_0)^2 - \frac{1}{2} \sin^2 u}},$$

and by the expression of  $\theta(l)$  above, we obtain that  $p_0$  and  $l$  are linked by the equality  $l = \mathcal{G}(p_0)$  with:

$$\mathcal{G}(p_0) = \int_0^{\arcsin \sqrt{\frac{1-2\gamma^2 \ell^2 (p_0)^2}{1-\gamma^2}}} \frac{du}{\sqrt{\ell^2 (p_0)^2 - \frac{1}{2} \sin^2 u}}.$$

The map  $\mathcal{G}$  is decreasing on  $]\frac{1}{\sqrt{2}}, \frac{1}{\ell\gamma\sqrt{2}}]$ ,  $\mathcal{G}(\frac{1}{\gamma\sqrt{2}}) = 0$  and  $\mathcal{G}(p_0)$  tends to  $+\infty$  when  $p_0$  tends to  $\frac{1}{\sqrt{2}}$ . Therefore, for all  $l > 0$ , there exists  $p_0 \in ]\frac{1}{\sqrt{2}}, \frac{1}{\gamma\sqrt{2}}]$  such that  $\mathcal{G}(p_0) = l$ . This  $p_0$  being given, we define  $\theta_0 : [0, l[ \rightarrow \mathbb{R}$  by solving the Cauchy problem:

$$\begin{cases} \theta_0(0) = 0, \theta'_0(0) = p_0, \\ \ell^2 \theta''_0 + \frac{1}{2} \sin \theta_0 \cos \theta_0 = 0. \end{cases}$$

We extend  $\theta_0$  on  $[l, +\infty[$  by solving the Cauchy problem:

$$\begin{cases} \ell^2 \theta''_0 + \frac{1}{2} \sin \theta_0 \cos \theta_0 = 0 \\ \theta_0(l) \text{ given by the resolution on } [0, l], \theta'_0(l) = \frac{1}{\ell\sqrt{2}} \cos \theta_0(l). \end{cases}$$

We extend now  $\theta_0$  in  $\mathbb{R}$  by oddness. This  $\theta_0$  is solution of our problem.  $\square$

#### 4. Stability of Domain Walls

In this section, we consider the model (23). We study the stability of the static solution  $\mathbf{m}_0$  built in the previous part for this model:

$$\mathbf{m}_0(x) = \begin{pmatrix} \sin \theta_0(x) \\ \cos \theta_0(x) \\ 0 \end{pmatrix}.$$

We aim to establish that the DW is pinned at the notch. We remark that the model is invariant by rotation around the wire axis: if  $m$  satisfies (23)-(22), then for all  $\psi \in \mathbb{R}$ ,  $(t, x) \mapsto R_\psi m(t, x)$  is also solution for (23)-(22), where

$$R_\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{pmatrix}.$$

Because of this invariance, we have no hope to obtain the asymptotic stability for  $\mathbf{m}_0$ . We will establish that  $\mathbf{m}_0$  is asymptotically stable modulo rotation. For the applications, the important point is that the DW position is asymptotically stable.

We will deal with strong solutions  $m$  for (23) such that:

- $m \in \mathcal{C}^0(\mathbb{R}_t^+ \times \mathbb{R}_x; \mathbb{R}^3)$  satisfies  $|m(t, x)| = 1$  for all  $(t, x)$ ,
- $m - \mathbf{m}_0 \in L^\infty(\mathbb{R}^+; H^1(\mathbb{R}))$  and  $\partial_t m \in L^2(\mathbb{R}^+; L^2(\mathbb{R}))$ ,
- $\mathbf{a} \partial_x m \in L^2(\mathbb{R}^+; H^1(\mathbb{R}))$ .

We prove the following stability result:

**Theorem 2.** *Let  $\mathbf{m}_0$  be the static solution given by Theorem 1. For all  $\varepsilon > 0$ , there exists  $\eta_0 > 0$  such that for all  $m$  being a strong solution for (23), if the initial data satisfy  $\|\mathbf{m}_0 - m(0, \cdot)\|_{H^1(\mathbb{R})} \leq \eta_0$ , then for all  $t \geq 0$ ,*

$$\|\mathbf{m}_0 - m(t, \cdot)\|_{H^1(\mathbb{R})} \leq \varepsilon,$$

and there exists  $\psi_\infty \in \mathbb{R}$  such that

$$\|R_{\psi_\infty} \mathbf{m}_0 - m(t, \cdot)\|_{H^1(\mathbb{R})} \text{ tends to 0 when } t \text{ tends to } +\infty.$$

**Remark 2.** The DW position is asymptotically stable, *i.e.* the DW is pinned at the notch. Without notch, as observed in Ref. 10, we only obtain the asymptotic stability modulo rotation and translation in the  $x$ -variable, so that DW drift is possible. In the case of a smoothly-notched nanowire (*i.e.* if  $x \mapsto r(x)$  is at least  $\mathcal{C}^1$ ), similar results are obtained in Ref. 12.

The proof of Theorem 2 is divided in five steps. First, in order to deal with perturbations of  $\mathbf{m}_0$  satisfying the saturation constraint, we write the Landau-Lifschitz System in a convenient system of spherical coordinates (see section 4.1). Because of the invariance of (23) by rotation around the wire axis  $\mathbb{R}e_1$ , 0 is an eigenvalue of the linearization. To overcome this difficulty, in Section 4.2, we write the perturbations  $m$  of  $\mathbf{m}_0$  as a time-dependent rotation of  $\mathbf{m}_0$  plus a perturbation  $V$  belonging to the orthogonal of the linearization kernel. We conclude the proof by variational estimates that yield that  $V$  tends exponentially fast to zero, so that  $m$  tends to a small rotation of  $\mathbf{m}_0$ . These variational estimates performed in Section 4.4 are obtained after proving coercivity for the linearization (see Section 4.3). The technical non-linear-terms estimates are postponed in Section 4.5.

We endow  $L^2(\mathbb{R})$  with the weighted inner product defined by:

$$\langle u|v \rangle_{\mathbf{a}} = \int_{x \in \mathbb{R}} \mathbf{a}(x) u(x) v(x) dx.$$

We denote by  $\|\cdot\|_{L_{\mathbf{a}}^2}$  the associated norm:

$$\|u\|_{L_{\mathbf{a}}^2} = \left( \int_{\mathbb{R}} \mathbf{a}(x) |u(x)|^2 dx \right)^{\frac{1}{2}}.$$

We denote by  $\mathbf{H}^2$  the set:

$$\mathbf{H}^2 = \{u \in H^1(\mathbb{R}), \mathbf{a}\partial_x u \in H^1(\mathbb{R})\},$$

endowed with the norm:

$$\|u\|_{\mathbf{H}^2} = \left( \|u\|_{H^1(\mathbb{R})}^2 + \|\partial_x(\mathbf{a}\partial_x u)\|_{L^2(\mathbb{R})}^2 \right)^{\frac{1}{2}}.$$

We remark that  $u \in \mathbf{H}^2$  if and only if  $u \in H^1(\mathbb{R})$ , in each interval  $I = ]-\infty, -l[, ]-l, l[$  or  $]l, +\infty[$ ,  $u|_I \in H^2(I)$ , and  $u$  satisfies the jump condition  $[\mathbf{a}\partial_x u] = 0$  at the points  $-l$  and  $l$ .

#### 4.1. Coordinates on the sphere.

We use the following system of spherical coordinates to describe the perturbations of  $\mathbf{m}_0$ : we define  $\mathcal{M}$  by

$$\begin{aligned} \mathcal{M} : ]-\pi, \pi[ \times ]-\frac{\pi}{2}, \frac{\pi}{2}[ &\rightarrow S^2 \setminus \Gamma \\ (\theta, \varphi) &\mapsto \begin{pmatrix} \sin \theta \cos \varphi \\ \cos \theta \cos \varphi \\ \sin \varphi \end{pmatrix}, \end{aligned}$$

where  $\Gamma = \{(0, y, z), y \leq 0, y^2 + z^2 = 1\}$ . The map  $\mathcal{M}$  is a  $\mathcal{C}^\infty$ -diffeomorphism. Since  $\mathbf{m}_0 = \mathcal{M}(\theta_0, 0)$  takes its values far from  $\Gamma$ , a small perturbation  $m$  of  $\mathbf{m}_0$  will take its values far from  $\Gamma$  and the norm of  $\mathbf{m}_0 - m$  can be controlled by the norms of  $\theta_0 - \theta$  and  $\varphi$ .

