Smoothing singular constant scalar curvature Kähler surfaces and minimal Lagrangians

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Abstract. Given a complex surface $\mathcal{X}$ with singularities of class T and no nontrivial holomorphic vector field, endowed with a Kähler class Ω_0, we consider smoothings $(\mathcal{M}_t, \Omega_t)$, where Ω_t is a Kähler class on $\mathcal{M}_t$ degenerating to Ω_0. Under an hypothesis of non degeneracy of the smoothing at each singular point, we prove that if $\mathcal{X}$ admits a constant scalar curvature Kähler metric in Ω_0, then $\mathcal{M}_t$ admits a constant scalar curvature Kähler metric in Ω_t for small t.

In addition, we construct small Lagrangian stationary spheres which represent Lagrangian vanishing cycles when t is small.

1. Introduction

Let $\mathcal{X}$ be a normal compact complex surface with singularities of class T. Such singularities are isolated and of orbifold type $\mathbb{C}^2/\Gamma$ for some finite group $\Gamma \subset \mathrm{U}_2$. The possible singularities are either rational double points ($\Gamma \subset \mathrm{SU}_2$) or cyclic quotient singularities of type $\frac{1}{dn^2}(1, dna-1)$, for positive integers d, n, a such that a and n are relatively prime.

Since there is no general answer to the existence problem of constant scalar curvature Kähler metrics (CSCK for short), numerous constructions relying on gluing techniques have been made (cf. for instance [10, 1, 2]). In the case of canonical singularities (rational double points), the idea is to start from a CSCK orbifold metric on $\mathcal{X}$ and to deduce a smooth CSCK metric on the minimal resolution $\widehat{\mathcal{X}}$ by perturbation theory. The exceptional divisor of $\widehat{\mathcal{X}} \to \mathcal{X}$ is a union of holomorphic -2 spheres, whose configuration is described by the Dynkin diagram of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ determined by Γ (this is a part of the McKay correspondence).

In this work we consider smoothings of singularities rather than resolutions. We prove, under natural hypotheses, that $\mathbb{Q}$-Gorenstein smoothings admit CSCK metrics.

In the special case of Kähler-Einstein metrics, the desingularizations cannot carry Kähler-Einstein metrics unless $c_1(\mathcal{X}) = 0$, since they contain holomorphic -2 spheres. But smoothings may carry Kähler-Einstein metrics, and our result gives a construction of those Kähler-Einstein metrics close to the singular ones. Also it is interesting to note that for singularities of class T which are not rational double points, the smoothings are not diffeomorphic to the minimal resolution. We give an explicit example starting from a quotient of $\mathbb{CP}^1 \times \mathbb{CP}^1$ by $\mathbb{Z}_4$.

If a -2 sphere collapses in the singular limit, and if its homology class is Lagrangian in the smoothing, we prove that it can be represented by a small Hamiltonian stationary sphere. This gives a concrete construction of the Lagrangian minimizer, whose existence is proved by Schoen and Wolfson [11].

1.1. **CSCK metrics.** We now give precise statements. Let $\mathcal{X}$ be a normal complex surface with quotient singularities. We consider a $\mathbb{Q}$-Gorenstein smoothing of $\mathcal{X}$, denoted $p : \mathcal{M} \to \Delta$, where Δ is an open disc centered at the origin in $\mathbb{C}$. This means that a multiple of the canonical divisor of $\mathcal{M}$ is Cartier and $\mathcal{M}$ is Cohen-Macaulay. Moreover, the central fiber is the given $\mathcal{X}$, and the general fiber $\mathcal{M}_t$ is smooth.

Kóllar and Shepherd-Barron [7] proved that the singularities which appear must be of class T; this class subdivides into two classes:

- canonical singularities of type $\mathbb{C}^2/\Gamma$ for a finite subgroup $\Gamma \subset \mathrm{SU}_2$; one gets the list A_d $(d \geq 1)$, D_d $(d \geq 4)$, E_d $(d = 6, 7, 8)$;
- cyclic quotients obtained by a quotient of a A_{dn-1} singularity by $\mathbb{Z}_n$.

The local theory of smoothings of such a singularity is well understood: there is an (explicit) hypersurface $\mathcal{Y} \subset \mathbb{C}^3 \times \mathbb{C}^d$, such that the projection $p : \mathcal{Y} \to \mathbb{C}^d$ is a $\mathbb{Q}$-Gorenstein smoothing of $\mathcal{Y}_0 \simeq \mathbb{C}^2/\Gamma$. Moreover, any $\mathbb{Q}$-Gorenstein smoothing of $\mathcal{Y}_0$ is isomorphic to the pull-back of p under a germ of holomorphic maps $f : \mathbb{C} \to \mathbb{C}^d$. The fibers $\mathcal{Y}_t = p^{-1}(t)$ over the discriminant locus $\mathcal{D} \subset \mathbb{C}^d$ are singular. This leads us to introduce a non degeneracy condition in the following way. As recalled below, there is a particular ramified covering map $\pi : \mathbb{C}^d \to \mathbb{C}^d$ associated to the singularity. This map is ramified exactly above $\mathcal{D}$, and moreover the ramification locus $\pi^{-1}(\mathcal{D})$ turns out to be a union of linear hyperplanes:

$$\pi^{-1}(\mathcal{D}) = \cup H_i.$$

Up to taking a ramified covering (which we choose of smallest possible degree), the germ $f : \mathbb{C} \to \mathbb{C}^d$ can be lifted to $\hat{f} : \mathbb{C} \to \mathbb{C}^d$ such that $f = \pi \circ \hat{f}$. The following definition is a way to say that the deformation is transversal to the discriminant locus:

Definition 1. We say that a smoothing $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ is non degenerate at the singular point $x \in \mathcal{X}$ if the corresponding $\hat{f}$ satisfies $\frac{\partial \hat{f}}{\partial t}(0) \notin \cup H_i$.

Example. Consider the A_k-singularity $w^{k+1} = xy$, with its family of deformations

$$xy = w^{k+1} + a_{k-1}w^{k-1} + a_{k-2}w^{k-2} + \cdots + a_0.$$

A deformation is given by a map $f(z) = (a_0(z), \ldots, a_{k-1}(z))$. It is easy to check that if $\frac{\partial a_0}{\partial z} \neq 0$ then the deformation is non degenerate.

To state the theorem, we need to fix a Kähler class. Along a ray to the origin in Δ, the Gauß-Manin connection identifies $H^2(\mathcal{M}_t, \mathbb{R})$ with a fixed vector space; going to the origin, we identify this vector space with

$$H^2_{\mathrm{orb}}(\mathcal{X}, \mathbb{R}) \oplus \bigoplus_{x \text{ singular}} \mathbb{R}^{e_x},$$

where e_x is the dimension of the real H^2 of the local smoothing of the singularity at x. As we shall see later, if d_x is the dimension of the space of smoothings of the singularity at x,

$$e_x = \begin{cases} d_x & \text{for a canonical singularity,} \\ d_x - 1 & \text{for other singularities.} \end{cases}$$

(One cannot get such an identification on the whole disk, because of the monodromy of the Gauß-Manin connection; but since the data of a Kähler class is real, it is natural to give it only on a ray).

The simplest form of our results can now be stated:

Theorem A. *Suppose that we are given*

- *a normal complex surface $\mathcal{X}$, with no holomorphic vector fields, and a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$, which is non degenerate at each singular point;*
- *along the ray $t \in \mathbb{R}_+ \cap \Delta$, a Kähler class $\Omega_t \in H^2(\mathcal{M}_t, \mathbb{R})$, depending smoothly on t, such that Ω_0 is an orbifold Kähler class on $\mathcal{X}$, containing an orbifold CSCK metric.*

Then for small $t > 0$ the class Ω_t contains a CSCK metric on $\mathcal{M}_t$.

By the classification results mentioned above, the $\mathbb{Q}$-Gorenstein condition is equivalent to the degeneration of the complex structure near the singularities being modelled by Ricci-flat Kähler ALE spaces. Given this, the main work in the proofs is to carry out a gluing construction of metrics in the spirit of Arezzo-Pacard [1].

In particular, the behavior of the CSCK metric for t small is well understood: outside the singularities it converges to the orbifold CSCK metric on $\mathcal{X}$, but near a singularity x some rescaling converges to an ALE Kähler Ricci flat space, which appears as the 'bubble' at x.

One special case is when the smoothing is given with a polarization Ω, resulting in a constant Ω_t. This covers in particular the Kähler-Einstein case: when $\Omega_t = \pm c_1(\mathcal{M}_t)$ our result gives a construction of the Kähler-Einstein metric on the smoothing of $\mathcal{X}$, as well as a concrete description of its degeneracy to an orbifold Kähler-Einstein metric. This Kähler-Einstein picture is recently obtained in the case of A_1 singularities by Spotti [12].

Complex orbifold singularities of codimension at least 3 are rigid by a result of Schlessinger. However, it would be interesting to extend our results in dimension 3 or more, allowing orbifold singularities of codimension 2. A natural problem would be to remove the triviality assumption of the automorphism group: one may expect the existence of the CSCK metric to be related to a K-stability property, like in [14].

One can replace the non degeneracy hypothesis on the smoothing $\mathcal{M}_t$ by a weaker non degeneracy hypothesis on the data $(\mathcal{M}_t, \Omega_t)$ of the smoothing with the Kähler class. This requires slightly more material that we now explain.

Let us come back to the $\mathbb{Q}$-Gorenstein smoothing $\mathcal{Y} \to \mathbb{C}^d$ of a canonical singularity. The parameter space $\mathbb{C}^d$ can be identified with $\mathfrak{h}_{\mathbb{C}}/W$, where $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ associated to Γ by the McKay correspondence ($\mathfrak{g}_{\mathbb{C}}$ is exactly the A_d, D_d or E_d simple Lie algebra), and W is the Weyl group. Here the canonical projection $\mathfrak{h}_{\mathbb{C}} \to \mathbb{C}^d$ is ramified over the discriminant locus $\mathcal{D} \subset \mathbb{C}^d$. The real 2-cohomology is isomorphic to the real Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ (actually, the smoothing is diffeomorphic to the minimal resolution, whose -2 spheres give a basis of $(\mathfrak{h}_{\mathbb{R}})^*$). But there is some ambiguity in this identification, because the monodromy of the Gauß-Manin connection is given by the action of W.

In the smoothing $\mathcal{Y} \to \mathbb{C}^d = \mathfrak{h}_{\mathbb{C}}/W$, the general fiber is smooth, but there are singular fibers, given by the walls of the Weyl chambers of $\mathfrak{h}_{\mathbb{C}}$. Nevertheless, there exists a simultaneous resolution of singularities $\widehat{\mathcal{Y}} \to \mathfrak{h}_{\mathbb{C}}$ (Brieskorn, Slodowy). Then all the fibers are smooth and their real 2-cohomology can be identified to $\mathfrak{h}_{\mathbb{R}}$.

The case of the other T singularities is similar: they are quotients of A_{dn-1} singularities by a $\mathbb{Z}_n$ action, and one obtains the same structure by taking the

$\mathbb{Z}_n$ fixed points of the A_{dn-1} simultaneous resolution: the space of parameters is therefore $\mathfrak{h}_{\mathbb{C}}^{\mathbb{Z}_n} \simeq \mathbb{C}^d$, and the real 2-cohomology is $\mathfrak{h}_{\mathbb{R}}^{\mathbb{Z}_n} \simeq \mathbb{R}^{d-1}$.

Now come back to our setup of a smoothing $\mathcal{M} \to \Delta$ with a family of Kähler classes Ω_t. At a singular point $x \in \mathcal{X}$, the smoothing is induced from the standard smoothing $\mathcal{Y}$ by a map (we denote the dependence in x by an index x)

$$f_x : \Delta_c \longrightarrow ((\mathfrak{h}_{\mathbb{C}})_x / W_x)^{G_x},$$

where $G_x = 1$ for a canonical singularity, and $\Delta_c \subset \Delta$ is a smaller disk of radius c. Up to a finite ramified covering $\Delta_b \to \Delta_c$, we can lift this map into a map into $(\mathfrak{h}_{\mathbb{C}})_x^{G_x}$:

$$(\zeta_c)_x : \Delta_b \longrightarrow (\mathfrak{h}_{\mathbb{C}})_x^{G_x}.$$

Up to taking a higher degree covering, we can take the same covering $\Delta_b \to \Delta_c$, say of order $\mathbf{d}$, for all singular points. Adding the data $(\zeta_r)_x$ of the Kähler class in a neighborhood of the singularity, we obtain at each singularity a map

$$\zeta_x : \Delta_b \longrightarrow (\mathfrak{h}_{\mathbb{R}})_x^{G_x} \oplus (\mathfrak{h}_{\mathbb{C}})_x^{G_x}, \quad \zeta_x = ((\zeta_r)_x, (\zeta_c)_x),$$

which represents the deformation of both the Kähler parameter and the complex parameter. We consider the Kähler class as depending smoothly on the parameter on Δ_b (since Δ_b is a covering, this is more general than the setting of Theorem A). Then at the origin we have for some integer p

$$\zeta_x(t) = t^p \dot{\zeta}_x + O(t^{p+1}).$$

We can now define an extended notion of non degeneracy:

Definition 2. We say that $(\mathcal{M}_t, \Omega_t)$ is non degenerate at the point x if $p \leq \mathbf{d}$ and $\dot{\zeta}_x$ does not belong to a wall of a Weyl chamber.

The above definition is clearly a generalization of Definition 1. The two conditions have different meanings: if $\dot{\zeta}_x$ is inside a wall of the Weyl chamber, then the local model for $\mathcal{M}_t$ near the singularity is a degenerating family of ALE spaces with cycles shrinking at different scales of time, and the smoothing would require an iterated construction and the use of orbifold ALE spaces; this looks possible but much more technical, so we prefer to restrict to the simplest case. On the other hand, the first condition guarantees that the order of deformation far from the singularity (which is $\mathbf{d}$) is not smaller than the order of deformation at the singularities; this is natural because one can imagine that in that case, one would first deform $\mathcal{M}_0$ keeping the singularities fixed, and then smooth the deformation.

Another comment on the definition is that the non degeneracy is stated in terms of the pair (ζ_r, ζ_c), which means that a smoothing may be degenerate in the sense of definition 1, but non degenerate in the sense of definition 2 thanks to a non trivial variation of the Kähler class. This means that the Ricci flat Kähler ALE space which models the degeneration may be, for example, a desingularization of $\mathbb{C}^2/\Gamma$ rather than a smoothing.

We now extend Theorem A under the form:

Theorem B. *Suppose that we are given*

- *a normal complex surface $\mathcal{X}$, with no holomorphic vector fields, and a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$;*
- *along the ray $t \in \mathbb{R}_+ \cap \Delta_b$, a Kähler class $\Omega_t \in H^2(\mathcal{M}_t, \mathbb{R})$, such that Ω_0 is an orbifold Kähler class on $\mathcal{X}$, containing an orbifold CSCK metric.*

If $(\mathcal{M}_t, \Omega_t)$ is non degenerate at each singular point in the sense of Definition 2, then for small $t > 0$ the class Ω_t contains a CSCK metric on $\mathcal{M}_t$.

There is a final extension of the results in the case of canonical singularities: then we do not need the initial family $\mathcal{M}$ to be a smoothing. Indeed suppose that $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ is any deformation of a surface with rational double points; this means that now the fibers may be singular. Then near each singularity x, the family is still induced by some map $\Delta_c \to (\mathfrak{h}_{\mathbb{C}})_x/W_x$ (because this is a semi-universal deformation), which can again be lifted to $(\zeta_c)_x : \Delta_b \to (\mathfrak{h}_{\mathbb{C}})_x$. Pulling back $\widehat{\mathcal{Y}} \to (\mathfrak{h}_{\mathbb{C}})_x$ near each singular point, we obtain a simultaneous resolution $\widehat{\mathcal{M}} \to \Delta_b$ of the singularities of $\mathcal{M} \to \Delta$ and we can apply our method in this setting to obtain:

Theorem C. *Suppose that we are given*

- *a normal complex surface $\mathcal{X}$, with no holomorphic vector fields, with only rational double points, and a deformation $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$; let then $\widehat{\mathcal{M}} \to \Delta_b$ the simultaneous resolution of singularities as above;*
- *along the ray $t \in \mathbb{R}_+ \cap \Delta_b$, a Kähler class $\Omega_t \in H^2(\widehat{\mathcal{M}}_t, \mathbb{R})$, such that Ω_0 is an orbifold Kähler class on $\mathcal{X}$, containing an orbifold CSCK metric.*

If $(\widehat{\mathcal{M}}_t, \Omega_t)$ is non degenerate at each singular point in the sense of Definition 2, then for small $t > 0$ the class Ω_t contains a CSCK metric on $\widehat{\mathcal{M}}_t$.

This result covers the known case of minimal resolutions, but also the case of partial resolutions which are not smoothings. The difference here between the case of rational double points and the other cases is that any small deformation of $\mathbb{C}^2/\Gamma$ with $\Gamma \subset \mathrm{SU}_2$ is $\mathbb{Q}$-Gorenstein, but this is not true if $\Gamma \not\subset \mathrm{SU}_2$, see an example of a non $\mathbb{Q}$-Gorenstein smoothing in [9]; our methods do not apply in that case.

1.2. **Hamiltonian stationary spheres.** Suppose we are under the hypothesis of Theorem B. As we have seen,

$$H^2(\mathcal{M}_t, \mathbb{R}) = H^2_{\mathrm{orb}}(\mathcal{X}, \mathbb{R}) \oplus \oplus_{x \text{ singular}} (\mathfrak{h}_{\mathbb{R}})_x^{G_x}.$$

Fix a singular point x, and a system $\mathfrak{R}_x^+ \subset (\mathfrak{h}_{\mathbb{C}})_x^*$ of positive roots. If x is a rational double point ($G_x = 1$), a root $\theta \in \mathfrak{R}_x^+$ represents an integral homology class on $\mathcal{M}_t$; in general, θ represents a homology class in the local G_x-covering, so after projection still represents a (possibly zero) homology class in $\mathcal{M}_t$.

Remind that the deformation $(\mathcal{M}_t, \Omega_t)$ is given near the singular point x by a map $\zeta_x = (\zeta_r)_x + (\zeta_c)_x : \Delta_b \to (\mathfrak{h}_{\mathbb{R}})_x^{G_x} \oplus (\mathfrak{h}_{\mathbb{C}})_x^{G_x}$, in which $(\zeta_r)_x(t)$ represents the class of the restriction of Ω_t near the singularity. Therefore the homology class defined by θ is Ω_t-Lagrangian if $\langle \theta, (\zeta_r)_x(t) \rangle \equiv 0$, which implies in particular $\langle \theta, (\dot{\zeta}_r)_x \rangle = 0$.

