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THE PLANCHEREL FORMULA FOR COUNTABLE GROUPS

BACHIR BEKKA

ABSTRACT. We discuss a Plancherel formula for countable groups, which provides a canonical decomposition of the regular representation λ_Γ of such a group Γ into a direct integral of factor representations. Our main result gives a precise description of this decomposition in terms of the Plancherel formula of the FC-center Γ_{fc} of Γ (that is, the normal subgroup of Γ consisting of elements with a finite conjugacy class); this description involves the action of an appropriate totally disconnected compact group of automorphisms of Γ_{fc} . As an application, we determine the Plancherel formula for linear groups. In an appendix, we use the Plancherel formula to provide a unified proof for Thoma's and Kaniuth's theorems which respectively characterize countable groups which are of type I and those whose regular representation is of type II.

1. INTRODUCTION

Given a second countable locally compact group G , a fundamental object to study is its **unitary dual space** \widehat{G} , that is, the set irreducible unitary representations of G up to unitary equivalence. The space \widehat{G} carries a natural Borel structure, called the Mackey Borel structure (see [Mac57, §6] or [Dix77, §18.6]). A classification of \widehat{G} is considered as being possible only if \widehat{G} is a standard Borel space; according to Glimm's celebrated theorem ([Gli61]), this is the case if and only if G is of type I in the following sense.

Recall that a von Neumann algebra is a self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$ which is closed for the weak operator topology of $\mathcal{L}(\mathcal{H})$, where \mathcal{H} is a Hilbert space. A von Neumann algebra \mathcal{M} is a factor if the center of \mathcal{M} consists of the scalar operators.

Let π be a unitary representation of G in a Hilbert space \mathcal{H} (as we will only consider representations which are unitary, we will often drop the adjective "unitary"). The von Neumann subalgebra generated by

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$\pi(G)$ coincides with the bicommutant $\pi(G)''$ of $\pi(G)$ in $\mathcal{L}(\mathcal{H})$; we say that π is a factor representation if $\pi(G)''$ of $\mathcal{L}(\mathcal{H})$ is a factor.

Definition 1. The group G is of **type I** if, for every factor representation π of G , the factor $\pi(G)''$ is of type I, that is, $\pi(G)''$ is isomorphic to the von Neumann algebra $\mathcal{L}(\mathcal{K})$ for some Hilbert space \mathcal{K} ; equivalently, the Hilbert space \mathcal{H} of π can be written as tensor product $\mathcal{K} \otimes \mathcal{K}'$ of Hilbert spaces in such a way that π is equivalent to $\sigma \otimes I_{\mathcal{K}'}$ for an irreducible representation σ of G on \mathcal{K} .

Important classes of groups are known to be of type I, such as semi-simple or nilpotent Lie groups. A major problem in harmonic analysis is to decompose the left regular representation λ_G on $L^2(G, \mu_G)$ for a Haar measure μ_G as a direct integral of irreducible representations. When G is of type I and unimodular, this is the content of the classical Plancherel's theorem: there exist a unique measure μ on \widehat{G} and a unitary isomorphism between $L^2(G)$ and the direct integral of Hilbert spaces $\int_{\widehat{G}}^{\oplus} (\mathcal{H}_{\pi} \otimes \overline{\mathcal{H}_{\pi}}) d\mu(\pi)$ which transforms λ_G into $\int_{\widehat{G}}^{\oplus} (\pi \otimes I_{\overline{\mathcal{H}_{\pi}}}) d\mu(x)$, where $\overline{\mathcal{H}_{\pi}}$ is the conjugate of the Hilbert space \mathcal{H}_{π} of π ; in particular, we have a Plancherel formula

$$\|f\|_2^2 = \int_{\widehat{G}} \text{Tr}(\pi(f)^* \pi(f)) d\mu(\pi) \quad \text{for all } f \in L^1(G) \cap L^2(G),$$

where $\|f\|_2$ is the L^2 -norm of f , $\pi(f)$ is the value at f of the extension of π to a representation of $L^1(G)$, and Tr denotes the standard trace on $\mathcal{L}(\mathcal{H}_{\pi})$; for all this, see [Dix77, 18.8.1].

When G is not type I, λ_G usually admits several integral decompositions into irreducible representations and it is not possible to single out a canonical one among them. However, when G is unimodular, λ_G does admit a canonical integral decomposition into factor representations; this is the content of a Plancherel theorem which we will discuss in the case of a discrete group (see Theorem A).

Let Γ be a countable group. As discussed below (see Theorem E), Γ is usually not of type I. In order to state the Plancherel theorem for Γ , we need to replace the dual space $\widehat{\Gamma}$ by the consideration of Thoma's dual space $\text{Ch}(\Gamma)$ which we now introduce.

Recall that a function $t : \Gamma \rightarrow \mathbf{C}$ is of positive type if the complex-valued matrix $(t(\gamma_j^{-1} \gamma_i))_{1 \leq i, j \leq n}$ is positive semi-definite for any $\gamma_1, \dots, \gamma_n$ in Γ .

A function of positive type t on Γ which is constant on conjugacy classes and normalized (that is, $t(e) = 1$) will be called a **trace** on G . The set of traces on Γ will be denoted by $\text{Tr}(\Gamma)$.

Let $t \in \text{Tr}(\Gamma)$ and $(\pi_t, \mathcal{H}_t, \xi_t)$ be the associated GNS triple (see [BHV08, C.4]). Then $\tau_t : \pi(\Gamma)'' \rightarrow \mathbf{C}$, defined by $\tau_t(T) = \langle T\xi_t \mid \xi_t \rangle$ is a trace on the von Neumann algebra $\pi_t(\Gamma)''$, that is, $\tau_t(T^*T) \geq 0$ and $\tau_t(TS) = \tau_t(ST)$ for all $T, S \in \pi_t(\Gamma)''$; moreover, τ_t is faithful in the sense that $\tau_t(T^*T) > 0$ for every $T \in \pi_t(\Gamma)''$, $T \neq 0$. Observe that $\tau_t(\pi(f)) = t(f)$ for $f \in \mathbf{C}[\Gamma]$, where t denotes the linear extension of t to the group algebra $\mathbf{C}[\Gamma]$.

The set $\text{Tr}(\Gamma)$ is a convex subset of the unit ball of $\ell^\infty(\Gamma)$ which is compact in the topology of pointwise convergence. An extreme point of $\text{Tr}(\Gamma)$ is called a **character** of Γ ; we will refer to $\text{Ch}(\Gamma)$ as **Thoma's dual space**.

Since Γ is countable, $\text{Tr}(\Gamma)$ is a compact metrizable space and $\text{Ch}(\Gamma)$ is easily seen to be a G_δ subset of $\text{Tr}(\Gamma)$. So, in contrast to $\widehat{\Gamma}$, Thoma's dual space $\text{Ch}(\Gamma)$ is always a standard Borel space.

An important fact is that $\text{Tr}(\Gamma)$ is a simplex (see [Tho64, Satz 1] or [Sak71, 3.1.18]); by Choquet theory, this implies that every $\tau \in \text{Tr}(\Gamma)$ can be represented as integral $\tau = \int_{\text{Ch}(\Gamma)} t d\mu(t)$ for a *unique* probability measure μ on $\text{Ch}(\Gamma)$.

As we now explain, the set of characters of Γ parametrizes the factor representations of finite type of Γ , up to quasi-equivalence; for more details, see [Dix77, §17.3] or [BH, §11.C].

Recall first that two representations π_1 and π_2 of Γ are quasi-equivalent if there exists an isomorphism $\Phi : \pi_1(\Gamma)'' \rightarrow \pi_2(\Gamma)''$ of von Neumann algebras such that $\Phi(\pi_1(\gamma)) = \pi_2(\gamma)$ for every $\gamma \in \Gamma$.

Let $t \in \text{Ch}(\Gamma)$ and π_t the associated GNS representation. Then $\pi_t(\Gamma)''$ is a factor of finite type. Conversely, let π be a representation of Γ such that $\pi(\Gamma)''$ is a factor of finite type and let τ be the unique normalized trace on $\pi(\Gamma)''$. Then $t := \tau \circ \pi$ belongs to $\text{Ch}(\Gamma)$ and only depends on the quasi-equivalence class $[\pi]$ of π .

The map $t \rightarrow [\pi_t]$ is a bijection between $\text{Ch}(\Gamma)$ and the set of quasi-equivalence classes of factor representations of finite type of Γ .

The following result is a version for countable groups of a Plancherel theorem due to Mautner [Mau50] and Segal [Seg50] which holds more generally for any unimodular second countable locally compact group; its proof is easier in our setting and will be given in Section 3 for the convenience of the reader.

Theorem A. (*Plancherel theorem for countable groups*) *Let Γ be a countable group. There exists a probability measure μ on $\text{Ch}(\Gamma)$, a measurable field of representations $t \mapsto (\pi_t, \mathcal{H}_t)$ of Γ on the standard Borel space $\text{Ch}(\Gamma)$, and an isomorphism of Hilbert spaces between $\ell^2(\Gamma)$*

and $\int_{\text{Ch}(\Gamma)}^{\oplus} \mathcal{H}_t d\mu(t)$ which transforms λ_{Γ} into $\int_{\text{Ch}(\Gamma)}^{\oplus} \pi_t d\mu(t)$ and has the following properties:

- (i) π_t is quasi-equivalent to the GNS representation associated to t ; in particular, the π_t 's are mutually disjoint factor representations of finite type, for μ -almost every $t \in \text{Ch}(\Gamma)$;
- (ii) the von Neumann algebra $L(\Gamma) := \lambda_{\Gamma}(\Gamma)''$ is mapped onto the direct integral $\int_{\text{Ch}(\Gamma)}^{\oplus} \pi_t(\Gamma)'' d\nu(t)$ of factors;
- (iii) for every $f \in \mathbf{C}[\Gamma]$, the following Plancherel formula holds:

$$\|f\|_2^2 = \int_{\text{Ch}(\Gamma)} \tau_t(\pi_t(f)^* \pi_t(f)) d\mu(t) = \int_{\text{Ch}(\Gamma)} t(f^* * f) d\mu(t).$$

The measure μ is the unique probability measure on $\text{Ch}(\Gamma)$ such that the Plancherel formula above holds.

The probability measure μ on $\text{Ch}(\Gamma)$ from Theorem A is called the **Plancherel measure** of Γ .

Remark 2. The Plancherel measure gives rise to what seems to be an interesting dynamical system on $\text{Ch}(\Gamma)$ involving the group $\text{Aut}(\Gamma)$ of automorphisms of Γ . We will equip $\text{Aut}(\Gamma)$ with the topology of pointwise convergence on Γ , for which it is a totally disconnected topological group. The natural action of $\text{Aut}(\Gamma)$ on $\text{Ch}(\Gamma)$, given by $t^g(\gamma) = t(g^{-1}(\gamma))$ for $g \in \text{Aut}(\Gamma)$ and $t \in \text{Ch}(\Gamma)$, is clearly continuous.

Since the induced action of $\text{Aut}(\Gamma)$ on $\ell^2(\Gamma)$ is isometric, the following fact is an immediate consequence of the uniqueness of the Plancherel measure μ of Γ :

the action of $\text{Aut}(\Gamma)$ on $\text{Ch}(\Gamma)$ preserves μ .