There exists  $\alpha_1 > 0$  and  $\alpha_2 > 0$  such that for all  $m = \mathcal{M}(\theta, \varphi) \in H_{loc}^1(\mathbb{R}; S^2)$  satisfying  $\|\theta - \theta_0\|_{L^\infty(\mathbb{R})} \leq \frac{\pi}{4}$  and  $\|\varphi\|_{L^\infty(\mathbb{R})} \leq \frac{\pi}{4}$ , then

$$\alpha_1 \|m - \mathbf{m}_0\|_{H^1(\mathbb{R})} \leq \|\theta - \theta_0\|_{H^1(\mathbb{R})} + \|\varphi\|_{H^1(\mathbb{R})} \leq \alpha_2 \|m - \mathbf{m}_0\|_{H^1(\mathbb{R})},$$

$$\text{and } \alpha_1 \|m - \mathbf{m}_0\|_{\mathbf{H}^2} \leq \|\theta - \theta_0\|_{\mathbf{H}^2} + \|\varphi\|_{\mathbf{H}^2} \leq \alpha_2 \|m - \mathbf{m}_0\|_{\mathbf{H}^2}.$$

We write now a time-dependent perturbation  $m$  of  $\mathbf{m}_0$  on the form

$$m(t, x) = \mathcal{M}(\theta(t, x), \varphi(t, x)),$$

where  $(\theta, \varphi)$  is a perturbation of  $(\theta_0, 0)$ . Plugging  $m$  in (23), we obtain that  $m$  satisfies (23) if and only if  $U = (\theta, \varphi)$  satisfies:

$$\partial_t U = \mathcal{F}(U), \tag{30}$$

where the operator  $\mathcal{F}$  is given by

$$\mathcal{F}(U) = A(\varphi) \frac{1}{\mathbf{a}} \partial_x(\mathbf{a} \partial_x U) + B_1(\varphi) (\partial_x \theta)^2 + B_2(\varphi) \partial_x \theta \partial_x \varphi + C(U) \tag{31}$$

with



$$\begin{aligned}
\bullet \quad A(\varphi) &= \ell^2 \begin{pmatrix} \alpha & -\frac{1}{\cos \varphi} \\ \cos \varphi & \alpha \end{pmatrix}, \\
\bullet \quad B_1(\varphi) &= \ell^2 \begin{pmatrix} -\sin \varphi \\ \alpha \cos \varphi \sin \varphi \end{pmatrix} \text{ and } B_2(\varphi) = -2\ell \begin{pmatrix} \alpha \tan \varphi \\ \sin \varphi \end{pmatrix}, \\
\bullet \quad C(\theta, \varphi) &= \begin{pmatrix} \frac{1}{2} \sin^2 \theta \sin \varphi + \frac{\alpha}{2} \cos \theta \sin \theta \\ -\frac{\alpha}{2} \sin^2 \theta \cos \varphi \sin \varphi + \frac{1}{2} \cos \theta \sin \theta \cos \varphi \end{pmatrix}.
\end{aligned}$$

In this new system of coordinates, the jump condition (22) writes:

$$[\theta] = [\varphi] = [\mathbf{a}\partial_x \theta] = [\mathbf{a}\partial_x \varphi] = 0 \text{ at the points } -l \text{ and } l. \quad (32)$$

While  $\|\theta - \theta_0\|_{L^\infty(\mathbb{R})} \leq \frac{\pi}{4}$  and  $\|\varphi\|_{L^\infty(\mathbb{R})} \leq \frac{\pi}{4}$ , Equation (30) is equivalent to (21) and the stability of  $\mathbf{m}_0$  for (21) is equivalent to the stability of  $(\theta_0, 0)$  for (30).

#### 4.2. Invariance by rotation.

In order to deal with the invariance by rotation, we already know that for all  $\psi \in \mathbb{R}$ ,  $x \mapsto R_\psi \mathbf{m}_0(x)$  is a static solution for (23)-(22). By writing this solution in the new coordinates, for  $\psi \in \mathbb{R}$  small enough, we define  $\mathcal{K}(\psi) : x \mapsto (\Theta(\psi)(x), \Phi(\psi)(x)) \in \mathbb{R}^2$  by:

$$\mathcal{M}(\Theta(\psi)(x), \Phi(\psi)(x)) = R_\psi \begin{pmatrix} \sin \theta_0(x) \\ \cos \theta_0(x) \\ 0 \end{pmatrix}, \quad (33)$$

which is a stationary solution for (30)-(32).

For  $\psi$  small enough and for all  $x \in \mathbb{R}$ ,  $R_\psi \begin{pmatrix} \sin \theta_0(x) \\ \cos \theta_0(x) \\ 0 \end{pmatrix}$  remains in  $S^2 \setminus \Gamma$  in which  $\mathcal{M}^{-1}$  is smooth, so  $\mathcal{K}$  is smooth for  $\psi$  small enough and  $x \in \mathbb{R} \setminus \{-l, l\}$ . By differentiating (33) with respect to  $\psi$ , we obtain that

$$\partial_\psi \mathcal{K}(\psi) = \cos \theta_0 \begin{pmatrix} \sin \psi \cos \Theta(\psi) \\ \cos \Theta(\psi) \sin \Phi(\psi) \sin \psi + \cos \Phi(\psi) \cos \psi \end{pmatrix}.$$

In particular,

$$\partial_\psi \mathcal{K}(0) = \begin{pmatrix} 0 \\ \cos \theta_0 \end{pmatrix}.$$

For  $U(t) = (\theta(t), \varphi(t))$  in a neighborhood of  $(\theta_0, 0)$ , we split  $U$  as:

$$U(t, x) = \mathcal{K}(\Psi(t))(x) + V(t, x), \quad (34)$$

where  $V = (v_1, v_2)$  satisfies the orthogonality condition:

$$\forall t \in \mathbb{R}^+, \quad \langle v_2(t, \cdot) | \cos \theta_0 \rangle_{\mathbf{a}} = 0.$$

We claim that if  $\|U(t, \cdot) - (\theta_0, 0)\|_{H^1}$  is small enough, the couple  $(\Psi(t), V(t, \cdot))$  is uniquely determined. Indeed, taking the inner product of  $U(t, \cdot)$  with  $(0, \cos \theta_0)$ , we obtain that

$$\langle u_2(t, \cdot) | \cos \theta_0 \rangle_{\mathbf{a}} = \langle \Phi(\Psi(t)) | \cos \theta_0 \rangle_{\mathbf{a}}. \quad (35)$$

We introduce the map  $f : \psi \mapsto \langle \Phi(\psi) | \cos \theta_0 \rangle_{\mathbf{a}}$ . We have  $f(0) = 0$  and  $f'(0) = \langle \cos \theta_0 | \cos \theta_0 \rangle_{\mathbf{a}} \neq 0$ . So there exist  $a_1$  and  $a_2$ ,  $a_1 < 0 < a_2$ , there exists  $a_3 > 0$ , such that  $f$  is a  $\mathcal{C}^1$ -diffeomorphism from  $]a_1, a_2[$  into  $] -a_3, a_3[$ . We introduce  $\nu_0 > 0$  such that if  $|\Psi(t)| \leq \nu_0$  and  $\|V(t, \cdot)\|_{H^1(\mathbb{R})} \leq \nu_0$ , then on the one hand, the second coordinate  $u_2$  of  $U = \mathcal{K}(\Psi) + V$  satisfies  $|\langle u_2(t, \cdot) | \cos \theta_0 \rangle_{\mathbf{a}}| < a_3$  (so that we remain on the domain of validity of the coordinates  $(\Psi, V)$ ), and on the other hand,  $\|u_1 - \theta_0\|_{L^\infty} \leq \frac{\pi}{4}$ , and  $\|u_2\|_{L^\infty} \leq \frac{\pi}{4}$ , so that  $U$  remains in the domain of validity of Equation (30).

We will deal now with the unknown  $(V, \Psi)$ , where  $V \in L^\infty(\mathbb{R}^+; (H^1(\mathbb{R}))^2) \cup \mathcal{C}^0(\mathbb{R}^+; (L^2(\mathbb{R}))^2)$  and  $\Psi \in \mathcal{C}^1(\mathbb{R}^+; \mathbb{R})$ . We aim to prove that if  $V(0)$  and  $\Psi(0)$  are small enough, then, on the one hand,  $V(t)$  and  $\Psi(t)$  remain small, and on the other hand,  $V(t)$  tends to zero and  $\Psi(t)$  tends to a finite limit  $\Psi_\infty$  when  $t$  tends to  $+\infty$ .