Theorem D. *Suppose that we are under the hypothesis of Theorem B or C. Fix a singular point $x \in \mathcal{X}$ and a root $\theta \in \mathfrak{R}_x^+$, such that $\langle \theta, (\dot{\zeta}_r)_x \rangle = 0$, and θ is primitive for this property (that is cannot decompose as $\theta_1 + \theta_2$ with $\theta_i \in \mathfrak{R}_x^+$ satisfying the same property). Finally suppose that for all $t > 0$, one has $\langle \theta, (\zeta_r)_x(t) \rangle \equiv 0$, that is the homology class represented by θ remains Ω_t-Lagrangian. Then:*

(1) *If x is a rational double point, then the homology class in $\mathcal{M}_t$ (or $\widehat{\mathcal{M}}_t$ in the setting of Theorem C) corresponding to θ is represented by a smooth*

Hamiltonian stationary sphere S_t, which is also a global Lagrangian minimizer of the area in its homotopy class.

(2) *If $G_x = \mathbb{Z}_n$, and if the homology class defined by θ is nonzero, then it is represented by a smooth Hamiltonian stationary sphere S_t.*

The theorem is obtained by a simple deformation argument from the corresponding sphere in the Ricci-flat Kähler ALE space. This implies (see section 5) that one gets an explicit description when $t \to 0$: after dilations, the Hamiltonian stationary representative of θ converges to a special Lagrangian sphere in the ALE Kähler Ricci flat space obtained at the limit.

In the second case ($G_x = \mathbb{Z}_n$), if the homology class defined by θ in the quotient vanishes, one can still get in some cases a Hamiltonian stationary embedded S^2 or $\mathbb{RP}^2$, see section 6.2. We do not state a minimizing property which is less clear, because the models in the ALE Kähler Ricci flat space are not calibrated.

For example, from Theorem A we recover Kähler-Einstein metrics on certain 4-point blowup of $\mathbb{CP}^2$ constructed by Tian [15], and Theorem D provides a construction of stationary Lagrangian spheres (cf §7.1).

1.3. **Organization of the paper.** In most of the article, we suppose that $\mathcal{X}$ has only canonical singularities and we prove directly Theorem C, which implies Theorem A, Theorem B) and Theorem D. At the end, we explain the case of general singularities of the class T: there is not much change, because they are obtained as cyclic quotients of canonical singularities and all our constructions pass to the quotient.

One of the main point in the paper is the construction of 'good' Kähler metrics on the deformations: this is not obvious, because the complex structure changes. It turns out that what is needed is an extension to the singular setting of the theorem of Kodaira on the stability of the Kähler condition under complex deformations. This is done in section 3, after the introduction of the 'tangent graviton' in section 2. Rather than adapting Kodaira's argument using a fourth order operator, we follow the method in Voisin's book [16, §9.3] which requires only the use of a second order operator. In section 4, we extend arguments in the literature to produce the CSCK metrics, so we only point out the new features in our situation. In section 5, we produce the Hamiltonian stationary Lagrangian spheres. The case of general singularities is explained in section 6, and some applications are given in section 7. In particular, we study the smoothing of the Del Pezzo orbifold $(\mathbb{CP}^1 \times \mathbb{CP}^1)/\mathbb{Z}_4$, thus obtaining a family of smooth Kähler-Einstein surfaces degenerating to a limit with singularities of the form $\mathbb{C}^2/\Gamma$ with $\Gamma \subset \mathrm{U}_2$ but $\Gamma \not\subset \mathrm{SU}_2$.

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2. Deformations of singular surfaces and Kähler classes

In this section, $\mathcal{X}$ will always denote a compact complex surface with canonical singularities endowed with an orbifold Kähler metric $\overline{g}$ with Kähler form $\overline{\omega}$ and Kähler class Ω_0. We consider a flat deformation $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ of $\mathcal{X}$.

2.1. **Complex deformations near singularities.** Our first goal is to understand a simultaneous resolution $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{M}} \to \Delta_d$ of the deformation, focusing near the singular points. For this purpose, we choose a set of adapted isomorphisms

$U_i \simeq \Delta^2_{\varepsilon_i}/\Gamma_i$ near each singularity $x_i \in \mathcal{X}$. Actually, for simplicity of notations we will suppose that there is only one singular point x_0, and we will point out from place to place what changes for several singular points. So near x_0, one can choose local coordinates z with values in Δ^2_ε defined modulo Γ, such that the pullback of $\overline{\omega}$ on the disc is expressed as

$$dd^c(|z|^2 + \eta) \tag{2.1}$$

where $|z|$ is the Euclidean distance to the origin in $\mathbb{C}^2$ and η is a smooth Γ-invariant real function such that $\eta = \mathcal{O}(|z|^4)$. Up to scaling the metric, and shrinking the open set U, we may assume that $\varepsilon = 1$. In the sequel, we shall always assume that the identification $U \simeq \Delta^2/\Gamma$ is chosen in such a way.

By restricting $\mathcal{M}$ to a suitable neighborhood of the singular point x_0, we deduce a flat deformation

$$\Delta^2/\Gamma \hookrightarrow \mathcal{N} \to \Delta_c$$

of the singularity $U = \mathcal{N}_0$. The singularity U admits a semi-universal flat family of deformations by a result of Kas-Schlessinger [6]: there is a deformation

$$\mathbb{C}^2/\Gamma \hookrightarrow \mathcal{Y} \to \mathfrak{h}_{\mathbb{C}}/W,$$

where $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of the complex semisimple Lie algebra associated to the Dynkin diagram of the singularity (this is the Lie algebra associated to the finite group Γ by the McKay correspondence), and W is the Weyl group. Then, $\mathcal{N}$ is induced by some holomorphic map $\psi : \Delta_c \to \mathfrak{h}_{\mathbb{C}}/W$. Thus, there is a holomorphic commutative diagram

$$\begin{array}{ccc}
\Delta^2/\Gamma & \longrightarrow & \mathbb{C}^2/\Gamma \\
\downarrow & & \downarrow \\
\mathcal{N} & \xrightarrow[\Psi]{} & \mathcal{Y} \\
\downarrow & & \downarrow \\
\Delta_c & \xrightarrow[\psi]{} & \mathfrak{h}_{\mathbb{C}}/W
\end{array}$$

such that the restriction $\Psi : \mathcal{N}_t \to \mathcal{Y}_{\psi(t)}$ is an embedding for every $t \in \Delta_c$.

A remarkable feature of canonical singularities is that $\mathbb{C}^2/\Gamma \hookrightarrow \mathcal{Y} \to \mathfrak{h}_{\mathbb{C}}/W$ admits a simultaneous resolution $\widehat{\mathbb{C}^2/\Gamma} \hookrightarrow \widehat{\mathcal{Y}} \to \mathfrak{h}_{\mathbb{C}}$ given by a diagram

$$\begin{array}{ccc}
\widehat{\mathbb{C}^2/\Gamma} & \longrightarrow & \mathbb{C}^2/\Gamma \\
\downarrow & & \downarrow \\
\widehat{\mathcal{Y}} & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathfrak{h}_{\mathbb{C}} & \longrightarrow & \mathfrak{h}_{\mathbb{C}}/W
\end{array} \tag{2.2}$$

where the map $\mathfrak{h}_{\mathbb{C}} \to \mathfrak{h}_{\mathbb{C}}/W$ is the canonical projection (Brieskorn, Slodowy).

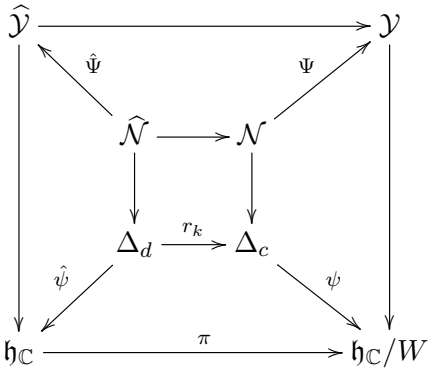
A priori, one cannot lift the map $\psi : \Delta_c \to \mathfrak{h}_{\mathbb{C}}/W$ to $\mathfrak{h}_{\mathbb{C}}$. The obstruction is the monodromy of ψ, which lies in W. By taking a ramified cover, with order $\mathbf{d}$ equal

to the order of the monodromy of ψ, we obtain a lifting

$$\begin{array}{ccc} \Delta_d & \xrightarrow{\hat{\psi}} & \mathfrak{h}_{\mathbb{C}} \\ {\scriptstyle r_{\mathbf{d}}}\downarrow & & \downarrow \\ \Delta_c & \xrightarrow{\psi} & \mathfrak{h}_{\mathbb{C}}/W \end{array} \tag{2.3}$$

If there are several singular points, one has to take the order to be the least common multiple of the orders at each point.

Thus, the family of deformations $\mathcal{N}' = \Delta_d \times_{r_k} \mathcal{N}$ admits a simultaneous resolution $\widehat{\mathcal{N}} \to \mathcal{N}'$ and we have a commutative holomorphic diagram



(2.4)

where the maps Ψ an $\hat{\Psi}$ restrict to fiberwise embeddings. Here we should point out that the maps also commute with the embeddings $\widehat{\Delta^2/\Gamma} \hookrightarrow \widehat{\mathcal{N}}$, $\Delta^2/\Gamma \hookrightarrow \mathcal{N}$, $\widehat{\mathbb{C}^2/\Gamma} \hookrightarrow \widehat{\mathcal{Y}}$, $\mathbb{C}^2/\Gamma \hookrightarrow \mathcal{Y}$ and the canonical inclusion $\Delta^2 \hookrightarrow \mathbb{C}^2$.

Remark 3. The fact that deformations of simple singularities do admit simultaneous resolutions after passing to a sufficiently high ramified cover, as recalled above, is the essential ingredient used in [6] to construct a simultaneous resolution for deformations of compact complex surfaces with canonical singularities.

Remark 4. Conversely, if the simultaneous resolution $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{M}} \to \Delta_d$ is already given as a data, the preimage $\widehat{\mathcal{N}}$ of $\mathcal{N}$ via the map $\widehat{\mathcal{M}} \to \mathcal{M}$ provides a simultaneous resolution of the deformation of the singularity for free. However, by a universal property of simultaneous resolutions of canonical singularities [4], it turns out that $\widehat{\Delta^2/\Gamma} \hookrightarrow \widehat{\mathcal{N}} \to \Delta_d$ must be given by the above construction.

2.2. **Kronheimer's gravitons.** The simultaneous resolution $\widehat{\mathcal{Y}} \to \mathfrak{h}_{\mathbb{C}}$ of the semi-universal family of deformations $\mathcal{Y} \to \mathfrak{h}_{\mathbb{C}}/W$ is explicitly constructed by Kronheimer. At this point, we need more details. Kronheimer actually constructs a family Y_ζ of hyperKähler manifolds, parameterized by a triple $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{h} \otimes \mathbb{R}^3$. To explain its properties, we choose a positive root system $\mathfrak{R}^+ \subset \mathfrak{h}^*$, and for a root $\theta \in \mathfrak{R}^+$ we define a hyperplane of $\mathfrak{h}$ by

$$D_\theta = \ker \theta.$$

Then the family (Y_ζ) has the following properties:

(1) Y_ζ is a smooth manifold if

$$\zeta \notin D = \cup_{\theta \in \mathfrak{R}^+} \mathbb{R}^3 \otimes D_\theta; \tag{2.5}$$

(2) all $(Y_\zeta)_{\zeta\notin D}$ are diffeomorphic to the minimal resolution $\widehat{\mathbb{C}^2/\Gamma}$ of $\mathbb{C}^2/\Gamma$; the diffeomorphism
$$F_\zeta : \widehat{\mathbb{C}^2/\Gamma} \xrightarrow{\sim} Y_\zeta$$
can be chosen so that the metric is asymptotic to the Euclidean metric: actually there is an asymptotic development at any order
$$F_\zeta^* \mathrm{g}_\zeta = g^{\mathrm{euc}} + \sum_{i=2}^{k-1} g_\zeta^i R^{-2i} + \mathcal{O}(R^{-2k}),$$
where g_ζ^i is a homogeneous polynomial in ζ of degree i; in the sequel, we will suppose that such a diffeomorphism is chosen (this is possible smoothly in the ζ parameter because $\mathfrak{h}\otimes\mathbb{R}^3 - D$ is simply connected);
(3) $H^2(Y_\zeta,\mathbb{R})$ is identified with $\mathfrak{h}$ in such a way that the homology classes of the -2 curves of the resolution get identified with the simple roots of $\mathfrak{h}$ (and so $H_2(Y_\zeta,\mathbb{Z})$ is identified with the root lattice of $\mathfrak{h}$); under this identification, the cohomology classes of the three Kähler forms $(\omega_1,\omega_2,\omega_3)$ of Y_ζ are ζ_1, ζ_2 and ζ_3;
(4) there is a SO_3 action which for $u\in SO_3$ identifies isometrically
$$Y_\zeta \longrightarrow Y_{u(\zeta)},$$
permuting $(\omega_1,\omega_2,\omega_3)$ to $u(\omega_1,\omega_2,\omega_3)$;
(5) when we want to underline the holomorphic symplectic structure $(I_1,\omega_2+i\omega_3)$ of Y_ζ, we use the notation Y_{ζ_r,ζ_c}, where $\zeta_r=\zeta_1$ and $\zeta_c=\zeta_2+i\zeta_3\in\mathfrak{h}_{\mathbb{C}}$; then there is a $\mathbb{C}^*$-action, giving for $\lambda\in\mathbb{C}^*$ an isomorphism of holomorphic symplectic manifolds, which is actually also an isometry,
$$H_\lambda : Y_{\zeta_r,\zeta_c} \longrightarrow \frac{1}{|\lambda|^2} Y_{|\lambda|^2\zeta_r,\lambda^2\zeta_c}; \tag{2.6}$$
Here the leading fraction means that the metric has been rescaled by a factor $\frac{1}{|\lambda|^2}$.
(6) if $\zeta_r\notin\cup_{\theta\in R_+} D_\theta$, then there is a map $Y_{\zeta_r,\zeta_c}\to Y_{0,\zeta_c}$ which is a minimal resolution of singularities, actually Y_{ζ_r,ζ_c} and Y_{0,ζ_c} are the fibers $\widehat{\mathcal{Y}}_{\zeta_c}$ and $\mathcal{Y}_{\zeta_c}$ of the simultaneous resolution (2.2).

Remark 5. In this paper, the notation $\mathcal{O}(R^k)$ for a function f on $\mathbb{C}^2$, or more generally a tensor, means that when R goes to ∞, then for any integer $l=0,1,\ldots$, one has $\nabla^l f=\mathcal{O}(R^{k-l})$.

We now point out the statement which will be central in this paper: it gives the geometric meaning of the walls D_θ.

Lemma 6. *Suppose θ is a root and $\zeta\notin D$, so θ corresponds to some homology class in $H_2(Y_\zeta,\mathbb{Z})$.*

1) If $\zeta_1\in\ker\theta$, then θ is a Lagrangian homology class for Y_{ζ_1,ζ_c}.

2) If $\zeta_c\in\ker\theta$, then θ is represented by a holomorphic cycle in Y_{ζ_1,ζ_c}; moreover if θ is primitive for this property (that is cannot be written as $\theta_1+\theta_2$, with $\theta_i\in\mathfrak{R}^+$ and $\zeta_c\in\ker\theta_i$), then θ is represented by a holomorphic sphere.

The condition $\zeta\notin D_\theta$ can now be understood in terms of this lemma: indeed, if $\zeta\in D_\theta$, then both $\zeta_r,\zeta_c\in\ker\theta$ which means that θ would represent at the same

time a Lagrangian class and a holomorphic cycle, which is impossible, so Y_{ζ_r,ζ_c} has to be singular.

The first part of the lemma is obvious from (iii). The second part is basically contained in the work of Brieskorn and Slodowy on Kleinian singularities. By property (vi), the holomorphic map $Y_{\zeta_r,\zeta_c} \to Y_{0,\zeta_c} = \mathcal{Y}_{\zeta_c}$ is a minimal resolution of singularities, but the semi-universal deformation $\mathcal{Y}$ is explicit and its singularities are completely understood: the discriminant locus $\mathcal{D} \subset \mathbb{C}^k = \mathfrak{h}/W$, that is the set of $v \in \mathbb{C}^k$ such that $\mathcal{Y}_v$ is singular, is exactly the branch locus of the projection $\mathfrak{h}_{\mathbb{C}} \to \mathfrak{h}_{\mathbb{C}}/W$, i.e. the projection of the kernels of the roots. (And the monodromy representation $\pi_1(\mathbb{C}^k - \mathcal{D}) \to \operatorname{Aut} H^2(\mathcal{Y}_{v_0}, \mathbb{C})$ is the standard representation of W on $\mathfrak{h}_{\mathbb{C}}$).

If $\zeta_c \in \ker\theta$ for only one root θ (this is the generic case), then $\mathcal{Y}_{\zeta_c}$ has a singular point with a singularity of type $\mathbb{C}^2/\mathbb{Z}_2$, giving a -2 holomorphic sphere in the minimal resolution Y_{ζ_r,ζ_c}, in the homology class corresponding to θ. For a general $\zeta_c \in \ker\theta$, then of course θ is still represented by a holomorphic cycle in Y_{ζ_r,ζ_c}, which might be a union of several curves if it can be decomposed as a sum of roots $\theta = \theta_1 + \cdots + \theta_\ell$ such that $\zeta_c \in \ker\theta_i$.

Remark 7. The A_k-gravitational instantons are known explicitly (multi-Eguchi-Hanson metrics given by the Gibbons-Hawking ansatz). In that case one can see explicitly the holomorphic cycles of Lemma 6.

2.3. **The tangent graviton.** We come back to our setting of a flat deformation $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ of a Kähler orbifold surface $\mathcal{X}$ with a simultaneous resolution $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{M}} \to \Delta_d$ after passing to a ramified cover. We deduce a family of deformations of the singularity $\Delta^2/\Gamma \hookrightarrow \mathcal{N} \to \Delta_c$ and a simultaneous resolution $\widehat{\Delta^2/\Gamma} \hookrightarrow \widehat{\mathcal{N}} \to \Delta_d$. As explained in §2.1 we have a morphism

$$\begin{array}{ccc} \widehat{\mathcal{N}} & \xrightarrow{\hat{\Psi}} & \widehat{\mathcal{Y}} \\ \downarrow & & \downarrow \\ \Delta_d & \xrightarrow{\hat{\psi}} & \mathfrak{h}_{\mathbb{C}} \end{array}$$

The cohomology class Ω_t defined for $t \in \Delta_d \cap \mathbb{R}^+$, restricted to $\widehat{\mathcal{N}}_t$ defines a class on $\widehat{\mathcal{Y}}_{\hat{\psi}(t)}$ identified to an element $\zeta_r(t) \in \mathfrak{h}$. So that the whole data $(\widehat{\mathcal{Y}}_{\hat{\psi}(t)}, \zeta_r(t))$ is exactly that of the Kronheimer graviton $Y_{\zeta_r(t),\zeta_c(t)}$ with the definition $\zeta_c(t) = \hat{\psi}(t)$. In view of Lemma 6, it is clear that the parameters $(\zeta_r(t), \zeta_c(t))$ satisfy condition (2.5), so $Y_{\zeta_r(t),\zeta_c(t)}$ is smooth.

