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# FROM $G_2$ GEOMETRY TO QUATERNIONIC KÄHLER METRICS

OLIVIER BIQUARD

ABSTRACT. We review a construction of quaternionic Kähler metrics starting from a rank 2 distribution in 5 dimensions. We relate it to a more general theory about Einstein deformations of symmetric metrics. Finally we ask some questions on complete metrics and relate them to a Zoll phenomenon.

*Dedicated to Paul Gauduchon for his 70th birthday.*

## INTRODUCTION

It is now a classical fact that the possible holonomy groups of non symmetric Riemannian metrics form a short list:  $SO(n)$  (general Riemannian),  $U(m)$  (Kähler),  $SU(m)$  (Ricci flat Kähler),  $Sp(m)$  (hyperKähler),  $Sp(m)Sp(1)$  (quaternionic Kähler), and the two exceptional  $G_2$  and  $Spin(7)$ . It is an important problem to have a general understanding of the four last holonomies. In this paper, I survey the following local construction of quaternionic Kähler metrics, which was done in the PhD thesis of my student Quentin Dufour [Duf14]:

**Theorem A.** *Let  $D$  be a generic real analytic rank 2 distribution in  $X^5$ . Then to  $(X, D)$  is canonically associated a quaternionic Kähler metric of dimension 8, and one can recover the initial distribution in five dimensions from the asymptotic behaviour of the quaternionic Kähler metric.*

Here the distribution  $D$  is generic if the brackets of vector fields in  $D$  generate distributions of the highest possible rank:

- $[D, D]$  has rank 3 at each point;
- $[D, [D, D]]$  has rank 5 at each point, that is

$$[D, [D, D]] = TX.$$

One can describe precisely the behaviour of the quaternionic Kähler metric: the 8-dimensional manifold is obtained as  $M^8 = F^7 \times (0, \epsilon)$ , where  $F^7$  is a disc bundle

$$p : F \rightarrow X$$

over  $X$ . If  $s$  is the projection on the second factor of  $F^7 \times (0, \epsilon)$ , then the metric has the following behaviour:

$$(1) \quad g \sim \frac{ds^2}{s^2} + \frac{\gamma_3}{s^3} + \frac{\gamma_2}{s^2} + \frac{\gamma_1}{s} + \gamma_0.$$

The  $\gamma_i$  are symmetric 2-tensors on  $TF$ . More accurately,  $\gamma_3$  is a symmetric 2-tensor on  $TF$ , but  $\gamma_2$  is defined only on  $\ker \gamma_3$  (this is the only part which

is independent of a choice of the equation  $s$  for  $F$ ),  $\gamma_1$  is defined only on  $\ker \gamma_2$  and finally  $\gamma_0$  on  $\ker \gamma_1$ , which turns out to be exactly the vertical tangent bundle of  $p$ . These kernels are given by:

$$\ker \gamma_3 = p^*[D, D] \supset \ker \gamma_2 = p^*D \supset \ker \gamma_1 = \ker p_*.$$

All the  $\gamma_i$  are explicit in terms of the initial data: one can identify the fiber  $p^{-1}(x)$  of  $p : F \rightarrow X$  with the space of conformal metrics on  $D_x$ , and then, after fixing a determinant of  $D \subset TX$ :

- $\gamma_0$  is the hyperbolic metric on the disc  $p^{-1}(x)$ , identified with the space of conformal metrics on  $D_x$ ;
- $\gamma_1$  is the tautological metric on  $p^*D$ ;
- $\gamma_2$  and  $\gamma_3$  can also be described explicitly in terms of  $D$ .

Changing the determinant of  $D$  amounts to multiplying the coordinate  $s$  by a nonvanishing function on  $X$ , so the asymptotic behaviour (1) is unchanged.