For example, when $\Gamma = \mathbf{Z}^d$, Thoma's dual $\text{Ch}(\Gamma)$ is the torus \mathbf{T}^d , the Plancherel measure μ is the normalized Lebesgue measure on \mathbf{T}^d which is indeed preserved by the group $\text{Aut}(\mathbf{Z}^d) = \text{GL}_d(\mathbf{Z})$. Dynamical systems of the form $(\Lambda, \text{Ch}(\Gamma), \mu)$ for a subgroup Λ of $\text{Aut}(\Gamma)$ may be viewed as generalizations of this example.

We denote by Γ_{fc} the FC-centre of Γ , that is, the normal subgroup of elements in Γ with a finite conjugacy class. It turns out (see Remark 5) that $t = 0$ on $\Gamma \setminus \Gamma_{\text{fc}}$ for μ -almost every $t \in \text{Ch}(\Gamma)$. In particular, when Γ is ICC, that is, when $\Gamma_{\text{fc}} = \{e\}$, the regular representation λ_{Γ} is factorial (see also Corollary 4) so that the Plancherel formula is vacuous in this case.

In fact, as we now see, the Plancherel measure of Γ is entirely determined by the Plancherel measure of Γ_{fc} . Roughly speaking, we will see that the Plancherel measure of Γ is the image of the Plancherel

measure of Γ_{fc} under the quotient map $\text{Ch}(\Gamma_{\text{fc}}) \rightarrow \text{Ch}(\Gamma_{\text{fc}})/K_\Gamma$, for a compact group K_Γ which we now define.

Let K_Γ be the closure in $\text{Aut}(\Gamma_{\text{fc}})$ of the subgroup $\text{Ad}(\Gamma)|_{\Gamma_{\text{fc}}}$ given by conjugation with elements from Γ . Since every Γ -conjugation class in Γ_{fc} is finite, K_Γ is a compact group. By a general fact about actions of compact groups on Borel spaces (see Corollary 2.1.21 and Appendix in [Zim84]), the quotient space $\text{Ch}(\Gamma_{\text{fc}})/K_\Gamma$ is a standard Borel space.

Given a function $t : H \rightarrow \mathbf{C}$ of positive type on a subgroup H of a group Γ , we denote by \tilde{t} the extension of t to Γ given by $\tilde{t} = 0$ outside H . Observe that \tilde{t} is of positive type on Γ (see for instance [BH, 1.F.10]).

Here is our main result.

Theorem B. (Plancherel measure: reduction to the FC-center)

Let Γ be a countable group. Let ν be the Plancherel measure of Γ_{fc} and $\lambda_{\Gamma_{\text{fc}}} = \int_{\text{Ch}(\Gamma_{\text{fc}})}^\oplus \pi_t d\mu(t)$ the integral decomposition of the regular representation of Γ_{fc} as in Theorem A. Let $\dot{\nu}$ be the image of ν under the quotient map $\text{Ch}(\Gamma_{\text{fc}}) \rightarrow \text{Ch}(\Gamma_{\text{fc}})/K_\Gamma$.

- (i) For every K_Γ -orbit \mathcal{O} in $\text{Ch}(\Gamma_{\text{fc}})$, let $m_{\mathcal{O}}$ be the unique normalized K_Γ -invariant probability measure on \mathcal{O} and let $\pi_{\mathcal{O}} := \int_{\mathcal{O}}^\oplus \pi_t m_{\mathcal{O}}(t)$. Then the induced representation $\widetilde{\pi}_{\mathcal{O}} := \text{Ind}_{\Gamma_{\text{fc}}}^\Gamma \pi_{\mathcal{O}}$ is factorial for $\dot{\nu}$ -almost every \mathcal{O} and we have a direct integral decomposition of the von Neumann algebra $L(\Gamma)$ into factors

$$L(\Gamma) = \int_{\text{Ch}(\Gamma_{\text{fc}})/K_\Gamma}^\oplus \widetilde{\pi}_{\mathcal{O}}(\Gamma)'' d\dot{\nu}(\mathcal{O}).$$

- (ii) The Plancherel measure of Γ is the image $\Phi_*(\nu)$ of ν under the map

$$\Phi : \text{Ch}(\Gamma_{\text{fc}}) \rightarrow \text{Tr}(\Gamma), \quad t \mapsto \int_{K_\Gamma} \tilde{t}^g dm(g),$$

where m is the normalized Haar measure on K_Γ .

It is worth mentioning that the *support* of μ was determined in [Tho67]. For an expression of the map Φ as in Theorem B.ii without reference to the group K_Γ , see Remark 6.

As we next see, the Plancherel measure on Γ_{fc} can be explicitly described in the case of a linear group Γ . We first need to discuss the Plancherel formula for a so-called **central group**, that is, a central extension of a finite group.

Let Λ be a central group. Then Λ is of type I (see Theorem E). In fact, $\widehat{\Lambda}$ can be described as follows; let $r : \widehat{\Lambda} \rightarrow \widehat{Z(\Lambda)}$ be the restriction map, where $Z(\Lambda)$ is the center of Λ . Then, for $\chi \in \widehat{Z(\Lambda)}$, every

$\pi \in r^{-1}(\chi)$ is equivalent to a subrepresentation of the finite dimensional representation $\text{Ind}_{\widehat{Z(\Lambda)}}^{\Lambda} \chi$, by a generalized Frobenius reciprocity theorem (see [Mac52, Theorem 8.2]); in particular, $r^{-1}(\chi)$ is finite. The Plancherel measure ν on $\widehat{\text{Ch}(\Lambda)}$ is, in principle, easy to determine: we identify every $\pi \in \widehat{\Lambda}$ with its normalized character given by $x \mapsto \frac{1}{\dim \pi} \text{Tr} \pi(x)$; for every Borel subset A of $\widehat{\text{Ch}(\Lambda)}$, we have

$$\nu(A) = \int_{\widehat{Z(\Lambda)}} \frac{\#(A \cap r^{-1}(\chi))}{\sum_{\pi \in r^{-1}(\chi)} (\dim \pi)^2} d\chi,$$

where $d\chi$ is the normalized Haar measure on the abelian compact group $\widehat{Z(\Lambda)}$.

Corollary C. (*The Plancherel measure for linear groups*) *Let Γ be a countable linear group.*

- (i) Γ_{fc} is a central group;
- (ii) K_{Γ} coincides with $\text{Ad}(\Gamma)|_{\Gamma_{\text{fc}}}$ and is a finite group;
- (iii) the Plancherel measure of Γ is image of the Plancherel measure of Γ_{fc} under the map $\Phi : \text{Ch}(\Gamma_{\text{fc}}) \rightarrow \text{Tr}(\Gamma)$ given by

$$\Phi(t) = \frac{1}{\#\text{Ad}(\Gamma)|_{\Gamma_{\text{fc}}}} \sum_{s \in \text{Ad}(\Gamma)|_{\Gamma_{\text{fc}}}} t^s.$$

When the Zariski closure of the linear group Γ is connected, the Plancherel measure of Γ has a particularly simple form.

Corollary D. (*The Plancherel measure for linear groups-bis*) *Let \mathbf{G} be a connected linear algebraic group over a field \mathbf{k} and let Γ be a countable Zariski dense subgroup of \mathbf{G} . The Plancherel measure of Γ is the image of the normalized Haar measure $d\chi$ on $\widehat{Z(\Gamma)}$ under the map*

$$\widehat{Z(\Gamma)} \rightarrow \text{Tr}(\Gamma), \quad \chi \mapsto \tilde{\chi}$$

and the Plancherel formula is given for every $f \in \mathbf{C}[\Gamma]$ by

$$\|f\|_2^2 = \int_{\widehat{Z(\Gamma)}} \mathcal{F}((f^* * f)|_{Z(\Gamma)})(\chi) d\chi,$$

where \mathcal{F} is the Fourier transform on the abelian group $Z(\Gamma)$.

The previous conclusion holds in the following two cases:

- (i) \mathbf{k} is a countable field of characteristic 0 and $\Gamma = \mathbf{G}(\mathbf{k})$ is the group of \mathbf{k} -rational points in \mathbf{G} ;
- (ii) \mathbf{k} is a local field (that is, a non discrete locally compact field), \mathbf{G} has no proper \mathbf{k} -subgroup \mathbf{H} such that $(\mathbf{G}/\mathbf{H})(\mathbf{k})$ is compact, and Γ is a lattice in $\mathbf{G}(\mathbf{k})$.

Corollary D generalizes the Plancherel theorem obtained in [CPJ94, Theorem 4] in the case $\Gamma = \mathbf{G}(\mathbf{Q})$ and in [PJ95, Theorem 3.6] in the case $\Gamma = \mathbf{G}(\mathbf{Z})$, where \mathbf{G} is a unipotent linear algebraic group over \mathbf{Q} ; indeed, \mathbf{G} is connected (since the exponential map identifies \mathbf{G} with its Lie algebra, as affine varieties) and these two results follow from (i) and (ii) respectively.

In an appendix to this article, we use Theorem A to give a unified proof of Thoma's and Kaniuth's results ([Tho64],[Tho68], [Kan69]) as stated in the following theorem. For a group Γ , we denote by $[\Gamma, \Gamma]$ its commutator subgroup. Recall that Γ is said to be virtually abelian if contains an abelian subgroup of finite index.

The regular representation λ_Γ is of type I (or type II) if the von Neumann $L(\Gamma)$ is of type I (or type II); equivalently (see Corollaire 2 in [Dix69, Chap. II, § 3, 5]), if $\pi_t(\Gamma)''$ is a finite dimensional factor (or a factor of type II) for μ -almost every $t \in \text{Ch}(\Gamma)$ in the Plancherel decomposition $\lambda_\Gamma = \int_{\text{Ch}(\Gamma)}^\oplus \pi_t d\mu(t)$ from Theorem A.

Theorem E (Thoma, Kaniuth). *Let Γ be a countable group. The following properties are equivalent:*

- (i) Γ is type I;
- (ii) Γ is virtually abelian;
- (iii) the regular representation λ_Γ is of type I.

Moreover, the following properties are equivalent:

- (iv) λ_Γ is of type II;
- (v) either $[\Gamma : \Gamma_{\text{fc}}] = \infty$ or $[\Gamma : \Gamma_{\text{fc}}] < \infty$ and $[\Gamma, \Gamma]$ is infinite.

Our proof of Theorem E is not completely new as it uses several crucial ideas from [Tho64] and especially from [Kan69] (compare with the remarks on p.336 after Lemma in [Kan69]); however, we felt it could be useful to have a short and common treatment of both results in the literature.

Theorem E holds for any discrete group, but the sake of simplicity we preferred to treat only countable ones. Observe that the equivalence between (i) and (iii) above does not carry over to non discrete groups (see [Mac61]).

This paper is organized as follows. In Section 2, we recall the well-known description of the center of the von Neumann algebra of a discrete group. Sections 3 and 4 contains the proofs of Theorems A and B. In Section 5, we prove Corollaries C and D. Section 6 is devoted to the explicit computation of the Plancherel formula for a few examples of countable groups. Appendix A contains the proof of Theorem E.