The fact that Ω_t converges to an orbifold Kähler class Ω_0 means that $\zeta_r(t)$ converges to 0, and we suppose that it depends smoothly on the parameter t, including at $t = 0$.

We assume that the map

$$\begin{aligned} \zeta : \Delta_d \cap \mathbb{R}^+ &\to \mathfrak{h} \oplus \mathfrak{h}_{\mathbb{C}} \\ t &\mapsto (\zeta_r(t), \zeta_c(t)) \end{aligned}$$

does not vanish at infinite order at $t = 0$ and introduce the first nonzero derivative $\dot{\zeta}$ for some order $p > 0$. This means that

$$\zeta(t) = t^p\dot{\zeta} + O(t^{p+1}) \tag{2.7}$$

for some $p > 0$ and $\dot{\zeta} \neq 0$.

The domain $\widehat{\mathcal{N}}_t$ identifies to a small domain of $Y_{\zeta(t)}$. Zooming by a factor $\varepsilon^{-1} = t^{-p/2}$, and multiplying the Kähler class by a factor $\varepsilon^{-2} = t^{-p}$, we obtain by (2.6) that this domain is identified via $H_{t^{-p/2}}$ with a larger and larger domain in $Y_{\varepsilon^{-2}\zeta(t)}$, which converges to $Y_{\dot{\zeta}}$ on compact subsets.

This discussion motivates:

Definition 8. The Kronheimer space $Y_{\dot{\zeta}}$ is called the tangent graviton to the deformation $(\widehat{\mathcal{N}}, \Omega_t|_{\widehat{\mathcal{N}}_t})$.

Remark 9. There are choices in the construction of $\dot{\zeta}$:

- the choice of lifting to the simultaneous resolution is done up to the action of the Weyl group W, acting on the parameter $\zeta_c(t)$: but this does not change the space $Y_{\zeta_r(t),\zeta_c(t)}$;
- the choice of a coordinate in the disc: since we have chosen a real ray, the ambiguity is the rescaling by a real $\lambda > 0$, but in view of (2.6), this amounts to rescale the graviton.

Remark 10. If the restriction of Ω_t to $\widehat{\mathcal{N}}_t$ is identically zero, then $\zeta_r(t) \equiv 0$. In this case, as ζ is a holomorphic map and the nonzero derivative $\dot{\zeta}$ can be defined without restricting to a particular ray in Δ_d. Thus, all the above definition make sense if we allow $t \in \Delta_d$. This property is of special interest in the case of polarized smoothings.

3. Representing Kähler classes

The one point blowup of a complex manifold endowed with a Kähler metric ω, carries a family of Kähler metric ω_ε. This is a very nice argument due to Kodaira in which the metric ω_ε can be constructed almost explicitly. The construction of ω_ε is done by cut and paste, where the Burns-Simanca metric defined on a neighborhood of the exceptional divisor is glued with the original metric. As $\varepsilon \to 0$, the metrics ω_ε converge smoothly away from the exceptional divisor toward the original metric ω.

The aim of the current section is to prove a similar result for families of deformations of a Kähler orbifold surface with isolated singularities. In particular, we shall prove the following proposition.

Proposition 11. *Let $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ be a family of deformations of a compact complex surface with canonical singularities. Let $\overline{\omega}$ be an orbifold Kähler metric on $\mathcal{X}$ with Kähler class Ω_0 and Ω a family of Kähler classes supported by a simultaneous resolution $\widehat{\mathcal{X}} \to \widehat{\mathcal{M}} \to \Delta_d$ degenerating toward Ω_0, such that the variation of Kähler class and complex structure is non degenerate in the sense of Definition 2.*

Then, there exists a family of Kähler metrics g_t with Kähler forms ω_t on $\widehat{\mathcal{M}}_t$ defined for $t \in \Delta_d \cap (0, +\infty)$ and a smooth trivialization $\phi : \Delta_d \times \widehat{\mathcal{X}} \to \widehat{\mathcal{M}}$ identifying all the fibers $\mathcal{M}_t$ with the property that

- *$[\omega_t] = \Omega_t$;*
- *the family of metrics g_t converges in the C^2-sense toward the orbifold metric $\overline{g}$ on every compact set of $X = \widehat{\mathcal{X}} \setminus E$, where E is the exceptional divisor of $\widehat{\mathcal{X}} \to \mathcal{X}$.*

Remark 12. The above proposition also holds if Ω is only assumed to be a family of $(1,1)$-classes (instead of Kähler). As a corollary, under the assumptions of admissibility, Ω_t must be a Kähler class for $t \in (0, +\infty)$ sufficiently small.

The rest of the section will be devoted to prove the proposition as well as giving more accurate results about the behavior of ω_t as $t \to 0$.

3.1. **Summary of the setup.** We use the notations introduced at §2. As in §2.1, we shall assume to keep the notations simple that there is exactly one singularity in $\mathcal{X}$ with a neighborhood U identified to Δ^2/Γ. The minimal resolution of the singularity will be denoted $\widehat{U} \to U$ and $\widehat{\mathcal{X}} \to \mathcal{X}$. The case when several singularities occur is a straightforward generalization of the constructions explained below.

The smooth manifold or orbifold deduced from $\widehat{\mathcal{X}}$ and $\mathcal{X}$ will be denoted $\widehat{X}$ and $\overline{X}$, and X will denote the complement of the singularity in $\overline{X}$, or equivalently, the complement of the exceptional divisor in $\widehat{X}$.

Using a smooth trivialization $\phi : \Delta_d \times \widehat{X} \to \widehat{\mathcal{M}}$ of the simultaneous resolution $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{M}} \to \Delta_d$, we have a family of complex structures J_t defined on $\widehat{X}$ deduced from $\widehat{\mathcal{M}}_t$ and ϕ. The degenerating family of Kähler classes Ω_t on $(\widehat{X}, J_t)$ also provides a cohomology class $\Omega_t|_{\widehat{U}} \in H^2(\widehat{U}, \mathbb{R}) \simeq H^2(\widehat{\mathbb{C}^2/\Gamma}, \mathbb{R})$. As explained in §2.3, this is the Kähler class of a Kähler Ricci-flat metric $\mathrm{g}_{\zeta(t)}$ compatible with the complex structure $\mathrm{I}_{\zeta(t)}$ on $\widehat{\mathbb{C}^2/\Gamma}$ provided by Kronheimer's construction.

If the variation of the deformation is non degenerate, there is a scaling parameter $\varepsilon = t^{p/2}$ defined for some $p \geq 1$, and homotheties $H_{\varepsilon^{-1}}$ by a factor $t^{-p/2} = \varepsilon^{-1}$ such that we have the following properties: $H_{\varepsilon^{-1}}$ induces a map $\widehat{U} \to \widehat{\Delta^2_{\varepsilon^{-1}}/\Gamma}$ such that the image of the Kähler structure $(\mathrm{I}_{\zeta(t)}, \mathrm{g}_{\zeta(t)})$ is $(\mathrm{I}_{\varepsilon^{-2}\zeta(t)}, \varepsilon^2 \mathrm{g}_{\varepsilon^{-2}\zeta(t)})$. Up to rescaling the metric by a factor ε^{-2}, the family of Kähler structure converges to to the tangent graviton graviton $Y_{\dot{\zeta}}$ which is smooth (not in Kronheimer's wall). Thus by 2.2 (ii), we have uniform estimates in t as $R \to +\infty$ of the form

$$\mathrm{g}_{\varepsilon^{-2}\zeta(t)} = g^{\mathrm{euc}} + \xi_t \quad \text{with } \xi_t = \mathcal{O}(R^{-4})$$

where g^{euc} is the Euclidean metric on $\mathbb{C}^2/\Gamma$, and R is the Euclidean distance to the origin. The function R is related to r by the scaling factor $r = \varepsilon R$. Similar estimates hold for the complex structures

$$\mathrm{I}_{\varepsilon^{-2}\zeta(t)} = \mathrm{euc} + \mathcal{O}(R^{-4}),$$

where euc is the canonical complex structure of $\mathbb{C}^2/\Gamma$.

3.2. **Background Hermitian metrics.** Let $\overline{g}$ be the orbifold Kähler metric on $(\overline{X}, J_0)$ with Kähler form $\overline{\omega}$. Recall that the isomorphism between U and Δ^2/Γ is chosen so that the Kähler form can be written as (2.1). We modify $\overline{g}$ so that it is flat near the the singularity. For this purpose, we need to choose a gluing scale of the form

$$b = \varepsilon^\beta = t^{p\beta/2}$$

where β is a constant very close to $\frac{1}{2}$ and such that $1 > \beta > \frac{1}{2}$. The precise value of this constant will be decided later on (cf. (3.4)).

We also need to define suitable cut-off functions. We fix a standard bump function $\chi : \mathbb{R} \to \mathbb{R}$, that is a smooth non decreasing function such that $\chi(x) = 0$

for $x \leq 0$ and $\chi(x) = 1$ for $x \geq 1$. Then we choose a pair of real parameters (x_1, x_2) such that $0 < x_1 < x_2$ and define

$$\chi_{(x_1,x_2)}(x) = \chi\left(\frac{x - x_1}{x_2 - x_1}\right) \tag{3.1}$$

Let r be the function on U corresponding to the Euclidean distance to the origin via the isomorphism $U \simeq \Delta^2/\Gamma$. Then we define

$$\begin{aligned}
&\omega_{B,t} = \overline{\omega} \quad \text{away from the domain } r \leq 4b \text{ of } U \text{ and}\\
&\omega_{B,t} = dd^c_{J_0}(r^2 + \chi_{(2b,4b)}(r)\eta) \quad \text{on the domain } r \leq 4b \text{ of } U
\end{aligned}$$

so that $\omega_{B,t}$ is the Kähler form of a Kähler orbifold metric on $(\overline{X}, J_0)$ for t small enough.

Similarly, we modify the model metric $\mathrm{g}_{\varepsilon^{-2}\eta(t)}$ on $\widehat{\mathbb{C}^2/\Gamma}$ as follows:

$$\begin{aligned}
&g_{A,t} = \mathrm{g}_{\varepsilon^{-2}\zeta(t)} \quad \text{on the domain } R \leq b/\varepsilon\\
&g_{A,t} = g^{\mathrm{euc}} \quad \text{on the domain } 2b/\varepsilon \leq R,\\
&g_{A,t} = g^{\mathrm{euc}} + (1 - \chi_{(b/\varepsilon,2b/\varepsilon)}(R))\xi_t \quad \text{on the annulus } b/\varepsilon \leq R \leq 2b/\varepsilon.
\end{aligned}$$

Hence $g_{A,t}$ defines a Riemannian metric on $\widehat{\mathbb{C}^2/\Gamma}$ for t small enough.

The homothety $H_{\varepsilon^{-1}}$ identifies the annuli $b \leq r \leq 4b$ and $b/\varepsilon \leq R \leq 4b/\varepsilon$. By construction $g_{B,t}$ is Euclidean at $r = 2b$ and so is $g_{A,t}$ near $R = 2b/\varepsilon$. Identifying the annuli via $H_{\varepsilon^{-1}}$, we define a Riemannian metric on $\widehat{X}$ by

$$\begin{aligned}
\tilde{h}_t &= \varepsilon^2 H^*_{\varepsilon^{-1}} g_{A,t} \quad \text{on the domain } r \leq 2b \text{ of } \widehat{U}\\
&= g_{B,t} \quad \text{outside the domain } r \leq 2b \text{ of } \widehat{U}
\end{aligned}$$

By definition, the metric $\tilde{h}_t$ is J_t-Hermitian on the domain $r \leq b$ of $\widehat{U}$ and J_0-Hermitian on the complement of $r \leq 2b$. We construct a globally J_t-Hermitian metric on $\widehat{X}$ by introducing its projection

$$h_t(u, v) = \frac{1}{2}(\tilde{h}_t(u, v) + \tilde{h}_t(J_t u, J_t v)). \tag{3.2}$$

Similarly, one can construct $\mathrm{I}_{\varepsilon^{-2}\zeta(t)}$-Hermitian metrics $h_{A,t}$ on $\widehat{\mathbb{C}^2/\Gamma}$ deduced from $g_{A,t}$.

3.3. **Hölder spaces.** Elliptic operators, Laplacians for instance, may not be Fredholm on a singular or noncompact manifold. At this point, we ought to introduce suitable weighted Hölder spaces to deal with this issue.

Hölder spaces for ALE spaces. We consider a *radius* function ρ_A on $\widehat{\mathbb{C}^2/\Gamma}$. That is a smooth function such that $\rho_A > 0$ with the property that $\rho_A = R$, say on the domain $R \geq 2$. For $\delta \in \mathbb{R}$, we define the weighted Hölder $C^{k,\alpha}_\delta$-norm given by

$$\|f\|_{C^{k,\alpha}_\delta(\widehat{\mathbb{C}^2/\Gamma},t)} = \sum_{j=0}^{k} \sup |\rho_A^{j-\delta}\nabla^j f| + \|\rho_A^{j-\delta-\alpha}\nabla^k f\|_\alpha$$

where ($i_0 > 0$ being fixed and chosen smaller than the injectivity radius)

$$\|f\|_\alpha = \sup_{d(x,y)\leq i_0} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}.$$

Here, f can be any tensor, and the pointwise norms are taken with respect to the metric $\mathrm{g}_{\varepsilon^{-2}\zeta(t)}$ (or alternatively $g_{A,t}$).

Remark 13. All the norms on $\widehat{\mathbb{C}^2/\Gamma}$ obtained in such a way are commensurate uniformly in t. Thus, any of these norms could be used in the estimating process.

Hölder spaces on orbifold. Similarly, we define a radius function ρ_B on $\overline{X}$, that is a smooth function such that $\rho_B > 0$ and $\rho_B = r$ on U. We define the $C^{k,\alpha}_\delta(\overline{X})$ with the same formula as above, using instead the metric $\overline{g}$ (or alternatively $g_{B,t}$).

Hölder spaces on the gluing. Finally a third weighted norm can be defined on $\widehat{X}$ itself. For this purpose, we define a weight function ρ (depending on the parameters t) on $\widehat{X}$ as follows: put $\rho = \rho_B$ on the complement of $\widehat{U}$, $\rho = r$ on the domain $b \leq r \leq 1$ of $\widehat{U}$ and $\rho = \varepsilon H^*_{\varepsilon^{-1}}\rho_A$ on the rest. Using the same formula as above with metrics h_t (or alternatively $\tilde{h}_t$), we define a norm $\|f\|_{C^{k,\alpha}_\delta(\widehat{X},t)}$ on $\widehat{X}$.

Given a m-form f on $\widehat{X}$, we can interpret a $C^{k,\alpha}_\delta(t)$-estimate on f as an estimate on each piece of the manifold. We decompose $f = f_A + f_B$ where $f_A = (1 - \chi_{(b,2b)}(\rho))f$ and $f_B = \chi_{(b,2b)}(\rho)f$. Then f_A is supported on the domain $\rho \leq 2b$ and $f_A = f$ on the domain $\rho \leq b$. Similarly f_B is supported in the domain $\rho \geq b$ and $f = f_B$ on the domain $\rho \geq 2b$. Then $\|f\|_{C^{k,\alpha}_\delta(\widehat{X},t)}$ is uniformly commensurate with the norms

$$\|f_B\|_{C^{k,\alpha}_\delta(\overline{X},t)} + \varepsilon^{-m-\delta}\|(H_{\varepsilon^{-1}})_* f_A\|_{C^{k,\alpha}_\delta(\widehat{\mathbb{C}^2/\Gamma},t)} \tag{3.3}$$

where m is the degree of the form f and $H_{\varepsilon^{-1}}$ is used to identify the domain $\rho \leq 2b$ with the domain $\rho_A \leq 2b/\varepsilon$.

Remark 14. The asymptotics of the metrics $\overline{g}$, $\mathrm{g}_{\varepsilon^{-2}\zeta(t)}$ and complex structures $\mathrm{I}_{\varepsilon^{-2}\zeta(t)}$, together with the naturality of the constructions of the metrics g_A, g_B, h_A, $\tilde{h}_t$, h_t and the fact that the functions $\chi_{(c,2c)}$ are uniformly bounded in C^k_0-norm imply that the Hölder norms defined using any of these metrics lead to uniformly commensurate norms. Therefore, we could use any of these metrics for estimating the $C^{k,\alpha}_\delta$-norms.

3.4. **Background Laplacian.** The Cauchy-Riemann operator on $(\widehat{X}, J_t)$ is denoted $\bar{\partial}_{J_t}$ or $\bar{\partial}_t$. Its adjoint deduced from the Hermitian metric h_t is denoted $\bar{\partial}^*_t$. Then the Dolbeault Laplacian is given by

$$\Box_t = \bar{\partial}^*_t\bar{\partial}_t + \bar{\partial}_t\bar{\partial}^*_t$$

for (p,q)-forms on $(\widehat{X}, J_t)$.

Error terms. It should be pointed out that the Dolbeault Laplacian $\Box_t$ need not agree with the Riemannian Laplacian Δ_t of h_t. Indeed, the metric h_t is not necessarily Kähler. Thus, h_t is Hermitian, by construction, but its Kähler form $\varpi_t = h_t(J_t\cdot,\cdot)$ is not a priori closed. In this section, we investigate how close h_t is to be Kähler. In particular, we prove the following lemma:

Lemma 15. *For every $k \geq 0$, there exist a constant $C_k > 0$ such that for every $\delta < 0$ sufficiently close to -2, $t > 0$ and $\beta = \frac{2}{2-\delta}$, we have*

$$\|d\varpi_t\|_{C^{k,\alpha}_{\delta-1}(t)} \leq C_k\varepsilon^2.$$

Similarly, we have an estimate

$$\|\nabla^{LC} J_t\|_{C^{k,\alpha}_{\delta-1}(t)} \leq C_k \varepsilon^2.$$

Proof. On the compact domain $\widehat{X}\backslash\widehat{U}$, the family of complex structure J_t converges smoothly to J_0, and $|J_t - J_0| = \mathcal{O}(t^{\mathbf{d}})$ where $\mathbf{d}$ is the order of the covering (2.3). By assumption, the variation is non degenerate ($\mathbf{d} \geq p$), hence $|J_t - J_0| = \mathcal{O}(\varepsilon^2)$ on the compact domain. It follows that ϖ_t converges smoothly to $\overline{\omega}$, the Kähler form of the orbifold metric $\overline{g}$. Since $d\overline{\omega} = 0$, the estimate $\|d\varpi_t\|_{C^{1,\alpha}} = \mathcal{O}(\varepsilon^2)$ on this domain follows.

