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► **To cite this version:**

Olivier Biquard, Farid Madani. A construction of conformal-harmonic maps. *Comptes Rendus. Mathématique*, 2012, 350 (21-22), pp.967-970. 10.1016/j.crma.2012.10.021 . hal-02928851

**HAL Id: hal-02928851**

**<https://hal.science/hal-02928851>**

Submitted on 30 Aug 2022

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# A CONSTRUCTION OF CONFORMAL-HARMONIC MAPS

OLIVIER BIQUARD AND FARID MADANI

ABSTRACT. Conformal harmonic maps from a 4-dimensional conformal manifold to a Riemannian manifold are maps satisfying a certain conformally invariant fourth order equation. We prove a general existence result for conformal harmonic maps, analogous to the Eells-Sampson theorem for harmonic maps. The proof uses a geometric flow and relies on results of Gursky-Viaclovsky and Lamm.

Let  $(M, g)$  and  $(N, h)$  be two compact Riemannian manifolds of dimension 4 and  $n$  respectively. Denote by  $R^M$ ,  $\text{Ric}^M$  and  $S^M$  the Riemann, Ricci and scalar curvatures associated to  $(M, g)$  respectively. Given a map  $u \in C^2(M, N)$ , we denote by  $u^*TN$  the pull-back on  $M$  of the tangent bundle of  $N$ . Then the tension field  $\tau(u) \in u^*TN$  is defined by:

$$\tau(u) = - \sum \tilde{\nabla}_{e_i} du(e_i),$$

where  $\{e_i\}$  is an orthonormal frame of  $TM$  and  $\tilde{\nabla}$ ,  $\bar{\nabla}$  are the Riemannian connections on  $T^*M \otimes u^*TN$  and  $u^*TN$  respectively. So  $u$  is a harmonic map if and only if  $\tau(u) = 0$ .

The following conformal energy functional for maps  $u : M \rightarrow N$  was introduced by Bérard [1]:

$$(1) \quad \mathcal{E}(u) = \int_M \left( |\tau(u)|^2 + \frac{2}{3} S^M |du|^2 - 2\text{Ric}^M(du, du) \right) dv_g$$

It is conformally invariant with respect to conformal changes of  $g$  (but not of  $h$ ). The map  $u$  is said to be a conformal-harmonic map (in short, C-harmonic map), if it is a critical point of  $\mathcal{E}$ . Namely, it is a solution of the following equation:

$$(2) \quad \mathcal{L}(u) := \bar{\Delta} \tau(u) + R^N(du(e_i), \tau(u)) du(e_i) + \bar{\nabla}^* \left\{ \left( \frac{2}{3} S^M - 2\text{Ric}^M \right) du \right\} = 0$$

where  $\bar{\Delta} = \bar{\nabla}^* \bar{\nabla}$  is the rough Laplacian,  $\bar{\nabla}^*$  is the  $L^2$ -adjoint of  $\bar{\nabla}$ . This C-harmonic equation (2) differs from the biharmonic equation by low order terms which make the C-harmonic equation conformally invariant.

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O.B. is supported by the ANR project ANR-10-BLAN-0105. F.M. is supported by the Alexander von Humboldt Foundation.

The Yamabe number  $\mu(M, [g])$  and the total Q-curvature  $\kappa(M, [g])$  are defined by

$$(3) \quad \mu(M, [g]) = \inf_{g' \in [g]} \frac{\int_M S_{g'} dv_{g'}}{(\int_M dv_{g'})^{1/2}}, \quad \kappa(M, [g]) = \frac{1}{12} \int_M (S_g^2 - 3|\text{Ric}_g|^2) dv_g.$$

Both are conformal invariants of  $[g]$ . The aim of this Note is to prove the following:

**Theorem 1.** *Let  $(M^4, [g])$  and  $(N^n, h)$  be compact manifolds equipped with a conformal metric  $[g]$  and a Riemannian metric  $h$  respectively. Assume that the curvature of  $(N, h)$  is non positive,  $\mu(M, [g]) > 0$  and  $\kappa(M, [g]) + \frac{1}{6}\mu^2(M, [g]) > 0$ . Then each homotopy class in  $C^\infty(M, N)$  can be represented by a C-harmonic map.*

This statement is analogous to the classical theorem of Eells and Sampson on the existence of harmonic maps, but no uniqueness is proved. Remark that C-harmonic maps are usually not harmonic for any metric in the conformal class  $[g]$ , so the theorem actually constructs new applications which depend only on the conformal geometry of  $[g]$ .