In order to write the system verified by  $(V, \Psi)$ , we plug (34) in (30), and we obtain:

$$\partial_t V + \partial_\psi \mathcal{K}(\Psi) \frac{d\Psi}{dt} = \mathcal{F}(\mathcal{K}(\Psi) + V).$$

Taking the inner product of this equation with  $\begin{pmatrix} 0 \\ \cos \theta_0 \end{pmatrix}$ , we get:

$$\langle \partial_\psi \Phi(\Psi) | \cos \theta_0 \rangle_{\mathbf{a}} \frac{d\Psi}{dt} = \langle \mathcal{F}_2(\mathcal{K}(\Psi) + V) | \cos \theta_0 \rangle_{\mathbf{a}},$$

where  $\mathcal{F}_2$  is the second coordinate of  $\mathcal{F}$ . We remark that since  $\partial_\psi \Phi(0) = \cos \theta_0$ , then for  $\Psi(t)$  small enough,  $\langle \partial_\psi \Phi(\Psi(t)) | \cos \theta_0 \rangle_{\mathbf{a}} \neq 0$ , so

$$\frac{d\Psi}{dt} = \mathcal{H}(\Psi, V) := \frac{\langle \mathcal{F}_2(\mathcal{K}(\Psi) + V) | \cos \theta_0 \rangle_{\mathbf{a}}}{\langle \partial_\psi \Phi(\Psi) | \cos \theta_0 \rangle_{\mathbf{a}}}, \quad (36)$$

and by subtraction, we will obtain the equation for  $V$ :

$$\partial_t V = \mathcal{F}(\mathcal{K}(\Psi) + V) - \mathcal{H}(\Psi, V) \partial_\psi \mathcal{K}(\Psi).$$

We remark that for a fixed  $\psi$ ,  $x \mapsto \mathcal{K}(\psi)$  is a static solution to (30). Thus for all  $t \geq 0$ ,  $\mathcal{F}(\mathcal{K}(\Psi(t))) = 0$ . So in the Taylor expansion at order one of  $\mathcal{F}$  around  $\mathcal{K}(\Psi(t))$ , the terms are at least linear in  $V$ . We define  $\Lambda_\psi$  by:

$$\Lambda_\psi V = A(\Phi(\Psi)) \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x V) + (\partial_x \Theta(\Psi))^2 B_1'(\Phi(\Psi)) v_2 + dC(\mathcal{K}(\Psi))(V), \quad (37)$$

Writing

$$\mathcal{F}(\mathcal{K}(\Psi) + V) = \Lambda_\Psi V + \mathcal{G}(\Psi, V),$$

we obtain that  $V$  satisfies:

$$\partial_t V = \Lambda_0 V + (\Lambda_\Psi - \Lambda_0) V + \mathcal{G}(\Psi, V) - \mathcal{H}(\Psi, V) \partial_\psi \mathcal{K}(\Psi), \quad (38)$$

where:

$$\Lambda_0(V) = \begin{pmatrix} -\alpha & 1 \\ -1 & -\alpha \end{pmatrix} \begin{pmatrix} L_1 v_1 \\ L_2 v_2 \end{pmatrix}$$

with

$$\begin{aligned} L_1 v_1 &= -\ell^2 \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x v_1) + \frac{1}{2} (\sin^2 \theta_0 - \cos^2 \theta_0) v_1, \\ L_2 v_2 &= -\ell^2 \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x v_2) + \left( \frac{1}{2} \sin^2 \theta_0 - \ell^2 (\theta_0')^2 \right) v_2. \end{aligned} \quad (39)$$

For the convenience of the reader, the exact expressions and the estimates of  $\mathcal{G}$  and  $\mathcal{H}$  are postponed in Section 4.5. System (36)-(38) remains valid while  $|\Psi(t)| \leq \nu_0$  and  $\|V\|_{H^1(\mathbb{R})} \leq \nu_0$ .

#### 4.3. Coercivity of $\Lambda_0$ .

We denote by  $\mathbf{H}^{2,\perp}$  the set of the  $v \in \mathbf{H}^2$  such that  $\langle v | \cos \theta_0 \rangle_{\mathbf{a}} = 0$ .

By integration by parts in each interval  $] -\infty, -l[$ ,  $] -l, l[$  and  $]l, +\infty[$ , using the jump condition (22), we obtain that  $L_1$  and  $L_2$  are self-adjoint for the weighted scalar product  $\langle \cdot | \cdot \rangle_{\mathbf{a}}$ . In addition, also by integration by part and using the jump conditions, we can factorize  $L_2$  since, for all  $u$  and  $v$  in  $\mathbf{H}$ :

$$\langle L_2 u | v \rangle_{\mathbf{a}} = \langle \lambda u | \lambda v \rangle_{\mathbf{a}},$$

where the order-one operator  $\lambda$  is given by:

$$\lambda u = \ell \partial_x u + \ell \theta_0' \tan \theta_0 u.$$

This factorization implies that  $L_2$  is a positive self-adjoint operator and that its kernel is one-dimensional and is generated by  $x \mapsto \cos \theta_0(x)$  (we

remark that this map satisfies the jump conditions since  $\theta_0$  does). Since  $(\frac{1}{2} \sin^2 \theta_0 - \ell^2(\theta'_0)^2)$  tends to  $\frac{1}{2}$  when  $x$  tends to  $\pm\infty$ , the essential spectrum of  $L_2$  is  $[\frac{1}{\sqrt{2}}, +\infty[$ . Let us prove that  $L_2 \geq \frac{1}{\sqrt{2}}$  on  $(\cos \theta_0)^\perp$  by showing that excepted 0, the eigenvalues of  $L_2$  are greater than  $\frac{1}{\sqrt{2}}$ . If it is not the case, let  $\beta \in ]0, \frac{1}{\sqrt{2}}]$  and  $u \in \mathbf{H}^{2,\perp}$  satisfy  $L_2 u = \beta u$ ,  $u \neq 0$ . We define  $v$  by  $v(x) = \mathbf{a}(x)(\lambda u)(x)$ . We remark that  $v \neq 0$ , since  $\lambda u = 0$  implies that  $u \in \mathbb{R} \cos \theta_0$ .

Since  $[u] = [\theta_0] = [\mathbf{a} \partial_x u] = [\mathbf{a} \theta'_0] = 0$  at the points  $-l$  and  $l$ ,  $[v] = 0$  at these points. In addition, in each interval  $] -\infty, -l[$ ,  $] -l, l[$  and  $]l, +\infty[$ , we write that  $\lambda^* \circ \lambda u = \beta u$ . We compose this relation by  $\lambda$  and we obtain that:

$$-\ell^2 \partial_{xx} v + (\ell^2(\theta'_0)^2(1 + 2 \tan^2 \theta_0) - \frac{1}{2} \sin^2 \theta_0) v = \beta v. \quad (40)$$

On  $]l, +\infty[$ , since  $\ell^2(\theta'_0)^2 + \frac{1}{2} \sin^2 \theta_0 = \frac{1}{2}$ , we obtain that

$$-\ell^2 \partial_{xx} v + (\frac{1}{2} - \beta) v = 0.$$

We have  $\frac{1}{2} - \beta > 0$ , so  $v \in L^2([l, +\infty[)$  is a linear combination of trigonometric functions, so  $v = 0$  in  $[l, +\infty[$ . In the same way,  $v = 0$  in  $] -\infty, -l[$ .

On  $[-l, l]$ , we have  $\ell^2(\theta'_0)^2 \geq \frac{1}{2} \cos^2 \theta_0$ , so

$$\ell^2(\theta'_0)^2(1 + 2 \tan^2 \theta_0) - \frac{1}{2} \sin^2 \theta_0 \geq \frac{1}{2}.$$

Therefore, with the jump condition  $[v] = 0$ , multiplying (40) by  $v$  and integrating on  $[-l, l]$ , we obtain that

$$\ell^2 \int_{[-l, l]} |\partial_x v|^2 + \frac{1}{2} \int_{[-l, l]} |v|^2 \leq \beta \int_{[-l, l]} |v|^2,$$

and since  $\beta < \frac{1}{2}$ , we conclude that  $v = 0$  on  $[-l, l]$ . Thus  $v = 0$  on  $\mathbb{R}$ , so  $\lambda u = 0$  on  $\mathbb{R}$ , and then  $u$  is collinear to  $\cos \theta_0$ , and we obtain a contradiction.