Theorem A fits in a long list of similar constructions:

- (1) the first result of this kind [LeB82] is LeBrun's construction of a 4-dimensional selfdual Einstein metric from a 3-dimensional conformal metric;
- (2) later [LeB89] LeBrun associated a  $4(k+1)$ -dimensional quaternionic Kähler metric to a conformal metric of signature  $(3, k)$ ;
- (3) to a quaternionic contact structure in dimension  $4k-1$  is associated a  $4k$ -dimensional quaternionic Kähler metric [Biq00];
- (4) to a CR structure in dimension 3 is associated a 4-dimensional selfdual Einstein metric [Biq07].

All these constructions have a homogeneous model of the following kind: let  $G$  be a real semisimple group such that the associated symmetric space of noncompact type  $G/H$  is quaternionic Kähler (so is a Wolff space), and  $P \subset G$  a parabolic subgroup. Then all the previous constructions have a model where  $X = G/P$  is the initial manifold and  $M = G/H$  the quaternionic Kähler metric. Let us list here the corresponding groups:

- (1)  $G = SO(4, 1)$ ,  $H = S(O(4) \times O(1))$ ,  $G/P = S^3$  and  $G/H$  is the 4-dimensional hyperbolic space;
- (2)  $G = SO(4, k+1)$ ,  $H = S(O(4) \times O(k+1))$ ,  $G/P$  is the quadric of isotropic lines in  $\mathbb{R}^{4, k+1}$  and  $G/H$  is the grassmannian of spacelike 4-planes;
- (3)  $G = Sp(k+1, 1)$ ,  $H = Sp(k+1)Sp(1)$ ,  $G/P = S^{4k-1}$  and  $G/H$  is the  $4k$ -dimensional quaternionic hyperbolic space;
- (4)  $G = SU(2, 1)$ ,  $H = S(U(2) \times U(1))$ ,  $G/P = S^3$  and  $G/H$  is complex hyperbolic 2-space.

Theorem A corresponds to a new addition to this list: the group is not obvious a priori, but it is well known since Élie Cartan [Car10] that the geometry of rank 2 distributions in 5 dimensions is related to the noncompact form  $G_2'$  of  $G_2$  and to the stabilizer  $P \subset G_2'$  of an isotropic line (this

makes sense because  $G_2^r \subset SO(4, 3)$ ; the corresponding symmetric space is  $G_2^r/SO(4)$ .

Of course this list suggests the following open question:

*Question 1.* Let  $G/H$  be a quaternionic Kähler symmetric space of non-compact type. Can one find a similar general construction of quaternionic Kähler metrics starting from any  $G/P$ , where  $P \subset G$  is any parabolic subgroup?

Each time one expects that the quaternionic Kähler metric admits a development similar to (1). Such a development is well-known for the symmetric metric itself, and relates the asymptotic behaviour of the metric to the geometric structure of the boundary  $G/P$ , which is the homogeneous model for the so-called “parabolic geometry” of type  $(G, P)$ . For parabolic geometries we will refer to the book [ČS09].

The various  $G/P$  when  $P$  varies among parabolic subgroups of  $G$  are known in the literature are “Furstenberg boundaries” of  $G/H$  and are related to the Poisson transform: in particular, if  $P_{min}$  denotes the minimal parabolic subgroup of  $G$ , it is a classical fact of harmonic analysis (see for example [Hel94, §V.3]) that bounded harmonic functions on the symmetric space  $M = G/H$  are in 1:1 correspondence with their “value at infinity”, which is a (generalized) function on the maximal Furstenberg boundary  $G/P_{min}$ ; the map taking the boundary value to the harmonic function is the Poisson transform.

The question above would fit in the program of Rafe Mazzeo and I on the construction of a “nonlinear Poisson transform” [BM06]: one expects that a certain type of deformation of the parabolic geometry of  $G/P_{min}$  remains the value at infinity of a deformation of an Einstein metric on  $M$ . Sometimes, such a geometry can be obtained from a lifting of a parabolic geometry on  $G/P$  via the projection  $G/P_{min} \rightarrow G/P$ , so one may hope to start from any boundary  $G/P$ . The asymptotic behaviour (1) is typical of the compatibility expected between the Einstein metric and the geometry on the boundary. The nonlinear Poisson transform is established for rank 1 spaces [GL91, Biq00] and reducible rank 2 spaces [BM11]. The author hopes to come back to the general problem at some point in the future. The simplest example is that of the real hyperbolic space  $\mathbb{R}H^4$ , whose boundary at infinity is the 3-sphere  $S^3$ : any small deformation of the conformal metric on  $S^3$  is the boundary at infinity of a complete Einstein metric which is a small deformation of the symmetric metric of  $\mathbb{R}H^4$ .

