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PARABOLIC HIGGS BUNDLES AND REPRESENTATIONS OF THE FUNDAMENTAL GROUP OF A PUNCTURED SURFACE INTO A REAL GROUP

OLIVIER BIQUARD, OSCAR GARCÍA-PRADA, AND IGNASI MUNDET I RIERA

ABSTRACT. We study parabolic G -Higgs bundles over a compact Riemann surface with fixed punctures, when G is a real reductive Lie group, and establish a correspondence between these objects and representations of the fundamental group of the punctured surface in G with arbitrary holonomy around the punctures. This generalizes Simpson's results for $\mathrm{GL}(n, \mathbb{C})$ to arbitrary complex and real reductive Lie groups. Three interesting features are the relation between the parabolic degree and the Tits geometry of the boundary at infinity of the symmetric space, the treatment of the case when the logarithm of the monodromy is on the boundary of a Weyl alcove, and the correspondence of the orbits encoding the singularity via the Kostant–Sekiguchi correspondence. We also describe some special features of the moduli spaces when G is a split real form or a group of Hermitian type.

*Dedicated with admiration and gratitude to Narasimhan and Seshadri
in the fiftieth anniversary of their theorem*

1. INTRODUCTION

The relation between representations of the fundamental group of a compact Riemann surface X into a compact Lie group and holomorphic bundles on X goes back to the celebrated theorem of Narasimhan and Seshadri [57], which implies that the moduli space of irreducible representations of $\pi_1(X)$ in the unitary group U_n and the moduli space of rank n and zero degree stable holomorphic vector bundles on X are homeomorphic. Of course, this generalises the classical case of representations in $U_1 = S^1$ and their relation with the Jacobian of X . The Narasimhan–Seshadri theorem has been a paradigm and an inspiration for more than 50 years now for many similar problems. The theorem was generalised by Ramanathan [60] to representations into any compact Lie group [60]. The gauge-theoretic point of view of Atiyah and Bott [1], and the new proof of the Narasimhan–Seshadri theorem given by Donaldson following this approach [25], brought new insight and new analytic tools into the problem.

The case of representations into a non-compact reductive Lie group G required the introduction of new holomorphic objects on the Riemann surface X called G -Higgs bundles. These were introduced by Hitchin [37, 38], who established a homeomorphism between the moduli space of reductive representation in $\mathrm{SL}_2\mathbb{C}$ and polystable $\mathrm{SL}_2\mathbb{C}$ -Higgs bundles. This correspondence was generalised by Simpson to any complex reductive Lie group (and in fact, to higher dimensional Kähler manifolds) [64, 66]. The correspondence in the case of non-compact G needed an extra ingredient — not present in

the compact case — having to do with the existence of twisted harmonic maps into the symmetric space defined by G . This theorem was provided by Donaldson for $G = \mathrm{SL}_2\mathbb{C}$ [26] and by Corlette [23] for arbitrary G . In fact Corlette’s theorem, which holds for any reductive real Lie group, can be combined with an existence theorem for solutions to the Hitchin equations for a G -Higgs bundle, given by the second and third authors in collaboration with Bradlow and Gothen [19, 29], to prove the correspondence for any real reductive Lie group G . In [66] Simpson gives an indirect proof of this by embedding G in its complexification when such embedding exists.

There is another direction in which the Narasimhan–Seshadri theorem has been generalised. This is by allowing punctures in the Riemann surface. Here one is interested in studying representations of the fundamental group of the punctured surface with fixed holonomy around the punctures. These representations now relate to the parabolic vector bundles introduced by Seshadri [62]. The correspondence in this case for $G = \mathrm{U}_n$ was carried out by Mehta and Seshadri [50]. A differential geometric proof modelled on that of Donaldson for the parabolic case was given by the first author in [6]. The case of a general compact Lie group is studied in [4, 68, 3, 2] under suitable conditions on the holonomy around the punctures. One of the main issues for general G is about the appropriate generalisation of parabolic principal bundles.

The non-compactness in the group and in the surface can be combined to study representations of the fundamental group of a punctured surface into a non-compact reductive Lie group G . Simpson considered this situation when $G = \mathrm{GL}_n\mathbb{C}$ in [65]. A new ingredient in his work is the study of filtered local systems. The aim of this paper is to extend this correspondence to the case of an arbitrary real reductive Lie group G (including the case in which G is complex). We establish a one-to-one correspondence between reductive representations of the fundamental group of a punctured surface X with fixed arbitrary holonomy around the punctures and polystable parabolic G -Higgs bundles on X .

One of the main technical issues to prove our correspondence, already present in [4, 68, 3], lies in the definition of parabolic principal bundles. If G is a non-compact reductive Lie group and $H \subset G$ is a maximal compact subgroup, we need to define parabolic $H^{\mathbb{C}}$ -bundles. This involves a choice for each puncture of an element in a Weyl alcove of H — the weights. Here it is crucial to fix an alcove, whose closure contains 0. If the element is in the interior of the alcove everything goes smoothly, but if the element is in a wall of the alcove, its adjoint may have eigenvalues with modulus equal to 1 (as opposed to the elements in the interior, whose eigenvalues has modulus strictly smaller than 1), and this introduces complications in the definition of the objects, as well as in the analysis to prove our existence theorems. However, we give a suitable definition of parabolic G -Higgs bundle including the case in which the elements are in a ‘bad’ wall of the alcove, which is appropriate to carry on the analysis and to prove the correspondence with representations. Of course the need of including elements in the walls of the Weyl alcove is determined by the fact that we want to have totally arbitrary fixed holonomy (conjugacy classes) around the punctures. Our approach for the bad weights is rather pedestrian, using holomorphic bundles and gauge transformations between them which can have meromorphic singularities, so that a representation corresponds not a single holomorphic bundle, but rather a class of holomorphic bundles

equivalent under such meromorphic transformations. A more formal algebraic point of view is that of parahoric torsors [3, 13, 36], but we preferred to stick to a more concrete definition which is sufficient to state completely the correspondence (see Section 3 for a comparison between the two points of view). The choice of an alcove whose closure contains 0 is important in order to make the parahoric structures to become parabolic. Our definition is also more natural from the differential geometry viewpoint which sees holomorphic bundles only outside the punctures, and defines the sheaves of sections at the singular points by growth conditions with respect to model metrics.

Our approach involves some features that we would like to point out. As in Simpson $\mathrm{GL}_n\mathbb{C}$ -case [65], we need to consider a slight extension of representations to the more general notion of filtered local systems, what we call parabolic G -local system. The definition of parabolic degree for both, parabolic G -local systems and parabolic G -Higgs bundles, involve the Tits geometry of the boundary at infinity of the symmetric spaces G/H and $H^{\mathbb{C}}/H$, respectively. Another new feature is given by the fact that relating the data at the punctures for the representation and the parabolic Higgs bundle implies a relation between G -orbits in \mathfrak{g} (the Lie algebra of G) and $H^{\mathbb{C}}$ -orbits in $\mathfrak{m}^{\mathbb{C}}$, where $\mathfrak{m}^{\mathbb{C}}$ is the complexification of \mathfrak{m} given by the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, which in the case of nilpotent orbits is known as the Kostant–Sekiguchi correspondence [63, 46, 47, 70]. For general orbits, this correspondence is proved in [5, 9].

We give now a brief description of the different sections of the paper. In Section 2 we define parabolic principal bundles and some important notions related to them, like their sheaves of (sometimes meromorphic) local automorphisms, or their parabolic degree. In Section 3 we explain the relation between parabolic and parahoric principal bundles, and we discuss Hecke transformations; this gives a transparent interpretation of the meromorphic local automorphisms of parabolic bundles. In Section 4 we introduce parabolic G -Higgs bundles and define the stability criteria. Section 5 is one of the technical sections where we carry out the analysis to prove the correspondence between polystable parabolic G -Higgs bundles and solutions of the Hermite–Einstein or Hitchin equation. As pointed out above, one of the main difficulties here is in dealing with parabolic structures lying in a ‘bad’ wall of the Weyl alcove. In Section 6 we introduce the notion of parabolic G -local system, prove an existence theorem for harmonic reductions and establish the relation with parabolic G -Higgs bundles completing the correspondence. In Section 7 we consider the moduli spaces of parabolic G -Higgs bundles, parabolic G -local systems and representations of the fundamental group of the punctured surface, and their various correspondences among them. We also show how they depend on the parabolic weights, residues, monodromy and conjugacy classes. For example, we show that if G is complex, for weights and residues of the Higgs field satisfying a certain genericity condition, all the moduli spaces of parabolic G -Higgs bundles are diffeomorphic. This generalises the case of $\mathrm{GL}_2\mathbb{C}$ -Higgs bundles studied by Nakajima [56]. In order to avoid too lengthy a paper, in this section we simply indicate the main ingredients for the proofs of the results, and address the reader to the relevant literature, where similar issues have been treated in detail.

In Section 8 we extend results of Hitchin [39] for split real forms and the theory of maximal Higgs bundles [18, 30, 11] to the punctured set up. We finish with two appendices containing Lie-theoretic background and some considerations on the Tits

geometry of the boundary of our symmetric spaces, to define the relative degree of two parabolic subgroups, needed for the definition of parabolic degree of Higgs bundles and local systems.

We believe that the results in this paper are a starting point for applying Higgs bundle methods to the systematic study of the topology of the moduli spaces of representations of the fundamental group of a punctured surface in non-compact reductive Lie groups. And, in particular, to exhibit the existence of higher Teichmüller components in the punctured case, as we briefly show in Section 8. This has been extremely successful in the case of a compact surface (see the survey paper [28]). For punctured surfaces, Betti numbers have been computed for parabolic $\mathrm{GL}_2\mathbb{C}$ -Higgs bundles with compact holonomy by Boden and Yokogawa [15], and Nasatyr and Steer [58] in the case of rational weights, and in [31] for parabolic $\mathrm{GL}_3\mathbb{C}$ -Higgs bundles with compact holonomy. Some partial results on representations in $U_{p,q}$ were obtained in [32] making some genericity assumptions, although the relation between representations and parabolic Higgs bundles which constitutes the main result in this paper was only sketched there. The results in the present paper may also be used to translate the results on the topology of parabolic $U_{2,1}$ -Higgs bundles obtained by Logares in [48] to the context of the moduli space of representations of the fundamental group.

One direction that can also be developed from this paper is the inclusion of higher order poles in the Higgs field, relating to wild non-abelian Hodge theory. This is a problem on which we plan to come back in the future.

A substantial amount of the content of this paper appeared in the notes of a course given by the first author at the CRM (Barcelona) in 2010 [10]. A survey on the subject has been given by the third author in [54]. We apologize for having taken so long to produce this paper.

We would like to thank the referees for their useful and helpful comments.

2. PARABOLIC PRINCIPAL BUNDLES

2.1. Definition of parabolic principal bundle. Let X be a compact connected Riemann surface and let $\{x_1, \dots, x_r\}$ be a finite set of different points of X . Let $D = x_1 + \dots + x_r$ be the corresponding effective divisor.

Let $H^{\mathbb{C}}$ be a reductive complex Lie group. We fix a maximal compact subgroup $H \subset H^{\mathbb{C}}$ and a maximal torus $T \subset H$ with Lie algebra \mathfrak{t} .

Let E be a holomorphic principal $H^{\mathbb{C}}$ -bundle over X .

If M is any set on which $H^{\mathbb{C}}$ acts on the left, we denote by $E(M)$ the twisted product $E \times_{H^{\mathbb{C}}} M$. If M is a vector space (resp. complex variety) and the action of $H^{\mathbb{C}}$ on M is linear (resp. holomorphic) then $E(M) \rightarrow X$ is a vector bundle (resp. holomorphic fibration). We denote by $E(H^{\mathbb{C}})$, the $H^{\mathbb{C}}$ -fibration associated to E via the adjoint action of $H^{\mathbb{C}}$ on itself. Recall that for any $x \in X$ the fibre $E(H^{\mathbb{C}})_x$ can be identified with the set of antiequivariant maps from E_x to $H^{\mathbb{C}}$:

$$(2.1) \quad E(H^{\mathbb{C}})_x = \{\phi : E_x \rightarrow H^{\mathbb{C}} \mid \phi(eh) = h^{-1}\phi(e)h \quad \forall e \in E_x, h \in H^{\mathbb{C}}\}.$$

We fix an alcove $\mathcal{A} \subset \mathfrak{t}$ of H such that $0 \in \bar{\mathcal{A}}$ (see Appendix A.1). Let $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$ and let $P_{\alpha_i} \subset H^{\mathbb{C}}$ be the parabolic subgroup defined by α_i as in Section B.1. By Proposition A.2, for $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$ the eigenvalues of $\text{ad}(\alpha_i)$ have an absolute value smaller or equal than 1, and \mathfrak{p}_{α_i} is the sum of the eigenspaces of $\text{ad}(\alpha_i)$ for nonpositive eigenvalues of $\text{ad} \alpha_i$. We shall distinguish the subalgebra $\mathfrak{p}_{\alpha_i}^1 \subset \mathfrak{p}_{\alpha_i}$ defined by

$$\mathfrak{p}_{\alpha_i}^1 = \ker(\text{ad}(\alpha_i) + 1),$$

and the associated unipotent group $P_{\alpha_i}^1 \subset P_{\alpha_i}$. Note that $P_{\alpha_i}^1$ is normal in P_{α_i} .

We distinguish the subset $\mathcal{A}' \subset \bar{\mathcal{A}}$ of elements α such that the eigenvalues of $\text{ad} \alpha$ have an absolute value smaller than 1 (see Appendix A.1). If $\alpha_i \in \sqrt{-1}\mathcal{A}'$ then $P_{\alpha_i}^1$ is trivial.

Definition 2.1. *We define a **parabolic structure of weight α_i on E over a point x_i** as the choice of a subgroup $Q_i \subset E(H^{\mathbb{C}})_{x_i}$ with the property that there exists some trivialization $e \in E_{x_i}$ for which $P_{\alpha_i} = \{\phi(e) \mid \phi \in Q_i\}$ (here we use (2.1) to regard the elements of Q_i as maps $E_x \rightarrow H^{\mathbb{C}}$).*

Note that the choice of Q_i in the previous definition is equivalent to choosing an orbit of the action of P_{α_i} on E_x , because the normalizer of P_{α_i} inside $H^{\mathbb{C}}$ is P_{α_i} itself.

Definition 2.2. *A local holomorphic trivialization of E on a neighbourhood of x_i is said to be compatible with a parabolic structure Q_i if, seen as a local section of E in the usual way, its value at x_i is an element $e \in E_{x_i}$ satisfying $P_{\alpha_i} = \{\phi(e) \mid \phi \in Q_i\}$.*

If the parabolic structure is clear from the context, we will often simply say that a given local trivialization is compatible.

Suppose that $Q_i \subset E(H^{\mathbb{C}})_{x_i}$ is a parabolic structure of weight α_i . Take any $e \in E_{x_i}$ such that $P_{\alpha_i} = \{\phi(e) \mid \phi \in Q_i\}$. Define the following subgroup of Q_i :

$$Q_i^1 = \{\phi \in E(H^{\mathbb{C}})_{x_i} \mid \phi(e) \in P_{\alpha_i}^1\}.$$

The subgroup Q_i^1 is intrinsic, i.e. it does not depend on the choice of e . Indeed, if $e' \in E_{x_i}$ is another element satisfying $P_{\alpha_i} = \{\phi(e') \mid \phi \in Q_i\}$ then $e' = eh$ for some $h \in P_{\alpha_i}$ because P_{α_i} is its own normalizer inside $H^{\mathbb{C}}$. Since $P_{\alpha_i}^1$ is normal in P_{α_i} , we have

$$\{\phi \in E(H^{\mathbb{C}})_{x_i} \mid \phi(e) \in P_{\alpha_i}^1\} = \{\phi \in E(H^{\mathbb{C}})_{x_i} \mid \phi(eh) \in P_{\alpha_i}^1\}.$$

We denote by \mathfrak{q}_i^1 the Lie algebra of Q_i^1 .

Definition 2.3. *Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a collection of elements in $\sqrt{-1}\bar{\mathcal{A}}$. A **parabolic principal bundle over (X, D) of weight α** is a (holomorphic) principal bundle with a choice, for any i , of a parabolic structure of weight α_i on x_i .*

We will usually not specify in the notation the parabolic structure, so we will refer to them by the same symbol denoting the underlying principal bundle. Similarly we will often avoid referring to the weight α .

Definition 2.4. *Let E be a parabolic principal bundle. The sheaf $PE(H^{\mathbb{C}})$ of **parabolic gauge transformations** is defined on $X \setminus D$, as the subsheaf of the sheaf $E(H^{\mathbb{C}})$ of holomorphic gauge transformations of the principal bundle, and near a marked point*

point x_i , by the sections of the form $g(z) \exp(n/z)$ in some trivialization near x_i , such that $n \in \mathfrak{q}_i^1$ and g is holomorphic near x_i with $g(0) \in Q_i$ (this is a group because Q_i^1 is a normal abelian subgroup of Q_i).

If $\alpha_i \in \sqrt{-1}\mathcal{A}'$ then $PE(H^{\mathbb{C}})$ is the sheaf of holomorphic sections of $E(H^{\mathbb{C}})$ such that $g(x_i) \in Q_i$, because $\mathfrak{q}_i^1 = 0$.

Of course we have the Lie algebra version $PE(\mathfrak{h}^{\mathbb{C}})$, whose sections are the sections u of $E(\mathfrak{h}^{\mathbb{C}})$ such that u is meromorphic with simple pole at x_i , $\text{Res}_{x_i} u \in \mathfrak{q}_i^1$ and the constant term $u(x_i) \in \mathfrak{q}_i$.

We next give a useful geometric interpretation of parabolic gauge transformations. With respect to any compatible holomorphic trivialization of E on a neighbourhood of x_i containing a disk Δ , and any holomorphic coordinate $z : \Delta \rightarrow \mathbb{C}$ satisfying $z(x_i) = 0$, the holomorphic sections of $E(H^{\mathbb{C}})$ (resp. $E(\mathfrak{h}^{\mathbb{C}})$) are identified with maps $\Delta \rightarrow H^{\mathbb{C}}$ (resp. $\mathfrak{h}^{\mathbb{C}}$). Denoting $\Delta^* = \Delta \setminus \{x\}$, one can check immediately:

$$(2.2) \quad \begin{aligned} \Gamma(\Delta, PE(\mathfrak{h}^{\mathbb{C}})) &= \{u : \Delta^* \rightarrow \mathfrak{h}^{\mathbb{C}} \mid \text{Ad}(|z|^{-\alpha_i})u(z) \text{ is uniformly bounded on } \Delta^*\}, \\ \Gamma(\Delta, PE(H^{\mathbb{C}})) &= \{g : \Delta^* \rightarrow H^{\mathbb{C}} \mid |z|^{-\alpha_i}g(z)|z|^{\alpha_i} \text{ is uniformly bounded on } \Delta^*\}. \end{aligned}$$

This means that parabolic transformations are holomorphic transformations on the punctured disc Δ^* , which remain bounded with respect to the metric $|z|^{-2\alpha_i}$. Recall that a metric for an $H^{\mathbb{C}}$ -bundle is a reduction of structure group to H (see discussion before Definition 2.12).

Remark 2.5. If $\alpha_i \in \sqrt{-1}(\bar{\mathcal{A}} \setminus \mathcal{A}')$ then there are local sections of $PE(H^{\mathbb{C}})$ which are strictly meromorphic. These can be understood as transforming the bundle E into another principal bundle of the same topological type but with a possibly different holomorphic structure. In this case the holomorphic bundle underlying a parabolic bundle is only defined up to such transformations, although the sheaves $PE(H^{\mathbb{C}})$ and $PE(\mathfrak{h}^{\mathbb{C}})$ are unique (this admits a transparent interpretation in terms of parahoric bundles, see Section 3 below).

Remark 2.6. The definition of parabolic bundle can be extended to arbitrary α 's, i.e., not necessarily in $\sqrt{-1}\bar{\mathcal{A}}$. However, this leads to more complicated objects that can be interpreted in terms of parahoric subgroups (see for example [13]) for which our analysis to prove the Hitchin–Kobayashi correspondence does not apply directly. But we do not need to go to these objects since by taking $\alpha \in \sqrt{-1}\bar{\mathcal{A}}$ we are able to parametrize all conjugacy classes of H and also of a non-compact real reductive group G with maximal compact H (see Appendix A.1) and hence obtain all possible monodromies at the punctures.

Example 2.7. The basic example is $H = U_n$ and $H^{\mathbb{C}} = \text{GL}_n\mathbb{C}$, so an $H^{\mathbb{C}}$ -bundle is in one-to-one correspondence with the holomorphic vector bundle $V = E(\mathbb{C}^n)$ associated to the fundamental representation of $\text{GL}_n\mathbb{C}$. The parabolic structure at the point x_i is given by a flag $V_{x_i} = V_i^1 \supset V_i^2 \supset \dots \supset V_i^{\ell_i}$, with corresponding weights $1 > \alpha_i^1 > \alpha_i^2 > \dots > \alpha_i^{\ell_i} \geq 0$. The matrix α_i is diagonal, with eigenvalues the α_i^j with multiplicity $\dim(V_i^j/V_i^{j+1})$, and the eigenvalues $\alpha_i^j - \alpha_i^k$ of $\text{ad}(\alpha_i)$ belong to $(-1, 1)$. The parabolic subalgebra consists of the endomorphisms of V_{x_i} , preserving the flag: it is the sum of the non positive eigenspaces of $\text{ad} \alpha_i$. Of course, in this case $P_{\alpha_i}^1 = \{1\}$.

Remark 2.8. It is important here to note that our convention for the parabolic weights is different from the usual convention in the literature, since we take a decreasing sequence of weights. This is somehow a more natural choice in the group theoretic context: in Kähler quotients, it is natural to have an action of the group on the symmetric space on the right; then, if α_i lies in a positive Weyl chamber, it defines a point on the boundary at infinity of the symmetric space $H \backslash H^{\mathbb{C}}$, whose stabilizer is the parabolic subgroup whose Lie algebra is exactly the one defined above. See Appendix B for details.

A consequence of our convention is that the monodromy corresponding to the parabolic structure is $\exp(2\pi\sqrt{-1}\alpha_i)$ instead of $\exp(-2\pi\sqrt{-1}\alpha_i)$. Also note later the change of sign in the definition of the parabolic degree.

Remark 2.9. If we have a morphism between two reductive groups $f : L^{\mathbb{C}} \rightarrow H^{\mathbb{C}}$, then given a parabolic principal $L^{\mathbb{C}}$ -bundle E , there is an induced principal $H^{\mathbb{C}}$ -bundle E_f . Suppose the parabolic structure of E has weight α at a point x , then the description (2.2) shows that one can define parabolic transformations on E_f ; but it is not completely obvious whether this comes from a parabolic structure on E_f at x with weight $f_*\alpha$.

The simplest case (and the only one that we shall use) is that when $f_*\alpha \in \sqrt{-1}\bar{\mathcal{A}}$: in that case we have standard parabolic groups $P_\alpha \subset L^{\mathbb{C}}$ and $P_{f_*\alpha} \subset H^{\mathbb{C}}$; there is a trivialization $e \in E_x$ in which the parabolic structure is given by P_α , and we use the same e seen as a trivialization of $(E_f)_x$ to define a parabolic structure given by the group $P_{f_*\alpha}$ on E_f at x . It is not difficult to check that this does not depend on the choice of e (as one may guess already from (2.2)).

We will not describe the general case, which requires to apply a Hecke transformation to E_f in order to get back $f_*\alpha$ in the Weyl alcove, see also Section 3.

2.2. Parabolic degree of parabolic reductions. Let E be a parabolic principal bundle over (X, D) of weight α and let $Q_i \subset E(H^{\mathbb{C}})_{x_i}$ denote the parabolic subgroups specified by the parabolic structure. For any standard parabolic subgroup $P \subset H^{\mathbb{C}}$, any antidominant character χ of \mathfrak{p} (see Appendix B), and any holomorphic reduction σ of the structure group of E from $H^{\mathbb{C}}$ to P we are going to define a number $\text{pardeg}(E)(\sigma, \chi) \in \mathbb{R}$, which we call the parabolic degree. This number will be the sum of two terms, one global and independent of the parabolic structure, and the other local and depending on the parabolic structure.

Before defining the parabolic degree, let us recall that the set of holomorphic reductions of the structure group of E from $H^{\mathbb{C}}$ to P is in one-to-one correspondence with the set of holomorphic sections σ of $E(H^{\mathbb{C}}/P)$ (the latter is the bundle associated to the action of $H^{\mathbb{C}}$ on the left on $H^{\mathbb{C}}/P$). Indeed, there is a canonical identification $E(H^{\mathbb{C}}/P) \simeq E/P$ and the quotient $E \rightarrow E/P$ has the structure of a P -principal bundle. So given a section σ of $E(H^{\mathbb{C}}/P)$ the pullback $E_\sigma := \sigma^*E$ is a P -principal bundle over X , and we can identify canonically $E \simeq E_\sigma \times_P H^{\mathbb{C}}$ as principal $H^{\mathbb{C}}$ -bundles. Equivalently, we can look at E_σ as a holomorphic subvariety $E_\sigma \subset E$ invariant under the action of $P \subset H^{\mathbb{C}}$ and inheriting a structure of principal bundle.

Now fix a parabolic subgroup $P = P_s \subset H^{\mathbb{C}}$, for $s \in \sqrt{-1}\mathfrak{h}$ (see Appendix B.1), an antidominant character χ of P , and a holomorphic reduction σ of the structure group of E from $H^{\mathbb{C}}$ to P .

The global term in $\text{pardeg}(E)(\sigma, \chi)$ is the degree $\text{deg}(E)(\sigma, \chi)$ defined in [30, (A.57)]. We introduce it here using Chern–Weil theory instead of the algebraic constructions of [op. cit.].

Let E_σ be the P -principal bundle corresponding to the reduction σ . Given an antidominant character $\chi : \mathfrak{p} \rightarrow \mathbb{C}$, where \mathfrak{p} is the Lie algebra of P , the degree is defined by the Chern–Weil formula

$$(2.3) \quad \text{deg}(E)(\sigma, \chi) := \frac{\sqrt{-1}}{2\pi} \int_X \chi_*(F_A)$$

for any P -connection A on E_σ . Here, $\chi_*(F_A)$ is the 2-form resulting from applying the character χ to the \mathfrak{p} -valued 2-form F_A . Since $P \cap H$ is a maximal compact subgroup of P and the inclusion $P \cap H \hookrightarrow P$ is a homotopy equivalence, one can evaluate (2.3) using a $P \cap H$ -connection, and it follows that $\text{deg}(E)(\sigma, \chi)$ is a real number. Recall that, by definition, an antidominant character is real (see Appendix B).

At each marked point x_i we have two parabolic subgroups of $E(H^\mathbb{C})_{x_i}$ equipped with an antidominant character:

- one coming from the parabolic structure, (Q_i, χ_{α_i}) , where χ_{α_i} is the antidominant character of \mathfrak{q}_i , defined in Appendix B.1;
- one coming from the reduction, $(E_\sigma(P)_{x_i}, \chi)$.

In appendix B, we define a relative degree $\text{deg}((Q_i, \alpha_i), (E_\sigma(P)_{x_i}, \chi))$ of such a pair. Then we define the parabolic degree as follows:

$$(2.4) \quad \text{pardeg}_\alpha(E)(\sigma, \chi) := \text{deg}(E)(\sigma, \chi) - \sum_i \text{deg}((Q_i, \alpha_i), (E_\sigma(P)_{x_i}, \chi)).$$

When it is clear from the context we will omit the subscript α in the notation of the parabolic degree.

The definition of parabolic degree of a reduction also makes sense if one takes as parabolic subgroup the whole $H^\mathbb{C}$. In this case the set of antidominant characters is simply $\text{Hom}_\mathbb{R}(\mathfrak{z}, \sqrt{-1}\mathbb{R})$, where \mathfrak{z} is the centre of \mathfrak{h} . Trivially, in this case there is a unique reduction of the structure group, which we denote by σ_0 . We then define for any $\chi \in \text{Hom}_\mathbb{R}(\mathfrak{z}, \sqrt{-1}\mathbb{R})$

$$\text{pardeg}_\chi E := \text{pardeg}_\alpha(E)(\sigma_0, \chi).$$

Of course, a priori the parabolic degree of a reduction does not seem to be well defined, because we can change the bundle by a meromorphic gauge transformation. Actually we will see that:

- after a meromorphic gauge transformation, the parabolic reductions are the same (proposition 3.7);
- there is an analytic formula for the degree (next section), which by its very definition is invariant by meromorphic gauge transformation.

So the parabolic reductions and their degree make perfect sense.