On the domain $r \leq b$, the metric $\tilde{h}_t$ is J_t-Hermitian. In fact it agrees with the model metric $\mathrm{g}_{\varepsilon^{-2}\zeta(t)}$ up to a scaling factor ε^2. Therefore $\tilde{h}_t = h_t$ and the J_t-Hermitian metric is Kähler on this domain. In particular $d\varpi_t = 0$.

On the domain $2b \leq \rho \leq 1$, we have $\tilde{h}_t = g_{B,t}$ and $J_t = \mathrm{I}_{\zeta(t)} = H_t^* \mathrm{I}_{\varepsilon^{-2}\zeta(t)}$. However $\mathrm{I}_{\varepsilon^{-2}\zeta(t)} = \mathrm{euc} + \mathcal{O}(R^{-4})$. It follows that $J_t = J_0 + \mathcal{O}(\varepsilon^4 r^{-4})$. Since $g_{B,t}$ is Kähler w.r.t. the complex structure J_0, we have $h_t = g_{B,t} + \mathcal{O}(\varepsilon^4 r^{-4})$ and $\varpi_t = \omega_{B,t} + \mathcal{O}(\varepsilon^4 r^{-4})$. Thus, $d\varpi_t = \mathcal{O}(\varepsilon^4 r^{-5})$ hence $r^{1-\delta} d\varpi_t = \mathcal{O}(\varepsilon^{4-\beta(5+\delta)})$ on the domain $2b \leq r \leq 1$. That is to say $\|d\varpi_t\|_{C^{k,\alpha}_{\delta-1}} = \mathcal{O}(\varepsilon^{4-\beta(5+\delta)})$ on the annulus $2b \leq r \leq 1$. We see that if delta is sufficiently close to -2 then the error term is a $\mathcal{O}(\varepsilon^2)$.

On the domain $b \leq \rho \leq 2b$, we can rescale via the homothety H_t and look at the construction on the annulus $b/\varepsilon \leq R \leq 2b/\varepsilon$ of $\widehat{\mathbb{C}^2/\Gamma}$. Up to a scaling factor ε^2, the metric $\tilde{h}_t$ corresponds to $g_{A,t}$. On the other had, we have $g_{A,t} = \mathrm{g}_{\varepsilon^{-2}\zeta(t)} + \mathcal{O}(R^{-4})$. It follows that the $\mathrm{I}_{\varepsilon^{-2}\zeta(t)}$-Hermitian metric $h_{A,t}$ deduced from $g_{A,t}$ satisfies the estimate $h_{A,t} = \mathrm{g}_{\varepsilon^{-2}\zeta(t)} + \mathcal{O}(R^{-4})$ and we have a similar estimate for the corresponding Kähler forms $\varpi_{A,t}$ and $\omega^{\mathrm{mod}}_{\varepsilon^{-2}\zeta(t)}$ defined using $\mathrm{I}_{\varepsilon^{-2}\zeta(t)}$. Hence $d\varpi_{A,t} = \mathcal{O}(R^{-5})$. Using the homothety again, we obtain the estimate $d\varpi_t = \mathcal{O}(\varepsilon^4 r^{-5})$ (there is an factor ε coming from the fact that we are taking the norm of a 3-form for the rescaled metric) on the annulus $b \leq r \leq 2b$. As in the previous case, we deduce an estimate $\|d\varpi_t\|_{C^{k,\alpha}_{\delta-1}} = \mathcal{O}(\varepsilon^{4-\beta(5+\delta)})$ on the annulus $b \leq r \leq 2b$ as well. □

From now on, we shall fix the gluing scale $b = \varepsilon^\beta$, with the convention

$$\boxed{\beta = \tfrac{2}{2-\delta}} \tag{3.4}$$

as in the above lemma. The point is that for $\delta \in (-2, 0)$, we have $\beta \in (\frac{1}{2}, 1)$, $\lim_{\delta \to -2} \beta = \frac{1}{2}$.

Then we deduce that the Kähler form ϖ_t is almost harmonic in the sense of the following corollary:

Corollary 16. *For $\delta < 0$ sufficiently close to -2 there exists a constant $C > 0$ such that for all $t > 0$*

$$\|\Box_t \varpi_t\|_{C^{0,\alpha}_{\delta-2}(t)} \leq C\varepsilon^2,$$

and

$$\|\Delta_t \varpi_t\|_{C^{0,\alpha}_{\delta-2}(t)} \leq C\varepsilon^2.$$

Proof. Using Lemma 15, together with the fact that $*\varpi_t = \varpi_t$ we deduce an estimate

$$\|d^*\varpi_t\|_{C^{1,\alpha}_{\delta-1}(t)} \leq C_1\varepsilon^2.$$

Therefore we have an estimate on $dd^*\varpi_t$ and $d^*d\varpi_t$ in $C^{0,\alpha}_{\delta-2}(t)$-norm and the result follows for $\Delta_t\varpi_t$.

The same proof works with the Dolbeault Laplacian. We merely use the fact that the norm of $*\bar\partial_t\varpi_t$ is controlled by the norm of $d\varpi_t$. The next step to deduce a control on $\bar\partial_t * \bar\partial_t\varpi_t$. The pointwise norm of this tensor is controlled by the pointwise norm of $\nabla^{Chern}_t * \bar\partial_t\varpi_t$. The metric being Hermitian, we only have $\nabla^{Chern}_t = \nabla^{LC}_t + T$ where T is a tensor such that $T = \mathcal{O}(\nabla^{LC}_t J_t)$. Using again 15, we conclude that

$$|\nabla^{Chern}_t * \bar\partial_t\varpi_t|\rho^{\delta-2} \leq |\nabla^{LC}_t * \bar\partial_t\varpi_t|\rho^{\delta-2} + (|T|\rho)(|*\bar\partial_t\varpi_t|\rho^{1-\delta}).$$

The estimate follows and we have the control on $\Box_t\varpi_t$ with the $C^{0,\alpha}_{\delta-2}$-norm. □

The Laplacian and gravitons. The space $\widehat{\mathbb{C}^2/\Gamma}$ is endowed with a complex structure $\mathrm{I}_{\varepsilon^{-2}\zeta(t)}$ and Kähler metric $\mathrm{g}_{\varepsilon^{-2}\zeta(t)}$. The parameter $t = 0$ corresponds to the tangent graviton. Each of these spaces has a corresponding Laplacian $\Box^A_t = \frac{1}{2}\Delta^A_t$. If δ is not an indicial root, the operator

$$\Box^A_t : C^{2,\alpha}_\delta \to C^{2,\alpha}_{\delta-2}$$

defined on (p,q)-forms with respect to $\mathrm{g}_{\varepsilon^{-2}\zeta(t)}$ is Fredholm.

Indicial roots are well understood for such operators

Lemma 17. *Every $\delta \in (-2,0)$ is not an indicial root. In the case of 1-forms every $\delta \in (-3,1)$ is not an indicial root.*

Proof. For the first part, see [10]. For the second part, one has to check that 0 is not an indicial root. This boils down to check that there are no Γ-invariant parallel 1-forms on $\mathbb{C}^2$. By duality, it follows that -2 is not an indicial root either. □

Harmonic forms on ALE spaces. The space of Harmonic forms of type (p,q) on the ALE space, denoted $\mathcal{H}^{p,q}_{A,t}$, is defined as the kernel of $\Box^A_t C^{2,\alpha}_\delta \to C^{2,\alpha}_{\delta-2}$, for $\delta \in (-2,0)$. Since there are no indicial root in the interval, it follows that the definition for $\mathcal{H}^{p,q}_{A,t}$ in independent of the choice of $\delta \in (-2,0)$. For 1-forms we could also choose $\delta \in (-3,1)$ as there are no indicial roots in this interval by Lemma 17.

If we choose δ sufficiently close to -2, we see that harmonic forms are in L^2. In fact, the decay is even better according to the following lemma

Lemma 18. *Any harmonic form $\gamma \in \mathcal{H}^{p,q}_{A,t}$ satisfies $\gamma = \mathcal{O}(R^{-3})$.*

Proof. It suffices to understand the case of a harmonic function $\gamma \in C^{2,\alpha}_\delta$. The standard theory for Laplacian on ALE spaces shows that $\gamma = cR^{-2} + \mathcal{O}(R^{-3})$ since there are no indicial roots in $(-3,-2)$. The coefficient c is a constant multiple of $\int \Box^A_t vol = 0$ (cf. [5, Theorem 8.3.6]) and the lemma follows. □

We recall some standard results for Hodge theory on ALE spaces (cf. [5] for instance): The canonical map $\mathcal{H}^{p,q}_{A,t} \to H^{p+q}(\widehat{\mathbb{C}^2/\Gamma}, \mathbb{C})$ is injective with image

denoted $H_t^{p,q}$. In addition

$$H^k(\widehat{\mathbb{C}^2/\Gamma},\mathbb{C}) = \bigoplus_{j=0}^{k} H_t^{j,j-k}$$

for all $0 < k < 4$. In particular we see that $\mathcal{H}^{1,0}_{A,t} = \mathcal{H}^{0,1}_{A,t} = 0$.

We also have the following result

Lemma 19.

$$\mathcal{H}^{2,0}_{A,t} = \mathcal{H}^{0,2}_{A,t} = 0$$

and

$$\mathcal{H}^{1,1}_{A,t} \simeq H^2(\widehat{\mathbb{C}^2/\Gamma},\mathbb{C})$$

for all t sufficiently small.

Proof. If t is sufficiently small, $\zeta' = \varepsilon^{-2}\zeta(t)$ does not belong to the wall D since the variation is assumed to be non degenerate. Similarly, let $\zeta'' = ((\zeta_r)'', (\zeta_c)'') \notin D$ be a parameter such that $\mathcal{Y}_{(\zeta_c)''}$ is isomorphic to $\widehat{\mathbb{C}^2/\Gamma}$ with its canonical complex structure. We have $H^{2,0}(\widehat{\mathbb{C}^2/\Gamma}) = H^{0,2}(\widehat{\mathbb{C}^2/\Gamma}) = 0$ (cf. [5, Theorem 8.4.2]). The semi-continuity of the dimension of the kernel for Fredholm operator forces this property to hold along the path from ζ'' to ζ' and we deduce the lemma. □

Laplacian and orbifold. One can consider the Kähler orbifold $(\overline{X}, J_0, \overline{g})$ together with its Laplacian. Alternatively, we can look at the manifold X, the smooth locus of $\overline{X}$ and the Laplacian defined between weighted Hölder spaces

$$\Box^B : C^{2,\alpha}_\delta \to C^{2,\alpha}_{\delta-2}$$

acting on (p,q)-forms with respect to J_0.

Like on the ALE space, this operator is Fredholm for $\delta \in (-2,0)$, and for $\delta \in (-3,1)$ in the case of 1-forms. Moreover, its kernel $\mathcal{H}^{p,q}_B$ corresponds to smooth harmonic forms on $\overline{X}$.

3.5. **Approximate kernel.** One can construct an approximate kernel of the operator $\Box_t$ on $(\widehat{X}, J_t, h_t)$ as follows. The spaces $\mathcal{H}^{1,1}_{A,t}$ are the fibers of a smooth (trivial) vector bundle over the base $t \in [0,d)$. Given $\gamma_{A,t} \in \mathcal{H}^{1,1}_{A,t}$ and $\gamma_B \in \mathcal{H}^{p,q}_B$, we construct a form γ'_t on $\widehat{X}$ by requiring that $\gamma'_t = H^*_{\varepsilon^{-1}}\gamma_{A,t}$ on the domain $\rho \leq b$, $\gamma'_t = \gamma_B$ on the domain $\rho \geq 4b$ and

$$\gamma'_t = (1 - \chi_{(b,2b)}(r))H^*_{\varepsilon^{-1}}\gamma_{A,t} + \chi_{(2b,4b)}(r)\gamma_B$$

Then we call γ_t the projection of γ'_t onto forms of type $(1,1)$ for the complex structure J_t. Thus we have constructed a linear map

$$\Phi_t : \mathcal{H}^{1,1}_{A,t} \oplus \mathcal{H}^{1,1}_B \to \Omega^{1,1}_{J_t}(\widehat{X}) \tag{3.5}$$

For every $t > 0$ small enough, the linear map (3.5) is injective and its image will be denoted $\mathcal{K}^{1,1}_t$.

Alternatively, using the isomorphisms $H^2(\widehat{\mathbb{C}^2/\Gamma},\mathbb{C}) \simeq \mathcal{H}^{1,1}_{A,t}$ and $H^{1,1}(\mathcal{X}) \simeq \mathcal{H}^{1,1}_B$, the map Φ_t can be though of as an isomorphism

$$\Psi_t : H^2(\widehat{\mathbb{C}^2/\Gamma},\mathbb{C}) \oplus H^{1,1}(\mathcal{X}) \to \mathcal{K}^{1,1}_t.$$

In the case of 1-forms, we have $\mathcal{H}^{1,0}_{A,t} = 0$ and $\mathcal{H}^{0,1}_{A,t} = 0$. A similar construction gives linear maps

$$\Phi_t : \mathcal{H}^{0,1}_B \to \Omega^{0,1}_{J_t}(\widehat{X}), \quad \Phi_t : \mathcal{H}^{1,0}_B \to \Omega^{1,0}_{J_t}(\widehat{X})$$

These maps are injective for t small enough and their images are denoted $\mathcal{K}^{0,1}_t$ and $\mathcal{K}^{1,0}_t$. Then, we define isomorphisms $\Psi_t : H^{0,1}(\mathcal{X}) \to \mathcal{K}^{0,1}_t$, $\Psi_t : H^{1,0}(\mathcal{X}) \to \mathcal{K}^{1,0}_t$.

Let $\|\cdot\|_A$ be an arbitrary norm on the cohomology $H^\bullet(\widehat{\mathbb{C}^2/\Gamma}, \mathbb{C})$ and $\|\cdot\|_B$ an arbitrary norm on the cohomology $H^\bullet(\overline{X}, \mathbb{C})$. We introduce a family of norms $\|\cdot\|_{\delta,t}$ on $H^2(\widehat{\mathbb{C}^2/\Gamma}, \mathbb{C}) \oplus H^2(\overline{X}, \mathbb{C}) \simeq H^2(\widehat{X}, \mathbb{C})$ given by

$$\|\Omega_A \oplus \Omega_B\|_{\delta,t} = \varepsilon^{-2-\delta}\|\Omega_A\| + \|\Omega_B\|$$

Lemma 20. *Given $k \geq 0$, there are constants $c_1, c_2 > 0$ such that for every $\delta \in (-2, 0)$ sufficiently close to -2 and every $t > 0$ sufficiently small, we have for all $\Omega \in H^2(\widehat{\mathbb{C}^2/\Gamma}, \mathbb{C}) \oplus H^{1,1}(\mathcal{X})$*

$$c_1\|\Omega\|_{\delta,t} \leq \|\Psi_t(\Omega)\|_{C^{k,\alpha}_\delta(\widehat{X},t)} \leq c_2\|\Omega\|_{\delta,t}.$$

For cohomology classes $\Xi \in H^{0,1}(\mathcal{X})$, we have a similar result with the estimate

$$c_1\|\Xi\|_B \leq \|\Psi_t(\Xi)\|_{C^{k,\alpha}_{\delta+1}(\widehat{X},t)} \leq c_2\|\Xi\|_B$$

Proof. The injection $\mathcal{H}^{1,1}_B \to \mathcal{K}^{1,1}_t$ induced by Φ_t allows to pull-back the $C^{k,\alpha}_\delta(\widehat{X}, t)$ norm. It is readily checked that this norm is uniformly commensurate with the $C^{k,\alpha}_\delta(\overline{X})$-norm.

Similarly, the injection $\mathcal{H}^{1,1}_{A,t} \to \mathcal{K}^{1,1}_t$ induced by Φ_t allows to pull-back the $C^{k,\alpha}_\delta(\widehat{\mathbb{C}^2/\Gamma}, t)$ norm. This norm is uniformly commensurate with the $C^{k,\alpha}_\delta(\overline{X})$-norm up to a factor $\varepsilon^{-\delta-2}$ by (3.3).

The proof of the second part of the statement goes along the same lines, except that we do not have to deal with harmonic forms on the ALE space. □

Uniform elliptic estimates. A key step in order to construct Kähler forms is Hodge theory. More precisely, we should control the first eigenvalues of the Dolbeault Laplacian between weighted Hölder spaces:

Proposition 21. *There exists a constant $c > 0$ such that for all $t > 0$ sufficiently small and every $(1,1)$-form γ on $(\widehat{X}, J_t)$, we have*

$$c\|\gamma\|_{C^{2,\alpha}_\delta(t)} \leq \|\gamma\|_{C^{0,\alpha}_\delta(t)} + \|\Box_t\gamma\|_{C^{0,\alpha}_{\delta-2}(t)}.$$

If in addition γ is L^2-orthogonal to the space $\mathcal{K}^{1,1}_t$ and $\delta \in (-2, 0)$ sufficiently close to -2, we have

$$c\|\gamma\|_{C^{2,\alpha}_\delta(t)} \leq \|\Box_t\gamma\|_{C^{0,\alpha}_{\delta-2}(t)}.$$

Similarly, if γ is a $(0,1)$-form orthogonal to $\mathcal{K}^{0,1}_t$, we have an estimate

$$c\|\gamma\|_{C^{2,\alpha}_{\delta+1}(t)} \leq \|\Box_t\gamma\|_{C^{0,\alpha}_{\delta-1}(t)}.$$

Proof. The first part of the proposition is standard. For the second part, let us argue by contradiction. If the proposition is not true, there are weights $\delta \in (-2, 0)$

arbitrarily close to -2 with and families of parameter $t_j \to 0$ and $(1,1)$-forms γ_j on $(\widehat{X}, J_t)$ such that

$$\|\gamma_j\|_{C^{0,\alpha}_{\delta}(t_j)} = 1, \text{ and } \|\Box_{t_j}\gamma_j\|_{C^{0,\alpha}_{\delta-2}(t_j)} \to 0. \tag{3.6}$$

The first part of the proposition provides a uniform $C^{2,\alpha}_{\delta}$ estimate on γ_j.

Then, up to extraction of a subsequence, we may assume that γ_j converges in the $C^{0,\alpha}$ sense on every compact set of X toward a form γ on X which is $\overline{g}$-harmonic and in $C^{2,\alpha}_{\delta}(\overline{X}, t_j)$. This implies that γ extends as a smooth orbifold form on the orbifold $\overline{X}$. Since each γ_j is orthogonal to $\mathcal{K}_t$ and $\delta > -3$, it follows that γ is orthogonal to $\mathcal{H}^{1,1}_B$. This forces $\gamma = 0$.