Theorem A can then be interpreted as a local version of this transform, where the resulting metric is better than Einstein: it is quaternionic Kähler. When the hypothesis of the nonlinear Poisson transform is satisfied, one can hope that the local quaternionic Kähler metric coincides with the global Einstein metric, and therefore extends into a complete quaternionic Kähler metric. There is no reason to think that this should always be true, as shows the example of the Positive Frequency phenomenon, predicted

by LeBrun [LeB91] and proved in [Biq02]: not all conformal metrics on  $S^3$  (close to the round one) are conformal infinities of selfdual Einstein metrics on the 4-ball, this is true only for a “half-dimensional” subspace. We will state later a more precise question about plane distributions in dimension 5.

In the first two sections of this article, we give some indications on the proof of the theorem. In the last section, we state some questions related to these metrics, and explain how they are related to a question on Zoll metrics.

### 1. GEODESIC FLOW FOR PLANE DISTRIBUTIONS IN DIMENSION 5

To understand the proof of theorem A, it is useful to remind the homogeneous situation. Recall that the group  $G_2^r$  is defined as the stabilizer of a certain 3-form  $\Phi$  on  $\mathbb{R}^7$  and is contained in  $SO(4, 3)$ . It has three parabolic subgroups,  $P_1$  (stabilizing an isotropic line),  $P_2$  (stabilizing a plane such that for some basis  $(e, f)$  of the plane one has  $(e \wedge f) \lrcorner \Phi = 0$ ), and the minimal parabolic  $P_{min} = P_1 \cap P_2$  if the plane defining  $P_2$  contains the line defining  $P_1$  (so  $P_{min}$  is the stabilizer of the corresponding partial flag). Then one has a double fibration, as is common in twistor theory (see [BE89])

$$\begin{array}{ccc} & G_2^r/P_{min} & \\ p \swarrow & & \searrow q \\ G_2^r/P_1 & & G_2^r/P_2 \end{array}$$

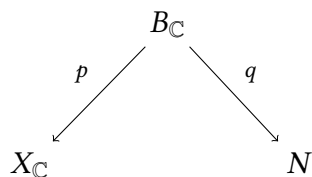
Here both  $G_2^r/P_1$  and  $G_2^r/P_2$  are 5-dimensional, and  $p$  and  $q$  are circle bundles.

It turns out that the complexified space  $G_2^{\mathbb{C}}/P_2^{\mathbb{C}}$  is the twistor space of the symmetric space  $G_2^r/SO(4)$ . Recall that a quaternionic Kähler metric in dimension  $4k$  is completely determined by its twistor space [Sal82], a  $(2k + 1)$ -dimensional complex manifold carrying

- a holomorphic contact distribution;
- a family of rational curves, transverse to the contact distribution, and with normal bundle  $\mathcal{O}(1) \otimes \mathbb{C}^{2k}$ ;
- a real structure preserving all the other structures.

Conversely, such twistor space determines a quaternionic Kähler metric.

Coming back to our problem, the homogeneous picture suggests, starting from a general  $X^5$  with a rank 2 distribution, to construct the following diagram:



where the maps  $p$  and  $q$  have 1-dimensional fibers, and  $N$  will be the twistor space of the expected quaternionic Kähler metric.

The space  $B$  is defined as the  $\mathbb{R}P^1$ -bundle over  $X$  of directions in the distribution  $D$ :

$$(2) \quad B = P(D).$$

**Proposition.** *The space  $B$  carries a canonical “geodesic flow”, that is a canonical rank 1 distribution  $\tilde{D}$  such that for any  $x \in B$  one has  $p_*\tilde{D}_x = D_{p(x)}$ .*

From the proposition, one will take  $N$  to be the space of geodesics in  $B$ , after complexifying the whole picture (then  $B_{\mathbb{C}}$  is a  $\mathbb{C}P^1$ -bundle over  $X_{\mathbb{C}}$ ).