2.3. Analytic formula for the degree. Let $s \in \sqrt{-1}\mathfrak{h}$ be any element, let $P = P_s \subset H^\mathbb{C}$ be the corresponding parabolic subgroup and let $\chi : \mathfrak{p} \rightarrow \mathbb{C}$ be the antidominant character defined as $\chi(\alpha) = \langle \alpha, s \rangle$, where $\langle \cdot, \cdot \rangle : \mathfrak{h}^\mathbb{C} \times \mathfrak{h}^\mathbb{C} \rightarrow \mathbb{C}$ is the extension of an

invariant scalar product on \mathfrak{h} to a Hermitian pairing. Note that the intersection of P and H can be identified with the centralizer of s in H :

$$(2.5) \quad P \cap H = Z_H(s) = \{h \in H \mid \text{Ad}(h)(s) = s\}.$$

Let σ be a holomorphic reduction of the structure group of E from $H^\mathbb{C}$ to P , and E_σ the corresponding P -principal bundle. Let $N \subset P$ be the unipotent part of P , and choose the Levi subgroup $L = Z_{H^\mathbb{C}}(s) \subset P$. Then there is a well-defined L -action on E_σ/N which turns it into a principal L -bundle, which we denote $E_{\sigma,L}$. (In the vector bundle case, the P -reduction is a flag of sub-vector bundles, and the L -bundle is the associated graded bundle.) Note that since $L = Z_{H^\mathbb{C}}(s)$, the element $s \in \sqrt{-1}\mathfrak{h}$ defines a canonical section

$$(2.6) \quad s_\sigma \in \Gamma(E_{\sigma,L}(\mathfrak{l} \cap \sqrt{-1}\mathfrak{h})).$$

Let h be a smooth metric on E , defined on the whole curve X : this is a reduction of the structure group of E from $H^\mathbb{C}$ to H . We denote by \mathbf{E} the resulting H -principal bundle. Combining σ and h we obtain a reduction of the structure group of E from $H^\mathbb{C}$ to $P \cap H$, and we denote by \mathbf{E}_σ the resulting bundle. But $P \cap H$ is a compact form of L , and the complexified bundle clearly identifies to $E_{\sigma,L}$, so we can also think of \mathbf{E}_σ as the bundle $E_{\sigma,L}$ equipped with the metric $h_{\sigma,L}$ induced by h . From this point of view, the section s_σ above can be seen as a section

$$(2.7) \quad s_{\sigma,h} \in \Gamma(\mathbf{E}_\sigma(\sqrt{-1}\mathfrak{h})) \simeq \Gamma(\mathbf{E}(\sqrt{-1}\mathfrak{h})).$$

Note the difference between (2.6) and (2.7): the section s_σ is canonical, while different choices of h lead to different sections $s_{\sigma,h}$ of $E(\mathfrak{h}^\mathbb{C})$.

We now introduce some notation. Let V be a Hermitian vector space, let $\rho : \mathfrak{h} \rightarrow \mathfrak{u}(V)$ be a morphism of Lie algebras, and denote also by $\rho : \mathfrak{h}^\mathbb{C} \rightarrow \text{End } V$ its complex extension. Choose elements $a \in \sqrt{-1}\mathfrak{h}$ and $v \in V$. Then $\rho(a)$ diagonalizes and has real eigenvalues, so that we may write $v = \sum v_j$ in such a way that $\rho(a)(v) = \sum l_j v_j$. Now, for any function $f : \mathbb{R} \rightarrow \mathbb{R}$ we define $f(a)(v) := \sum f(l_j) v_j$.

Lemma 2.10. *Define the function $\varpi : \mathbb{R} \rightarrow \mathbb{R}$ as $\varpi(0) = 0$ and as $\varpi(x) = x^{-1}$ if $x \neq 0$. Applying the previous definition to the adjoint representation, and extending it to sections of $E(\mathfrak{h}^\mathbb{C}) \otimes K$, we have:*

$$\text{deg}(E)(\sigma, \chi) = \frac{\sqrt{-1}}{2\pi} \int_X \langle F_h, s_{\sigma,h} \rangle - \langle \varpi(s_{\sigma,h})(\bar{\partial}s_{\sigma,h}), \bar{\partial}s_{\sigma,h} \rangle.$$

Here F_h is the curvature of the unique connection compatible with h and the holomorphic structure of E , the Chern connection [67]. The proof follows Chern–Weil theory and from identifying the RHS of the formula with the curvature $F_{h,L}$ of the Chern connection of the metric $h_{\sigma,L}$ on $E_{\sigma,L}$ defined by h . More precisely,

$$\text{Lemma 2.11.} \quad \langle F_{h,L}, s_{\sigma,h} \rangle = \langle F_h, s_{\sigma,h} \rangle - \langle \varpi(s_{\sigma,h})(\bar{\partial}s_{\sigma,h}), \bar{\partial}s_{\sigma,h} \rangle.$$

Proof. We can think of E and $E_{\sigma,L}$ as giving two holomorphic structures on the same principal bundle obtained by complexifying \mathbf{E} . Therefore the difference between the corresponding $\bar{\partial}$ operators is a $\mathbf{E}(\mathfrak{h}^\mathbb{C})$ -valued $(0, 1)$ -form: $\bar{\partial}^E - \bar{\partial}^{E_{\sigma,L}} = a$ with $a \in \Omega^{0,1}(\mathfrak{n})$ since $E_{\sigma,L}$ is a reduction of E .

Let τ denote the involution of $H^{\mathbb{C}}$ fixing H , and denote by the same symbol τ the induced involution of $\mathfrak{h}^{\mathbb{C}}$ fixing \mathfrak{h} pointwise. This induces a fiberwise involution of $\mathbf{E}(\mathfrak{h}^{\mathbb{C}})$, and via the identification $E(H^{\mathbb{C}}) \simeq \mathbf{E}(\mathfrak{h}^{\mathbb{C}})$ obtained from the reduction h we can transport this involution to an involution of $E(H^{\mathbb{C}})$ which we denote by τ_h .

Noting $\bar{\partial}^{E_{\sigma,L}} + \partial_h^{E_{\sigma,L}}$ the Chern connection of $E_{\sigma,L}$, we have

$$F_h = F_{h,L} + \partial_h^{E_{\sigma,L}} a + \bar{\partial}^{E_{\sigma,L}} \tau_h(a) + [a, \tau_h(a)],$$

and therefore

$$\langle F_{h,L}, s_{\sigma,h} \rangle = \langle F_h, s_{\sigma,h} \rangle + \langle [a, \tau(a)], s_{\sigma,h} \rangle = \langle F_h, s_{\sigma,h} \rangle + \langle a, [a, s_{\sigma,h}] \rangle.$$

The formula follows since $\bar{\partial}^E s_{\sigma,h} = [a, s_{\sigma,h}]$ hence $\varpi(s_{\sigma,h})(\bar{\partial} s_{\sigma,h})$ is the projection of $-a$ on nonzero eigenspaces of $\text{ad } s_{\sigma,h}$. \square

Our aim in the next section is to state and prove an analogue of Lemma 2.10 giving the parabolic degree. For that it will be necessary to replace the metric h (which was chosen to be smooth on the whole X) by a metric which blows up at the divisor D at a speed specified by the parabolic weights α_i .

2.4. α -adapted metrics and parabolic degree. It may be useful here to remind in a few words how we write local formulas for the metrics of principal bundles. There is a right action $h \mapsto h \cdot g$ of gauge transformations $g \in \Gamma(E(H^{\mathbb{C}}))$ on metrics $h \in \Gamma(E/H)$, which identifies in each fibre to the standard action of $H^{\mathbb{C}}$ on the symmetric space $H \backslash H^{\mathbb{C}}$. In concrete terms, a choice of $h \in \Gamma(E/H)$ is equivalent to a map $\chi : E \rightarrow H \backslash H^{\mathbb{C}}$ satisfying $\chi(e\gamma) = \chi(e)\gamma$ for $e \in E$ and $\gamma \in H^{\mathbb{C}}$ (any such χ corresponds to the section $h \in \Gamma(E/H)$ such that $h(x) = \{[e] \mid e \in E_x, \chi(e) \in H \in H \backslash H^{\mathbb{C}}\}$), and a gauge transformation $g \in \Gamma(E(H^{\mathbb{C}}))$ is equivalent to a map $\zeta : E \rightarrow H^{\mathbb{C}}$ satisfying $\zeta(e\gamma) = \gamma^{-1}\zeta(e)\gamma$. Then $h \cdot g$ is the section of E/H corresponding to the map $\chi\zeta : E \rightarrow H \backslash H^{\mathbb{C}}$.

Recall that τ denotes the involution of $H^{\mathbb{C}}$ fixing H . A local trivialization e of E defines a metric h_0 , such that e is h_0 -orthonormal. Any other metric is given by $h = h_0 \cdot g$ for some g with values in $H^{\mathbb{C}}$; then h depends only on $\tau(g)^{-1}g$ and we identify $h = \tau(g)^{-1}g$. Of course it is always possible to move g by an element of H so that $\tau(g)^{-1} = g$ and then $h = g^2$.

Summarizing, we have two equivalent methods to write down metrics in a h_0 -orthonormal trivialization e :

- $h = h_0 \cdot g$ for some $g \in H^{\mathbb{C}}$, where the action is that of $H^{\mathbb{C}}$ on $H \backslash H^{\mathbb{C}}$; or
- $h = \tau(g)^{-1}g$, which simplifies to $h = g^2$ if $\tau(g)^{-1} = g$.

Moreover, if we consider another trivialization $f = e\gamma$ for some $\gamma \in H^{\mathbb{C}}$, then in the trivialization f the same metric becomes $h \cdot \gamma = h_0 \cdot (g\gamma) = \tau(g\gamma)^{-1}g\gamma$.

The second method for writing metrics generalizes the local formula $h = g^*g$ in a complex vector bundle; in this way we will write formulas which are similar to the usual formulas in the bundle case.

After this small digression, let us come back to a parabolic bundle E , and a metric $h \in \Gamma(X \setminus D; E/H)$ defined away from the divisor D .

Definition 2.12. We say that h is an α -adapted metric if for any parabolic point x_i the following holds. Choose a local holomorphic coordinate z and a local holomorphic trivialization e_i of E near x_i compatible with the parabolic structure (see Definition 2.2). Then in the trivialization e_i one has

$$(2.8) \quad h = h_0 \cdot |z|^{-\alpha_i} e^c,$$

where h_0 is the standard constant metric, $\text{Ad}(|z|^{-\alpha_i})c = o(\log |z|)$ and $|F_h|_h \in L^1$.

The definition of h and the conditions on c are clarified by observing that in the trivialization $e_i \cdot |z|^{\alpha_i}$ (which is orthonormal for the metric $|z|^{-2\alpha_i} = h_0 \cdot |z|^{-\alpha_i}$), the metric h can be written as

$$h_0 \cdot |z|^{-\alpha_i} e^c |z|^{\alpha_i} = h_0 \cdot e^{\text{Ad}(|z|^{-\alpha_i})c},$$

where h_0 still denotes the standard constant metric in the trivialization $e_i \cdot |z|^{\alpha_i}$. If one chooses c so that $\tau(\text{Ad}(|z|^{-\alpha_i})c)^{-1} = \text{Ad}(|z|^{-\alpha_i})c$, then h can simply be written as

$$h = |z|^{-2\alpha_i} e^{2c}.$$

Note that the L^1 condition on F_h is conformally invariant.

Some discussion is in order on the role of the holomorphic gauge e_i . If we replace e_i by $g(e_i)$, where $g \in PE(H^C)$ is a meromorphic gauge transformation near x_i , then because of the interpretation (2.2) of meromorphic gauge transformations as holomorphic gauge transformations outside D which near each x_i are bounded with respect to the metric $|z|^{-2\alpha_i}$, we can still write h under the form (2.8) in the new gauge $g(e_i)$, with the new c satisfying the same condition: therefore the definition does not depend on any choice.

We are now ready to state and prove the analogue of Lemma 2.10 for the parabolic degree.

Lemma 2.13. Let $\varpi : \mathbb{R} \rightarrow \mathbb{R}$ be defined as in Lemma 2.10, and let h be an α -adapted metric. Then:

$$(2.9) \quad \text{pardeg}_\alpha(E)(\sigma, \chi) = \frac{\sqrt{-1}}{2\pi} \int_{X \setminus D} \langle F_h, s_{\sigma, h} \rangle - \langle \varpi(s_{\sigma, h})(\bar{\partial}s_{\sigma, h}), \bar{\partial}s_{\sigma, h} \rangle.$$

Proof. For any $v > 0$ let $X_v = \{x \in X \mid d(x, D) \geq e^{-v}\}$ and $B_v = X \setminus X_v$. Let h_v be a smooth metric on E (defined on the whole X) which coincides with h in a neighbourhood of $X_v \subset X$. The metrics h and h_v induce metrics h_L and $h_{L,v}$ on $E_{\sigma, L}$, and we denote by $\mathbf{E}_{\sigma, v}$ the resulting Hermitian holomorphic bundle. Denote the curvatures of h_L and $h_{L,v}$ by $F_{h, L}$ and $F_{h_v, L}$. It follows from the definition of $\text{deg}(E)(\sigma, \chi)$ that

$$\begin{aligned} \text{deg}(E)(\sigma, \chi) &= \frac{\sqrt{-1}}{2\pi} \int_X \langle F_{h_v, L}, s_\sigma \rangle \\ &= \frac{\sqrt{-1}}{2\pi} \left(\int_{X_v} \langle F_{h_v, L}, s_\sigma \rangle + \int_{B_v} \langle F_{h_v, L}, s_\sigma \rangle \right). \end{aligned}$$

By Lemma 2.11 we have

$$\begin{aligned} \lim_{v \rightarrow \infty} \int_{X_v} \langle F_{h_v, L}, s_\sigma \rangle &= \int_X \langle F_{h, L}, s_\sigma \rangle \\ &= \int_{X \setminus D} \langle F_h, s_{\sigma, h} \rangle - \langle \varpi(s_{\sigma, h})(\bar{\partial}s_{\sigma, h}), \bar{\partial}s_{\sigma, h} \rangle. \end{aligned}$$

Hence we need to prove that the remaining integral converges to the local terms in the definition of the parabolic degree, i.e.:

$$(2.10) \quad \lim_{v \rightarrow \infty} \frac{\sqrt{-1}}{2\pi} \int_{B_v} \langle F_{h_{v,L}}, s_\sigma \rangle = \sum_i \deg \left((P, \chi), (Q_i, \alpha_i) \right).$$

Observe that on a small ball, we can use a holomorphic trivialization of the bundle E_σ , and the connection form becomes $A_v = h_{L,v}^{-1} \partial h_{L,v}$ with curvature $F = dA_v$. It follows that the quantity we have to study is

$$\lim_{v \rightarrow \infty} \frac{\sqrt{-1}}{2\pi} \int_{\partial B_v} \langle A_v, s \rangle = \lim_{v \rightarrow \infty} \frac{\sqrt{-1}}{2\pi} \int_{\partial B_v} \langle h_L^{-1} \partial h_L, s \rangle,$$

since $h_{L,v}$ coincides with h_L in a neighbourhood of X_v .

We can choose a holomorphic trivialization of E in which the reduction σ has constant coefficients. This induces also a holomorphic trivialization of the L -bundle $E_{\sigma,L}$.

We begin by the case where the metric h on E in this trivialization can be written as

$$(2.11) \quad h = \tau(g)^{-1} g, \quad \text{where } g = |z|^{-\alpha_i} e^c,$$

and using $H^{\mathbb{C}} = HP = HLN$ we decompose

$$g = k(z)l(z)n(z), \quad \text{with } k(z) \in H, l(z) \in L, n(z) \in N.$$

The pair $(k(z), l(z))$ is defined only up to the action of $P \cap H$, so we can as well suppose that $\tau(l) = l^{-1}$. It then follows that $h = \tau(n(z))^{-1} l(z)^2 n(z)$, so the induced metric on $E_{\sigma,L}$ is

$$h_L = l(z)^2.$$

Then, because s is L -invariant, the limit to calculate becomes

$$\lim_{v \rightarrow \infty} \frac{\sqrt{-1}}{\pi} \int_{\partial B_v} \langle \partial l l^{-1}, s \rangle.$$

Now one has

$$\partial l l^{-1} = \text{Ad}(k)^{-1}(\partial g g^{-1}) - k^{-1} \partial k - \text{Ad}(l)(\partial n n^{-1}).$$

Observe that since $s \in \sqrt{-1}\mathfrak{h}$ and $k \in H$, the term $\frac{\sqrt{-1}}{\pi} \int_{\partial B_v} \langle k^{-1} \partial k, s \rangle$ reduces to $\frac{1}{2\pi} \int_{\partial B_v} \langle \sqrt{-1} k^{-1} \partial_\theta k, s \rangle$. Therefore the limit reduces to

$$(2.12) \quad \lim_{v \rightarrow \infty} \frac{\sqrt{-1}}{\pi} \int_{\partial B_v} \langle \text{Ad}(k)^{-1}(\partial g g^{-1}) - \frac{1}{2} k^{-1} \partial_\theta k, s \rangle.$$

On the other hand, decompose similarly $e^{t\alpha} = \tilde{k}(t)\tilde{p}(t)$ with $\tilde{k}(t) \in H$ and $\tilde{p}(t) \in P$ then $\langle s \cdot e^{-t\alpha}, \alpha \rangle = \langle \text{Ad}(\tilde{k}(t))s, \alpha \rangle$, so we obtain:

$$(2.13) \quad \deg \left((P, \chi), (Q_i, \alpha_i) \right) = \lim_{t \rightarrow \infty} \langle s \cdot e^{-t\alpha}, \alpha \rangle = \lim_{t \rightarrow \infty} \langle s, \text{Ad}(\tilde{k}(t))^{-1} \alpha \rangle.$$

If we have exactly the model behaviour $g = |z|^{-\alpha_i}$, then one has $k(z) = \tilde{k}(-\ln |z|)$ and $\partial g g^{-1} = -\frac{\alpha_i}{2} \frac{dz}{z}$, so $\partial_\theta k = 0$ and

$$\frac{\sqrt{-1}}{\pi} \int_{\partial B_v} \langle \text{Ad}(k(z))^{-1}(\partial g g^{-1}), s \rangle = \langle \text{Ad}(\tilde{k}(v)^{-1})\alpha_i, s \rangle,$$

so the two limits (2.12) and (2.13) are the same.

In the case we have a perturbation c , so that $g = |z|^{-\alpha_i} e^c$ and c satisfies the conditions in Definition 2.12, we will be brief here since the handling of the perturbation c is essentially similar to that in [64, Lemma 10.5] for the vector bundle case. We have just seen that the Lemma is true for a metric h which has exactly the behaviour $h = h_0 \cdot |z|^{-\alpha_i}$ in a holomorphic trivialization near each x_i , and we consider another metric $h' = h \cdot e^c$. The α -adapted hypothesis is $|c|_h = o(\log |z|)$ near each x_i and $|F_h|_h \in L^1$. We consider the family of metrics $h'_t = h \cdot e^{tc}$ for $0 \leq t \leq 1$ on the bundle E . For this calculation, since we have to use norms, it is more convenient to fix the metric h and to actually vary the holomorphic structure of the bundle E , this is a completely equivalent point of view which leads to the same curvature integral in (2.9): so we fix the metric h (and therefore the H -bundle \mathbf{E}) and consider the holomorphic bundle structures E_t defined on $X \setminus D$ by gauge transforming E :

$$\bar{\partial}^{E_t} = (e^{-tu})^* \bar{\partial}^E = \bar{\partial}^E - (\bar{\partial}^E e^{tu}) e^{-tu}.$$

As above, we can decompose $e^{tu} = \hat{k}(t) \hat{l}(t) \hat{n}(t)$ with $\hat{k}(t)$, $\hat{l}(t)$, $\hat{n}(t)$ sections of $\mathbf{E}(H)$, $\mathbf{E}_\sigma(L)$, $\mathbf{E}_\sigma(N)$, and $\tau(\hat{l}(t)) = \hat{l}(t)^{-1}$. The P -reduction σ of E gives a P -reduction σ_t of E_t , represented by the section $s_{\sigma_t, h}$ of $\mathbf{E}(\sqrt{-1}\mathfrak{h})$ given by

$$s_{\sigma_t, h} = \text{Ad}(\hat{k}_t) s_{\sigma, h}.$$

Similarly we decompose for each t

$$u = u_{\mathfrak{h}, t} + u_{\mathfrak{p}, t} \quad \text{with } u_{\mathfrak{h}, t} \in \mathbf{E}(\sqrt{-1}\mathfrak{h}), u_{\mathfrak{p}, t} \in \mathbf{E}_{\sigma_t}(\mathfrak{p}).$$

Write $\omega_t = \sqrt{-1}(\langle F_{E_t, h}, s_{\sigma_t, h} \rangle - \langle \varpi(s_{\sigma_t, h})(\bar{\partial}^{E_t} s_{\sigma_t, h}), \bar{\partial}^{E_t} s_{\sigma_t, h} \rangle)$ the integrand in the RHS of (2.9), then a standard calculation gives that $\frac{d}{dt} \omega_t = 2\sqrt{-1} \bar{\partial} \bar{\partial} \langle u_{\mathfrak{p}, t}, s_{\sigma_t, h} \rangle$, and therefore

$$\omega_1 - \omega_0 = 2\sqrt{-1} \bar{\partial} \bar{\partial} f, \quad \text{with } f = \int_0^1 \langle u_{\mathfrak{p}, t}, s_{\sigma_t, h} \rangle dt.$$

Since $s_{\sigma_1, h}$ provides a holomorphic P -reduction, $\bar{\partial}^{E_1} s_{\sigma_1, h}$ takes its values only in the negative eigenspaces of $\text{ad}(s_{\sigma_1, h})$ so we have $\omega_1 \geq \sqrt{-1} \langle F_{E_1, h}, s_{\sigma_1, h} \rangle$ which is L^1 since by hypothesis $|F_{E_1, h}|_h \in L^1$; therefore

$$(2.14) \quad 2\sqrt{-1} \bar{\partial} \bar{\partial} f \geq b \quad \text{with } b \in L^1.$$

Again from the hypothesis on u , we have near each puncture

$$(2.15) \quad f = o(\log |z|).$$

The conditions (2.14) (2.15) are enough to ensure that $2\sqrt{-1} \bar{\partial} \bar{\partial} f \in L^1$ and

$$\int_{X \setminus D} 2\sqrt{-1} \bar{\partial} \bar{\partial} f = 0.$$

(This was already used in [64], see Lemma 10.5, Proposition 2.2 and the remark following it). Therefore $\frac{1}{2\pi} \int_{X \setminus D} \omega_1 = \frac{1}{2\pi} \int_{X \setminus D} \omega_0 = \text{pardeg}_\alpha(E)(\sigma, \chi)$ and the Lemma is proved. \square

3. PARAHORIC BUNDLES

Our definition of parabolic bundles is suitable for prescribing the asymptotic behaviour near a divisor $D \subset X$ of reductions, defined on $X \setminus D$, of the structure group of an $H^{\mathbb{C}}$ -principal bundle to the maximal compact subgroup $H \subset H^{\mathbb{C}}$. But it has the obvious inconvenient that in the situations when $P_{\alpha_i}^1$ is nontrivial the sheaf $PE(H^{\mathbb{C}})$ contains meromorphic sections whose effect on E is somewhat unclear (see Remark 2.5 above), in that they transform E into a different bundle. This issue becomes transparent from the point of view of parahoric bundles introduced by Pappas and Rapoport [59] (see also [13]), and whose moduli problem was studied by Balaji and Seshadri in [3] in relation with local systems with compact structure group on punctured Riemann surfaces. In this section we recall the definition of parahoric bundles and we relate it to the objects we just defined. We would like to emphasize that this section is not strictly necessary for our arguments, except for the definition of meromorphic equivalence given in Definition 3.3 and the results in Section 3.4. The main purpose of most of this section is to clarify the nature of the objects we are going to work with.

We also show how the notion of parahoric bundles is the natural context for Hecke transformations. This is of course well known in the algebraic world (see e.g. [3, Section 8.2.1]), but our approach, much more analytic than that of [3, 13, 59], seems to be new (see Subsection 3.3 below).

The existence of Hecke transformations implies that even restricting to the case $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$ we cover, up to isomorphism, all possible choices of weights α . This is one reason for avoiding parahoric bundles in our approach. A second reason, not unrelated to the first, is that taking weights in $\sqrt{-1}\bar{\mathcal{A}}$ suffices to realize all possible monodromies around the punctures in the correspondence between parabolic Higgs bundles and local systems, as we prove in the paper. On the other hand one can use Hecke transformations to identify, for any choice of weights α , the category of parahoric bundles of weight α with the category of parabolic orbibundles on X for suitable choices of: weights in $\sqrt{-1}\mathcal{A}'$, orbifold structure on the divisor D , and topological type of orbibundles. This can be done thanks to Proposition A.2.

The Hecke transformations are compatible with the reductions of the structure group to standard parabolic subgroups, since they *use* only the action of the complexified maximal torus $T^{\mathbb{C}}$, which is contained in all standard parabolic subgroups. This, combined with an easy computation of degrees, implies that Hecke correspondence is compatible with the stability notions *à la* Ramanathan, see Subsection 4.2.

Finally, Hecke transformations can be used to extend the definition of parabolic Higgs bundles for a choice of parabolic structure of weights in $\sqrt{-1}\bar{\mathcal{A}}$ (see Section 4.1) to a notion of parahoric Higgs bundles for an arbitrary choice of weights α .

3.1. Definition of parahoric bundles. Recall that we are fixing an alcove $\mathcal{A} \subset \mathfrak{t}$ of H such that $0 \in \bar{\mathcal{A}}$ (see Appendix A.1). Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a collection of arbitrary elements of $\sqrt{-1}\mathfrak{t}$. Let $X^* = X \setminus D = X \setminus \{x_1, \dots, x_r\}$ and for any open subset $\Omega \subset X$ denote $\Omega^* := \Omega \cap X^*$. Choose for each j a local holomorphic coordinate z_j on a neighbourhood of x_j satisfying $z_j(x_j) = 0$.

Let \mathcal{G}_α be the sheaf of groups on X defined as follows: for any open subset $\Omega \subset X$, $\mathcal{G}_\alpha(\Omega)$ is equal to the set of all holomorphic maps $\phi : \Omega^* \rightarrow H^\mathbb{C}$ with the property that for every j such that $x_j \in \Omega$

$$|z_j(p)|^{-\alpha_j} \phi(p) |z_j(p)|^{\alpha_j} = \exp(-\ln |z_j(p)| \alpha_j) \phi(p) \exp(\ln |z_j(p)| \alpha_j)$$

stays in a compact subset of $H^\mathbb{C}$ as $p \rightarrow x_j$. It is easy to check that this definition is independent of the choice of local holomorphic coordinates. The sheaf \mathcal{G}_α is the holomorphic analogue of the Bruhat–Tits group scheme $\mathcal{G}_{\Theta, X}$ in [3].

A **parahoric bundle** over (X, D) of weight α is a sheaf of torsors over the sheaf of groups \mathcal{G}_α .

Let W denote the Weyl group of H and let $W\mathcal{A}' = \bigcup_{w \in W} w\mathcal{A}'$. We next prove that if $\alpha_j \in \sqrt{-1}W\mathcal{A}'$ for every j then a parahoric bundle of weight α is equivalent to a parabolic principal bundle in the sense of Definition 2.3. Define $\mathcal{G}_\alpha^{\text{std}}$ to be the sheaf of groups on X such that, for any open $\Omega \subset X$, $\mathcal{G}_\alpha^{\text{std}}(\Omega)$ is equal to the set of all holomorphic maps $\phi : \Omega \rightarrow H^\mathbb{C}$ satisfying $\phi(x_j) \in P_{\alpha_j}$ for each j such that $x_j \in \Omega$. We now have

$$(3.1) \quad \mathcal{G}_\alpha = \mathcal{G}_\alpha^{\text{std}} \iff \alpha_j \in \sqrt{-1}W\mathcal{A}'.$$

This is an immediate consequence of the characterisation of $\sqrt{-1}W\mathcal{A}'$ as the subset of $\sqrt{-1}\mathfrak{t}$ consisting of those elements β such that all eigenvalues of $\text{ad}(\beta)$ have absolute value smaller than 1 (see (3) in Proposition A.2). To conclude our argument, one has the following.

Proposition 3.1. *A sheaf of torsors over $\mathcal{G}_\alpha^{\text{std}}$ is the same thing as a holomorphic $H^\mathbb{C}$ -principal bundle E and a choice of parabolic structures $\{Q_j \subset E(H^\mathbb{C})_{x_j}\}_j$.*

Proof. Suppose given $(E, \{Q_j\})$. For each j define $R_j \subset E_{x_j}$ to be the set of all $e \in E_{x_j}$ such that $P_{\alpha_j} = \{\phi(e) \mid \phi \in Q_j\}$. Let \mathcal{E} be the sheaf whose sections on an open subset $U \subset X$ are the holomorphic sections of $E|_U$ whose value at each x_j belongs to R_j . Then \mathcal{E} is a sheaf of $\mathcal{G}_\alpha^{\text{std}}$ -torsors. For the converse, let \mathcal{G} denote the sheaf of local holomorphic maps to $H^\mathbb{C}$. Clearly $\mathcal{G}_\alpha^{\text{std}}$ is a subsheaf of \mathcal{G} , so if \mathcal{E} is a sheaf of $\mathcal{G}_\alpha^{\text{std}}$ -torsors then $\mathcal{E}' = \mathcal{E} \times_{\mathcal{G}_\alpha^{\text{std}}} \mathcal{G}$ is a sheaf of \mathcal{G} -torsors, which can be identified with the sheaf of local holomorphic sections of a holomorphic principal $H^\mathbb{C}$ -bundle E over X . Now, \mathcal{E} is naturally a subsheaf of \mathcal{E}' , and defining for every j the subset $R_j \subset E_{x_j}$ as the set of images at x_j of local sections of E contained in \mathcal{E} we obtain an orbit of the action of P_{α_j} at E_{x_j} . Then setting $Q_j = \{\phi \in E(H^\mathbb{C})_{x_j} \mid \phi(R_j) = P_{\alpha_j}\}$ we obtain a parabolic structure at x_j . This construction is clearly the inverse of the previous one. \square

3.2. Parahoric bundles vs. parabolic bundles with weights in $\sqrt{-1}\bar{\mathcal{A}}$. Our next aim is to understand the parabolic bundles defined in Subsection 2.1 (which may have weights in $\sqrt{-1}\bar{\mathcal{A}}$) from the viewpoint of parahoric bundles.