Similarly, we can transport γ_j on the domain $R \leq 4b_j/\varepsilon_j$ of $\widehat{\mathbb{C}^2/\Gamma}$ using the homothety $H_j = H_{\varepsilon_j^{-1}}$. Put $\mu_j = \varepsilon^{-2-\delta} H_j^* \gamma_j$. By definition of the norm we obtain a uniform $C^{2,\alpha}$ bound on μ_j. Up to extraction, we may assume that μ_j converge on every compact set of $\widehat{\mathbb{C}^2/\Gamma}$ to a harmonic form $\mu \in C^{2,\alpha}_{\delta}$ on the tangent graviton. Again, the fact that harmonic forms must have a strong decay forces $\mu \in C^{2,\alpha}_{-3}$ by Lemma 18. Then, the fact that γ_j is orthogonal to $\mathcal{K}^{1,1}t$ implies that μ is orthogonal to $\mathcal{K}^{1,1}_{A,t}$. We conclude $\mu = 0$.

Let m_j be a point where the function $|\rho_j^{-\delta}\gamma_j|$ is maximal equal to 1 (cf. assumption (3.6)). Up to extraction of a converging subsequence, we have either

(1) $\rho_j(m_j) \leq 2b_j$
(2) $\rho_j(m_j) \geq 2b_j$

for all j.

Case (ii): If $\rho_j(m_j)$ is bounded away from zero we clearly have a contradiction. Indeed, after further extraction, we may assume that m_j converges to a point in X. But we know that γ_j converges to 0 on every compact set of X hence hence $|\rho_j^{-\delta}\gamma_j|(m_j) \to 0$ which is impossible.

So we may assume up to extraction that $\lim \rho_j(m_j) = 0$. Using homotheties and rescaling for γ_j again, we can extract a converging subsequence on every compact set of the cone $\mathbb{C}^2 \setminus 0)/\Gamma$ such that the limit is nonvanishing, harmonic and in $C^{2,\alpha}_{\delta}$ on the cone. This is not possible for δ is not an indicial root of the Laplacian.

Case (i): If $\rho_j(m_j)/\varepsilon_j$ is bounded, we have a contradiction. The proof is the same as in the first case, using the rescaled forms μ_j instead.

If it is not bounded, we may assume that it goes to infinity after extraction. Then we use rescaling to extract a harmonic limit on the cone, exactly as in the first case. □

From approximate kernel to harmonic forms. The spaces $\mathcal{K}_t$ consist of forms which are approximately harmonic in the sense of the following lemma.

Lemma 22. *Fix $-2 < \delta < 0$ sufficiently close to -2. There is a constant $c > 0$ such that for all $\Omega_A \in H^2(\widehat{\mathbb{C}^2/\Gamma}, \mathbb{C})$, $\Omega_B \in H^{1,1}(\mathcal{X})$ and $\Omega = \Omega_A \oplus \Omega_B$, we have*

$$c\|\Box_t \Psi_t(\Omega_A)\|_{C^{0,\alpha}_{\delta-2}} \leq \varepsilon^{2-\beta(\delta+4)}\|\Omega_A\|_A$$

and

$$c\|\Box_t \Psi_t(\Omega_B)\|_{C^{0,\alpha}_{\delta-2}} \leq \varepsilon^{-\beta\delta}\|\Omega_B\|_B$$

In particular

$$c\|\Box_t\Psi_t(\Omega)\|_{C^{0,\alpha}_{\delta-2}} \leq \varepsilon^{\beta_1}\|\Omega\|_{\delta,t}$$

where $\beta_1 = \min(-\beta\delta, 4+\delta-\beta(\delta+4))$ *is very close to* 1*, by definition.*

Similarly, if $\Xi \in H^{0,1}(\mathcal{X})$*, we have*

$$c\|\Box_t\Psi_t(\Xi)\|_{C^{0,\alpha}_{\delta-1}} \leq \varepsilon^{-\beta(\delta+1)}\|\Xi\|_B.$$

Proof. The proof is similar to the one for Lemma 15 and Corollary 16. Let $\gamma_{A,t} \in \mathcal{H}^{1,1}_{A,t}$ be a family of harmonic $(1,1)$-forms on the family of ALE representing Ω_A. Then, we have uniform estimates $\gamma_{A,t} = \mathcal{O}(R^{-4})$ on the annulus $b/\varepsilon \leq R \leq 4b/\varepsilon$. Using the rescaled metric, we deduce an estimate $|\Psi_t(\Omega_A)| = \mathcal{O}(\varepsilon^2 r^{-4})$ on the annulus $b \leq r \leq 4b$. Hence $|\Box_t\Psi_t(\Omega_A)| = \mathcal{O}(\varepsilon^2 r^{-6})$ so $|r^{2-\delta}\Box_t\Psi_t(\Omega_A)| = \mathcal{O}(\varepsilon^2 r^{-4-\delta})$, that is to say $|r^{2-\delta}\Box_t\Psi_t(\Omega_A)| = \mathcal{O}(\varepsilon^{2-\beta(4+\delta)})$ on the annulus. The first part of the lemma follows.

For the second part of the statement, we start with the uniform estimate $\Psi_t(\Omega_B) = \mathcal{O}(1)$ on the annulus $b \leq r \leq 4b$. It follows that $|r^{2-\delta}\Box_t\Psi_t\Omega_B| = \mathcal{O}(\varepsilon^{-\beta\delta})$ on the annulus.

On the annulus $4b \leq r \leq 1$ we have the estimate $|J_0 - J_t| = \mathcal{O}(R^{-4}) = \mathcal{O}(\varepsilon^4 r^{-4})$. It follows that $|\Box_t\Psi_t(\Omega_B)| = \mathcal{O}(\varepsilon^4 r^{-6})$ on this annulus. Thus $|r^{2-\delta}\Box_t\Psi_t(\Omega_B)| = \mathcal{O}(\varepsilon^{4-\beta(4+\delta)})$ on the annulus $4b \leq r \leq 1$. We see that $4-\beta(4+\delta)$ goes to 3 as β is close to -2 and β close to $1/2$.

On the compact part, we have an estimate $J_0 - J_t = \mathcal{O}(\varepsilon^2)$. So the estimate $|\Box_t\Psi_t(\Omega_B)| = \mathcal{O}(\varepsilon^2)$ on the compact domain follows.

The second part of the lemma follows. The third inequality is obvious.

For the last statement, we just notice that the same estimates hold in the case of 1-forms. We merely have to replace δ by $\delta+1$). So we have the estimate $\mathcal{O}(\varepsilon^{-\beta(\delta+1)})$ on the annulus $b \leq r \leq 4b$, the estimate $\mathcal{O}(\varepsilon^{4-\beta(5+\delta)})$ on the annulus $4b \leq r \leq 1$ and $\mathcal{O}(\varepsilon^2)$ on the compact domain. □

Let $P_t : \mathcal{H}^{1,1}(\widehat{X}, J_t) \to \mathcal{K}^{1,1}_t$ be the L^2-orthogonal projection (deduced from h_t) on the space $\mathcal{K}^{1,1}_t$. Similarly, we denote $P_t : \mathcal{H}^{0,1}(\widehat{X}, J_t) \to \mathcal{K}^{0,1}_t$. This projection is very close to the identity in the sense of the following corollary.

Corollary 23. *Suppose* $-2 < \delta < 0$ *with* δ *sufficiently close to* -2*. There exists a constant* $c > 0$ *such that for all* $\gamma_t \in \mathcal{H}^{1,1}(\widehat{X}, J_t)$

$$\|\gamma_t - P_t(\gamma_t)\|_{C^{2,\alpha}_\delta} \leq c\varepsilon^{\beta_1}\|P_t(\gamma_t)\|_{C^{2,\alpha}_\delta}.$$

Similarly, if γ_t *denote a family* $\gamma_t \in \mathcal{H}^{0,1}(\widehat{X}, J_t)$ *we have*

$$\|\gamma_t - P_t(\gamma_t)\|_{C^{2,\alpha}_{\delta+1}} \leq c\varepsilon^{-\beta(\delta+1))}\|P_t(\gamma_t)\|_{C^{2,\alpha}_{\delta+1}}.$$

Proof. We write $P_t(\gamma_t) = \Psi_t(\Omega_t)$ and consider the form $\eta_t = P_t(\gamma_t) - \gamma_t$. By definition η_t is orthogonal to $\mathcal{K}_t$ and $\Box_t\eta_t = \Box_t P_t(\gamma_t) = \Box_t\Psi_t(\Omega_t)$. Lemma 22 applied to $\Psi_t(\Omega_t)$ followed by Lemma 20 and Proposition 21 give the result. □

In conclusion, P_t is an isomorphism and the operator norm $\|P_t - \mathrm{id}\|_{C^{2,\alpha}_\delta}$ is $\mathcal{O}(\varepsilon^{\beta_1})$ (and $\mathcal{O}(\varepsilon^{-\beta(\delta+1)})$ in the case of 1-forms) and the proposition below follows.

Proposition 24. *For all $k \geq 0$ there exists a constant $c > 0$ such that for all $\delta \in (-2, 0)$ sufficiently close to -2, $t > 0$ and every $(1,1)$-form β orthogonal to harmonic forms on $(\widehat{X}, J_t, h_t)$, and we have*

$$c\|\beta\|_{C^{2+k,\alpha}_{\delta}(t)} \leq \|\Box_t \beta\|_{C^{k,\alpha}_{\delta-2}(t)}.$$

In the case of $(0,1)$-forms orthogonal to harmonic forms, we have a similar estimate with

$$c\|\beta\|_{C^{2+k,\alpha}_{\delta+1}(t)} \leq \|\Box_t \beta\|_{C^{k,\alpha}_{\delta-1}(t)}.$$

3.6. **Background Kähler structure.** Recall that (J_t, h_t) is a Hermitian structure on $\widehat{X}$. However h_t is not a priori Kähler. We shall look for a nearby Hermitian metric which is Kähler.

For $t > 0$, the Kähler form ϖ_t of h_t admits a decomposition of the form

$$\varpi_t = \varpi_t^H + \varpi_t^\perp$$

where ϖ_t^H is a $\Box_t$-harmonic $(1,1)$-form on $(\widehat{X}, J_t)$ and $\varpi_t^\perp$ is L^2-orthogonal to $\Box_t$-harmonic forms.

>From this point, one can prove that the following proposition

Proposition 25. *Assuming that $\delta \in (-2, 0)$ and sufficiently close to -2, we have $\|\varpi_t - \varpi_t^H\|_{C^{2,\alpha}_{\delta}(\widehat{X},t)} = \mathcal{O}(\varepsilon^2)$.*

Proof. By definition, $\varpi_t - \varpi_t^H = \varpi_t^\perp$ and $\Box_t \varpi_t^\perp = \Box_t \varpi_t$. by Proposition 24, we have

$$\|\varpi_t - \varpi_t^H\|_{C^{2,\alpha}_{\delta}(\widehat{X},t)} = \mathcal{O}(\|\Box_t \varpi_t\|_{C^{0,\alpha}_{\delta-2}(\widehat{X},t)}).$$

The proposition follows from Corollary 16. □

By definition $\bar\partial_t \varpi_t^H = 0$ but it is not a priori d-closed. We remedy to this problem with the following lemma

Lemma 26. *There exists a $(1,1)$-form γ_t on $(\widehat{X}, J_t)$ in the cohomology class of ϖ_t^H such that $d\gamma_t = 0$ and $\|\gamma_t - \varpi_t^H\|_{C^{2,\alpha}_{\delta}(t)} = \mathcal{O}(\varepsilon^2)$.*

Proof. We now have the technical ingredients to apply the method in [16, §9.3.3]. Since $(\widehat{X}, J_t)$ is Kähler, the Frölicher exact sequence degenerates at the first page. In particular $\partial_t \varpi_t^H = \bar\partial_t \alpha_t$, for α_t a $(2,0)$-form w.r.t J_t. Here we may choose a form such that $\bar\partial_t^* \alpha_t = 0$ and α_t is orthogonal to $\Box_t$-harmonic forms.

So $\bar\partial_t \bar\alpha_t = 0$ because we are working in complex dimension 2. So $\bar\alpha_t$ defines a class in $H^{0,2}(\widehat{X}, J_t)$ and we can write $\bar\alpha_t = \bar\mu_t + \bar\partial_t \bar\beta_t$, where μ_t is a holomorphic $(2,0)$-form, β_t is a $(0,1)$-form orthogonal to harmonic forms which satisfies $\bar\partial_t^* \beta_t = 0$. Put $\gamma_t = \varpi_t^H + \bar\partial_t \bar\beta_t$. Then $\bar\partial_t \gamma_t = \bar\partial_t \varpi_t^H = 0$ and $\partial_t \gamma_t = \partial_t \varpi_t^H + \partial_t \bar\partial_t \bar\beta_t = 0$.

The next step is to estimate β_t. Using the identities $\bar\partial_t^* \partial_t \varpi_t^H = \bar\partial_t^* \bar\partial_t \alpha_t$ and $\bar\partial_t^* \alpha_t = 0$ we obtain $\bar\partial_t^* \partial_t \varpi_t^H = \Box_t \alpha_t$.

Now

$$\|\bar\partial_t^* \partial_t (\varpi_t^H - \varpi_t)\|_{C^{0,\alpha}_{\delta-2}} = \mathcal{O}(\varepsilon^2)$$

according to Proposition 25. Therefore

$$\|\bar\partial_t^* \partial_t \varpi_t^H\|_{C^{0,\alpha}_{\delta-2}} \leq \|\bar\partial_t^* \partial_t \varpi_t^H\|_{C^{0,\alpha}_{\delta-2}} + \mathcal{O}(\varepsilon^2).$$

Using Lemma 15 we obtain the estimate $\|\bar\partial_t^*\partial_t\varpi_t^H\|_{C^{0,\alpha}_{\delta-2}} = \mathcal{O}(\varepsilon^2)$ and we deduce

$$\|\Box_t\alpha_t\|_{C^{0,\alpha}_{\delta-2}} = \mathcal{O}(\varepsilon^2).$$

Proposition 24 provides the estimate

$$\|\alpha_t\|_{C^{2,\alpha}_{\delta}} = \mathcal{O}(\varepsilon^2). \tag{3.7}$$

Eventually, we would like to estimate the $(0,1)$-forms β_t. We have $\bar\partial_t^*\alpha_t = \bar\partial_t^*\bar\partial_t\beta_t = \Box_t\beta_t$ using the fact that $\bar\mu$ is harmonic and $\bar\partial_t^*\beta_t = 0$.

The estimate (3.7) provides an estimate $\|\bar\partial_t^*\alpha_t\|_{C^{1,\alpha}_{\delta-1}} = \mathcal{O}(\varepsilon^2)$ and it follows that

$$\|\Box_t\beta_t\|_{C^{1,\alpha}_{\delta-1}} = \mathcal{O}(\varepsilon^2).$$

Using the fact that β_t is orthogonal to harmonic forms and Proposition 24, we get the estimate

$$\|\beta_t\|_{C^{3,\alpha}_{\delta+1}} = \mathcal{O}(\varepsilon^2).$$

In particular this implies

$$\|\bar\partial_t\bar\beta_t\|_{C^{2,\alpha}_{\delta}} = \mathcal{O}(\varepsilon^2)$$

which proves the lemma. □

Thus we define the real closed form ω'_t of type $(1,1)$ on $(\widehat{X}, J_t)$ by taking

$$\omega'_t = \mathrm{Re}(\gamma_t)$$

were γ'_t is given by Lemma 26.

Corollary 27. *Fix $-2 < \delta < 0$ sufficiently close to -2. Define ω'_t to be the real part of γ_t. Then for all $t > 0$ sufficiently small, ω'_t defines a Kähler metric g'_t on $(\widehat{X}, J_t)$ such that $\|\varpi_t - \omega'_t\|_{C^{2,\alpha}_{\delta}(t)} = \mathcal{O}(\varepsilon^2)$.*

Proof. By definition we have $\|\varpi_t - \omega'_t\|_{C^{2,\alpha}_{\delta}(t)} = \mathcal{O}(\varepsilon^2)$. So we only need to check that the estimate is good enough to ensure that the real $(1,1)$-form ω'_t defines a metric.

On the domain $\rho \leq 4b$ of $\widehat{U}$, using the homothety $H_{\varepsilon^{-1}}$, we find an estimate

$$\|\omega^{\mathrm{mod}}_{\varepsilon^{-2}\eta(t)} - \varepsilon^{-2}(H_{\varepsilon^{-1}})_*\omega'_t\|_{C^{2,\alpha}_{\delta}(\rho_A \leq 4\varepsilon^{-\frac{1}{2}},t)} = \mathcal{O}(\varepsilon^{2+\delta})$$

Since $2+\delta > 0$ the form ω'_t is definite positive for t sufficiently small. A similar estimate on the domains $4\varepsilon^{\frac{1}{2}} \leq \rho \leq 1$ and $\widehat{X} \setminus \widehat{U}$ proves the lemma. □

Although they need not agree, the Kähler class $[\omega'_t]$ is very close to Ω_t according to the following result:

Lemma 28. *The cohomology class $[\omega'_t] \in H^2(\widehat{X}, \mathbb{R})$ satisfies*

$$\|\Omega_t - [\omega'_t]\|_{\delta,t} = \mathcal{O}(\varepsilon^2).$$

Proof. We have the decompositions $\Omega_t = \Omega_{t,A} + \Omega_{t,B}$ and $[\omega'_t] = [\omega'_t]_A + [\omega'_t]_B$ together with the estimate $\|\omega'_t - \varpi_t\|_{C^{2,\alpha}_{\delta}(t)} = \mathcal{O}(\varepsilon^2)$. On the ALE part, the form ϖ_t is closed and represents $\Omega_{A,t}$. From $|\omega'_t - \varpi_t| = \mathcal{O}(\varepsilon^{2+\delta})$ and the fact that the spheres representing the homology classes have ϖ_t-volume $\mathcal{O}(\varepsilon^2)$, we deduce that

$$|[\omega'_t]_A - \Omega_{t,A}| = \mathcal{O}(\varepsilon^{4+\delta}).$$

On the compact part, we have $|\omega'_t - \varpi_t| = \mathcal{O}(\varepsilon^2)$ and by assumption $|\Omega_{t,B} - \Omega_0| = \mathcal{O}(\varepsilon^2)$. Therefore

$$|[\omega'_t]_B - \Omega_{t,B}| = \mathcal{O}(\varepsilon^2).$$

These two estimates together prove the lemma. □

3.7. **Harmonic forms and Kähler form.** We shall now use the family of Kähler metrics g'_t with Kähler form ω'_t on $(\widehat{X}, J_t)$ provided by Corollary 27 as our favorite background metric for some $\delta \in (-2, 0)$ sufficiently close to -2. Its Laplacians will be denoted as well $\Box_t$ and $\Delta_t = 2\Box_t$.