2. ON THE CENTER OF THE GROUP VON NEUMANN ALGEBRA

Let Γ be a countable group. We will often use the following well-known description of the center $\mathcal{Z} = \lambda(\Gamma)'' \cap \lambda(\Gamma)'$ of $L(\Gamma) = \lambda_\Gamma(\Gamma)''$.

Observe that $\lambda_\Gamma(H)''$ is a von Neumann subalgebra of $L(\Gamma)$, for every subgroup H of Γ . For $h \in \Gamma_{\text{fc}}$, we set

$$T_{[h]} := \frac{1}{\#[h]} \sum_{x \in [h]} \lambda_\Gamma(h) \in \lambda_\Gamma(\Gamma_{\text{fc}})'',$$

where $[h]$ denotes the conjugacy class of h .

Lemma 3. *The center \mathcal{Z} of $L(\Gamma) = \lambda_\Gamma(\Gamma)''$ coincides with the closure of the linear span of $\{T_{[h]} \mid h \in \Gamma_{\text{fc}}\}$, for the strong operator topology; in particular, \mathcal{Z} is contained in $\lambda_\Gamma(\Gamma_{\text{fc}})''$.*

Proof. Let ρ_Γ be the right regular representation of Γ . Observe that $\rho_\Gamma(\Gamma)'' \subset \lambda_\Gamma(\Gamma)'$ and recall that the Dirac function $\delta_e \in \ell^2(\Gamma)$ at e is a cyclic vector for both λ_Γ and ρ_Γ . It follows that the linear and continuous map

$$\lambda_\Gamma(\Gamma)'' \rightarrow \ell^2(\Gamma), \quad T \mapsto T\delta_e$$

is injective.

Let J be the conjugate linear isometric involution on $\ell^2(\Gamma)$ given by $Jf(x) = \overline{f(x^{-1})}$ for $f \in \ell^2(\Gamma)$ and $x \in \Gamma$. Then, as is readily checked, we have $J\lambda_\Gamma(\gamma)J = \rho_\Gamma(\gamma)$ for all $\gamma \in \Gamma$ and so $J\lambda_\Gamma(\Gamma)''J = \rho_\Gamma(\Gamma)''$. Moreover, $JTJ\delta_e = T^*\delta_e$ for all $T \in \lambda_\Gamma(\Gamma)''$ or $T \in \rho_\Gamma(\Gamma)''$.

It is clear that $T_{[h]} \in \mathcal{Z}$ for every $h \in \Gamma_{\text{fc}}$. Observe also that the linear span of $\{T_{[h]} \mid h \in \Gamma_{\text{fc}}\}$ is a unital selfadjoint algebra; indeed, $T_{[h^{-1}]} = T_{[h]}^*$ for every $h \in \Gamma_{\text{fc}}$ and $\{\mathbf{1}_{[h]} \mid h \in \Gamma_{\text{fc}}\}$ is a vector space basis of the algebra $\mathbf{C}[\Gamma_{\text{fc}}]^\Gamma$ of Γ -invariant functions in $\mathbf{C}[\Gamma_{\text{fc}}]$.

Let $T \in \mathcal{Z}$. We have to show that $T \in \{T_{[h]} \mid h \in \Gamma_{\text{fc}}\}''$. For every $\gamma \in \Gamma$, we have

$$\lambda_\Gamma(\gamma)\rho_\Gamma(\gamma)(T\delta_e) = (\lambda_\Gamma(\gamma)\rho_\Gamma(\gamma)T)\delta_e = (T\lambda_\Gamma(\gamma)\rho_\Gamma(\gamma))\delta_e = T\delta_e.$$

and this shows that $T\delta_e$, which is a function in $\ell^2(\Gamma)$, is invariant under conjugation by γ . The support of $T\delta_e$ is therefore contained in Γ_{fc} .

Write $T\delta_e = \sum_{[h] \in \mathcal{C}} \frac{c_{[h]}}{\#[h]} \mathbf{1}_{[h]}$ for a sequence $(c_{[h]})_{[h] \in \mathcal{C}} \in \ell^2(\mathcal{C})$, where \mathcal{C} is a set of representatives for the Γ -conjugacy classes in Γ_{fc} .

Let $S \in \{T_{[h]} \mid h \in \Gamma_{\text{fc}}\}'$. Then

$$\begin{aligned}
 ST\delta_e &= S \left(\sum_{[h] \in \mathcal{C}} \frac{c_{[h]}}{\#[h]} \mathbf{1}_{[h]} \right) = S \left(\sum_{[h] \in \mathcal{C}} c_{[h]} T_{[h]} \delta_e \right) \\
 &= \sum_{[h] \in \mathcal{C}} (c_{[h]} T_{[h]} S \delta_e) = \sum_{[h] \in \mathcal{C}} (c_{[h]} T_{[h]} JS^* J \delta_e) \\
 &= JS^* J \left(\sum_{[h] \in \mathcal{C}} c_{[h]} T_{[h]} \delta_e \right) = JS^* J (T \delta_e) \\
 &= T(JS^* J) \delta_e = TS \delta_e
 \end{aligned}$$

and it follows that $ST = TS$. \square

The following well-known corollary shows that Plancherel measure is the Dirac measure at δ_e in the case where Γ is ICC group, that is, when $\Gamma_{\text{fc}} = \{e\}$.

Corollary 4. *Assume that Γ is ICC. Then $L(\Gamma) = \lambda_\Gamma(\Gamma)''$ is a factor.*

3. PROOF OF THEOREM A

Consider a direct integral decomposition $\int_X^\oplus \pi_x d\mu(x)$ of λ_Γ associated to the centre \mathcal{Z} of $L(\Gamma) = \lambda_\Gamma(\Gamma)''$ (see [8.3.2][Dix77]); so, X is a standard Borel space equipped with a probability measure μ and $(\pi_x, \mathcal{H}_x)_{x \in X}$ is measurable field of representations of Γ over X , such that there exists an isomorphism of Hilbert spaces

$$U : \ell^2(\Gamma) \rightarrow \int_X^\oplus \mathcal{H}_x d\mu(x)$$

which transforms λ_Γ into $\int_X^\oplus \pi_x d\mu(x)$ and for which $U\mathcal{Z}U^{-1}$ is the algebra of diagonal operators on $\int_X^\oplus \mathcal{H}_x d\mu(x)$. (Recall that a diagonal operator on $\int_X^\oplus \mathcal{H}_x d\mu(x)$ is an operator of the form $\int_X^\oplus \varphi(x) I_{\mathcal{H}_x} d\mu(x)$ for an essentially bounded measurable function $\varphi : X \rightarrow \mathbf{C}$.)

Then, upon disregarding a subset of X of μ -measure 0, the following holds (see [8.4.1][Dix77]):

- (1) π_x is a factor representation for every $x \in X$;
- (2) π_x and π_y are disjoint for every $x, y \in X$ with $x \neq y$.
- (3) we have $U\lambda_\Gamma(\Gamma)''U^{-1} = \int_X^\oplus \pi_x(\Gamma)'' d\mu(x)$.

Let ρ_Γ be the right regular representation of Γ . Let $\gamma \in \Gamma$. Then $U\rho_\Gamma(\gamma)U^{-1}$ commutes with every diagonalisable operator on $\int_X^\oplus \mathcal{H}_x d\mu(x)$, since $\rho_\Gamma(\gamma) \in L(\Gamma)'$. It follows (see [Dix69, Chap. II, §2, No 5, Théorème 1]) that $U\rho_\Gamma(\gamma)U^{-1}$ is a decomposable operator, that is, there exists a measurable field of unitary operators $x \mapsto \sigma_x(\gamma)$ such that

$U\rho_\Gamma(\gamma)U^{-1} = \int_X^\oplus \sigma_x(\gamma)d\mu(x)$. So, we have a measurable field $x \mapsto \sigma_x$ of representations of Γ in $\int_X^\oplus \mathcal{H}_x d\mu(x)$ such that

$$U\rho_\Gamma(\gamma)U^{-1} = \int_X^\oplus \sigma_x(\gamma)d\mu(x) \quad \text{for all } \gamma \in \Gamma.$$

Let $(\xi_x)_{x \in X} \in \int_X^\oplus \mathcal{H}_x d\mu(x)$ be the image of $\delta_e \in \ell^2(\Gamma)$ under U . We claim that ξ_x is a cyclic vector for π_x and σ_x , for μ -almost every $x \in X$. Indeed, since $\delta_e \in \ell^2(\Gamma)$ is a cyclic vector for both λ_Γ and ρ_Γ ,

$$\{(\pi_x(\gamma)\xi_x)_{x \in X} \mid \gamma \in \Gamma\} \quad \text{and} \quad \{(\sigma_x(\gamma)\xi_x)_{x \in X} \mid \gamma \in \Gamma\}$$

are countable total subsets of $\int_X^\oplus \mathcal{H}_x d\mu(x)$ and the claim follows from a general fact about direct integral of Hilbert spaces (see Proposition 8 in Chap. II, §1 of [Dix69]).

Since $\lambda_\Gamma(\gamma)\delta_e = \rho_\Gamma(\gamma^{-1})\delta_e$ for every $\gamma \in \Gamma$ and since Γ is countable, upon neglecting a subset of X of μ -measure 0, we can assume that

- (4) $\pi_x(\gamma)\xi_x = \sigma_x(\gamma^{-1})\xi_x$;
- (5) $\pi_x(\gamma)\sigma_x(\gamma') = \sigma_x(\gamma')\pi_x(\gamma)$ and
- (6) ξ_x is a cyclic vector for both π_x and σ_x ,

for all $x \in X$ and all $\gamma, \gamma' \in \Gamma$.

Let $x \in X$ and let φ_x be the function of positive type on Γ defined by

$$\varphi_x(\gamma) = \langle \pi_x(\gamma)\xi_x \mid \xi_x \rangle \quad \text{for every } \gamma \in \Gamma.$$

We claim that $\varphi_x \in \text{Ch}(\Gamma)$. Indeed, φ_x is a function of positive type. Using (4) and (5), we have, for every $\gamma_1, \gamma_2 \in \Gamma$,

$$\begin{aligned} \varphi_x(\gamma_2\gamma_1\gamma_2^{-1}) &= \langle \pi_x(\gamma_2\gamma_1\gamma_2^{-1})\xi_x \mid \xi_x \rangle = \langle \pi_x(\gamma_2\gamma_1)\sigma_x(\gamma_2)\xi_x \mid \xi_x \rangle \\ &= \langle \sigma_x(\gamma_2)\pi_x(\gamma_2\gamma_1)\xi_x \mid \xi_x \rangle = \langle \pi_x(\gamma_1)\xi_x \mid \pi_x(\gamma_2^{-1})\sigma_x(\gamma_2^{-1})\xi_x \rangle \\ &= \langle \pi_x(\gamma_1)\xi_x \mid \xi_x \rangle = \varphi_x(\gamma_1). \end{aligned}$$

So, φ_x is conjugation invariant and hence $\varphi_x \in \text{Tr}(\Gamma)$. Moreover, φ_x is an extreme point in $\text{Tr}(\Gamma)$, since π_x is factorial and ξ_x is a cyclic vector for π_x .