For any $\alpha = (\alpha_1, \dots, \alpha_r)$, define a sheaf of groups $\mathcal{G}_\alpha^{\text{wall}}$ over X by the prescription that, for any open subset $\Omega \subset X$, $\mathcal{G}_\alpha^{\text{wall}}(\Omega)$ is equal to the group of holomorphic maps $\Omega^* \rightarrow H^\mathbb{C}$ such that for every $x_j \in \Omega$ and any local holomorphic coordinate z defined on a neighbourhood of x_j and satisfying $z(x_j) = 0$ we have $\phi = g \exp(n/z)$, where g is

holomorphic map from a neighbourhood of x_j to $H^{\mathbb{C}}$ satisfying $g(x_j) \in P_{\alpha_j}$ and n is an element of $\mathfrak{p}_{\alpha_j}^1$. Similarly to (3.1), we have

$$(3.2) \quad \mathcal{G}_\alpha = \mathcal{G}_\alpha^{\text{wall}} \iff \alpha_j \in \sqrt{-1}W\bar{\mathcal{A}} = \sqrt{-1} \bigcup_{w \in W} w\bar{\mathcal{A}}.$$

Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a collection of elements of $\sqrt{-1}\bar{\mathcal{A}}$. Let $(E, \{Q_i\})$ be a parabolic bundle of weight α , and let \mathcal{E}^{std} be the corresponding sheaf of $\mathcal{G}_\alpha^{\text{std}}$ -torsors, defined in the subsection above. Now, $\mathcal{G}_\alpha^{\text{std}}$ is a subsheaf of \mathcal{G}_α (this is not true for arbitrary values of α : in fact, it is equivalent to the condition that α_j belongs to $\bar{\mathcal{A}}$ for each j). Hence, one can associate a parahoric bundle to $(E, \{Q_i\})$ by extending the structure group of \mathcal{E}^{std} . Namely,

$$\mathcal{E} = \mathcal{E}^{\text{std}} \times_{\mathcal{G}_\alpha^{\text{std}}} \mathcal{G}_\alpha.$$

Then equality (3.2) implies that the sheaf of groups $PE(H^{\mathbb{C}})$ is canonically isomorphic to the sheaf of automorphisms of \mathcal{E} .

It is interesting to understand how to go the other way round, since this explains why we need to identify different principal bundles. Passing from \mathcal{E} to \mathcal{E}^{std} is the same as reducing the structure group from \mathcal{G}_α to $\mathcal{G}_\alpha^{\text{std}}$, equivalently, choosing a section of $\mathcal{E}/\mathcal{G}_\alpha^{\text{std}}$. The latter is supported on D , and the stalk over x_j can be identified, using the residue map (hence, non canonically), with $(\mathcal{G}_\alpha)_{x_j}/(\mathcal{G}_\alpha^{\text{std}})_{x_j} \simeq \mathfrak{p}_{\alpha_j}^1$. Hence the collection of all reductions from \mathcal{E} to \mathcal{E}^{std} is parametrized by the vector space $\prod_j \mathfrak{p}_{\alpha_j}^1$. The set of all reductions defines a holomorphic principal bundle over $X \times \prod_j \mathfrak{p}_{\alpha_j}^1$, so all possible reductions have topologically equivalent underlying principal $H^{\mathbb{C}}$ -bundles. However, they will usually be different as holomorphic bundles.

Example 3.2. Suppose that $H = \text{SU}_2$, so that $H^{\mathbb{C}} = \text{SL}_2\mathbb{C}$ and assume that $V = L \oplus L^{-1}$ for some line bundle $L \rightarrow X$. Such V defines in the usual way a principal $\text{SL}_2\mathbb{C}$ -bundle E . Take $D = x$, and choose $\alpha \in \sqrt{-1}\mathfrak{su}_2$ to be the diagonal matrix with entries $1/2$ and $-1/2$. Let Q be the group of automorphisms of $L_x \oplus L_x^{-1}$ preserving the summand L_x . Then (E, Q) is a parabolic principal bundle with weight α . Let \mathcal{E}^{std} be the sheaf of $\mathcal{G}_\alpha^{\text{std}}$ -torsors associated to (E, Q) and let $\mathcal{E} = \mathcal{E}^{\text{std}} \times_{\mathcal{G}_\alpha^{\text{std}}} \mathcal{G}_\alpha$. We have

$$\mathfrak{p}^1 = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} : \lambda \in \mathbb{C} \right\}.$$

A choice of λ defines a reduction of \mathcal{E} whose underlying principal $\text{SL}_2\mathbb{C}$ -bundle corresponds to a holomorphic vector bundle V_λ sitting in a short exact sequence

$$0 \rightarrow L \rightarrow V_\lambda \rightarrow L^{-1} \rightarrow 0.$$

We claim that in general this sequence does not split. The relation between V and V_λ can be described explicitly in terms of local trivialization; namely, passing from V to V_λ consists on multiplying the patching map for suitable trivializations of V on $X \setminus \{x\}$ and a disk Δ centred at x by the map

$$\Delta \setminus \{0\} \rightarrow \text{SL}_2\mathbb{C}, \quad z \mapsto \begin{pmatrix} 1 & \lambda z^{-1} \\ 0 & 1 \end{pmatrix}.$$

Choosing L appropriately, the vector bundle V_λ is going to be (holomorphically) different from V when $\lambda \neq 0$.

Definition 3.3. Let E_0, E_1 be two holomorphic $H^\mathbb{C}$ -principal bundles and let $\{Q_{i,j} \subset E_i(H^\mathbb{C})_{x_j}\}$, be parabolic structures on E_i , $i = 0, 1$, defined along D . We say that $(E_0, \{Q_{0,i}\})$ and $(E_1, \{Q_{1,i}\})$ are **meromorphically equivalent** if, denoting by $\mathcal{E}_0^{\text{std}}$ and $\mathcal{E}_1^{\text{std}}$ the corresponding sheaves of $\mathcal{G}_\alpha^{\text{std}}$ -torsors, there is an isomorphism

$$\Psi : \mathcal{E}_0^{\text{std}} \times_{\mathcal{G}_\alpha^{\text{std}}} \mathcal{G}_\alpha \rightarrow \mathcal{E}_1^{\text{std}} \times_{\mathcal{G}_\alpha^{\text{std}}} \mathcal{G}_\alpha$$

of sheaves of \mathcal{G}_α -torsors. We call Ψ a **meromorphic equivalence** between $(E_0, \{Q_{0,i}\})$ and $(E_1, \{Q_{1,i}\})$.

A choice of meromorphic equivalence between $(E_0, \{Q_{0,i}\})$ and $(E_1, \{Q_{1,i}\})$ is the same thing as an isomorphism of principal bundles

$$\psi : E_0|_{X^*} \rightarrow E_1|_{X^*}$$

satisfying the following property, for every j . Let $R_{i,j} \subset E_{i,x_j}$ be defined as at the end of the previous subsection. Let $U \subset X$ be a small disk centred at x_j and disjoint from all other points of the support of D . Let $\sigma_i \in \Gamma(U, E_i)$, $i = 0, 1$, be holomorphic sections satisfying $\sigma_i(x_j) \in R_{i,j}$. Consider the holomorphic map $f : U \setminus \{x_j\} \rightarrow H^\mathbb{C}$ defined by the condition

$$\psi(\sigma_0(y)) = \sigma_1(y)f(y) \quad \text{for every } y \in U \setminus \{x_j\}.$$

Then $f \in \mathcal{G}_\alpha = \mathcal{G}_\alpha^{\text{wall}}$.

3.3. Hecke transformations. Consider for each j an element $\lambda_j \in \sqrt{-1}\mathfrak{t}$ such that $2\pi\sqrt{-1}\lambda \in \Lambda_{\text{cochar}}$ (see Appendix A.1) and let $\lambda = (\lambda_1, \dots, \lambda_r)$. In this subsection we define the Hecke transformation τ_λ , which is a natural 1–1 correspondence between sheaves of \mathcal{G}_α -torsors and sheaves of $\mathcal{G}_{\alpha+\lambda}$ -torsors.

Choose an isomorphism $T \simeq (S^1)^k$ and let $\theta_i : S^1 \rightarrow H$ be the composition of the inclusion of the i -th factor $S^1 \hookrightarrow T$ with the inclusion $T \hookrightarrow H$. Let $\chi_i \in \sqrt{-1}\Lambda_{\text{cochar}}$ be defined by the condition that $\theta_i(e^{2\pi\sqrt{-1}u}) = \exp(u\chi_i)$ and write

$$\lambda_j = \sum_i \frac{\sqrt{-1}}{2\pi} a_{ji} \chi_i,$$

where $a_{ji} \in \mathbb{Z}$. Consider the invertible sheaf $\mathcal{L}_i = \mathcal{O}(\sum_j a_{ji}x_j)$, and let $\sigma_i \in H^0(X, \mathcal{L}_i)$ satisfy $\sigma_i^{-1}(0) = \sum_j a_{ji}x_j$. Let $\mathcal{L}_i^* \subset \mathcal{L}_i$ be the subsheaf of local nowhere vanishing sections and let $\mathcal{L}^* = \prod_i \mathcal{L}_i^*$. Let \mathcal{O}^* be the sheaf of local nowhere vanishing holomorphic functions on X . Note that \mathcal{L}^* is in a natural way a sheaf of $(\mathcal{O}^*)^k$ -torsors.

Denote the complexification of θ_i by $\theta_i^\mathbb{C} : \mathbb{C}^* \rightarrow H^\mathbb{C}$. The morphisms $(\theta_1^\mathbb{C}, \dots, \theta_k^\mathbb{C})$ give a monomorphism of sheaves of groups

$$\Theta : (\mathcal{O}^*)^k \rightarrow \mathcal{G}$$

(recall that \mathcal{G} is the sheaf on X of local holomorphic maps to $H^\mathbb{C}$).

Fix some $\alpha = (\alpha_1, \dots, \alpha_r)$ with arbitrary components $\alpha_j \in \sqrt{-1}\mathfrak{t}$. Define the sheaf $\mathcal{L}^* \otimes_{\text{Ad}} \mathcal{G}_\alpha$ to be the quotient of $\mathcal{L}^* \times \mathcal{G}_\alpha$ by the relation that identifies

$$((l_1\psi_1, \dots, l_k\psi_k), \phi) \sim ((l_1, \dots, l_k), \Theta(\psi_1, \dots, \psi_k)\phi\Theta(\psi_1, \dots, \psi_k)^{-1})$$

for every $x \in X$ and elements $l_i \in (\mathcal{L}_i^*)_x$, $\psi_i \in \mathcal{O}_x^*$ and $\phi \in (\mathcal{G}_\alpha)_x$ (the subindex x denotes here as usual the stalk over x). We next define a structure of sheaf of groups on $\mathcal{L}^* \otimes_{\text{Ad}} \mathcal{G}_\alpha$. It suffices to describe the structure at the level of stalks. Given $x \in X$ and two elements $\zeta, \zeta' \in (\mathcal{L}^* \otimes_{\text{Ad}} \mathcal{G}_\alpha)_x$ one can take representatives of ζ and ζ' of the form $\zeta = [((l_1, \dots, l_k), \phi)]$ and $\zeta' = [((l_1, \dots, l_k), \phi')]$, and then we set $\zeta \zeta' = [((l_1, \dots, l_k), \phi \phi')]$. This operation is well defined and endows $\mathcal{L}^* \otimes_{\text{Ad}} \mathcal{G}_\alpha$ with the structure of sheaf of groups.

Lemma 3.4. *The sheaf of groups $\mathcal{L}^* \otimes_{\text{Ad}} \mathcal{G}_\alpha$ is isomorphic to $\mathcal{G}_{\alpha+\lambda}$.*

Proof. We construct a morphism of sheaves of groups $\Xi : \mathcal{L}^* \otimes_{\text{Ad}} \mathcal{G}_\alpha \rightarrow \mathcal{G}_{\alpha+\lambda}$ at the level of stalks. Given $\zeta = [((l_1, \dots, l_k), \phi)] \in (\mathcal{L}^* \otimes_{\text{Ad}} \mathcal{G}_\alpha)_x$ consider the expression

$$\mu = \Theta(l_1 \sigma_1^{-1}, \dots, l_k \sigma_k^{-1}) \cdot \phi \cdot \Theta(l_1 \sigma_1^{-1}, \dots, l_k \sigma_k^{-1})^{-1}.$$

Taking into account that each $l_i \sigma_i^{-1}$ is a germ of meromorphic function at x , we can view μ as a germ of meromorphic map from a neighbourhood U of x to $H^{\mathbb{C}}$. We claim that $\mu \in (\mathcal{G}_{\alpha+\lambda})_x$. If $x \notin \{x_1, \dots, x_r\}$ then this is obvious, since μ is actually holomorphic. Suppose now that $x = x_j$. Let $z = z_j$ be the local holomorphic coordinate near x chosen in Subsection 3.1. One checks that $\Theta(z^{-a_{j1}}, \dots, z^{-a_{jk}}) = \exp((\ln z)\lambda_j) = z^{\lambda_j}$. Now:

$$\begin{aligned} \mu \in (\mathcal{G}_{\alpha+\lambda})_x &\iff \Theta(z^{-a_{j1}}, \dots, z^{-a_{jk}}) \cdot \phi \cdot \Theta(z^{-a_{j1}}, \dots, z^{-a_{jk}})^{-1} \in (\mathcal{G}_{\alpha+\lambda})_x \\ &\iff z^{\lambda_j} \phi z^{-\lambda_j} \in (\mathcal{G}_{\alpha+\lambda})_x \\ &\iff |z_j|^{-(\alpha_j+\lambda_j)} z^{\lambda_j} \phi z^{-\lambda_j} |z_j|^{\alpha_j+\lambda_j} \quad \text{uniformly bounded near } x \\ &\iff |z_j|^{-\alpha_j} \phi |z_j|^{\alpha_j} \quad \text{uniformly bounded near } x \\ &\iff \phi \in (\mathcal{G}_\alpha)_x. \end{aligned}$$

This proves that setting $\Xi(\zeta) = \mu$ defines a morphism of sheaves from $\mathcal{L}^* \otimes_{\text{Ad}} \mathcal{G}_\alpha$ to $\mathcal{G}_{\alpha+\lambda}$. One checks easily that Ξ is an isomorphism of sheaves of groups. \square

Now let \mathcal{E} be a parahoric bundle of weight α . We define its λ -Hecke transformation to be

$$\tau_\lambda(\mathcal{E}) = \mathcal{E} \otimes_{(\mathcal{O}^*)^k} \mathcal{L}^*,$$

where $\mathcal{E} \otimes_{(\mathcal{O}^*)^k} \mathcal{L}^*$ denotes the quotient of $\mathcal{E} \times \mathcal{L}^*$ by the relation that identifies, at any point x ,

$$(\epsilon, (l_1 \psi_1, \dots, l_k \psi_k)) \sim (\epsilon \cdot \Theta(\psi_1, \dots, \psi_k)^{-1}, (l_1, \dots, l_k))$$

for every $\epsilon \in \mathcal{E}_x$, $l_i \in (\mathcal{L}_i^*)_x$ and $\psi_i \in \mathcal{O}_x^*$. This makes sense because $\Theta(\psi_1, \dots, \psi_k) \in (\mathcal{G}_\alpha)_x$ (in fact this holds for every α), and \mathcal{E} is a sheaf of torsors over \mathcal{G}_α . Now we define on $\mathcal{E} \otimes_{(\mathcal{O}^*)^k} \mathcal{L}^*$ a structure of sheaf of $\mathcal{L}^* \otimes_{\text{Ad}} \mathcal{G}_\alpha$ -torsors, by the condition that

$$(\epsilon, (l_1, \dots, l_k)) \cdot ((l_1, \dots, l_k), \phi) = (\epsilon \cdot \phi, (l_1, \dots, l_k)).$$

It is easy to prove that $\tau_{-\lambda}(\tau_\lambda(\mathcal{E}))$ is naturally isomorphic to \mathcal{E} . Taking into account Lemma 3.4, we obtain the following.

Theorem 3.5. *The λ -Hecke transformation τ_λ establishes a natural 1–1 correspondence between parahoric bundles of weight α and parahoric bundles of weight $\alpha + \lambda$.*

3.4. Meromorphic maps to $H^{\mathbb{C}}$. Let Δ be the unit disc in \mathbb{C} centred at 0, and let $\Delta^* = \Delta \setminus \{0\}$. A holomorphic map $g : \Delta^* \rightarrow H^{\mathbb{C}}$ is said to be **meromorphic** if for any holomorphic morphism $\rho : H^{\mathbb{C}} \rightarrow \mathrm{SL}_N\mathbb{C} \subset \mathrm{End}\mathbb{C}^N$ there exists some integer k such that

$$\Delta^* \ni z \mapsto z^k \rho(g(z)) \in \mathrm{End}\mathbb{C}^N$$

extends to a holomorphic map $\Delta \rightarrow \mathrm{End}\mathbb{C}^N$. By Riemann's extension theorem the sections of the sheaves \mathcal{G}_α are meromorphic in the previous sense.

Lemma 3.6. *Let $P \subset H^{\mathbb{C}}$ be a parabolic subgroup. Let $\sigma : \Delta \rightarrow H^{\mathbb{C}}/P$ be a holomorphic map and let $g : \Delta^* \rightarrow H^{\mathbb{C}}$ be a meromorphic map at 0. Then the map $g \cdot \sigma : \Delta^* \rightarrow H^{\mathbb{C}}/P$ defined by $g(z) \cdot \sigma(z)$ can be extended to a holomorphic map $\Delta \rightarrow H^{\mathbb{C}}/P$.*

Proof. Choose an embedding $\xi : H \rightarrow \mathrm{SU}_N$ for some big N , and denote by the same symbol both the holomorphic extension to the complexifications $\xi : H^{\mathbb{C}} \rightarrow \mathrm{SL}_N\mathbb{C}$ and the induced morphism $\xi : \mathfrak{h}^{\mathbb{C}} \rightarrow \mathfrak{sl}_N\mathbb{C}$ of Lie algebras. Suppose that $P = P_\beta$ for some $\beta \in \sqrt{-1}\mathfrak{h}$ and let $P' = P_{\xi(\beta)} \subset \mathrm{SL}_N\mathbb{C}$ be the corresponding parabolic subgroup. Identifying $H^{\mathbb{C}}$ with $\xi(H^{\mathbb{C}})$ we may write $P = P' \cap H^{\mathbb{C}}$, which implies that the inclusion $\xi : H^{\mathbb{C}} \hookrightarrow \mathrm{SL}_N\mathbb{C}$ induces a (holomorphic) inclusion $H^{\mathbb{C}}/P \hookrightarrow \mathrm{SL}_N\mathbb{C}/P'$ with closed image. Hence it suffices to consider the case $H^{\mathbb{C}} = \mathrm{SL}_N\mathbb{C}$. Then $\mathrm{SL}_N\mathbb{C}/P'$ is a partial flag variety. Via the Plücker embedding, we may further reduce our statement to the following one: if $g : \Delta^* \rightarrow \mathrm{SL}_N\mathbb{C}$ is a map whose composition with the inclusion $\mathrm{SL}_N\mathbb{C} \hookrightarrow \mathrm{End}\mathbb{C}^N$ is meromorphic, if $\sigma : \Delta \rightarrow \mathbb{P}^{N'}$ is a holomorphic map, and if there is a linear action of $\mathrm{SL}_N\mathbb{C}$ on $\mathbb{P}^{N'}$, then $g \cdot \sigma : \Delta^* \rightarrow \mathbb{P}^{N'}$ extends to a holomorphic map $\Delta \rightarrow \mathbb{P}^{N'}$. This last statement follows from an immediate computation, so the proof of the lemma is complete. \square

This has the following immediate consequence.

Lemma 3.7. *Suppose that two parabolic principal $H^{\mathbb{C}}$ -bundles $(E, \{Q_i\})$ and $(E', \{Q'_i\})$ are meromorphically equivalent in the sense of Definition 3.3. Then for any parabolic subgroup $P \subset H^{\mathbb{C}}$ the holomorphic reductions of the structure groups $\Gamma(E/P)$ and $\Gamma(E'/P)$ are in bijection. The bijection is given by extending across D the natural holomorphic isomorphism $E|_{X \setminus D} \rightarrow E'|_{X \setminus D}$ which exists in each of the two cases.*

This statement can be summarized by saying that P -reductions are not affected by meromorphic equivalence of bundles. It is interesting to note that if $L \subset P$ is a Levi subgroup, the same is not true for L -reductions. More precisely, if E and E' are as in the lemma, then a L -reduction for E may not give a L -reduction to E' (it is easy to construct an explicit example on a $\mathrm{SL}_2\mathbb{C}$ -bundle). This is because $H^{\mathbb{C}}/P$ is compact but $H^{\mathbb{C}}/L$ is not so the reduction may 'escape' at infinity.

4. PARABOLIC G -HIGGS BUNDLES

4.1. Definition of parabolic G -Higgs bundle. Following the definition and notation in Appendix A.2, let $G = (G, H, \theta, B)$ be a real reductive Lie group. Let X be a compact connected Riemann surface and let $\{x_1, \dots, x_r\}$ be a finite set of different points of X . Let $D = x_1 + \dots + x_r$ be the corresponding effective divisor. Let E be a parabolic

principal $H^{\mathbb{C}}$ -bundle over (X, D) . Let $E(\mathfrak{m}^{\mathbb{C}})$ be the bundle associated to E via the isotropy representation (see Appendix A.2).

As above, we fix an alcove $\mathcal{A} \subset \mathfrak{t}$ of H such that $0 \in \bar{\mathcal{A}}$ (see Appendix A.1). We now define the **sheaf $PE(\mathfrak{m}^{\mathbb{C}})$ of parabolic sections of $E(\mathfrak{m}^{\mathbb{C}})$** and the **sheaf $NE(\mathfrak{m}^{\mathbb{C}})$ of strictly parabolic sections of $E(\mathfrak{m}^{\mathbb{C}})$** . These consist of meromorphic sections of $E(\mathfrak{m}^{\mathbb{C}})$, holomorphic on $X \setminus D$, with singularities of a certain type on D . More precisely, choose a holomorphic trivialization e_i of E near x_i compatible with the parabolic structure. Let $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$ be the parabolic weight at x_i . In the trivialization e_i , we can decompose the bundle $E(\mathfrak{m}^{\mathbb{C}})$ under the eigenvalues of $\text{ad}(\alpha_i)$ (acting on $\mathfrak{m}^{\mathbb{C}}$),

$$E(\mathfrak{m}^{\mathbb{C}}) = \bigoplus_{\mu} \mathfrak{m}_{\mu}^{\mathbb{C}}.$$

Decompose accordingly a section φ of $E(\mathfrak{m}^{\mathbb{C}})$ as $\varphi = \sum \varphi_{\mu}$, then we say that φ is a section of the sheaf $PE(\mathfrak{m}^{\mathbb{C}})$ (resp. $NE(\mathfrak{m}^{\mathbb{C}})$) if φ is meromorphic at x_i , and φ_{μ} has order

$$(4.1) \quad v(\varphi_{\mu}) \geq -\lfloor -\mu \rfloor \quad (\text{resp. } v(\varphi_{\mu}) > -\lfloor -\mu \rfloor).$$

This means that if $a - 1 < \mu \leq a$ (resp. $a - 1 \leq \mu < a$) for some integer a , then $\varphi_{\mu} = O(z^a)$.

An equivalent way to define it is to say that a section of $PE(\mathfrak{m}^{\mathbb{C}})$ is a holomorphic section of the bundle

$$\bigoplus_{\mu} \mathfrak{m}_{\mu}^{\mathbb{C}}(\lfloor -\mu \rfloor x_i).$$

Of course, in general, if we take a holomorphic bundle with some decomposition at a point, this construction does not make sense, because the result depends on the extension of the decomposition near the point. However, the following lemma proves that, in our case, the definition does not depend on the choice of the trivialization.

Lemma 4.1. *The action of a section g of the sheaf $PE(H^{\mathbb{C}})$ preserves the set of sections of the sheaves $PE(\mathfrak{m}^{\mathbb{C}})$ and $NE(\mathfrak{m}^{\mathbb{C}})$.*

Proof. We write the proof only for $PE(\mathfrak{m}^{\mathbb{C}})$. The first case is that of a section $g = \exp(n/z)$ with $n \in \mathfrak{q}_i^1$. Then, for $\varphi \in E(\mathfrak{m}^{\mathbb{C}})$,

$$\begin{aligned} \text{Ad}(g)\varphi &= e^{\text{ad} \frac{n}{z}} \varphi \\ &= \varphi + \left[\frac{n}{z}, \varphi\right] + \frac{1}{2} \left[\frac{n}{z}, \left[\frac{n}{z}, \varphi\right]\right] + \cdots \end{aligned}$$

Since $[\alpha_i, n] = -n$, one has $[n, \mathfrak{m}_{\mu}^{\mathbb{C}}] \subset \mathfrak{m}_{\mu-1}^{\mathbb{C}}$, so $\text{Ad}(g)\varphi$ satisfies (4.1) if φ does.

The second case is that of a constant $g \in Q_i$: it is clear that nothing is changed if g belongs to the Levi subgroup L_{α_i} (see Section B.1), so we can suppose that g belongs to the unipotent part of Q_i : let us write $g = \exp(n)$ with

$$(4.2) \quad n \in \bigoplus_{\lambda < 0} \mathfrak{h}_{\lambda}^{\mathbb{C}},$$

where $\mathfrak{h}^{\mathbb{C}} = \bigoplus_{\lambda} \mathfrak{h}_{\lambda}^{\mathbb{C}}$ is the eigenspace decomposition of $\mathfrak{h}^{\mathbb{C}}$ under the action of $\text{ad}(\alpha_i)$. Then, as above,

$$\text{Ad}(g)\varphi = \varphi + [n, \varphi] + \frac{1}{2} [n, [n, \varphi]] + \cdots$$

Because of (4.2), if $\varphi \in \mathfrak{m}_\mu^{\mathbb{C}}$ then $[n, \varphi] \in \oplus_{\lambda < 0} \mathfrak{m}_{\lambda+\mu}^{\mathbb{C}}$, and more generally $\text{Ad}(g)\varphi - \varphi \in \oplus_{\mu' < \mu} \mathfrak{m}_{\mu'}^{\mathbb{C}}$, so that $\text{Ad}(g)\varphi$ again satisfies (4.1).

The third and last case consists in applying a holomorphic change of trivialization by a $g \in H^{\mathbb{C}}$ such that $g(x_i) = 1$. Let us write $g = \exp(zu)$ with $u \in \mathfrak{h}^{\mathbb{C}}$ holomorphic, and decompose $u = \oplus_{\lambda} u_{\lambda}$. Then, again,

$$\text{Ad}(g)\varphi = \varphi + z[u, \varphi] + \frac{z^2}{2}[u, [u, \varphi]] + \cdots$$

Here the important point is that all the eigenvalues λ satisfy $|\lambda| \leq 1$, so that if $\varphi \in \mathfrak{m}_\mu^{\mathbb{C}}$, then $\text{ad}(u)^k \varphi \in \oplus_{\mu' \leq \mu+k} \mathfrak{m}_{\mu'}^{\mathbb{C}}$ and $z^k \text{ad}(u)^k \varphi$ again satisfies (4.1). This concludes the proof that the sheaf $PE(\mathfrak{m}^{\mathbb{C}})$ is well defined. \square

Remark 4.2. In terms of meromorphic equivalences (see Section 3.2), the previous lemma can be stated as follows. If $(E, \{Q_i\})$ and $(E', \{Q'_i\})$ are two parabolic principal $H^{\mathbb{C}}$ -bundles, then any meromorphic equivalence $\psi : E|_{X \setminus D} \rightarrow E'|_{X \setminus D}$ induces an isomorphism of sheaves $PE(\mathfrak{m}^{\mathbb{C}}) \rightarrow PE'(\mathfrak{m}^{\mathbb{C}})$ and $NE(\mathfrak{m}^{\mathbb{C}}) \rightarrow NE'(\mathfrak{m}^{\mathbb{C}})$.