The Mayer-Vietoris isomorphism $H^2(\widehat{X}) \to H^2(\overline{X}) \oplus H^2(\widehat{\mathbb{C}^2/\Gamma})$, gives a corresponding decomposition $\Xi_A \oplus \Xi_B$ for each cohomology class $\Xi \in H^2(\widehat{X})$. The family of norms $\|\Xi_A \oplus \Xi_B\|_{\delta,t}$ defined at §3.3 provides a norm on $H^2(\widehat{X})$ denoted $\|\Xi\|_{\delta,t}$ as well.

Then we have the following result

Proposition 29. *There are constants $c_1, c_2 > 0$ such that for every family γ_t of g'_t-harmonic 2-forms on $\widehat{X}$ with cohomology class Ξ_t, we have*

$$c_1 \|\gamma_t\|_{C^{0,\alpha}_\delta(t)} \geq \|\Xi_t\|_{t,\delta} \geq c_2 \|\gamma_t\|_{C^{0,\alpha}_\delta(t)}.$$

Proof. The proof is done by contradiction. If the second inequality does not hold, there exists a family $t_j \to 0$ and g'_{t_j}-harmonic 2-forms γ_{t_j} such that $\|\Xi_{t_j}\|_{\delta,t_j} \to 0$ and $\|\gamma_{t_j}\|_{C^{0,\alpha}_\delta(t_j)} = 1$.

Elliptic regularity gives a uniforms $C^{2,\alpha}_\delta$ estimate on γ_{t_j} by Proposition 21. Arguing exactly as in the proof of of the second part of Proposition 21 by extracting converging subsequences, we show that either

(1) Ξ_{B,t_j} converges to a non vanishing cohomology class in $H^2(\widehat{X}, \mathbb{R})$
(2) or $\varepsilon_j^{-2-\delta}\Xi_{A,t_i}$ converges to a non vanishing cohomology class in $H^2(\widehat{X}, \mathbb{R})$.

This contradicts the fact that $\lim \|\Xi_{t_j}\|_{\delta,t_j} = 0$.

For the first part of the inequality, a similar proof gives the result. □

We do not control exactly the Kähler class $[\omega'_t]$ of the Kähler metric g'_t that was just constructed. We will construct a nearby Kähler metric g_t with the Kähler class Ω_t by perturbing g'_t.

Proposition 30. *For every $t > 0$ sufficiently small, the g'_t-harmonic representative ω_t of the Kähler class Ω_t defines a Kähler metric g_t, satisfying the estimate*

$$\|\omega_t - \omega'_t\|_{C^{2,\alpha}_\delta(t)} = \mathcal{O}(\varepsilon^2).$$

Proof. According to Lemma 28

$$\|\Omega_t - [\omega_t]\|_{\delta,t} = \mathcal{O}(\varepsilon^2). \tag{3.8}$$

Then by Proposition 29 one has $\|\omega'_t - \omega_t\|_{C^{2,\alpha}_\delta(t)} = \mathcal{O}(\varepsilon^2)$. The worst value of the weight is $\varepsilon^{-\delta}$, so it implies $\varepsilon^{-\delta}|\omega'_t - \omega_t| = \mathcal{O}(\varepsilon^2)$, that is

$$|\omega'_t - \omega_t| = \mathcal{O}(\varepsilon^{2+\delta}).$$

Since $-2 < \delta < 0$, this goes to zero and ω'_t is a Kähler form for t small enough. The proposition follows. □

We summarize the results of the current section in the following theorem

Theorem 31. *Let $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ be a family of deformations of a compact complex surfaces with canonical singularities. Let $\overline{\omega}$ be an orbifold Kähler metric on $\mathcal{X}$ with Kähler class Ω_0 and Ω a family of Kähler classes supported by a simultaneous resolution $\widehat{\mathcal{X}} \to \widehat{\mathcal{M}} \to \Delta_d$ degenerating toward Ω_0, such that the variation of Kähler class and complex structure non degenerate.*

Let $\phi : \Delta_d \times \widehat{X} \to \widehat{\mathcal{M}}$ be a smooth trivialization of the family and h_t the family of Hermitian metrics on $(\widehat{X}, J_t)$ with Kähler form ϖ_t constructed at §3.2 for all $t \in \Delta_d \cap \mathbb{R}^+$ sufficiently small.

Then, there exists a family of Kähler metrics g_t with Kähler form ω_t on $(\widehat{X}, J_t)$ defined for $t \in \Delta_d \cap \mathbb{R}^+$ for all t sufficiently small with the property that

- $[\omega_t] = \Omega_t$
- *for all $\delta \in (-2, 0)$ sufficiently close to -2, we have $\|\omega_t - \varpi_t\|_{C^{2,\alpha}_\delta} = \mathcal{O}(\varepsilon^2)$.*

Proof of Proposition 11. Since h_t converges smoothly toward $\overline{g}$ on every compact set of X, the proposition is an immediate corollary of Theorem 31. □

4. CSCK metrics

Let $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ be a flat deformation and $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{M}} \to \Delta_d$ a simultaneous resolution after passing to a ramified cover. Suppose that $\mathcal{X}$ is endowed with a CSCK orbifold metric $\overline{g}$ with Kähler class Ω_0. Let Ω be a family of Kähler classes supported by the simultaneous resolution degenerating toward Ω_0, with non degenerate variation at each singularity.

Let g_t be the family of Kähler metrics on $\mathcal{M}_t$ with Kähler class Ω_t obtained in § 3. In this section, we show that g_t can be perturbed into a CSCK metric. This kind of result is now well-known by the work of Arezzo-Pacard, see also Székelyhidi whose proof is closer to ours. Our setting is slightly different, because we vary the complex structure as well. Therefore we shall give quickly some steps of the proof, but omit most technical proofs since they are similar to that in the literature.

4.1. Scalar curvature estimates.

We begin by the following estimate on the scalar curvature of the Kähler metric g'_t.

Proposition 32. *Let κ be the (constant) scalar curvature of the orbifold Kähler metric on $\overline{X}$. For all $\delta \in (-2, 0)$, the Kähler metric g'_t with Kähler class Ω_t satisfies the estimate*

$$\|\mathrm{scal}(g'_t) - \kappa\|_{C^{0,\alpha}_{\delta-2}(t)} = \mathcal{O}(\varepsilon^2).$$

Proof. As an immediate consequence of Proposition 30 and Corollary 27, we have the estimate

$$\|\mathrm{scal}(g'_t) - \mathrm{scal}(h_t)\|_{C^{0,\alpha}_{\delta-2}(t)} = \mathcal{O}(\varepsilon^2).$$

Then the proposition will be the consequence of the estimate

$$\|\mathrm{scal}(h_t) - \kappa\|_{C^{0,\alpha}_{\delta-2}(t)} = \mathcal{O}(\varepsilon^2), \tag{4.1}$$

that we now prove.

By construction $\mathrm{scal}(g_B) = \kappa$ on the domain $\rho_B \geq 4b$ and $\mathrm{scal}(g_B) = \mathcal{O}(1)$ on the annulus $2b \leq r \leq 4b$. On the other hand $\mathrm{scal}(g_{A,t}) = 0$ on the domain $R \leq b/\varepsilon$ and $\mathrm{scal}(g_{A,t}) = \mathcal{O}(\varepsilon^6 b^{-6})$ on the annulus $b/\varepsilon \leq R \leq 2b/\varepsilon$. Hence the metric $\tilde{h}_t$ obtained by gluing together g_A and g_B satisfies $|\mathrm{scal}(\tilde{h}_t)| = \mathcal{O}(\varepsilon^4 b^{-6}) + \mathcal{O}(1)$

on the annulus $b \leq r \leq 2b$. Therefore $r^{2-\delta}|\mathrm{scal}(\tilde{h}_t)| = \mathcal{O}(\varepsilon^4 b^{4-\delta}) + \mathcal{O}(b^{2-\delta}) = \mathcal{O}(\varepsilon^{4-\beta(2+\delta)}) + \mathcal{O}(\varepsilon^{\beta(2-\delta)})$ on the annulus. By definition $\beta\delta = 2$, hence

$$\rho^{2-\delta}|\mathrm{scal}(\tilde{h}_t) - \kappa| = \mathcal{O}(\varepsilon^{4-\beta(2+\delta)}) + \mathcal{O}(\varepsilon^2)$$

on the annulus $b \leq \rho \leq 4b$ which give an estimate

$$\|\mathrm{scal}(\tilde{h}_t) - \kappa\|_{C^{0,\alpha}_{\delta-2}(t)} = \mathcal{O}(\varepsilon^2)$$

on $\widehat{X}$ for δ sufficiently close to -2.

The metric h_t is obtained by projecting $\tilde{h}_t$ onto its J_t-invariant component. The estimate (4.1) follows from the estimates on $J_t - J_0$ with a proof along the same lines of Lemma 22. □

4.2. **Construction of the metrics.** It is convenient to work with the family of complex structure J_t on a fixed smooth manifold $\widehat{X}$ as in § 3. Then, we consider

$$\omega_{t,\phi} = \omega_t + dd^c_{J_t}\phi.$$

where ϕ is a function on $\widehat{X}$. If ϕ is small enough, $\omega_{t,\phi}$ is also the Kähler form of a Kähler metric $g_{t,\phi}$ on $(\widehat{X}, J_t)$ representing Ω_t. More specifically, we have the following result:

Lemma 33. *Let C be a positive constant and $\delta \in (-2, 0)$ sufficiently close to -2. Then for every $t > 0$ sufficiently small and every function ϕ such that $\|\phi\|_{C^{4,\alpha}_{\delta+2}} \leq C\varepsilon^{2-\beta(\delta+2)}$, the form $\omega_{t,\phi}$ is definite positive.*

Proof. We deduce an estimate $\|\omega_t - \omega_{t,\phi}\|_{C^{2,\alpha}_{\delta}} = \mathcal{O}(\varepsilon^{2-\beta(\delta+2)})$. The worst value of the weight is $\varepsilon^{-\delta}$ so we have an estimate $\varepsilon^{-\delta}|\omega_t - \omega_{t,\phi}| = \mathcal{O}(\varepsilon^{2-\beta(\delta+2)})$, hence $|\omega_t - \omega_{t,\phi}| = \mathcal{O}(\varepsilon^{(1-\beta)(\delta+2)})$. Since $(1-\beta)(\delta+2) > 0$, $\omega_{\phi,t}$ is definite positive for t sufficiently small. □

We want to solve the equation

$$\mathrm{scal}(g_{t,\phi}) = cst \tag{4.2}$$

where $\mathrm{scal}(g_{t,\phi})$ is the scalar curvature of the metric $g_{t,\phi}$. The linearization of this equation at $\phi = 0$ is given by a fourth order elliptic operator L_t, the Lichnerowicz operator. The idea is to apply a suitable version of the implicit function theorem in order to solve (4.2).

Proposition 34. *Suppose that $\mathcal{X}$ does not carry any nontrivial holomorphic vector field. If $-2 < \delta < 0$, then for sufficiently small $t > 0$ the operator*

$$\begin{aligned} P_t : \mathbb{R} \times C^{4,\alpha}_{\delta+2}(\widehat{X}, t) &\to C^{0,\alpha}_{\delta-2}(\widehat{X}, t) \\ (v, \phi) &\mapsto v + L_t\phi \end{aligned}$$

admits a right inverse Q_t with norm satisfying $\|Q_t\| \leq c\varepsilon^{-\beta(\delta+2)}$ for some constant c independent of t.

>From that and the initial control on the scalar curvature, it follows:

Corollary 35. *Suppose that $\mathcal{X}$ does not carry any nontrivial holomorphic vector field. For all $\delta \in (-2, 0)$ sufficiently close to -2, there exists a constant $C > 0$ such that for all $t > 0$ sufficiently small, there is a unique solution ϕ_t, up to a constant, to the equation $\mathrm{scal}(g_{t,\phi}) = cst$ with the condition that $\|\phi_t\|_{C^{4,\alpha}_{\delta+2}} \leq C\varepsilon^{2-\beta(\delta+2)}$.*

Proof. We solve the problem via the fixed point method. The equation we are interested in can be written $\mathrm{scal}(g_{t,\phi}) + v = \kappa_0$, which can be written

$$P_t(v,\phi) + N(v,\phi) = \kappa - \mathrm{scal}(h_t) \tag{4.3}$$

where N is the non linear term of the equation.

We are looking for a solution of the form $(v,\phi) = Q_t(f)$ and the equation reads

$$f = \kappa - \mathrm{scal}(h_t) - N_t \circ Q_t(f) =: T_t(f). \tag{4.4}$$

Applying the fixed point theorem, the Corollary is a consequence of the following claim.

Claim. There exists $C > 0$, such that for all $t > 0$ sufficiently small, the operator T_t maps the ball $\|f\|_{C^{0,\alpha}_{\delta-2}} \leq C\varepsilon^2$ to itself and is $\frac{1}{2}$-contractant.

Let us now prove the claim. The map $N_t(v,\phi)$ depends only on ϕ so we can write it $N_t(\phi)$. Then there exists $c_2, C_2 > 0$ such that if

$$\|\phi\|_{C^{4,\alpha}_2}, \|\psi\|_{C^{4,\alpha}_2} \leq c_2,$$

then

$$\|N_t(\phi) - N_t(\psi)\|_{C^{0,\alpha}_{\delta-2}} \leq C_2(\|\phi\|_{C^{4,\alpha}_2} + \|\psi\|_{C^{4,\alpha}_2})\|\phi - \psi\|_{C^{4,\alpha}_{\delta+2}} \tag{4.5}$$

(cf. [14, Lemma 19] and notice that the condition $\delta < 0$ required there is not needed).

By Proposition 32,

$$\|\mathrm{scal}(h_t) - \kappa\|_{C^{0,\alpha}_{\delta-2}} \leq \frac{1}{2}C\varepsilon^2$$

for some constant $C > 0$. Using Proposition 34, the bound $\|f\|_{C^{0,\alpha}_{\delta-2}} \leq C\varepsilon^2$ gives on $\phi = Q_t f$ a bound $\|\phi\|_{C^{4,\alpha}_{\delta+2}} \leq C\varepsilon^{2-\beta(\delta+2)}$ and we deduce that

$$\|\phi\|_{C^{4,\alpha}_2} \leq C\varepsilon^{2+\delta-\beta(\delta+2)} = C\varepsilon^{(\delta+2)(1-\beta)}.$$

Since $\delta + 2 > 0$ and $1 - \beta > 0$, we conclude that $\|\phi\|_{C^{4,\alpha}_2} = o(1)$. Using (4.5), we deduce that for $t > 0$ small enough, the map T_t is $1/2$-contractant on the ball $\|f\|_{C^{0,\alpha}_{\delta-2}} \leq C\varepsilon^2$. The map T_t preserves the ball since

$$\begin{aligned}
\|T_t(f)\|_{C^{0,\alpha}_{\delta-2}} &\leq \|T_t(f) - T_t(0)\|_{C^{0,\alpha}_{\delta-2}} + \|T_t(0)\|_{C^{0,\alpha}_{\delta-2}} \\
&\leq \frac{1}{2}\|f\|_{C^{0,\alpha}_{\delta-2}} + \|\mathrm{scal}(h_t) - \kappa\|_{C^{0,\alpha}_{\delta-2}} \\
&\leq C\varepsilon^2.
\end{aligned}$$

□

Proof of Proposition 34. This proposition is close to [14, Proposition 20], with the difference that the complex structure is deformed and the sign of the weight $\delta+2$ is opposite to that of [14]. The change of complex structure just gives an additional error term in the estimates so is not a substantial change. The choice of opposite sign of the weight is more important, and we explain briefly how to deal with it.

In this kind of problem, one obtains a right inverse for P_t by gluing a right inverse on the ALE space Y_t with a right inverse on the orbifold part $\mathcal{X}$. The

point is that we consider the weight $\delta+2>0$. This immediately implies that on the ALE space Y_t, the operator

$$L : C^{4,\alpha}_{\delta+2} \longrightarrow C^{\alpha}_{\delta-2}$$

is surjective (by duality the cokernel is the kernel of L in $C^{4,\alpha}_{-\delta-2}$, which is 0 because $-\delta-2<0$).

Dually, the same operator $L : C^{4,\alpha}_{\delta+2} \longrightarrow C^{\alpha}_{\delta-2}$ on the orbifold has no kernel since $\delta+2>0$ rules out the constants near the punctures, and $\mathcal{X}$ has no holomorphic vector field. But L has a cokernel: since L is selfadjoint, index theory in weighted spaces gives that the index of L is the opposite of the number of punctures k. Define a space $\hat{C}^{4,\alpha}_{\delta+2} = C^{4,\alpha}_{\delta+2} \oplus \mathbb{R}^k$ of functions of the form

$$u + \sum_1^k \lambda_i \chi_i, \quad u \in C^{4,\alpha}_{\delta+2}, (\lambda_i) \in \mathbb{R}^k, \tag{4.6}$$

where χ_i is a cutoff function which vanishes outside a small ball around the puncture x_i. For example, we can equip the space $\hat{C}^{4,\alpha}_{\delta+2}$ with the norm

$$\|f\|_{\hat{C}^{4,\alpha}_{\delta+2}} = \sum_1^k |f(x_i)| + \|df\|_{C^{3,\alpha}_{\delta+1}}. \tag{4.7}$$

Then saying that L has index $-k$ translates to the fact that

$$\tilde{L} : \mathbb{R}^k \oplus \hat{C}^{4,\alpha}_{\delta+2} \longrightarrow C^{\alpha}_{\delta-2}$$

has index 0. Since its kernel is now reduced to the constants, its cokernel is also reduced to the constants.

Now an inverse for this operator (orthogonally to the constants), combined with the inverses of the operators at the punctures, can be used to construct an approximate right inverse for P_t. One deduces that P_t has a right inverse Q_t. The only tricky point is to estimate the norm $\|Q_t\|$, because of the constants at the punctures which appear in the space $\hat{C}^{4,\alpha}_{\delta+2}$. These constants are bounded in the space $\hat{C}^{4,\alpha}_{\delta+2}$, but on the glued manifold, they blow up in the $C^{4,\alpha}_{\delta+2}$ norm. Since they are cut around the radius $r=\varepsilon^{\beta}$, they contribute at most by a factor $(\varepsilon^{\beta})^{-2-\delta} = \varepsilon^{-\beta(\delta+2)}$ which explains the norm estimate given for Q_t in the statement of the proposition. □

5. Hamiltonian stationary spheres

We now construct the Hamiltonian stationary spheres and prove Theorem D in the case of canonical singularities. The spheres are obtained as deformations of a Lagrangian sphere in the tangent graviton, which is holomorphic for another complex structure in the hyperKähler family, so is Hamiltonian stationary.

