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# Examples of Hidden Convexity in Nonlinear PDEs

Yann Brenier

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Yann Brenier. Examples of Hidden Convexity in Nonlinear PDEs. Doctoral. France. 2020. hal-02928398

**HAL Id: hal-02928398**

**<https://hal.science/hal-02928398>**

Submitted on 2 Sep 2020

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# EXAMPLES OF HIDDEN CONVEXITY IN NONLINEAR PDEs

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September 1, 2020



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## Acknowledgements

I would like to express my warmest gratitude to the Forschung Institut Mathematik (FIM) of the ETHZ, and more particularly to his director Tristan Rivière, for inviting me to deliver a "Nachdiplomvorlesung" course in the academic year 2009-2010. Traditionally, such a course is supposed to be followed by the writing of a short book based on the notes taken by a dedicated graduate student. In my case, this preliminary work was very nicely and quickly done by Mircea Petrache and I would like to thank him very warmly. Unfortunately, I was extremely slow in completing the work and many years have passed at the point that I almost gave up the project. However, thanks to Tristan Rivière and also to his successor at the direction of FIM, Alessio Figalli, I got a second chance to finish the work in the summer of 2019 at ETHZ. This was a great opportunity to add a lot of more recent contributions. (The original course roughly corresponds to Chapters 3-4-6-8 while Chapters 1-5-7-9-10 are new and Chapter 2 has been substantially expanded.) I am also very grateful to Thomas Kappeler, Craig (LC) Evans and Michael Struwe for their encouragements to complete this project.



# Chapter 1

## Few examples of hidden convexity, away from PDEs

### 1.1 Two elementary examples

**Theorem 1.1.1.** *Let  $K$  be a compact metric space and  $f$  be a continuous real function on  $K$ . We denote by  $P(K)$  the convex space of all Borel probability measures on  $K$ . Then, it is equivalent to say that  $f$  achieves its minimum at some point  $x_0$  in  $K$  and that  $\delta_{x_0}$  achieves on  $P(K)$  the minimum of the linear functional*

$$\mu \in P(K) \rightarrow F(\mu) = \int_K f(x) d\mu(x)$$

*Proof.* Since  $x_0$  achieves the minimum of  $f$  on  $K$ , then, for every  $\mu \in P(K)$ , one has on one hand,

$$F(\mu) \geq \int_K f(x_0) d\mu(x) = f(x_0)$$

and, on the other hand

$$F(\delta_{x_0}) = f(x_0).$$

Thus  $\delta_{x_0}$  minimizes  $F$  on  $P(K)$ . Conversely, if  $\delta_{x_0}$  minimizes  $F$  on  $P(K)$ , we get for every  $x \in K$ ,

$$f(x_0) = F(\delta_{x_0}) \leq F(\delta_x) = f(x),$$

which shows that the minimum of  $f$  is achieved by  $x_0$ .

*Remark :* observe that if the minimum of  $f$  is achieved at once by several points  $x_0, \dots, x_N$  then the minimum of  $F$  is achieved by any convex combination of the  $\delta_{x_i}$ .

*Remark :* this result extends to the case when  $f$  is only l.s.c on  $K$  and valued in  $] -\infty, +\infty]$ , but not identically equal to  $+\infty$ . In that case,  $F$  can no longer be considered as a linear functional but rather as an l.s.c convex functional (with respect to weak-\* convergence on  $P(K)$ ), valued in  $] -\infty, +\infty]$  and not identically equal to  $+\infty$ .

**Theorem 1.1.2.** *Let  $H$  be a separable Hilbert space of infinite dimension. Then, the closed unit ball of  $H$  is the weak closure of the unit sphere.*



*Remark* : in finite dimension, there is no difference between the concepts of weak and strong convergence. Therefore, the unit sphere is weakly closed and certainly not weakly dense in the unit ball.

*Proof* : In infinite dimension, we can find an infinite sequence of orthonormal vectors  $u_n \in H$ , i.e. such that  $(u_n|u_m) = \delta_{nm}$ . This sequence weakly converges to zero. Indeed, for each  $x \in H$ , one has:

$$0 \leq |x - \sum_{i=1}^N (x|u_i)u_i|^2 = |x|^2 - \sum_{i=1}^N (x|u_i)^2.$$

Thus the series of the  $(x|u_n)^2$  is sommable. Therefore, its generic term  $(x|u_n)^2$  goes to zero which is enough to show that  $u_n$  weakly goes to zero. Let us now fix  $x$  such that  $|x| \leq 1$ . For each  $n$ , let us introduce  $x_n = x + r_n u_n$  where  $r_n \in \mathbb{R}$  is chosen so that  $|x_n| = 1$ . This is possible, since it amounts to solving

$$|x|^2 + 2r_n(x|u_n) + r_n^2 = 1,$$

i.e.

$$(r_n + (x|u_n))^2 = 1 - |x|^2 + (x|u_n)^2,$$

and a solution is given by

$$r_n = -(x|u_n) + \sqrt{1 - |x|^2 + (x|u_n)^2}$$

(since  $|x| \leq 1$ ). As a consequence,

$$|r_n| \leq |x| + 1,$$

which shows that, up to the extraction of a subsequence, still labelled by  $n$  for notational simplicity, we may assume  $r_n \rightarrow r$  for some real  $r$ . So, we have found a sequence  $x_n$  of points of the unit sphere that weakly converges to  $x$ . Indeed, for each  $y \in H$ , one has

$$(x_n - x|y) = (r_n u_n|y) = (r_n - r)(u_n|y) + r(u_n|y)$$

where  $|(r_n - r)(u_n|y)| \leq |r_n - r||y| \rightarrow 0$  and  $(u_n|y) \rightarrow 0$  since  $u_n$  weakly converges to zero. So, we may weakly approximate any point of the unit ball by a sequence of points of the unit sphere. This has been possible because the infinite dimension of  $H$  has left a lot of room available to us!

## 1.2 Convexity and Combinatorics: the Birkhoff theorem

**Theorem 1.2.1.** *Let  $DS_N$  be the convex set of all  $N \times N$  real matrices with nonnegative entries such that every row and every column add up to one. (Such matrices are frequently called doubly stochastic matrices). Then  $DS_N$  exactly is the convex hull of the subset of all permutation matrices, i.e. of all doubly stochastic matrices with entries in  $\{0, 1\}$ .*

*Proof.*

It is obvious that the convex hull of all permutation matrices is a subset of  $DS_N$ . The converse part, as shown by G. Birkhoff [65], is a rather direct consequence of the famous "marriage lemma" in combinatorics. that asserts that a necessary and sufficient condition to marry  $N$  girls to  $N$  boys without dissatisfaction is that, for all subset of  $r \leq N$  girls, there are at least  $r$  convenient boys. Now, let us consider a doubly stochastic matrix  $(\nu_{ij})$ . There is a permutation  $\sigma$  such that  $\inf_i \nu_{i,\sigma(i)}$  is a positive number  $\alpha > 0$ . (In other words the "support" of  $\sigma$  is contained in the support of  $\nu$ .) Then, we have the following alternative. Either  $\alpha = 1$  and  $\nu$  is automatically a permutation matrix. Or  $\alpha < 1$  and

$$\nu'_{ij} = (\nu_{ij} - \alpha \delta_{j,\sigma(i)}) \frac{1}{1 - \alpha}$$

defines a new doubly stochastic matrix with a strictly smaller support and  $\nu$  is a convex combination of  $\nu'$  and a permutation matrix. Recursively, after a finite number of steps,  $\nu$  is written as a convex combination of permutation matrices which completes the proof.

## Application to combinatorial optimization

**Theorem 1.2.2.** *Let  $c_{ij}$  be a real  $N \times N$  fixed matrix. Then it is equivalent to solve*

- 1) *The so-called "linear assignment problem"*

$$\inf_{\sigma \in \mathfrak{S}_N} \sum_{i=1, N} c_{i\sigma_i}$$

where  $\mathfrak{S}_N$  denotes the symmetric group (i.e. the group of all permutations of the first  $N$  integers);

- 2) *The "linear program"*

$$\inf_{s \in DS_N} \sum_{i,j=1}^N c_{ij} s_{ij}.$$

This result is striking since it reduces a combinatorial optimization problem to a simple "linear program" (i.e. the minimization of a linear functional with linear equality or inequality constraints) [433].

*Remark :* There are algorithms of sequential computational cost  $O(N^3)$  for this problem [29], which is usually considered as very simple in Combinatorial Optimization. Just to quote an example of a "hard" combinatorial optimization problem that *cannot* be reduced to a convex optimization problem, let us mention the "quadratic assignment problem", where a second  $N \times N$  real matrix  $\gamma_{ij}$  is given, which amounts to solving:

$$\inf_{\sigma \in \mathfrak{S}_N} \sum_{i,j=1, N} c_{ij} \gamma_{\sigma_i \sigma_j}.$$

This "NP" problem contains as a particular case the famous traveling salesman problem. (Nevertheless, in some special cases, related problems discussed in [108] can be addressed by somewhat conventional "gradient flow" strategies related to the "Brockett" flow [67].)

### 1.3 The Least Action Principle for 2nd order ODEs

Let us consider the 2nd order ODE, typical of Classical Mechanics,

$$X''(t) = -(\nabla p)(t, X(t)).$$

where  $X = X(t) \in \mathbb{R}^d$  describes the trajectory of a particle of unit mass moving under the action of a time-dependent potential  $p = p(t, x) \in \mathbb{R}$ . We may, as well, write this ODE as a 1st order system of ODEs:

$$X'(t) = V(t), \quad V'(t) = -(\nabla p)(t, X(t)).$$

In order to keep our discussion as simple as possible, let us assume that  $p$  is smooth and that its second order derivatives in  $x$  are uniformly bounded in  $(t, x)$ . This is enough, according to the Cauchy-Lipschitz theorem, to justify the global existence of a unique solution  $t \in \mathbb{R} \rightarrow (X(t), V(t))$ , once its value  $(X(t_0), V(t_0))$  is known at some fixed time  $t_0 \in \mathbb{R}$ .

As a matter of fact, this 2nd order ODE  $X''(t) = -(\nabla p)(t, X(t))$  obeys the famous "Least Action Principle" (LAP), which means, in modern words, that, for every fixed  $t_0 < t_1$ , its solutions  $X$  are critical points of "functional"

$$u \in C^1([t_0, t_1]; \mathbb{R}^d) \rightarrow J_{t_0, t_1, p}[u] = \int_{t_0}^{t_1} \left( \frac{1}{2} |u'(t)|^2 - p(t, u(t)) \right) dt$$

subject to  $u(t_0) = X(t_0)$  and  $u(t_1) = X(t_1)$ . By critical point, we simply mean that for any "perturbation"  $y \in C^1([t_0, t_1]; \mathbb{R}^d)$  such that  $y(t_0) = y(t_1) = 0$ , the derivative of

$$s \in \mathbb{R} \rightarrow f(s) = J_{t_0, t_1, p}[X + sy] = \int_{t_0}^{t_1} \left( \frac{1}{2} |X'(t) + sy'(t)|^2 - p(t, X(t) + sy(t)) \right) dt$$

vanishes at  $s = 0$ , which exactly means

$$\int_{t_0}^{t_1} (X'(t) \cdot y'(t) - \nabla p(t, X(t)) \cdot y(t)) dt = 0$$

i.e., after integration by part

$$\int_{t_0}^{t_1} (-X''(t) \cdot y(t) - \nabla p(t, X(t)) \cdot y(t)) \cdot y(t) dt = 0.$$

Since  $y$  has been arbitrarily chosen, we therefore have exactly recovered the 2nd order EDO  $X''(t) = -(\nabla p)(t, X(t))$ . (To check it, just observe that a dense subset of  $L^2([t_0, t_1]; \mathbb{R}^d)$  is formed by all  $y \in C^1([t_0, t_1]; \mathbb{R}^d)$  such that  $y(t_0) = y(t_1) = 0$ .)

(The discovery of the LAP was attributed by Euler [230], when he was a member of the "Académie Royale des Sciences de Berlin", to Maupertuis, who currently was the president of the same Academy. At some stage, a mathematician, Koenig, claimed that he had a letter proving that the LAP had been discovered earlier by Leibniz. The Academy, and Euler himself, accused Koenig of fraud and a violent dispute started for a while. Voltaire took advantage of the situation to

write a pamphlet -where Maupertuis was nicknamed as Dr. Akakia- which became very popular in France. Furious, Friedrich the second, king of Prussia, decided to destroy all copies available in his kingdom.)

The LAP has been extended to many PDEs of Physics and Mechanics: solutions are characterized as critical points of some suitable functional. In most examples, this critical points are not minimizers of the functional and it would be more accurate to speak of "Critical Action Principle", although the expression LAP has been kept since the 18th century. However, in the very special case of our 2nd order ODE, it turns out that solutions are really minimizers *provided* the time interval  $[t_0, t_1]$  is sufficiently short. This follows from the fact that function  $s \rightarrow f(s)$ , as defined above, is *convex* for small values of  $t_1 - t_0$ . More precisely

**Theorem 1.3.1.** *Let  $p = p(t, x)$  be a smooth function on  $\mathbb{R} \times \mathbb{R}^d$  for which we assume that the 2nd order derivatives in  $x$  are uniformly bounded, so that*

$$K(p) = \sup_{t,x,|y|=1} \sum_{i,j=1}^d \frac{\partial^2 p(t,x)}{\partial x^i \partial x^j} y_i y_j$$

or, in short,

$$K(p) = \sup_{t,x,|y|=1} D_x^2 p(t,x) : y \otimes y,$$

is finite. Let  $X$  be a solution of  $X''(t) = -(\nabla p)(t, X(t))$ . Then, provided that  $(t_1 - t_0)^2 K(p) < \pi^2$ , any curve  $u \in C^1([t_0, t_1]; \mathbb{R}^d)$ , different from  $X$ , such that  $u(t_0) = X(t_0)$ ,  $u(t_1) = X(t_1)$ , satisfies

$$J_{t_0, t_1, p}[u] > J_{t_0, t_1, p}[X]$$

where

$$J_{t_0, t_1, p}[u] = \int_{t_0}^{t_1} \left( \frac{1}{2} |u'(t)|^2 - p(t, u(t)) \right) dt.$$

The proof is an easy consequence of the 1D Poincaré inequality

**Lemma 1.3.2.** *Assume  $t_0 < t_1$ . Then, for every curve  $C^1$*

$$[t_0, t_1] \rightarrow y(t) \in \mathbb{R}^d,$$

such that  $y(t_0) = y(t_1) = 0$ ,

$$\pi^2 \int_{t_0}^{t_1} |y(t)|^2 dt \leq (t_1 - t_0)^2 \int_{t_0}^{t_1} |y'(t)|^2 dt.$$

*Proof.*

It is enough to expand  $y$  as a series of sine functions:

$$y(t) = \sum_{k=1}^{+\infty} y_k \sin\left(k\pi \frac{t - t_0}{t_1 - t_0}\right)$$

and use Parseval's identity. (Saturation is obtained as all  $y_k$  vanish but  $y_1$ .)

### Proof of Theorem 1.3.1

Let us compute the 2nd derivative of

$$s \in \mathbb{R} \rightarrow f(s) = J_{t_0, t_1, p}[X + sy] = \int_{t_0}^{t_1} \left( \frac{1}{2} |X'(t) + sy'(t)|^2 - p(t, X(t) + sy(t)) \right) dt,$$

where  $y$  is a non vanishing perturbation such that  $y(t_0) = 0$ ,  $y(t_1) = 0$ . We first get

$$f'(s) = \int_{t_0}^{t_1} \left( (X'(t) + sy'(t)) \cdot y'(t) - \nabla p(t, X(t) + sy(t)) \cdot y(t) \right) dt,$$

next

$$f''(s) = \int_{t_0}^{t_1} \left( |y'(t)|^2 - D^2 p(t, X(t) + sy(t)) : y(t) \otimes y(t) \right) dt,$$

and, therefore,

$$f''(s) \geq \int_{t_0}^{t_1} \left( |y'(t)|^2 - K(p) |y(t)|^2 \right) dt.$$

From the Poincaré inequality, we deduce

$$f''(s) \geq \left( \frac{\pi^2}{(t_1 - t_0)^2} - K(p) \right) \int_{t_0}^{t_1} |y(t)|^2 dt > 0$$

as soon as  $K(p)(t_1 - t_0)^2 < \pi^2$ , since  $y$  is not identically null. So,  $f(s)$  is a strictly convex function of  $s$ . We already saw that  $f'(0) = 0$ . So  $s = 0$  is a strict minimum for  $f$ , which completes the proof. Finally observe that the "hidden" convexity is directly related to the Poincaré inequality.

## 1.4 A continuous version of the Birkhoff theorem

Let us consider the unit cube  $D = [0, 1]^d$ . We may split it in  $N = 2^{nd}$  dyadic subcubes of equal volume  $D_\alpha$  for  $\alpha = 1, \dots, N$  and attach to each permutation  $\pi \in \mathfrak{S}_N$  the map  $T_\pi : D \rightarrow D$  which rigidly translates the interior of each subcube  $D_\alpha$  to the interior of  $D_{\pi(\alpha)}$ . This makes  $T_\pi$  an element of the set  $VPM(D)$  of all volume preserving maps  $T : D \rightarrow D$ , defined as follows:

**Definition 1.4.1.** *Let  $D = [0, 1]^d$ . We define  $VPM(D)$  as the set of all Borel maps  $T : D \rightarrow D$  such that*

$$\mathcal{L}(T^{-1}(A)) = \mathcal{L}(A),$$

*for all Borel subset  $A$  of  $D$ , where  $\mathcal{L}$  denotes the Lebesgue measure restricted to  $D$ , i.e. in short  $\mathcal{L} \circ T^{-1} = \mathcal{L}$ . Equivalently, this means*

$$\int_D f(T(x)) dx = \int_D f(x) dx,$$

*for every function  $f \in C(\mathbb{R}^d)$ .*

It is fairly easy to check the following properties of  $VPM(D)$ :

1)  $VPM(D)$  can be seen as a closed subset of the Hilbert space  $H = L^2(D; \mathbb{R}^d)$ , contained in the sphere

$$\{ T \in H; \int_D |T(x)|^2 dx = \int_D |x|^2 dx \}$$

and, therefore, cannot be a convex set.

2)  $VPM(D)$  is a semi-group for the composition rule. However, it is not a group since it contains many non invertible maps  $T$ , such as, for example in the case  $d = 1$ ,

$$T(x) = 2x \text{ mod. } 1.$$

As a matter of fact, the subset of all invertible maps in  $VPM(D)$  forms a group but is not a closed subset of  $H$ .

3)  $VPM(D)$  contains the group  $P_N(D)$  of all "permutation maps"  $T_\pi$  constructed as above, for each permutation  $\pi \in \mathfrak{S}_N$ , after splitting  $D$  in  $N = 2^{nd}$  dyadic subcubes. The collection of all these  $P_N(D)$  forms a group  $P(D)$ .

4)  $VPM(D)$  also contains the group  $SDiff(D)$  of all orientation and volume preserving diffeomorphisms  $T$  of  $D$ , in the sense that  $T$  is the restriction of a diffeomorphism of  $\mathbb{R}^d$ , still denoted by  $T$ , such that  $T(D) = D$  and

$$\det(DT(x)) = 1, \quad \forall x \in D.$$

This group is trivially reduced to the identity map as  $d = 1$ .

Nevertheless,  $VPM(D)$  in spite of being a closed bounded subset of the Hilbert space  $H = L^2(D; \mathbb{R}^d)$ , is not compact. However, there is a natural "compactification" of  $VPM(D)$  [381, 117] which involves the convex set  $DS(D)$ , defined as follows.

**Definition 1.4.2.** We define the space of doubly stochastic measures  $DS(D)$  as the set of all Borel probability measures  $\mu \in Prob(D \times D)$  such that

$$\mu(D \times A) = \mu(A \times D) = \mathcal{L}(A),$$

for each Borel subset  $A \subset D$ , or, equivalently,

$$\int_{D \times D} f(x) d\mu(x, y) = \int_{D \times D} f(y) d\mu(x, y) = \int_D f(x) dx, \quad \forall f \in C^0(D).$$

$Prob(D \times D)$  is a weak-\* compact subset of the space of all bounded Borel measures on  $D \times D$ , namely the dual Banach space of  $C^0(D \times D; \mathbb{R})$ . Thus,  $DS(D)$ , as a weak-\* closed subset of  $Prob(D \times D)$ , is also weak-\* compact.

There is a natural injection  $\mathfrak{i}$  of  $VPM(D)$  in  $DS(D)$

$$\mathfrak{i} : T \in VPM(D) \rightarrow \mu_T \in DS(D),$$

defined by setting

$$\int_{D \times D} f(x, y) d\mu_T(x, y) = \int_D f(x, T(x)) dx, \quad \forall f \in C^0(D \times D).$$

**Theorem 1.4.3.** *The space of doubly stochastic measures  $DS(D)$  is the weak- $*$  closure of  $\mathbf{i}(P(D))$  -and therefore of  $\mathbf{i}(VPM(D))$ -. In other words, any  $\mu \in DS(D)$  can be approximated by a sequence of "permutation maps"  $T_n \in P(D)$  in the sense*

$$\int_{D \times D} f(x, y) d\mu(x, y) = \lim_n \int_D f(x, T_n(x)) dx, \quad \forall f \in C^0(D \times D).$$

**Corollary 1.4.4.**  *$VPM(D)$  is the closure, in  $L^2$  norm, of  $P(D)$ .*

This Corollary is a straightforward consequence of the easy lemma:

**Lemma 1.4.5.** *A sequence  $T_n \in VPM(D)$  converges to  $T \in VPM(D)$  in  $L^2$  norm, if and only if*

$$\int_D f(x, T_n(x)) dx \rightarrow \int_D f(x, T(x)) dx, \quad \forall f \in C^0(D \times D).$$

*which exactly means that  $\mathbf{i}(T_n)$  weak- $*$  converges to  $\mathbf{i}(T)$  in  $DS(D)$ .*

Observe the similarity of Theorem 1.4.3 with Theorem 1.1.2,  $DS(D)$  and  $VPM(D)$  somehow playing the respective role of the unit ball and the unit sphere. We also see here another manifestation of the concept of "hidden convexity", where behind  $VPM(D)$ , we have exhibited the convex set  $DS(D)$  as a natural weak- $*$  compactification through injection  $\mathbf{i}$ .

Finally, Theorem 1.4.3 can be interpreted as a continuous version of the Birkhoff theorem where the concept of weak- $*$  closure substitutes for the concept of convex hull. However, notice that  $\mathbf{i}(VPM(D))$  is *strictly* contained in the set of all extremal points of the convex set  $DS(D)$ . Indeed, each time  $T \in VPM(D)$  is not invertible, we get automatically two extremal points  $\mu, \tilde{\mu}$  of  $DS(D)$ , respectively defined by

$$\int_{D \times D} f(x, y) d\mu(x, y) = \int_D f(x, T(x)) dx, \quad \forall f \in C^0(D \times D),$$

$$\int_{D \times D} f(x, y) d\tilde{\mu}(x, y) = \int_D f(T(x), x) dx, \quad \forall f \in C^0(D \times D),$$

but only  $\mu$  belongs to  $\mathbf{i}(VPM(D))$ !