The proof of the proposition is not a priori obvious. A powerful tool to handle the geometry of  $X$  is the parabolic geometry machinery [ČS00, ČS09], which says that such  $X$  admits a canonical normal Cartan connection

$$\omega \in \Omega^1(\mathcal{G}, \mathfrak{g}_2^r),$$

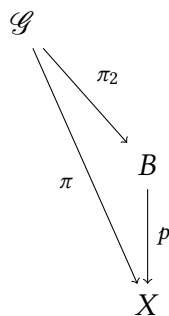
where

$$\pi : \mathcal{G} \rightarrow X$$

is the  $P$ -principal bundle induced by the plane distribution  $D$  (here and in the following, we note  $P = P_1$  to simplify notation). Without giving details let us just say that:

- for the homogeneous model,  $\mathcal{G} = G_2^r$  and  $\omega$  is the Maurer-Cartan form;
- in general,  $\omega_g : T_g\mathcal{G} \rightarrow \mathfrak{g}_2^r$  is an isomorphism, sending the vertical directions of  $\mathcal{G}$  to  $\mathfrak{p}$ , and therefore identifying a tangent space  $TX$  with  $\mathfrak{g}_2^r/\mathfrak{p}$  modulo the action of  $P$ .

This Cartan connection is used to define the distribution  $\tilde{D}$  on  $B$ : actually the principal bundle  $\pi : \mathcal{G} \rightarrow X$  factorizes as



where  $\pi_2$  is a  $P_{min}$ -principal bundle. Given  $g \in \mathcal{G}$ , we can now define a lift of a vector  $\xi \in D_{\pi(g)}$  in the following way: decompose

$$\mathfrak{g}_2^r = \mathfrak{p} \oplus \mathfrak{p}_-$$

where  $\mathfrak{p}_-$  is an opposite nilpotent algebra (some compatibility with the choice of  $P_{min}$  is required), then  $\xi = \pi_* A$  for a unique  $A \in T_g \mathcal{G}$  such that  $\omega(A) \in \mathfrak{p}_-$ , and we can choose

$$(3) \quad \tilde{\xi} = (\pi_2)_* A.$$

It turns out that  $\tilde{\xi}$  does not depend on the various choices involved here, and this gives the distribution expected on  $B$ .

A more sophisticated proof of the proposition consists in lifting the parabolic geometry of  $X$  to a parabolic geometry on  $B$  for the parabolic group  $P_{min}$  (this implies the existence of the rank 1 distribution, because  $\mathfrak{g}_2^r/\mathfrak{p}_{min}$  has a  $P_{min}$  invariant line). There exists an algebraic criterion for this to be possible [Čap05, ČS09], and one can check that it is satisfied here. This remark can also be used to prove that the space  $N$  of geodesics (which is at least locally defined) carries a contact structure, which is the projection of the distribution given at a point  $b \in B$  by  $p^*(b^\perp)$ , where  $b$  is considered as a line in  $T_{p(b)}X$  and the orthogonal is taken with respect to the conformal structure of signature  $(3, 2)$  on  $X$  induced by the  $(G_2^r, P)$  structure.

Of course one can complexify the whole structure if  $(X, D)$  is real analytic. Summarizing, we then obtain

- (1) a  $\mathbb{C}P^1$ -bundle  $p : B_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ ;
- (2) a rank 1 complex distribution on  $B_{\mathbb{C}}$ ;
- (3) a space of leaves  $N$  which carries a holomorphic contact structure.

It is an interesting question to understand when a parabolic geometry of type  $(G, P)$  admits a natural flow defined on some bundle with fiber  $P/Q$ , where  $Q \subset P$  is some smaller parabolic group. We call such a flow a “generalized geodesic flow”. We have just seen that the  $(G_2^r, P)$  geometry indeed admits such a generalized geodesic flow. Of course all the conformal pseudo-Riemannian metrics admit a geodesic flow (the null geodesics). In [Duf14] is obtained a classification of the parabolic geometries which:

- are not rigid, so admit local deformations which are not isomorphic to the model;
- admit a generalized geodesic flow.