The sheaves $PE(\mathfrak{m}^{\mathbb{C}})$ and $NE(\mathfrak{m}^{\mathbb{C}})$ have a much simpler description when $\alpha_i \in \sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$, where

$$(4.3) \quad \mathcal{A}'_{\mathfrak{g}} = \{\alpha \in \bar{\mathcal{A}} \text{ such that the eigenvalues } \lambda \text{ of } \text{ad}(\alpha) \text{ on } \mathfrak{g} \text{ satisfy } |\lambda| < 1\}.$$

So the eigenvalues of $\text{ad}(\alpha)$ have modulus smaller than 1, not only on \mathfrak{h} , but on the whole \mathfrak{g} , and in particular on \mathfrak{m} (one can often choose \mathcal{A} so that this happens). To show this, consider for $\alpha \in \sqrt{-1}\mathfrak{h}$ the subspaces of $\mathfrak{m}^{\mathbb{C}}$ defined by

$$\begin{aligned} \mathfrak{m}_{\alpha} &= \{v \in \mathfrak{m}^{\mathbb{C}} : \text{Ad}(e^{t\alpha})v \text{ is bounded as } t \rightarrow \infty\} \\ \mathfrak{m}_{\alpha}^0 &= \{v \in \mathfrak{m}^{\mathbb{C}} : \text{Ad}(e^{t\alpha})v = v \text{ for every } t\}. \end{aligned}$$

We have that $\mathfrak{m}_{\alpha}^0 \subset \mathfrak{m}_{\alpha}$ and we can choose a complement \mathfrak{n}_{α} so that $\mathfrak{m}_{\alpha} = \mathfrak{m}_{\alpha}^0 \oplus \mathfrak{n}_{\alpha}$

Recall that when $\alpha_i \in \sqrt{-1}\mathcal{A}'$, the parabolic structure at x_i is given by a parabolic subgroup $Q_i \subset E(H^{\mathbb{C}})_{x_i}$ isomorphic to P_{α_i} . This determines an isomorphism of $E(\mathfrak{m}^{\mathbb{C}})_{x_i}$ with $\mathfrak{m}^{\mathbb{C}}$. We can then define the subspaces \mathfrak{m}_i , \mathfrak{m}_i^0 and \mathfrak{n}_i of $E(\mathfrak{m}^{\mathbb{C}})_{x_i}$ corresponding to \mathfrak{m}_{α_i} , $\mathfrak{m}_{\alpha_i}^0$ and \mathfrak{n}_{α_i} , respectively. Then, when $\alpha_i \in \sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$ the **sheaf $PE(\mathfrak{m}^{\mathbb{C}})$ of parabolic sections of $E(\mathfrak{m}^{\mathbb{C}})$** is the sheaf of local holomorphic sections ψ of $E(\mathfrak{m}^{\mathbb{C}})$ such that $\psi(x_i) \in \mathfrak{m}_i$. Similarly, the **sheaf $NE(\mathfrak{m}^{\mathbb{C}})$ of nilpotent sections of $E(\mathfrak{m}^{\mathbb{C}})$** is the sheaf of local holomorphic sections ψ of $E(\mathfrak{m}^{\mathbb{C}})$ such that $\psi(x_i) \in \mathfrak{n}_i$. We then have short exact sequences of sheaves

$$0 \rightarrow PE(\mathfrak{m}^{\mathbb{C}}) \rightarrow E(\mathfrak{m}^{\mathbb{C}}) \rightarrow \bigoplus_i E(\mathfrak{m}^{\mathbb{C}})_{x_i}/\mathfrak{m}_i \rightarrow 0,$$

and

$$0 \rightarrow NE(\mathfrak{m}^{\mathbb{C}}) \rightarrow E(\mathfrak{m}^{\mathbb{C}}) \rightarrow \bigoplus_i E(\mathfrak{m}^{\mathbb{C}})_{x_i}/\mathfrak{n}_i \rightarrow 0.$$

After these preliminaries, we can define a **parabolic G -Higgs bundle** to be a pair of the form (E, φ) , where E is a parabolic $H^{\mathbb{C}}$ -principal bundle over (X, D) and φ —

the **Higgs field** — is a holomorphic section of $PE(\mathfrak{m}^{\mathbb{C}}) \otimes K(D)$. We shall say that (E, φ) is **strictly parabolic** if in addition φ is a section of $NE(\mathfrak{m}^{\mathbb{C}}) \otimes K(D)$.

Remark 4.3. It is worth point out that these definitions can be extended to more general Higgs pairs, where we replace $\mathfrak{m}^{\mathbb{C}}$ by an arbitrary representation of $H^{\mathbb{C}}$ (see [29]).

We now define the residue of φ at the points x_i . This is again much simpler if the weights $\alpha_i \in \sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$. The Higgs field φ is then a meromorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$ with a simple pole at $x_i \in D$ and the **residue** of φ at x_i is hence an element

$$\text{Res}_{x_i} \varphi \in \mathfrak{m}_i.$$

We denote the projection of $\text{Res}_{x_i} \varphi$ in \mathfrak{m}_i^0 by $\text{Gr Res}_{x_i} \varphi$. The space \mathfrak{m}_i^0 is invariant under the action of $L_i \subset Q_i$, the subgroup corresponding to the Levi subgroup $L_{\alpha_i} \subset P_{\alpha_i}$. As we will see, the orbit of this projection under L_i is what is relevant in relation to local systems and the construction of the appropriate moduli space.

More generally, if $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$, φ is a section of $PE(\mathfrak{m}^{\mathbb{C}}) \otimes K(D)$, with $PE(\mathfrak{m}^{\mathbb{C}}) \simeq \bigoplus_{\mu} \mathfrak{m}_{\mu}^{\mathbb{C}}([- \mu]x_i)$ in a neighbourhood of x_i . Choosing a holomorphic trivialization of $\mathcal{O}(D)$ at the point x_i , we can identify the fibre of $PE(\mathfrak{m}^{\mathbb{C}})$ at x_i with $\mathfrak{m}^{\mathbb{C}}$, and then project the residue of φ at x_i to the space $\widetilde{\mathfrak{m}}_i^0 \subset E(\mathfrak{m}^{\mathbb{C}})_{x_i}$ corresponding to

$$(4.4) \quad \widetilde{\mathfrak{m}}_{\alpha_i}^0 := \ker_{\mathfrak{m}^{\mathbb{C}}}(\text{Ad}(\exp 2\pi\sqrt{-1}\alpha_i) - 1).$$

We denote again this projection by $\text{Gr Res}_{x_i} \varphi$. (In the case where $\alpha_i \in \sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$, $\widetilde{\mathfrak{m}}_i^0 = \mathfrak{m}_i^0$, and this is just the projection on the Levi part \mathfrak{m}_i^0 as mentioned above.) Now what will be relevant is the orbit of $\text{Gr Res}_{x_i} \varphi$ under the group \widetilde{L}_i corresponding to

$$(4.5) \quad \widetilde{L}_{\alpha_i} := \text{Stab}_{H^{\mathbb{C}}}(\exp 2\pi\sqrt{-1}\alpha_i)$$

under the isomorphism of Q_i with P_{α_i} given by the parabolic structure of E .

In concrete terms, if we have a local coordinate z near x_i and a holomorphic trivialization of E near x_i , we can write

$$\varphi = \sum_{\mu} \frac{\varphi_{\mu}}{z^{[-\mu]}} \frac{dz}{z},$$

with φ_{μ} holomorphic, and then

$$\text{Gr Res}_{x_i} \varphi = \sum_{\mu \in \mathbb{Z}} \varphi_{\mu}(0).$$

If we change the coordinate, $z' = fz$, then

$$\varphi = \sum_{\mu} f^{[-\mu]} \frac{\varphi_{\mu}}{(z')^{[-\mu]}} \left(\frac{dz'}{z'} - \frac{df}{f} \right),$$

and $\sum_{\mu \in \mathbb{Z}} \varphi_{\mu}(0)$ is changed into

$$\sum_{\mu \in \mathbb{Z}} f(0)^{-\mu} \varphi_{\mu}(0) = \text{Ad}(e^{-\alpha_i \ln f(0)}) \sum_{\mu \in \mathbb{Z}} \varphi_{\mu}(0),$$

for any choice of logarithm of $f(0)$. So we deduce that $\text{Gr Res}_{x_i} \varphi$ is well defined up to the action of the 1-parameter group generated by α_i . Note that this ambiguity exists

only in the case where $\text{ad}(\alpha_i)$ has non zero integer eigenvalues on $\mathfrak{m}^{\mathbb{C}}$. In particular, the orbit of $\text{Gr Res}_{x_i} \varphi$ is well-defined under the action of \widetilde{L}_i .

We must also verify that the definition of $\text{Gr Res}_{x_i} \varphi$ does not depend on the choice of a gauge $g \in PE(H^{\mathbb{C}})$. The only significant case is that of a $g(z) = \exp \frac{n}{z}$ with $n \in \mathfrak{q}_i^1$. Then

$$\text{Ad}(g(z))\varphi = \varphi + \frac{1}{z}[n, \varphi] + \frac{1}{2z^2}[n, [n, \varphi]] + \dots$$

Since $[n, \mathfrak{m}_{\mu}^{\mathbb{C}}] \subset \mathfrak{m}_{\mu-1}^{\mathbb{C}}$, it follows that $\text{Gr Res}_{x_i} \varphi$ is transformed into

$$\text{Ad}(e^n) \text{Gr Res}_{x_i} \varphi.$$

The other cases are left to the reader.

Let us now see what the definition means in three simple cases.

Example 4.4. The first case is the one of Example 2.7, that is $G = \text{GL}_n \mathbb{C}$. Here a G -Higgs bundle is just an ordinary Higgs bundle. In that case, the ambiguity on $\text{Gr Res}_{x_i} \varphi$ does not show up. The Higgs field φ is a meromorphic 1-form with simple poles at the x_i such that $\text{Res}_{x_i} \varphi$ preserves the flag, and $\text{Gr Res}_{x_i} \varphi$ is the endomorphism induced by $\text{Res}_{x_i} \varphi$ on the associated graded space (hence our notation $\text{Gr Res}_{x_i} \varphi$).

Example 4.5. The second example, in which the ambiguity shows up, is $G = \text{SL}_2 \mathbb{R}$ and $H = U_1$. A G -Higgs bundle in this case is given by an $H^{\mathbb{C}}$ -bundle E , which is equivalent to the data of the line bundle $L = E(\mathbb{C})$, and a Higgs field φ which is a 1-form with values in $E(\mathfrak{m}^{\mathbb{C}}) = L^2 \oplus L^{-2}$. The parabolic structure at a point x is given by a weight α which we can take in the interval $(-\frac{1}{2}, \frac{1}{2}]$. The eigenvalues of $\text{ad}(\alpha)$ on $\mathfrak{m}^{\mathbb{C}}$ are $\pm 2\alpha$, and integer eigenvalues ± 1 appear only for $\alpha = \frac{1}{2}$. Let us examine this case more carefully. If we represent $\alpha \in \sqrt{-1}\mathfrak{u}_1$ as the matrix

$$(4.6) \quad \alpha = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},$$

then we obtain, for holomorphic φ_{\pm} ,

$$(4.7) \quad \varphi = \begin{pmatrix} 0 & z\varphi_+ \\ \frac{1}{z}\varphi_- & 0 \end{pmatrix} \frac{dz}{z}, \quad \text{Gr Res}_x \varphi = \begin{pmatrix} 0 & \varphi_+(0) \\ \varphi_-(0) & 0 \end{pmatrix},$$

and $\text{Gr Res}_{x_i} \varphi$ is defined up to

$$\begin{pmatrix} 0 & \varphi_+(0) \\ \varphi_-(0) & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & c\varphi_+(0) \\ c^{-1}\varphi_-(0) & 0 \end{pmatrix}.$$

Of course there is another way to consider the same object: we can think of (E, φ) as a $\text{GL}_2 \mathbb{C}$ -Higgs bundle, the underlying holomorphic bundle of which is $L \oplus L^{-1}$, with parabolic weights $(\frac{1}{2}, -\frac{1}{2})$. Actually, to get weights in a correct interval, it is better to consider $L \oplus L^{-1}(D)$, with weights $(\frac{1}{2}, \frac{1}{2})$. If (e_1, e_2) is a trivialization of $L \oplus L^{-1}$, then $(e_1, z^{-1}e_2)$ is a trivialization of $L \oplus L^{-1}(D)$ and the Higgs field (4.7) becomes

$$\varphi = \begin{pmatrix} 0 & \varphi_+ \\ \varphi_- & 0 \end{pmatrix} \frac{dz}{z},$$

so that $\text{Gr Res}_{x_i} \varphi$ coincides with the one obtained in (4.7), with the same ambiguity coming from the choice of a trivialization of $\mathcal{O}(D)$.

Example 4.6. The third and last example is just our second example considered for the group $G = \text{SL}_2\mathbb{C}$, so $H = \text{SU}_2$. Then $H^\mathbb{C} = \text{SL}_2\mathbb{C}$ and $\mathfrak{m}^\mathbb{C} = \mathfrak{sl}_2\mathbb{C}$. The difference is now that the weight (4.6) has nonzero integer eigenvalues on $\mathfrak{h}^\mathbb{C}$ itself, so one must consider gauge transformations which are meromorphic:

$$g = \begin{pmatrix} O(1) & O(z) \\ O(\frac{1}{z}) & O(1) \end{pmatrix}.$$

This gauge transformation becomes holomorphic in the $\text{GL}_2\mathbb{C}$ -gauge $(e_1, z^{-1}e_2)$ considered above, and all the data of the G -Higgs bundle is equivalent to that of the $\text{GL}_2\mathbb{C}$ -Higgs bundle obtained after this Hecke transformation. Nevertheless, it is useful to have a general definition which does not require to change the group.

4.2. Stability of parabolic G -Higgs bundles. The notion of stability, semistability and polystability of a parabolic G -Higgs bundle depends on an element of $\sqrt{-1}\mathfrak{z}$, where \mathfrak{z} is the centre of \mathfrak{h} . We will develop the theory here for any element in $\sqrt{-1}\mathfrak{z}$. However, in order to relate parabolic G -Higgs bundles to G -local systems, one requires this element to lie also in the centre of \mathfrak{g} , and actually to be 0. This is always the case in particular if G is semisimple.

Let $s \in \sqrt{-1}\mathfrak{h}$. We consider the parabolic subgroup P_s of $H^\mathbb{C}$, and the corresponding Levi subgroup L_s as defined in Section B.1. Let χ_s be the corresponding antidominant character of \mathfrak{p}_s , where \mathfrak{p}_s is the Lie algebra of P_s . We consider

$$\begin{aligned} \mathfrak{m}_s &= \{v \in \mathfrak{m}^\mathbb{C} : \text{Ad}(e^{ts})v \text{ is bounded as } t \rightarrow \infty\} \\ \mathfrak{m}_s^0 &= \{v \in \mathfrak{m}^\mathbb{C} : \text{Ad}(e^{ts})v = v \text{ for every } t\}. \end{aligned}$$

One has that \mathfrak{m}_s is invariant under the action of P_s and \mathfrak{m}_s^0 is invariant under the action of L_s . If G is complex, $\mathfrak{m}^\mathbb{C} = \mathfrak{g}$ and hence the isotropy representation ι coincides with the adjoint representation, then $\mathfrak{m}_s = \mathfrak{p}_s$ and $\mathfrak{m}_s^0 = \mathfrak{l}_s$.

We need to consider the subalgebra of \mathfrak{h} defined by

$$(4.8) \quad \mathfrak{h}_0 = \mathfrak{h} \cap (\cap_{\chi \text{ character of } \mathfrak{g}} \ker \chi),$$

or, equivalently, by $\mathfrak{h}_0 = (\mathfrak{h} \cap \mathfrak{z}(\mathfrak{g}))^\perp$.

Let (E, φ) be a parabolic G -Higgs bundle over (X, D) . Let \mathfrak{z} be the Lie algebra of $Z(H)$, and let $c \in \sqrt{-1}\mathfrak{z}$. We say that (E, φ) is c -**semistable** if for every $s \in \sqrt{-1}\mathfrak{h}$ and any holomorphic reduction of the structure group of E to P_s , σ , such that $\varphi|_{X \setminus D} \in H^0(X \setminus D, E_\sigma(\mathfrak{m}_s) \otimes K)$, where E_σ is the principal P_s -bundle obtained from the reduction σ , we have

$$(4.9) \quad \text{pardeg } E(\sigma, \chi_s) - \langle c, s \rangle \geq 0.$$

If we always have strict inequality for $s \in \sqrt{-1}\mathfrak{h}_0$ we say that (E, φ) is c -**stable**. The Higgs bundle (E, φ) is c -**polystable** if it is semistable and if, whenever equality occurs for some $s \in \sqrt{-1}\mathfrak{h}_0$ and σ , there is:

- a Higgs bundle (E', φ') meromorphically equivalent to (E, φ) ; by Lemma 3.7, (E', φ') inherits from σ a holomorphic reduction to P_s of its structure group; and
- a further holomorphic reduction σ_{L_s} of the structure group of E' to L_s ,

so that

- (1) $\varphi'|_{X \setminus D} \in H^0(X \setminus D, E_{\sigma_{L_s}}(\mathfrak{m}_s^0) \otimes K)$;
- (2) the bundle E_{L_s} has a parabolic structure at the points of D which is compatible with that of E' in the sense that the parabolic bundle E' is induced from E_{L_s} through the injection $L_s \hookrightarrow H^{\mathbb{C}}$ as in Remark 2.9; this implies in particular that the parabolic weights α_i of E at the punctures actually lie in \mathfrak{l}_s .

In this definition, in contrast to parabolic reductions, it is important to allow the reduction to the Levi to exist on a meromorphically equivalent bundle, see Lemma 3.7 and the remarks following it. It then follows that our various stability conditions are invariant by meromorphic equivalence.

We will refer to 0-stability simply as stability.

Remark 4.7. If the weights are in $\sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$ we can define the sheaf $PE_{\sigma}(\mathfrak{m}_s)$ of parabolic sections of $E_{\sigma}(\mathfrak{m}_s)$ as the sheaf of holomorphic sections ψ of $E_{\sigma}(\mathfrak{m}_s)$ such that $\psi(x_i) \in \mathfrak{m}_i \cap E_{\sigma}(\mathfrak{m}_s)_{x_i}$, and require that $\varphi \in H^0(X, PE_{\sigma}(\mathfrak{m}_s) \otimes K(D))$.

Remark 4.8. (Semi)stability can be formulated as above in terms of any parabolic subgroup $P \subset H^{\mathbb{C}}$ conjugated to a parabolic subgroup of the form P_s , and any antidominant character χ of \mathfrak{p} , the Lie algebra of P .

5. HITCHIN–KOBAYASHI CORRESPONDENCE

Let X be a compact Riemann surface and let D a divisor of X as above. Choose a smooth 2-form ω on $X \setminus D$. Suppose either that ω extends smoothly across D or that it blows up near D less rapidly than the Poincaré metric in $X \setminus D$ (given by 5.17). In any case we assume that $\int \omega = 2\pi$. Let $G = (G, H, \theta, B)$ be a real reductive Lie group. Let (E, φ) be a parabolic G -Higgs bundle on (X, D) . Let $c \in \sqrt{-1}\mathfrak{z}$, as in Section 4.2. We are looking for a metric h on E outside the divisor D , i.e. $h \in \Gamma(X \setminus D, E(H \setminus H^{\mathbb{C}}))$, satisfying the c -Hermite–Einstein equation:

$$(5.1) \quad R(h) - [\varphi, \tau_h(\varphi)] + \sqrt{-1}c\omega = 0,$$

where $R(h)$ is the curvature of the unique connection $A(h)$ compatible with the holomorphic structure of E and the metric h , and τ_h is the conjugation on $\Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}}))$ defined by combining the metric h and the standard conjugation on X from $(1, 0)$ -forms to $(0, 1)$ -forms. We shall denote

$$F(h) = R(h) - [\varphi, \tau_h(\varphi)] + \sqrt{-1}c\omega.$$

5.1. Initial metric. We begin by constructing a singular metric h_0 which gives an approximate solution to the equations. This metric is α -adapted, in the sense of Section 2.4.

We first construct a model metric near each singular point x_i . We decompose $\text{Gr Res}_{x_i} \varphi$ into its semisimple part and its nilpotent part,

$$(5.2) \quad \text{Gr Res}_{x_i} \varphi = s_i + Y_i.$$

Let $e_i \in E_{x_i}$ be an element belonging to the P_{α_i} orbit specified by the parabolic structure. Choose a local holomorphic coordinate z , and extend the trivialization e_i into a holomorphic trivialization of E around x_i , so we can identify locally the metric with a map into $H \setminus H^{\mathbb{C}}$. If $Y_i = 0$, then the model metric is

$$(5.3) \quad h_0 = |z|^{-2\alpha_i} (= e^{-2\alpha_i \ln |z|}).$$

If we change the trivialization by a gauge transformation $g \in \Gamma(X, PE(H^{\mathbb{C}}))$, the resulting metric will remain at bounded distance of h_0 in $H \setminus H^{\mathbb{C}}$, so h_0 defines a *quasi-isometry* class of metrics on E near x_i .

If $Y_i \neq 0$, then consider the reductive subalgebra

$$(5.4) \quad \mathfrak{r}_i = \ker(\text{Ad}(e^{2\pi\sqrt{-1}\alpha_i}) - 1) \cap \ker(\text{ad } s_i)$$

of $\mathfrak{g}^{\mathbb{C}}$. Recall that $\text{Gr Res}_{x_i} \varphi$ is the projection of the residue of φ to $\widetilde{\mathfrak{m}}_i^0$ the centralizer of $e^{2\pi\sqrt{-1}\alpha_i}$, defined in (4.4), and hence Y_i belongs to \mathfrak{r}_i . We can complete Y_i into a Kostant–Rallis \mathfrak{sl}_2 -triple (H_i, X_i, Y_i) (see Appendix A.3) such that

$$(5.5) \quad H_i \in \mathfrak{h}^{\mathbb{C}} \cap \mathfrak{r}_i, \quad X_i, Y_i \in \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{r}_i.$$

Moreover, maybe after conjugation (which means that we change the trivialization), we can assume that

$$(5.6) \quad H_i \in \sqrt{-1}\mathfrak{h}, \quad X_i = -\tau(Y_i),$$

where τ is the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to a compact form so that θ and τ commute. Now the Higgs field has the form

$$(5.7) \quad \varphi = \left(s_i + \text{Ad}(z^{\alpha_i})(Y_i) + \psi \right) \frac{dz}{z}, \quad \psi \in NE(\mathfrak{m}^{\mathbb{C}}).$$

This is well defined because $Y_i \in \ker(\text{Ad}(e^{2\pi\sqrt{-1}\alpha_i}) - 1)$ so that Y_i decomposes on eigenspaces of $\text{ad } \alpha_i$ corresponding to the integral eigenvalues. The model for the metric is

$$(5.8) \quad h_0 = |z|^{-\alpha_i} (-\ln |z|^2)^{\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i} |z|^{-\alpha_i}.$$

Again this is well defined because $H_i \in \ker(\text{Ad}(e^{2\pi\sqrt{-1}\alpha_i}) - 1)$. In the case where $\alpha_i \in \mathcal{A}'$ (that is all eigenvalues of $\text{ad } \alpha_i$ in $\mathfrak{h}^{\mathbb{C}}$ have modulus less than 1) then $[\alpha_i, H_i] = 0$ and the formula simplifies to $h_0 = |z|^{-2\alpha_i} (-\ln |z|^2)^{H_i}$. The general formula (5.8) is obtained from the gauge z^{α_i} , which exists only on the universal cover of the punctured disc: in that gauge the Higgs field writes $(s_i + Y_i) \frac{dz}{z}$ and the metric h_0 is simply $(-\ln |z|^2)^{H_i}$.

Again, a different choice of trivialization leads to a metric which remains quasi-isometric to h_0 . Also note that h_0 is α_i -adapted in the sense of definition 2.1.

Now extend the local model to some global metric h_0 on E . The quasi-isometry class of h_0 is well-defined. We shall now prove that the metric h_0 gives an approximate

solution of the Hermite–Einstein equation near the marked point x_i . Instead of working in the holomorphic gauge e of E , we choose the unitary gauge

$$(5.9) \quad \mathbf{e} = eg_0, \quad g_0 = |z|^{\alpha_i} (-\ln |z|^2)^{-\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i/2},$$

so that the $\bar{\partial}$ -operator of E can be written locally as

$$(5.10) \quad \bar{\partial}^E = \bar{\partial} + g_0^{-1} \bar{\partial} g_0 = \bar{\partial} + \left(\alpha_i - \frac{\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i}{\ln |z|^2} \right) \frac{d\bar{z}}{2\bar{z}}$$

and the associated H -connection is

$$(5.11) \quad A_0 = d - \sqrt{-1} \left(\alpha_i - \frac{\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i}{\ln |z|^2} \right) d\theta.$$

Here by a $\bar{\partial}$ -operator on E we mean a holomorphic structure on the underlying smooth $H^{\mathbb{C}}$ -bundle. The space of holomorphic structures on this smooth bundle is an affine space modelled on the space of $(0, 1)$ -forms with values in the adjoint bundle $E(\mathfrak{h}^{\mathbb{C}})$. The $\bar{\partial}$ appearing in (5.10) represents an origin in this affine space. After choosing a faithful representation of $H^{\mathbb{C}}$, one obtains a true $\bar{\partial}$ -operator on the associated vector bundle.

On the other hand, still in the unitary gauge \mathbf{e} , we obtain the expression for the Higgs field by the action of $\text{Ad}(g_0^{-1})$, which gives

$$(5.12) \quad \varphi = \left(s_i - \frac{\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})Y_i}{\ln |z|^2} \right) \frac{dz}{z} + O(|z|^\epsilon) \frac{dz}{z}.$$

As in formula (5.7) this is well defined. Therefore we obtain, using (5.6),

$$(5.13) \quad \begin{aligned} R(h_0) &= \text{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i \frac{dz \wedge d\bar{z}}{|z|^2 (\ln |z|^2)^2}, \\ [\varphi, \tau_{h_0}(\varphi)] &= \text{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i \frac{dz \wedge d\bar{z}}{|z|^2 (\ln |z|^2)^2} + O(|z|^\epsilon) \frac{dz \wedge d\bar{z}}{|z|^2}, \end{aligned}$$

and we clearly have an approximate solution of the Hermite–Einstein equation (5.1).

We can regard the unitary gauge \mathbf{e} as defining a different extension of E near the punctures, which is a unitary extension. The resulting H -principal bundle on X will be denoted \mathbf{E} .

5.2. The correspondence. We are now in a position to state the main theorem in this section. First remark that if χ is a character of G , then $\chi([\varphi, \theta(\varphi)]) = 0$ and therefore the Hermite–Einstein equation (5.1) implies $\text{pardeg}_\chi E = \chi(c)$. In particular, if $c = 0$, $\text{pardeg}_\chi E = 0$. It is important to note that this is no longer true in general for a character of H alone, so we cannot conclude that the total parabolic degree of E must vanish. This justifies the topological condition in the following theorem.

Theorem 5.1. *Let (E, φ) be a parabolic G -Higgs bundle, equipped with an adapted initial metric h_0 . Let $c \in \sqrt{-1}\mathfrak{z}(\mathfrak{h})$ such that $\text{pardeg}_\chi E = \chi(c)$ for all characters χ of \mathfrak{g} . Then (E, φ) admits a c -Hermite–Einstein metric h quasi-isometric to h_0 and α_i -adapted at each puncture x_i , if and only if (E, φ) is c -polystable. Moreover, any two*

such Hermite-Einstein metrics are related by an automorphism of (E, φ) , that is, an element of $PE(H^{\mathbb{C}})$ fixing φ .

It seems difficult to reduce Theorem 5.1 to the theorem of Simpson [65] for the case $G = \mathrm{GL}_n \mathbb{C}$ by taking a faithful representation, since in particular it is not clear how the stability conditions would relate. Instead, we prefer to give a direct proof by checking that the proof in [8] still applies here.

We prove the Theorem in the next sections. For clarity, we restrict to the case $c = 0$ (it is well-known how to modify the proof to handle nonzero c , see for example [19]), and we will refer to 0-polystability as polystability.

5.3. The polystability condition is necessary. Suppose that $s \in \sqrt{-1}\mathfrak{h}$ and we have a holomorphic reduction σ of the structure group of E to the parabolic group P_s , such that

$$(5.14) \quad \varphi|_{X \setminus D} \in H^0(X \setminus D, E_{\sigma}(\mathfrak{m}_s) \otimes K).$$

Using the operator D'' defined by $D''s = \bar{\partial}s + [\varphi, s]$, we can rewrite the formula in Lemma 2.13 as

$$\mathrm{pardeg}_{\alpha}(E)(\sigma, \chi) = \frac{\sqrt{-1}}{2\pi} \int_{X \setminus D} \langle R_h - [\varphi, \tau_h(\varphi)], s_{\sigma, h} \rangle - \langle \varpi(s_{\sigma, h})(D''s_{\sigma, h}), D''s_{\sigma, h} \rangle.$$

(The additional terms involving the Higgs field cancel out). If h is a Hermite-Einstein metric, then the first term in the sum vanishes and there remains

$$\mathrm{pardeg}_{\alpha}(E)(\sigma, \chi) = \frac{\sqrt{-1}}{2\pi} \int_{X \setminus D} -\langle \varpi(s_{\sigma, h})(D''s_{\sigma, h}), D''s_{\sigma, h} \rangle \geq 0.$$

The inequality comes from the fact that for a holomorphic reduction satisfying (5.14), then $D''s_{\sigma, h}$ lives in the negative eigenspaces of $\mathrm{ad} s_{\sigma, h}$. Equality therefore occurs if and only if $D''s_{\sigma, h} = 0$. Since $s \in \sqrt{-1}\mathfrak{h}$, this implies that $s_{\sigma, h}$ is parallel:

$$\nabla_h s_{\sigma, h} = [\varphi, s_{\sigma, h}] = [\tau_h(\varphi), s_{\sigma, h}] = 0.$$

It is now clear that $s_{\sigma, h}$ induces a reduction of the Higgs bundle (E, φ) to the Levi subgroup L_s over $X \setminus D$, and there remains to understand the behavior at the punctures x_i .