*Remark.* It turns out [381] (see also [117]) that  $DS(D)$  is also the weak- $*$  closure of  $\mathbf{i}(SDiff(D))$  provided that  $d \geq 2$ , and, as a consequence  $VPM(D)$  is the closure of  $SDiff(D)$  with respect to the  $L^2$  norm. This has the disturbing consequence that any *orientation reversing* volume-preserving diffeomorphism of  $D$  (which clearly belongs to  $VPM(D)$ ) -such as

$$T(x) = (1 - x_1, x_2), \quad x = (x_1, x_2) \in [0, 1]^d, \quad d = 2,$$

can be approximated in  $L^2$  norm by a sequence of orientation and volume preserving diffeomorphism of  $D$ .

### Proof of Theorem 1.4.3

Given  $\mu \in DS(D)$ , we want to find a sequence of “permutation” maps  $p$  such that the corresponding doubly stochastic measures  $\mathbf{i}(p)$  weak- $*$  converge to  $\mu$ .

Let  $n > 0$  be a fixed integer. We split  $D = [0, 1]^d$  into  $N = 2^{nd}$  subcubes of equal volume denoted by  $D_{n,i}$  for  $i = 1, \dots, N$ . We set

$$\nu_{ij} = N\mu(D_{n,i} \times D_{n,j}),$$

for  $i, j = 1, \dots, N$  so that  $\nu$  is a doubly stochastic matrix. By Birkhoff’s theorem, such a matrix always can be written as a convex combination of at most  $K = K(N)$  (where, as a matter of fact,  $K(N) = O(N^2)$ ) permutation matrices. Thus, there are coefficients  $\theta_1, \dots, \theta_K \geq 0$  and permutations  $\sigma_1, \dots, \sigma_K$  such that

$$\sum_{k=1}^K \theta_k = 1, \quad \nu_{ij} = \sum_{k=1}^K \theta_k \delta_{j, \sigma_k(i)}.$$

Let us introduce  $L = 2^{ld}$ , where  $l$  will be chosen later, and set

$$\theta'_k = \frac{1}{L}([L\theta_k] + \epsilon_k),$$

where  $[\cdot]$  denotes the integer part of a real number and  $\epsilon_k \in [0, 1[$  is chosen so that

$$\sum_{k=1}^K \theta'_k = 1, \quad \sup_k |\theta_k - \theta'_k| \leq \frac{1}{L}.$$

By setting

$$\nu'_{ij} = \sum_{k=1}^K \theta'_k \delta_{j, \sigma_k(i)},$$

we get a new doubly stochastic matrix which satisfies

$$\sum_{i,j} |\nu'_{ij} - \nu_{ij}| \leq \frac{NK}{L}.$$

Up to a relabelling of the list of permutations, with possible repetitions, we may assume all coefficients  $\theta'_k$  to be equal to  $1/L$  and get a new expression

$$\nu'_{ij} = \frac{1}{L} \sum_{k=1}^L \delta_{j, \sigma_k(i)}.$$

Now, we can split again each  $D_{n,i}$  into  $L$  subcubes, denoted by  $D_{n+l,i,m}$ , for  $i = 1, \dots, N$ ,  $m = 1, \dots, L$ , with size  $2^{-(n+l)}$  and volume  $2^{-(n+l)d}$ . Then, we define

$$p(x) = x - x_{n+l,i,m} + x_{n+l, \sigma_m(i), m},$$

for each  $x \in D_{n+l,i,m}$ . By construction,  $(i, m) \rightarrow (\sigma_m(i), m)$  is one-to-one. Thus,  $p$  belongs to  $P_{n+l}(D)$ . Let us now estimate, for any fixed  $f \in C(D)$ ,

$$I_1 - I_2 = \int_{D^2} f(x, y) \mu(dx, dy) - \int_D f(x, p(x)) dx.$$



We denote by  $\eta$  the modulus of continuity of  $f$ .  $I_1$  is equal, up to an error of  $\eta(2^{-n+d/2})$ , to

$$I_3 = \frac{1}{N} \sum_{i,j} f(x_{n,i}, x_{n,j}) \nu_{ij}.$$

$I_3$  is equal, up to an error of  $\sup |f|K/L$  to

$$I_4 = \frac{1}{N} \sum_{i,j} f(x_{n,i}, x_{n,j}) \nu'_{ij} = \frac{1}{NL} \sum_{i,m} f(x_{n,i}, x_{n,\sigma_m(i)}).$$

Up to  $\eta(2^{-n+d/2})$ ,  $I_4$  is equal to

$$I_5 = \frac{1}{NL} \sum_{i,m} f(x_{n+l,i,m}, x_{n+l,\sigma_m(i),m}).$$

$I_5$ , up to  $\eta(2^{-n-l+d/2})$ , is equal to

$$I_6 = \sum_{i,m} \int_{D_{n+l,i,m}} f(x, x - x_{n+l,i,m} + x_{n+l,\sigma_m(i),m}),$$

which is exactly  $I_2$ , by definition of  $p$ . Finally, we have shown

$$|I_1 - I_2| \leq \sup |f| 2^{(2n-l)d} + 3\eta(2^{-n-l+d/2}),$$

since  $L = 2^{ld}$ ,  $K = N^2 = 2^{2nd}$ . This completes the proof, after letting first  $l$  and then  $n$  to  $+\infty$ .

# Chapter 2

## Hidden convexity in the Euler equations of incompressible fluids

### 2.1 The central place of the Euler equations among PDEs

This section, where we discuss the importance of the Euler equations of fluids among PDEs, can be skipped by the reader in a hurry who may go directly to section 2.2...Anyway, as Laplace used to say:

**"Lisez Euler, il est notre maître à tous !"**

In our opinion, it is very difficult to question the priority and the centrality of the Euler equations of fluids in Mathematics, Mechanics, Physics and Geometry:

1) Euler's theory of fluids, entirely described in terms of density, velocity and pressure fields, governed by a self-consistent set of partial differential equations, provides the first "Field Theory" ever in Physics, before the theories later developed by Maxwell (Electromagnetism), Einstein (Gravitation), Schrödinger and Dirac (Quantum Mechanics).

2) The Euler model is the backbone of a very large part of Natural Sciences (Fluid Mechanics, Oceanography, Weather Forecast, Climatology, Convection Theory, Dynamo Theory...).

3) To the best of our knowledge, Euler's equations form the first self-consistent system of PDEs ever written, in 1755-57 [230], except the 1D linear wave equation which was introduced and solved by d'Alembert few years earlier in 1746 [4]. (See also [131].) It is striking to compare the style of [4] and [230]. Euler introduced remarkably modern notations that are still easily readable. On top of that, while the 1D wave equation is now considered as a rather trivial equation (which in no way diminishes the merit of d'Alembert for his elegant solution at such an early stage of mathematical Analysis!), the solution of the Euler equations, after a quarter of millennium, is still considered as one of the most challenging problem in PDEs (typ-

ically, together with the solution of the Einstein and the Navier-Stokes equations).

4) The Euler equations already (implicitly) contain the wave, heat and Poisson equations, which are the basic equations of respectively hyperbolic, parabolic and elliptic type, according to the traditional terminology of PDEs [231, 293, 307, 442], and, also, the advection equation (which is just an ODE rephrased as a PDE).

5) The Euler model of incompressible fluids admits a remarkable geometric interpretation due to Arnold [21, 22, 220] that makes it an archetype of Geometry in infinite dimension (for which we may refer, among many others, to [19, 20, 176, 220, 221, 248, 250, 264, 367, 371, 410, 439, 448]...). Indeed, in the case of a fluid moving in a compact Riemannian manifold  $\mathcal{M}$ , the Euler equations just describe constant speed geodesic curves along the (formal) Lie group  $SDiff(\mathcal{M})$  of all volume and orientation preserving diffeomorphisms of  $\mathcal{M}$ , with respect to the  $L^2$  norm on its (formal) Lie Algebra, made of all divergence-free vector fields along  $\mathcal{M}$ . In the case of a fluid moving inside the unit cube,  $D = [0, 1]^d$ , this amounts, in more elementary terms, to looking for curves  $t \in \mathbb{R} \rightarrow X_t \in SDiff(D) \subset H = L^2(D; \mathbb{R}^d)$  that minimize

$$\int_{t_0}^{t_1} \left\| \frac{dX_t}{dt} \right\|_H^2 dt,$$

on short enough intervals  $[t_0, t_1]$ , as the time-boundary values  $X_{t_0}, X_{t_1}$  are fixed. These geodesic curves can also be seen as "harmonic maps" from  $\mathbb{R}$  to  $SDiff(D)$ .

*Remark*

This immediately suggests a generalization to "harmonic maps" or, rather, "wave maps" from an open set of  $\mathbb{R}^2$  to  $SDiff(D)$ , which, as a matter of fact, corresponds to the particular ideal incompressible model in the wider field of Electromagnetohydrodynamics for which we refer, among many other references, to [22, 59, 255, 300, 374]. We may also consider the corresponding "harmonic heat flow" which more or less correspond to the model of magnetic relaxation [22, 102, 107, 373]. To the best of our knowledge, "harmonic maps" valued in the infinite dimensional group  $SDiff(D)$  have never been investigated so far, in spite of the paramount importance in geometric analysis of harmonic maps when they are valued in finite dimensional Riemannian manifolds [132, 289, 408, 417].

## The Euler equations

Here below are the equations written by Euler in 1755/57 [230], where we use the familiar notation  $\nabla$  for the partial derivatives. (They are denoted more explicitly by Euler, with a notation already modern. See below a fac simile of [230].)

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \partial_t v + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla(p(\rho))$$

where  $(\rho, p, v) \in \mathbb{R}^{1+1+3}$  denote the density, pressure and velocity fields of the fluid, the pressure being assumed by Euler to be a given function of the density. They can also be written in "conservation form"

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) = -\nabla(p(\rho))$$

and also "in coordinates" (which can be easily extended to the framework of Riemannian manifolds)

$$\partial_t \rho + \partial_j(\rho v^j) = 0, \quad \partial_t(\rho v_i) + \partial_j(\rho v^j v_i) = -\partial_i(p(\rho)).$$

(In the Euclidean case  $v_i$  is just a notation for  $\delta_{ij}v^j$ , but in the Riemannian case  $v_i = g_{ij}v^j$  definitely involves the metric tensor  $g$ .) It is important to emphasize that, in the same paper, Euler also addresses the case of incompressible fluids, for which

$$\partial_t v + \nabla \cdot (v \otimes v) + \nabla p = 0, \quad \nabla \cdot v = 0,$$

or, equivalently,

$$\partial_t v_i + \partial_j(v^j v_i) = -\partial_i p, \quad \partial_i v^i = 0,$$

which corresponds, grosso modo, to a constant unit density field and where  $p$  becomes an unknown field that balances the divergence-free condition on  $v$ . As a matter of fact,  $p$  can be eliminated (up to boundary conditions that we do not discuss at this stage) by applying the divergence operator, which leads to the Poisson equation for  $p$

$$-\Delta p = (\nabla \otimes \nabla) \cdot (v \otimes v),$$

(Note that the passage from the compressible case to the incompressible case is now very well understood at the mathematical level [308, 313, 365].)

In fact, it is important for many applications, in particular in Geophysics, to consider incompressible *inhomogeneous* fluids. This means that the velocity is still considered to be divergence-free but the density may vary. The resulting equation are

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) + \nabla p + \rho \nabla \Phi = 0, \quad \nabla \cdot v = 0,$$

where we have included an external potential  $\Phi$  (typically the gravity potential). Note that, due to the divergence-free condition, such a potential has no effect in the homogeneous case when  $\rho$  is constant. (This is why the feeling of gravity is so weak for us when we are swimming under water because our density is essentially the same as water.) However, for inhomogeneous fluid, the impact of  $\Phi$  may be considerable. As a matter of fact, this is the origin of convective phenomena, which play an amazingly important role in Natural Sciences (climate, volcanism, earthquakes, continental drift, terrestrial magnetism,..) and daily life (weather, heating, boiling etc...) and will be considered in Chapter 7.

XXI. Nous n'avons donc qu'à éгалer ces forces accélératrices avec les accélérations actuelles que nous venons de trouver, & nous obtiendrons les trois équations suivantes :

$$P - \frac{1}{q} \left( \frac{dp}{dx} \right) = \left( \frac{du}{dt} \right) + u \left( \frac{du}{dx} \right) + v \left( \frac{du}{dy} \right) + w \left( \frac{du}{dz} \right)$$

$$Q - \frac{1}{q} \left( \frac{dp}{dy} \right) = \left( \frac{dv}{dt} \right) + u \left( \frac{dv}{dx} \right) + v \left( \frac{dv}{dy} \right) + w \left( \frac{dv}{dz} \right)$$

$$R - \frac{1}{q} \left( \frac{dp}{dz} \right) = \left( \frac{dw}{dt} \right) + u \left( \frac{dw}{dx} \right) + v \left( \frac{dw}{dy} \right) + w \left( \frac{dw}{dz} \right)$$

Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la considération de la continuité du fluide :

$$\left( \frac{dq}{dt} \right) + \left( \frac{d.qu}{dx} \right) + \left( \frac{d.qv}{dy} \right) + \left( \frac{d.qw}{dz} \right) = 0.$$

Si le fluide n'étoit pas compressible, la densité  $q$  seroit la même en  $Z$ , & en  $Z'$ , & pour ce cas on auroit cette équation :

$$\left( \frac{du}{dx} \right) + \left( \frac{dv}{dy} \right) + \left( \frac{dw}{dz} \right) = 0.$$

qui est aussi celle sur laquelle j'ai établi mon Mémoire latin allégué ci-dessus.

qui obeit à son action. Cette idée de l'effort est de la dernière importance dans toute la Théorie, tant de l'équilibre que du mouvement, ayant fait voir, que la somme de tous les efforts est toujours un *maximum* ou *minimum*. Cette belle propriété convient admirablement avec le beau principe de la moindre action ; dont nous devons la découverte à notre Illustre Président, M. de *Maupertuis*.

tombent dans la surface même. Or nous voyons par là suffisamment, combien nous sommes encore éloignés de la connoissance complète du mouvement des fluides, & que ce que je viens d'expliquer, n'en contient qu'un foible commencement. Cependant tout ce que la Théorie des fluides renferme, est contenu dans les deux équations rapportées cy-dessus (§. XXXIV.), de sorte que ce ne sont pas les principes de Méchanique qui nous manquent dans la poursuite de ces recherches, mais uniquement l'Analyse, qui n'est pas encore assez cultivée, pour ce dessein : & partant on voit clairement, quelles découvertes nous restent encore à faire dans cette Science, avant que nous puissions arriver à une Théorie plus parfaite du mouvement des fluides.

## The Euler system as a master equation

Let us now formally check that the most basic PDEs (heat, wave, Poisson and advection equations [231, 293, 307, 442]) are hidden behind the Euler equations.

### From Euler to the heat equation

We may recover the heat equation (and more generally the "porous medium" equation) from the Euler equations of *compressible* fluids, through a very simple process that does not seem to be so well-known in the PDE literature, just by a straightforward, quadratic, change of time. This technique will be used later in this book, in Chapter 9. We start from a solution, denoted by  $(\tilde{\rho}, \tilde{v})(t, x)$ , of the Euler equations

$$\partial_t \tilde{\rho} + \nabla \cdot (\tilde{\rho} \tilde{v}) = 0, \quad \partial_t (\tilde{\rho} \tilde{v}) + \nabla \cdot (\tilde{\rho} \tilde{v} \otimes \tilde{v}) = -\nabla(p(\tilde{\rho}))$$

(where, following Euler, the pressure  $p$  is a known function of the density). We perform the quadratic change of time:

$$t \rightarrow \tau = t^2/2, \quad \frac{d\tau}{dt} = t, \quad (\tilde{\rho}, \tilde{v})(t, x) = (\rho(\tau, x), \frac{d\tau}{dt} v(\tau, x)),$$

(so that  $\tilde{v}(t, x)dt = v(\tau, x)d\tau$ ). We easily obtain

$$\partial_\tau \rho + \nabla \cdot (\rho v) = 0, \quad \rho v + 2\tau (\partial_\tau (\rho v) + \nabla \cdot (\rho v \otimes v)) = -\nabla(p(\rho)).$$

For very short times  $\tau \ll 1$ , we get an asymptotic equation by withdrawing all terms in factor of  $\tau$ . We are left with

$$\partial_\tau \rho + \nabla \cdot (\rho v) = 0, \quad \rho v = -\nabla(p(\rho))$$

which, in the "isothermal" case when  $p$  is linear in  $\rho$ , i.e.  $p = \gamma^2 \rho$ , with "sound speed"  $\gamma$ , is nothing but the heat equation (solved by Fourier in the 19th century, half of a century after Euler's work on fluids):

$$\partial_\tau \rho = \gamma^2 \Delta \rho, \quad \Delta = \nabla \cdot \nabla.$$

In the general case, we get the so-called "porous medium" equation [449]

$$\partial_\tau \rho = \Delta(p(\rho)),$$

that will be addressed later in this book, in Chapter 5.

### From Euler to the wave equation

By inputting

$$(\tilde{\rho}, \tilde{v})(t, x) = (\rho^* + \epsilon \rho(t, x), \epsilon v(t, x)),$$

(where  $\epsilon$  is small and  $\rho^*$  is a constant density of reference) in the Euler equations of compressible fluids

$$\partial_t \tilde{\rho} + \nabla \cdot (\tilde{\rho} \tilde{v}) = 0, \quad \partial_t (\tilde{\rho} \tilde{v}) + \nabla \cdot (\tilde{\rho} \tilde{v} \otimes \tilde{v}) = -\nabla(p(\tilde{\rho})),$$

we find

$$\partial_t \rho + \nabla \cdot ((\rho^* + \epsilon \rho)v) = 0$$

$$\partial_t((\rho^* + \epsilon\rho)v) + \nabla \cdot ((\rho^* + \epsilon\rho)\epsilon v \otimes v) = -\nabla \left( \frac{p(\rho^* + \epsilon\rho) - p(\rho^*)}{\epsilon} \right).$$

In the regime  $\epsilon \ll 1$ , for "small density and velocity fluctuations", one obtains an asymptotic equation by dropping the smallest terms and using

$$p(\rho^* + \epsilon\rho) = p(\rho^*) + \epsilon p'(\rho^*)\rho + O(\epsilon^2).$$

We are left with

$$\partial_t \rho + \rho^* \nabla \cdot v = 0, \quad \rho^* \partial_t v + p'(\rho^*) \nabla \rho = 0$$

which is nothing but the famous wave equation (that d'Alembert had solved in one space dimension, few years before Euler's work on fluids [4]) :

$$\partial_{tt}^2 \rho = \gamma^2 \Delta \rho$$

(after eliminating  $v$ ), with "sound speed"  $\gamma = \sqrt{p'(\rho^*)}$ .

## 2D Euler equations as a *coupling* of two linear PDEs

In the case of incompressible fluids, where  $\nabla \cdot v = 0$ , and in two space dimensions, we may write (at least locally)

$$v = (-\partial_2 \psi, \partial_1 \psi)$$

for some scalar function  $\psi = \psi(t, x)$  (usually called "stream function"). By setting  $\omega = \partial_2 v_1 - \partial_1 v_2$ , we easily get both

$$-\Delta \psi = \omega$$

and

$$\partial_t \omega + (v \cdot \nabla) \omega = 0.$$

In this case, the Euler equations can be interpreted as a non-trivial coupling of two elementary linear PDEs:

1) The Poisson equation, prototype of elliptic PDEs,

$$-\Delta \psi = \omega$$

where  $\psi$  is unknown and  $\omega$  given;

2) The transport (or advection) equation

$$\partial_t \omega + (v \cdot \nabla) \omega = 0.$$

where  $\omega$  is unknown while  $v = (-\partial_2 \psi, \partial_1 \psi)$  is given.

## Euler equations and ODEs

By integrating the velocity field  $v$  of the fluid, we may recover the trajectory of each fluid parcel, labeled by  $a$ , through

$$\frac{dX_t}{dt}(a) = v(t, X_t(a)).$$



(It is common, but not necessary, to use the initial position as a label, so that  $X_0(a) = a$ .) Thanks to the chain rule, we immediately see that the Euler equation

$$\partial_t v + (v \cdot \nabla)v = -\frac{\nabla p}{\rho}$$

has no other meaning than the 2nd order ODE

$$\frac{d^2 X_t}{dt^2}(a) = -\left(\frac{\nabla p}{\rho}\right)(t, X_t(a)).$$

In the case of homogeneous incompressible fluids of unit density, we just get

$$\frac{d^2 X_t}{dt^2}(a) = -(\nabla p)(t, X_t(a)).$$

As a matter of fact, in his paper [230], Euler *starts* from this 2nd order EDO and gets his famous equations after introducing the key concept of velocity field. (This fact is frequently ignored in the literature.) The link with ODEs is even more striking in the special case of homogeneous incompressible fluids in two space dimensions. Indeed, the "vorticity equation"

$$\partial_t \omega + (v \cdot \nabla)\omega = 0$$

just means that  $\Omega(t, a) = \omega(t, X_t(a))$  is time independent. Indeed, the vorticity equation is just equivalent to the trivial ODE

$$\frac{d\Omega}{dt} = 0.$$

This can be very fruitfully exploited at the computational level [168, 183, 400].

### Few words on the analysis of the Euler equations

So far, we have not addressed the Euler equations from the Analysis viewpoint. This is somewhat consistent with the prophetic conclusion of Euler's paper [230]:

*“Tout ce que la théorie des fluides renferme est contenu dans les deux équations rapportées ci-dessus, de sorte que ce ne sont pas les principes de Mécanique qui nous manquent dans la poursuite de ces recherches, mais uniquement l'Analyse, qui n'est pas encore assez cultivée, pour ce dessein.”*

A quarter of millennium later, progresses have been indeed significant but not yet conclusive (cf. [162, 171, 331, 352, 353]...). So, the Analysis of the Euler equations, which are essentially the first PDEs ever written, persists as a major challenge in the field of nonlinear PDEs. Let us start with the case of homogeneous incompressible fluids and quote what we believe to be some of the most noticeable results obtained so far (mostly in the case  $D = \mathbb{T}^d$ , for simplicity):

1) A unique smooth classical solution always exists for a short while, as long as the initial velocity field  $v_0$  is smooth (i.e. with Hölder continuous derivatives) and this solution is global in the 2D case  $d = 2$  [326, 462]. However the vorticity gradient may exhibit a double exponential growth in time (at least as  $D$  is a disk) [304]. In

addition, the trajectories of the fluid are known to be time-analytic [419] (see [60] for a recent account).

2) In the 2D case, a unique global solution exists (in a suitable generalized sense) as soon as the initial vorticity  $\omega_0$  (i.e. the curl of  $v_0$ ) is essentially bounded on  $D$  [464]. Moreover, the smoothness of the vorticity level sets is preserved during the evolution [161] (which has been a very striking result going very much against numerical simulations which predicted formation of singularities in finite time). There are always global weak solutions in the special class of vorticity fields  $\omega(t, x)$  that stay, at any time  $t$ , a nonnegative bounded measure up to the addition of an  $L^1$  function in  $x$  [199].