Finally, since $U: \ell^2(\Gamma) \rightarrow \int_X^\oplus \mathcal{H}_x d\mu(x)$ is an isometry, we have for every $f \in \mathbf{C}[\Gamma]$,

$$\begin{aligned} \|f\|^2 &= f^* * f(e) = \langle \lambda_\Gamma(f^* * f)\delta_e \mid \delta_e \rangle \\ &= \|\lambda_\Gamma(f)\delta_e\|^2 = \|U(\lambda_\Gamma(f)\delta_e)\|^2 \\ &= \int_X \|\pi_x(f)\xi_x\|^2 d\mu(x) = \int_X \varphi_x(f^* * f) d\mu(x). \end{aligned}$$

The measurable map $\Phi: X \rightarrow \text{Ch}(\Gamma)$ given by $\Phi(x) = \varphi_x$ is injective, since π_x and π_y are disjoint by (2) and hence $\varphi_x \neq \varphi_y$ for $x, y \in X$ with

$x \neq y$. It follows that $\Phi(X)$ is a Borel subset of $\text{Ch}(\Gamma)$ and that Φ is a Borel isomorphism between X and $\Phi(X)$ (see [Mac57, Theorem 3.2]). Pushing forward μ to $\text{Ch}(\Gamma)$ by Φ , we can therefore assume without loss of generality that $X = \text{Ch}(\Gamma)$ and that μ is a probability measure on $\text{Ch}(\Gamma)$. With this identification, it is clear that Items (i), (ii) and (iii) of Theorem A are satisfied and that the Plancherel formula holds.

It remains to show the uniqueness of μ . Let ν any probability measure on $\text{Ch}(\Gamma)$ such that the Plancherel formula. By polarization, we have then $\delta_e = \int_{\text{Ch}(\Gamma)} t d\nu(t)$, which is an integral decomposition of $\delta_e \in \text{Tr}(\Gamma)$ over extreme points of the convex set $\text{Tr}(\Gamma)$. The uniqueness of such a decomposition implies that $\nu = \mu$.

Remark 5. (i) For μ -almost every $t \in \text{Ch}(\Gamma)$, we have $t = 0$ on $\Gamma \setminus \Gamma_{\text{fc}}$. Indeed, let $\gamma \notin \Gamma_{\text{fc}}$. Then $\langle \lambda_\Gamma(\gamma) \lambda_\Gamma(h) \delta_e \mid \delta_e \rangle = 0$ for every $h \in \Gamma_{\text{fc}}$ and hence

$$(*) \quad \langle \lambda_\Gamma(\gamma) T \delta_e \mid \delta_e \rangle = 0 \quad \text{for all } T \in \lambda_\Gamma(\Gamma_{\text{fc}})''.$$

With the notation as above, let E be a Borel subset of X . Then $T_E := U^{-1} P_E U$ is a projection in \mathcal{Z} , where P_E is the diagonal operator $\int_X^\oplus \mathbf{1}_E(x) I_{\mathcal{H}_x} d\mu(x)$. It follows from Lemma 3 and (*) that

$$\langle \lambda_\Gamma(\gamma) T_E \delta_e \mid \delta_e \rangle = \langle T_E \lambda_\Gamma(\gamma) \delta_e \mid \delta_e \rangle = 0$$

and this means that $\int_E \varphi_x(\gamma) d\mu(x) = 0$. Since this holds for every Borel subset E of X , this implies that $\varphi_x(\gamma) = 0$ for μ -almost every $x \in X$.

As Γ is countable, for μ -almost every $x \in X$, we have $\varphi_x(\gamma) = 0$ for every $\gamma \notin \Gamma_{\text{fc}}$.

(ii) Let $\lambda_\Gamma = \int_{\text{Ch}(\Gamma)}^\oplus \pi_t d\mu(t)$, $\rho_\Gamma = \int_{\text{Ch}(\Gamma)}^\oplus \sigma_t d\mu(t)$, and $\delta_e = (\xi_t)_{t \in \text{Ch}(\Gamma)}$ be the decompositions as above. For μ -almost every $t \in \text{Ch}(\Gamma)$, the linear map

$$\pi_t(\Gamma)'' \rightarrow \mathcal{H}_t, T \mapsto T \xi_t$$

is injective. Indeed, this follows from the fact that ξ_t is cyclic for σ_t and that $\sigma_t(\Gamma) \subset \pi_t(\Gamma)'$.

4. PROOF OF THEOREM B

Set $N := \Gamma_{\text{fc}}$ and $X := \text{Ch}(N)$. Consider the direct integral decomposition $\int_X^\oplus \pi_t d\nu(t)$ of λ_N into factor representations (π_t, \mathcal{K}_t) of N with corresponding traces $t \in X$, as in Theorem A.

Let K_Γ be the compact group which is the closure in $\text{Aut}(N)$ of $\text{Ad}(\Gamma)|_N$. Since the quotient space X/K_Γ is a standard Borel space, there exists a Borel section $s : X/K_\Gamma \rightarrow X$ for the projection map $X \rightarrow X/K_\Gamma$. Set $\Omega := s(X/K_\Gamma)$. Then Ω is a Borel transversal for

X/K_Γ . The Plancherel measure ν can accordingly be decomposed over Ω : we have

$$\nu(f) = \int_{\Omega} \int_{\mathcal{O}_\omega} f(t) dm_\omega(t) d\nu(\omega)$$

for every bounded measurable function f on $\text{Ch}(\Gamma_{\text{fc}})$, where m_ω be the unique normalized K_Γ -invariant probability measure on the K_Γ -orbit \mathcal{O}_ω of ω and $\dot{\nu}$ is the image of ν under s .

For every $\omega \in \Omega$, set

$$\pi_{\mathcal{O}_\omega} := \int_{\mathcal{O}_\omega}^{\oplus} \pi_t dm_\omega(t).$$

which is a unitary representation of N on the Hilbert space

$$\mathcal{K}_\omega := \int_{\mathcal{O}_\omega}^{\oplus} \mathcal{K}_t dm_\omega(t).$$

For $g \in \text{Aut}(N)$, let $\pi_{\mathcal{O}_\omega}^g$ be the conjugate representation of N on \mathcal{K}_ω given by $\pi_{\mathcal{O}_\omega}^g(h) = \pi_{\mathcal{O}_\omega}(g(h))$ for $h \in N$.

Step 1 There exists a unitary representation $U_\omega : g \mapsto U_{\omega,g}$ of K_Γ on \mathcal{K}_ω such that

$$U_{\omega,g} \pi_{\mathcal{O}_\omega}(h) U_{\omega,g}^{-1} = \pi_{\mathcal{O}_\omega}(g(h)) \quad \text{for all } g \in K_\Gamma, h \in N.$$

in particular, $\pi_{\mathcal{O}_\omega}^g$ is equivalent to $\pi_{\mathcal{O}_\omega}$ for every $g \in K_\Gamma$.

Indeed, observe that the representations π_t for $t \in \mathcal{O}_\omega$ are conjugate to each other (up to equivalence) and may therefore be considered as defined on the same Hilbert space.

Let $g \in K_\Gamma$. Then $\pi_{\mathcal{O}_\omega}^g$ is equivalent to $\int_{\mathcal{O}_\omega}^{\oplus} \pi_t^g dm_\omega(t)$. Define a linear operator $U_{\omega,g} : \mathcal{K}_\omega \rightarrow \mathcal{K}_\omega$ by

$$U_{\omega,g}((\xi_t)_{t \in \mathcal{O}_\omega}) = (\xi_{t^g})_{t \in \mathcal{O}_\omega} \quad \text{for all } (\xi_t)_{t \in \mathcal{O}_\omega} \in \mathcal{K}_\omega.$$

Then $U_{\omega,g}$ is an isometry, by K_Γ -invariance of the measure m_ω . It is readily checked that U_ω intertwines $\pi_{\mathcal{O}_\omega}$ and $\pi_{\mathcal{O}_\omega}^g$ and that U_ω is a homomorphism. To show that U_ω is a representation of K_Γ , it remains to prove that $g \mapsto U_{\omega,g} \xi$ is continuous for every $\xi \in \mathcal{K}_\omega$.

For this, observe that \mathcal{K}_ω can be identified with the Hilbert space $L^2(\mathcal{O}_\omega, m_\omega) \otimes \mathcal{K}$, where \mathcal{K} is the common Hilbert space of the π_t 's for $t \in \mathcal{O}_\omega$; under this identification, U_ω corresponds to $\kappa \otimes I_{\mathcal{K}}$, where κ is the Koopman representation of K_Γ on $L^2(\mathcal{O}_\omega, m_\omega)$ associated to the action $K_\Gamma \curvearrowright \mathcal{O}_\omega$ (for the fact that κ is indeed a representation of K_Γ , see [BHV08, A.6]) and the claim follows.

Next, let

$$\widetilde{\pi_{\mathcal{O}_\omega}} := \text{Ind}_N^\Gamma \pi_{\mathcal{O}_\omega}$$

be the representation of Γ induced by $\pi_{\mathcal{O}_\omega}$.

We recall how $\widetilde{\pi_{\mathcal{O}_\omega}}$ can be realized on $\ell^2(R, \mathcal{K}_\omega) = \ell^2(R) \otimes \mathcal{K}_\omega$, where $R \subset \Gamma$ is a set of representatives for the cosets of N with $e \in R$. For every $\gamma \in \Gamma$ and $r \in R$, let $c(r, \gamma) \in N$ and $r \cdot \gamma \in R$ be such that $r\gamma = c(r, \gamma)r \cdot \gamma$. Then $\widetilde{\pi_{\mathcal{O}_\omega}}$ is given on $\ell^2(R, \mathcal{K}_\omega)$ by

$$(\widetilde{\pi_{\mathcal{O}_\omega}}(\gamma)F)(r) = \pi_{\mathcal{O}_\omega}(c(r, \gamma))(F(r \cdot \gamma)) \quad \text{for all } F \in \ell^2(R, \mathcal{K}_\omega).$$

Step 2 We claim that there exists a unitary map

$$\widetilde{U}_\omega : \ell^2(R, \mathcal{K}_\omega) \rightarrow \ell^2(R, \mathcal{K}_\omega)$$

which intertwines the representation $I_{\ell^2(R)} \otimes \pi_{\mathcal{O}_\omega}$ and the restriction $\widetilde{\pi_{\mathcal{O}_\omega}}|_N$ of $\widetilde{\pi_{\mathcal{O}_\omega}}$ to N ; moreover, $\omega \rightarrow \widetilde{U}_\omega$ is a measurable field of unitary operators on Ω .

Indeed, we have an orthogonal decomposition

$$\ell^2(R, \mathcal{K}_\omega) = \bigoplus_{r \in R} (\delta_r \otimes \mathcal{K}_\omega)$$

into $\widetilde{\pi_{\mathcal{O}_\omega}}(N)$ -invariant; moreover, the action of N on every copy $\delta_r \otimes \mathcal{K}_\omega$ is given by $\pi_{\mathcal{O}_\omega}^r$. For every $r \in R$, the unitary operator $U_{\omega, r} : \mathcal{K}_\omega \rightarrow \mathcal{K}_\omega$ from Step 1 intertwines $\pi_{\mathcal{O}_\omega}$ and $\pi_{\mathcal{O}_\omega}^r$. In view of the explicit formula of $U_{\omega, r}$, the field $\omega \rightarrow U_{\omega, r}$ is measurable on Ω .