So  $L_2 \geq \frac{1}{\sqrt{2}}$  on  $(\cos \theta_0)^\perp$ , and:

$$\text{for all } u \in \mathbf{H}^{2,\perp}, \quad \langle L_2 u | u \rangle_{\mathbf{a}} \geq \frac{1}{2} \|u\|_{L_{\mathbf{a}}^2}^2. \quad (41)$$

Concerning  $L_1$ , we remark that  $L_1 = L_2 + g$  with  $g = \ell^2(\theta'_0)^2 - \frac{1}{2} \cos^2 \theta_0$ . Outside  $] -l, l[$ ,  $g = 0$  and inside  $[-l, l]$ ,  $g$  is a non negative constant  $\gamma_0$ . The essential spectrum of  $L_1$  in  $[\frac{1}{\sqrt{2}}, +\infty[$ . Let us assume that the smallest eigenvalue of  $L_1$  is zero, and let  $v \in \mathbf{H}^2$  be in the kernel of  $L_1$ , with  $v \neq 0$ . Then we have, by taking the inner product of  $L_1 v$  with  $v$  and by integration by parts:

$$0 = \langle L_1 v | v \rangle_{\mathbf{a}} = \langle L_2 v | v \rangle_{\mathbf{a}} + \int_{[-l, l]} \mathbf{a}(x) \gamma_0 |v|^2.$$

Both terms of this equality are positive so both vanish. Thus on the one hand  $\langle L_2 v | v \rangle_{\mathbf{a}} = 0$  so  $\lambda v = 0$ , *i.e.*  $v = \nu \cos \theta_0$ ,  $\nu \in \mathbb{R}$ . On the other hand,  $v$  vanishes on  $[-l, l]$ , so  $\nu = 0$ . Therefore  $v = 0$ , which leads to a contradiction. So the smallest eigenvalue of  $L_1$  is strictly positive, and there exists a constant  $c_0 > 0$  such that

$$\text{for all } v \in \mathbf{H}^2, \quad \langle L_1 v | v \rangle_{\mathbf{a}} \geq c_0 \|v\|_{L_{\mathbf{a}}^2}^2. \quad (42)$$

We denote  $\mathbf{H} = \mathbf{H}^2 \times \mathbf{H}^{2,\perp}$  and  $\mathcal{L}$  the operator defined by

$$\text{for } V = (v_1, v_2), \quad \mathcal{L}V = \begin{pmatrix} L_1 v_1 \\ L_2 v_2 \end{pmatrix}.$$

For  $U = (u_1, u_2) \in (L^2(\mathbb{R}))^2$  and  $V = (v_1, v_2) \in (L^2(\mathbb{R}))^2$ , we denote:

$$\langle U | V \rangle_{\mathbf{a}} = \langle u_1 | v_1 \rangle_{\mathbf{a}} + \langle u_2 | v_2 \rangle_{\mathbf{a}},$$

and

$$\|U\|_{L_{\mathbf{a}}^2} = \left( \|u_1\|_{L_{\mathbf{a}}^2}^2 + \|u_2\|_{L_{\mathbf{a}}^2}^2 \right)^{\frac{1}{2}}.$$

From (41) and (42), with Cauchy-Schwartz inequality, we obtain that

$$\text{for all } V \in \mathbf{H}, \quad c_0 \langle \mathcal{L}V | V \rangle_{\mathbf{a}} \leq \|\mathcal{L}V\|_{L_{\mathbf{a}}^2}^2. \quad (43)$$

We prove also the following proposition:

**Proposition 2.** *There exist  $\alpha_1 > 0$  and  $\alpha_2 > 0$  such that for all  $U \in \mathbf{H}$ ,*

$$\alpha_1 \|U\|_{H^1(\mathbb{R})} \leq (\langle \mathcal{L}U | U \rangle_{\mathbf{a}})^{\frac{1}{2}} \leq \alpha_2 \|U\|_{H^1(\mathbb{R})}$$

and

$$\alpha_1 \|U\|_{\mathbf{H}^2} \leq \|\mathcal{L}U\|_{L_{\mathbf{a}}^2} \leq \alpha_2 \|U\|_{\mathbf{H}^2}$$

**Proof.** We have

$$\langle L_1 u_1 | u_1 \rangle_{\mathbf{a}} = \int_{\mathbb{R}} \mathbf{a} \ell^2 |\partial_x u_1|^2 + \frac{1}{2} \int_{\mathbb{R}} \mathbf{a} (\sin^2 \theta_0 - \cos^2 \theta_0) u_1^2,$$

so

$$\begin{aligned} \ell^2 \|\partial_x u_1\|_{L_{\mathbf{a}}^2}^2 &\leq \langle L_1 u_1 | u_1 \rangle_{\mathbf{a}} + \frac{1}{2} \|u_1\|_{L_{\mathbf{a}}^2}^2 \\ &\leq \left(1 + \frac{1}{2c_0}\right) \langle L_1 u_1 | u_1 \rangle_{\mathbf{a}} \end{aligned}$$

using (42). This provides the claimed bound for the  $H^1$  norm. We obtain the  $H^2$  estimate writing that

$$\frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x u_1) = \frac{1}{\ell^2} \left( -L_1 u_1 + \frac{1}{2} (\sin^2 \theta_0 - \cos^2 \theta_0) u_1 \right),$$

so

$$\|\partial_x(\mathbf{a}\partial_x u_1)\|_{L_{\mathbf{a}}^2} \leq c(\|L_1\|_{L_{\mathbf{a}}^2} + \|u_1\|_{L_{\mathbf{a}}^2}).$$

Now, by Cauchy Schwarz inequality, (42) yields

$$c_0\|u_1\|_{L_{\mathbf{a}}^2}^2 \leq \langle L_1 u_1 | u_1 \rangle_{\mathbf{a}} \leq \|L_1 u_1\|_{L_{\mathbf{a}}^2} \|u_1\|_{L_{\mathbf{a}}^2},$$

so  $\|u_1\|_{L_{\mathbf{a}}^2} \leq \frac{1}{c_0} \|L_1 u_1\|_{L_{\mathbf{a}}^2}$ . This conclude the proof of the  $H^2$  estimate for  $u_1$ . Using now (41), we prove in the same way the  $H^1$  and  $H^2$  estimates for  $u_2$ .  $\square$

#### 4.4. Last Step. Variational estimates on $V$ .

We recall that System (36)-(38) is valid while  $|\Psi(t)| \leq \nu_0$  and  $\|V(t, \cdot)\|_{H^1(\mathbb{R})} \leq \nu_0$ . By Proposition 2, we introduce  $\nu_1$  such that  $0 < \nu_1 \leq \nu_0$  and such that

$$\langle \mathcal{L}V | V \rangle_{\mathbf{a}}^{\frac{1}{2}} \leq \nu_1 \implies \|V\|_{H^1(\mathbb{R})} \leq \nu_0,$$

so that System (36)-(38) is valid while  $|\Psi(t)| \leq \nu_1$  and  $\langle \mathcal{L}V(t) | V(t) \rangle_{\mathbf{a}}^{\frac{1}{2}} \leq \nu_1$ .