3) Weak solutions  $v \in L^2$  in the sense of distributions globally exist for any fixed initial condition  $v_0 \in L^2(D; \mathbb{R}^d)$ , but there are uncountably many of them [459]! This is a rather direct consequence of the analysis by "convex integration" of the Euler equations performed in [196, 197]; through similar methods, there exist weak solutions  $v(t, x)$  that are Hölder continuous of exponent  $\alpha$  less than  $1/3$  in  $x$  and do not preserve their kinetic energy (resolution of the so-called "Onsager conjecture" [297, 135]) although, whenever  $\alpha > 1/3$ , the kinetic energy is necessarily conserved [177, 235].

4) Global generalized solutions, called "dissipative solutions", always exist in  $C^0(\mathbb{R}_+; L_w^2(D))$ , as soon as  $v_0 \in L^2(D; \mathbb{R}^d)$  [331]; they are not necessarily weak solutions but their kinetic energy cannot exceed its initial value and they enjoy the "weak-strong uniqueness principle" in the sense that if there is a classical solution with initial condition  $v_0$  then this solution is unique in the class of dissipative solutions starting from  $v_0$ . (See [114, 210, 216] for related concepts of generalized solutions.)

5) From a more geometric viewpoint, the geodesic flow on the group  $SDiff(D)$  is well defined, in a classical sense, but only in a tiny neighborhood of the identity map for a very fine (Sobolev) topology [220]. Nevertheless, as  $d = 3$ , one can prove the existence of many orientation and volume preserving diffeomorphisms, that are trivial in the third space coordinate, i.e. of form  $h(x_1, x_2, x_3) = (H(x_1, x_2), x_3)$ , that can be connected by smooth paths of finite length to the identity map but none of them has minimal length [429] (see also the related work [225]). In this case, the minimizing geodesic problem can be *relaxed* as a *convex* minimization problem in a suitable space of measures, which always admits generalized solutions, with the additional property that there is a *unique* pressure gradient attached to them, that only depends on  $H$  and approximately "accelerates" all paths of approximately minimal length [91]. Thanks to an appropriate density result [430], this result still applies to more general data, in particular to all  $h$  in  $SDiff(D)$  [11].

In the case of compressible fluids, the results are less complete. Roughly speaking, the 4th first results extend, except that the second one, proving global existence of suitable "entropy" solutions, is valid only for  $d = 1$  and for initial data that are small enough in total variation. Both the existence part [271] and the well-posedness (uniqueness and stability) part [63, 128] are remarkable achievements of the theory of hyperbolic nonlinear systems of conservation laws.

## 2.2 Hidden convexity in the Euler equations: The geometric viewpoint

A simple geometric definition (going back to Arnold [21]) of the Euler equations of an incompressible fluid, confined in a compact domain  $D \subset \mathbb{R}^d$  without any external force, amounts to finding curves

$$t \in \mathbb{R} \rightarrow X_t \in SDiff(D) \subset H = L^2(D; \mathbb{R}^d)$$

that minimize

$$\int_{t_0}^{t_1} \left\| \frac{dX_t}{dt} \right\|_H^2 dt,$$

on any sufficiently short time intervals  $[t_0, t_1]$ , as  $X_{t_0}, X_{t_1}$  are fixed. Here  $SDiff(D)$  is the group of all orientation and volume preserving diffeomorphisms of  $D$ . (Alternately, we could consider the larger semi-group  $VPM(D)$  of all volume preserving Borel maps of  $D$ , which is the  $L^2$  completion of  $SDiff(D)$ , as long as  $d \geq 2$ , as already discussed in Chapter 1.) In other words, the Euler equations obey to the Least Action Principle (LAP) ("le beau principe de (la) moindre action", as expressed by Euler himself [230]), the "configuration space" being  $SDiff(D)$ .

We have already discussed at the beginning of this book, in Chapter 1, at least in the case  $D = [0, 1]^d$ , the completion of  $SDiff(D)$  and  $VPM(D)$  by the convex compact set  $DS(D)$  of all doubly stochastic measures on  $D \times D$ . So, it is tempting to get a generalized version of the LAP by substituting  $DS(D)$  for  $SDiff(D)$ , taking into account that  $SDiff(D)$ , viewed as a subset of the ambient Hilbert space  $H = L^2(D; \mathbb{R}^d)$ , is neither compact nor convex. As a matter of fact, it is not difficult to attach to any curve  $t \rightarrow X_t \in SDiff(D)$  a corresponding curve of doubly stochastic measures  $t \rightarrow c_t \in DS(D)$ , just by setting

$$\int_{D^2} f(x, a) dc_t(x, a) = \int_D f(X_t(a), a) da, \quad \forall f \in C^0(D^2)$$

or, in short,

$$dc_t(x, a) = \delta(x - X_t(a)) da.$$

However, this is not enough to define a reasonable dynamical system describing geodesics on  $DS(D)$ . So we also attach a curve of vector-valued Borel measures

$$t \rightarrow q_t \in (C^0(D^2; \mathbb{R}^d))'$$

by setting

$$\int_{D^2} f(x, a) \cdot dq_t(x, a) = \int_D \frac{dX_t}{dt}(a) \cdot f(X_t(a), a) da, \quad \forall f \in C^0(D^2; \mathbb{R}^d)$$

where  $\frac{dX_t}{dt}(a)$  just denotes the partial derivative  $\partial_t X_t(a)$ . We may also write, more briefly,

$$dq_t(x, a) = \frac{dX_t}{dt}(a) \delta(x - X_t(a)) da.$$

Notice that  $q_t$  is automatically absolutely continuous with respect to  $c_t$  so that we can write its Radon-Nikodym derivative as  $(x, a) \rightarrow v_t(x, a) \in \mathbb{R}^d$  and denote:

$$dq_t(x, a) = v_t(x, a) dc_t(x, a).$$

(This idea is not new, it is just an avatar of the concept of "current", familiar in Geometric Measure Theory. See [237, 376] and [15] as a recent reference. Let us also quote the related concept of Young's measures [31, 440, 463].) An important property of measures  $c$  and  $q$  is their link through the following linear PDE

$$\partial_t c_t + \nabla_x \cdot q_t = 0,$$

satisfied in the sense of distributions. Indeed, for every test function  $f = f(x, a)$  defined on  $D \times D$ , we have

$$\begin{aligned} \frac{d}{dt} \int_{D^2} f(x, a) dc_t(x, a) &= \frac{d}{dt} \int_D f(X_t(a), a) da \\ &= \int_D (\nabla_x f)(X_t(a), a) \cdot \frac{dX_t}{dt}(a) da = \int_{D^2} f(x, a) dq_t(x, a). \end{aligned}$$

Another key point is that we can rewrite the "kinetic energy" just in terms of  $c$  and  $q = cv$ :

$$\frac{1}{2} \left\| \frac{dX_t}{dt} \right\|_H^2 = \frac{1}{2} \int_{D^2} |v_t(x, a)|^2 dc_t(x, a).$$

To check this identity, let us write the right-hand side in a dual way as:

$$\begin{aligned} &\frac{1}{2} \int_{D^2} |v_t(x, a)|^2 dc_t(x, a) = \\ &\sup \left\{ \int_{D^2} \left( -\frac{1}{2} |f(x, a)|^2 + f(x, a) \cdot v_t(x, a) \right) dc_t(x, a); \quad f \in C^0(D^2; \mathbb{R}^d) \right\} \end{aligned}$$

(here we use the density of continuous functions in the space of  $L^2$  functions with respect to measure  $c_t$ )

$$\begin{aligned} &= \sup \left\{ \int_{D^2} \left( -\frac{1}{2} |f(x, a)|^2 dc_t(x, a) + f(x, a) \cdot dq_t(x, a) \right); \quad f \in C^0(D^2; \mathbb{R}^d) \right\}. \\ &= \sup \left\{ \int_D \left( -\frac{1}{2} |f(X_t(a), a)|^2 + f(X_t(a), a) \cdot \frac{dX_t}{dt}(a) \right) da; \quad f \in C^0(D^2; \mathbb{R}^d) \right\}. \end{aligned}$$

(by definition of  $c$  and  $q = cv$ )

$$= \frac{1}{2} \int_D \left| \frac{dX_t}{dt}(a) \right|^2 da$$

(by completion of squares, using that  $a \rightarrow X_t(a)$  is one-to-one since  $X_t$  belongs to  $SDiff(D)$ ). These relations are of particular interest, since they provide a *convex* expression in terms of  $(c, q)$ :

$$\sup \left\{ \int_{D^2} \left( -\frac{1}{2} |f(x, a)|^2 dc_t(x, a) + f(x, a) \cdot dq_t(x, a) \right); \quad f \in C^0(D^2; \mathbb{R}^d) \right\}.$$

We may even go a little further, in defining for a any pair  $(c, q) \in (C^0(D^2; \mathbb{R} \times \mathbb{R}^d))'$

$$K(c, q) = \sup \left\{ \int_{D^2} A(x, a) dc(x, a) + B(x, a) \cdot dq(x, a); \right.$$

$$(A, B) \in C^0(D^2; \mathbb{R} \times \mathbb{R}^d) \text{ s.t. } 2A + |B|^2 \leq 0\},$$

which defines a l.s.c convex function valued in  $] - \infty, +\infty]$  without any restriction on  $(c, q) \in (C^0(D^2; \mathbb{R} \times \mathbb{R}^d))'$ , not even that  $c \geq 0$ . Indeed, it can be shown that  $K(c, q)$  takes the value  $+\infty$  unless  $c \geq 0$ ,  $q$  is absolutely continuous with respect to  $c$ , with Radon-Nikodym derivative  $v$ , square integrable in  $c$ , in which case

$$\frac{1}{2} \int_{D^2} |v(x, a)|^2 dc(x, a).$$

(This can be shown by elementary arguments of Measure Theory. See [90] for more details.) So, we are now ready to formulate the LAP entirely in terms of  $(c, q)$  by requiring the minimization on each sufficiently short time interval  $[t_0, t_1]$  of

$$\int_{t_0}^{t_1} K(c_t, q_t) dt,$$

under the constraints that  $c_t$  is doubly stochastic, i.e.  $c_t \in DS(D)$ , and satisfies, together with  $q_t$  the linear PDE

$$\partial_t c_t + \nabla_x \cdot q_t = 0,$$

while the time-boundary values  $c_{t_0}, c_{t_1}$  are fixed in  $DS(D)$ . The novelty of this formulation is that we may now ignore that  $c$  and  $q$  have been derived from some curve  $t \rightarrow X_t \in SDiff(D)$ . In other words, we have a possible relaxed version of the LAP, with the remarkable advantage that the formulation is now entirely convex! In a more geometric wording, we can interpret this relaxed problem as the "minimizing geodesic" problem along  $DS(D)$  between two given points of  $DS(D)$ . Although the detailed study of this problem will be done in Chapter 4, we may already at this stage provide a synthesis of the results obtained in [91], extended and improved in [10, 11, 32, 35, 105, 137, 344].

For notational simplicity, it is convenient to normalize  $t_0 = 0, t_1 = 1$  and denote  $c_{t_0}, c_{t_1}$  by  $c_0, c_1$ . We will also use the following notations:

- i)  $c(t, x, a), q(t, x, a), v(t, x, a)$  instead of  $c_t(x, a), q_t(x, a), v_t(x, a)$ ;
- ii)  $\int_{x,a} f(x, a) c(t, x, a)$  rather than  $\int_{D^2} f(x, a) dc_t(x, a)$ , etc...

**Theorem 2.2.1.** *Let  $D$  be the periodic cube  $D = \mathbb{T}^d$ . Given any data  $c_0$  and  $c_1$  in the convex compact set of all doubly stochastic measures on  $D$ , the relaxed minimizing geodesic problem always admits at least one solution  $(c, cv)$  and there is a unique pressure gradient  $(t, x) \in ]0, 1[ \times D \rightarrow \nabla p(t, x) \in \mathbb{R}$ , depending only on  $c_0$  and  $c_1$  such that*

$$\partial_t \int_a (cv)(t, x, a) + \nabla_x \cdot \int_a (cv \otimes v)(t, x, a) = -\nabla p(t, x)$$

whatever solution  $(c, cv)$  is.

In addition,  $\nabla p$  has some limited regularity: it is locally square integrable in time with values in the space of bounded measures on  $D = \mathbb{T}^d$ .

Moreover, whenever  $d \geq 2$ , each optimal solution  $(c, cv)$  can be weakly-\* approximated by a family of smooth curves  $t \in [0, T] \rightarrow T_t^\epsilon \in SDiff(D)$ , in the sense that, denoting

$$v^\epsilon = \frac{dT_t^\epsilon}{dt} \circ (T_t^\epsilon)^{-1},$$

the corresponding measures

$$\left(1, \frac{dT_t^\epsilon}{dt}(a)\right)\delta(x - T_t^\epsilon(a))$$

weakly-\* converge to  $(c, cv)(t, x, a)$  and without gap of energy, in the sense

$$\int_0^1 \int_D |v^\epsilon(t, x)|^2 dx dt \rightarrow \int_0^1 \int_{x,a} (c|v|^2)(t, x, a) dt.$$

Finally, the  $v^\epsilon$  are almost solutions to the Euler equations in the sense that

$$\partial_t v^\epsilon + \nabla \cdot (v^\epsilon \otimes v^\epsilon) \rightarrow -\nabla p,$$

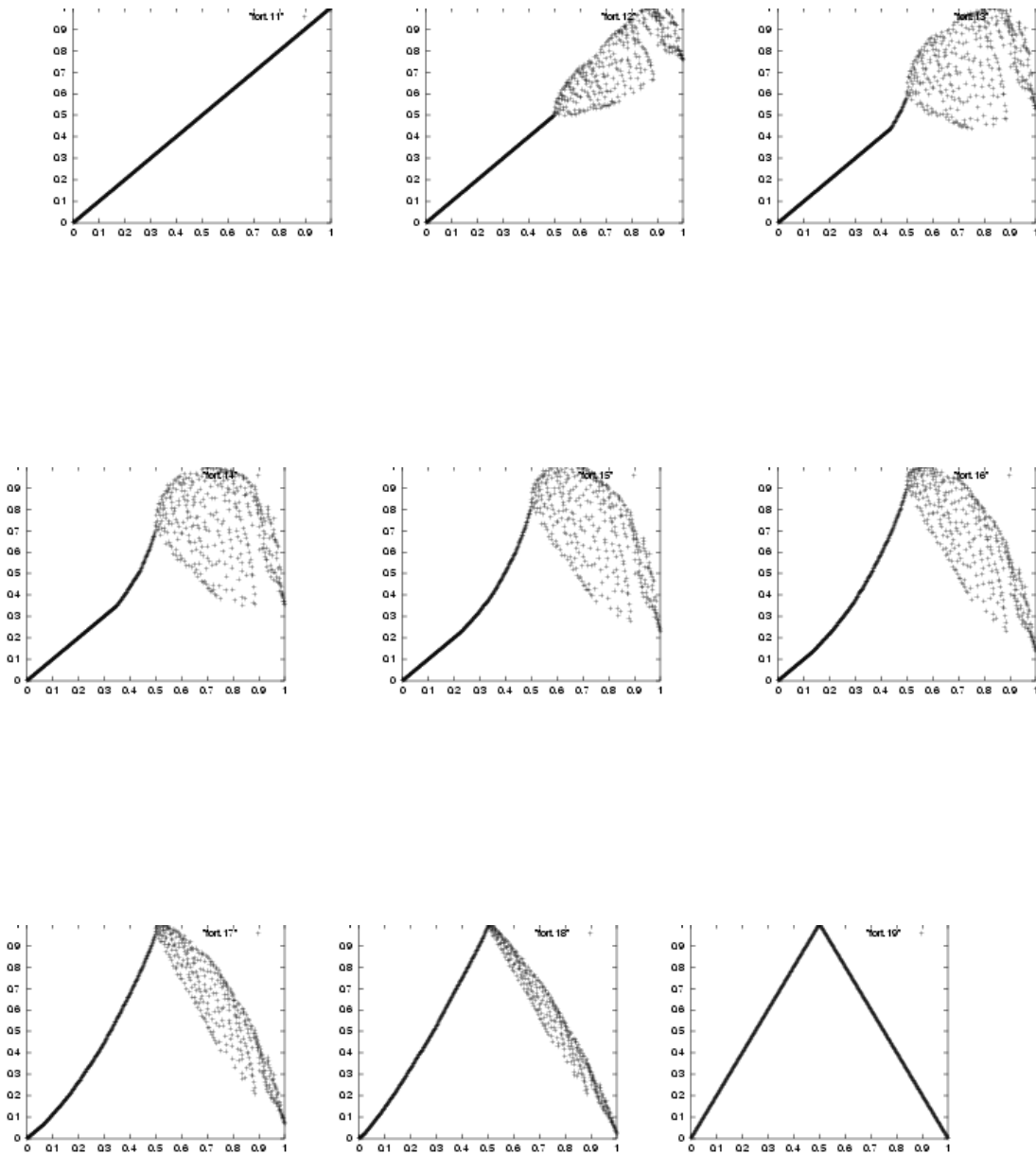
where  $\nabla p$  is the unique pressure gradient attached to the data  $(c_0, c_1)$ .

Let us emphasize that it is very surprising that the pressure gradient is uniquely determined by the data. Indeed, let us consider, as Arnold did in his founding paper [21], the finite dimensional counterpart of the Euler model of incompressible fluids, namely the model of rigid bodies, where the finite dimensional Lie group  $SO(3)$  substitutes for  $SDiff(D)$ , and a non-degenerate quadratic form (corresponding to the matrix of inertia of the rigid body) substitute for the  $L^2$  metric. Then the geodesic curves precisely describe the motion of a perfect rigid body moving in vacuum (without external forces). There is also a substitute for the pressure gradient, which turns out to be a  $3 \times 3$  symmetric time dependent matrix which is attached to each geodesic, and acts in order to preserve the rigidity of the body. Then one can find examples of two minimizing geodesics having the same end-points for which these matrices are not the same [105]. As a matter of fact, the uniqueness of the pressure gradient is, in our opinion, a striking manifestation of “hidden convexity” due to the infinite dimension of  $SDiff(D)$  and the convexity of its weak completion  $DS(D)$ . So, in some sense, we have a rather sophisticated avatar of Theorem 1.1.2 (stating that, in Hilbert spaces, the unit ball is the right weak completion of the unit ball if only if the dimension of the space is infinite). There is certainly some room to improve the results we have just mentioned. In particular, it would be very useful to know the precise regularity of the pressure field. There is some evidence [105] that the pressure  $p(t, x)$  should be, locally in time in  $]0, 1[$ , semi-concave in  $x$ , and not more in general, which means that the derivatives in  $x$  of  $p$  should be Borel measures up to *second* order and not only to first order as in the Theorem!

To conclude this sub-section, let us just us mention a striking additional property: the "Boltzmann entropy"

$$\int_{x,a} (c \log c)(t, x, a)$$

is convex in  $t$  along every generalized minimizing geodesic. This has been conjectured in [96] and proven first by Lavenant [318] (with some restrictions) and then by Baradat-Monsaingeon [35]. In our opinion, this convexity might be an indication that  $SDiff(D)$  has, in some suitable sense, a nonnegative Ricci curvature (in the spirit of Lott-Sturm-Villani [345, 436]). This would be another striking manifestation of “hidden convexity”, since, in the classical framework, the measures  $c(t, x, a)$  are delta measures and their Boltzmann entropy is *always* infinite!



Example of a minimizing geodesic along  $DS(D)$ ,  $D = [0, 1]$ . Note that only the end points belong to  $VPM(D)$ .  
 (Numerical approximation using permutation maps.)

## 2.3 Hidden convexity in the Euler equations: the Eulerian viewpoint

Let us go back to the classical setting, where the Euler equations of incompressible fluids read

$$\partial_t v + \nabla \cdot (v \otimes v) + \nabla p = 0, \quad \nabla \cdot v = 0,$$

and mention the remarkable results of De Lellis et Székelyhidi [196, 197, 198], based on the concepts of differential inclusions and convex integration that go back to the work of Gromov, Nash et Tartar [284, 380, 440]. (See also [191].) They follow earlier works by Constantin-E-Titi, Eyink, Scheffer, Shnirelman, about the so-called "Onsager conjecture" [177, 235] and the existence of non trivial space-time compactly supported weak solutions [416, 431]. Let also quote subsequent papers [135, 197, 198, 297] among many others.

A key point in the analysis is the *convex* concept of subsolution to the Euler equations. We say that a pair  $(V, M)$  is such a subsolution if

1) There is a scalar function  $p$  (the "pressure") such that

$$\partial_t V + \nabla \cdot M + \nabla p = 0, \quad \nabla \cdot V = 0$$

holds true, in the sense of distributions. In coordinates, this reads

$$\partial_t V^i + \partial_j M^{ij} + \partial^i p = 0, \quad \partial_i V^i = 0.$$

ii)  $M \geq V \otimes V$  holds true in the sense of distributions and symmetric matrices.

We immediately note that a subsolution  $(V, M)$  becomes a weak solution as soon as inequality  $M \geq V \otimes V$  is saturated:  $M = V \otimes V$ .

In terms of functional spaces, the concept of subsolution requires a very limited amount of regularity. Typically, in the simple case when  $Q = [0, T] \times D$  with  $D = \mathbb{T}^d$ , it makes sense as soon as  $V \in L^2(Q; \mathbb{R}^d)$  and  $M$  is a bounded Borel measure valued in the convex cone of all nonnegative symmetric matrices. We may add an initial condition  $V_0$ , typically an  $L^2$  divergence-free vector field, to the concept of subsolution  $(V, M)$  by requiring

$$\int_Q \partial_t A_i(t, x) V^i(t, x) dt dx + \partial_j A_i(t, x) M^{ij}(dt dx) + \int_D V_0^i(x) A_i(0, x) dx = 0,$$

for all smooth divergence-free vector field  $A = A(t, x) \in \mathbb{R}^d$  such that  $A(T, x) = 0$ . Notice, however, that since a priori  $M$  is just a measure,  $V(t, x)$  may not depend continuously on  $t$  (just enjoying a bounded variation) and, therefore, there is no reason that  $V(t, x)$  achieves  $V_0$  as  $t \downarrow 0$ . We will discuss this kind of problem later in Chapter 5. This is also a situation that specialists of hyperbolic conservation laws have to face when they deal with space boundary conditions, as discussed in the classical paper by Bardos, Le Roux et Nédélec [38]. Let us also observe that the set of subsolutions with initial condition  $V_0$  is trivially convex.