Define a unitary operator $\widetilde{U}_\omega : \ell^2(R, \mathcal{K}_\omega) \rightarrow \ell^2(R, \mathcal{K}_\omega)$ by

$$\widetilde{U}_\omega(\delta_r \otimes \xi) = \delta_r \otimes U_{\omega, r}(\xi) \quad \text{for all } \xi \in \mathcal{K}_\omega.$$

Then \widetilde{U}_ω intertwines $I_{\ell^2(R)} \otimes \pi_{\mathcal{O}_\omega}$ and $\widetilde{\pi_{\mathcal{O}_\omega}}|_N$; moreover, $\omega \rightarrow \widetilde{U}_\omega$ is a measurable field on Ω .

Observe that the representation λ_N is equivalent to $\int_\Omega^\oplus \pi_{\mathcal{O}_\omega} d\nu(\omega)$. Since λ_Γ is equivalent to $\text{Ind}_N^\Gamma \lambda_N$, it follows that λ_Γ is equivalent to $\int_\Omega^\oplus \widetilde{\pi_{\mathcal{O}_\omega}} d\nu(\omega)$.

In the sequel, we will identify the representations λ_N on $\ell^2(N)$ and λ_Γ on $\ell^2(\Gamma)$ with respectively the representations

$$\int_\Omega^\oplus \pi_{\mathcal{O}_\omega} d\nu(\omega) \quad \text{on } \mathcal{K} := \int_\Omega^\oplus \mathcal{K}_\omega d\nu(\omega)$$

and

$$\int_\Omega^\oplus \widetilde{\pi_{\mathcal{O}_\omega}} d\nu(\omega) \quad \text{on } \mathcal{H} := \int_\Omega^\oplus \ell^2(R, \mathcal{K}_\omega) d\nu(\omega).$$

Step 3 The representations $\widetilde{\pi}_{\mathcal{O}_\omega}$ are factorial and are mutually disjoint, outside a subset of Ω of ν -measure 0.

To show this, it suffices to prove (see [Dix77, 8.4.1]) that the algebra \mathcal{D} of diagonal operators in $\mathcal{L}(\mathcal{H})$ coincides with the center \mathcal{Z} of $\lambda_\Gamma(\Gamma)''$. For this, we only have to prove that $\mathcal{D} \subset \lambda_\Gamma(\Gamma)''$, since it is clear that $\mathcal{D} \subset \lambda_\Gamma(\Gamma)'$.

By Step 2, for every $\omega \in \Omega$, there exists a measurable field $\omega \rightarrow \widetilde{U}_\omega$ of unitary operators $\widetilde{U}_\omega : \ell^2(R, \mathcal{K}_\omega) \rightarrow \ell^2(R, \mathcal{K}_\omega)$ intertwining $I_{\ell^2(R)} \otimes \pi_{\mathcal{O}_\omega}$ and $\pi_{\mathcal{O}_\omega}|_N$. So,

$$\widetilde{U} := \int_{\Omega}^{\oplus} \widetilde{U}_\omega d\nu(\omega)$$

is a unitary operator on \mathcal{H} which intertwines $I_{\ell^2(R)} \otimes \lambda_N$ and $\lambda_\Gamma|_N$; it is obvious that \widetilde{U} commutes with the diagonal operators on \mathcal{H} .

Let $\varphi : \Omega \rightarrow \mathbf{C}$ be a measurable essential bounded function on Ω . By Theorem A.ii, the corresponding diagonal operator

$$T = \int_{\Omega}^{\oplus} \varphi(\omega) I_{\mathcal{K}_\omega} d\nu(\omega)$$

on \mathcal{K} belongs to $\lambda_N(N)''$. For the corresponding diagonal operator

$$\widetilde{T} = \int_{\Omega}^{\oplus} \varphi(\omega) I_{\ell^2(R, \mathcal{K}_\omega)} d\nu(\omega)$$

on \mathcal{H} , we have $\widetilde{T} = I_{\ell^2(R)} \otimes T$. So, \widetilde{T} belongs to $(I_{\ell^2(R)} \otimes \lambda_N)(N)''$. Since, \widetilde{U} commutes with \widetilde{T} and intertwines $I_{\ell^2(R)} \otimes \lambda_N$ and $\lambda_\Gamma|_N$, it follows that

$$\widetilde{T} = \widetilde{U}(I_{\ell^2(R)} \otimes T)\widetilde{U}^{-1} \in \lambda_\Gamma(N)'' \subset \lambda_\Gamma(\Gamma)''.$$

As a result, we have a decomposition

$$\lambda_\Gamma = \int_{\Omega}^{\oplus} \widetilde{\pi}_{\mathcal{O}_\omega} d\nu(\omega)$$

of λ_Γ as a direct integral of pairwise disjoint factor representations. By the argument of the proof of Theorem A, it follows that, for ν -almost every $\omega \in X$, there exists a cyclic unit vector $\xi_\omega \in \ell^2(R, \mathcal{K}_\omega)$ for $\widetilde{\pi}_{\mathcal{O}_\omega}$ so that

$$\varphi_\omega := \langle \widetilde{\pi}_{\mathcal{O}_\omega}(\cdot)\xi_\omega \mid \xi_\omega \rangle$$

belongs to $\text{Ch}(\Gamma)$. In particular,

$$\widetilde{\mathcal{M}}_\omega := \widetilde{\pi}_{\mathcal{O}_\omega}(\Gamma)''$$

is a factor of type II_1 and its normalized trace is the extension of φ_ω to $\widetilde{\mathcal{M}}_\omega$, which we again denote by φ_ω . Our next goal is to determine φ_ω in terms of the character $\omega \in \text{Ch}(N)$.

Fix $\omega \in \Omega$ such that $\widetilde{\mathcal{M}}_\omega$ is a factor. We identify the Hilbert space \mathcal{K}_ω of $\pi_{\mathcal{O}_\omega}$ with the subspace $\delta_e \otimes \mathcal{K}_\omega$ and so $\pi_{\mathcal{O}_\omega}$ with a subrepresentation of the restriction of $\widetilde{\pi_{\mathcal{O}_\omega}}$ to N .

Let $\eta_\omega \in \mathcal{K}_\omega$ be a cyclic vector for $\pi_{\mathcal{O}_\omega}$ such that

$$\omega = \langle \pi_{\mathcal{O}_\omega}(\cdot)\eta_\omega \mid \eta_\omega \rangle.$$

For $g \in K_\Gamma$, define a normal state $\psi_{\omega,g}$ on $\widetilde{\mathcal{M}}_\omega$ by the formula

$$\psi_{\omega,g}(\widetilde{T}) = \langle \widetilde{T}U_{g,\omega}^{-1}\eta_\omega \mid U_{g,\omega}^{-1}\eta_\omega \rangle \quad \text{for all } \widetilde{T} \in \widetilde{\mathcal{M}}_\omega,$$

where $U_{g,\omega}$ is the unitary operator on \mathcal{K}_ω from Step 1.

Consider the linear functional $\psi_\omega : \widetilde{\mathcal{M}}_\omega \rightarrow \mathbf{C}$ given by

$$\psi_\omega(\widetilde{T}) = \int_{K_\Gamma} \psi_{\omega,g}(\widetilde{T})dm(g) \quad \text{for all } \widetilde{T} \in \widetilde{\mathcal{M}}_\omega,$$

where m is the normalized Haar measure on K_Γ .

Step 4 We claim that ψ_ω is a normal K_Γ -invariant trace on $\widetilde{\mathcal{M}}_\omega$

Indeed, the fact that ψ_ω is a K_Γ -invariant state on $\widetilde{\mathcal{M}}_\omega$ is clear. To show that ψ_ω is normal, let $(\widetilde{T}_n)_n$ be an increasing sequence of positive operators in $\widetilde{\mathcal{M}}_\omega$ with $\widetilde{T} = \sup_n \widetilde{T}_n \in \widetilde{\mathcal{M}}_\omega$.

For every $g \in K_\Gamma$, the sequence $(\psi_{\omega,g}(\widetilde{T}_n))_n$ is increasing and its limit is $\psi_{\omega,g}(\widetilde{T})$. It follows from the monotone convergence theorem that

$$\lim_n \psi_\omega(\widetilde{T}_n) = \lim_n \int_{K_\Gamma} \psi_{\omega,g}(\widetilde{T}_n)dm(g) = \int_{K_\Gamma} \psi_{\omega,g}(\widetilde{T})dm(g) = \psi_\omega(\widetilde{T}).$$

So, ψ_ω is normal, as $\widetilde{\mathcal{M}}_\omega$ acts on a separable Hilbert space. Moreover, ψ_ω is a trace on $\widetilde{\mathcal{M}}_\omega$, since it is $\text{Ad}(\Gamma)$ -invariant and normal.

Step 5 We claim that ψ_ω coincides with the trace φ_ω on $\widetilde{\mathcal{M}}_\omega$ from above. Moreover, writing γ instead of $\widetilde{\pi_{\mathcal{O}_\omega}}(\gamma)$ for $\gamma \in \Gamma$, we have

$$\psi_\omega(\gamma) = \begin{cases} \int_{K_\Gamma} \omega^g(\gamma)dm(g) & \text{if } \gamma \in N \\ 0 & \text{if } \gamma \notin N. \end{cases}$$

Indeed, $\widetilde{\mathcal{M}}_\omega$ is a factor; so, the fact that $\widetilde{\psi}_\omega = \varphi_\omega$ follows from the uniqueness of normal traces on factors (see Corollaire p. 92 and Corollaire 2 p.83 in [Dix69, Chap. I, §6]).

Let $\gamma \in N$. Since

$$U_{\omega,g}\pi_{\mathcal{O}_\omega}(\gamma)U_{\omega,g}^{-1} = \pi_{\mathcal{O}_\omega}(g(\gamma)),$$

we have

$$\begin{aligned} \psi_\omega(\gamma) &= \int_{K_\Gamma} \langle \tilde{\pi}_{\mathcal{O}_\omega}(g(\gamma))\eta_\omega \mid \eta_\omega \rangle dm(g) = \int_{K_\Gamma} \langle \pi_{\mathcal{O}_\omega}(g(\gamma))\eta_\omega \mid \eta_\omega \rangle dm(g) \\ &= \int_{K_\Gamma} \omega^g(\gamma) dm(g). \end{aligned}$$

Let $\gamma \in \Gamma \setminus N$. Then, by the usual properties of an induced representation, $\tilde{\pi}_{\mathcal{O}_\omega}(\gamma)(\mathcal{K}_\omega)$ is orthogonal to \mathcal{K}_ω . It follows that

$$\psi_{\omega,g}(\gamma) = \langle \tilde{\pi}_{\mathcal{O}_\omega}(\gamma)U_{g,\omega}^{-1}\eta_\omega \mid U_{g,\omega}^{-1}\eta_\omega \rangle = 0$$

and hence $\psi_\omega(\gamma) = 0$.

Step 6 The Plancherel measure μ on Γ is the image of ν under the map Φ as in the statement of Theorem B.ii.