The answer is that there is only one additional example: the geometry of type  $(SO(4, 3), P)$  where  $P$  is the parabolic group fixing a totally isotropic 3-space. This means there might be a further extension to that case of the construction of quaternionic Kähler metrics using generalized geodesic flow, the only other examples being LeBrun’s constructions and the present construction for the group  $G_2^r$ .

## 2. RATIONAL CURVES

We have now constructed a space  $N$  which has all the properties of the twistor space of a quaternionic Kähler metric, except the family of rational curves that we now consider.

Recall we have a diagram

$$\begin{array}{ccc} & B_{\mathbb{C}} & \\ p \swarrow & & \searrow q \\ X_{\mathbb{C}} & & N \end{array}$$

For each  $x \in X$  the fiber  $p^{-1}(x)$  is a rational curve with trivial normal bundle  $\mathbb{C}^5$ , so the projection

$$(4) \quad \mathcal{C}_x = q(p^{-1}(x))$$

is again a rational curve, now with normal bundle  $\mathbb{C}^3 \oplus \mathcal{O}(1)$  (because along each curve  $p^{-1}(x)$  we quotient by the tautological bundle  $\mathcal{O}(-1) \subset p^*D = \mathbb{C}^2$ ). So we get in this way a 5-dimensional family of rational curves.

We want to extend this 5-dimensional family to a 8-dimensional family. The normal bundle  $\mathbb{C}^3 \oplus \mathcal{O}(1)$  seems to indicate that the space of deformations is 5-dimensional, therefore we already have all the deformations of the curves  $\mathcal{C}_x$ . This is contrast with usual twistor construction, where usually one finds other directions of deformation. Fortunately, this problem is solved when one looks at the homogeneous model: it turns out that the curves  $\mathcal{C}_x$  are not reduced. They are actually given by equations with a multiplicity 2 in the  $\mathcal{O}(1)$  direction.

As a toy model to understand the deformations of a non reduced curve, we can consider a line  $\mathbb{C}P^1 \subset \mathbb{C}P^2$  with multiplicity 2, that is a degenerate quadric. The space of quadrics is 5-dimensional, when the space of lines is 2-dimensional, so deforming a  $\mathbb{C}P^1$  as a quadric gives 3 extra dimensions of deformations, and these are exactly the 3 missing dimensions above.

More precisely, suppose the

$$\mathbb{C}P^1 = \{z_0^2 = 0\},$$

where we have chosen homogeneous coordinates  $(z_0, z_1, z_2)$ . Then a 3-dimensional family of reduced deformations is given by

$$(5) \quad z_0^2 = az_1^2 + 2bz_1z_2 + cz_2^2.$$

If  $b^2 - ac = 0$  we get singular curves; for  $b^2 - ac \neq 0$ , the curve is smooth.

Coming back to our curves  $\mathcal{C}_x$  in  $N$ , one can now prove that, if we consider them as non reduced curves, with multiplicity 2 in the  $\mathcal{O}(1)$  direction, then the space of deformations is actually 8-dimensional (the local parameters being  $(x, a, b, c)$ ). One still needs these deformations to have normal bundle  $\mathcal{O}(1) \otimes \mathbb{C}^4$ . There is no abstract reason why this should be true,



because if we deform the above  $\mathbb{C}P^1 \subset \mathbb{C}P^2 \times \mathbb{C}P^3$  lying in the first factor, the deformations have obviously a normal bundle  $\mathcal{O}(4) \oplus \mathbb{C}^3$ .