Take a local holomorphic trivialization of E near x_i , such that the α_i -adapted metric h can be written $h = h_0 \cdot |z|^{-\alpha_i} e^c$, where c satisfies the conditions of Definition 2.12. Since $s_{\sigma, h}$ is parallel, it has constant norm, which implies that in the trivialization the coefficients of $s_{\sigma, h}$ satisfy $\mathrm{Ad}(|z|^{-\alpha_i})s_{\sigma, h} = O(|z|^{-\epsilon})$ for every $\epsilon > 0$, which implies that actually $s_{\sigma, h}$ is a section of the sheaf $PE(\mathfrak{h}^{\mathbb{C}})$. Decomposing $\mathfrak{h}^{\mathbb{C}} = \bigoplus_{\lambda} \mathfrak{h}_{\lambda}^{\mathbb{C}}$ according to the eigenvalues $\lambda \in (-1, 1)$ of $\mathrm{ad} \alpha_i$, we therefore obtain that $s_{\sigma, h} = \mathrm{Ad}(z^{\alpha_i})\sigma_0 + \sigma_- + z\sigma_+ + s_1$, where $\sigma_0 \in \ker(\mathrm{Ad}(e^{2\sqrt{-1}\pi\alpha_i}) - 1)$ (so that $\mathrm{Ad}(z^{\alpha_i})\sigma_0$ makes sense), $\sigma_- \in \bigoplus_{-1 < \lambda < 0} \mathfrak{h}^{\mathbb{C}}$, $\sigma_+ \in \bigoplus_{0 < \lambda < 1} \mathfrak{h}^{\mathbb{C}}$ and $\frac{s_1}{z}$ is a section of the sheaf $PE(\mathfrak{h}^{\mathbb{C}})$.

We can conjugate σ_0 to a $\sigma'_0 \in \ker(\mathrm{ad} \alpha_i)$ under an element u of the centralizer of $e^{2\pi\sqrt{-1}\alpha_i}$. Writing $u = e^v$ with $v \in \ker(\mathrm{Ad}(e^{2\sqrt{-1}\pi\alpha_i}) - 1)$, the meromorphic gauge transformation $g_i = e^{\mathrm{Ad}(z^{\alpha_i})v}$ is an element of $PE(\mathfrak{h}^{\mathbb{C}})$, and

$$\mathrm{Ad}(g_i) \mathrm{Ad}(z^{\alpha_i})\sigma_0 = \mathrm{Ad}(z^{\alpha_i}) \mathrm{Ad}(e^v)\sigma_0 = \mathrm{Ad}(z^{\alpha_i})\sigma'_0 = \sigma'_0.$$

If we decompose $v = v_1 + v_0 + v_{-1}$, then $\text{Ad}(z^{\alpha_i})v = zv_1 + v_0 + \frac{v_{-1}}{z}$ and $\text{Ad}(g_i) = \exp(\text{ad}(zv_1 + v_0 + \frac{v_{-1}}{z}))$; the same arguments as in Lemma 4.1 show that $\text{Ad}(g_i)(\sigma_- + z\sigma_+ + s_1) = \sigma'_- + z\sigma'_+ + s'_1$ with again $\sigma'_- \in \oplus_{-1 < \lambda < 0} \mathfrak{h}^{\mathbb{C}}$, $\sigma'_+ \in \oplus_{0 < \lambda < 1} \mathfrak{h}^{\mathbb{C}}$ and $\frac{s'_1}{z}$ a section of the sheaf $PE(\mathfrak{h}^{\mathbb{C}})$. So finally we obtain that $s_i = \text{Ad}(g_i)s_{\sigma,h}$ is holomorphic and $s_i(0) \in \mathfrak{p}_{\alpha_i}$. By conjugating further by an element of P_{α_i} , we can suppose that the semisimple element $s_i(0)$ actually satisfies $[\alpha_i, s_i(0)] = 0$.

It follows that the collection of the local meromorphic gauges (g_i) gives a holomorphic $H^{\mathbb{C}}$ -bundle E' over X which is meromorphically equivalent to E , in which $s_{\sigma,h}$ extends holomorphically, and therefore defines a L_s -reduction of E' , which is a parabolic bundle over X with parabolic weights α_i at x_i . This finishes the proof of polystability.

5.4. Preliminaries: functional spaces. We now pass to the existence of the Hermite–Einstein metric. The basic idea to prove that polystability is sufficient is common to a whole collection of results extending the original Hitchin–Kobayashi correspondence on existence of Hermite–Einstein metrics on holomorphic vector bundles. Recall that when looking for Hermite–Einstein metrics one considers the Donaldson functional $M(h, h')$, defined for two metrics h and h' on E such that

$$(5.15) \quad M(h, h'') = M(h, h') + M(h', h'')$$

$$(5.16) \quad \left. \frac{d}{dt} M(h, he^{ts}) \right|_{t=0} = \int_X h(\sqrt{-1}F(h), s)$$

(see Section 5.5 below for some details and references). In particular the critical points of M are the Hermite–Einstein metrics. Roughly speaking, polystability enables to prove C^0 -convergence of a sequence of metrics $\{h_i\}$ such that $M(h, h_i)$ converges to $\inf_{h'} M(h, h')$, and then convergence in stronger functional spaces follows. Finally the limit is proved to be a solution of the Hermite–Einstein equation.

We shall not give full details of the argument in our situation since the proof follows the one in [8], and we refer to this reference for details. We give only the general setup.

The equation to solve does not depend on the metric on the Riemann surface. Because of the calculation (5.13), it is natural to work with a cusp metric near the punctures, that is equal to

$$(5.17) \quad ds^2 = \frac{|dz|^2}{|z|^2 \ln^2 |z|^2}$$

in some fixed local coordinate z near each puncture. Writing $z = |z|e^{\sqrt{-1}\theta}$ and $t = \ln(-\ln |z|^2)$, this can be written

$$ds^2 = dt^2 + e^{-2t}d\theta^2.$$

Extend t by a smooth function in the interior of X . We define weighted C^0 and L^p spaces by

$$C^0_{\delta} = e^{-\delta t} C^0, \quad L^p_{\delta} = e^{-(\delta + \frac{1}{p})t} L^p.$$

The curious choice for L^p is motivated by the compatibility with C^0 : indeed, with this choice, we have $C^0_{\delta} \subset L^p_{\delta'}$ as soon as $\delta > \delta'$, since $\text{vol} = e^{-t} dt d\theta$ near the punctures. The exponent p is thought as being very large—a replacement of ∞ because C^k spaces are not suitable for elliptic analysis.

Consider the H -connection ∇^+ induced by h_0 on E , and define

$$\nabla = \nabla^+ \oplus \text{ad}(\varphi) : \mathbf{E}(\mathfrak{g}^{\mathbb{C}}) \rightarrow \Omega^1 \otimes (\mathbf{E}(\mathfrak{g}^{\mathbb{C}}) \oplus \mathbf{E}(\mathfrak{g}^{\mathbb{C}})).$$

Define now the weighted spaces C_δ^k (resp. the weighted Sobolev spaces $L_\delta^{k,p}$) of sections f of $\mathbf{E}(\mathfrak{g}^{\mathbb{C}})$ such that $\nabla^j f \in C_\delta^0$ (resp. L_δ^p) for $j \leq k$. We will also use the refinement $\hat{C}_\delta^k(E(\mathfrak{g}^{\mathbb{C}}))$ (resp. $\hat{L}_\delta^{k,p}(\mathbf{E}(\mathfrak{g}^{\mathbb{C}}))$) of sections f of $\mathbf{E}(\mathfrak{h}^{\mathbb{C}})$ such that $\nabla f \in C_\delta^{k-1}$ (resp. $L_\delta^{k-1,p}$), but we ask nothing on f itself.

Recall from (5.4) the subalgebra

$$(5.18) \quad \mathfrak{r}_i = \ker(\text{Ad}(e^{2\pi\sqrt{-1}\alpha_i}) - 1) \cap \ker(\text{ad } s_i),$$

and define furthermore \mathfrak{r}'_i the commutator of the \mathfrak{sl}_2 -triple (H_i, X_i, Y_i) ,

$$(5.19) \quad \mathfrak{r}'_i = \ker(\text{ad } H_i) \cap \ker(\text{ad } X_i) \cap \ker(\text{ad } Y_i).$$

For $\delta > 0$, it is easy to see that, near a puncture, an element f of \hat{C}_δ^k (resp. $\hat{L}_\delta^{k,p}(\mathfrak{g}^{\mathbb{C}})$) can be decomposed as

$$f = \text{Ad}(e^{\sqrt{-1}\theta\alpha_i})f(0) + f_1, \quad f(0) \in \mathfrak{r}_i \cap \mathfrak{r}'_i, \quad f_1 \in C_\delta^k \text{ (resp. } L_\delta^{k,p}).$$

As before, this is well defined since $f(0) \in \ker(\text{Ad}(e^{2\pi\sqrt{-1}\alpha_i}) - 1)$. Therefore the elements of \hat{C}_δ^2 or $\hat{L}_\delta^{2,p}$ do not go to zero at the punctures. Furthermore one checks easily that $\hat{L}_\delta^{2,p}(\mathbf{E}(\mathfrak{g}^{\mathbb{C}}))$ is a Lie algebra.

Finally the space of metrics in which we look for a solution of the Hermite–Einstein equation is, for a small $\delta > 0$ and a large p ,

$$(5.20) \quad \mathcal{H} = \{h = h_0 e^s, s \in \hat{L}_\delta^{2,p}(\mathbf{E}(\sqrt{-1}\mathfrak{h}))\}.$$

From equation (5.13) it is clear that $F(h_0) \in L_\delta^p$ for any p and any $\delta > 0$. We choose any fixed $\delta \in (0, 1)$.

5.5. Donaldson's functional. For a pair of metrics $h_0, h_0 e^s \in \mathcal{H}$ let

$$(5.21) \quad M(h_0, h_0 e^s) = \int_X \langle \sqrt{-1}\Lambda F(h_0), s \rangle_{h_0} + \int_0^1 (1-t) \| \text{Ad}(e^{ts/2}) D'' s \|_{L^2(X)}^2 dt,$$

where $D'' s = \bar{\partial}^E s + [\varphi, s]$. Note that $\bar{\partial}^E s$ belongs to $\Omega^{0,1}(E(\mathfrak{h}^{\mathbb{C}}))$ and $[\varphi, s]$ belongs to $\Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}}))$. Hence the two summands are orthogonal and we can write

$$\| \text{Ad}(e^{ts/2}) D'' s \|_{L^2(X)}^2 = \| \text{Ad}(e^{ts/2}) \bar{\partial}^E s \|_{L^2(X)}^2 + \| \text{Ad}(e^{ts/2}) [\varphi, s] \|_{L^2(X)}^2.$$

This allows to view M as a particular case of the functionals defined in [51, 19] using a symplectic point of view (they are instances of integrals of a moment map), see in particular [51, (4.10)]. The arguments given in [op.cit.] imply that M satisfies the cocycle condition (5.15) and property (5.16).

The functional M is also an analogue of Donaldson's functional considered by Simpson in [64]. Indeed, we have the following formula

$$\int_0^1 (1-t) \| \text{Ad}(e^{ts/2}) D'' s \|_{L^2(X)}^2 dt = \int_X (\psi(s)(D'' s), D'' s)_{h_0},$$

where we apply here the notation introduced in Section 2.3 to the adjoint and the isotropy representations, and extend it to sections of twists of $E(\mathfrak{h}^{\mathbb{C}})$ and $E(\mathfrak{m}^{\mathbb{C}})$, for

the function $\psi(t) = (e^t - t - 1)/t^2$. This follows by decomposing $D''s$ on eigenspaces of the adjoint action of s , and from the calculation $\int_0^1 (1-t)e^{t\lambda} dt = \psi(\lambda)$ for any eigenvalue λ of $\text{ad } s$. Also, (\cdot, \cdot) is distinguished from $\langle \cdot, \cdot \rangle$: the latter is the Hermitian product on the bundle only, while the former also includes the Hermitian product on forms, so the result is a scalar (which is implicitly integrated against the volume form).

Hence Donaldson's functional can be rewritten in the following form

$$(5.22) \quad M(h_0, h_0 e^s) = \int_X \langle \sqrt{-1} \Lambda F(h_0), s \rangle_{h_0} + \int_X (\psi(s)(D''s), D''s)_{h_0},$$

which makes evident the relation to Simpson's definition.

5.6. Reduction to the stable and semisimple case. Before solving the equation, we make two reductions.

Jordan-Hölder reduction. Using similar arguments to the ones used in the non-parabolic case [29], one can prove that if a G -Higgs bundle (E, φ) is polystable there is a canonical reduction (E', φ') of structure group to a G' -Higgs bundle, up to isomorphism, such that the reduced Higgs bundle is stable. This is done by iterating the process, mentioned in definition of polystability, of reduction to a Levi given by a parabolic reduction for which the degree vanishes. Moreover, this reduction also satisfies $\text{pardeg}_\chi E = 0$ for all characters of \mathfrak{g}' . (Recall that we are in the case $c = 0$).

Therefore, to prove existence of the Hermite-Einstein metric, we can suppose that (E, φ) is actually stable.

Central part. The second step is to reduce to the semisimple part of G . Indeed, as is well-known, the Hermite-Einstein equation (5.1) decouples on the direct sum

$$\mathfrak{h} = (\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{h}) \oplus \mathfrak{h}_0,$$

where \mathfrak{h}_0 was defined in (4.8), into

$$(5.23) \quad \chi(R(h)) = 0 \text{ for all characters } \chi \text{ of } \mathfrak{g},$$

$$(5.24) \quad R(h)_{\mathfrak{h}_0} - [\varphi, \tau_h(\varphi)] = 0.$$

We have a trivial subbundle $E(\mathfrak{z}(\mathfrak{g})) \subset E(\mathfrak{h})$, and for a section s of $E(\sqrt{-1}\mathfrak{z}(\mathfrak{g}))$ one has $\chi(R(h e^s)) = \chi(R(h)) + \bar{\partial} \partial \chi(s)$. Therefore, starting from the initial metric h_0 , the equation (5.23) is achieved by $h = h_0 e^s$ if one can solve the equation, for all characters χ of \mathfrak{g} ,

$$\Delta \chi(s) = 2i \Lambda \chi(R(h)).$$

This is just the Laplace equation on the trivial bundle $\sqrt{-1}\mathfrak{z}(\mathfrak{g})$. Since by hypothesis $\int \chi(R(h)) = 0$ for all characters χ of \mathfrak{g} , this equation is solvable in the space $\hat{L}_\delta^{2,p}$ since $F(h_0) \in L_\delta^p$. (The non uniqueness of the solution of the Hermite-Einstein equation comes from the non uniqueness of such s , since one can add any constant section of $\sqrt{-1}\mathfrak{z}(\mathfrak{g})$).

Finally we can suppose that we are in the case of a stable bundle (E, φ) , and that our initial metric h_0 satisfies (5.23). We look for a solution of (5.24) of the form $h = h_0 e^s$ with s a section of $E(\sqrt{-1}\mathfrak{h}_0)$.

5.7. Proof that polystability is sufficient. The method consists in minimizing the Donaldson functional $M(h_0, h)$ for $h \in \mathcal{H}$ under the constraint $\|F(h)\|_{L_\delta^{2,p}} \leq B$ for some large B .

At the end, the solution h will satisfy an elliptic equation, so have additional regularity:

$$(5.25) \quad h \in \mathcal{H}^\infty = \{h_0 e^s, s \in \hat{C}_\delta^\infty(\mathbf{E}(\sqrt{-1}\mathfrak{h}_0))\}.$$

Actually it can be shown that there is a stronger decay of the components of s which are orthogonal to \mathfrak{r}_i .

Take some big B and define

$$\mathcal{S}^\infty(B) = \{s \in \hat{C}_\delta^\infty(\mathbf{E}(\sqrt{-1}\mathfrak{h}_0)), \|F(h_0 e^s)\|_{L_\delta^p} \leq B\}.$$

Stability implies the following C^0 -estimate:

Proposition 5.2. *Suppose (E, φ) is given as in Theorem 5.1 and (E, φ) is stable. Then there exist constant C and C' such that*

$$(5.26) \quad \sup |s| \leq C + C' M(h_0, h_0 e^s) \quad \text{for any } s \in \mathcal{S}^\infty(B).$$

The technical results in [8] reduce the proof of existence of the solution h to Proposition 5.2. Therefore in the rest of the subsection we only prove the Proposition.

Proof of Proposition 5.2. Assume that there do not exist any constants C, C' satisfying (5.26). The bound on the curvature implies that

$$\sup |s| \leq \text{const} \cdot (1 + \|s\|_{L^1})$$

for some constant depending on B (see Lemma 8.4 in [8]). Hence our assumption implies that there neither exist constants C, C' such that

$$\|s\|_{L^1} \leq C + C' M(h_0, h_0 e^s) \quad \text{for any } s \in \mathcal{S}^\infty(B).$$

It follows that we can take a sequence $\{s_i\} \subset \mathcal{S}^\infty(B)$ and positive real numbers $C_i \rightarrow \infty$ such that

$$\|s_i\|_{L^1} \rightarrow \infty, \quad C_i M(h_0, h_0 e^{s_i}) \leq |s_i|_{L^1} =: \lambda_i.$$

Define $u_i = \lambda_i^{-1} s_i$, so that $\|u_i\|_{L^1} = 1$. Now the arguments in Section 5 of [64] imply that, up to passing to a subsequence, the sections u_i converge (weakly and locally in $L^{1,2}$) to a nonzero and locally $L^{1,2}$ section u_∞ of $\mathbf{E}(\sqrt{-1}\mathfrak{h}_0)|_{X \setminus D}$. This section satisfies the following properties:

- the L^2 norm of $D''u_\infty$ is finite, and the projection of $D''u_\infty$ on eigenspaces of $\text{ad } u_\infty$ corresponding to nonnegative eigenvalues of $\text{ad } u_\infty$ is zero;
- there exist locally $L^{1,2}$ orthogonal projections $(\pi_j)_{j=1 \dots k}$ of $\text{End}(E(\mathfrak{h}^{\mathbb{C}}))$, satisfying $[\pi_i, \pi_j] = 0$, such that $\text{ad}(u_\infty) = \sum l_j \pi_j$, where $l_1 < \dots < l_k$ are the eigenvalues of $\text{ad } u_\infty$ on $E(\mathfrak{h}^{\mathbb{C}})$;
- for each j let $\Pi^j = \pi_1 + \dots + \pi_j$. Then $\Pi^j \circ \Pi^j = \Pi^j$ and $(1 - \Pi^j)D''\Pi^j = 0$.

The regularity result of Uhlenbeck and Yau [69] for locally $L^{1,2}$ subbundles implies that the image of Π^j is a holomorphic subbundle $E^j \subset E(\mathfrak{h}^{\mathbb{C}})|_{X \setminus D}$ (see Proposition 5.8 in [64]). This implies that u_∞ is a smooth section of $\mathbf{E}(\sqrt{-1}\mathfrak{h})|_{X \setminus D} \subset E(\mathfrak{h}^{\mathbb{C}})|_{X \setminus D}$, since h_0 is smooth on $X \setminus D$.

Lemma 5.3. *Take any $x, y \in X \setminus D$, choose trivializations of \mathbf{E}_x and \mathbf{E}_y , and use them to identify $\mathbf{E}(\mathfrak{h}^{\mathbb{C}})_x$ and $\mathbf{E}(\mathfrak{h}^{\mathbb{C}})_y$ with $\mathfrak{h}^{\mathbb{C}}$. Then the elements $u_{\infty}(x), u_{\infty}(y) \in \mathfrak{h}^{\mathbb{C}}$ are conjugate.*

Proof. Denote by $\mathfrak{h}^{\mathbb{C}}//H^{\mathbb{C}}$ the affine quotient of the adjoint action. Since semisimple elements have closed adjoint orbits, two semisimple elements $x, y \in \mathfrak{h}^{\mathbb{C}}$ are conjugate if and only if their images by the projection map $\mathfrak{h}^{\mathbb{C}} \rightarrow \mathfrak{h}^{\mathbb{C}}//H^{\mathbb{C}}$ coincide. The later is equivalent to showing that for any invariant polynomial $p \in \text{Hom}_{\mathbb{C}}(S^*\mathfrak{h}^{\mathbb{C}}, \mathbb{C})^{H^{\mathbb{C}}}$ we have $p(x) = p(y)$. Let p be any such invariant polynomial. If $u \in \sqrt{-1}\mathfrak{h}^{\mathbb{C}}$ and $n \in \mathfrak{h}^{\mathbb{C}}$ is concentrated on eigenspaces for negative eigenvalues of $\text{ad } u$, then $p(u + n) = p(\text{Ad}(tu)(u + n)) = p(u + \text{Ad}(tu)n) \xrightarrow{t \rightarrow +\infty} p(u)$, so $p(u + n) = p(u)$ and $d_u p(n) = 0$. Applying this to u_{∞} and $D''u_{\infty}$, we obtain

$$(5.27) \quad \bar{\partial}p(u_{\infty}) = d_{u_{\infty}}p(D''u_{\infty}) = 0.$$

Hence $p(u_{\infty})$ is a holomorphic function. On the other hand using the isomorphisms

$$\begin{aligned} \text{Hom}_{\mathbb{C}}(S^*\mathfrak{h}^{\mathbb{C}}, \mathbb{C})^{H^{\mathbb{C}}} &\simeq \text{Hom}_{\mathbb{C}}(S^*\mathfrak{h}^{\mathbb{C}}, \mathbb{C})^H \quad (H \subset H^{\mathbb{C}} \text{ is Zariski dense}) \\ &\simeq \text{Hom}_{\mathbb{R}}(S^*(\sqrt{-1}\mathfrak{h}), \mathbb{R})^H \otimes_{\mathbb{R}} \mathbb{C} \end{aligned}$$

we may write $p = a + \sqrt{-1}b$, where $a(\sqrt{-1}\mathfrak{h}) \subset \mathbb{R}$ and $b(\sqrt{-1}\mathfrak{h}) \subset \mathbb{R}$ and both a and b are $H^{\mathbb{C}}$ invariant. Then $a(u_{\infty})$ and $b(u_{\infty})$ are holomorphic functions and since u_{∞} is a section of $\mathbf{E}(\sqrt{-1}\mathfrak{h})$ the function $a(u_{\infty})$ (resp. $b(u_{\infty})$) is real (resp. imaginary) valued. Hence both $a(u_{\infty})$ and $b(u_{\infty})$ vanish. We have thus proved that $p(u_{\infty})$ is constant for any invariant polynomial p . Since u_{∞} takes semisimple values, the lemma is proved. \square

Let $x \in X \setminus D$ be any point, take a trivialization of \mathbf{E}_x , use it to identify $\mathbf{E}(\sqrt{-1}\mathfrak{h})_x$ with $\sqrt{-1}\mathfrak{h}$, and let $s \in \sqrt{-1}\mathfrak{h}_0$ be the element corresponding to $u_{\infty}(x)$. From the Lemma, the section u_{∞} defines a reduction σ of the structure group of $E|_{X \setminus D}$ to the parabolic subgroup $P = P_s$, such that $u_{\infty} = s_{\sigma, h_0}$ as in (2.7). Furthermore, this section is holomorphic, because the bundles $E^j \subset E(\mathfrak{h}^{\mathbb{C}})$ are holomorphic.

Lemma 5.4. *The reduction σ extends into a holomorphic reduction of E to P on the whole X .*

Proof. Denote by \mathfrak{p}_s the Lie algebra of P_s . Let Gr denote the Grassmannian variety parametrizing $\dim \mathfrak{p}_s$ -dimensional subspaces of $\mathfrak{h}^{\mathbb{C}}$. The adjoint representation $H^{\mathbb{C}} \rightarrow \text{GL}(\mathfrak{h}^{\mathbb{C}})$ induces an action of $H^{\mathbb{C}}$ on Gr . By [40, Proposition 7.83] the stabilizer of $\mathfrak{p}_s \in \text{Gr}$ is equal to P_s , so there is a unique $H^{\mathbb{C}}$ -equivariant map $\iota_0 : F = H^{\mathbb{C}}/P_s \rightarrow \text{Gr}$ sending P_s to \mathfrak{p}_s , and the $H^{\mathbb{C}}$ -orbit through \mathfrak{p}_s can be identified with the image of ι_0 . Let $\iota : E(F) \rightarrow E(\text{Gr})$ be the map induced by ι_0 .

We know that the L^2 norm of $D''(u_{\infty}) = \bar{\partial}^E u_{\infty} + [\phi, u_{\infty}]$ is finite. Since, as already pointed out, $\bar{\partial}^E u_{\infty}$ and $[\phi, u_{\infty}]$ are orthogonal, it follows that the L^2 norm of $\bar{\partial}^E u_{\infty}$ is finite. Denote as before $\text{ad}(u_{\infty}) = \sum l_j \pi_j$. Since each projector π_j can be expressed as a polynomial of u_{∞} , and u_{∞} is bounded (because the eigenvalues l_j are constant), the boundedness of $\|\bar{\partial}^E u_{\infty}\|_{L^2}$ implies that the L^2 norm of $\bar{\partial}^E \pi_j$ is finite. This L^2 norm is computed with respect to the Hermitian metric h'_0 on $E(\mathfrak{h}^{\mathbb{C}})|_{X \setminus D}$ induced by h_0 .

Let E^j be the image of $\Pi^j = \pi_1 + \cdots + \pi_j$. Since $(1 - \Pi^j)D''\Pi^j = 0$ the same orthogonality argument as before implies that $(1 - \Pi^j)\bar{\partial}^E \Pi^j = 0$, from which it follows that E^j is a holomorphic subbundle of $E(\mathfrak{h}^{\mathbb{C}})|_{X \setminus D}$. We claim that we can apply [64, Lemma 10.6] to E^j and thus get a holomorphic extension of E^j as a holomorphic subbundle \bar{E}^j of $E(\mathfrak{h}^{\mathbb{C}})$ defined over the entire X . Indeed, the requirements for [64, Lemma 10.6] to apply are that h'_0 has polynomial growth when expressed in terms of a local trivialisation of $E(\mathfrak{h}^{\mathbb{C}})$ in a neighborhood of D , and that the curvature of h'_0 has bounded L^1 norm. Now the first property follows from (5.8), and the second one follows from the first formula in (5.13).

Since u_∞ is a nowhere vanishing section of $E(\mathfrak{h}^{\mathbb{C}})$ and $\text{ad}(u_\infty)(u_\infty) = 0$ there exists some m such that $l_m = 0$. Then the fibers of $E^m \subset E(\mathfrak{h}^{\mathbb{C}})|_{X \setminus D}$ can be identified with Lie algebras isomorphic to \mathfrak{p}_s , so E^m defines a holomorphic section ρ_0 of $E(\text{Gr})$ along $X \setminus D$ which actually comes from a section ρ_1 of $E(F)$ along $X \setminus D$. Hence, $\rho_0 = \iota \circ \rho_1$. The fact that E^m extends to a holomorphic subbundle \bar{E}^m of $E(\mathfrak{h}^{\mathbb{C}})$ implies that ρ_0 can be extended to a holomorphic section $\bar{\rho}_0$ of $E(\text{Gr})$. Now, F is closed in Gr (because $H^{\mathbb{C}}/P_s$ is compact, precisely since P_s is a parabolic subgroup of $H^{\mathbb{C}}$), and consequently there exists a holomorphic extension $\bar{\rho}_1$ of ρ_1 satisfying $\bar{\rho}_0 = \iota \circ \bar{\rho}_1$.

For any $y \in X$ let $Q_y := \{a \in E(H^{\mathbb{C}})_y \mid a \cdot \bar{E}_y^m \subseteq \bar{E}_y^m\}$. Then Q_y is conjugate to P_s for any trivialization of E_y (because the normalizer of P_s is P_s itself) and $Q = \bigcup_{y \in X} Q_y$ is a holomorphic subbundle of $E(H^{\mathbb{C}})$. By construction, Q is a holomorphic extension of P to the whole X . \square

The element $s \in \sqrt{-1}\mathfrak{h}_0$ which was used to define $P = P_s$ also provides a strictly antidominant character $\chi : \mathfrak{p} \rightarrow \mathbb{C}$, see Appendix B.1. The same arguments as in Lemma 5.4 in [64] allow to prove that $\varphi \in H^0(X \setminus D, E_\sigma(\mathfrak{m}_s) \otimes K)$, and that the section u_∞ satisfies (recall that $D''u_\infty$ is concentrated on negative eigenspaces of $\text{ad } u_\infty$)

$$(5.28) \quad \begin{aligned} & \sqrt{-1} \int_{X \setminus D} \langle \Lambda F(h_0), u_\infty \rangle - \langle \varpi(u_\infty)(\bar{\partial}u_\infty), \bar{\partial}u_\infty \rangle \\ & \leq \int_{X \setminus D} \langle \sqrt{-1}\Lambda F(h_0), u_\infty \rangle - (\varpi(u_\infty)(D''u_\infty), D''u_\infty) \\ & \leq 0. \end{aligned}$$

The function ϖ appears because the rescaling $u_i = \lambda_i^{-1}s_i$ implies to replace in the Donaldson functional the function $\psi(t)$ by the rescaled function $\lambda_i\psi(\lambda_it)$ which converge when $\lambda_i \rightarrow \infty$ to the function $-\varpi(t) = -t^{-1}$ for $t < 0$ and $+\infty$ for $t \geq 0$.