As inequality  $M \geq V \otimes V$  is always strict, we speak of strict subsolutions. Conversely, when this inequality is saturated, we recover standard weak solutions. So the situation reminds us very much of Theorem 1.1.2 that we discussed at the beginning of this book in Chapter 1. As a consequence of the works by De Lellis et Székelyhidi [196], we have the following result [198] :



**Theorem 2.3.1.** *Let  $(V, M)$  be a strict smooth subsolution to the Euler equations on  $[0, T] \times \mathbb{T}^d$ . Then, there exists a sequence of weak solutions  $v_n(t, x)$  (which we can even assume to be Hölder continuous in  $x$  of small exponent -no more than  $1/3$  anyway-) such that  $(v_n - V)(t, x)$  and  $(v_n \otimes v_n - M)(t, x)$  weak-\* converge to zero in  $L^\infty(\mathbb{T}^d)$ , uniformly in  $t$ . We may further assume that, for all  $t \in [0, T]$ ,*

$$\int_{\mathbb{T}^d} (v_n \otimes v_n)(t, x) dx = \int_{\mathbb{T}^d} M(t, x) dx.$$

This is a highly non-trivial result which requires a large amount of Analysis. We will not even try to sketch a proof and we invite the interested reader to look at De Lellis et Székelyhdi papers [196, 198].

As already mentioned, this result can be seen as a very sophisticated version of Theorem 1.1.2 in Chapter 1, strict subsolutions and weak solutions playing respectively the role of the points lying in the interior of the unit ball and the points of the unit sphere.

## 2.4 More results on the Euler equations

In this section, that can be skipped at a first stage, we provide more informations on the Euler equations. We start by describing various formulations of the Euler equations.

### The trajectorial viewpoint

It is very instructive to look at the Euler equations of incompressible homogeneous fluids at the level of trajectories (in so-called "Lagrangian coordinates"). As we already saw, they just read

$$\frac{d^2 X_t}{dt^2}(a) = -(\nabla p)(t, X_t(a)), \quad \forall t \quad \mathcal{L} \circ X_t^{-1} = \mathcal{L} = \text{Lebesgue},$$

where  $a$  denotes the label of a typical fluid particle and  $X_t(a)$  its location in the domain  $D$  at time  $t$ . (Let us recall that this is the very starting point of Euler's paper [230]! The main point of his paper was *precisely* the derivation of the *Eulerian* equations that have become so popular that many people ignore their origin which is definitely on the trajectorial -or so-called "Lagrangian"- side.) Indeed, Euler postulated the existence of a vector field  $v = v(t, x)$ , the so-called "Eulerian velocity field" such that

$$v(t, X_t(a)) = \frac{dX_t}{dt}(a).$$

Thus, by the chain rule and assuming  $X_t$  to be one-to-one in  $D$ , one easily gets, as Euler did,

$$(\partial_t + v \cdot \nabla)v + \nabla p = 0, \quad \nabla \cdot v = 0,$$

which is the "non-conservative" form of the Euler equations, usually written as

$$\partial_t + \nabla \cdot (v \otimes v) + \nabla p = 0, \quad \nabla \cdot v = 0.$$

Very much as we did in the geometrical framework, let us introduce the "mixed Eulerian-Lagrangian" measures

$$c(t, x, a) = \delta(x - X_t(a)), \quad q(t, x, a) = \frac{dX_t}{dt}(a)\delta(x - X_t(a)),$$

which are defined on the space  $[0, T] \times D \times \mathcal{A}$ , where  $\mathcal{A}$  is the space of "fluid particle labels". (It is customary, but in no way necessary, as will be seen later on, to define  $\mathcal{A}$  as  $D$  itself, with the convention that  $a$  is nothing but the "initial position"  $X_0(a)$  of the particle with label  $a$ . We just assume  $\mathcal{A}$  to be a compact metric space with a probability measure on it, denoted by  $da$  for simplicity.) As observed before, from its very definition,  $q$  is absolutely continuous with respect to  $c$  and therefore it makes sense to consider its Radon-Nikodym derivative that will be denoted by  $v = v(t, x, a)$ , so that we will write

$$q(t, x, a) = v(t, x, a)c(t, x, a) = (cv)(t, x, a).$$

With such notations, we may write

$$\begin{aligned} \int_{x,a} f(x, a)c(t, x, a) &= \int_{\mathcal{A}} f(X_t(a), a)da, \\ \int_{x,a} f(x, a)q(t, x, a) &= \int_{x,a} f(x, a)(cv)(t, x, a) = \int_{\mathcal{A}} \frac{dX_t}{dt}(a)f(X_t(a), a)da, \end{aligned}$$

for all continuous function  $f$  on  $D \times \mathcal{A}$  and all  $t \in [0, T]$ . By standard differential calculus, we can get a consistent system of PDEs for  $(c, v)$  together with  $\nabla p$ . The following computations are perfectly rigorous as long as  $\nabla p(t, x)$  is sufficiently smooth, say Lipschitz continuous in  $x \in D$  with a Lipschitz constant integrable in  $t \in [0, T]$ :

**Proposition 2.4.1.** *Let  $\nabla p(t, x)$  be sufficiently smooth, say Lipschitz continuous in  $x$ , for  $(t, x) \in [0, T] \times D$ , where  $D = \mathbb{T}^d$ . Assume that  $(X_t, t \in [0, T])$  is a family of measure-preserving maps in the sense that*

$$\int_{\mathcal{A}} f(X_t(a))da = \int_D f(x)dx,$$

for all  $f \in C(D)$  and all  $t \in [0, T]$ . Further assume, that

$$\frac{d^2 X_t}{dt^2}(a) = -(\nabla p)(t, X_t(a)),$$

holds true for all  $a \in \mathcal{A}$  and  $t \in [0, T]$ .

Then the measures  $(c, q = cv)$ , associated with  $(X_t, t \in [0, T])$  through

$$\begin{aligned} \int_{x,a} f(x, a)c(t, x, a) &= \int_{\mathcal{A}} f(X_t(a), a)da, \\ \int_{x,a} f(x, a)q(t, x, a) &= \int_{x,a} f(x, a)(cv)(t, x, a) = \int_{\mathcal{A}} \frac{dX_t}{dt}(a)f(X_t(a), a)da, \end{aligned}$$

for all continuous function  $f$  on  $D \times \mathcal{A}$  and all  $t \in [0, T]$ , satisfy the following set of equations

$$\int_a c(t, x, a) = 1, \quad \partial_t c(t, x, a) + \nabla_x \cdot (cv(t, x, a)) = 0,$$

$$(\partial_t(cv) + \nabla_x \cdot (cv \otimes v))(t, x, a) = -c(t, x, a)\nabla_x p(t, x).$$

In addition, by integrating these equations in  $a$ , we also have

$$\nabla \cdot \int_a (cv)(t, x, a) = 0, \quad -\Delta_x p(t, x) = \nabla_x \otimes \nabla_x \cdot \int_a (cv \otimes v)(t, x, a).$$

**Proof:**

First, since  $X_t$  is volume-preserving, we get for all test functions  $f = f(x)$ :

$$\int_{x,a} f(x)c(t, x, a) = \int_D f(X_t(a))da = \int_D f(x)dx.$$

Thus:  $\int_a c(t, x, a) = 1$  immediately follows. Next,

$$\begin{aligned} \frac{d}{dt} \int_{x,a} f(x, a)c(t, x, a) &= \frac{d}{dt} \int f(X_t(a), a)da = \int (\nabla_x f)(X_t(a), a) \cdot \frac{dX_t}{dt}(a)da \\ &= \int_{x,a} \nabla_x f(x, a) \cdot (cv)(t, x, a), \quad \text{for all test functions } f = f(x, a). \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \int_{x,a} f(x, a)(cv)(t, x, a) &= \frac{d}{dt} \int f(X_t(a), a) \frac{dX_t}{dt}(a)da \\ &= \int (\nabla_x f)(X_t(a), a) \cdot \left( \frac{dX_t}{dt} \otimes \frac{dX_t}{dt} \right)(a)da - \int f(X_t(a), a) (\nabla_x p)(t, X_t(a))da \\ &= \int_{x,a} \nabla_x f(x, a) \cdot (cv \otimes v)(t, x, a) - f(x, a)c(t, x, a)\nabla_x p(t, x). \end{aligned}$$

as announced. Finally,

$$-\Delta p(t, x) = \nabla_x \otimes \nabla_x \cdot \int_a (cv \otimes v)(t, x, a).$$

just follows from

$$\int_a c(t, x, a) = 1, \quad \nabla \cdot \int_a (cv)(t, x, a) = 0.$$

*End of proof.*

So, the relaxed equations we have derived by pure differential calculus from the *original* Euler's model, written in terms of trajectories rather than in terms of "eulerian" fields, are nothing but the optimality conditions we have stated for the relaxed version of the minimizing geodesic, as just seen in section 2.2. Let us recall that this relaxed problem reads, in short,

$$\inf \left\{ \int_0^1 dt \int_{x,a} c|v|^2 ; \partial_t c + \nabla_x \cdot (cv) = 0, \quad \int_a c = 1 \right\}$$

with  $c(t, x, a)$  prescribed at  $t = 0$  and  $t = 1$ , and is *convex* in  $(c, cv)$ .

*Remark.*

As a matter of fact (we will go back to that later on), the optimality conditions contain an extra condition:  $\nabla_x \times v(t, x, a) = 0$ , that has a variational interpretation in terms of principle of least action (in relationship with Noether's celebrated

invariance theorem) and says that the velocity field  $v(\cdot, \cdot, a)$  attached to the label  $a$  is curl-free. This does not contradict that the averaged velocity

$$\int_a (cv)(t, x, a)$$

is divergence-free. As a matter of fact, this provides a striking example of a *macroscopic* divergence-free vector field that can be written as a linear superposition of a family of curl-free vector fields.

*End of remark.*

### Relaxed solutions versus sub-solutions

By averaging out the relaxed solutions of the Euler equations, we immediately get some sub-solutions of the Euler equations, just by setting

$$V(t, x) = \int_a (cv)(t, x, a), \quad M(t, x) = \int_a (cv \otimes v)(t, x, a).$$

Indeed,

$$\partial_t V + \nabla \cdot M + \nabla p = 0, \quad \nabla \cdot v = 0,$$

just follow from the relaxed equations

$$\begin{aligned} \partial_t c(t, x, a) + \nabla_x \cdot (cv)(t, x, a) &= 0, \quad \int_a c(t, x, a) = 1, \\ (\partial_t (cv)(t, x, a) + \nabla_x \cdot (cv \otimes v))(t, x, a) &= -c(t, x, a) \nabla_x p(t, x), \end{aligned}$$

after integration in  $a$  and,

$$M \geq V \otimes V$$

is just a consequence of Jensen's inequality since  $\int_a c(t, x, a) = 1$ . Notice that these sub-solutions have no reason to be strict and, therefore, the De Lellis-Székelyhidi Theorem 2.3.1 a priori does not apply to them.

### Relaxed versus kinetic solutions

There is a parallel formulation of the relaxed equation, of Vlasov or "kinetic" type, involving the "kinetic" "phase-density"

$$f(t, x, \xi) = \int_a \delta(\xi - v(t, x, a)) c(t, x, a), \quad (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d.$$

(Here  $f$  is a traditional notation in kinetic theory for the phase density and the letter  $f$  should not be used to denote test functions!) It is easy to get a self-consistent system of equations for  $f$  together with the pressure gradient, provided we go back, as we did for the relaxed equations, to the trajectorial formulation of the Euler equations,

$$\frac{d^2 X_t}{dt^2}(a) = -(\nabla p)(t, X_t(a)),$$

where  $X_t$  is volume-preserving in the sense that

$$\int_{\mathcal{A}} \phi(X_t(a)) da = \int_{\mathbb{T}^d} \phi(x) dx,$$

for all test functions  $\phi$  on  $\mathbb{T}^d$ . Setting

$$f(t, x, \xi) = \int_{\mathcal{A}} \delta(\xi - \frac{dX_t}{dt}(a)) \delta(x - X_t(a)) da,$$

we get

$$\partial_t f(t, x, \xi) + \nabla_x \cdot (\xi f(t, x, \xi)) = \nabla_\xi \cdot (\nabla_x p(t, x) f(t, x, \xi)), \quad \int_{\xi \in \mathbb{R}^d} f(t, x, \xi) = 1.$$

Once again, this is an easy consequence of the chain rule, and we only need  $\nabla p(t, x)$  to be Lipschitz in  $x \in \mathbb{T}^d$  to make it rigorous. Indeed, for every test  $\phi$  function depending only on  $x$ , we first find

$$\int_{(x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d} \phi(x) f(t, x, \xi) = \int_{\mathcal{A}} \phi(X_t(a)) da = \int_{\mathbb{T}^d} \phi(x) dx,$$

and, therefore,

$$\int_{\xi \in \mathbb{R}^d} f(t, x, \xi) = 1.$$

Next, we get for any test function  $\phi$  depending on both  $x$  and  $\xi$ ,

$$\begin{aligned} \frac{d}{dt} \int_{(x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d} \phi(x, \xi) f(t, x, \xi) &= \frac{d}{dt} \int_{\mathcal{A}} \phi(X_t(a), \frac{dX_t}{dt}(a)) da \\ &= \int_{\mathcal{A}} \frac{dX_t}{dt}(a) \cdot (\nabla_x \phi)(X_t(a), \frac{dX_t}{dt}(a)) da \\ &\quad - \int_{\mathcal{A}} (\nabla p)(t, X_t(a)) \cdot (\nabla_\xi \phi)(X_t(a), \frac{dX_t}{dt}(a)) da \\ &= \int_{(x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d} (\xi \cdot \nabla_x \phi(x, \xi) - (\nabla p)(t, x) \cdot \nabla_\xi \phi(x, \xi)) f(t, x, \xi). \end{aligned}$$

This ‘‘kinetic formulation’’ of the Euler equations was already introduced in [84] and was, in some sense, the departure points of [87, 89, 90, 91].

## Well-posedness issues

As we have seen, the relaxed Euler equations:

$$\partial_t c(t, x, a) + \nabla_x \cdot (cv(t, x, a)) = 0, \quad \int_a c(t, x, a) = 1,$$

$$(\partial_t (cv) + \nabla_x \cdot (cv \otimes v))(t, x, a) = -c(t, x, a) \nabla_x p(t, x),$$

are very well suited for the ‘‘minimizing geodesic problem’’. It is therefore tempting to think that the relaxed Euler equations, or their kinetic counterpart,

$$\partial_t f(t, x, \xi) + \nabla_x \cdot (\xi f(t, x, \xi)) = \nabla_\xi \cdot (\nabla_x p(t, x) f(t, x, \xi)), \quad \int_{\xi \in \mathbb{R}^d} f(t, x, \xi) = 1,$$

might be good candidates to substitute for the usual Euler equations when we address the initial value problem (IVP), i.e. when we try to get a solution  $(c, cv)$  (or  $f$ ,

in kinetic terms), just by prescribing its value at time 0. Unfortunately, it turns out that the relaxed Euler equations are not even well-posed in short time, unless severe restrictions are imposed to the initial conditions  $(c_0, c_0 v_0)$  (or  $f_0$  in kinetic terms). Positive and negative results have been obtained in the last 20 years, with many contributors such as Baradat, Bardos and Besse, Brenier, Grenier, Han-Kwan and Iacobelli, Han-Kwan and Rousset, Masmoudi and Wong [33, 36, 282, 286, 287, 355]. Strictly speaking some of these papers, in particular [36, 287], are rather devoted to the "compressible" version of the relaxed Euler equations, which reads, in kinetic terms,

$$\partial_t f(t, x, \xi) + \nabla_x \cdot (\xi f(t, x, \xi)) = \nabla_\xi \cdot \left( \frac{\nabla_x p}{\rho}(t, x) f(t, x, \xi) \right), \quad \rho(t, x) = \int_\xi f(t, x, \xi),$$

where the pressure  $p$  is a given function of the density  $\rho$ .

### Comparison with the Muskat equations

The Euler equations of incompressible inhomogeneous fluids admit a "friction dominated" version which reads (in terms of trajectories)

$$\frac{dX_t}{dt}(a) = -\rho_0(a)G - (\nabla p)(t, X_t(a)), \quad \mathcal{L} \circ X_t^{-1} = \mathcal{L}, \quad \forall t,$$

where we assume, for a moment, that each  $X_t$  belongs to  $SDiff(D)$ . Here, the external force, denoted by  $G$ , is a given constant vector in  $\mathbb{R}^d$  (typically along the vertical axis, if one considers the gravity force in the simplest possible situation). Notice that the density  $\rho_0$  exclusively features in front of the external force. This corresponds to the so-called "Boussinesq approximation" (see [187, 394]). As a matter of fact, assuming the existence of a velocity field  $v$  and a density field  $\rho$  such that

$$\frac{dX_t}{dt}(a) = v(t, X_t(a)), \quad \rho(t, X_t(a)) = \rho_0(a),$$

then the equations admit the following "Eulerian" version:

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \nabla \cdot v = 0, \quad v = -\rho G - \nabla p.$$

This set of equations is sometimes called "incompressible porous media equations" or "Muskat's equations" [180, 437], and we will come back to them in section 9.3. Notice that they get trivial when there is no external force. (Indeed, in such a case  $v$  is both potential and divergence-free.) These equations are very useful for applications (typically, they are the basic equations for "reservoir simulations" in Civil Engineering and Oil Industry [160, 120]). They have been studied in many different ways recently in the mathematical literature, in particular in the framework of convex integration theory. Note that the concept of sub-solutions is not so clearly defined as for the Euler equations, as explained in [437] (that we also quote for the many references it contains).

Anyway, following what we did for the Euler equations, we can easily get a relaxed version for these equations:

**Proposition 2.4.2.** *The Muskat equations admit the following relaxed formulation:*

$$\begin{aligned} \partial_t c(t, x, a) &= \nabla_x \cdot (c(t, x, a)(\rho_0(a)G + \nabla p(t, x))) \\ \int_a c(t, x, a) &= 1, \quad -\Delta p(t, x) = \nabla_x \cdot \left( \int_a c(t, x, a) \rho_0(a) G \right). \end{aligned}$$

*Proof* (just as before): For all test functions  $f = f(x, a)$ , we have

$$\begin{aligned}
\frac{d}{dt} \int_{(x,a)} f(x, a) c(t, x, a) &= \frac{d}{dt} \int f(X_t(a), a) da \\
&= \int (\nabla_x f)(X_t(a), a) \cdot \frac{dX_t}{dt}(a) da \\
&= \int (\nabla_x f)(X_t(a), a) \cdot (-\rho_0(a)G - (\nabla p)(t, X_t(a))) \\
&= \int_{(x,a)} c(t, x, a) \nabla_x f(x, a) \cdot (-\rho_0(a)G - \nabla p(t, x)).
\end{aligned}$$

leading to

$$\partial_t c(t, x, a) = \nabla_x \cdot (c(t, x, a)(\rho_0(a)G + \nabla p(t, x))),$$

as announced. Then

$$-\Delta p(t, x) = \nabla_x \cdot \left( \int_a c(t, x, a) \rho_0(a) G \right),$$

immediately follows from  $\int_a c(t, x, a) = 1$  by integrating the previous equation with respect to  $a$ .

*End of proof.*

In sharp contrast with the relaxed Euler equations, the relaxed Muskat equations enjoy a well-posedness property for the IVP. This follows from:

**Proposition 2.4.3.** *The relaxed Muskat system admits an extra conservation law for the Boltzmann entropy  $\int_a c(t, x, a)$ , namely*

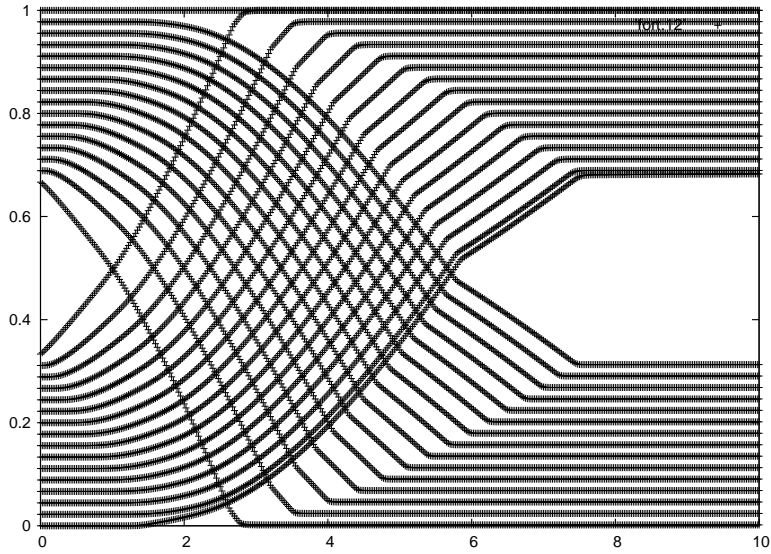
$$\partial_t \int_a (c \log c)(t, x, a) + \nabla_x \cdot \left( \int_a c(t, x, a) \rho_0(a) G \right) = 0.$$

[This is just a straightforward calculation, since:

$$\begin{aligned}
\partial_t \int_a (c \log c)(t, x, a) &= \int_a (1 + \log c(t, x, a)) \nabla_x \cdot ((\rho_0(a)G + \nabla p(t, x))c(t, x, a)) \\
&= - \int_a \nabla_x c(t, x, a) \cdot (\rho_0(a)G + \nabla p(t, x)) = - \nabla_x \cdot \left( \int_a c(t, x, a) \rho_0(a) G \right),
\end{aligned}$$

using that  $\int_a c(t, x, a) = 1$ .]

Since the Boltzmann entropy is strictly convex in  $c$ , the existence of this extra conservation law essentially suffices to guarantee the local well-posedness of the relaxed Muskat equations (at least as label  $a$  is discrete), following the general theory of entropic system of conservation laws [193] that we will discuss later in Chapter 6.3.



Relaxed Muskat equations:

A solution featuring three "phases" (heavy, neutral, light) on top of each other.

Trajectories are drawn for the heavy and the light phases only.

Observe the final rearrangement of the phases in stable order.

(Horizontal axis:  $t \in [0, 10]$ , vertical axis:  $x \in [0, 1]$ .)