The rest of the paper is devoted to the proof of Theorem 1. The geometric hypothesis is explained by the following crucial proposition, which is adapted from the theorem of Gursky and Viaclovsky [3] giving conditions for the Paneitz operator to be positive (this corresponds to the case  $N = \mathbb{R}$ ). The conditions on  $(M, [g])$  are the same as in [3], but we add a non positive curvature assumption on  $(N, h)$ :

**Proposition 2.** *Let  $(M, g)$  and  $(N, h)$  be compact Riemannian manifolds of dimension 4 and  $n$  respectively. Assume that the curvature of  $(N, h)$  is non positive,  $\mu(M, [g]) > 0$  and  $\kappa(M, [g]) + \frac{1}{6}\mu^2(M, [g]) > 0$ . Then there exists a positive constant  $c$  such that  $\mathcal{E}(u) \geq c\|du\|_2^2$ .*

It is important to note that the RHS of the inequality is not conformally invariant, so the constant  $c$  depends on the metric  $g$  itself. Nevertheless, since the LHS is conformally invariant, it is sufficient to prove the inequality just for one special metric in the conformal class. *Proof.* We adapt [3, § 6]. For any map  $u \in C^\infty(M, N)$ , we have the Bochner-Weitzenböck formula:

$$(4) \quad \|\tau(u)\|_2^2 = \|\tilde{\nabla} du\|_2^2 - \int_M \left( \sum_{i,j} \langle R_{du(e_i), du(e_j)}^N du(e_j), du(e_i) \rangle - \text{Ric}^M(du, du) \right) dv_g.$$

We combine (1) and  $\frac{4}{3}$  times (4) and use the fact that the curvature of  $N$  is non positive, to get

$$(5) \quad \mathcal{E}(u) \geq \frac{4}{3} \left( \|\tilde{\nabla} du\|_2^2 - \frac{1}{4} \|\tau(u)\|_2^2 \right) + \frac{2}{3} \int_M \left( S^M |du|^2 - \text{Ric}^M(du, du) \right) dv_g.$$

Again by [3], under the hypothesis, there exists a metric  $g$  in the conformal class such that the scalar curvature and the second symmetric function of the eigenvalues of  $\text{Ric}$  are positive, which implies  $\text{Ric}^M < S^M g$ . Combining with the fact that  $|\tilde{\nabla} du|_2^2 - \frac{1}{4}|\tau(u)|_2^2 = |\tilde{\nabla}_0 du|^2$ , where  $\tilde{\nabla}_0 du$  is the trace free part of  $\tilde{\nabla} du$ , we deduce that there exists  $c > 0$  such that  $\mathcal{E}(u) \geq c\|du\|_2^2$ .  $\square$

In order to prove Theorem 1, we study the gradient flow of the energy:

$$(6) \quad \partial_t u = -\mathcal{L}(u), \quad u(\cdot, 0) = u_0.$$

Equation (6) is a fourth order strongly parabolic equation. So, there exists  $T > 0$  such that (6) has a solution  $u \in C^\infty(M \times [0, T], N)$ . We have the following estimates:

**Lemma 3.** *Under the assumptions of Theorem 1, there is a constant  $c = c(\mathcal{E}_0, M, N)$ , such that and if  $u \in C^\infty(M \times [0, T], N)$  is a solution of (6) with  $\mathcal{E}(u_0) \leq \mathcal{E}_0$ , then we have for all  $t \in [0, T]$ :*

$$(7) \quad \mathcal{E}(u(\cdot, t)) + 2 \int_0^t \|\partial_t u\|_2^2 dt = \mathcal{E}(u_0),$$

$$(8) \quad \|du(\cdot, t)\|_2^2 + 2 \int_0^t \|\bar{\nabla} \tau(u)\|_2^2 dt \leq \|du_0\|_2^2 + ct,$$

$$(9) \quad \int_M |\tilde{\nabla} du|^2 dv_g \leq c,$$

$$(10) \quad \int_M |du|^4 dv_g \leq c.$$

*Proof.* Let  $u$  be a solution of (6). We have  $\frac{d}{dt} \mathcal{E}(u) = d\mathcal{E}(\partial_t u) = 2\langle \mathcal{L}(u), \partial_t u \rangle = -2\langle \partial_t u, \partial_t u \rangle$ . Equality (7) holds after an integration over time.

Using Proposition 2 and (7), we obtain  $\|du\|_2 \leq c$ . Using again (7), we conclude that  $\|\tau(u)\|_2 \leq c$ . Using (4) and the fact that  $N$  has non positive curvature, we have  $\|\tilde{\nabla} du\|_2^2 \leq \|\tau(u)\|_2^2 + c\|du\|_2^2$ . Hence we obtain (9).