5.1. **Deformation theory.** As we shall now see, the deformation theory of Hamiltonian stationary spheres is our case is very simple. Let us remind some basic facts about Hamiltonian stationary surfaces in a Kähler 4-manifold (X,ω). A Hamiltonian stationary surface is a Lagrangian surface which is a critical point of the area for Hamiltonian deformations. This gives an equation that can be written in the following way: given an embedded surface $\iota_S : S \subset X$, let H be its mean

curvature vector and $\alpha = H \lrcorner \omega$, then $\alpha_S = \iota_S^* \alpha$ is a 1-form on S satisfying the equation $d\alpha_S = \iota_S^* \mathrm{Ric}$. Then S is Hamiltonian stationary if on S one has

$$\delta \alpha_S = 0.$$

In the hyperKähler case, one has $\alpha_S = -d\theta$, where θ is the phase defined from the holomorphic symplectic form Ω by $\iota_S^* \Omega = e^{i\theta} dVol_{g_S}$, and the equation is equivalent to $\Delta \theta = 0$. If S is holomorphic for another complex structure in the hyperKähler family, then S is minimal so obviously Hamiltonian stationary. Moreover, in that case, one obtains readily that the linearization of the equation is given by $f \mapsto \Delta_S^2 f$, where the infinitesimal deformations are parameterized by a function f on S (the graph of df in T^*S giving the infinitesimal Lagrangian deformation). The important point here is that this linearization is automatically an isomorphism

$$C^{k,\alpha}(S)/\mathbb{R} \longrightarrow C_0^{k-4,\alpha}(S),$$

where C_0 denote the functions f on S such that $\int_S f dVol_{g_S} = 0$, and the quotient by $\mathbb{R}$ is natural since constants give trivial deformations. From this we deduce immediately the following lemma:

Lemma 36. *Suppose S is a Lagrangian sphere in a hyperKähler 4-manifold (X, ω), which is holomorphic with respect to one of the complex structures of X. If (Y, ξ) is a Kähler manifold, sufficiently close to (X, ω) in $C^{2,\alpha}$ norm, such that $[S]$ remains a Lagrangian homology class for ξ ($[\xi][S] = 0$), then in (Y, ξ), in a small $C^{3,\alpha}$ neighborhood of S, there exists a unique Hamiltonian stationary sphere T such that $[T] = [S]$.*

Proof. The proof relies on two facts: if the homology class remains Lagrangian, it can be represented by a nearby Lagrangian surface ; and the linearization under Hamiltonian deformations is an isomorphism (see above). So the proof is standard, but we give a short argument where the two aspects are treated simultaneously.

We look at maps $f : S \to X$ which are deformations of the given inclusion $\iota_S : S \subset X$, and consider the operator

$$\Phi(f, \xi) = (f^* \xi, \delta_{f^* g_\xi}(H_{f,\xi} \lrcorner \xi)),$$

for ξ a nearby Kähler structure (the complex structure is also deformed), and $H_{f,\xi}$ denotes the mean curvature vector of $f(S)$ for the metric ξ. It is clear that $f(S)$ is Lagrangian stationary if and only if $\Phi(f) = 0$.

A tangent vector to the space of maps $f : S \to X$ is a section n of the normal bundle of S, but we find more convenient to represent it by the 1-form $\alpha = n \lrcorner \omega$ on S. Then at the inclusion ι_S one has

$$\frac{\partial \Phi}{\partial f}(\alpha) = (d\alpha, \delta \Delta \alpha).$$

To avoid the difference of the orders of the differential operators, we consider instead of Φ the operator

$$\Psi(f, \xi) = (f^* \xi, \Delta^{-1}_{f^* g_\xi} \delta_{f^* g_\xi}(H_{f,\xi} \lrcorner \xi)),$$

so that

$$\frac{\partial \Psi}{\partial f}(\alpha) = (d\alpha, \Delta^{-1} \delta \Delta \alpha) = (d\alpha, \delta \alpha).$$

Then Ψ is a smooth operator $C^{3,\eta} \times C^{2,\eta} \to C_0^{2,\eta} \times C_0^{2,\eta}$, where each time the index 0 means with zero integral over S (for the metric $f^* g_\xi$); here we have used

the hypothesis that Σ remains Lagrangian, so $\int_S f^*\xi = 0$. The differential $\frac{\partial \Psi}{\partial f}$ is obviously an isomorphism at $(f,\xi) = (\iota_S, \omega)$, since it identifies with $d + \delta : \Omega^1(S) \to \Omega^2(S)_0 + \Omega^0(S)_0$. The result is a consequence of the implicit function theorem. □

This lemma is useful because of the following remark:

Lemma 37. *If Y_ζ is a gravitational instanton for some $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{h} \otimes \mathbb{R}^3$, and θ is a positive root such that $\zeta_1 \in \ker \theta$, then the Lagrangian homology class corresponding to θ is represented by a holomorphic cycle for a complex structure orthogonal to I_1 (and therefore Lagrangian for I_1).*

Proof. Because $\theta(\zeta_1) = 0$, there exists an angle φ, such that

$$u = \begin{pmatrix} 0 & -\cos\varphi & \sin\varphi \\ 1 & 0 & 0 \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \in SO_3$$

sends ζ to $\xi = (\xi_1, \xi_c)$ such that $\theta(\xi_c) = 0$. By the second statement in Lemma 6, the homology class corresponding to θ is represented by a holomorphic cycle for the complex structure $u^{-1}(I_1) = -\cos\varphi I_2 + \sin\varphi I_3$. □

Together with Lemma 36 we deduce:

Corollary 38. *Under the hypotheses of Theorem C, fix a singular point $x \in \mathcal{X}$, with tangent graviton Y_{ζ_r,ζ_c}. Let θ be a positive root, such that $\zeta \in \ker\theta$ and θ is primitive for this property, so that Σ is represented in Y_{ζ_r,ζ_c} by a Hamiltonian stationary sphere S_0 (Lemma 6). Finally suppose that the 2-homology class $\Sigma \in H_2(\widehat{\mathcal{M}}_t, \mathbb{Z})$ defined by θ remains Lagrangian, that is $\Omega_t \cdot \Sigma = 0$.*

If ω_t is the CSCK metric on X_t in the class Ω_t, then for t small enough, Σ can be represented by a Lagrangian stationary sphere, close to S_0.

Here we use that ω_t is a CSCK metric only through the estimates that it satisfies. The conclusion holds also for every metric in the class Ω_t satisfying the same estimates; the CSCK metric is a canonical example of such a metric.

Proof. We blow up the metrics ω_t: from Corollary 35, on every compact, the Kähler metrics $\frac{\omega_t}{t^p}$ converge to the gravitational instanton Y_{ζ_r,ζ_c} in $C^{2,\alpha}$ (actually in C^∞). Then we apply Lemma 36. □

5.2. **Minimizing property.** We now prove the minimizing property of the spheres that we constructed, which is stated in Theorem D. The idea is that a sequence of minimizers in the homotopy class must converge to a minimizer in the tangent graviton Y_{ζ_1,ζ_c}. But since our minimizer $S_0 \subset Y_{\zeta_1,\zeta_c}$ is calibrated, it is unique, so the sequence of minimizers must converge to S_0. But then in a neighborhood of S_0 we have a uniqueness statement for our Lagrangian stationary sphere. Of course the whole process relies strongly on the fundamental results of Schoen-Wolfson [11].

Let us now give more details. For each $t > 0$, by [11] the free homotopy class of Σ can be represented by a ω_t Hamiltonian stationary, weakly conformal map $s_t : S^2 \to X_t$, each being area minimizing in the homotopy class.

Of course the same map works for $\varpi_t = \frac{\omega_t}{t^p}$, which we now choose as the metric on $\widehat{\mathcal{M}}_t$. Since the area of the Hamiltonian stationary sphere that we constructed

is $O(t)$ for ω_t, it is bounded for ϖ_t, and so is the area of the collection s_t which is not bigger.

The local geometry of $(\widehat{\mathcal{M}}_t, \varpi_t)$ is controlled: indeed, on the ALE part, $(\widehat{\mathcal{M}}_t, \varpi_t)$ converges to Y_{ζ_1,ζ_c}, while on the rest of the manifold, the curvature of $\varpi_t = \frac{\omega_t}{t^p}$ goes to zero. Moreover the injectivity radius of ϖ_t remains bounded below. It follows that the local regularity results in [11] apply uniformly in t, in particular [11, Theorem 2.8] there is a uniform Hölder bound on the s_t. Since $\Sigma^2 \neq 0$, the image of s_t must cut S_0, and it follows that for t small enough, the image of s_t is completely included in a bounded domain of the ALE part.

Now we again claim that the compactness theorem [11, Theorem 5.8] applies in our context, because the geometry is controlled. This implies that some sequence s_{t_j} for $t_j \to 0$ converges to a Lagrangian stationary, weakly conformal $W^{1,2}$ map $s_0 : S^2 \to Y_{\zeta_1,\zeta_c}$, still representing the same homotopy class. Since S_0 is calibrated, the family must identify to the sphere $S_0 \subset Y_{\zeta_1,\zeta_c}$. Since all s_t are Lagrangian stationary, by regularity [11, Theorem 4.10] the convergence is smooth. Therefore for each $t > 0$ small enough, $s_{t,1} : S^2 \to \widehat{\mathcal{M}}_t$ is an embedding converging to the standard embedding $S_0 \to Y_{\zeta_1,\zeta_c}$. By the uniqueness statement in Lemma 36 it must coincide with the Hamiltonian stationary sphere that we constructed.

6. T-singularities and $\mathbb{Q}$-Gorenstein smoothings

6.1. **CSCK metrics.** We extend our results in the setting of $\mathbb{Q}$-Gorenstein smoothings. The singularities that can appear are rational double points on one hand, on the other hand cyclic singularities of type $\frac{1}{dn^2}(1, dnm-1)$, where n and m are coprime integers. The last one is actually a $\mathbb{Z}_n$ quotient of the rational double point $\frac{1}{dn}(1,-1)$ by the action generated by (ξ, ξ^{dnm-1}), for $\xi = e^{\frac{2\pi i}{dn^2}}$.

We now explain why the results of the previous sections extend. Roughly speaking, the deformation theory of $\mathbb{Q}$-Gorenstein smoothings is the $\mathbb{Z}_n$ invariant part of the deformation theory for A_{dn-1} singularities, and the models we glue are the $\mathbb{Z}_n$ quotients of A_{dn-1} gravitational instantons. These models (the tangent gravitons) are no more hyperKähler ALE spaces, but only Kähler Ricci flat ALE spaces, and are given explicitly the Gibbons-Hawking ansatz, see [13]. Nevertheless everything done in sections 2–5 extends just by quotienting the local models by the action of $\mathbb{Z}_n$. This gives immediately Theorem A and Theorem B in the general case of T singularities.

6.2. **Hamiltonian stationary spheres.** Here there are some interesting phenomenons happening in the case of T singularities. The starting point is the same: in the setting of Theorem D, near a singular point $x \in \mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$, we have, up to a covering of group $\mathbb{Z}_n$, a graviton Y_{ζ_1,ζ_c}, where $\zeta_1 \in \mathfrak{h}_{\mathbb{R}}^{\mathbb{Z}_n}$, $\zeta_c \in \mathfrak{h}_{\mathbb{C}}^{\mathbb{Z}_n}$, and the space $\mathcal{M}_t$ is made by gluing $Y_{\zeta_1,\zeta_c}/\mathbb{Z}_n$ with the orbifold $\mathcal{X}$. In particular, given a large $\mathbb{Z}_n$ invariant region $V \subset Y_{\zeta_1,\zeta_c}$, one can identify $V/\mathbb{Z}_n$ with some region in $U \subset \mathcal{M}_t$ such that the metric $\frac{\omega_t}{t^p}$ converges to the restriction to $V/\mathbb{Z}_n$ of the ALE Ricci flat metric. Denote this projection $p : V \to U$.

If we have a root $\theta \in \mathfrak{R}^+$, such that $\langle \theta, \dot{\zeta} \rangle = 0$ and θ is primitive for this property, and moreover $\langle \zeta_1(t), \theta \rangle = 0$ for all t, then in the local $\mathbb{Z}_n$-covering V the root θ represents a $p^*\Omega_t$ Lagrangian class. Corollary 38 applies as well, and θ can be represented by a $p^*\omega_t$ Hamiltonian stationary sphere $S_t \subset V$, converging to a sphere S_0 of the graviton Y_{ζ_1,ζ_c}, which is holomorphic with respect to a complex

structure, orthogonal to I_1. Then $p(S_t)$ is a Hamiltonian stationary surface in $(\mathcal{M}_t, \omega_t)$, but $p(S_t)$ might be not embedded. This depends only on the model: $p(S_t)$ is embedded if $p(S_0)$ is, so we have to analyze the model.

Denote $\vartheta = \frac{1}{n}\sum_{g\in\mathbb{Z}_n} g\cdot\theta$. We claim that, if $\vartheta \neq 0$, then $p(S_0)$ is embedded, and we get a Hamiltonian stationary sphere in the homology class $p_*\vartheta$. This will end the proof of Theorem D. At the end of the section, we will also see some examples of behaviors when $\vartheta = 0$, resulting in the construction of a $\mathbb{R}P^2$ or a S^2 with a double point.

Fortunately the possible spheres S_0 are explicit in the Gibbons-Hawking ansatz, so we merely have to check the above claim. Therefore we remind briefly what we need [13, §5].

We consider $k+1$ distinct points $p_0, \dots, p_k \in \mathbb{R}^3$, and the harmonic function $V(x) = \frac{1}{2}\sum_i \frac{1}{|x-p_i|}$. Then $*dV$ is a closed 2-form on $\mathbb{R}^3\setminus\{p_i\}$, which is furthermore integral (indeed, the integral of $*dV$ on a small sphere around p_i is 1). Therefore $*dV = d\eta$, where η is the connection 1-form on the total space of a circle bundle $L \to \mathbb{R}^3 \setminus \{p_i\}$. Because of the topology of L near each p_i, the restriction of L to a small 2-sphere around p_i is diffeomorphic to a 3-sphere, and one can compactify L into a smooth manifold M, equipped with a projection $\pi : M \to \mathbb{R}^3$, by adding just one point above each p_i. Then it turns out that

$$g = V(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}\eta^2$$

is a smooth hyperKähler metric on M, whose three complex structures are given by

$$I_i dx_i = V^{-1}\eta, \quad I_i dx_j = dx_k,$$

where (i,j,k) is a circular permutation of $(1,2,3)$. More generally, for any $\xi = (\xi_1,\xi_2,\xi_3) \in S^2$, we have a complex structure I_ξ such that $I_\xi \sum \xi_i dx_i = V^{-1}\eta$ and I_ξ is a rotation of angle $\frac{\pi}{2}$ in the plane $(\sum \xi_i dx_i)^\perp \subset (\mathbb{R}^3)^*$; the corresponding Kähler form is $\omega_\xi = \sum_i \xi_i(dx_i\wedge\eta + V dx_j\wedge dx_k)$. All the structures are invariant under the circle action.

If the segment $[p_a, p_b]$ does not contain another point p_c, then $\pi^{-1}[a,b]$ is a 2-sphere, which is holomorphic for the complex structure I_ξ, where $\xi = \frac{a-b}{|a-b|}$, and Lagrangian for the Kähler forms ω_ζ for $\zeta \perp \xi$. It follows immediately that $\pi^{-1}[a,b]$ is Hamiltonian stationary for the Kähler metric $(M, I_\zeta, \omega_\zeta)$, where $\zeta \perp \xi$, and this gives our model spheres in the case of A_k singularities.

It is also interesting to describe the cohomology in this model: define the 2-form

$$\chi_i = df_i\wedge\eta + *df_i, \quad f_i(x) = \frac{1}{2V(x)|x-p_i|}.$$

It is easy to check that χ_i is a smooth closed antiselfdual 2-form on M, and $\sum\chi_i = 0$. This is the only relation and one gets the representation of the cohomology of M by harmonic forms:

$$H^2(M,\mathbb{R}) = \Big\{\sum_i c_i\chi_i, \sum_i c_i = 0\Big\}. \tag{6.1}$$

(This actually describes the L^2 cohomology of M, which turns out to be equal to the ordinary cohomology.) The form χ_i evaluated on the 2-sphere $\pi^{-1}[p_a,p_b]$ gives

$$\langle\chi_i, \pi^{-1}[p_a,p_b]\rangle = 2\pi(f_i(p_a) - f_i(p_b)) = 2\pi(\delta_{ia} - \delta_{ib}).$$

This formula justifies the equality (6.1).

In the case of T singularities, we start from a A_{dn-1} gravitational instanton, given from the Gibbons-Hawking ansatz with $k+1 = dn$ points. For a careful choice of the points p_a, there is a free isometric action of $\mathbb{Z}_n$ which is holomorphic for one of the complex structures: denote $z = x_1 + ix_2$ (this a I_3 holomorphic function), we consider the action of a generator $\varpi = e^{\frac{2i\pi}{n}}$ of $\mathbb{Z}_n$ given by

$$\varpi \cdot z = \varpi z, \quad \varpi \cdot \theta = \varpi^m \theta,$$

where θ is the angular coordinate, and n and m are coprime. Of course this action is well defined only if the configuration of the dn points is invariant under $z \mapsto \varpi z$, that is if they are organized into a collection of d regular centered n-polygons in planes orthogonal to the vector ∂_{x_3}. This gives all the ALE Ricci flat models that we need for singularities of class T. Incidentally, the parameters are a collection of d points (one in each polygon), that is $3d$ real parameters, modulo the translation in the ∂_{x_3} direction, so finally $3d-1$ parameters as expected (d complex parameters and $d-1$ real parameters). In particular, from (6.1), the $\mathbb{Z}_n$ invariant part of $H^2(M, \mathbb{R})$ has dimension $d-1$.

We can now describe Hamiltonian stationary spheres: as we have seen, these are given by $\pi^{-1}[p_a, p_b]$, where $p_a - p_b \perp \partial_{x_3}$, that is by the segments in the planes of the polygons. Here there are two cases:

- a segment $[p_a, p_b]$ between two points of two distinct polygons (which is possible only if the two polygons are in the same plane): if we change the point p_b in the same polygon, we do not change the homology class in the quotient, because the sides of the polygon go to zero in the homology of the quotient; therefore we can choose p_b to be the closest vertex to p_a, so that the images under $\mathbb{Z}_n$ of the segment $[p_a, p_b]$ are disjoint; therefore in the quotient we still obtain a 2-sphere with nonzero class in homology (indeed the pairing with $\frac{1}{n}\sum_{g\in\mathbb{Z}_n} g \cdot (\chi_a - \chi_b)$ is nonzero);
- a segment $[p_a, p_b]$ between two points of the same polygon: we consider only the two following cases (the images of the other spheres have more complicated crossings):
  - $[p_a, p_b]$ is an edge of the polygon; then in the quotient, the points p_a and p_b represent the same point and we obtain an immersed 2-sphere with one double point;
  - $n = 2$ and $p_b = -p_a$: then the image of $\pi^{-1}[p_a, p_b]$ is an embedded $\mathbb{R}P^2$.