## Solution of the IVP by convex minimization

It is now quite clear that the relaxed Euler equations are much more adequate for the generalized minimizing geodesic problem, where  $c$  is prescribed at the end points  $t = 0$  and  $t = 1$ , for which the solutions are successfully obtained by convex minimization (with a very convincing existence and uniqueness result for the pressure gradient), than for the initial value problem (IVP), when  $(c, cv)$  is prescribed at time 0, which is very likely to be ill-posed. Anyway, it seems foolish to solve the IVP problem by a space-time convex minimization technique. Indeed, this way, we are very likely to get optimality equations of space-time elliptic type and therefore ill-posed, although there is a little room left if the convexity is sufficiently degenerate (which is, by the way, the case of the generalized minimizing geodesic problem where the convex functional to be minimized is homogeneous of degree one and, therefore, degenerate). However, as will be discussed later in Chapter 5, there is a (limited) possibility of that sort which actually involves the cruder concept of sub-solutions we have discussed in the framework of “convex integration” à la De Lellis-Székelyhidi. The idea amounts to minimizing, on a given time interval  $[0, T]$ ,

$$\int_{[0,T] \times \mathbb{T}^d} (\text{trace } M)(dtdx)$$

among all  $(V, M)$ , where  $V$  is square-integrable space-time and  $M$  is a bounded Borel space-time measure valued in the set of semi-definite symmetric  $d \times d$  matrices, which satisfy  $M \geq V \otimes V$  and solve

$$\partial_t V + \nabla \cdot M + \nabla p = 0, \quad \nabla \cdot V = 0,$$

with given initial condition  $V_0$  in the sense

$$\int_Q \partial_t A_i(t, x) V^i(t, x) dtdx + \partial_j A_i(t, x) M^{ij}(dtdx) + \int_D V_0^i(x) A_i(0, x) dx = 0,$$

for all smooth divergence-free vector-fields  $A = A(t, x) \in \mathbb{R}^d$  that vanish at  $t = T$ . It will be shown that:

- 1) Any smooth solution of the Euler equations can be obtained this way, at least for small enough  $T$ .
- 2) It may happen that the optimal solution is a classical solution to the Euler equations, but for a *different* initial condition than  $V_0$ ! This strange phenomenon is related to the fact that  $M$  is just a space-time measure which prevents  $V(t, x)$  to be weakly continuous at  $t = 0$ . Interestingly enough, in some special situations, the resulting solution at time  $T$  can be seen as a “relaxed solution”, not in the sense we have discussed so far, but rather in the sense developed by Otto [386] for incompressible fluid motions in porous media and recently revisited in [269, 437]. Let us just give an explicit example, due to Helge Dietert [204], with  $d = 2$ , not on  $\mathbb{T}^2$  but rather on  $\mathbb{T} \times [-1/2, 1/2]$  (to make the example easier to handle) and we assume  $T \leq 1/2$ . We take as initial condition

$$V_0(x_1, x_2) = (\text{sign}(x_2), 0),$$

which is an exact, time-independent, discontinuous, trivial solution to the Euler equations, but well known to be “physically unstable” (“Kelvin-Helmholtz instability”). Then, the convex optimization problem provides a completely different solution, which is stationary (i.e. time independent), Lipschitz continuous and explicitly

depends on the final time  $T$ , namely

$$V_T(x_1, x_2) = \left( \max(-1, \min(\frac{x_2}{T}, 1)), 0 \right).$$

This looks non sense. However, if we consider this family of stationary solutions as a time dependent solution (the final time  $T$  playing the role of the current time), we recover the kind of relaxed solutions advocated by Otto in the (quite different but closely related) framework of incompressible fluid motion in porous media [386, 437]. These topics will be discussed in Chapter 5.



# Chapter 3

## Hidden convexity in the Monge-Ampère equation and Optimal Transport Theory

As we have seen earlier in this book, the Euler model of incompressible fluids crucially relies on the ODE

$$\frac{d^2 X(t)}{dt^2} = -(\nabla p)(t, X(t)),$$

where  $p$  is the pressure field and adjusts itself in order to enforce the incompressibility condition. In the simpler case when  $p = p(t, x)$  is a given potential, this ODE can be derived from the Least Action principle (LAP) as explained in Chapter 1. As a matter of fact, the LAP also applies to many PDEs and not only to ODEs (see, for instance, [22, 211, 354, 434, 442, 460]...). More precisely, many PDEs can be interpreted as optimality equation of a suitable optimization problem. One of the simplest example is the Poisson (or Laplace) equation

$$\Delta u = f$$

where  $f$  is a given function on a compact domain  $D \subset \mathbb{R}^d$  with suitable boundary conditions, typically for the unknown  $u = u(x) \in \mathbb{R}$  to vanish along the boundary, i.e. as  $x \in \partial D$ . It is very well known that the solution can be obtained as the unique minimizer of the functional

$$\int_D \left( \frac{|\nabla u(x)|^2}{2} + f(x)u(x) \right) dx$$

on a suitable functional space. (Typically, the Sobolev space  $H_0^1(D)$ .) As we are going to see in the present chapter, such a variational principle may apply, in a not so obvious way, to fully nonlinear equations such as the Monge-Ampère equation (MAE),

$$\det D^2 u = f.$$

where

$$D^2 u(x) = \left( \frac{\partial^2 u}{\partial x^i \partial x^j}(t, x), \quad i, j = 1, \dots, d \right),$$

at least for some suitable boundary conditions. Surprisingly enough, this variational structure of the MAE may be suggested by the study of the Euler equations of

incompressible fluids! (So that we may add the MAE to the long list of PDEs that can be derived from the Euler equations, such as the wave or the heat equations, as we have seen in Chapter 2.)

### 3.1 The Least Action Principle for the Euler equations

Let us go back for a short while to the Euler equations of incompressible fluids. Inspired by Arnold's geometric interpretation (as seen in section 2.2), we introduce the functional

$$\mathbb{J}_{t_0, t_1}[X] = \int_{t_0}^{t_1} \int_D \frac{1}{2} |\partial_t X_t(a)|^2 dx dt$$

where  $D \subset \mathbb{R}^d$  is a compact convex domain,  $t_0 < t_1$  are given,  $t \rightarrow X_t \in VPM(D)$  is prescribed at  $t = t_0$  and  $t = t_1$ , where  $VPM(D)$  is the semi-group of all volume-preserving maps of  $D$ , i.e. all Borel maps  $X : D \rightarrow \mathbb{R}^d$  such that

$$\int_D \phi(X(a)) da = \int_D \phi(x) dx, \quad \forall \phi \in C^0(\mathbb{R}^d).$$

Then we have the following version of the LAP:

**Theorem 3.1.1.** *Let  $(X, p)$  be a solution of the Euler equations, in the sense:*

$$\frac{d^2}{dt^2} X_t(a) = -(\nabla p)(t, X_t(a)),$$

$$\int_D \phi(t, X_t(a)) dx = \int_D \phi(x) dx, \quad \forall \phi \in C^0(\mathbb{R}^d), \quad \forall t.$$

*Assume that the pressure field  $p$  is smooth enough so that  $K(p)$  is finite, where*

$$K(p) = \sup_{(t, x) \in [t_0, t_1] \times D} \sup_{k=1, \dots, d} \lambda_k(t, x),$$

*where we denote by  $\lambda_k \in \mathbb{R}$  the eigenvalues of  $D_x^2 p(t, x)$ . Then, if the time interval  $[t_0, t_1]$  is small enough so that*

$$\frac{(t_1 - t_0)^2}{\pi^2} K(p) < 1,$$

*then, for all curves  $t \in [t_0, t_1] \rightarrow \tilde{X}_t \in VPM(D)$  such that*

$$\tilde{X}_{t_0} = X_{t_0}, \quad \tilde{X}_{t_1} = X_{t_1},$$

*different from  $X$ , one has*

$$\mathbb{J}_{t_0, t_1}[\tilde{X}] > \mathbb{J}_{t_0, t_1}[X].$$

**Proof**

The proof follows almost immediately from Theorem 1.3.1 already seen in Chapter 1. Indeed, for (almost) every fixed  $a \in D$ , we have, by setting  $u(t) = X_t(a)$  and  $\tilde{u}(t) = \tilde{X}_t(a)$ ,

$$\int_{t_0}^{t_1} [-p(t, u(t)) + \frac{1}{2}|u'(t)|^2]dt \leq \int_{t_0}^{t_1} [-p(t, \tilde{u}(t)) + \frac{1}{2}|\tilde{u}'(t)|^2]dt,$$

and, thus,

$$\int_{t_0}^{t_1} [-p(t, X_t(a)) + \frac{1}{2}|\partial_t X_t(a)|^2]dt \leq \int_{t_0}^{t_1} [-p(t, \tilde{X}_t(a)) + \frac{1}{2}|\partial_t \tilde{X}_t(a)|^2]dt,$$

with equality only if  $u = \tilde{u}$ . Then integrating in  $a \in D$  and using that both  $X$  and  $\tilde{X}$  are valued in  $VPM(D)$ , we get

$$\int_D \int_{t_0}^{t_1} \frac{1}{2}|\partial_t X_t(a)|^2 dt da \leq \int_D \int_{t_0}^{t_1} \frac{1}{2}|\partial_t \tilde{X}_t(a)|^2 dt da$$

with equality only if  $\tilde{X} = X$ , which completes the proof.

**A dual Least Action Principle**

We can go a little further by observing that the pressure field itself obeys a sort of LAP in the following sense:

**Theorem 3.1.2.** *Let us use the same notations as in Theorem 3.1.1 and assume*

$$\frac{(t_1 - t_0)^2}{\pi^2} K(p) \leq 1.$$

*Then the pressure field  $p$  is a maximizer of functional*

$$\mathbb{K}_{t_0, t_1}[p] = \int_{t_0}^{t_1} \int_D p(t, x) dx dt + \int_D K_{t_0, t_1, p}(X_{t_0}(a), X_{t_1}(a)) da,$$

where

$$K_{t_0, t_1, p}(u_0, u_1) = \inf \left\{ \int_{t_0}^{t_1} \left( \frac{1}{2}|u'(t)|^2 - p(t, u(t)) \right) dt, u \in C^1([0, T], D), u(t_0) = u_0, u(t_1) = u_1 \right\}$$

**Proof**

Let  $\tilde{p}$  be a "competitor" for  $p$ . By definition, we have

$$K_{t_0, t_1, \tilde{p}}(u_0, u_1) = \inf \left\{ \int_{t_0}^{t_1} \left( \frac{1}{2}|u'(t)|^2 - \tilde{p}(t, u(t)) \right) dt, u \in C^1([0, T], D), u(t_0) = u_0, u(t_1) = u_1 \right\},$$

so that, for each fixed  $a \in D$ ,

$$K_{t_0, t_1, \tilde{p}}(X_{t_0}(a), X_{t_1}(a)) \leq \int_{t_0}^{t_1} \left( \frac{1}{2}|\partial_t X_t(a)|^2 - \tilde{p}(t, X_t(a)) \right) dt.$$

By integration in  $a \in D$ , we get

$$\begin{aligned} \int_D K_{t_0, t_1, \tilde{p}}(X_{t_0}(a), X_{t_1}(a)) da &\leq \int_D \int_{t_0}^{t_1} \left( \frac{1}{2} |\partial_t X_t(a)|^2 - \tilde{p}(t, X_t(a)) \right) dt da \\ &= \int_D \int_{t_0}^{t_1} \left( \frac{1}{2} |\partial_t X_t(a)|^2 - \tilde{p}(t, a) \right) dt da \end{aligned}$$

(using that  $X_t$  is volume preserving). For  $p$  itself, we get *equality*:

$$K_{t_0, t_1, p}(X_{t_0}(a), X_{t_1}(a)) = \int_{t_0}^{t_1} \left( \frac{1}{2} |\partial_t X_t(a)|^2 - p(t, X_t(a)) \right) dt$$

(because of Theorem 1.3.1) and, therefore, integrating in  $a$ ,

$$\int_D K_{t_0, t_1, p}(X_{t_0}(a), X_{t_1}(a)) da = \int_D \int_{t_0}^{t_1} \left( \frac{1}{2} |\partial_t X_t(a)|^2 - p(t, a) \right) dt da.$$

So, by subtraction, we have obtained

$$\begin{aligned} &\int_{t_0}^{t_1} \int_D \tilde{p}(t, x) dx dt + \int_D K_{t_0, t_1, \tilde{p}}(X_{t_0}(a), X_{t_1}(a)) da \\ &\leq \int_{t_0}^{t_1} \int_D p(t, x) dx dt + \int_D K_{t_0, t_1, p}(X_{t_0}(a), X_{t_1}(a)) da, \end{aligned}$$

which completes the proof.

*Remark.*

So, we have obtained a "dual" optimization problem that enjoys two remarkable properties:

- i) it is concave in  $p$ , which shows that, behind the original optimization problem in  $X$ , which was definitely not convex in  $X$ , we have exhibited some *hidden* convexity;
- ii) it does not involve any partial derivatives in  $p$ !

## 3.2 Monge-Ampère equation and Optimal Transport

The maximization problem solved by the pressure field in the framework of the Euler equations of incompressible fluid suggests the study of a very similar but simpler problem, namely the maximization of functional

$$\phi \rightarrow \int_{\mathbb{R}^d} \phi(x) \rho_0(x) dx + \int_{\mathbb{R}^d} \inf_{x \in \mathbb{R}^d} \left( \frac{1}{2} |y - x|^2 - \phi(x) \right) \rho_1(y) dy,$$

where  $\rho_0 \geq 0$  and  $\rho_1 \geq 1$  are given compactly supported functions of unit Lebesgue integral on  $\mathbb{R}^d$ . Remarkably enough, this simpler problem is related to the famous, fully nonlinear, real Monge-Ampère equation, well known in both Riemannian and Kählerian geometries [51, 164]:

$$\rho_1(x + \nabla \phi(x)) \det(\mathbb{I}_d + D^2 \phi(x)) = \rho_0(x)$$

(which was considered by Minkowski more than a century ago, to show that convex hypersurfaces in  $\mathbb{R}^d$  can be recovered just by the knowledge of their Gaussian curvature). The variational study of the MAE relies on techniques borrowed from the Monge-Kantorovich Theory of OptimalTransport.

*Remark.*

Optimal transport theory has been a very flourishing field of pure and applied Mathematics in the last 30 years (cf. books and surveys [13, 202, 232, 262, 399, 405, 414, 450, 451, 452]), with applications and generalizations in all kind of directions. Let us just quote few examples:

Cosmology [116, 245, 343],  
 General Relativity [360, 375] ,  
 Quantum Chemistry [175, 182, 260, 324, 397],  
 Quantum particles [148, 275, 277],  
 Free Probability and Noncommutative Geometry [64, 195, 285],  
 Random matrices [240] ,  
 Geometrical Mechanics [301],  
 Continuum Mechanics [74, 123, 296, 325, 338],  
 Kinetic Theory [227, 341, 438]  
 Statistical Mechanics [52, 53, 75, 144, 154, 212, 388],  
 Markov and stochastic processes [24, 228, 236, 368, 385]  
 Functional Analysis[178, 224, 251, 280, 303, 450],  
 Functional and Geometric Inequalities [28, 41, 142, 159, 241, 357, 358, 389],  
 Gradient flows and Parabolic PDEs [13, 18, 58, 66, 145, 150, 154, 173, 213, 356],  
 Elliptic PDEs [132, 140, 141, 201, 348, 447],  
 Dynamical Systems [55, 234]  
 Riemannian Geometry [14, 268, 309, 343, 345, 359, 361, 435, 436, 456],  
 Subriemannian Geometry [16, 242],  
 Kählerian Geometry [50],  
 Computational Geometry [152, 143, 229, 305, 323, 337, 362, 412],  
 Inverse Problems and Optimization [205, 226, 272, 366, 384],  
 Data Analysis and Data Assimilation [2, 398, 399, 406].  
 Economics [151, 165, 166, 246].

*End of remark.*

It is quite amazing that a fully non-linear equation such as the Monge-Ampère equation can be solved by a concave optimization problem which does not involve any partial derivative!

**Theorem 3.2.1.** *Let  $B$  a closed ball in  $\mathbb{R}^d$  centered at 0. Let  $\mu_0$  and  $\mu_1$  be to Borel probability measures on  $B$ . Assume that  $\mu_0$  is absolutely continuous with respect to the Lebesgue measure, i.e. there exists  $\rho_0 \geq 0$  in  $L^1(B)$  such that  $\mu_0(dx) = \rho_0(x)dx$ . Then, there is a unique Borel map  $T : B \rightarrow B$  that transports  $\mu_0$  toward  $\mu_1$  and can be written  $T(x) = \nabla a(x)$ ,  $\rho_0(x)dx$  almost everywhere, where  $a$  is a Lipschitz convex function on  $B$ .*



## Remark

This result tells us, at least in the simpler case, where  $\mu_1(dy) = \rho_1(y)dy$  for some  $\rho_1 \in L^1(B)$ , that the MAE

$$\rho_1(\nabla a(x))\det(D^2a(x)) = \rho_0(x),$$

is solved, in a generalized sense, for some convex Lipschitz function  $a$  on  $B$ . Indeed, assuming the change of variable

$$x \in B \rightarrow y = \nabla a(x) \in B, \quad dy = \det(D^2a(x))dx$$

to be valid, we get for each  $u \in C^0(B)$ ,

$$\int_B u(y)\rho_1(y)dy = \int_B u(\nabla a(x))\rho_1(\nabla a(x))\det(D^2a(x))dx = \int_B u(\nabla a(x))\rho_0(x)dx,$$

which means that  $x \rightarrow \nabla a(x)$  transports  $\rho_0(x)dx$  toward  $\rho_1(y)dy$  as the MAE is satisfied.

Theorem 3.2.1, that admits many variations (see for instance [83, 88, 249, 310, 359]) can be proven through the study of the "Monge-Kantorovich" problem [405, 414, 451, 452]

$$\inf\left\{\int_B a(x)\mu_0(dx) + \int_B b(y)\mu_1(dy), \quad (a, b) \in C^0(B) \times C^0(B)\right\},$$

under constraint  $a(x) + b(y) \geq x \cdot y, \forall x, y \in B$ . So, the solution of a fully nonlinear geometric PDE will be obtained by solving a "linear program" without any partial derivative!

## 3.3 Nonlinear Helmholtz decomposition and polar factorization of maps

A rather direct application of Theorem 3.2.1 can be obtained in the special case when:

- i)  $\mu_0$  is just the (normalized) Lebesgue measure restricted to a compact subdomain  $D$  of  $B$ ;
- ii)  $\mu_1$  is the image measure of  $\mu_0$  by a given bounded Borel map  $Y : D \rightarrow B$ .

**Theorem 3.3.1.** *Let  $D$  be a compact domain in  $\mathbb{R}^d$  contained in a ball  $B$  and let  $Y : D \rightarrow B$  be a Borel map. Assume the image measure of the Lebesgue measure on  $D$  by  $Y$ , that we denote by  $\nu$ , to be absolutely continuous with respect to the Lebesgue measure on  $B$  (in which case, map  $Y$  is called a "non-degenerate" map). Then, there is a unique "polar factorization" (or "nonlinear Helmholtz decomposition") of  $Y$  of form  $Y = T \circ X$  where*

- 1)  $X : D \rightarrow D$  is a Lebesgue measure-preserving Borel map;
  - 2)  $T : D \rightarrow \mathbb{R}^d$  has a "convex potential", in the sense that there exists a Lipschitz convex function  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  such that for a.e.  $x \in D$   $T(x) = \nabla\Phi(x)$ .
- Moreover,  $X$  is characterized as the unique  $L^2$  projection of  $Y$  on the set  $VPM(D)$  of all volume-preserving Borel maps of  $D$ , i.e.

$$\int_D |Y(x) - X(x)|^2 dx < \int_D |Y(x) - \tilde{X}(x)|^2 dx,$$

for each  $\tilde{X} \in VPM(D)$  different from  $X$ .

In addition,  $T : D \rightarrow \mathbb{R}^d$  is characterized as the unique map with a convex potential such that sending the Lebesgue measure on  $D$  to  $\nu$ .

This result [83, 88] deserves to be called "nonlinear Helmholtz decomposition" for the following reason. The usual Helmholtz decomposition asserts that every vector field  $z \in L^2(D; \mathbb{R}^d)$  can be uniquely written  $z = w + \nabla p$ , where  $w$  is some  $L^2$  divergence-free vector field on  $D$ , parallel to  $\partial D$ , and  $p$  some scalar function on  $D$ . This can be seen as the linearization of the "polar factorization" of maps about the identity map. Indeed, at least formally, the factorization  $Y = \nabla \Phi \circ X$ , for a map  $Y$  close to the identity map, so that  $Y(x) = x + \epsilon z(x)$ , with  $\epsilon$  small, first returns  $\Phi(x) = |x|^2/2 + \epsilon p(x)$ ,  $X(x) = x + \epsilon w(x) + O(\epsilon^2)$ , with  $z = \nabla p + w$ . Next, since  $X$  is volume-preserving, one has, for all test function  $f$ ,

$$0 = \int_D f(x + \epsilon w(x) + O(\epsilon^2)) dx - \int_D f(x) dx = \int_D \nabla f(x) \cdot w(x) dx + O(\epsilon^2)$$

which means, in a weak sense, that  $w$  is divergence-free and parallel to  $\partial D$ .

Furthermore, the name "polar factorization" comes from the fact that, in the very special case, when  $D = B$  is the unit ball and  $Y(x) = Ax$ ,  $\forall x \in D$ , for some real  $d \times d$  matrix  $A$ , one has

$$T = \nabla \Phi \circ X, \quad \Phi(x) = \frac{1}{2} x \cdot \sqrt{AA^t} x,$$

and, whenever  $A$  is non-degenerate (i.e. invertible),

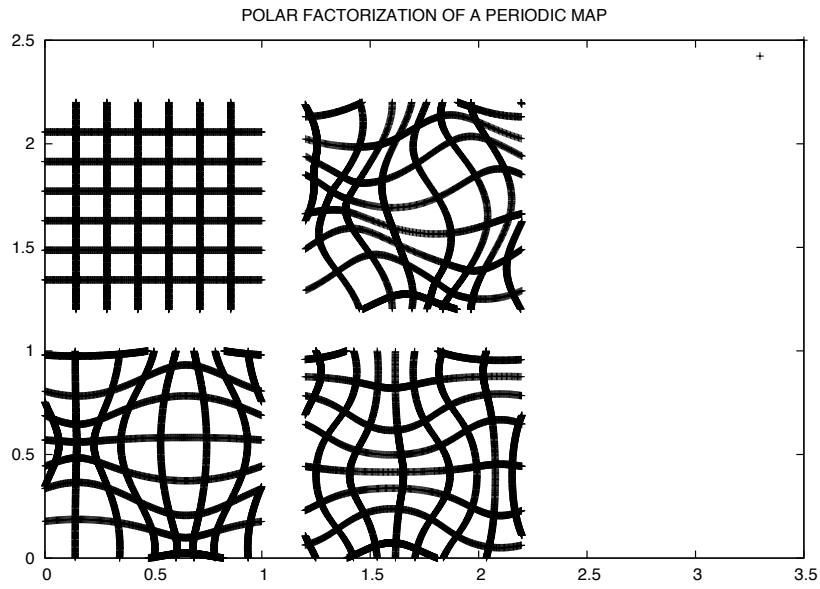
$$X(x) = Ux, \quad U = (AA^t)^{-1/2} A,$$

where  $U$  is an orthogonal matrix since

$$UU^t = (AA^t)^{-1/2} AA^t (AA^t)^{-1/2} = \mathbb{I}_d = U^t U,$$

$\mathbb{I}_d$  denoting the identity matrix. (By the way, in this very peculiar case,  $X$  is not only a volume-preserving map of  $B$ , but also an isometry!)

Note that the polar factorization theorem, established in [83, 88], admits an important generalization to compact Riemannian manifolds due to R. McCann [359]. Finally, let us also mention [139] and finally [261] as a non trivial generalization of the concept of polar factorization.



Polar factorization of a given map  $Y : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ ,  
drawn on the upper right corner.  
The volume (area)-preserving factor lies on the lower right corner.  
The map with convex potential features on the lower left corner.