Now observe that

$$\mathcal{L}(u) = \bar{\Delta} \tau(u) + R^N(du(e_i), \tau(u))du(e_i) + (\nabla R^M) \odot du + R^M \odot \tilde{\nabla} du,$$

where the  $\odot$ 's are fixed bilinear operations. Since  $N$  has nonnegative curvature, using (9) and the bounds on  $\|du\|_2$  and  $\|\tau(u)\|_2$ , it follows that  $\langle \mathcal{L}(u), \tau(u) \rangle \geq \|\bar{\nabla} \tau(u)\|_2^2 - c$  for some constant  $c$ . Therefore,

$$\frac{1}{2} \frac{d}{dt} \|du(\cdot, t)\|_2^2 = \langle \partial_t u, \tau(u) \rangle = -\langle \mathcal{L}(u), \tau(u) \rangle \leq -\|\bar{\nabla} \tau(u)\|_2^2 + c.$$

Hence (8) holds after an integration over time. Inequality (10) is obtained from the bound on  $\|du\|_2$  by using (9) and Sobolev embedding.  $\square$

Given the estimates of the lemma, the arguments for proving the long time existence of the flow and its convergence are very similar to

that developed by Lamm [4] in its study of the flow for biharmonic maps. Below we briefly explain how to use its proof to finish the proof of Theorem 1.

Biharmonic maps are solutions of the equation

$$(11) \quad \bar{\Delta}\tau(u) + R^N(du(e_i), \tau(u))du(e_i) = 0$$

which are critical points of the functional

$$(12) \quad \mathcal{F}(u) = \int_M |\tau(u)|^2 dv_g.$$

If we compare the functionals  $\mathcal{E}$ ,  $\mathcal{F}$ , we note that there exists  $c > 0$  such that

$$\mathcal{F}(u) - c\|du\|_2^2 \leq \mathcal{E}(u) \leq \mathcal{F}(u) + c\|du\|_2^2.$$

Since  $\|du\|_2$  is estimated in Lemma 3, all the estimates obtained by Lamm for the solutions of the biharmonic map gradient flow hold for the  $C$ -harmonic ones. Therefore, we give a series of results where the proofs are given in [4] (itself building on deep regularity results of Chang-Yang and Struwe). To state them, we need to introduce a smooth nonnegative cut-off function  $\eta$  on  $M$ , such that

$$(13) \quad \begin{aligned} \eta &= 1 \text{ on } B_R(P_0), \\ \eta &= 0 \text{ on } M - B_{2R}(P_0), \\ \|\nabla^i \eta\|_\infty &\leq \frac{c}{R^{2i}} \text{ for } i = 1 \text{ and } 2, \end{aligned}$$

where  $B_R(P_0)$  is a geodesic ball with centre  $P_0 \in M$  and radius  $R < \min(1, \frac{1}{4}\text{inj}_M(P_0))$ . Let  $E$  be the (conformally invariant) total energy defined by

$$(14) \quad E(u) = \mathcal{E}(u) + \left( \int_M |du|^4 dv_g \right)^{\frac{1}{2}}.$$

The local energy  $E(u; B_R(P_0))$  is defined by an analogous formula as  $E$ , where the integration is taken over the ball  $B_R(P_0)$ .

**Lemma 4.** *There exists  $\varepsilon_0 = \varepsilon_0(\mathcal{E}_0, M, N)$  such that if  $u \in C^\infty(M \times [0, T], N)$  is a solution of (6) with  $\mathcal{E}(u_0) \leq \mathcal{E}_0$  and*

$$\sup_{0 \leq t < T} E(u(\cdot, t); B_{2R}(P_0)) < \varepsilon_0,$$

*then for all  $t < T$  and  $k = 2, 3$ ,*

$$\int_0^t \int_M \eta^4 |\tilde{\nabla}^k du|^2 dv_g dt \leq c(1 + \frac{t}{R^4}).$$

**Lemma 5.** *If  $v : \mathbb{R}^4 \rightarrow N$  is a weak  $C$ -harmonic map (i.e. a critical point of (1) among the  $v \in W_{loc}^{2,2}(\mathbb{R}^4, N)$  such that  $dv \in W^{1,2}$ ), and  $E(v) < +\infty$ , then  $E(v) = 0$ .*

**Lemma 6.** *There exists  $\varepsilon_0 = \varepsilon_0(\mathcal{E}_0, M, N)$  such that if  $u \in C^\infty(M \times [0, T], N)$  is a solution of (6) with  $\mathcal{E}(u_0) \leq \mathcal{E}_0$  and*

$$\sup_{(P,t) \in M \times [0,T]} E(u(\cdot, t); B_{2R}(P)) < \varepsilon_0,$$

*then there exists  $\delta \in (0, \min(T, cR^4))$  such that the Hölder norms of  $u$  and all derivatives of  $u$  are uniformly bounded on  $M \times [t - \frac{\delta}{4}, t]$  for all  $t \in [\frac{\delta}{2}, T)$  by constants which depend only on  $\mathcal{E}_0, M, N, R$  and the order of derivatives of  $u$ .*