Indeed, for $f \in \mathbf{C}[\Gamma]$, we have

$$\begin{aligned} \|f\|_2^2 &= \int_\Omega \|\widetilde{\pi_{\mathcal{O}_\omega}}(f)\|^2 d\nu(\omega) = \int_\Omega \psi_\omega(f^* * f) d\nu(\omega) \\ &= \int_\Omega \int_{K_\Gamma} \psi_{\omega,g}(f^* * f) dm(g) d\nu(\omega) = \int_\Omega \int_{K_\Gamma} \psi_{\omega,g}((f^* * f)|_N) dm(g) d\nu(\omega) \\ &= \int_\Omega \int_{K_\Gamma} \omega^g((f^* * f)|_N) dm(g) d\nu(\omega) \\ &= \int_\Omega \int_{\mathcal{O}_\omega} \left(\int_{K_\Gamma} \omega^g((f^* * f)|_N) dm(g) \right) dm_\omega(t) d\nu(\omega) \\ &= \int_{\text{Ch}(\Gamma_{\text{fc}})} \int_{K_\Gamma} t^g((f^* * f)|_N) d\nu(t) \\ &= \int_{\text{Ch}(\Gamma)} \Phi(t)(f^* * f) d\nu(t) \end{aligned}$$

and the claim follows.

Remark 6. The map Φ in Theorem B.ii can be described without reference to the group K_Γ as follows. For $t \in \text{Ch}(\Gamma_{\text{fc}})$ and $\gamma \in \Gamma$, we have

$$\Phi(t)(\gamma) = \begin{cases} \frac{1}{\#[\gamma]} \sum_{x \in [\gamma]} t(\gamma) & \text{if } \gamma \in \Gamma_{\text{fc}} \\ 0 & \text{if } \gamma \notin \Gamma_{\text{fc}} \end{cases},$$

where $[\gamma]$ denotes the Γ -conjugacy class of γ . Indeed, it suffices to consider the case where $\gamma \in \Gamma_{\text{fc}}$. The stabilizer K_0 of γ in K_Γ is an

open and hence cofinite subgroup of K_Γ ; in particular, the K_Γ -orbit of γ coincides with the $\text{Ad}(\Gamma)$ -orbit of γ and so $\{\text{Ad}(x) \mid x \in [\gamma]\}$ is a system of representatives for K_Γ/K_0 . Let m_0 be the normalized Haar measure on K_0 . The normalized Haar measure m on K_Γ is then given by $m(f) = \frac{1}{[\gamma]} \sum_{x \in [\gamma]} \int_{K_0} f(\text{Ad}(x^{-1}g)) dm_0(g)$ for every continuous function f on K_Γ . It follows that

$$\Phi(t)(\gamma) = \int_{K_\Gamma} t(g^{-1}(\gamma)) dm(g) = \frac{1}{[\gamma]} \sum_{x \in [\gamma]} t(x).$$

5. PROOFS OF COROLLARY C AND COROLLARY D

Let Γ be a countable linear group. So, Γ is a subgroup of $GL_n(\mathbf{k})$ for a field \mathbf{k} , which may be assumed to be algebraically closed. Let \mathbf{G} be the closure of Γ in the Zariski topology of $GL_n(\mathbf{k})$ and let \mathbf{G}_0 be the irreducible component of \mathbf{G} . As is well-known, \mathbf{G}_0 has finite index in \mathbf{G} and hence $\Gamma_0 := \mathbf{G}_0 \cap \Gamma$ is a normal subgroup of finite index in Γ .

Let $\gamma \in \Gamma_{\text{fc}}$. On the one hand, the centralizer Γ_γ of γ in Γ is a subgroup of finite index of Γ ; therefore, the irreducible component of the Zariski closure of Γ_γ coincides with \mathbf{G}_0 . On the other hand, the centralizer \mathbf{G}_γ of γ in \mathbf{G} is clearly a Zariski-closed subgroup of \mathbf{G} . It follows that the irreducible component of \mathbf{G}_γ contains (in fact coincides with) \mathbf{G}_0 and hence

$$\Gamma_0 = \mathbf{G}_0 \cap \Gamma \subset \mathbf{G}_\gamma \cap \Gamma = \Gamma_\gamma.$$

As a consequence, we see that Γ_0 acts trivially on Γ_{fc} and hence $\text{Ad}(\Gamma)|_{\Gamma_{\text{fc}}}$ is a finite group. In particular, $\Gamma_0 \cap \Gamma_{\text{fc}}$ is contained in the center $Z(\Gamma_{\text{fc}})$ of Γ_{fc} ; so $Z(\Gamma_{\text{fc}})$ has finite index in Γ_{fc} which is therefore a central group. This proves Items (i) and (ii) of Corollary C. Item (iii) follows from Theorem B.ii.

Assume now that \mathbf{G} is connected, that is $\mathbf{G} = \mathbf{G}_0$. Then $\Gamma = \Gamma_0$ acts trivially on Γ_{fc} and so Γ_{fc} coincides with the center $Z(\Gamma)$ of Γ . This proves the first part of Corollary D.

It remains to prove that the assumption $\mathbf{G} = \mathbf{G}_0$ is satisfied in Cases (i) and (ii) of Corollary D:

- (i) Let \mathbf{G} be a connected linear algebraic group over a countable field \mathbf{k} of characteristic 0. Then $\Gamma = \mathbf{G}(\mathbf{k})$ is Zariski dense in \mathbf{G} , by [Ros57, Corollary p.44].
- (ii) Let \mathbf{G} be a connected linear algebraic group \mathbf{G} over a local field \mathbf{k} . Assume that \mathbf{G} has no proper \mathbf{k} -subgroup \mathbf{H} such that

$(\mathbf{G}/\mathbf{H})(\mathbf{k})$ is compact. Then every lattice Γ in $\mathbf{G}(\mathbf{k})$ is Zariski dense in \mathbf{G} , by [Sha99, Corollary 1.2].

6. THE PLANCHEREL FORMULA FOR SOME COUNTABLE GROUPS

6.1. Restricted direct product of finite groups. Let $(G_n)_{n \geq 1}$ be a sequence of finite groups. Let $\Gamma = \prod'_{n \geq 1} G_n$ be the **restricted direct product** of the G_n 's, that is, Γ consists of the sequences $(g_n)_{n \geq 1}$ with $g_n \in G_n$ for all n and $g_n \neq e$ for at most finitely many n .

Set $X_n := \text{Ch}(G_n)$ for $n \geq 1$ and let $X = \prod_{n \geq 1} X_n$ be the cartesian product equipped with the product topology, where each X_n carries the discrete topology. Define a map $\Phi : X \rightarrow \text{Tr}(\Gamma)$ by

$$\Phi((t_n)_{n \geq 1})((g_n)_{n \geq 1}) = \prod_{n \geq 1} t_n(g_n) \quad \text{for all } (t_n)_{n \geq 1} \in X, (g_n)_{n \geq 1} \in \Gamma$$

(observe that this product is well-defined since $g_n = e$ and hence $t_n(g_n) = 1$ for almost every $n \geq 1$). Then $\Phi(X) = \text{Ch}(\Gamma)$ and $\Phi : X \rightarrow \text{Ch}(\Gamma)$ is a homeomorphism (see [Mau51, Lemma 7.1])

For every $n \geq 1$, let ν_n be the measure on X_n given by

$$\nu_n(\{t\}) = \frac{d_t^2}{\#G_n} \quad \text{for all } t \in X_n,$$

where d_t is the dimension of the irreducible representation of G_n with t as character; observe that ν_n is a probability measure, since $\sum_{t \in X_n} d_t^2 = \#G_n$.

Let $\nu = \otimes_{n \geq 1} \nu_n$ be the product measure on the Borel subsets of X . The Plancherel measure on Γ is the image of ν under Φ (see Equation (5.6) in [Mau51]). The regular representation λ_Γ is of type II if and only if infinitely many G_n 's are non abelian (see *loc.cit.*, Theorem 1 or Theorem E below).

6.2. Infinite dimensional Heisenberg group. Let \mathbf{F}_p be the field of order p for an odd prime p and let $V = \oplus_{i \in \mathbf{N}} \mathbf{F}_p$ be a vector space over \mathbf{F}_p of countable infinite dimension. Denote by ω the symplectic form on $V \oplus V$ given by

$$\omega((x, y), (x', y')) = \sum_{i \in \mathbf{N}} (x_i y'_i - y_i x'_i) \quad \text{for } (x, y), (x', y') \in V \oplus V.$$

The ‘‘infinite dimensional’’ Heisenberg group over \mathbf{F}_p is the group Γ with underlying set $V \oplus V \oplus \mathbf{F}_p$ and with multiplication defined by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \omega((x, y), (x', y')))$$

for $(x, y, z), (x', y', z') \in \Gamma$.

The group Γ is an FC-group; since $p \geq 3$, its center Z coincides with $[\Gamma, \Gamma]$ and consists of the elements of the form $(0, 0, z)$ for $z \in \mathbf{F}_p$. Observe that Γ is not virtually abelian.

Let z_0 be a generator for the cyclic group Z of order p . The unitary dual \widehat{Z} consists of the unitary characters defined by $\chi_\omega(z_0^j) = \omega^j$ for $j \in \{0, 1, \dots, p-1\}$ and $\omega \in C_p$, where C_p is the group of p -th roots of unity in \mathbf{C} .

For $\omega \in C_p$, the subspace

$$\mathcal{H}_\omega = \{f \in \ell^2(\Gamma) \mid f(z_0x) = \omega f(x) \text{ for every } x \in \Gamma\}.$$

is left and right translation invariant and we have an orthogonal decomposition of $\ell^2(\Gamma) = \bigoplus_{\omega \in C_p} \mathcal{H}_\omega$. The orthogonal projection P_ω on \mathcal{H}_ω belongs to the center of $L(\Gamma)$ and is given by

$$P_\omega(f)(x) = \frac{1}{p} \sum_{i=0}^{p-1} \omega^{-i} f(z_0^i x) \quad \text{for all } f \in \ell^2(\Gamma), x \in \Gamma.$$

One checks that $\|P_\omega(\delta_e)\|^2 = 1/p$.

Let π_ω be the restriction of λ_Γ to \mathcal{H}_ω . Observe that \mathcal{H}_1 can be identified with $\ell^2(\Gamma/Z)$ and π_1 with $\lambda_{\Gamma/Z}$.

For $\omega \neq 1$, the representation π_ω is factorial of type II_1 and the corresponding character is $\widetilde{\chi}_\omega$ (for more details, see the proof of Theorem 7.D.4 in [BH]).

The integral decomposition of δ_e is

$$\delta_e = \frac{1}{p} \int_{\widehat{\Gamma/Z}} \chi d\nu(\chi) + \frac{1}{p} \sum_{\omega \in C_p \setminus \{1\}} \widetilde{\chi}_\omega,$$

with the corresponding Plancherel formula for every $f \in \mathbf{C}[\Gamma]$

$$\|f\|_2^2 = \frac{1}{p} \int_{\widehat{\Gamma/Z}} |\mathcal{F}(P_1(f))(\chi)|^2 d\nu(\chi) + \frac{1}{p} \sum_{\omega \in C_p \setminus \{1\}} \sum_{j=0}^{p-1} (f^* * f)(z_0^j) \omega^j,$$

where ν is the normalized Haar measure of the compact abelian group $\widehat{\Gamma/Z}$ and \mathcal{F} the Fourier transform. In particular, $\lambda_\Gamma(\Gamma)''$ is a direct sum of an abelian von Neumann algebra and $p-1$ factors of type II_1 . For a more general result, see [Kap51, Theorem 2].