We take the inner product of (38) with  $\mathcal{L}V = \begin{pmatrix} L_1 v_1 \\ L_2 v_2 \end{pmatrix}$ . Since  $L_1$  and  $L_2$  are self-adjoint, we obtain that:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle \mathcal{L}V | V \rangle_{\mathbf{a}} + \alpha \|\mathcal{L}V\|_{L_{\mathbf{a}}^2}^2 &= \langle (\Lambda_{\Psi} - \Lambda_0) V + \mathcal{G}(\Psi, V) | \mathcal{L}V \rangle_{\mathbf{a}} \\ &\quad - \mathcal{H}(\Psi, V) \langle \partial_{\psi} \mathcal{K}(\Psi) | \mathcal{L}V \rangle_{\mathbf{a}}. \end{aligned}$$

The right-hand-side terms are estimated as follows:

**Proposition 3.** *There exists a constant  $c$  such that for all  $\psi \in \mathbb{R}$  and  $W \in \mathbf{H}$ , if the following hypothesis holds:*

$$|\psi| \leq \nu_1 \text{ and } \langle \mathcal{L}W | W \rangle_{\mathbf{a}}^{\frac{1}{2}} \leq \nu_1, \quad (44)$$

then

$$\|\mathcal{G}(\psi, W)\|_{L_{\mathbf{a}}^2} \leq c(\langle \mathcal{L}W | W \rangle_{\mathbf{a}}^{\frac{1}{2}} + |\psi|) \|\mathcal{L}W\|_{L_{\mathbf{a}}^2},$$

$$\|\Lambda_0 - \Lambda_{\psi}\|_{L_{\mathbf{a}}^2} \leq c|\psi| \|\mathcal{L}W\|_{L_{\mathbf{a}}^2},$$

$$|\mathcal{H}(\psi, W)| \leq c \langle \mathcal{L}W | W \rangle_{\mathbf{a}}^{\frac{1}{2}},$$

$$\text{and } |\langle \partial_{\psi} \mathcal{K}(\psi) | \mathcal{L}W \rangle_{\mathbf{a}}| \leq c|\psi| \|\mathcal{L}W\|_{L_{\mathbf{a}}^2}.$$

We postpone the proof of Proposition 3 in Section 4.5. Using this Proposition, while  $|\Psi(t)| \leq \nu_1$  and  $\langle \mathcal{L}V(t)|V(t) \rangle_{\mathbf{a}}^{\frac{1}{2}} \leq \nu_1$ , we have:

$$| \langle (\Lambda_\Psi - \Lambda_0) V | \mathcal{L}V \rangle_{\mathbf{a}} | \leq c |\Psi| \|\mathcal{L}V\|_{L_{\mathbf{a}}^2}^2,$$

$$| \langle \mathcal{G}(\Psi, V) | \mathcal{L}V \rangle_{\mathbf{a}} | \leq c (\langle \mathcal{L}V | V \rangle_{\mathbf{a}}^{\frac{1}{2}} + |\Psi|) \|\mathcal{L}V\|_{L_{\mathbf{a}}^2},$$

$$| \mathcal{H}(\Psi, V) \langle \partial_\psi \mathcal{K}(\Psi) | \mathcal{L}V \rangle_{\mathbf{a}} | \leq c^2 \langle \mathcal{L}V | V \rangle_{\mathbf{a}}^{\frac{1}{2}} |\Psi| \|\mathcal{L}V\|_{L_{\mathbf{a}}^2} \leq c^2 \frac{1}{c_0} |\Psi| \|V\|_{L_{\mathbf{a}}^2}^2.$$

Therefore, there exists  $c_1$  such that while  $|\Psi(t)| \leq \nu_1$  and  $\langle \mathcal{L}V(t)|V(t) \rangle_{\mathbf{a}}^{\frac{1}{2}} \leq \nu_1$ , then

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{L}V | V \rangle_{\mathbf{a}} + \alpha \|\mathcal{L}V\|_{L_{\mathbf{a}}^2}^2 \leq c_1 \|\mathcal{L}V\|_{L_{\mathbf{a}}^2}^2 \left( |\Psi| + \langle \mathcal{L}V | V \rangle_{\mathbf{a}}^{\frac{1}{2}} \right),$$

that is:

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{L}V | V \rangle_{\mathbf{a}} + \|\mathcal{L}V\|_{L_{\mathbf{a}}^2}^2 \left( \alpha - c_1 |\Psi| + c_1 \langle \mathcal{L}V | V \rangle_{\mathbf{a}}^{\frac{1}{2}} \right) \leq 0.$$

Let  $\nu_2$  be defined by  $\nu_2 = \min\{\frac{\alpha}{3c_1}, \nu_1\}$ . While  $|\Psi| \leq \nu_2$  and  $\langle \mathcal{L}V | V \rangle_{\mathbf{a}}^{\frac{1}{2}} \leq \nu_2$ ,

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{L}V | V \rangle_{\mathbf{a}} + \frac{\alpha}{3} \|\mathcal{L}V\|_{L_{\mathbf{a}}^2}^2 \leq 0,$$

and using (43),

$$\frac{d}{dt} \langle \mathcal{L}V | V \rangle_{\mathbf{a}} + c_0 \frac{2\alpha}{3} \langle \mathcal{L}V | V \rangle_{\mathbf{a}} \leq 0.$$

Thus, while  $|\Psi| \leq \nu_2$  and  $\langle \mathcal{L}V | V \rangle_{\mathbf{a}}^{\frac{1}{2}} \leq \nu_2$ ,

$$\langle \mathcal{L}V | V \rangle_{\mathbf{a}} \leq \langle \mathcal{L}V(0) | V(0) \rangle_{\mathbf{a}} e^{-\frac{2c_0\alpha t}{3}}. \quad (45)$$

Using again Proposition 3, this estimate yields that while  $|\Psi| \leq \nu_2$  and  $\langle \mathcal{L}V | V \rangle_{\mathbf{a}}^{\frac{1}{2}} \leq \nu_2$ ,

$$\left| \frac{d\Psi}{dt} \right| \leq c_1 \langle \mathcal{L}V(0) | V(0) \rangle_{\mathbf{a}}^{\frac{1}{2}} e^{-\frac{c_0\alpha t}{3}}, \quad (46)$$

so, by integration:

$$|\Psi(t)| \leq |\Psi(0)| + c_1 \langle \mathcal{L}V(0) | V(0) \rangle_{\mathbf{a}}^{\frac{1}{2}} \frac{3}{c_0\alpha}. \quad (47)$$

We set  $\nu_3 = \frac{\nu_2}{3} \min\{1, \frac{c_0\alpha}{3c_1}\}$ . We assume that  $|\Psi(0)| \leq \nu_3$  and  $\langle \mathcal{L}V(0) | V(0) \rangle_{\mathbf{a}}^{\frac{1}{2}} \leq \nu_3$ . Then for all times,  $|\Psi(t)|$  and  $\langle \mathcal{L}V(t) | V(t) \rangle_{\mathbf{a}}^{\frac{1}{2}}$  remain smaller than  $\nu_2$ . Indeed if it is not the case, we introduce the first time in

which one of this estimate is false. Now, on  $[0, t_1[$ , (45) and (47) remain valid, so:

$$\forall t \in [0, t_1[, \quad \langle \mathcal{L}V|V \rangle_{\mathbf{a}}^{\frac{1}{2}} \leq \frac{\nu_2}{3} \text{ and } |\Psi(t)| \leq \frac{2\nu_2}{3},$$

which leads to a contradiction by continuity when  $t$  tends to  $t_1$ .

So, (45), (46) and (47) remain valid for all time, thus  $V(t)$  tends to zero in  $H^1(\mathbb{R})$ ,  $\Psi$  remains small, and since  $\frac{d\Psi}{dt}$  is integrable (by (47)),  $\Psi(t)$  admits a finite limit when  $t$  tends to  $+\infty$ . In order to complete the proof of Theorem 1, it remains to establish Proposition 3.

#### 4.5. Nonlinear estimates.

First we write the Taylor expansion of  $\mathcal{F}(\mathcal{K}(\psi) + W)$  around  $\mathcal{K}(\psi) = (\Theta(\psi), \Phi(\psi))$ , with  $W = (w_1, w_2)$ . Using the fact that  $\mathcal{F}(\mathcal{K}(\psi)) = 0$ , we obtain that

$$\mathcal{F}(\mathcal{K}(\psi) + W) = \Lambda_\psi W + \mathcal{G}(\psi, W),$$

where  $\Lambda_\psi$  is given by:

$$\Lambda_\psi W = A(\Phi(\psi)) \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x W) + (\partial_x \Theta(\psi))^2 B_1'(\Phi(\psi)) w_2 + dC(\mathcal{K}(\psi))(W),$$

and  $\mathcal{G}$  is given by

$$\mathcal{G}(\psi, W) = T_1 + \dots + T_6, \tag{48}$$

with

$$T_1 = \overline{A}(\Phi(\psi), w_2) \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x (\mathcal{K}(\psi) + W))$$

$$T_2 = \tilde{B}_1(\Phi(\psi), w_2) (\partial_x \Theta(\psi))^2$$

$$T_3 = 2B_1(\Phi(\psi) + w_2) (\partial_x \Theta(\psi)) \partial_x w_1$$

$$T_4 = B_1(\Phi(\psi) + w_2) (\partial_x w_1)^2$$

$$T_5 = \overline{B}_2(\Phi(\psi), w_2) (\partial_x \Theta(\psi)) (\partial_x \Phi(\psi))$$