So one has to use the geometry of the situation to understand the normal bundle. Here we use a feature of all parabolic geometries: the model homogeneous geometry has a family of dilations at each point; the pull-back of a curved geometry by this family of dilations converges to the model geometry. (This is similar to the pull-back of a Riemannian metric by homotheties converging to the flat metric). The meaning is that if we look at the geometry at larger and larger scale, it becomes closer and closer to the model geometry. The dilations can be lifted to the twistor space  $N$ , and this means that after parabolic dilations the twistor space converges to the twistor space of the model; in particular the rational curves converge to the rational curves of the model, so must have normal bundle  $\mathcal{O}(1) \otimes \mathbb{C}^4$  when they converge to a smooth curve of the model (since these have normal bundle  $\mathcal{O}(1) \otimes \mathbb{C}^4$ ). The same argument shows that they are transverse to the holomorphic contact structure.

Let us see more precisely what it means in the parameters  $(x, a, b, c)$ : it turns out that the action of the dilations on  $(a, b, c)$  is homogeneous. So near  $x$ , the transverse directions with

$$b^2 - ac \neq 0$$

correspond to a nonsingular rational curve in the model, so one can deduce that for each  $(a, b, c)$  such that  $\Delta \neq 0$ , and for small enough  $t$ , the curve indexed by  $(x, ta, tb, tc)$  is a nonsingular rational curve with normal bundle  $\mathcal{O}(1) \otimes \mathbb{C}^4$ , which is transverse to the holomorphic contact distribution of  $N$ .

We have now all the ingredients of a quaternionic Kähler metric: a twistor space with a family of transverse rational curves with normal bundle  $\mathcal{O}(1) \otimes \mathbb{C}^4$ . Therefore, there is a quaternionic Kähler metric defined on the parameter space of real curves. Let us see where exactly this metric is defined.

It turns out that the real structure acts on the parameters  $(a, b, c)$  by usual complex conjugation. The metric is then defined (positive) in the direction of the  $(a, b, c)$  such that  $b^2 - ac < 0$ . We can write

$$(6) \quad (a, b, c) = s(a_0, b_0, c_0), \quad b_0^2 - a_0c_0 = -1.$$

The space of  $(x, a_0, b_0, c_0)$  such that  $b_0^2 - a_0c_0 = -1$  is the 7-dimensional manifold  $F$  mentioned in the introduction, we recover the bundle

$$(7) \quad p : F \longrightarrow X$$

by projection on the first factor, and the fibers are discs (and the equation  $b_0^2 - a_0c_0 = -1$  already suggests that these will be hyperbolic discs). The asymptotic behaviour (1) is then obtained performing inverse twistor transform. The precise calculations require the use of “normal coordinates” adapted to the parabolic geometry (see [ČM13, §1], or [Duf14] for precise

statements on the order of approximation of the parabolic geometry by the model).

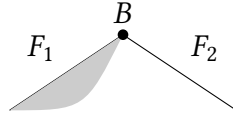
It is interesting to understand more precisely the region on which the quaternionic Kähler metric is constructed, by comparing to the homogeneous model  $G_2^r/SO(4)$ . Recall that  $G_2^r/SO(4)$  has three boundaries, two 5-dimensional boundaries ( $G_2^r/P_1, G_2^r/P_2$ ) and one 6-dimensional maximal boundary ( $B = G_2^r/P_{min}$ ). There are two 7-dimensional faces  $F_1$  and  $F_2$  associated to the two 5-dimensional boundaries, which are disc bundles over the corresponding boundary. Both  $F_1$  and  $F_2$  have  $B$  as a common boundary, and the Furstenberg compactification is obtained by adding to the symmetric space these three faces: the union

$$(8) \quad G_2^r/SO(4) \cup F_1 \cup F_2 \cup B.$$

From the above point of view, the boundary  $B = G/P_{min}$  is obtained as the circle bundle over  $X$  obtained at the boundary of the disc bundle, that is the projectivized bundle of the null cone

$$(9) \quad b^2 - ac = 0.$$

These are exactly the directions which correspond to reduced, but singular, curves in the twistor space. This means that we get a metric in the neighbourhood of the face  $F_1$  which does not cover a neighbourhood of the codimension 2 face  $B$ : this domain is illustrated by the following diagram:



In particular, one cannot expect to have a metric defined near any point of  $F_2$ . Obtaining a *complete* metric near the whole infinity of the symmetric space does not follow from this method.