6.3. An example involving $SL_d(\mathbf{Z})$. Let $\Lambda = SL_d(\mathbf{Z})$ for an odd integer d . Fix a prime p and for $n \geq 1$, let $G_n = SL_3(\mathbf{Z}/p^n\mathbf{Z})$, viewed as (finite) quotient of Λ . Let Γ be the semi-direct product $\Lambda \ltimes \prod'_{n \geq 1} G_n$, where Λ acts diagonally in the natural way on the restricted direct product $G := \prod'_{n \geq 1} G_n$ of the G_n 's.

Since Λ is an ICC-group, it is clear that $\Gamma_{\text{fc}} = G$. The group K_Γ as in Theorem B can be identified with the projective limit of the groups G_n 's, that is, with $SL_d(\mathbf{Z}_p)$, where \mathbf{Z}_p is the ring of p -adic integers. Since Λ acts trivially on $\text{Ch}(G)$, the same is true for the action of K_Γ on $\text{Ch}(G)$.

Let $\lambda_G = \int_{\text{Ch}(G)}^\oplus \pi_t d\nu(t)$ be the Plancherel decomposition of λ_G (see Example 6.1). It follows from Theorem B that the Plancherel decomposition of λ_Γ is

$$\lambda_\Gamma = \int_{\text{Ch}(G)}^\oplus \text{Ind}_G^\Gamma \pi_t d\nu(t).$$

APPENDIX A. PROOF OF THEOREM E

A.1. Easy implication: virtually abelian groups are type I. Let Γ be a group containing an abelian normal subgroup N of finite index. Let (π, \mathcal{H}) be a factorial representation of Γ . We have to show that π is a multiple of an irreducible representation.

Denote by \mathcal{B} the set of Borel subsets of the dual group \widehat{N} and by $\text{Proj}(\mathcal{H})$ the set of orthogonal projections in $\mathcal{L}(\mathcal{H})$. Let $E: \mathcal{B}(\widehat{N}) \rightarrow \text{Proj}(\mathcal{H})$ be the projection-valued measure on \widehat{N} associated with the restriction $\pi|_N$ by the SNAG Theorem (see [BHV08, D.3.1]); so, we have

$$\pi(n) = \int_{\widehat{G}} \chi(n) dE(\chi) \quad \text{for all } n \in N.$$

The commutant $\pi(N)'$ coincides with the commutant of $\{E(B) \mid B \in \mathcal{B}\}$; in particular $E(B) \in \pi(\Gamma)''$ for every $B \in \mathcal{B}$.

The dual action of Γ on \widehat{N} , given by $\chi^\gamma(n) = \chi(\gamma^{-1}n\gamma)$ for $\chi \in \widehat{N}$ and $\gamma \in \Gamma$, factorizes through Γ/N . Moreover, the following covariance relation holds

$$\pi(\gamma)E(B)\pi(\gamma^{-1}) = E(B^\gamma) \quad \text{for all } B \in \mathcal{B}(\widehat{N}),$$

where $B^\gamma = \{\chi^\gamma \mid \chi \in B\}$.

Let $S \in \mathcal{B}(\widehat{N})$ be the support of E , that is, S is the complement of the largest open subset U of \widehat{N} with $E(U) = 0$. We claim that S consists of a single Γ -orbit.

Indeed, let $\chi_0 \in S$ and let $(U_n)_{n \geq 1}$ be a sequence of open neighbourhoods of χ_0 with $\bigcap_{n \geq 1} U_n = \{\chi_0\}$. Fix $n \geq 1$. The set U_n^Γ is Γ -invariant and hence $E(U_n^\Gamma) \in \pi(\Gamma)'$, by the covariance relation. As $E(U_n^\Gamma) \in \pi(\Gamma)''$, it follows that the projection $E(U_n^\Gamma)$ belongs to the center of $\pi(\Gamma)''$. Since π is factorial and $E(U_n) \neq 0$, we have therefore

$E(U_n^\Gamma) = I_{\mathcal{H}}$. By the usual properties of a projection-valued measure, this implies that

$$E(\chi_0^\Gamma) = E\left(\bigcap_{n \geq 1} U_n^\Gamma\right) = I_{\mathcal{H}}.$$

and the claim is proved.

Since S is finite, we have $\mathcal{H} = \bigoplus_{\chi \in S} \mathcal{H}^\chi$, where

$$\mathcal{H}^\chi := \{\xi \in \mathcal{H} \mid \pi(n)\xi = \chi(n)\xi \quad \text{for all } n \in N\};$$

moreover, since N is a normal subgroup, we have $\pi(\gamma)\mathcal{H}^\chi = \mathcal{H}^{\chi^\gamma}$ for every $\chi \in S$ and every $\gamma \in \Gamma$.

Let H be the stabilizer of χ_0 and let $T \subset \Gamma$ be a set of representatives of the right T -cosets of H . Then \mathcal{H}^{χ_0} is invariant under $\pi(H)$ and we have

$$\mathcal{H} = \bigoplus_{t \in T} \mathcal{H}^{\chi_0^t} = \bigoplus_{t \in T} \pi(t)\mathcal{H}^{\chi_0}.$$

This shows that π is equivalent to the induced representation $\text{Ind}_H^\Gamma \sigma$, where σ is the subrepresentation of $\pi|_H$ defined on \mathcal{H}^{χ_0} .

We claim that $\text{Ind}_H^\Gamma \sigma$ is contained in a multiple of $\text{Ind}_N^\Gamma \chi_0$. Indeed, as is well-known (see [BHV08, E.2.5]), $\text{Ind}_N^H(\sigma|_N)$ is equivalent to the tensor product representation $\sigma \otimes \lambda_{H/N}$, where $\lambda_{H/N}$ is the quasi-regular representation on $\ell^2(\Gamma/H)$. Since H/N is finite, 1_H is contained in $\lambda_{H/N}$ and therefore σ is contained in $\text{Ind}_N^H(\sigma|_N)$. Notice that $\sigma|_N$ is a multiple $n\chi_0$ of χ_0 , for some cardinal n . We conclude that $\pi = \text{Ind}_H^\Gamma \sigma$ is contained in $\text{Ind}_H^\Gamma(\text{Ind}_N^H n\chi_0) = n \text{Ind}_N^\Gamma \chi_0$, as claimed.

Now, $\text{Ind}_N^\Gamma \chi_0$ is a finite dimensional representation of Γ , since N has finite index. So, $n \text{Ind}_N^\Gamma \chi_0$ decomposes as direct sum $\bigoplus_{i=1}^r n_i \pi_i$ of multiple $n_i \pi_i$ of pairwise non equivalent irreducible (finite dimensional) representations π_i of Γ . It follows that the factorial subrepresentation π of $\text{Ind}_N^\Gamma \chi_0$ is a multiple of π_i for some $i \in \{1, \dots, r\}$.

A.2. Proof of the other implications. In the sequel, Γ will be a countable group and $\lambda_\Gamma = \int_{\text{Ch}(\Gamma)}^\oplus \pi_t d\mu(t)$ the direct integral decomposition given by the Plancherel theorem A. Recall (see Section 3) that, if we write $\delta_e = \int_{\text{Ch}(\Gamma)}^\oplus \xi_t d\mu(t)$, then ξ_t is a cyclic vector in the Hilbert space \mathcal{H}_t of π_t and $t = \langle \pi_t(\cdot)\xi_t \mid \xi_t \rangle$, for μ -almost every $t \in \text{Ch}(\Gamma)$.

A.2.1. Case where the FC-centre of Γ has infinite index. We assume that $[\Gamma: \Gamma_{\text{fc}}]$ is infinite; we claim that λ_Γ is of type II.

By Remark 5.i, there exists a subset X of $\text{Ch}(\Gamma)$ with $\mu(X) = 1$ such that $t = 0$ outside Γ_{fc} for every $t \in X$.

Let $t \in X$. Then the factor $\pi_t(\Gamma)''$ is infinite dimensional. Indeed, since $[\Gamma: \Gamma_{\text{fc}}]$ is infinite, we can find a sequence $(\gamma_n)_{n \geq 1}$ in Γ with $\gamma_m^{-1}\gamma_n \notin \Gamma_{\text{fc}}$ for every m, n with $m \neq n$. Then

$$\langle \pi_t(\gamma_n)\xi_t \mid \pi_t(\gamma_m)\xi_t \rangle = \langle \pi_t(\gamma_m^{-1}\gamma_n)\xi_t \mid \xi_t \rangle = t(\gamma_m^{-1}\gamma_n) = 0,$$

for $m \neq n$; so, $(\pi_t(\gamma_n)\xi_t)_{n \geq 1}$ is an orthonormal sequence in \mathcal{H}_t . This implies that $(\pi_t(\gamma_n))_{n \geq 1}$ is a linearly independent sequence in $\pi_t(\Gamma)''$ and the claim is proved.

Observe that we have proved, in particular, that Γ is not of type I.

A.2.2. Reduction to FC-groups. Let $\text{Ch}(\Gamma)_{\text{fd}}$ be the set of $t \in \text{Ch}(\Gamma)$ for which $\pi_t(\Gamma)''$ is finite dimensional. Then

$$\text{Ch}(\Gamma)_{\text{fd}} = \bigcup_{n \geq 1} \text{Ch}(\Gamma)_n,$$

where $\text{Ch}(\Gamma)_n$ is the set of t such that $\dim \pi_t(\Gamma)'' = n$. We claim that $\text{Ch}(\Gamma)_n$ and hence $\text{Ch}(\Gamma)_{\text{fd}}$ is a measurable subset of $\text{Ch}(\Gamma)$.

Indeed, let $t \in \text{Ch}(\Gamma)$. Let \mathcal{F} be collection of finite subsets of Γ . For every $F \in \mathcal{F}$, let C_F be the set of $t \in \text{Ch}(\Gamma)$ such that the family $(\pi_t(\gamma))_{\gamma \in F}$ is linearly independent, equivalently (see Remark 5.ii), such that $(\pi_t(\gamma)\xi_t)_{\gamma \in F}$ is linearly independent. Since $t = \langle \pi_t(\cdot)\xi_t \mid \xi_t \rangle$, it follows that

$$C_F := \{t \in \text{Ch}(\Gamma) \mid \det(t(\gamma^{-1}\gamma')) \neq 0 \quad \text{for all } (\gamma, \gamma') \in F \times F\}$$

and this shows that C_F is measurable. Since

$$(*) \quad \text{Ch}(\Gamma)_n = \bigcup_{F \in \mathcal{F}: \#F=n} C_F \setminus \left(\bigcup_{F' \in \mathcal{F}: \#F'>n} C_{F'} \right)$$

and \mathcal{F} is countable, it follows that $\text{Ch}(\Gamma)_n$ is measurable.