Given these three lemmas, we now finish the proof of Theorem 1. This still follows Lamm, but it may be useful for the reader to follow how one quickly deduces the result from the previous technical lemmas. As mentioned above, there exists a maximal  $T > 0$  and  $u \in C^\infty(M \times [0, T], N)$  solution of (6). Fix  $\varepsilon_0$  smaller than the ones given by Lemmas 4 and 6, then two cases can occur:

- (1) for all  $R > 0$ , there exists  $P \in M$  and  $t \in [0, T)$  such that

$$E(u(\cdot, t); B_R(P)) \geq \varepsilon_0;$$

- (2) there exists  $R > 0$  such that

$$\sup_{(P,t) \in M \times [0,T]} E(u(\cdot, t); B_R(P)) < \varepsilon_0.$$

The first case is when we have concentration. We can choose a sequence  $(P_m, t_m, R_m)$  such that  $t_m \rightarrow T$ ,  $R_m \rightarrow 0$ ,  $E(u(t_m), B_{R_m}(P_m)) = \frac{\varepsilon_0}{2}$ , and  $E(u(t_m), B_{R_m}(P)) \leq \frac{\varepsilon_0}{2}$  for all  $P \in M$ . So  $P_m$  is the point where the energy is most concentrated. Up to extracting, one can suppose  $P_m \rightarrow P_0$ . Now consider the blowup metrics  $g_m = \frac{g}{R_m^2}$ , and the new solution  $u_m(x, t) = u(x, t_m + R_m^4 t)$  to the flow for the metric  $g_m$ . Fix some  $R_0 > 0$  and  $t_0 < 0$ . From (7) and Lemma 4 it follows that

$$\int_{t_0}^0 \|\partial_t u_m\|_{L^2(B_{R_0}^{g_m}(P_m))}^2 \longrightarrow 0, \quad \int_{t_0}^0 \|\nabla^4 u_m\|_{L^2(B_{R_0}^{g_m}(P_m))}^2 \leq C.$$

The second estimate holds because the LHS is invariant by rescaling of space and time. The balls  $(B_{R_0}^{g_m}(P_m), g_m)$  converge to the standard Euclidean ball  $(B_{R_0}(P_0), \text{euc})$  in  $\mathbb{R}^4$ . From the above estimates, there exist  $s_m \in [t_0, 0]$  such that  $\partial_t u_m(\cdot, s_m) \rightarrow 0$  in  $L^2$ , and  $u_m(\cdot, s_m)|_{B_{R_0}^{g_m}(P_m)}$  converges weakly in  $W^{4,2}$  to some  $\tilde{u}$  on  $B_{R_0}(P_0) \subset \mathbb{R}^4$ . Doing this for any  $R_0 > 0$  we extract a limit  $\tilde{u}$  on  $\mathbb{R}^4$ , with bounded total energy (since this is a conformal invariant). Since  $\partial_t u_m(\cdot, s_m) \rightarrow 0$  in  $L^2$  and  $u_m$  is a solution of the flow, it follows that at the limit  $\tilde{u}$  satisfies the C-harmonic equation (weakly), which contradicts Lemma 5.

So we must be in the second case: Lemma 6 then implies that the flow can be continued smoothly beyond  $T$ . Therefore we conclude that the heat flow doesn't have any singularities and exists for all times. Fix

some  $\tau > 0$ . It then follows from (7) that  $\lim_{T \rightarrow +\infty} \int_T^{T+\tau} \|\partial_t u\|_2^2 dt = 0$ . From Lemma 4 we have that for  $T$  sufficiently large

$$\int_T^{T+\tau} \int_M |\nabla^4 u|^2 dv_g dt \leq c(\mathcal{E}_0, M, N)\tau.$$

Therefore, there exists  $t_m \rightarrow +\infty$  such that  $u_m := u(\cdot, t_m)$  is bounded in  $W^{4,2}(M, N)$  and  $\partial_t u_m$  converges to 0 in  $L^2(M, N)$ . We extract a subsequence converging to some  $u_\infty$  weakly in  $W^{4,2}(M, N)$ . From Lemma 6, we have that  $u_m$  is uniformly bounded in  $C^k$  for all  $k \in \mathbb{N}$ . Up to subsequence,  $u_m$  converges to  $u_\infty$  in  $C^k$ . Taking a limit in (6), we deduce that  $u_\infty$  is a smooth C-harmonic map.

**Acknowledgements.** The authors thank the referee for useful comments which resulted in a better presentation of the paper.

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