## Proof of Theorem 3.2.1

The proof relies on the Rademacher theorem that asserts that any Lipschitz function on  $\mathbb{R}^d$  is Lebesgue-almost everywhere differentiable [233] and on a well-known result of Convex Analysis, which is a rather direct consequence of the Hahn-Banach Theorem, namely the Fenchel-Rockafellar duality theorem, as stated in [130].

### The Fenchel-Rockafellar duality theorem

**Theorem 3.3.2.** *Let  $E$  be a real Banach space and consider two functions  $K_1, K_2 : E \rightarrow \mathbb{R} \cup \{+\infty\}$  which are both convex. Assume that there exists a point  $u_0 \in E$  such that both  $K_1$  and  $K_2$  are finite at  $u_0$  while  $K_2$  is continuous at  $u_0$ . Then we have the duality equality*

$$\sup_{u \in E} (-K_1(u) - K_2(u)) = \inf_{f \in E'} (K_1^*(-f) + K_2^*(f)),$$

where  $E'$  is the dual of  $E$  and the Legendre-Fenchel dual  $K^* : E' \rightarrow \mathbb{R} \cup \{+\infty\}$  of a function  $K : E \rightarrow \mathbb{R} \cup \{+\infty\}$  is defined by

$$K^*(f) = \sup_{u \in E} [\langle f, u \rangle_{E', E} - K(u)].$$

Moreover, the infimum in the duality equality is achieved by some point  $f \in E'$ .

*Remark.*

Surprisingly enough, this duality theorem is quite similar to the Plancherel formula in harmonic analysis. Indeed, at least formally, one can consider the correspondence between the algebraic structures with operations, respectively,  $[+, \cdot]$  and  $[\max, +]$  (sometimes in this correspondence, inequalities can show up instead of equalities). Then, the Legendre-Fenchel transform is analogous to the Fourier transform and the duality equality just corresponds to the Plancherel formula:

$$\int u \cdot v = \int \hat{u} \cdot \hat{v},$$

where  $u \rightarrow \hat{u}$  stands for the Fourier transform. This "Fenchel-Fourier" dictionary is now well established in Mathematics ("Tropical Geometry" in Algebraic Geometry being probably the most famous example [155], see also [85, 181, 346].)

### Application of the Fenchel-Rockafellar theorem

We introduce

$$E = C^0(B \times B),$$

which is a Banach space for the sup norm. We are given a continuous function  $c$  on  $B \times B$  (that later will be simply taken as  $c(x, y) = x \cdot y$ ). We define two convex functions  $\Phi, \Psi$  on  $E$ , valued in  $] -\infty, +\infty]$  and respectively given, for each  $w \in E$  by:

$$\begin{aligned} \Phi(w) &= 0, \text{ if } w \geq c, \quad +\infty \text{ otherwise,} \\ \Psi(w) &= \int_B a(x) \mu_0(dx) + \int_B b(y) \mu_1(dy) \text{ if } w = a \oplus b, \end{aligned}$$

for some continuous functions  $a, b$  on  $B$ , and  $+\infty$  otherwise. [Note that  $\Psi$  is defined without ambiguity since  $\mu_0$  and  $\mu_1$  have the same, unit, mass.] Observe that there

is at least one point  $w \in E$  where  $\Phi$  is continuous and  $\Psi$  finite. [Take, for instance, the constant function  $w = 1 + \sup c$  on  $B \times B$ .] Since  $\Phi$  and  $\Psi$  are obviously convex, we may apply the Fenchel-Rockafellar theorem 3.3.2 and get:

$$\inf\{\Phi(w) + \Psi(w), w \in E\} = \max\{-\Phi^*(-\mu) - \Psi^*(\mu), \mu \in E'\}$$

where the dual space  $E'$  is just the space of all real-valued bounded Borel measures  $\mu$  on  $B \times B$  (By Riesz' Theorem), and  $\Phi^*$ ,  $\Psi^*$  are the Legendre-Fenchel transforms:

$$\Phi^*(\mu) = \sup\{\langle \mu, w \rangle - \Phi(w), w \in \mathcal{W}\}$$

$$\Psi^*(\mu) = \sup\{\langle \mu, w \rangle - \Psi(w), w \in \mathcal{W}\},$$

where the duality bracket is defined by

$$\langle \mu, w \rangle = \int_{B \times B} w(x, y) \mu(dx, dy), \quad \forall w \in \mathcal{W}, \quad \forall \mu \in \mathcal{W}'.$$

Observe that notation "max" is used on purpose to emphasize that the sup is achieved on the right-hand side (which is a priori not true for the infimum on the left-hand side).

Let us now compute  $\Phi^*$  and  $\Psi^*$ . We first get  $\Phi^*(-\mu) = +\infty$ , unless  $\mu \geq 0$ , in which case

$$\Phi^*(-\mu) = - \int_{B \times B} c(x, y) \mu(dx, dy).$$

Next,  $\Psi^*(\mu) = +\infty$ , unless both projections of  $\mu$  on  $B$  are respectively  $\mu_0$  and  $\mu_1$ , in which case  $\Psi^*(\mu) = 0$ . So, we have obtained the existence of  $\mu_{\text{opt}} \geq 0$ , with projections  $\mu_0$  and  $\mu_1$ , that maximizes  $\int_{B \times B} c(x, y) \mu(dx, dy)$  among all nonnegative Borel measures on  $B \times B$  with projections  $\mu_0, \mu_1$ . Furthermore, we have the duality equality:

$$\int_{B \times B} c(x, y) \mu_{\text{opt}}(dx, dy) = \inf\left\{ \int_B a(x) \mu_0(dx) + \int_B b(y) \mu_1(dy), \quad a \oplus b \geq c \right\}.$$

### Existence part of Theorem 3.2.1

A priori, the inf is not achieved in the duality equality. So, we consider a minimizing sequence  $(a_n, b_n)$ . Remarkably enough, we may get a new minimizing sequence  $(\tilde{a}_n, \tilde{b}_n)$  with better performances, just by setting

$$\tilde{b}_n(y) = \sup_{x \in B} c(x, y) - a_n(x),$$

$$\tilde{a}_n(x) = \sup_{y \in B} c(x, y) - \tilde{b}_n(y).$$

(Note that  $\tilde{b}_n \leq b_n$ ,  $\tilde{a}_n \leq a_n$  and  $\tilde{a}_n \oplus \tilde{b}_n \geq c$ .) This new sequence is uniformly equicontinuous on the compact set  $B \times B$ . For notational simplicity, let us denote it again by  $(a_n, b_n)$ . Since we may add an arbitrarily chosen constant to  $a_n$  and subtract the same constant from  $b_n$ , we may assume that the  $\tilde{a}_n$  and  $\tilde{b}_n$  are uniformly bounded on  $B$ . (Indeed, we may adjust  $a_n$  so that the supremum of  $x \rightarrow c(x, 0) - a_n(x)$  on  $B$  is equal to 0, which guarantees that  $a_n \geq \inf c$  and  $\tilde{b}_n(0) = 0$ . It follows that the

$|\tilde{b}_n|$  are uniformly by some constant  $R$ , since they are uniformly equicontinuous. By definition, the  $\tilde{a}_n$  are bounded away from above by  $R + \sup c$  and bounded away from below by  $\inf c$ .) At this stage, we apply the Ascoli Theorem to get a subsequence, still denoted by  $(a_n, b_n)$ , that converges in sup norm to some limit  $(a, b)$  on  $B$ . We may further ensure that

$$a(x) = \sup_{y \in B} c(x, y) - b(y)$$

(by using the same process as above). We immediately see that  $(a, b)$  minimizes the continuous functional on  $C(B) \times C(B)$  defined by:

$$(a, b) \rightarrow \int_B a(x) \mu_0(dx) + \int_B b(y) \mu_1(dy)$$

among all pairs  $(a, b)$  such that  $a \oplus b \geq c$ . Therefore, we have obtained

$$\int_{B \times B} c(x, y) \mu_{\text{opt}}(dx, dy) = \int_B a(x) \mu_0(dx) + \int_B b(y) \mu_1(dy),$$

from which we deduce

$$\int_{B \times B} (a(x) + b(y) - c(x, y)) \mu_{\text{opt}}(dx, dy) = 0,$$

since  $\mu_0, \mu_1$  are projections of  $\mu_{\text{opt}}$ . Since  $\mu_{\text{opt}}$  is a nonnegative measure, this implies

$$a(x) + b(y) = c(x, y)$$

for  $\mu_{\text{opt}}$ -every  $x, y$  in  $B$ .

At this stage, we limit ourself to the special choice  $c(x, y) = x \cdot y$  and assume that  $\mu_0$  is absolutely continuous with respect to the Lebesgue measure and can be written

$$\mu_0(dx) = \rho_0(x) dx,$$

for some Lebesgue integrable function  $\rho_0 \geq 0$  on  $B$ , with integral 1. Thus, we may write

$$a(x) = \sup_{y \in B} x \cdot y - b(y)$$

which shows that  $a$  is *both* Lipschitz continuous and convex on  $B$ . The Rademacher Theorem [233] tells us that  $a$  is almost everywhere integrable in the interior of  $B$ . Since  $B$  is smooth, its boundary  $\partial B$  is a set of zero Lebesgue measure in  $\mathbb{R}^d$ . Therefore the set of all points  $x$  in  $B$  which either lie on  $\partial B$  or in the interior of  $B$  without being a point of differentiability for  $a$  is of zero  $\mu_0$  measure (since  $\mu_0$  is absolutely continuous with respect to the Lebesgue measure). Since  $\mu_{\text{opt}}$  admits  $\mu_0$  as first projection, we deduce that, for  $\mu_{\text{opt}}$ -almost every point  $(x^*, y^*) \in B \times B$ ,  $x^*$  belongs to the interior of  $B$  and is a differentiability point for  $a$ . We may further assume que

$$a(x^*) + b(y^*) = x^* \cdot y^*,$$

since, as already seen, thus property is true  $\mu_{\text{opt}}$ -almost everywhere. Since

$$a(x) + b(y^*) \geq x \cdot y^*$$

is true for every  $x \in B$ , we see that  $x^*$  is a minimizer for function  $x \rightarrow a(x) - x \cdot y^*$ . Thus, by differentiation, we have

$$\nabla a(x^*) = y^*.$$

This property is therefore true  $\mu_{\text{opt}}$ -almost everywhere, which implies

$$\mu_{\text{opt}}(dx, dy) = \delta(y - \nabla a(x))\rho_0(x)dx,$$

in the precise sense that

$$\int_{B \times B} w(x, y)\mu_{\text{opt}}(dx, dy) = \int_B w(x, \nabla a(x))\rho_0(x)dx, \quad \forall w \in C(B \times B).$$

(Observe that this already enforces the uniqueness of the optimal solution  $\mu_{\text{opt}}$ .) By projection (i.e. by setting  $w(x, y) = u(y)$ ), we deduce

$$\int_B u(y)\mu_1(dy) = \int_B u(\nabla a(x))\rho_0(x)dx, \quad \forall u \in C(B),$$

which exactly tells that  $x \rightarrow \nabla a(x)$  transports  $\rho_0(x)dx$  toward  $\mu_1(dy)$ . Since  $a$  is Lipschitz continuous and convex, we have already proven the existence part of Theorem 3.2.1.

### Uniqueness part of Theorem 3.2.1

Assume the existence of a convex Lipschitz function  $\tilde{a}$  such that  $x \rightarrow y = \nabla \tilde{a}(x)$  transports  $\rho_0(dx)$  toward  $\mu_1(dy)$  and set

$$\tilde{b}(y) = \sup_{x \in B} x \cdot y - \tilde{a}(x), \quad y \in B.$$

We first observe that

$$\tilde{a}(x) + \tilde{b}(\nabla \tilde{a}(x)) = x \cdot \nabla \tilde{a}(x)$$

holds true for Lebesgue-almost every  $x \in B$ . [Indeed, by the Rademacher theorem, almost every  $x^* \in B$  lies in the interior of  $B$  and is a differentiability point for  $\tilde{a}$ . Let us set  $y^* = \nabla \tilde{a}(x^*)$ . Note that  $y^*$  lies in  $B$ , since  $\nabla \tilde{a}$  transports  $\rho(x)dx$  toward  $\mu_1(dy)$  and both measures are supported in the compact set  $B$ . The Lipschitz concave function on  $B$   $x \in B \rightarrow x \cdot y^* - \tilde{a}(x)$  is differentiable in  $x = x^*$ , which lies in the interior of  $B$ , with zero derivative. Thus its maximum is achieved in  $x^*$ , which, by definition, is nothing but  $\tilde{b}(y^*)$ . Therefore, we have  $\tilde{b}(y^*) = x^* \cdot y^* - \tilde{a}(x^*)$ . Since  $y^* = \nabla \tilde{a}(x^*)$ , we have obtained the required equality.]

Let us now set

$$\mu(dx, dy) = \delta(y - \nabla \tilde{a}(x))\rho_0(x)dx.$$

We have

$$\begin{aligned} \int_{B \times B} x \cdot y \mu(dx, dy) &= \int_B x \cdot \nabla \tilde{a}(x) \rho_0(x) dx \\ &= \int_B (\tilde{a}(x) + \tilde{b}(\nabla \tilde{a}(x))) \rho_0(x) dx = \int_{B \times B} (\tilde{a}(x) + \tilde{b}(y)) \mu(dx, dy) \\ &= \int_{B \times B} (\tilde{a}(x) + \tilde{b}(y)) \mu_{\text{opt}}(dx, dy) \end{aligned}$$

(since  $\mu_{\text{opt}}$  and  $\mu$  have the same projections)

$$\geq \int_{B \times B} x \cdot y \mu_{\text{opt}}(dx, dy)$$

(because  $\tilde{a}(x) + \tilde{b}(y) \geq x \cdot y$ ). Thus  $\mu$  is optimal, just as  $\mu_{\text{opt}}$ , which is, as already noticed, is the unique optimal solution. We therefore have, by definition of  $\mu$ :

$$\delta(y - \nabla \tilde{a}(x)) \rho_0(x) dx = \mu(dx, dy) = \mu_{\text{opt}}(dx, dy) = \delta(y - \nabla a(x)) \rho_0(x) dx,$$

and this is possible only if  $\nabla \tilde{a}(x) = \nabla a(x)$  for  $\rho_0(x) dx$ -almost every  $x$ , which is exactly the uniqueness part of our Theorem. So, the proof of Theorem 3.2.1 is now complete.

### 3.4 An application to the best Sobolev constant problem

In this section, that can be skipped without affecting the rest of the book, we sketch just one remarkable application of the Monge-Ampère equation in the framework of Optimal transportation. We are motivated by the non-convex minimization problem

$$I(U, p, q) = \inf \left\{ \int_U |\nabla u(x)|^p dx, \quad u \in C_c^\infty(U), \quad \text{t.q.} \quad \int_U |u(x)|^q dx = 1 \right\}$$

where  $p, q \in ]1, +\infty[$  and  $U$  is an open subset of  $\mathbb{R}^d$ .

It is rather straightforward, by using linear changes of variable of type  $x \rightarrow rx + a$  with  $r > 0$  and  $a \in \mathbb{R}^d$  on functions  $u \in C_c^\infty(U)$ , to see that:

i) in case  $U = \mathbb{R}^d$ ,  $I(U, p, q) = 0$  except if  $1 - d/p = 0 - d/q$ ;

ii) whenever  $U$  is bounded (in which case, we only use retractions for which  $r > 1$ )  $I(U, p, q) = 0$  unless if

$$1 - d/p \geq 0 - d/q.$$

When  $U$  is bounded and  $1 - d/p > 0 - d/q$ , traditional compactness methods may be used and we rather easily get the existence of an optimal generalized solution in the Banach space obtained by completion of  $C_c^\infty(D)$  for the norm

$$u \rightarrow \|u\|_{L^q(U)} + \|\nabla u\|_{L^p(U)}.$$

Such a solution can be easily shown to satisfy, in the sense of distributions in  $U$ ,

$$-\nabla(|\nabla u|^{p-2} \nabla u) = \lambda u |u|^{q-2}$$

where constant  $\lambda$  has to be chosen so that  $\|u\|_{L^q(U)} = 1$ . In particular, in the most usual case  $p = 2$ , we find the semi-linear PDE

$$-\Delta u = \lambda u |u|^{q-2}.$$

In the critical case,  $1 - d/p = 0 - d/q$ , it is also easy to see that  $I(U, p, q)$  does not depend on  $U$ ! It is more subtle (and this is strongly connected to the "concentration-compactness" theory [332]) to figure out, that when  $U$  is bounded, there is *no* optimal solution, even in the completed space! Furthermore, one can prove that the



minimizing sequences  $u_n$  have the strange property that, up to the extraction of a subsequence, they concentrate in the sense that one can find a point  $x_\infty$  in  $U$  such that  $|u_n|^q$  converges as a Borel nonnegative measure to the Dirac mass at point  $x_\infty$ . (This is a prototype of the "bubble" phenomenon, that occurs so often in Geometric Analysis [434].)

For a more positive result, we limit ourself to the simplest case when  $U$  is unbounded, namely  $U = \mathbb{R}^d$ . Then:

**Theorem 3.4.1.** *In the critical case  $1 - d/p = 0 - d/q$ ,*

$$I(\mathbb{R}^d, p, q) = \inf \left\{ \int_{\mathbb{R}^d} |\nabla u(x)|^p dx, \quad u \in C_c^\infty(\mathbb{R}^d), \quad \text{t.q.} \quad \int_{\mathbb{R}^d} |u(x)|^q dx = 1 \right\}$$

*is achieved by a unique (up to translations and dilations) solution  $u$  in the Banach  $E$  obtained by completion of  $C_c^\infty(\mathbb{R}^d)$  with respect to the norm*

$$\|u\|_E = \|u\|_{L^q(\mathbb{R}^d)} + \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

*As a consequence, equation*

$$-\nabla(|\nabla u|^{p-2} \nabla u) = \lambda u |u|^{q-2}$$

*admits a unique (up to translations and dilations) solution in  $E$ , where constant  $\lambda$  has to be fixed so that*

$$\|u\|_{L^q(\mathbb{R}^d)} = 1.$$

There are several possible proof, in particular by the "concentration-compactness" method [332]. A remarkable and very simple proof follows directly (up to a lot of technicalities) from Theorem 3.2.1 and is due to Dario Cordero-Erausquin, Bruno Nazaret and Cédric Villani [179]. Let us sketch this proof (while skipping many technicalities).

Consider two functions  $u$  et  $v$  dans  $C_c^\infty(\mathbb{R}^d)$  such that  $\|u\|_{L^q(\mathbb{R}^d)} = \|v\|_{L^q(\mathbb{R}^d)} = 1$  and consider the Borel probability measures

$$F(x)dx = |u(x)|^q dx, \quad G(y)dy = |v(y)|^q dy.$$

According to Theorem 3.2.1, there is a unique Borel map  $T$  that transports the first measure to second one and can be written, for  $F(x)dx$ -almost every  $x$ ,

$$T(x) = \nabla \Phi(x),$$

where  $\Phi$  is a convex Lipschitz function on  $\mathbb{R}^d$ . In addition, in the generalized sense of Theorem 3.2.1,  $\Phi$  satisfies the Monge-Ampère equation

$$G(\nabla \phi(x)) \det(D^2 \Phi(x)) = F(x).$$

Let us now simply evaluate

$$J = \int_{\mathbb{R}^d} G(y)^{1-1/d} dy$$

and, remarkably enough, all the results we are interested in (existence, uniqueness and explicit formulae for a solution to best Sobolev constant problem) will follow from two elementary inequalities, namely Young's inequality

$$\frac{|a|^p}{p} + \frac{|b|^{p'}}{p'} \geq a \cdot b, \quad \forall a, b \in \mathbb{R}^d, \quad 1/p' + 1/p = 1, \quad p \in ]1, \infty[$$

(with equality if and only if  $b = a|a|^{p-2}$  or  $a = b|b|^{p'-2}$ ) and the domination of the geometric mean by the arithmetic mean for any finite sequence of nonnegative real numbers, with equality only if all these numbers are equal.)

By construction of  $T = \nabla\Phi$ , we first get

$$\begin{aligned} J &= \int_{\mathbb{R}^d} G(y)^{1-1/d} dy = \int_{\mathbb{R}^d} G(\nabla\Phi(x))^{-1/d} F(x) dx \\ &= \int_{\mathbb{R}^d} \det(D^2\Phi(x))^{1/d} F(x)^{1-1/d} dx. \end{aligned}$$

(Here the proof is only formal, since the Monge-Ampère equation is a priori not satisfied in the classical sense. For a rigorous proof, more work is needed, as in [179].) Since  $\Phi$  is convex, the eigenvalues of  $D^2\Phi$  are nonnegative which leads to the point-wise inequality

$$\det(D^2\Phi(x))^{1/d} \leq 1/d \Delta\Phi(x).$$

We deduce  $J \leq \tilde{J}$  where

$$\begin{aligned} \tilde{J} &= 1/d \int_{\mathbb{R}^d} \Delta\Phi(x) F(x)^{1-1/d} dx \\ &= -1/d \int_{\mathbb{R}^d} \nabla\Phi(x) \cdot \nabla(F(x)^{1-1/d}) dx \end{aligned}$$

(by integration by part)

$$= -s/d \int_{\mathbb{R}^d} \nabla\Phi(x) \cdot u(x) |u(x)|^{s-2} \nabla u(x) dx$$

(by setting  $s = (1 - 1/d)q$  and by definition of  $F = |u|^q$ )

$$\leq s/d \|\nabla u\|_{L^p(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |u(x)|^{(s-1)p'} |\nabla\Phi(x)|^{p'} dx \right)^{1/p'}$$

(by Young-Hölder, with  $1/p' = 1 - 1/p$ )

$$= s/d \|\nabla u\|_{L^p(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} F(x) |\nabla\Phi(x)|^{p'} dx \right)^{1/p'},$$

(using that  $(s-1)p' = q$  and  $F = |u|^q$ )

$$= s/d \|\nabla u\|_{L^p(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} G(y) |y|^{p'} dy \right)^{1/p'},$$

(since  $G(y)dy$  is the image measure by  $T = \nabla\Phi$  of  $F(x)dx$ ). So, we have obtained that, for all  $u, v$  of unit norm in  $L^q$ ,

$$\int_{\mathbb{R}^d} |v(y)|^s dy \leq s/d \|\nabla u\|_{L^p(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |v(y)|^q |y|^{p'} \right)^{1/p'}$$

with  $s = (1 - 1/d)q$ , which extends by completion to all  $u, v$  in the completed Banach space  $E$ . Observe, furthermore, that this inequality becomes an equality if only if the geometric-arithmetic inequality and the Hölder inequality are both saturated. Then, one finds (after some calculations) a constant  $r > 0$  and a point  $x_0$  such that  $T(x) = (x - x_0)r$ ,  $u(x) = r^{-d}v((x - x_0)r)$  and, finally,  $u(x) = (\mu + |x - x_0|^\alpha)^\beta \nu$  for some constants  $\alpha, \beta, \mu, \nu$  to be fixed in terms of  $p$  and  $d$  via  $q$  and  $s$  (in fact,  $\alpha = p'$  and  $\beta = 1 - d/p = -d/q$ ). [Observe that, concerning  $u$  and  $v$ , we have exited space  $C_c^\infty(\mathbb{R}^d)$  and entered the completed space  $E$ .]