$$T_6 = B_2(\Phi(\psi) + w_2) (\partial_x w_1 \partial_x \Phi(\psi) + \partial_x \Theta(\psi) \partial_x w_2)$$

$$T_7 = B_2(\Phi(\psi) + w_2) \partial_x w_1 \partial_x w_2$$

$$T_8 = \tilde{C}(\mathcal{K}(\psi), W).$$



In the previous formula,  $\bar{A}$ ,  $\tilde{B}_1$ ,  $\bar{B}_2$  and  $\tilde{C}$  are the remainder terms arising in the Taylor expansions at order 1 or 2:

$$\bar{A}(a, b) = \int_0^1 A'(a + \tau b) b \, d\tau, \quad a \in \mathbb{R}, b \in \mathbb{R},$$

$$\tilde{B}_1(a, b) = \int_0^1 (1 - \tau) B_1''(a + \tau b) b^2 \, d\tau, \quad a \in \mathbb{R}, b \in \mathbb{R},$$

$$\bar{B}_2(a, b) = \int_0^1 B_2(a + \tau b) b \, d\tau, \quad a \in \mathbb{R}, b \in \mathbb{R},$$

$$\tilde{C}(U, V) = \int_0^1 (1 - \tau) d^2 C(U + \tau V)(V, V) \, d\tau, \quad U \in \mathbb{R}^2, V \in \mathbb{R}^2.$$

Concerning  $\mathcal{K}(\psi) = (\Theta(\psi), \Phi(\psi))$ , by differentiating (33) with respect to  $x$ , we obtain that:

$$\partial_x(\Theta(\psi)) = \theta'_0 \left( \cos \theta_0 \cos \Theta(\psi) \cos \Phi(\psi) + \sin \Theta(\psi) \cos \Phi(\psi) \cos \psi \sin \theta_0 \right)$$

$$\begin{aligned} \partial_x(\Phi(\psi)) = \theta'_0 \left( -\cos \theta_0 \sin \Phi(\psi) \sin \Theta(\psi) + \sin \psi(\psi) \cos \Theta(\psi) \cos \psi \sin \theta_0 \right. \\ \left. - \sin \psi \sin \theta_0 \cos \Phi(\psi) \right) \end{aligned}$$

So there exists  $K$  such that for all  $\psi \in [-\frac{\pi}{4}, \frac{\pi}{4}]$  and all  $x \in \mathbb{R}$

$$|\partial_x \mathcal{K}(\psi)(x)| \leq K |\theta'_0(x)|. \quad (49)$$

By differentiating (33) once again with respect to  $x$ , we obtain that

$$|\partial_x(\mathbf{a} \partial_x \mathcal{K}(\psi))(x)| \leq K (|\partial_x(\mathbf{a} \partial_x \theta_0)(x)| + |\theta'_0(x)|^2). \quad (50)$$

In addition,

$$|\mathcal{K}(\psi)(x) - \mathcal{K}(0)(x)| \leq K |\psi| \quad \text{and} \quad |\partial_x \mathcal{K}(\psi)(x) - \partial_x \mathcal{K}(0)(x)| \leq K |\theta'_0(x)| |\psi|. \quad (51)$$

Using the formulations of  $A$ ,  $B_1$ ,  $B_2$ , there exists a constant  $K$  such that under Hypothesis (44), then:

$$|A(\Phi(\psi))| \leq K, \quad |A(\Phi(\psi)) - A(0)| \leq K |\psi|,$$

$$|\bar{A}(\Phi(\psi), w_2)| \leq K |w_2| (|\psi| + |w_2|),$$

$$|B_1(\Phi(\psi) + w_2)| \leq K (|\psi| + |w_2|), \quad |B'_1(\Phi(\psi))| \leq K, \quad |\tilde{B}_1(\Phi(\psi), w_2)| \leq K |w_2|^2,$$

$$|B_2(\Phi(\psi) + w_2)| \leq K (|\psi| + |w_2|), \quad |\bar{B}_2(\Phi(\psi), w_2)| \leq K |w_2|,$$

$$|\mathrm{d}C(\mathcal{K}(\psi)) - \mathrm{d}C(\mathcal{K}(0))| \leq K|\psi|, \text{ and } |\tilde{C}(\mathcal{K}(\psi), W) \leq K|W|^2.$$

Thus, under the condition (44),

$$\begin{aligned} \|T_1\|_{L_{\mathbf{a}}^2} &\leq \|A(\Phi(\psi), w_2)\|_{L^\infty} \|\frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x (\mathcal{K}(\psi) + W))\|_{L_{\mathbf{a}}^2} \\ &\leq K \|w_2\|_{L^\infty} (|\psi| + \|w_2\|_{L^\infty}) (\|\mathcal{K}(\psi)\|_{\mathbf{H}^2} + \|W\|_{\mathbf{H}^2}) \\ &\leq K \|w_2\|_{H^1} (|\psi| + \|w_2\|_{H^1}) (K + \|V\|_{\mathbf{H}^2}) \\ &\leq K \|W\|_{\mathbf{H}^2} (|\psi| + \|W\|_{H^1}), \\ \|T_2\|_{L_{\mathbf{a}}^2} &\leq \|\tilde{B}_1(\Phi(\psi), w_2)\|_{L_{\mathbf{a}}^2} \|\partial_x \Theta(\psi)\|_{L^\infty}^2 \\ &\leq K \|w_2\|_{L_{\mathbf{a}}^2} \|w_2\|_{L^\infty} \leq K \|W\|_{H^1}^2, \\ \|T_3\|_{L_{\mathbf{a}}^2} &\leq 2 \|B_1(\Phi(\psi) + w_2)\|_{L^\infty} \|\partial_x \Theta(\psi)\|_{L^\infty} \|\partial_x w_1\|_{L_{\mathbf{a}}^2} \\ &\leq K (|\psi| + \|W\|_{L^\infty}) \|W\|_{H^1} \leq K (|\psi| + \|W\|_{H^1}) \|W\|_{H^1}, \\ \|T_4\|_{L_{\mathbf{a}}^2} &\leq \|B_1(\Phi(\psi) + w_2)\|_{L^\infty} \|\partial_x w_1\|_{L_{\mathbf{a}}^2} \|\partial_x w_1\|_{L^\infty} \leq K \|W\|_{H^1} \|W\|_{H^2} \\ \|T_5\|_{L_{\mathbf{a}}^2} &\leq \|\bar{B}_2(\Phi(\psi), w_2)\|_{L^\infty} \|\partial_x \Theta(\psi)\|_{L^\infty} \|\partial_x \Phi(\psi)\|_{L_{\mathbf{a}}^2} \\ &\leq K |\psi| \|w_2\|_{L^\infty} \leq K |\psi| \|W\|_{H^1}, \\ \|T_6\|_{L_{\mathbf{a}}^2} &\leq \|B_2(\Phi(\psi) + w_2)\|_{L^\infty} \|\partial_x w_1\|_{L_{\mathbf{a}}^2} \|\partial_x \Phi(\psi)\|_{L^\infty} \\ &\quad + \|B_2(\Phi(\psi) + w_2)\|_{L^\infty} \|\partial_x w_2\|_{L_{\mathbf{a}}^2} \|\partial_x \Theta(\psi)\|_{L^\infty} \\ &\leq K (|\psi| + \|W\|_{L^\infty}) \|W\|_{H^1} \leq K (|\psi| + \|W\|_{H^1}) \|W\|_{H^1}, \\ \|T_7\|_{L_{\mathbf{a}}^2} &\leq \|B_2(\Phi(\psi) + w_2)\|_{L^\infty} \|\partial_x w_1\|_{L_{\mathbf{a}}^2} \|\partial_x w_2\|_{L^\infty} \\ &\leq K (|\psi| + \|W\|_{H^1}) \|W\|_{\mathbf{H}^2} \|W\|_{H^1}, \\ \|T_8\|_{L_{\mathbf{a}}^2} &= \|\tilde{C}(\mathcal{K}(\psi), V)\|_{L_{\mathbf{a}}^2} \\ &\leq K \|W\|_{L^\infty} \|V\|_{L_{\mathbf{a}}^2} \leq K \|W\|_{H^1}^2. \end{aligned}$$

Therefore adding up the previous estimates and using the norm equivalences

in Proposition 2, we obtain that there exists a constant  $c$  such that under hypothesis (44),

$$\|\mathcal{G}(\psi, W)\|_{L_{\mathbf{a}}^2} \leq c \left( |\psi| + \langle \mathcal{L}W | W \rangle_{\mathbf{a}}^{\frac{1}{2}} \right) \|\mathcal{L}W\|_{L_{\mathbf{a}}^2}.$$