  The other segments $[p_a, p_b]$ are more complicated since for a $g \in \mathbb{Z}_n$, the segment $g \cdot [p_a, p_b]$ might meet $[p_a, p_b]$ in an interior point, which means that one gets a double circle.

7. Applications

7.1. **Del Pezzo surfaces.** Our result is applied to produce extremal metrics on $\mathbb{Q}$-Gorenstein smoothings of singular extremal Del Pezzo surfaces with no nontrivial holomorphic vector field.

More precisely, let $\mathcal{X}$ be a normal Del Pezzo surface. If $\mathcal{X}$ admits a $\mathbb{Q}$-Gorenstein smoothing, then all the singularities of $\mathcal{X}$ must be of class T. So it is natural to assume that every singularity of $\mathcal{X}$ is of such type. Locally, we may pick a one parameter $\mathbb{Q}$-Gorenstein smoothing for each singularity of $\mathcal{X}$. It turns out that such local smoothing can always be globalized. In other words, one can find a

$\mathbb{Q}$-Gorenstein smoothing $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ such that the germs of deformations of the singularities are the one we started with. This result is due to the fact that $H^2(\mathcal{X}, T\mathcal{X}) = 0$, hence there is no *local to global obstruction* to deformation theory as proved in [3, Proposition 3.1]. This means that local deformations of the singularities can always be integrated into global deformations of the surface $\mathcal{X}$.

In particular, we can always construct in this way a one parameter $\mathbb{Q}$-Gorenstein smoothing $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ satisfying the non degeneracy condition in the sense of Definition 1. Suppose that $\mathcal{X}$ admits an extremal Kähler metric and no nontrivial holomorphic vector field (hence the metric must be CSCK) and that Ω_t is a family of Kähler classes on $\mathcal{M}_t$ that degenerates toward the orbifold Kähler class. Then, by Theorem A, the smoothing $\mathcal{M}_t$ admits a CSCK metric with Kähler class Ω_t for all $t > 0$ sufficiently small.

For instance, consider the Del Pezzo orbifold $\mathcal{X} = (\mathbb{CP}^1 \times \mathbb{CP}^1)/\mathbb{Z}_4$, where the action of $\mathbb{Z}_4$ on the product is spanned by

$$([u_1 : v_1], [u_2, v_2]) \mapsto ([iu_1 : v_1], [iu_2, v_2]).$$

The quotient contains exactly four singularities. Two of them are A_3 singularities whereas the two others are of type T, modelled on cyclic quotients of the form $\frac{1}{4}(1,1)$. Let V_1 be the space $\mathbb{C}^{d_1}$ of parameters for semi-universal deformations of an A_3 singularity and V_2 be the space $\mathbb{C}^{d_2}$ for the $\frac{1}{4}(1,1)$ singularity. We denote by $V_i' \simeq V_i$ (for $i = 1, 2$) the parameter spaces for deformations of the second singularities of type A_3 and $\frac{1}{4}(1,1)$ in $\mathcal{X}$.

Any 1-parameter flat family of deformations $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ induces a morphism

$$\Delta \longrightarrow V_1 \oplus V_1' \oplus V_2 \oplus V_2' \tag{7.1}$$

deduced from the deformations of each of the four singularities in $\mathcal{X}$. Conversely, by [3], any sufficiently small disc $\Delta \subset V_1 \oplus V_1' \oplus V_2 \oplus V_2'$ around the origin contained in a complex line gives rise to 1-parameter family of complex deformations $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ where the morphism (7.1) is the canonical inclusion map. Furthermore, the deformation $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ is well defined up to isomorphism for there are no nontrivial complex deformations of $\mathcal{X}$ preserving the four orbifold singularities.

There is another $\mathbb{Z}_2$ action on $\mathbb{CP}^1 \times \mathbb{CP}^1$ spanned by

$$([u_1 : v_1], [u_2 : v_2]) \mapsto ([v_1 : u_1], [v_2 : u_2]).$$

This action descends to the quotient $\mathcal{X} = (\mathbb{CP}^1 \times \mathbb{CP}^1)/\mathbb{Z}_4$. Such a $\mathbb{Z}_2$-action on $\mathcal{X}$ extends as a $\mathbb{Z}_2$ action on the space of semi-universal deformations of $\mathcal{X}$ that sends fibers onto fibers. Since the $\mathbb{Z}_2$ action flips the singularities of the same type in $\mathcal{X}$, the induced action on $V_1 \oplus V_1' \oplus V_2 \oplus V_2'$ is clear : the action merely exchanges the factors V_i and V_i' which are identified by a linear isomorphism. We choose a complex line $\mathbb{C}v \subset V_1 \oplus V_2$ corresponding to a smoothing of the underlying A_3 and $\frac{1}{4}(1,1)$ singularities. Let $v' \in V_1' \oplus V_2'$ be the image of v under the $\mathbb{Z}_2$-action. Then $v \oplus v'$ is an invariant vector under the $\mathbb{Z}_2$-action. The one parameter space of deformation $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$ induced by the choice $\Delta \subset \mathbb{C} \cdot v \oplus v'$ is acted on by $\mathbb{Z}_2$ and the fibers are invariant under the action.

The product $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be endowed with a CSCK (or Kähler-Einstein) metric by choosing a multiple of the standard Fubini-Study metric on each factor. The group $\mathbb{Z}_4$ acts isometrically on the product, thus we obtain a CSCK orbifold metric on $\mathcal{X}$.

Then we pick any family of Kähler classes Ω_t on $\mathcal{M}_t$ for $t > 0$ that degenerates toward the orbifold Kähler class. At this point, we are ready to apply Theorem A. Unfortunately, $\mathcal{X}$ does admit nontrivial holomorphic vector fields. To get around this issue, we work equivariantly, modulo the additional $\mathbb{Z}_2$ symmetry. The key point here is that $\mathcal{X}$ does not carry any nontrivial holomorphic vector field invariant under the $\mathbb{Z}_2$ action. The proof of Theorem A can be done in this $\mathbb{Z}_2$-equivariant context. It follows that $\mathcal{M}_t$ admits a CSCK metric with Kähler class Ω_t for all $t > 0$ sufficiently small.

Quite interestingly, a smoothing $\mathcal{M}_t$ for $t \neq 0$ is not diffeomorphic to the minimal resolution $\widehat{\mathcal{X}}$ of $\mathcal{X}$. In the example discussed above, $\mathcal{M}_t$, as a smooth manifold, is obtained by removing a neighborhood of the -4 exceptional spheres in $\widehat{\mathcal{X}}$ and gluing back two copies of the rational homology ball $T^*S^2/\mathbb{Z}_2 = T^*\mathbb{RP}^2$. Such an operation is known under the name of rational blowdown. Thus, $\mathcal{M}_t$ is the rational blowdown of $\widehat{\mathcal{X}}$ for $t \neq 0$. As $\widehat{\mathcal{X}}$ is an eight-point iterated blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$, one can show that $\mathcal{M}_t$ is a six-point blow up of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Working with a Kähler-Einstein orbifold metric on $\mathcal{X}$ and $\Omega_t = c_1(\mathcal{M}_t)$ we recover Tian's metric for certain special Fano surfaces very close to the boundary of the moduli space, diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ blown up six times. In addition Theorem D applies in this setting. Let E the homology class of an exceptional holomorphic sphere in the resolution $\widehat{\mathcal{X}}$. Then E can be represented by a stationary Lagrangian sphere in $\mathcal{M}_t$ for $t > 0$ sufficiently small with respect to the Kähler-Einstein metric. It is also possible to prove that $\mathcal{M}_t$ contains two stationary Lagrangian $\mathbb{RP}^2$ obtained by perturbing the zero section of the tangent graviton $T^*\mathbb{RP}^2$.

It was pointed out to us by the anonymous referee that the example $\mathcal{X} = (\mathbb{CP}^1 \times \mathbb{CP}^1)/\mathbb{Z}_4$ discussed above appears naturally in [8, Example 5.7], where the Gromov-Hausdorff compactification of the moduli space of Kähler-Einstein Del Pezzo surfaces is investigated.

Similar examples can be constructed by considering the CSCK orbifold $\mathcal{X} = (\mathbb{CP}^1 \times \mathbb{CP}^1)/\mathbb{Z}_2$, an example considered by Spotti [12]. In this case $\mathcal{X}$ has four A_1 singularities and the smoothings are diffeomorphic to the minimal resolution $\widehat{\mathcal{X}}$. The construction above provides a construction of CSCK metrics on certain four-point blowups of $\mathbb{CP}^1 \times \mathbb{CP}^1$. In particular, we can apply this to the Kähler-Einstein case. Again, we prove that the -2 exceptional spheres can be deformed to get stationary Lagrangian spheres in the smoothing endowed with its Kähler-Einstein metric.

7.2. **Geometrically ruled CSCK orbifold surfaces.** A large class of geometrically ruled CSCK orbifolds can be constructed via representation theory. The idea is to consider an orbifold Riemann surface $\overline{\Sigma}$ and a twisted product $\mathcal{X} = \overline{\Sigma} \times_\rho \mathbb{CP}^1$ where ρ is a morphism $\rho : \pi_1^{orb}(\overline{\Sigma}) \to \mathrm{SU}_2/\mathbb{Z}_2$, where $\pi_1^{orb}(\overline{\Sigma})$ is the orbifold fundamental group. Here we require that if p_i is a singular point of order q_i in $\overline{\Sigma}$ and l_i is the homotopy class of a small loop around p_i, then $\rho(l_i)$ is of order q_i.

If $\overline{\Sigma}$ has only orbifold points of order 2, then $\mathcal{X}$ has isolated singularities of type A_1. Suppose that $\overline{\Sigma}$ carries a CSCK metric (we just have to exclude the case of a teardrop with exactly one singularity of order 2). Then the local product metric provides a CSCK orbifold metric on $\mathcal{X}$.

Under the assumption that $\pi_1^{orb}(\overline{\Sigma})$ acts transitively on $\mathbb{CP}^1$ via ρ and that $\overline{\Sigma}$ is not the football, the orbifold $\mathcal{X}$ does not carry any nontrivial holomorphic vector field [10].

In the next section, we show that is is always possible to find a nondegenerate $\mathbb{Q}$-Gorenstein smoothing of $\mathcal{X}$. In particular Theorem A and Theorem D apply and we may construct some new CSCK metrics on blownup ruled surfaces with stationary Lagrangian spheres.

7.3. **Smoothing model for orbifold ruled surfaces.** The example of ruled orbifold $\mathcal{X} \to \overline{\Sigma}$ given in the previous section has singularities that come by pair above each orbifold point in $\overline{\Sigma}$.

More precisely, the local model is given by an orbifold surface $\mathcal{Y} \to \mathbb{C}/\mathbb{Z}_2$ where $\mathcal{Y} = (\mathbb{C}\times\mathbb{CP}^1)/\mathbb{Z}_2$, the action of $\mathbb{Z}_2$ is spanned by $(u,[v:w]) \mapsto (-u,[-v:w])$ and the projection map $\mathcal{Y} \to \mathbb{C}/\mathbb{Z}_2$ is just induced by the first canonical projection.

Using the coordinates $(u,[v:w])$ on the ramified cover $\mathbb{C}\times\mathbb{CP}^1$, we see that $\mathbb{Z}_2$-invariant polynomials in the chart $w \neq 0$ are generated by

$$\left\{\begin{array}{rcl} x_1 &=& u^2 \\ y_1 &=& (\frac{v}{w})^2 \\ z_1 &=& \frac{uv}{w} \end{array}\right. \tag{7.2}$$

and the equation of $\mathcal{Y}$ in this chart is given by

$$x_1 y_1 = z_1^2,$$

which is the equation of an A_1 singularity. Similarly, in the chart $v \neq 0$, we have the invariant polynomials

$$\left\{\begin{array}{rcl} x_2 &=& u^2 \\ y_2 &=& (\frac{w}{v})^2 \\ z_2 &=& \frac{uw}{v} \end{array}\right. \tag{7.3}$$

and the equation

$$x_2 y_2 = z_2^2,$$

giving a second A_1 singularity.

Putting together (7.2) and (7.3), we conclude that $\mathcal{Y}$ is the subvariety of $\mathbb{C}\times\mathbb{CP}^2$ given by the equation

$$x\alpha\beta = \gamma^2$$

where $(x,[\alpha:\beta:\gamma]) \in \mathbb{C}\times\mathbb{CP}^2$, $x = x_1 = x_2$, $\frac{\alpha}{\beta} = y_1 = \frac{1}{y_2}$, $\frac{\gamma}{\beta} = z_1$ and $\frac{\gamma}{\alpha} = z_2$.

We introduce a family of deformation $\mathcal{Y} \hookrightarrow \mathcal{N} \to \mathbb{C}^2$ where $\mathcal{N}$ is the subvariety with points $(x,[\alpha:\beta:\gamma],\varepsilon_1,\varepsilon_2) \in \mathbb{C}\times\mathbb{CP}^2\times\mathbb{C}^2$ given by the equation

$$\varepsilon_1\alpha^2 + \varepsilon_2\beta^2 + x\alpha\beta = \gamma^2$$

and the map $\mathcal{N} \to \mathbb{C}^2$ is induced by the canonical projection $(x,[\alpha:\beta:\gamma],\varepsilon_1,\varepsilon_2) \mapsto \varepsilon = (\varepsilon_1,\varepsilon_2)$. The singular locus of $\mathcal{N}$ is contained in the hypersurface $\varepsilon_1\varepsilon_2 = 0$ and if neither ε_1 nor ε_2 vanish, then the fiber $\mathcal{N}_\varepsilon$ is a smooth deformation of $\mathcal{Y}$.

Using affine coordinates, one can check that the parameters ε_i correspond to the parameters of the semi-universal family of smoothings for the A_1 singularity. In particular, if we choose a line in $\mathbb{C}^2$ distinct of the lines $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$, we obtain a nondegenerate family of smoothings of $(\mathbb{C}\times\mathbb{CP}^1)/\mathbb{Z}_2$.

We have a canonical projection $\mathcal{N} \to \mathbb{C}$ given by the coordinate x. The restriction of this map $\mathcal{N}_\varepsilon \to \mathbb{C}$ defines a ruled surface over $\mathbb{C}$ and it is smooth unless $\varepsilon_1\varepsilon_2 = 0$. One can also check that the fibers of the ruling are all $\mathbb{CP}^1$ except when

$x^2 = 4\varepsilon_1\varepsilon_2$ where the polynomial $\varepsilon_1\alpha^2 + \varepsilon_2\beta^2 + x\alpha\beta - \gamma^2$ splits as a product of two polynomials of degree 1 in (α, β, γ). If $\varepsilon_1\varepsilon_2 \neq 0$, there are two distinct fibers that consist of a union of two $\mathbb{CP}^1$ with normal crossing at one point. Topological considerations imply that the curves have selfintersection -1. In conclusion, assuming $\varepsilon_1\varepsilon_2 \neq 0$, the ruled surface $\mathcal{N}_\varepsilon \to \mathbb{C}$ is a two point blowup of $\mathbb{C} \times \mathbb{CP}^1$ at two distinct fibers.

Let $\Delta^2_{1/2}$ be the ball of radius $1/2$ is $\mathbb{C}^2$, so that $|\varepsilon_1\varepsilon_2| < 1/4$ for $\varepsilon \in \Delta^2_{1/2}$. Let $\mathcal{N}'$ (resp. $\mathcal{N}''$) be the restriction of $\mathcal{N}$ to the domain $|x| \leq 2$ and $\varepsilon \in \Delta^2_{1/2}$ (resp. $x \in A = \{1 < |x| \leq 2\}$ and $\varepsilon \in \Delta^2_{1/2}$). Accordingly, we shall denote $\mathcal{Y}' = \mathcal{N}'_0$ and $\mathcal{Y}'' = \mathcal{N}''_0$.

By definition, $\mathcal{N}''$ is equipped with a projection $\mathcal{N}'' \to A \times \Delta^2_{1/2}$. This projection is a geometric ruling (a submersive holomorphic map with fibers $\mathbb{CP}^1$) for reducible curves appear only for $x^2 = 4\varepsilon_1\varepsilon_2$. Hence there exists a biholomorphism

$$\phi : \mathcal{N}'' \to A \times \Delta^2_{1/2} \times \mathbb{CP}^1$$

which commutes with the projection maps to $A \times \Delta^2_{1/2}$.

The restriction of the map ϕ induces an isomorphism $\mathcal{Y}'' \to A \times \mathbb{CP}^1$. Taking the product with the identity, we deduce an isomorphism

$$\psi : \Delta^2_{1/2} \times \mathcal{Y}'' \to A \times \Delta^2_{1/2} \times \mathbb{CP}^1.$$

We shall the maps ϕ and ψ to construct deformations of orbifold ruled surfaces in the next section.

7.4. **Construction of deformations.** Let $\mathcal{X}$ be a compact complex orbifold surface with a holomorphic embedding

$$j : \mathcal{Y}' \hookrightarrow \mathcal{X}$$

where $\mathcal{Y}' = \mathcal{N}'_0$. We shall construct a smoothing of $\mathcal{X}$ using the smoothing $\mathcal{N}'$ described in the previous section.

Let $\mathcal{X}'$ be the complement in $\mathcal{X}$ of the domain $|x| \leq 1$ of $\mathcal{Y}'$. In particular, we have the restriction $j : \mathcal{Y}'' \hookrightarrow \mathcal{X}'$. We introduce

$$\mathcal{M} = (\Delta^2_{1/2} \times \mathcal{X}') \sqcup \mathcal{N}' / \sim$$

The equivalence relation is given as follows: if $(\varepsilon, m) \in \Delta^2_{1/2} \times \mathcal{X}'$ is such that $m = j(y)$ for some $y \in \mathcal{Y}''$, we identify the point (ε, m) with $z \in \mathcal{N}'' \subset \mathcal{N}'$ provided $\psi(\varepsilon, m) = \phi(z)$.

The complex variety $\mathcal{M}$ endowed with the canonical maps $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta^2_{1/2}$ is a flat deformation of $\mathcal{X}$ and it is non degenerate in the sense of Definition 1.

We deduce the following proposition by applying Theorem A and Theorem D to the family of smoothings $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$.

Proposition 39. *Let $\mathcal{X} = \overline{\Sigma} \times_\rho \mathbb{CP}^1$ be a CSCK geometrically ruled surface with singularities of type A_1 and no nontrivial holomorphic vector fields as described in § 7.2*

Then there exists one parameter families of nondegenerate smoothings $\mathcal{X} \hookrightarrow \mathcal{M} \to \Delta$. In particular, $\mathcal{X}$ admits CSCK smoothings with stationary Lagrangian spheres representing the vanishing Lagrangian cycles.

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