This amounts to the following *non* convex duality equality

$$\max_{v \in S_1(L^q)} \frac{\int_{\mathbb{R}^d} |v(y)|^s dy}{\left( \int_{\mathbb{R}^d} |v(y)|^q |y|^{p'} dy \right)^{1/p'}} = s/d \min_{u \in S_1(L^p)} \|\nabla u\|_{L^p(\mathbb{R}^d)},$$

$$s = (1 - 1/d)q, \quad 1 - d/p = -d/q,$$

where  $S_1(L^q)$  denotes the unit sphere of  $L^q$  intersected with  $E$ . Existence, uniqueness (up to translations and dilations) of solutions in the completed Banach space  $E$  to the best Sobolev constant problem are just direct corollary of this truly remarkable *non* convex duality formula.

# Chapter 4

## The optimal incompressible transport problem

This chapter is entirely devoted to the analysis of the relaxed minimizing geodesic problem, already presented in section 2.2, that we can call, as well, the “optimal incompressible transport” (OIT). This problem is substantially more complicated than the *regular* optimal transport problem, which is related to the Monge-Ampère equation, as discussed in Section 3.2. However, there are many similarities, in particular the crucial use of convexity tools, as the Fenchel-Rockafellar duality theorem.

We consider pairs of measures  $(c, q) \in (C^0([t_0, t_1] \times D^2; \mathbb{R} \times \mathbb{R}^d))'$  and use systematically the following notation for duality brackets:

$$\langle c, A \rangle + \langle q, B \rangle = \int_{t,x,a} A(t, x, a)c(t, x, a) + B(t, x, a) \cdot q(t, x, a)$$

for all  $(A, B) \in C^0([t_0, t_1] \times D^2; \mathbb{R} \times \mathbb{R}^d)$ . The OIT problem amounts to finding such a pair  $(c, q)$  that minimizes

$$K(c, m) = \frac{1}{2} \int_{t,x,a} |v(t, x, a)|^2 c(t, x, a),$$

subject to the following constraints:

i)  $(c, q)$  satisfies the "microscopic continuity equation"

$$\partial_t c(t, x, a) + \nabla \cdot q(t, x, a) = 0$$

and  $c(t_0, \cdot, \cdot)$  and  $c(t_1, \cdot, \cdot)$  are given in  $DS(D)$  and are respectively denoted by  $c_{t_0}$  and  $c_{t_1}$ . (The word microscopic refers to the variable  $a$  which plays the role of a parameter in the equation and the  $\nabla$  operator only involves the space variable  $x$ .)

This can be expressed in weak form by

$$\begin{aligned} & \int_{t,x,a} \partial_t \varphi(t, x, a)c(t, x, a) + \nabla \varphi(t, x, a)q(t, x, a) \\ &= \int_{x,a} \varphi(T, x, a)c_{t_1}(x, a) - \varphi(0, x, a)c_{t_0}(x, a), \end{aligned}$$

for all  $\varphi = \varphi(t, x, a) \in \mathbb{R}$  which are continuous and  $C^1$  in  $(t, x)$ .

ii) at each  $t \in [t_0, t_1]$ ,  $c(t, \cdot, \cdot)$  is doubly stochastic, i.e.

$$\int_x c(t, x, a) = 1, \quad \int_a c(t, x, a) = 1.$$

This first constraint is automatically satisfied because of the continuity equation (to check it, just take  $\varphi$  in the weak formulation as a function of  $t$  and  $a$  only), while the second one can be simply expressed by

$$\int_{t,x,a} p(t,x)c(t,x,a) + \int_{[t_0,t_1] \times D} p(t,x)dxdt, \quad \forall p \in C^0([t_0,t_1] \times D).$$

Let us now recall the precise definition of  $K$ :

$$K(c, q) = \sup \left\{ \int_{t,x,a} A(t, x, a)c(t, x, a) + B(t, x, a) \cdot q(t, x, a); \right. \\ \left. (A, B) \in C^0([t_0, t_1] \times D^2; \mathbb{R} \times \mathbb{R}^d) \text{ s.t. } 2A + |B|^2 \leq 0 \right\},$$

which is a l.s.c. function (with respect to the weak-\* topology), valued in  $] - \infty, +\infty]$ , with value  $K(c, q) = +\infty$ , *unless*  $c \geq 0$ ,  $q$  is absolutely continuous with respect to  $c$ , with a vector-valued Radon-Nikodym density  $v$  square-integrable in  $c$ , in which case

$$K(c, m) = \frac{1}{2} \int_{t,x,a} |v(t, x, a)|^2 c(t, x, a).$$

(The proof of this fact is a rather elementary exercise in measure theory. See [90] for a detailed proof.)

## 4.1 Saddle-point formulation and convex duality

Using Lagrangian multipliers, our optimization problem can therefore be written as the following "inf-sup" problem:  $K_{opt}(t_0, t_1, c_{t_0}, c_{t_1}) =$

$$\inf_{c,q} \sup_{A,B,\varphi,p} \int_{[0,T] \times D} p(t,x)dxdt + \int_{x,a} \varphi(T, x, a)c_{t_1}(x, a) - \varphi(0, x, a)c_{t_0}(x, a) \\ + \int_{t,x,a} (A(t, x, a) - \partial_t \varphi(t, x, a) - p(t, x))c(t, x, a) + (B(t, x, a) - \nabla \varphi(t, x, a)) \cdot q(t, x, a),$$

subject to

$$A(t, x, a) + \frac{|B(t, x, a)|^2}{2} \leq 0, \quad \forall (t, x, a) \in [t_0, t_1] \times D^2.$$

Notice that the optimal value can be easily rescaled, by homogeneity and translation invariance in  $t$ , as

$$K_{opt}(t_0, t_1, c_{t_0}, c_{t_1}) = (t_1 - t_0)^{-1} K_{opt}(0, 1, c_{t_0}, c_{t_1})$$

so that we may consider only the case  $t_0 = 0$ ,  $t_1 = 1$  and, consistently, denote  $c_{t_0}$  and  $c_{t_1}$  by  $c_0$  and  $c_1$  and  $K_{opt}(0, 1, c_0, c_1)$  just by  $K_{opt}(c_0, c_1)$ , as will be done subsequently. Notice that the sup-inf problem can be trivially computed (because we just have to minimize in  $(c, q)$  without any constraint thanks to the Lagrange multipliers  $(A, B, \varphi, p)$ ), which leads to the maximization problem in  $(\varphi, p)$ :

$$\sup_{\varphi,p} \int_{[0,1] \times D} p(t,x)dxdt + \int_{x,a} \varphi(1, x, a)c_1(x, a) - \varphi(0, x, a)c_0(x, a),$$

where

$$\partial_t \varphi(t, x, a) + \frac{|\nabla \varphi(t, x, a)|^2}{2} + p(t, x) \leq 0, \quad \forall (t, x, a) \in [0, 1] \times D^2.$$

(after elimination of  $A = \partial_t \varphi + p$  and  $B = \nabla \varphi$ ). Notice that we keep using the  $\nabla$  notation only for the derivation in  $x$ . As a matter of fact, there will be subsequently never any derivation performed in the "microscopic" variable  $a$ . The first step in our analysis is now to justify that the inf-sup and the sup-inf coincide, thanks to the Fenchel-Rockafellar duality theorem 3.3.2 that we have already used for the Monge-Ampère equation in Chapter 3.

### Rockafellar duality

We introduce

$$E = C^0([0, 1] \times D^2; \mathbb{R} \times \mathbb{R}^d),$$

which is a Banach space for the sup norm, and define two convex functions  $K_1$  and  $K_2$  on  $E$ , valued in  $]0, +\infty]$ , as follows. We first set

$$K_1(A, B) = - \int_{[0,1] \times D} p(t, x) dx dt - \int_{D^2} \varphi(1, x, a) dc_1(x, a) - \varphi(0, x, a) dc_0(x, a),$$

whenever there are  $p \in C([0, 1] \times D)$  and  $\varphi \in C([0, 1] \times D^2)$ , which is  $C^1$  in  $(t, x)$  such that

$$A(t, x, a) = \partial_t \varphi(t, x, a) + p(t, x), \quad B(t, x, a) = \nabla \varphi(t, x, a),$$

and  $K_1(A, B) = +\infty$  otherwise. Then, we define

$$K_2(A, B) = 0, \quad \text{if } A(t, x, a) + \frac{|B(t, x, a)|^2}{2} \leq 0, \quad \forall (t, x, a) \in [0, 1] \times D^2.$$

and  $K_2(A, B) = +\infty$  otherwise.

Notice that the first definition is consistent, in the sense that if  $A, B$  are represented as above by two different couples  $(\varphi, p)$ ,  $(\tilde{\varphi}, \tilde{p})$ , then the value of  $K_1(A, B)$  is unchanged.

**Lemma 4.1.1.** *The functionals  $K_1, K_2 : E \rightarrow \mathbb{R} \cup \{+\infty\}$  verify the hypotheses of Theorem 3.3.2.*

*Proof.* The convexity condition is clear. Next, we have to find a function  $u_0$  in  $E$  having the required properties in the Theorem. We observe here that there is no chance that  $K_1$  is continuous (for the  $C^0$ -norm) because arbitrarily near any function where  $K_1 < +\infty$  there is some function with  $K_1 = +\infty$ . On the other side, in the point  $(A_0, B_0) = (-1, 0)$  we have  $A_0 = \partial_t \varphi_0 + p_0, B_0 = \nabla \varphi_0$  for  $\varphi_0 = 0, p_0 = -1$ , so  $K_1$  is finite at this point. On the other side,  $K_2(A_0, B_0) = 0$  and this condition is preserved for small perturbations of  $(A_0, B_0)$  in the  $C^0$ -norm. Therefore the assumptions of Theorem 3.3.2 are satisfied.  $\square$

We now want to exploit Theorem 3.3.2 in our setting. We start by noticing that

$$K_2^*(c, q) = K(c, q),$$

where  $K$  is nothing but the functional introduced at the beginning of this chapter. Let us now compute  $K_1^*(-c, -q)$ . By definition,

$$\begin{aligned} K_1^*(-c, -q) &= \sup_{\varphi, p} \int_{t,x,a} (-\partial_t \varphi(t, x, a) - p(t, x)) c(t, x, a) - \nabla \varphi(t, x, a) \cdot q(t, x, a) \\ &\quad + \int_{t,x} p(t, x) dx dt + \int_{x,a} \varphi(1, x, a) dc_1(x, a) - \varphi(0, x, a) dc_0(x, a). \end{aligned}$$

This exactly means that  $K_1^*(-c, -q)$  takes value  $\infty$  unless

$$\int_a c(t, x, a) = 1, \quad \partial_t c + \nabla \cdot q = 0, \quad c(0, x, a) = c_0(x, a), \quad c(1, x, a) = c_1(x, a),$$

in which case  $K_1^*(-c, -q) = 0$ . So, we conclude that

$$\sup_{c,q} K_1^*(-c, -q) + K_2^*(c, q) = K_{opt}(c_0, c_1)$$

which corresponds to the inf-sup problem while

$$\sup_{A,B} -K_1(A, B) - K_2(A, B)$$

is (almost by definition) just the value of the sup-inf problem that we have computed earlier. So the inf-sup and the sup-inf have the same optimal value and we can state:

**Theorem 4.1.2.** *The optimal incompressible transport (OIT) problem can be successively written in primal (sup) and dual (inf) form:*

$$\sup_{\varphi, p} \int_{[0,1] \times D} p(t, x) dx dt - \int_{D^2} \varphi(1, x, a) dc_1(x, a) - \varphi(0, x, a) dc_0(x, a),$$

subject to

$$\partial_t \varphi(t, x, a) + \frac{|\nabla \varphi(t, x, a)|^2}{2} + p(t, x) \leq 0, \quad \forall (t, x, a) \in [0, 1] \times D^2$$

and

$$\inf_{c,q} K(c, q), \quad K(c, q) = \frac{1}{2} \int_{t,x,a} |v(t, x, a)|^2 c(t, x, a), \quad q = cv,$$

subject to

$$\partial_t c + \nabla \cdot q = 0, \quad \int_a c(t, x, a) = 1, \quad c(0, x, a) = c_0(x, a), \quad c(1, x, a) = c_1(x, a),$$

and there is at least an optimal solution  $(c, q)$  to the second one.

## 4.2 Existence and uniqueness of the pressure gradient

**Theorem 4.2.1.** *There is a unique distribution,  $\nabla p$  depending only on the data  $c_0, c_1$  such that  $\nabla p^\varepsilon \rightarrow \nabla p$  in the sense of distributions in the interior of  $[0, 1] \times D$ , for any  $(\varphi^\varepsilon, p^\varepsilon)$   $\varepsilon$ -solution to the primal problem. In addition,  $\nabla p$  is characterized by*

$$\nabla p(t, x) = -\partial_t \int_a (cv)(t, x, a) - \nabla \cdot \int_a (cv \otimes v)(t, x, a)$$

for all optimal solutions  $(c, q = cv)$  of the dual problem.

We introduce a short notation for the boundary data:

$$BT(f) = \int_{x,a} f(1, x, a)c_1(x, a) - f(0, x, a)c_0(x, a)$$

and denote

$$J(p, \varphi) = \int_{[0,1] \times D} p(t, x) dx dt - \int_{D^2} \varphi(1, x, a) dc_1(x, a) - \varphi(0, x, a) dc_0(x, a).$$

We consider a minimizer  $(c, q = cv)$  for the dual problem, which exists by Rockafellar's duality theorem, and we denote by  $(CE)$  the ‘‘continuity equation’’ with boundary data, namely, in weak form,

$$\forall f, \quad BT(f) = \int_{t,x,a} (\partial_t f + (v \cdot \nabla) f) c,$$

and by  $(IC)$  the ‘‘incompressibility’’ constraint  $\int_a c = 1$ .

**Lemma 4.2.2.** *For all optimal pairs  $(c, cv)$ , for all pairs  $(\tilde{c}, \tilde{v}\tilde{c})$  satisfying  $(CE)$  but not necessarily  $(IC)$  and for any  $\varepsilon$ -solution  $(p^\varepsilon, \varphi^\varepsilon)$  of the primal problem, we have (with  $\int$  meaning  $\int_{t,x,a}$ )*

$$\begin{aligned} \int p^\varepsilon(c - \tilde{c}) + \frac{1}{2} \int \tilde{c} |\nabla \varphi^\varepsilon - \tilde{v}|^2 + \int \tilde{c} \left| \partial_t \varphi^\varepsilon + \frac{1}{2} |\nabla \varphi^\varepsilon|^2 + p^\varepsilon \right| \\ \leq \frac{1}{2} \int \tilde{c} |\tilde{v}|^2 - \frac{1}{2} \int c |v|^2 + \varepsilon^2 \end{aligned}$$

*Proof.* We use inequality  $\partial_t \varphi^\varepsilon + \frac{1}{2} |\nabla \varphi^\varepsilon|^2 + p^\varepsilon \leq 0$ , defining the  $\varepsilon$ -solutions, together with the fact that  $\tilde{c} \geq 0$ , and rewrite

$$\begin{aligned} -BT(\varphi^\varepsilon) &= - \int (\partial_t \varphi^\varepsilon + (\tilde{v} \cdot \nabla) \varphi^\varepsilon) \tilde{c} = \int \left| \partial_t \varphi^\varepsilon + \frac{1}{2} |\nabla \varphi^\varepsilon|^2 + p^\varepsilon \right| \tilde{c} \\ &\quad + \frac{1}{2} \int |\nabla \varphi^\varepsilon - \tilde{v}|^2 \tilde{c} - \frac{1}{2} \int |\tilde{v}|^2 \tilde{c} + \int p^\varepsilon \tilde{c}. \end{aligned}$$

By definition of an  $\varepsilon$ -solution, and since  $(c, cv)$  realizes the supremum in the dual problem, we have

$$-BT(\varphi^\varepsilon) - \int p^\varepsilon = -J(p^\varepsilon, \varphi^\varepsilon) \leq -\frac{1}{2} \int |v|^2 c + \varepsilon^2,$$

which inserted in the previous inequality gives the wanted result.  $\square$

If in Lemma 4.2.2 we take  $(\tilde{c}, \tilde{v}) = (c, v)$  we obtain

$$\frac{1}{2} \int c |v - \nabla \varphi^\varepsilon|^2 + \int c \left| \partial_t \varphi^\varepsilon + \frac{1}{2} |\nabla \varphi^\varepsilon|^2 + p^\varepsilon \right| \leq \varepsilon^2. \quad (4.2.1)$$

If we were able to pass to the limit in this inequality, we would obtain, as optimality conditions for the OIT problem:

$$\left\{ \begin{array}{l} v = \nabla \phi, \quad \partial_t \varphi + \frac{1}{2} |\nabla \varphi|^2 + p = 0, \quad c - \text{a.e.}, \\ \partial_t c + \nabla \cdot (cv) = 0, \quad \int_a c(t, x, a) = 1, \\ \partial_t \varphi(t, x, a) + \frac{1}{2} |\nabla \varphi(t, x, a)|^2 + p(t, x) \leq 0, \quad \forall (t, x, a) \in [0, 1] \times D^2 \\ c(0, x, a) = c_0(x, a), \quad c(1, x, a) = c_1(x, a). \end{array} \right. \quad (4.2.2)$$



Unfortunately, it is unclear that the limit  $\phi$  can be defined in a reasonable sense (this is an open question in the OIT theory). However, we will be shortly able to prove the convergence of  $\nabla p^\varepsilon$  to a definite limit  $\nabla p$ . To achieve this goal, we first perform smooth deformations of a given pair  $(c, v)$  (typically a solution of the dual OIT problem) into another pair  $(\tilde{c}, \tilde{v})$  satisfying (CE) but not necessarily (IC). This turns out to be a good way to “feel” how  $p^\varepsilon$  acts on test functions. We use a definition by duality, requiring that, for all test functions  $f(t, x, a) \in \mathbb{R}$  and  $B(t, x, a) \in \mathbb{R}^d$ ,

$$\begin{aligned} & \int_{t,x,a} f(t, x, a) \tilde{c}(t, x, a) + B(t, x, a) \cdot (\tilde{c}\tilde{v})(t, x, a) \\ &= \int (f(t, M(t, x), a) + B(t, M(t, x), a) \cdot [(\partial_t + v(t, x, a) \cdot \nabla)M(t, x)]) c(t, x, a), \end{aligned}$$

where  $(t, x) \in [0, T] \times D \rightarrow M(t, x) \in D$  is smooth and so that  $M(t, x) = x$  near  $\partial([0, T] \times D)$  and  $M(t, \cdot)$  is a diffeomorphism of  $D$  for all  $t \in [0, T]$ .

We first observe that under such hypotheses  $(\tilde{c}, \tilde{v})$  satisfies (CE) as soon as  $(c, v)$  satisfies it. Indeed, denoting  $\tilde{f}(t, x, a) = f(t, M(t, x), a)$ , we find:

$$\begin{aligned} & \int [\partial_t f + (\tilde{v} \cdot \nabla) f] \tilde{c} = \int ((\partial_t f)(t, M(t, x), a) \\ & + (\nabla f)(t, M(t, x), a) \cdot [\partial_t + v(t, x, a) \cdot \nabla] M(t, x)) c(t, x, a) \\ &= \int [\partial_t \tilde{f} + v \cdot \nabla \tilde{f}] c = BT(\tilde{f}) = BT(f), \end{aligned}$$

where we have used (CE) for  $(c, v)$  and the chain rule for  $\tilde{f}$ .

Now, let us rewrite the conclusion of Lemma 4.2.2 where  $(\tilde{c}, \tilde{v})$  is as above. We first treat the term:

$$\int p^\varepsilon \tilde{c} = \int p^\varepsilon(t, M(t, x)) c(t, x, a) = \int p^\varepsilon(t, M(t, x)) dt dx,$$

where we used the (IC) condition for  $c$ .

Next, we write

$$\begin{aligned} & \frac{1}{2} \int \tilde{c} |\tilde{v}|^2 = \sup_{A + \frac{1}{2}|B|^2 \leq 0} \int A(t, x, a) \tilde{c} + B(t, x, a) \cdot \tilde{c}\tilde{v} = \sup_B \int \left( -\frac{1}{2}|B|^2 + B \cdot \tilde{v} \right) \tilde{c} \\ &= \sup_B \int \left[ -\frac{1}{2}|B(t, M(t, x), a)|^2 + B(t, M(t, x), a) \cdot (\partial_t + v(t, x, a) \cdot \nabla)M(t, x) \right] c(t, x, a) \\ &= \sup_{\tilde{B}} \int \left[ -\frac{1}{2}|\tilde{B}|^2 + \tilde{B} \cdot (\partial_t + v \cdot \nabla)M \right] c \\ &= \frac{1}{2} \int |(\partial_t M(t, x) + (v(t, x, a) \cdot \nabla)M(t, x))|^2 c(t, x, a), \end{aligned}$$

where  $\tilde{B}(t, x, a) = B(t, M(t, x), a)$ .

So we have obtained

**Lemma 4.2.3.** *For all optimal pairs  $(c, cv)$ , for all smooth function  $(t, x) \in [0, T] \times D \rightarrow M(t, x) \in D$  such that  $M(t, x) = x$  near  $\partial([0, T] \times D)$  and  $M(t, \cdot)$  is a diffeomorphism of  $D$  for all  $t \in [0, T]$ , we have*

$$\begin{aligned} \int_{t,x} (\tilde{p}^\varepsilon - p^\varepsilon) + \int_{t,x,a} c |\widetilde{\partial_t \varphi^\varepsilon} + \frac{1}{2} |\widetilde{\nabla \varphi^\varepsilon}|^2 + \tilde{p}^\varepsilon| + \frac{1}{2} \int_{t,x,a} |\widetilde{\nabla \varphi^\varepsilon} - \partial_t M - (v \cdot \nabla) M|^2 c \\ \leq \frac{1}{2} \int_{t,x,a} c |\partial_t M + (v \cdot \nabla) M|^2 - \frac{1}{2} \int_{t,x,a} c |v|^2 + \varepsilon^2, \end{aligned}$$

where we still use notation  $\tilde{f}(t, x, a) = f(t, M(t, x), a)$  for generic functions  $f$ .

Although less general, this Lemma is much more tractable than Lemma 4.2.2, since, the dependence on  $(\tilde{c}, \tilde{v})$  we had is now substituted for by the dependence on the simpler smooth function  $M$ .