We estimate now  $(\Lambda_\psi - \Lambda_0)V$ :

$$\begin{aligned} \|(\Lambda_\psi - \Lambda_0)W\|_{L_{\mathbf{a}}^2} &\leq \|A(\Phi(\psi)) - A(0)\|_{L^\infty} \left\| \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x W) \right\|_{L_{\mathbf{a}}^2} \\ &\quad + \|(\partial_x \Theta(\psi))^2 B_1'(\Phi(\psi)) - (\theta_0')^2 B_1'(0)\|_{L^\infty} \|w_2\|_{L_{\mathbf{a}}^2} \\ &\quad + \|\mathrm{d}C(\mathcal{K}(\psi)) - \mathrm{d}C(\mathcal{K}(0))\|_{L^\infty} \|W\|_{L_{\mathbf{a}}^2} \\ &\leq K|\psi| \|W\|_{\mathbf{H}^2} + K|\psi| \|W\|_{L_{\mathbf{a}}^2}, \end{aligned}$$

So there exists  $c$  such that under Hypothesis (44),

$$\|(\Lambda_\psi - \Lambda_0)W\|_{L_{\mathbf{a}}^2} \leq c|\psi| \|\mathcal{L}W\|_{L_{\mathbf{a}}^2}.$$

In order to estimate  $\mathcal{H}(\psi, W)$  given in (36), we split the second coordinate of  $\mathcal{F}(\mathcal{K}(\psi) + W)$  in the following way:

$$\mathcal{F}_2(\mathcal{K}(\psi) + W) = \mathcal{T}_1 + \mathcal{T}_2 + T_{2,2} + \dots + T_{6,2},$$

where

$$\begin{aligned} \mathcal{T}_1 &= \ell^2 \frac{1}{\mathbf{a}} \left( \cos \Phi(\psi) \partial_x (\mathbf{a} \partial_x w_1) + \alpha \partial_x (\mathbf{a} \partial_x w_2) \right) \\ \mathcal{T}_2 &= (\partial_x \Theta(\psi))^2 \ell^2 \alpha \left( \cos^2 \Phi(\psi) - \sin^2 \Phi(\psi) \right) w_2 + \mathrm{d}C_2(\mathcal{K}(\psi))(W) \\ &\quad + \overline{A}_{21}(\Phi(\psi), w_2) \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x \Theta(\psi)), \end{aligned}$$

( $\overline{A}_{21}$  is the entry  $(2, 1)$  of the matrix  $\overline{A}$ ) and  $T_{2,2}, \dots, T_{6,2}$  are the second coordinates of the  $T_i$ 's given above. By integration by parts, we have:

$$\langle \mathcal{T}_1 | \cos \theta_0 \rangle_{\mathbf{a}} = -\ell^2 \langle \partial_x w_1 | \partial_x (\cos \Phi(\psi) \cos \theta_0) \rangle_{\mathbf{a}} - \alpha \ell^2 \langle \partial_x w_1 | \partial_x \cos \theta_0 \rangle_{\mathbf{a}},$$

so

$$|\langle \mathcal{T}_1 | \cos \theta_0 \rangle_{\mathbf{a}}| \leq K \|\partial_x W\|_{L_{\mathbf{a}}^2}.$$

In addition,

$$\|\mathcal{T}_2\|_{L_{\mathbf{a}}^2} \leq K \|W\|_{L_{\mathbf{a}}^2}.$$

From the estimates obtain on the  $T_i$ 's, we have for  $i \in \{2, 3, 5, 6, 8\}$ :

$$|\langle T_{i,2} | \cos \theta_0 \rangle_{\mathbf{a}}| \leq \|T_{i,2}\|_{L_{\mathbf{a}}^2} \|\cos \theta_0\|_{L_{\mathbf{a}}^2} \leq K \|W\|_{H^1}.$$

We estimate the terms arising from  $T_4$  and  $T_7$  in the following way:

$$\begin{aligned} |\langle T_{4,2} | \cos \theta_0 \rangle_{\mathbf{a}}| &\leq \|\mathbf{a}\|_{L^\infty} \|B_1(\Phi(\psi) + w_2)\|_{L^\infty} \|\partial_x w_1\|_{L^2}^2 \|\cos \theta_0\|_{L^\infty} \\ &\leq K \|W\|_{H^1}^2, \end{aligned}$$

$$\begin{aligned} |\langle T_{7,2} | \cos \theta_0 \rangle_{\mathbf{a}}| &\leq \|\mathbf{a}\|_{L^\infty} \|B_2(\Phi(\psi) + w_2)\|_{L^\infty} \|\partial_x w_1\|_{L^2}^2 \|\cos \theta_0\|_{L^\infty} \\ &\leq K \|W\|_{H^1}^2. \end{aligned}$$

Therefore, using the equivalent of norms established in Proposition 2, under assumption (44), we obtain that there exists  $K$  independent of  $\psi$  and  $V$  such that:

$$|\langle \mathcal{F}_2(\mathcal{K}(\psi) + W) | \cos \theta_0 \rangle_{\mathbf{a}}| \leq K \langle \mathcal{L}W | W \rangle_{\mathbf{a}}^{\frac{1}{2}}.$$

Since  $\langle \partial_\psi \Phi(\psi) | \cos \theta_0 \rangle_{\mathbf{a}}$  is bounded by below by a non negative constant for  $|\psi| \leq \nu_1$ , we obtain the claimed estimate for  $\mathcal{H}(\psi, W)$ .

We finish the proof of Proposition 3 by estimating  $\langle \partial_\psi \mathcal{K}(\psi) | \mathcal{L}W \rangle_{\mathbf{a}}$ : we remark that

$$\langle \partial_\psi \mathcal{K}(0) | \mathcal{L}W \rangle_{\mathbf{a}} = \langle \cos \theta_0 | L_2 w_2 \rangle_{\mathbf{a}} = 0$$

since  $L_2$  is self-adjoint and  $L_2 \cos \theta_0 = 0$ . Thus:

$$\begin{aligned} |\langle \partial_\psi \mathcal{K}(\psi) | \mathcal{L}W \rangle_{\mathbf{a}}| &= |\langle \partial_\psi \mathcal{K}(\psi) - \partial_\psi \mathcal{K}(0) | \mathcal{L}W \rangle_{\mathbf{a}}| \\ &\leq \|\partial_\psi \mathcal{K}(\psi) - \partial_\psi \mathcal{K}(0)\|_{L_{\mathbf{a}}^2} \|\mathcal{L}W\|_{L_{\mathbf{a}}^2} \\ &\leq K |\psi| \|\theta'_0\|_{L_{\mathbf{a}}^2} \|\mathcal{L}W\|_{L_{\mathbf{a}}^2} \text{ with (51).} \end{aligned}$$

Therefore,

$$|\langle \partial_\psi \mathcal{K}(\psi) | \mathcal{L}W \rangle_{\mathbf{a}}| \leq c |\psi| \|\mathcal{L}W\|_{L_{\mathbf{a}}^2}.$$

This concludes the proof of Proposition 3.  $\square$

## 5. DW depinning

We consider in this section a notched infinite ferromagnetic nanowire with a non vanishing applied field  $\mathbf{h}_a$  oriented in the direction of the wire:  $\mathbf{h}_a = h e_1$ . This situation is modelled by:

$$\begin{cases} \partial_t m = -m \times \mathcal{H} - \alpha m \times (m \times \mathcal{H}), \\ \mathcal{H} = \frac{1}{\mathbf{a}} \ell^2 \partial_x (\mathbf{a} \partial_x m) - \frac{1}{2} (m_2 e_2 + m_3 e_3) + h e_1. \end{cases} \quad (52)$$

First, we prove that for  $h$  small enough, (52) admits a stationary solution describing a DW:

**Theorem 3.** *There exists  $h^* > 0$  such that for all  $h$  with  $|h| < h^*$ , then (52) admits a stationary solution  $\mathbf{m}_h$  of the form:*

$$\mathbf{m}_h(x) = \begin{pmatrix} \sin \theta_h \\ \cos \theta_h \\ 0 \end{pmatrix}$$

where  $\theta_h : \mathbb{R} \rightarrow \mathbb{R}$  tends to  $-\frac{\pi}{2}$  (resp.  $+\frac{\pi}{2}$ ) when  $x$  tends to  $-\infty$  (resp.  $+\infty$ ).