### 3. COMPLETE METRICS AND THE ZOLL CONDITION

We end this paper by some speculations about the possible extension of the quaternionic Kähler metrics constructed here to complete quaternionic Kähler metrics.

As mentioned in the introduction, [BM06] proposes a general program for constructing complete Einstein metrics from certain deformations of the parabolic geometry of  $B = G/P_{min}$ . Let us restrict to the case of rank 2, where we have the same diagram as above:  $G/H$  has two codimension 1 faces  $F_1$  and  $F_2$ , which intersect in the codimension 2 face  $B$ . As above, each face  $F_i$  is fibered over a boundary  $B_i = G/P_i$  by rank one symmetric spaces  $X_i$ , and  $B$  is fibered over  $B_i$  by the boundaries of these symmetric spaces. For rank one spaces the boundaries are spheres. Denote by  $D_i \subset TB$  the distribution induced by the fibers of  $\partial F_i \rightarrow B$ .

It is proposed in [BM06] that a deformation of the parabolic geometry of  $B$  which remains regular will still be the boundary at infinity of a complete Einstein metric, provided an additional assumption is satisfied: the distributions  $D_i$  should be deformed to distributions which remain tangent to the boundary of a fiber space

$$(10) \quad p_i : F_i \rightarrow B.$$

Indeed the construction requires an extension of the initial data on  $B$  on the faces  $F_i$  along the fibers of  $p_i$ , to obtain an approximate metric in all directions going to infinity.

When  $\dim D_i \geq 2$ , this condition is equivalent to the distribution  $D_i$  remaining integrable, because the spheres  $S^k$  ( $k \geq 2$ ) are simply connected, so that any integrable deformation of the underlying distribution will integrate to spheres. But this is no more the case if  $k = 1$ , so that an important assumption here is that, if  $\dim D_i = 1$ , the trajectories of  $D_i$  should remain closed. Filling them by discs will then reconstruct the disc fibration (10).

Let us now see what this means in the geometric example that we have studied in this paper. We started from the boundary  $B_1 = G_2^r/P_1$ . The parabolic geometry on  $B_1$  amounts to giving a generic rank 2 distribution on  $B_1$ , and we have constructed the fiber bundle  $B \rightarrow B_1$ , which carries a deformation of the model parabolic geometry of  $G_2^r/P_{min}$ . For this deformed geometry, the distribution  $D_1$  is the vertical space of the fibration  $B \rightarrow B_1$ , but the distribution  $D_2$  is exactly the distribution tangent to the generalized geodesic flow that we constructed in section 1. From these data on  $B$  there is a hope to construct a global Einstein metric if this flow has closed trajectories, so the following question is important:

*Question 2.* Call a generic rank 2 distribution in dimension 5 Zoll if its generalized geodesics are circles.

The model  $G_2^r/P_1$  is Zoll. Are there other Zoll rank 2 distributions ?

If the answer to the question is yes, then this leads to the next question:

*Question 3.* Which Zoll 2-distributions in dimension 5 are the boundaries at infinity of 8-dimensional complete quaternionic Kähler metrics ?

Recall from the introduction that there is no reason why the Einstein metric in general should be automatically quaternionic Kähler.

Of course there are other cases where similar questions are open. A particularly fascinating one is obtained from the group  $SO(4, 2)$ : then a boundary of the group is the compactified Minkowski space with its flat conformal Lorentzian structure. It is of course Zoll. Nothing seems to be known on the Zoll problem for Lorentzian metrics in dimension 4, unlike dimension 3 (see [Gui89]):

*Question 4.* Are there other 4-dimensional Zoll conformal Lorentzian metrics ?

If yes, which Zoll conformal Lorentzian metrics are the boundaries at infinity of a complete quaternionic Kähler metric in dimension 8 ?

These questions can be written down purely in terms of twistor spaces. It seems likely that in this case, a “Positive Frequency phenomenon” (see the introduction) may appear.

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