This contradicts stability since $u_\infty = s_{\sigma, h_0}$ so by Lemma 2.13 the first expression in (5.28) is

$$2\pi \text{pardeg}(E)(\sigma, \chi).$$

Proposition 5.2 is proved. \square

5.8. Uniqueness up to automorphism. Suppose that h and $he^s \in \mathcal{H}$ are two critical points of the Donaldson functional $M(h_0, \cdot)$. Then (5.15) and (5.21) imply that $M(h, he^{ts})$ is a constant function of t . Here we need to use the fact that $M(h, he^{ts})$ is a convex function of t , which follows from (5.21). Now formulas (5.15), (5.16) and (5.21)

imply that $D''s = 0$, and therefore s is an element of $PE(\mathfrak{h}^{\mathbb{C}})$ and $[\varphi, s] = 0$. Finally $e^s \in PE(H^{\mathbb{C}})$ and stabilizes φ .

6. PARABOLIC LOCAL SYSTEMS

In this section we take $c = 0$ and we refer to c -(poly)stability of a parabolic G -Higgs bundle simply as (poly)stability. As above, we fix an alcove $\mathcal{A} \subset \mathfrak{t}$ of H such that $0 \in \bar{\mathcal{A}}$ (see Appendix A.1).

6.1. From Higgs bundles to parabolic local systems. Let (E, φ) be a polystable parabolic G -Higgs bundle with $\text{pardeg}_{\chi} E = 0$ for all characters χ of G . By Theorem 5.1, we get an Hermite–Einstein metric h on E , which is quasi-isometric to some given adapted metric h_0 . Equation 5.1 simply means that

$$D = A(h) + \varphi - \tau_h(\varphi)$$

is a flat G -connection on the G -bundle obtained by extending the structure group of the H -bundle given by the Hermite–Einstein metric. We therefore obtain a G -local system on $X' := X \setminus \{x_1, \dots, x_r\}$. Recall that a G -local system on a manifold can be equivalently seen as a representation of the fundamental group of the manifold in G , a G -bundle over the manifold equipped with a flat G -connection, or a G -bundle with locally constant transition functions.

Let us now examine the behaviour of the local system near the puncture. Let us see that just on the local model: from (5.11) and (5.12), we deduce that, in the orthonormal frame \mathbf{e} ,

$$(6.1) \quad \begin{aligned} D = d + & \left(s_i - \tau(s_i) - \text{Ad}(e^{\sqrt{-1}\theta\alpha_i}) \frac{Y_i + X_i}{\ln r^2} \right) \frac{dr}{r} \\ & + \sqrt{-1} \left(-\alpha_i + s_i + \tau(s_i) - \text{Ad}(e^{\sqrt{-1}\theta\alpha_i}) \frac{Y_i - H_i - X_i}{\ln r^2} \right) d\theta. \end{aligned}$$

Observe that $Y_i - H_i - X_i = \text{Ad}(e^{-X_i})Y_i$ is nilpotent. The monodromy around x_i is

$$(6.2) \quad e^{2\pi\sqrt{-1}\alpha_i} e^{2\pi\sqrt{-1}(-s_i - \tau(s_i) + Y_i - H_i - X_i)}.$$

In this formula the monodromy appears as the product of two commuting elements of G (the first is compact, the second is non compact). The $\frac{dr}{r}$ term has also an interpretation: taking

$$(6.3) \quad f = \mathbf{e}g, \quad g = r^{-s_i + \tau(s_i)} (-\ln r^2)^{\frac{1}{2}} \text{Ad}(e^{\sqrt{-1}\theta\alpha_i})(X_i + Y_i),$$

we get a G -trivialization, which is parallel along rays from the origin. The metric along these parallel rays has the form

$$(6.4) \quad h = r^{2(-s_i + \tau(s_i))} (-\ln r^2)^{\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})(X_i + Y_i)}.$$

To calculate D in the new trivialization f , we write formally $g = e^{\sqrt{-1}\theta\alpha_i} g_0 e^{-\sqrt{-1}\theta\alpha_i}$ with $g_0 = r^{-s_i + \tau(s_i)} (-\ln r^2)^{\frac{1}{2}} (X_i + Y_i)$, and we use the fact that $d + a$ in the trivialization e becomes $d + g^{-1}dg + \text{Ad}(g^{-1})a$ in the trivialization eg . Therefore in the trivialization $e_1 = \mathbf{e}e^{\sqrt{-1}\theta\alpha_i}$ we have for D the same formula as (6.1) with α_i replaced by 0; since $\text{ad}(X_i + Y_i)$ acts on $Y_i - H_i - X_i$ with eigenvalue -2 , we obtain in the trivialization

$e_2 = e_1 g_0 = e e^{\sqrt{-1}\theta\alpha_i} g_0$ the formula $d + \sqrt{-1}(s_i + \tau(s_i) - (Y_i - H_i - X_i))d\theta$, which gives finally in the trivialization $f = e_2 e^{-\sqrt{-1}\theta\alpha_i}$ the formula:

$$(6.5) \quad D = d - \sqrt{-1}(\alpha_i - s_i - \tau(s_i) + \text{Ad}(e^{\sqrt{-1}\theta\alpha_i})(Y_i - H_i - X_i))d\theta.$$

The logarithmic part of h in (6.4) gives no new information, because the triple (H_i, X_i, Y_i) is already encoded in the unipotent part $\exp(2\pi\sqrt{-1}(Y_i - H_i - X_i))$ of the monodromy. But the semisimple part $s_i - \tau(s_i)$ is an additional information: it gives the polynomial order of growth of the harmonic metric h near the point x_i , along parallel rays. In the case $G = \text{GL}_n\mathbb{C}$, this additional structure transforms the local system into a **filtered local system** in the sense of Simpson, that is the fibre over the ray has a filtration with weights induced by $s_i - \tau(s_i)$.

In our case, the metric on the ray is a map to the symmetric space of non compact type G/H , and $\exp((s_i - \tau(s_i))u)$, where $u = -\ln r$, describes a geodesic in G/H , with some fixed speed, depending on s_i . The corresponding geometric data is a point on the geodesic boundary of G/H , and a positive real number describing the (constant) speed of the geodesic. This data is equivalent to that of a parabolic subgroup P of G with an antidominant character χ of the Lie algebra of P . This leads to the following definition.

Definition 6.1. *Let $\beta_i \in \mathfrak{m}$ be semisimple and $\beta = (\beta_1, \dots, \beta_r)$. A **parabolic G -local system on $X \setminus \{x_1, \dots, x_r\}$ of weight β** is defined by the following data:*

- (1) *a G -local system F on $X \setminus \{x_1, \dots, x_r\}$,*
- (2) *on a ray ρ_i going to x_i , a choice of a parabolic subgroup P_i of $F(G)|_{\rho_i}$ isomorphic to P_{β_i} , invariant under the monodromy transformation around x_i , with a strictly antidominant character χ_i of the Lie algebra of P_i , where the $F_x(G)$ for $x \in \rho_i$ are identified by parallel transport.*

Recall from Appendix B.1, that pairs (P, χ) consisting of a parabolic subgroup P of G and a strictly antidominant character of the Lie algebra of \mathfrak{p} are in one-to-one correspondence with elements in \mathfrak{m} . In the definition we take χ_i to be the character corresponding to χ_{β_i} under the isomorphism of $F(G)|_{\rho_i}$ with P_{β_i} . Note that, in contrast with a parabolic bundle, at x_i the weights of the parabolic G -local system are not constrained to lie in a Weyl alcove.

Remark 6.2. Note that a choice of $P_i \subset F(G)|_{\rho_i}$ isomorphic to P_{β_i} is equivalent to choosing an orbit of the action of P_{β_i} on F_{ρ_i} , i.e. a point in the flag manifold F_{ρ_i}/P_{β_i} . The second condition in the definition is asking that this point be fixed by the monodromy group, that is, the image of the representation $\pi_1(X, x_i) \rightarrow G$ corresponding to the local system. This condition makes the data independent of the choice of ρ_i .

Let us come back to the G -local system coming from a G -Higgs bundle with Hermite–Einstein metric h . The formula (6.4) gives the behaviour of h for the model, but the formula remains valid for the adapted metric h_0 , up to lower order terms, and also for the Hermite–Einstein metric $h \in \mathcal{H}^\infty$, up to replacing h_0 by $h_0 e^s$, for some constant $s \in \mathfrak{t}_i \cap \mathfrak{t}'_i$. Thus we see that we have a well defined parabolic structure induced at the point x_i , with the parabolic subgroup and the character of its Lie algebra defined by $s_i - \tau(s_i)$. Recall that for a parabolic G -Higgs bundle (E, φ) , $\text{Gr Res}_{x_i} \varphi$ is defined in Section 4.1, after (4.4).

Proposition 6.3. *Let (E, φ) be a polystable parabolic G -Higgs bundle with $\text{pardeg}_\chi E = 0$ for all characters χ of G . Then the G -local system induced by the Hermite–Einstein metric constructed by Theorem 5.1 carries naturally a structure of parabolic G -local system.*

If $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$ is the parabolic weight of E at x_i , and $\text{Gr Res}_{x_i} \varphi = s_i + Y_i$, then the weight of the parabolic G -local system is given by the element $s_i - \tau(s_i) \in \mathfrak{m}$, and the projection of the monodromy around x_i on the Levi group defined by $s_i - \tau(s_i)$ is

$$\exp(2\pi\sqrt{-1}\alpha_i) \exp(2\pi\sqrt{-1}(-s_i - \tau(s_i) + Y_i - H_i - X_i)).$$

To be precise, the compatibility between the metric and the parabolic structure is the following: on a ray going to x_i , we compare the metric h , seen as an application into G/H , with a geodesic given by the (Q_i, χ_i) , parametrized by $(-\ln r)$, and the condition is that the distance between them should grow at most like $|\ln r|^N$. For $G = \text{GL}_n \mathbb{C}$, this is the property referred by Simpson in [65] as **tameness**.

Proof of Proposition 6.3. We have already seen the model behaviour. In general we have a perturbed flat connection $D + a$, where D is the model (6.1) and the perturbation $a \in C_\delta^\infty$, which implies $a = a_r \frac{dr}{r} + a_\theta d\theta$, with $|a_r|, |a_\theta| = O(|\ln r|^{-1-\delta})$. The radial trivialization (6.3) is then modified by a bounded transformation (this does not change the parabolic weight $s_i - \tau(s_i)$ of the model), while the formula (6.5) for the connection in this radial trivialization comes with an additional term, depending only on the angle θ ,

$$b(\theta)d\theta = \text{Ad}(g^{-1})a_\theta d\theta.$$

The term $r^{-s_i + \tau(s_i)}$ in f and the initial bound on a_θ imply the vanishing of the components of b on the eigenspaces of $\text{ad}(s_i - \tau(s_i))$ corresponding to the nonnegative eigenvalues. On the contrary, there can be a nonzero contribution from the eigenspaces for the negative eigenvalues, which is an additional unipotent term in the monodromy, in the unipotent subgroup associated to the parabolic subgroup. Therefore, only the Levi part is fixed and is equal to that of the model. \square

The converse of Proposition 6.3 is given in the next section.

6.2. Harmonic metrics and polystability of G -parabolic local systems. Given a flat G -bundle (F, D) over X' and a metric h on F , that is, a reduction of structure group to an H -bundle, we decompose $D = D_h^+ + \psi_h$, where D_h^+ is an H -connection and ψ_h is a section of $\Omega^1 \otimes E(\mathfrak{m})$. The metric h is said to be **harmonic** if

$$(6.6) \quad (D_h^+)^* \psi_h = 0.$$

If we regard the flat G -bundle as a representation $\rho : \pi_1(X') \rightarrow G$, then a harmonic metric is the same as a harmonic map from the universal cover of X' to the symmetric space G/H , which is equivariant with respect to the action of the fundamental group on both sides.

A harmonic metric on a parabolic G -local system is a harmonic metric on the local system, which is tamed in the sense defined in the previous section. The existence of a harmonic metric on the a parabolic G -local system is governed, like for the Hermite–Einstein equation, by a stability condition. To define this we first define the degree.

This is simpler than for G -Higgs bundles, since the global term of the degree vanishes here due to the flatness of the connection.

Let F be a parabolic G -local system with (P_i, χ_i) defining the parabolic structure at the point x_i . Let $Q \subset G$ be a parabolic subgroup of G and χ be a strictly antidominant character of its Lie algebra \mathfrak{g} . Let σ be a reduction of structure group of F to a Q -bundle, which is constant (invariant under the flat connection). Using the relative degree of two parabolic subgroups with antidominant characters, we define the parabolic degree of F with respect to the reduction to (Q, χ) by the formula

$$(6.7) \quad \text{pardeg}(F)(Q, \chi, \sigma) := - \sum_i \deg \left((P_i, \chi_i), (Q, \chi) \right).$$

This makes sense since both P_i and Q can be identified to subgroups of $F(G)$ near the marked point x_i .

We say that F is **semistable** if for any such reduction of the local system one has

$$(6.8) \quad \text{pardeg}(F)(Q, \chi, \sigma) \geq 0.$$

It is **stable** if the inequality is strict for any non-trivial reduction, and **polystable** if it is semistable and equality happens in (6.8) if and only if there is a reduction of the local system to a Levi subgroup $L \subset Q$ (as for Higgs bundles, this means that there is a parabolic L -local system which induces F through the inclusion $L \hookrightarrow G$). In particular this definition implies a compatibility of the parabolic structure with the reduction. When there is no parabolic structure, the condition only says that there is no reduction of the local system to a parabolic subgroup, unless there is a reduction to a Levi subgroup: the local system is reductive.

Remark 6.4. Like in the case of G -bundles when G is complex, in the case of parabolic G -local systems (for real or complex G) it is enough to check (6.7) for characters that lift to Q . In fact, it suffices to verify the condition for maximal parabolic subgroups and a certain character χ_Q .

Theorem 6.5. *Let F be a parabolic G -local system, with vanishing parabolic degree with respect to all characters of \mathfrak{g} . Then F admits a harmonic metric h (compatible with the parabolic structure near the marked points) if and only if F is polystable. Moreover, any two such harmonic metrics are related by a parallel automorphism of F , preserving the parabolic structure at the marked points (i.e. the automorphism belongs to the parabolic group $P_i \subset F(G)|_{\rho_i}$ on the chosen ray ρ_i going to the marked point x_i).*

The harmonic metric induces a polystable parabolic G -Higgs bundle, and the relation between the weights at the marked points is the same as in Proposition 6.3.

The proof of this theorem can be made formally similar to that of Theorem 5.1, see [65, Theorem 6] and [8, Section 11], by replacing the symbols $(D'', D' = \partial^E - \tau(\varphi))$ by $(D, D^c = \sqrt{-1}((D^+)^{0,1} + \psi^{1,0} - (D^+)^{1,0} - \psi^{0,1}))$ and the curvature $F = (D'' + D')^2$ by the pseudocurvature $G = -\frac{1}{2}(D - \sqrt{-1}D^c)^2$, so we will not give the details of the proof, but just sketch a few steps.

The first step is to construct an initial metric h_0 (a section of $E(G/H)$) near a puncture x_i : start with a trivialization f where the flat connection D is given by formula (6.5), up to terms in the nilpotent part of the parabolic, and define the initial metric

h_0 by formula (6.4). Implicit in this construction is the choice of appropriate Kostant–Sekiguchi \mathfrak{sl}_2 -triples (see Appendix Section A.3). In the orthonormal trivialization $\mathbf{e} = fh_0^{-\frac{1}{2}}$, the flat connection has then the form (6.1), up to C_δ^∞ terms. Choose any extension of h_0 in the interior of X .

The second step is to define the functional space of metrics that we want to use: the relevant choice here is

$$(6.9) \quad \mathcal{H} = \{h = h_0 e^s, s \in \hat{L}_\delta^{2,p}(\mathfrak{m})\}.$$

As in Section 5.2, this space preserves the fact that h_0 can change at the points x_i , since $s(x_i) \in \mathfrak{r}_i \cap \mathfrak{r}'_i$. Of course, this is only a technical space needed for the proof, since at the end, we shall get by local elliptic regularity, as in (5.25),

$$(6.10) \quad h \in \mathcal{H}^\infty = \{h_0 e^s, s \in \hat{C}_\delta^\infty(\mathfrak{m})\}.$$

To solve the problem in \mathcal{H} , one first solves the abelian equation on the central part, therefore reducing to the semisimple part of G . Then one minimizes the relative energy

$$(6.11) \quad N(h_0, h) = \int_X (|\psi_h|_h^2 - |\psi_{h_0}|_{h_0}^2)$$

on \mathcal{H} , under the constraint $\|(D_h^+)^* \psi_h\|_{L_\delta^p} \leq B$ (note the inaccuracy in [8, p. 88], where the second term was forgotten). It turns out that using the formalism (D, D^e) , the functional $N(h_0, h)$ coincides with the Donaldson functional $M(h_0, h)$ in formula (5.21). Of course the reason to introduce this relative energy is that the usual harmonic map energy $\int_X |\psi_h|^2$ can be infinite in our case, while $N(h_0, h)$ is well defined. The fact that $N(h_0, h)$ might be unbounded below explains why a stability condition appears here, replacing Corlette’s semisimplicity condition for harmonic maps.

The coincidence between (6.11) and the Donaldson functional can be proved just by checking that they have the same gradient, but since this fact does not seem well known, we also give a direct proof:

Lemma 6.6. *One has $N(h_0, h_0 e^s) = \int_X -((D_{h_0}^+)^* \psi_{h_0}, s)_{h_0} + \frac{1}{2}(\eta(s)(Ds), Ds)_{h_0}$, where $\eta(t) = \frac{e^t - t - 1}{t^2}$.*

Proof. We use the formulas from [8, Section 11]. For $h = h_0 e^s$, one has

$$\begin{aligned} \psi_h &= -\frac{1}{2}h^{-1}Dh \\ &= \text{Ad}(e^{-s})\psi_{h_0} - \frac{1}{2}e^{-s}D(e^s) \\ &= \text{Ad}(e^{-s})\psi_{h_0} - \frac{1}{2}\frac{1-e^{-t}}{t}(s)(Ds). \end{aligned}$$

Since $\text{Ad}(e^{\frac{s}{2}})\psi_h$ is a section of $E(\mathfrak{m})$ and $\text{ad}(s)$ exchanges \mathfrak{h} and \mathfrak{m} , we deduce

$$\begin{aligned} \text{Ad}(e^{\frac{s}{2}})\psi_h &= \text{Ad}(e^{-\frac{s}{2}})\psi_{h_0} - \frac{\sinh \frac{t}{2}}{t}(s)(Ds) \\ &= \cosh(\frac{t}{2})(s)(\psi_{h_0}) - \frac{\sinh \frac{t}{2}}{t}(s)(D_{h_0}^+ s), \end{aligned}$$

and, since $|\psi_h|_h^2 = |\text{Ad}(e^{\frac{s}{2}})\psi_h|_{h_0}^2$,

$$(6.12) \quad |\psi_h|_h^2 - |\psi_{h_0}|_{h_0}^2 = \left(\sinh^2\left(\frac{t}{2}\right)(s)(\psi_{h_0}), \psi_{h_0} \right) + \left(\frac{\sinh^2 \frac{t}{2}}{t^2}(s)(D_{h_0}^+ s), D_{h_0}^+ s \right) \\ - 2 \left(\cosh\left(\frac{t}{2}\right)(s)(\psi_{h_0}), \frac{\sinh \frac{t}{2}}{t}(s)(D_{h_0}^+ s) \right).$$

On the other hand, decomposing $Ds = D_{h_0}^+ s + [\psi_{h_0}, s]$, and taking the even and odd parts of $\eta(t)$, one has

$$\left(\eta(s)(Ds), Ds \right) = \left((\cosh t - 1)(s)\psi_{h_0}, \psi_{h_0} \right) + \left(\frac{\cosh t - 1}{t^2}(s)(D_{h_0}^+ s), D_{h_0}^+ s \right) \\ - 2 \left(\frac{\sinh t - t}{t}(s)(D_{h_0}^+ s), \psi_{h_0} \right).$$

Comparing with (6.12), we obtain

$$\frac{1}{2} \left(\eta(s)(Ds), Ds \right) = |\psi_h|_h^2 - |\psi_{h_0}|_{h_0}^2 + \left(D_{h_0}^+ s, \psi_{h_0} \right).$$

The lemma follows. \square

The heart of the proof of Theorem 6.5 consists in proving that non convergence of a minimizing sequence would imply the existence of a reduction of E to a parabolic subgroup P , appearing with an antidominant character, which would contradict stability, and this is done exactly as in the Higgs bundle case, except that the $L^{1,2}$ holomorphic subbundles are replaced by $L^{1,2}$ parallel subbundles, so no subtle regularity theory is needed here.

The harmonic metric h induces a G -Higgs bundle (E, φ) , where the $\bar{\partial}$ -operator of E is $(D^+)^{0,1}$ and the Higgs field $\varphi = \psi_h^{1,0}$. A priori, this defines (E, φ) only in the interior of X , and we have to extend it over the points x_i . Fortunately, because of the good control (6.10) on the solution, this can be done relatively easily. Because $h \in \mathcal{H}^\infty$, we have, in an orthonormal trivialization \mathbf{e} ,

$$\bar{\partial}^E = \bar{\partial}_0 + a, \quad \bar{\partial}_0 = \bar{\partial} + \left(\alpha_i - \frac{\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i}{\ln|z|^2} \right) \frac{d\bar{z}}{2\bar{z}}, \\ \varphi = \varphi_0 + b, \quad \varphi_0 = \left(s_i - \frac{\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})Y_i}{\ln|z|^2} \right) \frac{dz}{z},$$

where the perturbations a and b belong to the space C_δ^∞ .

Lemma 6.7. *There exists a gauge transformation $g \in \hat{C}_\delta^\infty(E(H^\mathbb{C}))$, defined in a neighbourhood of x_i , such that*

$$g(\bar{\partial}^E) = \bar{\partial}_0.$$

Proof. The equation to solve is

$$g^{-1}\bar{\partial}_0 g = \text{Ad}(g^{-1})a.$$

Writing $g = \exp(u)$, the equation becomes $e^{-u}\bar{\partial}_0 e^u = e^{-\text{ad} u}a$, with linear part $\bar{\partial}_0 u = a$. One can deduce an explicit solution for the linear problem from the Cauchy kernel, with suitable estimates [8, Lemma 9.1]. Shrinking to a smaller ball if necessary, a fixed point argument gives the expected solution. \square

We now have a \hat{C}_δ^∞ -gauge $f = \mathbf{e}g$ of \mathbf{E} (seen as a $H^\mathbb{C}$ -bundle) in which $\bar{\partial}^E$ is exactly the model $\bar{\partial}_0$, and $\varphi = \varphi_0 + b'$ for some $b' \in C_\delta^\infty$. We deduce an explicit holomorphic gauge,

$$e = f(-\ln r^2)^{\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i/2}r^{-\alpha_i},$$

which we use to extend the holomorphic bundle E over x_i . Moreover, in the gauge e , the Higgs field becomes

$$\varphi = \text{Ad}\left(r^{\alpha_i}(-\ln r^2)^{-\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i/2}\right)(\varphi_0 + b').$$

Here

$$\text{Ad}(r^{\alpha_i}(-\ln r^2)^{-\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i/2})\varphi_0 = (s_i + \text{Ad}(z^{\alpha_i})Y_i)\frac{dz}{z}$$

is just our model for the Higgs field in the holomorphic trivialization e , with $\text{Gr Res}_{x_i} = s_i + Y_i$, so we have to analyse the behaviour of the remainder

$$\varphi' = \text{Ad}\left(r^{\alpha_i}(-\ln r^2)^{-\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i/2}\right)b', \quad b' = O\left(\frac{1}{|\ln r|^\delta}\right)\frac{dz}{z \ln r}.$$

Here remind that b' is holomorphic outside the origin, and $0 < \delta < 1$. This implies that φ' is meromorphic: decompose

$$\varphi' = \sum_{\mu} \varphi'_\mu \frac{dz}{z}$$

along the eigenvalues μ of $\text{ad}(\alpha_i)$ on $\mathfrak{m}^\mathbb{C}$, and let us analyse the pole of φ'_μ . We certainly have

$$\varphi'_\mu = O(r^\mu |\ln r|^N)$$

which implies $v(\varphi'_\mu) \geq -\lfloor -\mu \rfloor$, as wanted for a parabolic G -Higgs bundle. We can say something more on $\text{Gr Res}_{x_i} \varphi'$, that is on the components φ'_μ for $\mu \in \mathbb{Z}$: decompose further with respect to the eigenvalues η of $\text{ad}(H_i)$ on $\mathfrak{m}^\mathbb{C}$, and we get

$$\varphi'_{\mu,\eta} = O(r^\mu |\ln r|^{-\frac{\eta}{2}-1-\delta}).$$

This implies that the (μ, η) -component for $\mu \in \mathbb{Z}$ can be non zero only if $\eta < -2$, which implies that $Y_i + \text{Gr Res}_{x_i} \varphi'$ is conjugate to Y_i , that is $\text{Gr Res}_{x_i} \varphi$ remains conjugate to $s_i + Y_i$. This finishes the proof of Theorem 6.5. \square

Table 1 gives the relation between the singularities for the Higgs bundle and for the corresponding local systems, in a similar way to Simpson's table in [65]. The interesting feature here is that they correspond in a way which extends the Kostant–Sekiguchi correspondence, see Appendix A.3. More specifically, in the nilpotent case, one gets exactly the correspondence between $H^\mathbb{C}$ -nilpotent orbits in $\mathfrak{m}^\mathbb{C}$ and nilpotent G -orbits in \mathfrak{g} which is the Kostant–Sekiguchi correspondence (this was turned by Vergne [70] into a diffeomorphism which can be seen as a toy model of our correspondence theorem). The more general case with semisimple residues corresponds to an extension of Kostant–Sekiguchi–Vergne [5, 9].

We summarize the discussion in the following way. Choose a fundamental domain $A^f \subset \bar{A}$ as in Proposition A.1, then:

Proposition 6.8. *Table 1 gives a 1:1 correspondence between:*

	Weight	Monodromy (projected to the Levi)
(E, φ)	α	$s + Y = \text{Gr Res}_x \varphi$
(F, ∇)	$s - \tau(s)$	$\exp(2\pi\sqrt{-1}\alpha) \exp(2\pi\sqrt{-1}(-s - \tau(s) + Y - H - X))$

TABLE 1. Table of relations for weights and monodromies

- couples (α, \mathcal{L}) of a weight $\alpha \in \mathcal{A}^f$ and a \tilde{L} -orbit $\mathcal{L} \subset \tilde{\mathfrak{m}}^0$, where \tilde{L} and $\tilde{\mathfrak{m}}^0$ are associated to α by (4.4) and (4.5);
- couples (β, \mathcal{C}) of a weight $\beta \in \mathfrak{m}$ and a conjugacy class \mathcal{C} in the Levi group of the parabolic subgroup $P_\beta \subset G$.

In particular, all representations of the fundamental group of the punctured surface are obtained in our correspondance.

Proof. We have already seen how to go from (α, \mathcal{L}) to (β, \mathcal{C}) . Conversely, given (β, \mathcal{C}) , we must prove that this can be put in the form $\beta = s - \tau(s)$ with \mathcal{C} being the orbit of some $\exp(2\pi\sqrt{-1}\alpha) \exp(2\pi\sqrt{-1}(-s - \tau(s) + Y - H - X))$. We choose the Levi subgroup $L_\beta \subset G$ as in Appendix B and $u \in L_\beta$ representing \mathcal{C} . Factorize $u = u_s u_n$ with u_s semisimple, u_n unipotent, and $u_s u_n = u_n u_s$. Up to changing u in \mathcal{C} we can arrange so that u_s is in a maximal torus of G which contains the maximal torus T of H , and the Cartan subalgebra contains β ; then $u_s = ke^v$ with $k \in T$ and $v \in \mathfrak{m}$ commuting; by definition there exists a unique $\alpha \in \mathcal{A}^f$ such that $k = \exp(2\pi\sqrt{-1}\alpha)$; one has $[\alpha, v] = 0$. We can then define uniquely s by $\beta = s - \tau(s)$ and $v = -2\pi\sqrt{-1}(s + \tau(s))$. There remains to write the unipotent part u_n as $\exp(2\pi\sqrt{-1}(Y - H - X))$ for some \mathfrak{sl}_2 -triple (H, X, Y) in the Lie algebra of the commutator of (k, v, β) , up to conjugation by this commutator group: this is provided by the theory of Kostant-Sekiguchi triples, see Propositions A.4 and A.5. \square

6.3. Deformations of the harmonicity equation and polystability. As for parabolic G -Higgs bundles, there is also a more general stability condition for parabolic G -local systems depending on a parameter. In this case the parameter is an element of the subspace of fixed points of \mathfrak{m} under the isotropy action of H , that is

$$\mathfrak{m}^H = \{v \in \mathfrak{m} \text{ such that } \text{Ad}(h)(v) = v \text{ for every } h \in H\}.$$

Let F be a parabolic G -local system with (P_i, χ_i) defining the parabolic structure. Let $Q \subset G$ be a parabolic subgroup of G and χ be an antidominant character of its Lie algebra \mathfrak{q} . Let σ be a reduction of structure group of F to a Q -bundle, which is invariant under the flat connection. Given an element $\zeta \in \mathfrak{m}^H$, we define ζ -**polystability** of F by the condition

$$(6.13) \quad \text{pardeg}(F)(Q, \chi, \sigma) - \langle \zeta, s \rangle \geq 0,$$

for every such (Q, χ) and reduction σ , where $s \in \mathfrak{m}$ is the element corresponding to (Q, χ) (see Appendix B.1).