**Proof.** The map  $\mathbf{m}_h$  is a static solution of (52) is and only if

$$\ell^2 \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x \theta_h) + \frac{1}{2} \sin \theta \cos \theta_h + h \cos \theta_h = 0. \quad (53)$$

We look for  $\theta_h$  on the form:  $\theta_h = \theta_0 + v_h$ , where  $v_h \in \mathbf{H}^2$ . We define  $\chi : \mathbb{R} \times \mathbf{H}^2 \rightarrow L^2(\mathbb{R})$  by:

$$\chi(h, v) = \ell^2 \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x (\theta_0 + v)) + \frac{1}{2} \sin(\theta_0 + v) \cos(\theta_0 + v) + h \cos(\theta_0 + v),$$

We remark that  $\chi(0, 0) = 0$ , and that

$$\partial_v \chi(0, 0)(w) = \ell^2 \frac{1}{\mathbf{a}} \partial_x (\mathbf{a} \partial_x w) + \frac{1}{2} (\sin^2 \theta_0 - \cos^2 \theta_0) w = L_1 w,$$

where  $L_1$  is defined by (39) (see Section 4). We proved that  $L_1$  is coercive, so we can apply the implicit function theorem: there exists  $h^* > 0$  and a  $\mathcal{C}^1$  function  $h \in ]-h^*, h^*[ \mapsto v_h \in \mathbf{H}^2$  such that for all  $h \in ]-h^*, h^*[$ ,  $\chi(h, v_0) = 0$ , that is  $\theta_0 + v_h$  is a static solution to (53) tending to  $+\frac{\pi}{2}$  (resp.  $-\frac{\pi}{2}$ ) when  $x$  tends to  $+\infty$  (resp.  $-\infty$ ).  $\square$

Using the same method as in Part 4, we can prove that for  $h$  small enough, the solution constructed in Theorem 3 is stable modulo rotations. In this case, the linear operators arising in the linearization will be:

$$L_1^h = -\frac{1}{\mathbf{a}} \ell^2 \partial_x (\mathbf{a} \partial_x) + \frac{1}{2} (\sin^2 \theta_h - \cos^2 \theta_h) - h \sin \theta_h,$$

$$L_h^2 = -\frac{1}{\mathbf{a}} \ell^2 \partial_x (\mathbf{a} \partial_x) + \frac{1}{2} \sin^2 \theta_h - \ell^2 (\theta_h')^2 - h \sin \theta_h.$$

In particular, since  $\theta_h$  tends to  $\pm \frac{\pi}{2}$  at the infinity, since both  $\frac{1}{2} (\sin^2 \theta_h - \cos^2 \theta_h)$  and  $\frac{1}{2} \sin^2 \theta_h - \ell^2 (\theta_h')^2$  tend to  $\frac{1}{2}$  at the infinity, the essential spectrum of  $L_1$  and  $L_2$  will be  $[\frac{1}{2} - |h|, +\infty[$ . Therefore, even if there exists a static solution for  $h > \frac{1}{2}$ , it will be unstable. In fact, in the case  $h > \frac{1}{2}$  (resp.  $h < -\frac{1}{2}$ ), a  $-e_1$ -domain (resp. a  $+e_1$ -domain) is unstable.

In order to illustrate DW depinning, we prove that if the applied field is sufficiently large, there exists no static stable solution describing one DW:

**Theorem 4.** *Whatever the length  $l$  of the notch, if  $\frac{1}{2} \frac{1-\gamma}{1+\gamma} < |h| \leq \frac{1}{2}$ , there is no  $\theta_h$  satisfying (53) and tending to  $-\frac{\pi}{2}$  (resp.  $+\frac{\pi}{2}$ ) when  $x$  tends to  $-\infty$  (resp.  $+\infty$ ).*

We recall that  $\gamma$ , defined by (28), is the ratio of the cross-section areas.

**Proof.** We fix  $h$  such that  $\frac{1}{2} \frac{1-\gamma}{1+\gamma} < |h| \leq \frac{1}{2}$ . We assume that there exists  $\theta_h$  satisfying (53) and tending to  $-\frac{\pi}{2}$  (resp.  $+\frac{\pi}{2}$ ) when  $x$  tends to  $-\infty$  (resp.  $+\infty$ ). We introduce the quantity  $\mathcal{F}_h$  defined by:

$$\mathcal{F}_h(x) = \ell^2(\theta'_h(x))^2 + \frac{1}{2} \sin^2 \theta_h(x) + 2h \sin \theta_h(x).$$

By (53),  $\mathcal{F}_h$  is constant in each interval  $]-\infty, -l[$ ,  $] -l, l[$  and  $]l, +\infty[$ . We precise this constant with the limit conditions:

$$\begin{aligned} \forall x \in ]-\infty, -l[, \quad \mathcal{F}_h(x) &= \frac{1}{2} - 2h, \\ \forall x \in ]l, +\infty[, \quad \mathcal{F}_h(x) &= \frac{1}{2} + 2h. \end{aligned} \tag{54}$$

We denote  $\theta^l = \theta_h(-l)$  and  $\theta^r = \theta_h(l)$ . We denote by  $\theta'((-l)^-)$  (resp.  $\theta'((-l)^+)$ ) the limit of  $\theta'(x)$  when  $x$  tends to  $-l$ ,  $x < -l$  (resp.  $x > -l$ ). In the same way, we denote by  $\theta'(l^-)$  (resp.  $\theta'(l^+)$ ) the limit of  $\theta'(x)$  when  $x$  tends to  $l$ ,  $x < l$  (resp.  $x > l$ ). By the jump conditions, we have  $\theta'((-l)^-) = \gamma \theta'((-l)^+)$  and  $\theta'(l^+) = \gamma \theta'(l^-)$ . In particular, we have, using also (54):

$$\ell^2(\theta'((-l)^+))^2 = \frac{1}{\gamma^2} \ell^2(\theta'((-l)^-))^2 = \frac{1}{\gamma^2} \left( \frac{1}{2} - 2h - \frac{1}{2} \sin^2 \theta^l - 2h \sin \theta^l \right)$$

$$\ell^2(\theta'(l^-))^2 = \frac{1}{\gamma^2} \ell^2(\theta'(l^+))^2 = \frac{1}{\gamma^2} \left( \frac{1}{2} + 2h - \frac{1}{2} \sin^2 \theta^r - 2h \sin \theta^r \right).$$

Thus, using the conservation of  $\mathcal{F}_h$  on  $] -l, l[$ , we have:

$$\begin{aligned} \frac{1}{\gamma^2} \left( \frac{1}{2} - 2h - \frac{1}{2} \sin^2 \theta^l - 2h \sin \theta^l \right) + \frac{1}{2} \sin^2 \theta^l + 2h \sin \theta^l &= \\ \frac{1}{\gamma^2} \left( \frac{1}{2} + 2h - \frac{1}{2} \sin^2 \theta^r - 2h \sin \theta^r \right) + \frac{1}{2} \sin^2 \theta^r + 2h \sin \theta^r, \end{aligned}$$

that is

$$v_h(\sin \theta^r) - v_h(\sin \theta^l) = \frac{4h}{1 - \gamma^2},$$

with

$$v_h(z) = \frac{1}{2}z^2 + 2hz.$$

For  $|h| < \frac{1}{2}$ , we have  $\max_{[-1,1]} v_h = \frac{1}{2} + 2|h|$  and  $\min_{[-1,1]} v_h = -2h^2$ . Therefore,

$$\left| \frac{4h}{1-\gamma^2} \right| \leq \frac{1}{2} + 2|h| + 2h^2,$$

that is

$$0 \leq \frac{1}{2} - 2\frac{1+\gamma^2}{1-\gamma^2}|h| + 2h^2.$$

So  $|h|$  is outside the roots of  $\xi \mapsto 2\xi^2 - 2\frac{1+\gamma^2}{1-\gamma^2}\xi + \frac{1}{2}$ . The roots are  $\xi^- = \frac{1}{2}\frac{1-\gamma}{1+\gamma}$  and  $\xi^+ = \frac{1}{2}\frac{1+\gamma}{1-\gamma}$ . Since  $|h| \leq \frac{1}{2} < \xi^+$ , then  $0 \leq |h| \leq \xi^-$ , which leads to a contradiction. This concludes the proof of Theorem 4.  $\square$

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