Assume now that $[\Gamma: \Gamma_{\text{fc}}]$ is finite. Observe that, for a cyclic representation π of Γ_{fc} , the induced representation $\text{Ind}_{\Gamma_{\text{fc}}}^{\Gamma} \pi$ is cyclic and so $(\text{Ind}_{\Gamma_{\text{fc}}}^{\Gamma} \pi)(\Gamma)''$ is finite dimensional if and only if $\pi(\Gamma_{\text{fc}})''$ is finite dimensional.

let ν be the Plancherel measure of Γ_{fc} . It follows from Theorem B that $\mu(\text{Ch}(\Gamma)_{\text{fd}}) = \nu(\text{Ch}(\Gamma_{\text{fc}})_{\text{fd}})$; in particular, we have $\mu(\text{Ch}(\Gamma)_{\text{fd}}) = 0$ (or $\mu(\text{Ch}(\Gamma)_{\text{fd}}) = 1$) if and only if $\nu(\text{Ch}(\Gamma_{\text{fc}})_{\text{fd}}) = 0$ (or $\nu(\text{Ch}(\Gamma_{\text{fc}})_{\text{fd}}) = 1$), that is, λ_{Γ} is of type I (or of type II) if and only if $\lambda_{\Gamma_{\text{fc}}}$ is of type I (or of type II).

Observe also that Γ is virtually abelian if and only if Γ_{fc} . As a consequence, we see that it suffices to prove Theorem E in the case where $\Gamma = \Gamma_{\text{fc}}$.

A.2.3. *Case of an FC-group.* We assume from now on that $\Gamma = \Gamma_{\text{fc}}$.

Step 1 We claim that the regular representation λ_Γ is of type II if and only if $[\Gamma, \Gamma]$ is infinite.

We have to show that $\mu(\text{Ch}(\Gamma)_{\text{fd}}) > 0$ if and only if $[\Gamma, \Gamma]$ is finite.

Assume first that $[\Gamma, \Gamma]$ is finite. The representation $\lambda_{\Gamma/[\Gamma, \Gamma]}$, lifted to Γ , is a subrepresentation of λ_Γ , since $\ell^2(\Gamma/[\Gamma, \Gamma])$ can be viewed in an obvious way as Γ -invariant subspace of $\ell^2(\Gamma)$. As $\Gamma/[\Gamma, \Gamma]$ is abelian, $\lambda_{\Gamma/[\Gamma, \Gamma]}$ is of type I and so $\mu(\text{Ch}(\Gamma)_{\text{fd}}) > 0$.

Conversely, assume that $\mu(\text{Ch}(\Gamma)_{\text{fd}}) > 0$. Since Γ is an FC-group, it suffices to show that Γ has a subgroup of finite index with finite commutator subgroup (see [Neu55, Lemma 4.1]).

As $\mu(\text{Ch}(\Gamma)_{\text{fd}}) > 0$ and $\text{Ch}(\Gamma)_{\text{fd}} = \bigcup_{n \geq 1} \text{Ch}(\Gamma)_n$, we have $\mu(\text{Ch}(\Gamma)_n) > 0$ for some $n \geq 1$. It follows from (*) that there exists $F \in \mathcal{F}$ with $|F| = n$ such that $\mu(C_F \cap \text{Ch}(\Gamma)_n) > 0$.

Let Λ be the subgroup of Γ generated by F . Since Γ is an FC-group and Λ is finitely generated, the centralizer $H := \text{Cent}_\Gamma(\Lambda)$ of Λ in Γ has finite index.

Let $t \in C_F \cap \text{Ch}(\Gamma)_n$ and $\gamma_0 \in H$. On the one hand, since $(\pi_t(\gamma))_{\gamma \in F}$ is a basis of the vector space $\pi_t(\Gamma)$, we have $\pi_t(\Lambda)'' = \pi_t(\Gamma)''$. On the other hand, as γ_0 centralizes Λ , we have $\pi_t(\gamma_0) \in \pi_t(\Lambda)'$. Hence, $\pi_t(\gamma_0)$ belongs to the center $\pi_t(\Gamma)' \cap \pi_t(\Gamma)''$ of the factor $\pi_t(\Gamma)''$ and so $\pi_t(\gamma_0)$ is a scalar multiple of $I_{\mathcal{H}_t}$. It follows in particular that π_t is trivial on $[H, H]$. As a result, the subrepresentation $\int_{C_F \cap \text{Ch}(\Gamma)_n}^\oplus \pi_t d\mu(t)$ of λ_Γ is trivial on $[H, H]$. Since the matrix coefficients of λ_Γ vanish at infinity, it follows that $[H, H]$ is finite and the claim is proved.

In view of what we have shown so far, we may and will assume from now on that $[\Gamma, \Gamma]$ is finite and that λ_Γ is of type I. We are going to show that Γ is a virtually abelian (in fact, a central) group and this will finish the proof of Theorem E.

Set $N := [\Gamma, \Gamma]$ and let $r : \text{Ch}(\Gamma) \rightarrow \text{Tr}(N)$ be the restriction map.

Step 2 We claim that there exist finitely many functions s_1, \dots, s_m in $\text{Tr}(N)$ such that $r(t) \in \{s_1, \dots, s_m\}$, for μ -almost every $t \in \text{Ch}(\Gamma)$.

Indeed, since λ_Γ is of type I, there exists a subset X of $\text{Ch}(\Gamma)$ with $\mu(X) = 1$ such that $\pi_t(\Gamma)''$ is finite dimensional for every $t \in X$.

Let $t \in X$. The Hilbert space \mathcal{H}_t of π_t is finite dimensional and π_t is a (finite) multiple of an irreducible representation σ_t of Γ . As π_t and σ_t have the same normalized character, we may assume that π_t is irreducible.

Let \mathcal{K} be an irreducible N -invariant subspace of \mathcal{H}_t and let ρ be the corresponding equivalence class of representation of N . For $g \in \Gamma$, the subspace $\pi_t(g)\mathcal{K}$ is N -invariant with ρ^g as corresponding representation of N . Since π_t is irreducible, we have $\mathcal{H} = \sum_{g \in \Gamma} \pi_t(g)\mathcal{K}$.

Let L be the stabilizer of ρ ; observe that L has finite index in Γ , since L contains the centralizer of N and Γ is an FC-group. Let g_1, \dots, g_r be a set of representatives for the coset space Γ/L . Then $\mathcal{H}_t = \bigoplus_{j=1}^r \pi_t(g_j)\mathcal{K}_\rho$, where \mathcal{K}_ρ is the sum of all N -invariant subspaces of \mathcal{H}_t with corresponding representation equivalent to ρ . The normalized trace of the representation of N on $\pi_t(g_j)\mathcal{K}_\rho$ is $\chi_\rho^{g_j}$, where χ_ρ is the normalized character of ρ . It follows that, for every $g \in N$, we have

$$t(g) = \frac{1}{r} \sum_{j=1}^r \chi_\rho^{g_j}(g).$$

Since $[\Gamma, \Gamma]$ is finite, $[\Gamma, \Gamma]$ has only finitely many equivalence class of irreducible representations and the claim follows.

We will identify $\text{Ch}(\Gamma/[\Gamma, \Gamma])$ with the set $\{s \in \text{Ch}(\Gamma) \mid r(s) = 1_{[\Gamma, \Gamma]}\}$, that is, with the set of unitary characters of Γ . Observe that, for every $s \in \text{Ch}(\Gamma/[\Gamma, \Gamma])$ and $t \in \text{Ch}(\Gamma)$, we have $st \in \text{Ch}(\Gamma)$.

Step 3 Let $t, t' \in \text{Ch}(G)$ be such that $r(t) = r(t')$. We claim that there exists $s \in \text{Ch}(\Gamma/[\Gamma, \Gamma])$ such that $t' = st$.

Indeed, the integral decomposition of $\mathbf{1}_{[\Gamma, \Gamma]} \in \text{Tr}(\Gamma)$ into characters is given by

$$\mathbf{1}_{[\Gamma, \Gamma]} = \int_{\text{Ch}(\Gamma/[\Gamma, \Gamma])} s d\nu(s),$$

where ν is the Haar measure of $\Gamma/[\Gamma, \Gamma]$. By assumption, we have $t\mathbf{1}_{[\Gamma, \Gamma]} = t'\mathbf{1}_{[\Gamma, \Gamma]}$ and hence

$$t\mathbf{1}_{[\Gamma, \Gamma]} = \int_{\text{Ch}(\Gamma/[\Gamma, \Gamma])} t s d\nu(s) = \int_{\text{Ch}(\Gamma/[\Gamma, \Gamma])} t' s d\nu(s).$$

By uniqueness of integral decomposition, it follows that the images ν_t and $\nu_{t'}$ of ν under the maps $\text{Ch}(\Gamma/[\Gamma, \Gamma]) \rightarrow \text{Ch}(\Gamma)$ given respectively by multiplication with t and t' coincide. In particular, the supports of ν_t and $\nu_{t'}$ are the same, that is, $t\text{Ch}(\Gamma/[\Gamma, \Gamma]) = t'\text{Ch}(\Gamma/[\Gamma, \Gamma])$ and the claim follows.

Step 4 The center $Z(\Gamma)$ has finite index in Γ .

Indeed, by Step 2, there exists a subset X of $\text{Ch}(\Gamma)$ with $\mu(X) = 1$ and finitely many $t_1, \dots, t_m \in X$ such that $r(t) \in \{r(t_1), \dots, r(t_m)\}$ and such that \mathcal{H}_t is finite dimensional for every $t \in X$.

By Step 3, for every $t \in X$, there exists $s \in \text{Ch}(\Gamma/[\Gamma, \Gamma])$ such that $t = st_i$ for some $i \in \{1, \dots, m\}$ and hence $\dim \pi_t(\Gamma)'' = \dim \pi_{t_i}(\Gamma)''$. As a result, we can find a finitely generated normal subgroup M of Γ such that $\dim \pi_t(\Gamma)'' = \dim \pi_t(M)''$ for every $t \in X$.

Since the centralizer C of M in Γ has finite index, it suffices to show that C contains $Z(\Gamma)$.

For $g \in C, \gamma \in \Gamma$ and $x \in M$, we have $t(x^{-1}g\gamma g^{-1}) = t(x^{-1}\gamma)$, that is,

$$\langle \pi_t(g\gamma g^{-1})\xi_t \mid \pi_t(x)\xi_t \rangle = \langle \pi_t(\gamma)\xi_t \mid \pi_t(x)\xi_t \rangle.$$

Since $\dim \pi_t(\Gamma)'' = \dim \pi_t(M)''$, the linear span of $\pi_t(H)\xi_t$ is dense in \mathcal{H}_t and this implies that $\pi_t(g\gamma g^{-1})\xi_t = \pi_t(\gamma)\xi_t$ for all $g \in C, \gamma \in \Gamma$, and $t \in X$. It follows that $\lambda_\Gamma(g\gamma g^{-1})\delta_e = \lambda_\Gamma(\gamma)\delta_e$ and hence $g\gamma g^{-1} = \gamma$ for all $g \in C$ and $\gamma \in \Gamma$; so, $C \subset Z(\Gamma)$.

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