This condition corresponds to a deformation of the harmonicity equation (6.6) given by

$$(6.14) \quad (D_h^+)^* \psi_h = \zeta.$$

Note that this equation is indeed H -gauge invariant since $\zeta \in \mathfrak{m}^H$. Using the methods above one can prove the following.

Proposition 6.9. *A parabolic G -local system admits a reduction h to H satisfying (6.14) if and only if it is ζ -polystable.*

For $\zeta \neq 0$ a ζ -polystable parabolic G -local system no longer defines a G -Higgs bundle since now φ is not a holomorphic section. The equation $\bar{\partial}\varphi = 0$ is replaced by

$$\bar{\partial}\varphi = \eta\omega,$$

with $\eta \in (\mathfrak{m}^{\mathbb{C}})^{H^{\mathbb{C}}}$ and $\zeta = \eta - \tau(\eta)$, where τ is the compact conjugation in $\mathfrak{g}^{\mathbb{C}}$, and ω is a Kähler form on X . However the Higgs field φ defines a holomorphic section if we replace the bundle associated to $\mathfrak{m}^{\mathbb{C}}$ by the bundle associated to the $H^{\mathbb{C}}$ -representation $\mathfrak{m}^{\mathbb{C}}/(\mathfrak{m}^{\mathbb{C}})^{H^{\mathbb{C}}}$. Strictly speaking this is no longer a G -Higgs bundle but it is a Higgs pair in the more general sense mentioned in Remark 4.3.

Remark 6.10. The equation (6.14) has been recently studied by Collins–Jacob–Yau [22] for $G = \mathrm{GL}_n\mathbb{C}$, where they prove in particular Proposition 6.9 in this case.

7. MODULI SPACES

7.1. Moduli spaces of parabolic G -Higgs bundles. We follow the notation of Section 4. Let X be a compact connected Riemann surface and let $S = \{x_1, \dots, x_r\}$ be a finite set of different points of X . Let $D = x_1 + \dots + x_r$ be the corresponding effective divisor. Let (G, H, θ, B) be a real reductive Lie group (see Appendix A.2). We fix an alcove $\mathcal{A} \subset \mathfrak{t}$ of H such that $0 \in \bar{\mathcal{A}}$ (see Appendix A.1). Consider parabolic weights $\alpha = (\alpha_1, \dots, \alpha_r)$ with $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$, and let $c \in \mathfrak{z}$.

Let $\mathcal{M}_c(\alpha) := \mathcal{M}_c(X, D, G, \alpha)$ be the **moduli space of meromorphic equivalence classes of c -polystable parabolic G -Higgs bundles (E, φ) on (X, D) with parabolic weights α** . (See Definition 3.3 for the notion of meromorphic equivalence.) Note that if none of the α_i 's is contained in a bad wall then we are simply considering isomorphism classes in the definition of $\mathcal{M}_c(\alpha)$. The moduli space for $c = 0$ will be simply denoted by $\mathcal{M}(\alpha)$. For the moment, we are considering $\mathcal{M}_c(\alpha)$ just as a set without any additional structure.

We use the same notation as in Section 4.1. Let L_i the Levi subgroup of Q_i and \widetilde{L}_i the subgroup corresponding to (4.5). Consider the spaces \mathfrak{m}_i^0 and $\widetilde{\mathfrak{m}}_i^0$ corresponding to (4.4). Recall that if $\alpha_i \in \sqrt{-1}\mathcal{A}'_g$ (see 4.3), $\widetilde{L}_i = L_i$ and $\widetilde{\mathfrak{m}}_i^0 = \mathfrak{m}_i^0$. We denote by $\widetilde{\mathfrak{m}}_i^0/\widetilde{L}_i$ the set of \widetilde{L}_i -orbits.

There is a map

$$\varrho : \mathcal{M}_c(\alpha) \longrightarrow \prod_i (\widetilde{\mathfrak{m}}_i^0/\widetilde{L}_i)$$

defined by taking the \widetilde{L}_i -orbit of $\mathrm{Gr} \mathrm{Res}_{x_i} \varphi \in \widetilde{\mathfrak{m}}_i^0$.

We fix now orbits $\mathcal{L}_i \in \widetilde{\mathfrak{m}}_i^0 / \widetilde{L}_i$ and denote $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_r)$. We consider the moduli space

$$\mathcal{M}_c(\alpha, \mathcal{L}) := \varrho^{-1}(\mathcal{L}).$$

One can give a gauge-theoretic analytic construction of the moduli space $\mathcal{M}_c(\alpha, \mathcal{L})$ by means of the identification of this moduli space with the moduli space of solutions of the Hitchin equations given in Section 5.2. This endows the moduli space with the structure of a complex analytic space. We do not give any details here since this is by now fairly standard (see [6, 43, 56]), using the weighted Sobolev spaces for the connections, the Higgs fields and the gauge group. The appropriate functional spaces to do this are defined in Section 5.4 (see also [8]). One can then develop a Kuranishi construction for this situation using the L^2 -cohomology of a suitable complex. A crucial step to do this is the invertibility of the Laplacian (see [8]).

To ensure smoothness of $\mathcal{M}_c(\alpha)$ at a point (E, φ) one generally needs that (E, φ) be stable and simple. A parabolic G -Higgs bundle (E, φ) is said to be **simple** if $\text{Aut}(E, \varphi) = Z(H^{\mathbb{C}}) \cap \ker \iota$, where ι is the isotropy representation (see Appendix A.2), and $\text{Aut}(E, \varphi)$ is the group of parabolic automorphisms of (E, φ) . If G is complex we hence require $\text{Aut}(E, \varphi) = Z(G)$. Except for Lie groups of type A_n stability does not generally imply simplicity. For a general reductive real Lie group G , to have smoothness of $\mathcal{M}_c(\alpha)$ at a point (E, φ) one also needs the vanishing of a certain obstruction class in the second L^2 -cohomology of the appropriate deformation complex associated to (E, φ) (see [29]). This condition is always satisfied if G is complex — it follows from stability and Serre duality. This is also satisfied for $c = 0$ when G is a real form of a complex reductive group $G^{\mathbb{C}}$ if the extension of (E, φ) to a parabolic $G^{\mathbb{C}}$ -Higgs bundle is stable.

From the previous discussion one has the following.

Proposition 7.1. *Let $\mathcal{M}_c^*(\alpha) \subset \mathcal{M}_c(\alpha)$ and $\mathcal{M}_c^*(\alpha, \mathcal{L}) \subset \mathcal{M}_c(\alpha, \mathcal{L})$ be the subspaces of parabolic G -Higgs bundles of weight α that are stable, simple and have vanishing obstruction class. Then*

(1) $\mathcal{M}_c^*(\alpha)$ is a smooth Poisson manifold foliated by symplectic leaves $\mathcal{M}_c^*(\alpha, \mathcal{L})$, admitting a Kähler structure.

(2) If G is complex $\mathcal{M}_c^*(\alpha)$ is a holomorphic Poisson manifold (in fact a hyperpoisson manifold) foliated by holomorphic symplectic leaves $\mathcal{M}_c^*(\alpha, \mathcal{L})$, admitting a hyperkähler structure.

For the existence of the Poisson structure and the Kähler structure of these moduli spaces one can look at [16, 49, 43].

Recall from Remark 4.8 that (semi)stability can be formulated in terms of any parabolic subgroup $P \subset H^{\mathbb{C}}$ conjugated to a parabolic subgroup of the form P_s , and any antidominant character χ of \mathfrak{p} , the Lie algebra of P . The set of characters of P is an abelian free group, and the subset of antidominant characters is a cone A inside of this group. The characters for which the Higgs field φ satisfies $\varphi|_{X \setminus D} \in H^0(X \setminus D, E_\sigma(\mathfrak{m}_s) \otimes K)$ is a subcone $B \subset A$. It is hence enough to check the numerical condition for the elements of the 1-dimensional faces of B , and hence for a finite number of antidominant characters of P . If G is complex, these antidominant characters define characters of maximal

parabolic subgroups (not merely subalgebras). In view of this, we will now make the following assumption.

Assumption 7.2. We will assume that the reductive structure (G, H, θ, B) is such that for a parabolic G -Higgs bundle to be (semi)stable it is enough to check the numerical condition (4.9) when the antidominant character χ_s of the subalgebra \mathfrak{p}_s of a parabolic subgroup P_s lifts to a character $\widetilde{\chi}_s$ of P_s . In this situation $\deg(E)(\sigma, \widetilde{\chi}_s)$ is the degree of the line bundle associated to E via $\widetilde{\chi}_s : P_s \rightarrow \mathbb{C}^*$, and hence an integer. This is satisfied, in particular if G is complex.

Remark 7.3. Assumption 7.2 is needed in order to give a GIT construction of the moduli space of ordinary G -Higgs bundles (see [61]).

Under Assumption 7.2 we can define a genericity condition for the weights. We say that $\alpha = (\alpha_1, \dots, \alpha_r)$ is **generic** if for any parabolic subgroup $P \subset H^{\mathbb{C}}$ and any antidominant character of P in the finite collection mentioned above, the relative degree satisfies

$$\sum_i \deg((P_i, \alpha_i), (P, \chi)) \notin \mathbb{Z}.$$

Since the choice of conjugacy classes of parabolic subgroups P is also finite (see Appendix B.1), this defines walls dividing $\sqrt{-1}\bar{A}^r$ in chambers.

If we now take the stability parameter $c = 0$ (which is the relevant value in relation to local systems and representations) then, since the first term in (4.9) is an integer, we have that every semistable G -Higgs bundle is actually stable and more over its extension to a $G^{\mathbb{C}}$ -Higgs bundle is also stable, which implies the vanishing of the obstruction second cohomology class, and hence every point in $\mathcal{M}(\alpha)$ is smooth if it is simple or possibly an orbifold singularity if it is not simple (recall that $\mathcal{M}(\alpha)$ denotes the moduli space for $c = 0$). We thus have the following

Proposition 7.4. *If (G, H, θ, B) satisfies Assumption 7.2, and α is generic, then*

(1) $\mathcal{M}(\alpha)$ is a Poisson manifold (possibly with orbifold singularities) foliated by symplectic leaves $\mathcal{M}(\alpha, \mathcal{L})$, admitting a Kähler structure. Moreover, if α and α' are in the same chamber $\mathcal{M}(\alpha) = \mathcal{M}(\alpha')$ and $\mathcal{M}(\alpha, \mathcal{L}) = \mathcal{M}(\alpha', \mathcal{L})$ as real and complex orbifolds, respectively (the Poisson and symplectic structures, respectively, depend on α).

(2) If G is complex $\mathcal{M}(\alpha)$ is a holomorphic Poisson manifold (possibly with orbifold singularities) foliated by holomorphic symplectic leaves $\mathcal{M}(\alpha, \mathcal{L})$, admitting a hyperkähler structure. Moreover, if α and α' are in the same chamber $\mathcal{M}(\alpha) = \mathcal{M}(\alpha')$ as holomorphic Poisson orbifolds and $\mathcal{M}(\alpha, \mathcal{L}) = \mathcal{M}(\alpha', \mathcal{L})$ as holomorphic symplectic orbifolds.

Remark 7.5. Result (2) in Proposition 7.4 is a consequence of the fact that in the hyperkähler structure defined as a hyperkähler quotient by the Hitchin equations, the symplectic form ω_1 depends on α , while the complex structure I_1 , and the I_1 -holomorphic symplectic form $\Omega_1 = \omega_2 + i\omega_3$ depend on \mathcal{L} .

Remark 7.6. As mentioned in Proposition 7.4, the orbifold singularities take place possibly at the stable but not simple points. If (E, φ) is stable but not simple, there is a reduction of structure group of (E, φ) to a reductive subgroup —the centralizer in

G of $\text{Aut}(E, \varphi)$ (see [33]). It is not clear whether one could define an extra genericity condition for α and \mathcal{L} to avoid this phenomenon.

There should be a GIT construction of these moduli spaces giving $\mathcal{M}_c(\alpha, \mathcal{L})$ the structure of a quasiprojective variety. As far as we are aware this has only been done for $G = \text{GL}_n\mathbb{C}$ (see [71]). In the generality considered here, this will need most likely to involve parahoric torsors (see [3, 13, 36]).

7.2. Moduli spaces of parabolic G -local systems and representations. Let X be a compact connected Riemann surface and let $S = \{x_1, \dots, x_r\}$ be a finite set of different points of X . Let (G, H, θ, B) be a real reductive Lie group. We use the notation of Section 6. Let $\beta = (\beta_1, \dots, \beta_r)$ be as in Definition 6.1, and $\zeta \in \mathfrak{m}^H$.

We define $\mathcal{S}_\zeta(\beta) := \mathcal{S}_\zeta(X, S, G, \beta)$ to be the **moduli space of isomorphism classes of ζ -polystable parabolic G -local systems F on (X, S) with parabolic weights β** . The moduli space for $\zeta = 0$ will be simply denoted by $\mathcal{S}(\beta)$. Let P_β be the parabolic subgroup of G defined by β , and $L_\beta \subset P_\beta$, the Levi subgroup. We fix loops c_i enclosing the marked points x_1, \dots, x_r simply. The monodromy of the loop c_i around x_i takes values in P_{β_i} , and we consider its projection to L_{β_i} . Its conjugacy class \mathcal{C}_i in L_{β_i} is independent of the simple loop that we have taken. This defines a map

$$\mu : \mathcal{S}_\zeta(\beta) \longrightarrow \prod_i \text{Conj}(L_{\beta_i})$$

where $\text{Conj}(L_{\beta_i})$ is the set of conjugacy classes of L_{β_i} .

Let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_r)$ with $\mathcal{C}_i \in \text{Conj}(L_{\beta_i})$, and consider the moduli space

$$\mathcal{S}_\zeta(\beta, \mathcal{C}) = \mu^{-1}(\mathcal{C}).$$

As in the case of parabolic Higgs bundles, using the same techniques, one can give an analytic gauge-theoretic construction of $\mathcal{S}_\zeta(\beta)$ and $\mathcal{S}_\zeta(\beta, \mathcal{C})$, via the identification provided by Theorem 6.5 of these spaces with the spaces of solutions to the Hermitian–Einstein equations (see also Proposition 6.9 for $\zeta \neq 0$). This construction endows the spaces $\mathcal{S}_\zeta(\beta)$ and $\mathcal{S}_\zeta(\beta, \mathcal{C})$ with a real analytic structure (complex, if G is complex). This is done by Nakajima [56] for the case $G = \text{GL}_2\mathbb{C}$.

We say that a parabolic G -local system is **irreducible** if its group of automorphisms coincides with the centre of G . Similarly to Proposition 7.1, one has the following.

Proposition 7.7. *Let $\mathcal{S}_\zeta^*(\beta) \subset \mathcal{S}_\zeta(\beta)$ and $\mathcal{S}_\zeta^*(\beta, \mathcal{C}) \subset \mathcal{S}_\zeta(\beta, \mathcal{C})$ be the subsets of stable and irreducible elements. Then*

(1) $\mathcal{S}_\zeta^*(\beta)$ is a smooth Poisson manifold foliated by symplectic leaves $\mathcal{S}_\zeta^*(\beta, \mathcal{C})$, admitting a Kähler structure.

(2) If G is complex $\mathcal{S}_\zeta^*(\beta)$ is a holomorphic Poisson manifold foliated by holomorphic, symplectic leaves $\mathcal{S}_\zeta^*(\beta, \mathcal{C})$, admitting a hyperkähler structure.

As in the case of Higgs bundles, by Remark 6.4, we can define a genericity condition for β . We say that $\beta = (\beta_1, \dots, \beta_r)$ is **generic** if for any maximal parabolic subgroup

$Q \subset G$, and its corresponding preferred character $\chi = \chi_Q$ (see Remark 6.4) we have

$$\sum_i \deg \left((P_i, \chi_i), (Q, \chi_Q) \right) \neq 0.$$

We observe that the function μ defining the relative degree in (B.1), as a function on $Q \backslash G \subset \mathfrak{m}$, has a finite number of values, and since the number of types of maximal parabolic subgroups Q is finite, the condition $\sum_i \deg \left((P_i, \chi_i), (Q, \chi_Q) \right) = 0$ defines a finite number of walls dividing \mathfrak{m}^r in chambers. By (6.7) for generic β a semistable parabolic G -local system is actually stable, and we thus have the following.

Proposition 7.8. *If β is generic, then*

(1) $\mathcal{S}(\beta)$ is a Poisson manifold (possibly with orbifold singularities at the stable but not irreducible points) foliated by symplectic leaves $\mathcal{S}(\beta, \mathcal{C})$, admitting a Kähler structure. Moreover, if β and β' are in the same chamber $\mathcal{S}(\beta) = \mathcal{S}(\beta')$ and $\mathcal{S}(\beta, \mathcal{C}) = \mathcal{S}(\beta', \mathcal{C})$ as real and complex orbifolds, respectively (the Poisson and symplectic structures, respectively, depend on β).

(2) If G is complex $\mathcal{S}(\beta)$ is a holomorphic Poisson manifold (possibly with orbifold singularities at the stable but not irreducible points) foliated by holomorphic symplectic leaves $\mathcal{S}(\beta, \mathcal{C})$, admitting a hyperkähler structure. Moreover, if β and β' are in the same chamber $\mathcal{S}(\beta) = \mathcal{S}(\beta')$ as holomorphic Poisson orbifolds and $\mathcal{S}(\beta, \mathcal{C}) = \mathcal{S}(\beta', \mathcal{C})$ as holomorphic symplectic orbifolds.

We consider now the moduli space of representations of $\pi_1(X \setminus S)$. By a **representation** of $\pi_1(X \setminus S)$ in G we mean a homomorphism $\rho: \pi_1(X \setminus S) \rightarrow G$. A representation is **reductive** if composed with the adjoint representation in the Lie algebra of G decomposes as a sum of irreducible representations. If G is algebraic this is equivalent to saying that the Zariski closure of the image of $\pi_1(X \setminus S)$ in G is a reductive group. Define the **moduli space of reductive representations of $\pi_1(X \setminus S)$ in G** to be the orbit space

$$\mathcal{R} := \mathcal{R}(X, S, G) = \text{Hom}^+(\pi_1(X \setminus S), G)/G,$$

where $\text{Hom}^+(\pi_1(X \setminus S), G)$ is the set of reductive representations and G acts by conjugation. This is a real analytic variety (algebraic if G is algebraic). If G is complex \mathcal{R} is the affine GIT quotient $\text{Hom}(\pi_1(X \setminus S), G) // G$. Let c_i be a loop enclosing x_i simply. Fix conjugacy classes $\mathcal{C}_i \in \text{Conj}(G)$, $i = 1, \dots, r$, and let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_r)$. We define the moduli space of representations of $\pi_1(X \setminus S)$ in G with fixed conjugacy classes \mathcal{C} as the subvariety

$$\mathcal{R}(\mathcal{C}) := \{[\rho] \in \mathcal{R} : \rho([c_i]) = \mathcal{C}_i, i = 1, \dots, r\}.$$

Similarly to the case $G = \text{GL}(n, \mathbb{C})$ studied by Simpson [65], one has the following.

Proposition 7.9. *Let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_r)$ with $\mathcal{C}_i \in \text{Conj}(L_{\beta_i})$ like in Section 7.2 and let $\mathcal{C}' = (\mathcal{C}'_1, \dots, \mathcal{C}'_r)$ with $\mathcal{C}'_i \in \text{Conj}(P_{\beta_i})$. Let $\pi_i: P_{\beta_i} \rightarrow L_{\beta_i}$ be the projection to the Levi subgroup. Then there is a forgetful map*

$$\mathcal{S}(\beta, \mathcal{C}) \rightarrow \bigcup_{\mathcal{C}' : \pi_i(\mathcal{C}'_i) = \mathcal{C}_i} \mathcal{R}(\mathcal{C}').$$

In particular, if $\beta = 0$, $L_{\beta_i} = G$, $\mathcal{C}_i \in \text{Conj}(G)$ and

$$\mathcal{S}(0, \mathcal{C}) = \mathcal{R}(\mathcal{C}).$$

7.3. Correspondences of moduli spaces. We formulate now in terms of moduli spaces the correspondences that we have proved in this paper. We follow the notation of Sections 7.1 and 7.2. Using similar arguments to those in [42, 43, 65] to show that the correspondences are homeomorphisms or diffeomorphisms (in fact real analytic isomorphisms), from Proposition 6.3 and Theorem 6.5 one has the following two theorems.

Theorem 7.10. *Let (G, H, θ, B) be a real reductive group. Let (α, \mathcal{L}) and (β, \mathcal{C}) be related as in Table 1 (where $\beta = s - \tau(s)$). Then, $\mathcal{M}(\alpha, \mathcal{L})$ and $\mathcal{S}(\beta, \mathcal{C})$ are homeomorphic. In particular, if $\beta = 0$, $\mathcal{M}(\alpha, \mathcal{L})$ and $\mathcal{R}(\mathcal{C})$ are homeomorphic. Moreover, $\mathcal{M}^*(\alpha, \mathcal{L})$ and $\mathcal{S}^*(\beta, \mathcal{C})$ are diffeomorphic. In particular, if $\beta = 0$, $\mathcal{M}^*(\alpha, \mathcal{L})$ and $\mathcal{R}^*(\mathcal{C})$ are diffeomorphic, where $\mathcal{R}^*(\mathcal{C})$ is the subvariety of irreducible representations.*

Theorem 7.11. *Let (G, H, θ, B) satisfy Assumption 7.2. Let (α, \mathcal{L}) and (β, \mathcal{C}) be related by Table 1, with $s = \tau(s)$. Then if α and β are generic $\mathcal{M}(\alpha, \mathcal{L})$ and $\mathcal{S}(\beta, \mathcal{C})$ are diffeomorphic.*

Now, since the combined walls defining genericity for α and β have codimension bigger or equal than 2 in the space of parameters defining a connected chamber for the generic parameters, Theorem 7.11, combined with Propositions 7.4 and 7.8, imply the following.

Theorem 7.12. *Let (G, H, θ, B) satisfy Assumption 7.2. Let (α, \mathcal{L}) and (β, \mathcal{C}) , and (α', \mathcal{L}') and (β', \mathcal{C}') , be related by Table 1. Then if α, α', β and β' are generic*

- (1) $\mathcal{M}(\alpha, \mathcal{L})$ and $\mathcal{M}(\alpha', \mathcal{L}')$ are diffeomorphic.
- (2) $\mathcal{S}(\beta, \mathcal{C})$ and $\mathcal{S}(\beta', \mathcal{C}')$ are diffeomorphic.

Remark 7.13. It would be interesting to explore the possible extension of this correspondence to the moduli spaces $\mathcal{M}_c(\alpha, \mathcal{L})$ and $\mathcal{S}_\zeta(\beta, \mathcal{C})$ for non-zero values of $c \in \mathfrak{z}(\mathfrak{h})$ and $\zeta \in \mathfrak{m}^H$.

8. EXAMPLES

We give some illustrations of the use of parabolic Higgs bundles for defining certain classical components in the moduli space of representations $\mathcal{R}(G)$. We do not give details since this follows more or less directly from the arguments in the compact case and the parabolic machinery developed here.

8.1. Teichmüller–Hitchin component of split groups. We begin by the case $G = \text{PSL}_2\mathbb{R}$: this gives a parametrization of the space of hyperbolic metrics with cusps at the marked points. We have $H = \text{U}_1/\mathbb{Z}_2$ where the U_1 is seen as a maximal compact subgroup of $\text{SL}_2\mathbb{R}$. The construction on an unpunctured surface requires a square root of K ; in the punctured case we need a square root of $K(D)$. The \mathbb{C}^* -bundle $K(D)$ has not always a square root (this requires the degree of D to be even), but as a $\mathbb{C}^*/\mathbb{Z}_2$ bundle it always has such a square root: let E be such a choice (here E is a principal

holomorphic $\mathbb{C}^*/\mathbb{Z}_2$ bundle). Equip E with a trivial parabolic structure at the marked points. Then

$$E(\mathfrak{m}^{\mathbb{C}}) = K(D) \oplus K(D)^{-1},$$

and we consider the meromorphic Higgs field $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K(D))$

$$\varphi = q_2 \oplus 1, \quad q_2 \in H^0(X, K^2(D)),$$

where 1 has a simple pole at the marked points, while q_2 appears to be holomorphic at the marked points. In particular $\text{Res}_{x_i} \varphi$ is nilpotent and nonzero.

The Higgs bundles (E, φ) for $q_2 \in H^0(X, K^2(D))$ are stable, and, as in [37], the corresponding solutions of the Hitchin selfduality equations (Hermite–Einstein) provide hyperbolic metrics on $X \setminus D$, whose monodromy is the representation $\pi_1(X \setminus D) \rightarrow \text{PSL}_2\mathbb{R}$ corresponding to (E, φ) . In particular, its monodromy around x_i is unipotent so we obtain a cusp at x_i . This gives a complete parametrization of the Teichmüller space for the punctured surface (X, D) by the space of quadratic differentials $H^0(X, K^2(D))$.

If the degree of D is even, then a choice of square root L of $K(D)$ gives a lifting of this $\text{PSL}_2\mathbb{R}$ component to $\text{SL}_2\mathbb{R}$, with unipotent monodromies at the punctures. It is also well known that for all degrees, it is possible to lift the component to $\text{SL}_2\mathbb{R}$ with minus unipotent monodromies: in the Higgs bundle formalism, this amounts to considering a parabolic structure with weight at the boundary of the alcove, in the following way. One considers a square root L of K with a parabolic weight $-1/2$ at each puncture (this is morally a square root of $K(D)$, in particular $\text{pardeg } L = g - 1 + \frac{1}{2} \deg D$). Then the same Higgs field as above,

$$\varphi = \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix}$$

is now a parabolic Higgs bundle in our sense, with nilpotent residue $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, this example is discussed as the end of Section 4.1. This bundle induces the previous $\text{PSL}_2\mathbb{R}$ Higgs bundle after applying the Hecke transform $z^{1/2}$ near each puncture (this exists in $\mathbb{C}^*/\mathbb{Z}_2$). (See [58, 12] for a related description.)

Now pass to other split real groups. The generalisation of [39] is straightforward. We work with G a split real group of adjoint type, and we consider an irreducible representation $\rho : \text{SL}_2\mathbb{R} \rightarrow G$, sending U_1 in the maximal compact H of G . Choose as above a $\mathbb{C}^*/\mathbb{Z}_2$ principal holomorphic bundle E , square root of $K(D)$. We have a decomposition $\mathfrak{g}^{\mathbb{C}} = \bigoplus_1^l V_j$ into irreducible pieces under ρ , such that V_1 is the image of ρ , and we choose a highest weight vector $e_j \in V_j$. If (H, X, Y) is a standard \mathfrak{sl}_2 basis, with $H \in \sqrt{-1}\mathfrak{u}_1$, then $e_1 = \rho(X)$, and we also define $e_{-1} = \rho(Y)$. Moreover there exist a basis (p_j) of invariant polynomials on $\mathfrak{g}^{\mathbb{C}}$, of degrees $m_j + 1$ (determined by the fact that $\text{ad } H$ acts with eigenvalue m_j on e_j), such that for any element $f = e_{-1} + f_1 e_1 + \dots + f_l e_l$ one has $f_j = p_j(f)$. Now, adapting [39] to the punctured case, we consider the Higgs bundle $\rho(E)$ with trivial parabolic structure at the punctures, and Higgs field

$$\varphi = e_{-1} + \sum_1^l q_j e_j, \quad q_j \in H^0(K^{m_j+1}(m_j D)).$$

It turns out that φ is meromorphic with simple poles, and $\text{Res}_{x_j} \varphi = e_{-1}$ is regular nilpotent. This provides the expected Teichmüller–Hitchin component.

One can remark that there is a natural deformation keeping the same Higgs bundles but changing the parabolic structure: one can consider at each marked point the parabolic group $P_{\rho(H)}$, with some strictly antidominant character α_i . Then $e_{-1} \in \mathfrak{p}_{\rho(H)}$ so the space of Higgs fields with $\text{Res}_{x_i} \varphi \in \mathfrak{p}_{\alpha_i}$ remain the same. This gives a deformation of the Teichmüller–Hitchin component to a space of representations with fixed compact monodromy around the punctures. One can obtain similarly any regular semisimple monodromy at the punctures by allowing $q_j \in H^0(K^{m_j+1}((m_j+1)D))$ and by fixing the highest order term of each q_j at each puncture: this modifies the residue of φ , hence the noncompact part of the monodromy.

8.2. Hermitian groups and the Milnor–Wood inequality. Another case where there is a distinguished component in the space of representations of $\pi_1(X \setminus D)$ into G is when G is Hermitian, that is G/H is a Hermitian symmetric space of noncompact type. The general theory from the Higgs bundle viewpoint is done in [11] and can be generalised to the punctured case. In particular one recovers the Milnor–Wood inequality of Burger–Iozzi–Wienhard [21] in the punctured case. Again we do not give details of the proofs, which can be adapted from [11] to the parabolic case.

In the Hermitian case, \mathfrak{h} has a 1-dimensional centre, generated by an element J which induces the complex structure of G/H . In particular it decomposes $\mathfrak{m}^{\mathbb{C}}$ into $\pm i$ eigenspaces: $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$. Now consider a parabolic G -Higgs bundle (E, φ) , so φ decomposes as $\varphi = \varphi^+ + \varphi^-$. We can define a Toledo invariant in the following way: it is proved in [11] that there exists a character, called the Toledo character, $\chi_T : H^{\mathbb{C}} \rightarrow \mathbb{C}^*$ and a polynomial $\det : \mathfrak{m}^+ \rightarrow \mathbb{C}$ of degree $r = \text{rk}(G/H)$, such that for any $h \in H^{\mathbb{C}}$ one has $\det(h \cdot x) = \chi_T(h) \det(x)$. The Toledo invariant on a compact surface is $\deg E(\chi_T)$, and on a punctured surface we define the Toledo invariant by

$$\tau(E) = \text{pardeg}(E, \chi_T),$$

where the reduction of E is E itself. This is actually equal to the parabolic degree of the line bundle $E(\chi_T)$ equipped with the parabolic weight $\chi_T(\alpha_i)$ at each puncture x_i .

The proof of Theorem 4.5 in [11] extends to give the following Milnor–Wood inequality: if (E, φ) is a semistable parabolic G -Higgs bundle on the Riemann surface (X, D) with n punctures, then

$$(8.1) \quad -\text{rk}(\varphi^+)(2g - 2 + n) \leq \tau(E) \leq \text{rk}(\varphi^-)(2g - 2 + n),$$

where the rank of φ^{\pm} is the generic rank (there is a well defined notion of rank for elements of \mathfrak{m}^{\pm} , it is bounded by r). In particular, one has $|\tau(E)| \leq r(2g - 2 + n)$, which gives another proof of the Milnor–Wood inequality of [21] in the punctured case, when the Higgs bundle comes from a representation.

The case of equality in the Milnor–Wood inequality is of interest (the corresponding representations are called maximal representations), and leads to a nice description of the moduli space. Restrict to the case G/H is of tube type. One way to state this condition is to say that the Shilov boundary of G/H is itself a symmetric space H/H' . It is proved in [11] that there is a Cayley transform, that is the moduli space is isomorphic to a moduli space of K^2 -twisted H^* -Higgs bundles, where H^*/H' is the noncompact dual of H/H' . Here K^2 -twisted means that the Higgs field takes values in $E(\mathfrak{m}^{\mathbb{C}}) \otimes K^2$ rather than $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$. This can be extended to the punctured case in the following

way. For simplicity, suppose that the parabolic weights lie in $\sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$, which means that all eigenvalues of $\text{ad } \alpha_i$ on $\mathfrak{m}^{\mathbb{C}}$ have modulus smaller than 1. Then one can similarly prove that in the maximal case, one has $\alpha_i \in \mathfrak{h}'$ and the moduli space of polystable G -Higgs bundle is isomorphic to a moduli space of $K(D)^2$ -twisted polystable H^* -Higgs bundles, with parabolic structure α_i at the punctures. Such an isomorphism remains true if one drops the condition that $\alpha_i \in \sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$, but then one has only $e^{2\pi\sqrt{-1}\alpha_i} \in H'$ and a Hecke transformation is needed to obtain the parabolic weights of the Cayley transformed bundle.

This fact on α_i also implies that the monodromy around x_i fixes a point of the Shilov boundary, a fact also known from [21].

APPENDIX A. LIE THEORY

A.1. Weyl alcoves and conjugacy classes of a compact Lie group. For the following see e.g. [20].

Let H be a compact Lie group with Lie algebra \mathfrak{h} . Let $\langle \cdot, \cdot \rangle$ be a H -invariant inner product on \mathfrak{h} . Let $T \subset H$ be a Cartan subgroup, i.e. a maximal torus, and $\mathfrak{t} \subset \mathfrak{h}$ be its Lie algebra (a Cartan subalgebra). Fix a system of real simple roots (see e.g. [20, Chap V, Def 1.3]) and denote by R_+ the set of positive roots. Consider the family of affine hyperplanes in \mathfrak{t}

$$\mathcal{H}_{\lambda n} = \lambda^{-1}(n), \quad \lambda \in R^+, \quad n \in \mathbb{Z}$$

together with the union $\mathfrak{t}_s = \cup_{\lambda, n} \mathcal{H}_{\lambda n}$. As shown in [20], this family is given by the critical points of the exponential map

$$(A.1) \quad \exp : \mathfrak{t} \longrightarrow T.$$

The set $\mathfrak{t} - \mathfrak{t}_s$ decomposes into convex connected components which are called the **alcoves** of H (sometimes also referred as **Weyl alcoves**). Note that, by definition, alcoves are open. So, a choice of an alcove is basically a choice of a logarithm for (A.1). A **wall** of an alcove \mathcal{A} is one of the subsets of $\bar{\mathcal{A}} \cap \mathcal{H}_{\lambda n}$ of \mathfrak{t} that have dimension $k - 1$, where $\bar{\mathcal{A}}$ is the closure of \mathcal{A} and $k = \text{rank}(H)$.

Let $W := N(T)/T$ be the Weyl group of H . The H -invariant inner product on \mathfrak{h} induces a W -invariant inner product in \mathfrak{t} . The **co-character lattice** $\Lambda_{\text{cochar}} \subset \mathfrak{t}$ is defined as the kernel of the exponential map (A.1). Recall that the **co-roots** are the elements of \mathfrak{t} defined by

$$\lambda^* = 2b^{-1}(\lambda)/\langle \lambda, \lambda \rangle,$$

where b is the isomorphism $b : \mathfrak{t} \longrightarrow \mathfrak{t}^*$ defined by the inner product $\langle \cdot, \cdot \rangle$.

The co-roots define a lattice $\Lambda_{\text{coroot}} \subset \mathfrak{t}$. We have that $\Lambda_{\text{coroot}} \subset \Lambda_{\text{cochar}}$ and $\pi_1(H) = \Lambda_{\text{cochar}}/\Lambda_{\text{coroot}}$. In particular, $\Lambda_{\text{coroot}} = \Lambda_{\text{cochar}}$ if H is simply connected.

The **affine Weyl group** is defined as

$$W_{\text{aff}} = \Lambda_{\text{cochar}} \rtimes W$$

where Λ_{cochar} acts on \mathfrak{t} by translations.

The alcoves of H are important for us because of their relation with conjugacy classes of H . If H is connected, every element of H is conjugate to an element of T , in particular

every element of H lies in a Cartan subgroup. If $\text{Conj}(H)$ is the space of **conjugacy classes** of H we have then homeomorphisms

$$\text{Conj}(H) \simeq T/W \simeq \mathfrak{t}/W_{\text{aff}}.$$

We have the following.

Proposition A.1. *Let H be a connected compact Lie group. The closure $\bar{\mathcal{A}}$ of any alcove \mathcal{A} contains a fundamental domain \mathcal{A}^f for the action of W_{aff} on \mathfrak{t} , i.e. every W_{aff} -orbit meets \mathcal{A}^f in exactly one point. Hence the space of conjugacy classes of H is in bijection with \mathcal{A}^f .*

Define

$$\sqrt{-1}\mathcal{A}' = \{\alpha \in \sqrt{-1}\bar{\mathcal{A}} \mid \text{Spec}(\text{ad}(\alpha)) \subset (-1, 1)\},$$

and let

$$\sqrt{-1}W\mathcal{A}' = \bigcup_{w \in W} \sqrt{-1}w\mathcal{A}'.$$

The following will play a crucial role in our definition of parabolic Higgs bundle and in the analysis involved in the Hitchin–Kobayashi correspondence.

Proposition A.2. *Let $\mathcal{A} \subset \mathfrak{t}$ be an alcove of H such that $0 \in \bar{\mathcal{A}}$. Then:*

- (1) *If $\alpha \in \sqrt{-1}\bar{\mathcal{A}}$ then $\text{Spec}(\text{ad}(\alpha)) \subset [-1, 1]$.*
- (2) *We have $\sqrt{-1}\mathcal{A} \subset \sqrt{-1}\mathcal{A}'$.*
- (3) *We have $\sqrt{-1}W\mathcal{A}' = \{\alpha \in \sqrt{-1}\mathfrak{t} \mid \text{Spec}(\text{ad}(\alpha)) \subset (-1, 1)\}$.*
- (4) *For any $\alpha \in \sqrt{-1}\bar{\mathcal{A}}$ there exist $k \in \mathbb{Z}$ and $\lambda \in \sqrt{-1}(2\pi)^{-1}\Lambda_{\text{cochar}}$ such that*

$$k\alpha + \lambda \in \sqrt{-1}W\mathcal{A}'.$$

Proof. Items (1), (2) and (3) are immediate consequences of the definitions. We now prove (4). Let $\Gamma = \sqrt{-1}(2\pi)^{-1}\Lambda_{\text{cochar}}$, and note that $\sqrt{-1}\mathfrak{t}/\Gamma$ is compact. For any $\beta \in \sqrt{-1}\mathfrak{t}$ let $[\beta]$ denote its class in $\sqrt{-1}\mathfrak{t}/\Gamma$. Given $\alpha \in \sqrt{-1}\mathfrak{t}$ consider the sequence $[\alpha], [2\alpha], [3\alpha], \dots$. By compactness this sequence must accumulate somewhere. So one may take elements of the form $[\mu\alpha]$ and $[\nu\alpha]$ with $\mu \neq \nu$ in such a way that $[\nu\alpha] - [\mu\alpha] = [(\nu - \mu)\alpha]$ is arbitrary close to $[0]$. In particular, since $\sqrt{-1}W\mathcal{A}'$ is a neighborhood of 0 we can find k and λ such that $k\alpha + \lambda \in \sqrt{-1}W\mathcal{A}'$. \square

Proposition A.3. *Let G be the complexification of a connected compact Lie group H , and let $T^{\mathbb{C}}$ be the complexification of a Cartan subgroup of H . Then every semisimple element of G is conjugate to an element of $T^{\mathbb{C}}$, which in particular can be written as $\exp(\alpha)\exp(s)$ with $\alpha \in \bar{\mathcal{A}}$, $s \in \sqrt{-1}\mathfrak{t}$ and $[\alpha, s] = 0$.*

A.2. Real reductive Lie groups. Following Harisch-Chandra [34] and Knapp [40, p. 384], a **real reductive Lie group** is defined as a 4-tuple (G, H, θ, B) , where G is a real Lie group, $H \subset G$ is a maximal compact subgroup, $\theta: \mathfrak{g} \rightarrow \mathfrak{g}$ is a Cartan involution, and B is a non-degenerate bilinear form on \mathfrak{g} , which is $\text{Ad}(G)$ -invariant and θ -invariant. The data (G, H, θ, B) has to satisfy in addition that

- the Lie algebra \mathfrak{g} of G is reductive

- θ gives a decomposition (the Cartan decomposition)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

into its ± 1 -eigenspaces, where \mathfrak{h} is the Lie algebras of H , so we have

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h},$$

- \mathfrak{h} and \mathfrak{m} are orthogonal under B , and B is positive definite on \mathfrak{m} and negative definite on \mathfrak{h} ,
- multiplication as a map from $H \times \exp \mathfrak{m}$ into G is a diffeomorphism,
- every automorphism $\text{Ad}(g)$ of $\mathfrak{g}^{\mathbb{C}}$ is inner for $g \in G$, i.e., is given by some x in Intg .

Of course a compact real Lie group G whose Lie algebra is equipped with a non-degenerate $\text{Ad}(G)$ -invariant bilinear form belongs to a reductive tuple (G, H, θ, B) with $H = G$ and $\theta = \text{Id}$. Also the underlying real structure of the complexification G of a compact Lie group H , whose Lie algebra \mathfrak{h} is equipped with a non-degenerate $\text{Ad}(H)$ -invariant bilinear form can be endowed with a natural reductive structure.

If G is semisimple, the data (G, H, θ, B) can be recovered (to be precise, the quadratic form B can only recovered up to a scalar but this will be sufficient for everything we do in this paper) from the choice of a maximal compact subgroup $H \subset G$. There are other situations where less information is enough, e.g. for certain linear groups (see [40, p. 385]).

The bilinear form B does not play any role in the definition of a parabolic G -Higgs bundle but is essential for defining the stability condition and the gauge equations involved in the Hitchin–Kobayashi correspondence.

Note that the compactness of H together with the second to last property above say that G has only finitely many components.

Let $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$ be the complexifications of \mathfrak{g} and \mathfrak{h} respectively, and let $H^{\mathbb{C}}$ be the complexification of H . Let

$$(A.2) \quad \mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$$

be the complexification of the Cartan decomposition. The group H acts linearly on \mathfrak{m} through the adjoint representation, and this action extends to a linear holomorphic action of $H^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}}$, the **isotropy representation** that we will denote by

$$\iota : H^{\mathbb{C}} \longrightarrow \text{Aut}(\mathfrak{m}^{\mathbb{C}}),$$

or sometimes by Ad since it is obtained by restriction of the adjoint representation of G .

If G is complex with maximal compact subgroup H , then $\mathfrak{g} = \mathfrak{h} \oplus \sqrt{-1}\mathfrak{h}$. We thus have that $\mathfrak{m} = \sqrt{-1}\mathfrak{h}$, and the isotropy representation coincides with the adjoint representation $\text{Ad} : G \longrightarrow \text{Aut}(\mathfrak{g})$

If G is a reductive group, then the map $\Theta : G \longrightarrow G$ defined by

$$(A.3) \quad \Theta(h \exp A) = h \exp(-A) \quad \text{for } h \in H \text{ and } A \in \mathfrak{m}$$

is an automorphism of G and its differential is θ . This is called the **global Cartan involution**.

A.3. \mathfrak{sl}_2 -triples and orbit theory. We consider a reductive group G as defined in Appendix A.2.

An ordered triple of elements (x, e, f) in \mathfrak{g} (or $\mathfrak{g}^{\mathbb{C}}$) is called a **\mathfrak{sl}_2 -triple** if the bracket relations $[x, e] = 2e$, $[x, f] = -2f$, and $[e, f] = x$ are satisfied. One has that the elements e and f are nilpotent. An \mathfrak{sl}_2 -triple (x, e, f) in $\mathfrak{g}^{\mathbb{C}}$ is called **normal** if $e, f \in \mathfrak{m}^{\mathbb{C}}$ and $x \in \mathfrak{h}^{\mathbb{C}}$. Some times we refer to a normal triple as a **Kostant–Rallis triple** (see [45]).

Let $\mathcal{N}(\mathfrak{g})$ and $\mathcal{N}(\mathfrak{m}^{\mathbb{C}})$ be the set of nilpotent elements in \mathfrak{g} and $\mathfrak{m}^{\mathbb{C}}$ respectively. One has the following.

Proposition A.4. (1) *Every element $0 \neq e \in \mathcal{N}(\mathfrak{g})$ can be embedded in a \mathfrak{sl}_2 -triple (x, e, f) , establishing a 1–1 correspondence between the set of all G -orbits in $\mathcal{N}(\mathfrak{g})$ and the set of all G -conjugacy classes of \mathfrak{sl}_2 -triples in \mathfrak{g} .*

(2) *Every element $0 \neq e \in \mathcal{N}(\mathfrak{m}^{\mathbb{C}})$ can be embedded in a normal \mathfrak{sl}_2 -triple (x, e, f) , establishing a 1–1 correspondence between the set of all $H^{\mathbb{C}}$ -orbits in $\mathcal{N}(\mathfrak{m}^{\mathbb{C}})$ and the set of all $H^{\mathbb{C}}$ -conjugacy classes of normal \mathfrak{sl}_2 -triples in $\mathfrak{g}^{\mathbb{C}}$.*

Statement (1) is a real version of a refinement of the Jacobson–Morozov theorem, proved in [44]. For (2) see [45, Prop 4, 38]

We say that an \mathfrak{sl}_2 -triple in \mathfrak{g} is a **Kostant–Sekiguchi triple** if $\theta(e) = -f$, and hence $\theta(x) = -x$. A normal \mathfrak{sl}_2 -triple in $\mathfrak{g}^{\mathbb{C}}$ is called **Kostant–Sekiguchi triple** if $f = \sigma(e)$ where σ is the conjugation of $\mathfrak{g}^{\mathbb{C}}$ defining \mathfrak{g} .

One has the following (see [63]).

Proposition A.5. (1) *Every \mathfrak{sl}_2 -triple in \mathfrak{g} is conjugate under G to a Kostant–Sekiguchi triple in \mathfrak{g} . Two Kostant–Sekiguchi triples are G -conjugate if and only if they are conjugate under H .*

(2) *Every normal triple in $\mathfrak{g}^{\mathbb{C}}$ is conjugate under $H^{\mathbb{C}}$ to a Kostant–Sekiguchi triple. Two Kostant–Sekiguchi triples are conjugate under $H^{\mathbb{C}}$ to the same Kostant–Rallis triple if and only if they are conjugate under H .*

Propositions A.4 and A.5 can be combined, together with a linear transformation sometimes called Cayley transform (see e.g. [24, p. 579]) to obtain the Kostant–Sekiguchi correspondence:

Proposition A.6. *There is a one-to-one correspondence (see [63, 70]).*

$$\mathcal{N}(\mathfrak{g})/G \longleftrightarrow \mathcal{N}(\mathfrak{m}^{\mathbb{C}})/H^{\mathbb{C}}.$$

A similar correspondence can be established for orbits of arbitrary elements (see [5, 9]).

A.4. Conjugacy classes of a real reductive Lie group. Now let (G, H, θ, B) be a reductive group as defined in Appendix A.2. We can also give in this case a description of conjugacy classes of G . A **Cartan subgroup** of G is defined as the centralizer in G

of a Cartan subalgebra of \mathfrak{g} . When G is non-compact, it is no longer true that every element of G lies in a Cartan subgroup. For example, for $G = \mathrm{SL}_2\mathbb{R}$, the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ does not lie in any Cartan subgroup (see [40], p. 487).

We know (see [40]) that any Cartan subalgebra is conjugate via $\mathrm{Int}\mathfrak{g}$ to a θ -invariant Cartan subalgebra, and that there are only finitely many conjugacy classes of Cartan subalgebras. Consequently any Cartan subgroup of G is conjugate via G to a Θ -stable Cartan subgroup, where Θ is given in (A.3), and there are only finitely many conjugacy classes of Cartan subgroups. Moreover, a Θ -invariant Cartan subgroup is reductive.

Cartan subgroups of a non-compact real reductive Lie group can be non-connected even if the group G is connected. This already happens for $\mathrm{SL}_2\mathbb{R}$ (see [40]). In the following we will see how alcoves can be useful to deal with elements of a Cartan subgroup that are not in the identity component.

Proposition A.7. *Let T' be a Θ -invariant Cartan subgroup. Every element $t \in T'$ is conjugate to an element of the form $\exp(\alpha)\exp(s)$ where $\alpha \in \bar{\mathcal{A}}$ and $s \in \mathfrak{m}$, such that $\exp(\alpha)$ and $\exp(s)$ commute.*

Proof. Let $t \in T'$. Then by the reductivity and Θ -invariance of T' we have that $t = h \exp(s')$ with $h \in T' \cap H$ and $s' \in \mathfrak{t}' \cap \mathfrak{m}$. We can extend \mathfrak{t}' to a Cartan subalgebra \mathfrak{t} of \mathfrak{h} with corresponding Cartan subgroup T , and choose an alcove \mathcal{A} containing 0 such that h is conjugate (via an element of the Weyl group of T) to an element of the form $\exp(\alpha)$ with $\alpha \in \bar{\mathcal{A}}$. So $h = k^{-1} \exp(\alpha) k$ with $k \in H$. Then $ktk^{-1} = \exp(\alpha) k \exp(s') k^{-1}$. The element $k \exp(s') k^{-1}$ is in $\exp(\mathfrak{m})$, and can thus be written $k \exp(s') k^{-1} = \exp(s)$ with $s \in \mathfrak{m}$. Clearly, since h and $\exp(s')$ (both are elements in T') we have that $\exp(\alpha)$ and $\exp(s)$ commute. □

For the correspondences proved in this paper we will be considering conjugacy classes of G in which one can find a representative of the form

$$g = g_e g_h g_u$$

so that all the factors commute two by two, with

- $g_e = \exp(2\sqrt{-1}\pi\alpha)$, with $\alpha \in \sqrt{-1}\bar{\mathcal{A}}$ (**elliptic** element);
- $g_h = \exp(s - \tau(s))$ with $s \in \mathfrak{a}$ (**hyperbolic** element), where τ is a conjugation of $\mathfrak{g}^{\mathbb{C}}$ defining a compact real form, and $\mathfrak{a} \subset \mathfrak{m}$ is a maximal abelian subalgebra;
- $g_u = \exp(n)$, where $n \in \mathfrak{m}$ is a nilpotent element of the form $i(Y - X - H)$ where (H, X, Y) is an appropriate \mathfrak{sl}_2 -triple (**unipotent** element).

The decomposition $g = g_e g_h g_u$ is the **multiplicative Jordan decomposition**, which is proved in Helgason [35] for $\mathrm{GL}_n\mathbb{R}$ and in Eberlein [27] for the connected component of the isometry group of a symmetric space of non-compact type, hence for the connected component of the adjoint group.

APPENDIX B. PARABOLIC SUBGROUPS AND RELATIVE DEGREE

Let (G, H, θ, B) be a real reductive Lie group as defined in Appendix A.2. In this section we recall some basics on parabolic subgroups of G and define the relative degree of two parabolic subgroups.

B.1. Parabolic subgroups. Let $\Sigma = H \backslash G$ be the symmetric space of non compact type. (The action is taken to be a right action, because this fits better with the way symmetric spaces arise in Kähler quotients.) Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the Cartan decomposition, and $\mathfrak{a} \subset \mathfrak{m}$ a maximal abelian subalgebra (the dimension of \mathfrak{a} is the rank of Σ). Let $\Phi \subset \mathfrak{a}^*$ the roots of \mathfrak{g} , so that $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}_\lambda$, where \mathfrak{g}_0 is the centralizer of \mathfrak{a} .

Choose a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and let $\Phi^\pm \subset \Phi$ (resp. $\Delta \subset \Phi$) be the set of positive/negative roots (resp. simple roots). For $I \subset \Delta$, denote $\Phi^I \subset \Phi$ the set of roots which are linear combinations of elements of I , then we define a standard parabolic subalgebra

$$\mathfrak{p}_I = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Phi^I \cup \Phi^-} \mathfrak{g}_\lambda,$$

and $P_I \subset G$ the corresponding subgroup. Any parabolic subalgebra of \mathfrak{g} is conjugate to one of the standard parabolic subalgebras.

Given an element $s \in \mathfrak{m}$, one has a parabolic subgroup P_s and its Lie algebra defined as follows:

$$\begin{aligned} P_s &= \{g \in G, e^{ts} g e^{-ts} \text{ is bounded as } t \rightarrow \infty\}, \\ \mathfrak{p}_s &= \{x \in \mathfrak{g}, \text{Ad}(e^{ts})x \text{ is bounded as } t \rightarrow \infty\}. \end{aligned}$$

When $t \rightarrow \infty$, the geodesic $t \mapsto *e^{ts}$ (where $*$ is the base point, fixed by H), hits the visual boundary $\partial_\infty \Sigma$ in a point, whose stabilizer in G is precisely P_s . If G is connected every parabolic subgroup P_s obtained in this way is conjugate a standard parabolic subgroup P_I . The element s defines also a Levi subgroup $L_s \subset P_s$ and a Levi subalgebra $\mathfrak{l}_s \subset \mathfrak{p}_s$ by

$$L_s = \{g \in G, \text{Ad}(g)(s) = s\}, \quad \mathfrak{l}_s = \{x \in \mathfrak{g}, [s, x] = 0\}.$$

A character χ of \mathfrak{p}_s defines an element in the dual of \mathfrak{g} and hence in \mathfrak{g} via the invariant metric, which by projection defines an element s_χ in \mathfrak{m} . When $\mathfrak{p}_s \subset \mathfrak{p}_{s_\chi}$ we say that χ is **antidominant**. When $\mathfrak{p}_s = \mathfrak{p}_{s_\chi}$ we say that χ is **strictly antidominant**. The mapping $\chi_s : \mathfrak{p}_s \rightarrow \mathbb{R}$ defined by

$$\chi_s(x) = \langle s, x \rangle$$

is hence a strictly antidominant character of \mathfrak{p}_s .

Remark B.1. If G is complex then $\mathfrak{g} = \mathfrak{h} \oplus \sqrt{-1}\mathfrak{h}$ and the subgroups P_s , L_s and corresponding Lie subalgebras \mathfrak{p}_s , \mathfrak{l}_s are of course complex.

B.2. Relative degree. We now define a function which is important in the paper, since it calculates the contribution to the parabolic degree at the punctures. The setting is the following.

Let $\mathcal{O}_H \subset \mathfrak{m}$ be an H -orbit in \mathfrak{m} . As is well known, \mathcal{O}_H is also a G -homogeneous space. This can be seen in the following way: given $s \in \mathcal{O}_H$, one can consider $\eta(s) =$

$\lim_{t \rightarrow +\infty} *e^{ts} \in \partial_\infty \Sigma$. It turns out that the image of \mathcal{O}_H under η is a G -orbit in $\partial_\infty \Sigma$. Of course the stabilizer of $\eta(s)$ is the parabolic group P_s defined above, so one gets an identification

$$\eta : \mathcal{O}_H = H/(P_s \cap H) \subset \mathfrak{m} \longrightarrow P_s \backslash G \subset \partial_\infty \Sigma.$$

The action of $g \in G$ on \mathcal{O}_H will be denoted by $s \cdot g$; if one decomposes $g = ph$ with $h \in H$ and $p \in P$, then $s \cdot g = s \cdot h = \text{Ad}(h^{-1})s$.

Now take another element $\sigma \in \mathfrak{m}$. As we shall see below in the proof of the proposition, the function

$$t \mapsto \langle s \cdot e^{-t\sigma}, \sigma \rangle$$

is a nonincreasing function of t , so we can define a function

$$\mu_s : \mathfrak{m} \longrightarrow \mathbb{R}, \quad \mu_s(\sigma) = \lim_{t \rightarrow +\infty} \langle s \cdot e^{-t\sigma}, \sigma \rangle.$$

This function is actually (up to a normalization) a function defined on $\partial_\infty \Sigma$, as follows from the following proposition.

Proposition B.2. *Suppose s and σ normalized so that $|s| = |\sigma| = 1$. Then one has*

$$\mu_s(\sigma) = \cos \angle_{\text{Tits}}(\eta(\sigma), \eta(s)),$$

where \angle_{Tits} is the Tits distance on $\partial_\infty \Sigma$. In particular, one has the reciprocity $\mu_\sigma(s) = \mu_s(\sigma)$.

Proof. Decompose $e^{-t\sigma} = p_t h_t$ with $h_t \in H$ and $p_t \in P_s$. Then

$$\langle s \cdot e^{-t\sigma}, \sigma \rangle = \langle \text{Ad}(h_t^{-1})s, \sigma \rangle = \cos \angle(\text{Ad}(h_t^{-1})s, \sigma).$$

On the other hand, the distance

$$d(*e^{u \text{Ad}(h_t^{-1})s} e^{t\sigma}, *e^{us}) = d(*e^{us} p_t^{-1}, *e^{us})$$

is bounded when $u \rightarrow +\infty$, so $u \rightarrow *e^{u \text{Ad}(h_t^{-1})s} e^{t\sigma}$ is the geodesic emanating from $*e^{t\sigma}$ and going to $\eta(s)$. So the angle between the geodesics emanating from $*e^{t\sigma}$ and converging to $\eta(\sigma)$ and $\eta(s)$ is the angle between $\text{Ad}(h_t^{-1})s$ and σ . It is well known that this angle is increasing and converges when $t \rightarrow \infty$ to the Tits distance between $\eta(s)$ and $\eta(\sigma)$, and the proposition follows. \square

The function $-\mu_\sigma$ is the ‘asymptotic slope’ of [41]. In [53], the complex case is studied: if $G = H^\mathbb{C}$ then the adjoint orbit \mathcal{O}_H is a Kähler manifold, and the asymptotic slope can be reinterpreted in terms of maximal weights of the action of G on \mathcal{O}_H .

We will use this notion to define a relative degree. From the proposition, $\mu_s(\sigma)$ depends only on giving two pairs (P, s) and (Q, σ) of a parabolic subgroup of G and an antidominant character on the parabolic subgroup. So we can define the **relative degree** of (P, s) and (Q, σ) as

$$(B.1) \quad \text{deg} \left((P, s), (Q, \sigma) \right) = \mu_s(\sigma).$$

Observe, again from the proposition, that deg is a symmetric function of its two arguments.

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