### Application of Moser's lemma

Let us now use the following variant of ‘‘Moser's Lemma’’ [377, 192, 409]

**Lemma 4.2.4** (Moser's Lemma for  $\mathbb{T}^d$ ). *Let  $\sigma_0, \sigma_1 \in C^\infty(\mathbb{T}^d)$  be strictly positive probability densities (i.e.  $\sigma_i > 0$ ,  $\int_{\mathbb{T}^d} \sigma_i dx = 1$  for  $i = 0, 1$ ). Then there exists a diffeomorphism  $M : \mathbb{T}^d \rightarrow \mathbb{T}^d$  with  $\det(DM) > 0$  such that for all continuous test functions  $\varphi$  there holds*

$$\int_{\mathbb{T}^d} \varphi(M(x)) \sigma_0(x) dx = \int_{\mathbb{T}^d} \varphi(x) \sigma_1(x) dx.$$

*Proof.* We will find an expression of  $M$  as the flow  $N(t, x)$  at time  $t = 1$  of a vectorfield  $z(t, x)$ :

$$\begin{cases} \partial_t N(t, x) = z(t, N(t, x)) \\ N(0, x) = x \end{cases}$$

To impose the right conditions on  $z$ , we express the pushforward density obtained from  $\sigma_0(x) dx$  via  $N(t, \cdot)$ :

$$\int \varphi(N(t, x)) \sigma_0(x) dx = \int \varphi(x) \sigma(t, x) dx \text{ for all } t, \varphi \in C^\infty(\mathbb{T}^d)$$

The flow equation then gives us the evolution equation  $\partial_t \sigma + \nabla \cdot (z\sigma) = 0$  for  $\sigma(t, x)$ . If we ask that  $\sigma(t, x) = (1 - t)\sigma_0(x) + t\sigma_1(x)$ , then the above equation assumes a much simpler form:  $(\sigma_1 - \sigma_0)(x) = -\nabla \cdot [\sigma(t, x)z(t, x)] = -\nabla \cdot Z(x)$ . We make the extra Ansatz that  $Z = \nabla \zeta$ , and we obtain the equation

$$\Delta \zeta + \sigma_1 - \sigma_0 = 0 \text{ on } \mathbb{T}^d.$$

The integrability condition for this equation is  $\int (\sigma_1 - \sigma_0) = 0$ , which is satisfied in our case. Therefore we obtain a smooth solution  $\zeta$ . The vectorfield  $z$  can now be expressed in terms of  $\zeta, \sigma_0, \sigma_1$  and it is bounded because of the strict positivity condition on  $\sigma_0, \sigma_1$ :

$$z(t, x) = \frac{\nabla \zeta(x)}{(1 - t)\sigma_0(x) + t\sigma_1(x)},$$

and since  $z$  is smooth and bounded, also  $N$  is smooth, therefore  $M(x) = N(1, x)$  is smooth, as wanted.  $\square$

**Remark 4.2.5.** • For this version of Moser's Lemma, we needed  $\sigma_0, \sigma_1$  to be strictly positive.

- In [192] a richer variant of the lemma is done on a compact domain  $D \subset \mathbb{R}^d$  and is followed by a second step where the boundary condition  $M(x) = x$  on  $\partial D$  is ensured. This somehow hints at the fact that the possible constructions are more flexible, and that the results could be ameliorated as done in [409].

We will need the following refinement of Moser's Lemma:

**Lemma 4.2.6.** Let  $\theta \in C_c^\infty([0, 1])$  be a nonnegative function and  $w \in C^\infty(D, \mathbb{R}^d)$ . If  $\|\theta\|_{L^\infty}$  is small enough, we can find a family of diffeomorphisms  $M(t, x)$  such that  $M(t, x) = x$  near  $\partial([0, 1] \times D)$  and for all  $\varphi \in C_c^1(\mathbb{R}^d)$  there holds

$$\int_D \varphi(M(t, x)) dx = \int_D \varphi(x) dx + \theta(t) \int_D \nabla \varphi(x) \cdot w(x) dx.$$

Moreover  $M$  will be representable as a flow, i.e. there will hold

$$\partial_t M(t, x) = z(t, M(t, x)),$$

where  $z(t, x) = \frac{\theta'(t)w(x)}{1 - \theta(t)[\nabla \cdot w(x)]}$ .

*Proof.* Call  $S = \|\theta\|_{L^\infty}$ , so that  $\theta([0, 1]) = [0, S]$ . We observe that since  $\theta$  has compact support,  $\theta(0) = 0$ . We start by defining

$$\begin{aligned} \tilde{\sigma}(s, x) &= 1 - s \nabla \cdot w(x) \\ \tilde{w}(s, t) &= \frac{w(x)}{\tilde{\sigma}(s, x)}, \end{aligned}$$

so that  $\partial_s \tilde{\sigma} + \nabla \cdot (\tilde{w} \tilde{\sigma}) = 0$ . We then consider the flow of  $\tilde{w}$ . We define

$$\begin{cases} \partial_s \tilde{M}(s, x) = \tilde{w}(s, \tilde{M}(s, x)) \text{ for } s \in [0, S] \\ \tilde{M}(0, x) = x \end{cases}$$

Then clearly  $\tilde{M}(s, x) = x$  for  $x$  near  $\partial D$ . We observe that  $\tilde{\sigma}(0, x) = 1$  and that for all  $\varphi \in C^0(D)$

$$\int \varphi(\tilde{M}(s, x)) dx = \int \varphi(x) \tilde{\sigma}(s, x) dx.$$

We then define  $M(t, x) = \tilde{M}(\theta(t) - \theta(0), x) = \tilde{M}(\theta(t), x)$ , and we have

$$\begin{aligned} \partial_t M(t, x) &= \partial_t \tilde{M}(\theta(t), x) = \tilde{w}(\theta(t), M(t, x)) \theta'(t) \\ &= \frac{w(M(t, x))}{\tilde{\sigma}(\theta(t), M(t, x))} \theta'(t) \\ &= \frac{w(M(t, x))}{1 - \theta(t) \nabla \cdot w(M(t, x))} \theta'(t) \\ &= z(t, M(t, x)) \end{aligned}$$

We can also compute

$$\begin{aligned}
\int \varphi(M(t, x)) dx &= \int \varphi(\tilde{M}(\theta(t), x)) dx \\
&= \int \varphi(x) \tilde{\sigma}(\theta, x) dx \\
&= \int \varphi(x) dx - \theta(t) \int \varphi(x) (\nabla \cdot w(x)) dx \\
&= \int \varphi(x) dx + \theta(t) \int \nabla \varphi(x) \cdot w(x) dx,
\end{aligned}$$

as wanted.  $\square$

Now we can rewrite the pressure terms in Lemma 4.2.3 as

$$\int [p^\varepsilon(t, M(t, x)) - p(t, x)] dx = \theta(t) \int \nabla p^\varepsilon(t, x) \cdot w(x) dx.$$

Thus, we deduce from Lemma 4.2.3:

**Lemma 4.2.7.**  $\nabla p^\varepsilon$ , viewed as a distribution on the interior of  $[0, 1] \times D$ , satisfies

$$\langle \nabla p^\varepsilon, \theta \otimes \omega \rangle = \int_{t,x} \nabla p^\varepsilon(t, x) \theta(t) \cdot w(x) \leq \varepsilon^2 + \frac{1}{2} \int_{t,x,a} (|\partial_t M + v \cdot \nabla M|^2 - |v|^2) c.$$

So, we see that, as a distribution,  $\nabla p^\varepsilon$  is bounded in the interior of  $[0, 1] \times D$  uniformly in  $\varepsilon$ . Up to a subsequence we then have  $\nabla p^\varepsilon \rightharpoonup \nabla p$  in the sense of distributions, combining Banach-Steinhaus and Banach-Alaoglu theorems.

### Uniqueness of the limit $\nabla p$

Let us use again the inequality in Lemma 4.2.7, but we now take a limit in the time-dependent test function  $\theta(t)$  more carefully:

$$\theta(t) = \delta \zeta(t) \text{ for } \zeta \in C_c^\infty(]0, T[), \text{ and for } |\delta| \text{ small}$$

therefore  $M(t, x) = \delta \zeta(t) w(x)$ . We now want to take the limit as  $\delta \rightarrow 0$ . Therefore we start by computing:

$$\begin{aligned}
M(t, x) - x &= O(\delta) \\
\partial_t M(t, x) &= \delta \zeta'(t) w(x) + O(\delta^2) \\
M(t, x) &= x + \delta \zeta(t) w(x) + O(\delta^2) \\
\frac{\partial}{\partial x_j} M(t, x) &= \delta_{ij} + \delta \zeta(t) \frac{\partial w}{\partial x_j}(x) + O(\delta^2),
\end{aligned}$$

and inserting this in the integrand in the right hand side of the inequality of Lemma 4.2.7, we obtain

$$\begin{aligned}
&|\partial_t M + v \cdot \nabla M|^2 - |v|^2 = \\
&\frac{1}{2} \left( \left| \delta \zeta'(t) w_j(x) + v_i + \sum_j v_j \delta \zeta(t) \partial_j w_i + O(\delta^2) \right|^2 - |v|^2 \right)
\end{aligned}$$

$$= \sum_i \delta \left[ \zeta'(t)w_i(x) + \sum_j v_j \zeta \partial_j w_i \right] v_i + O(\delta^2),$$

and since the inequality should hold along the subsequence  $\varepsilon_n \rightarrow 0$  such that  $\nabla p^{\varepsilon_n} \rightharpoonup \nabla p$  found in the previous section and for all  $\delta$  small enough, we obtain (first passing  $n \rightarrow \infty$  then  $\delta \rightarrow 0$ )

$$\begin{aligned} \langle \nabla p, \theta \otimes w \rangle &= \sum_i \int_{t,x,a} \left[ \zeta' w_i + \sum_j v_j \partial_j w_i \zeta \right] c v_i \\ &= - \sum_i \langle \partial_t \int_a c v_i + \sum_j \partial_j \int_a c v_i v_j, \zeta \otimes w_i \rangle, \end{aligned}$$

which means that in the sense of distributions,

$$\nabla p = -\partial_t \int_a c v - \nabla \cdot \int_a c v \otimes v.$$

Since this is true for *every* optimal solution  $(c, cv)$ ,  $\nabla p$  is uniquely defined. This means that the limit  $\nabla p$  is unique as a distribution, and in particular it does not depend on the sequence  $\nabla p^{\varepsilon_n}$  which we choose. Therefore  $\nabla p^\varepsilon \rightarrow \nabla p$ .

**Remark 4.2.8** (regularity of the pressure field). *From the above discussion we obtain that  $\nabla p$  is the derivative of a measure. By working substantially harder, in [91],  $\nabla p(t, x)$  was shown to be itself a locally bounded measure in the interior of  $[0, 1] \times D$ , and an improvement on the time integrability was achieved in [11, 10], where  $\nabla p(t, x)$  is an  $L^2_{loc}$  function of  $t$  valued in the set of bounded measures in  $x \in D$ .  $\nabla p \in L^2([0, T[, C^0(D; \mathbb{R}^d)')$  was shown.*

### 4.3 Convergence of approximate solutions

**Definition 4.3.1.** *We say that a couple  $(c^\varepsilon, q^\varepsilon) \in E'$  (we recall that  $E = C^0([0, 1] \times D^2; \mathbb{R} \times \mathbb{R}^d)$ ), is an approximate solution if:*

*i)  $c^\varepsilon \geq 0$ ,  $q^\varepsilon \ll c^\varepsilon$ ,  $q^\varepsilon = c^\varepsilon v^\varepsilon$  and*

$$K(c^\varepsilon, q^\varepsilon) = \frac{1}{2} \int_{t,a,x} |v^\varepsilon(t, x, a)|^2 c^\varepsilon(t, x, a) < +\infty$$

*ii) the continuity equation and the incompressibility constraint -we denote them respectively by (ACE) and (AIC)- hold in the limit  $\varepsilon \rightarrow 0$  (in the sense of distributions);*

*ii)  $K(c^\varepsilon, q^\varepsilon) \rightarrow K_{opt}(c_0, c_1)$  as  $\varepsilon \rightarrow 0$ .*

**Theorem 4.3.2.** *There is a unique pressure gradient  $\nabla p$  which depends only on the data  $(c_0, c_1)$ , such that, for all approximate solutions  $(c^\varepsilon, q^\varepsilon = c^\varepsilon v^\varepsilon)$ , we have in the sense of definition (4.3.1),*

$$\partial_t \int_a c^\varepsilon v^\varepsilon + \nabla \cdot \int_a c^\varepsilon v^\varepsilon \otimes v^\varepsilon \rightarrow -\nabla p,$$

*as  $\varepsilon \rightarrow 0$ , in the sense of distributions. This pressure gradient is precisely the one just found in the study of the OIT problem.*

*Proof.* We first observe that, from the assumption, the positive measures  $c^\varepsilon$  form a precompact set since (because of condition (ACI))

$$\int_{t,x,a} c^\varepsilon(t, x, a) \rightarrow \int_{[0,T] \times D} dxdt = 1.$$

For the measures  $|q^\varepsilon|$  we get

$$\int |q^\varepsilon| \leq \sqrt{\int \frac{|q^\varepsilon|^2}{c^\varepsilon}} \sqrt{\int c^\varepsilon} = \sqrt{2K(c^\varepsilon, q^\varepsilon)} \sqrt{\int c^\varepsilon} \rightarrow \sqrt{2K_{opt}(c_1, c_0)}.$$

From the above two boundedness results it follows that up to extracting a subsequence we may assume that  $(c^\varepsilon, q^\varepsilon)$  converge to a measure  $(c, q)$  weakly. Passing to the limit in the equations (ACE) and (AIC) we obtain (CE), (IC), which makes  $(c, q)$  an admissible solution for the OIT problem. Next, by lower semicontinuity (looking at  $K$  in its dual formulation), we obtain

$$K(c, q) \leq \liminf K(c^\varepsilon, q^\varepsilon) = K_{opt}(c_1, c_0),$$

which is the optimal value of the OIT problem. Since  $(c, q)$  is an admissible solution, we obtain that the equality should hold and, therefore,  $(c, q)$  is an optimal solution of the OIT problem.

Now, let us show the convergence of  $\int_a c^\varepsilon v^\varepsilon \otimes v^\varepsilon$  to  $\int_a cv \otimes v$  in the sense of distributions. To do this we first observe that by compactness, there exist a symmetric-matrix valued measure  $\nu(t, x, a)$  and a subsequence  $\varepsilon_n \rightarrow 0$  such that

$$c^{\varepsilon_n} v^{\varepsilon_n} \otimes v^{\varepsilon_n} \rightarrow \nu \text{ weakly.}$$

Then by lower semicontinuity we have  $cv \otimes v \leq \nu$  in the sense of symmetric-matrix valued measures. But since we already know that

$$\int_{t,x} \text{tr}(\nu) = \lim \int_{t,a,x} c^{\varepsilon_n} |v^{\varepsilon_n}|^2 = 2K(c, q) = \int_a cv \otimes v,$$

we get  $\nu = \int_a cv \otimes v$ . So

$$\nabla \cdot \int_a c^\varepsilon v^\varepsilon \otimes v^\varepsilon \rightarrow \nabla \cdot \int_a cv \otimes v.$$

Since we have  $c^\varepsilon v^\varepsilon = q^\varepsilon \rightarrow q = cv$ , we deduce

$$\partial_t \int_a c^\varepsilon v^\varepsilon + \nabla \cdot \int_a c^\varepsilon v^\varepsilon \otimes v^\varepsilon \rightarrow \partial_t \int_a cv + \nabla \cdot \int_a cv \otimes v.$$

But, as we have seen,  $(c, q = cv)$  is optimal and therefore satisfies

$$\partial_t \int_a cv + \nabla \cdot \int_a cv \otimes v = \nabla p,$$

where  $\nabla p$  is unique pressure gradient of the OIT problem. This completes the proof.  $\square$

## 4.4 Shnirelman's density theorem

In this section, we want to show how the convex OIT problem is a good way to treat the minimizing geodesic problem leading to the Euler equation according to Arnold [22]. We consider two maps  $X_0$  and  $X_1$  in  $VPM(D)$ , the semi-group of volume preserving maps of  $D$ , and associate the corresponding doubly stochastic measures  $c_0$  and  $c_1$  defined by

$$c_0(x, a) = \delta(x - X_0(a)), \quad c_1(x, a) = \delta(x - X_1(a)).$$

For simplicity we assume  $X_0(a) = a$  and simply denote  $X_1$  by  $X$ . This is not a restriction from the geometric viewpoint. Indeed, in that case, we restrict ourself to two maps  $X_0, X_1$  in the group  $SDiff(D)$ , and see that the minimizing geodesic problem from  $X_0$  to  $X_1$  is strictly equivalent to the one from  $I_d$  to  $X_1 \circ X_0^{-1}$ .

Let us now quote a crucial result due to Shnirelman [430] (or, more precisely, the version used in [11])

**Theorem 4.4.1** (Shnirelman's approximation theorem). *Assume  $d \geq 2$ .*

*Let  $(c, q) \in E'$  be an admissible solution to the OIT problem with data  $c_0, c_1$*

$$c_0(x, a) = \delta(x - a), \quad c_1(x, a) = \delta(x - X(a)), \quad X \in VPM(D),$$

*i.e. satisfying (IC) and (CE) conditions with  $K(c, q) < +\infty$ . Then, we can find, for every small  $\varepsilon > 0$ , a smooth divergence-free vector field  $v^\varepsilon(t, x)$ , compactly supported in the interior of  $[0, 1] \times D$ , with associated volume-preserving flow  $g_t^\varepsilon(x)$ , defined by*

$$\frac{d}{dt}g_t^\varepsilon(x) = v^\varepsilon(t, g_t^\varepsilon(x)), \quad g_0^\varepsilon(x) = x,$$

*such that*

$$\begin{cases} \int_D |X(a) - g_1^\varepsilon(a)|^2 da \leq \varepsilon^2, \\ \frac{1}{2} \int_0^1 \int_D |v(t, x)|^2 dx dt \leq K(c, q) + \varepsilon^2. \end{cases}$$

From this result, we immediately obtain approximate solutions as in Definition 4.3.1, by setting:

$$\begin{cases} c^\varepsilon(t, x, a) = \delta(x - g_t^\varepsilon(a)) \\ q^\varepsilon(t, x, a) = \partial_t g_t^\varepsilon(a) c^\varepsilon(t, x, a) = v^\varepsilon(t, g_t^\varepsilon(a)) c^\varepsilon(t, x, a) = v^\varepsilon(t, x) c^\varepsilon(t, x, a) \end{cases}$$

We easily verify (ACE):

$$\begin{aligned} \int_{t,x,a} [\partial_t f + v^\varepsilon \cdot \nabla f] c^\varepsilon &= \int_{t,a} [\partial_t f(t, g_t^\varepsilon(a), a) + \partial_t g_t^\varepsilon(a) \cdot (\nabla f)(t, g_t^\varepsilon(a), a)] \\ &= \int_a [f(1, g_1^\varepsilon(a), a) - f(0, g_0^\varepsilon(a), a)] \\ &= \int_a [f(1, g_1^\varepsilon(a), a) - f(0, a, a)] \\ &\rightarrow \int_a f(1, X(a), a) - \int_a f(0, a, a) = \langle c_1, f(1, \cdot, \cdot) \rangle - \langle c_0, f(0, \cdot, \cdot) \rangle, \end{aligned}$$

as wanted. As for the verification of (AIC), we have:

$$\begin{aligned} \int_{t,x,a} f(t,x)c^\varepsilon(t,x,a) &= \int_{t,a} f(t,g_t^\varepsilon(a)) \\ &= \int_{t,x} f(t,x), \end{aligned}$$

since  $g_t^\varepsilon$  is volume preserving. Finally, we verify the convergence of the energy:

$$\begin{aligned} K(c^\varepsilon, q^\varepsilon) &= \inf_{A+\frac{1}{2}|B|^2 \leq 0} \int_{t,x,a} [Ac^\varepsilon + B \cdot q^\varepsilon] \\ &= \inf_{A+\frac{1}{2}|B|^2 \leq 0} \int_{t,x} [A(t, g_t^\varepsilon(a), a) + \partial_t g_t^\varepsilon(a) \cdot B(t, g_t^\varepsilon(a), a)] \\ &= \frac{1}{2} \int_{t,a} |\partial_t g_t^\varepsilon(a)|^2 \\ &= \frac{1}{2} \int_{t,a} |v^\varepsilon(t, g_t^\varepsilon(a))|^2 = \frac{1}{2} \int_{t,x} |v^\varepsilon(t, x)|^2 \\ &\rightarrow K_{opt}(c_1, c_0). \end{aligned}$$

From the existence of such “Shnirelman” approximate solutions, combined with the convergence theorem 4.3.2, we conclude that the OIT problem provides the correct "relaxation" of the minimizing geodesic problem. (Here, we use the word "relaxation" in the sense that we have substituted, for a given optimization problem, a suitable extended problem set up in a larger framework where solutions can be more easily obtained and shown to be the correct limits of all approximate solutions of the original problem. Let us mention, just as an example, the theory of "optimal design" where such techniques have been used [5, 311].)

For the sake of completeness we provide in the next section a rather explicit ersatz of Shnirelman’s theorem, for admissible solutions  $(c, m)$  to the OIT problem on  $D = \mathbb{T}^3$  such that  $m \cdot e = 0$  where  $e$  is the vertical direction,  $e = (0, 0, 1)$ , of the unit torus. Let us call them “flat” admissible solutions. (Actually they can be identified to the admissible solutions of the OIT problem in one less space dimension, i.e. on  $\mathbb{T}^2$ .) This flatness property allows us to play with the vertical coordinate to construct, rather explicitly, a smooth time-dependent vector field  $u$  on  $D$  which, in general, needs a tiny but non-trivial component  $e \cdot v$  to do the approximation correctly. As a matter of fact, the flatness condition is sufficient to cover all data  $X$  that are trivial in the third coordinate  $e$ , namely:  $e \cdot (X(a) - a) = 0$ . This is precisely for this kind of data that Shnirelman was able in 1985 to prove the non-existence of classical solutions to the minimizing geodesic problem [429]. Therefore, the flatness condition is perfectly meaningful with respect to this fundamental negative result of Shnirelman. In addition, from the physical point of view, the flatness condition is directly related to the popular “hydrostatic approximation” of the Euler equations used in geo-sciences to describe fluid motions in thin domains, such as lakes, oceans or the atmosphere [187, 394], as will be discussed subsequently.



## 4.5 Approximation of a generalized flow by introduction of an extra dimension

This section is devoted to the proof of a variant of Shnirelman's density theorem 4.4.1, using the introduction of an additional space dimension. More precisely, we consider here an optimal solution of the OIT (or generalized geodesic) problem,  $(c, m)(t, x, a)$ , where  $t$  is valued in  $[0, 1]$  and the space variable  $x$  belongs to  $D = \mathbb{T}^d$ , with typically  $d = 2$ . So far, the space of labels  $a$  has always been considered to be  $D$  itself. However, since in the OIT theory, there is never any differential calculus performed in the  $a$  variable, but only integrations, we may use any abstract space of labels  $\mathcal{A}$  instead of  $D$ . It turns out to be very convenient to take  $\mathcal{A} = \mathbb{T}$ , the one dimensional torus  $\mathbb{T}$ , instead of  $D$ . This will allow us to substitute for  $a$  an extra space variable  $z \in \mathbb{T}$  and, through a rather explicit construction, to approximate  $(c, m)$  by a classical flow of volume preserving diffeomorphisms living no longer on the former spatial domain  $D = \mathbb{T}^d$  but rather on the new domain  $D \times \mathbb{T}$  with an extra dimension. From a physical viewpoint, this approach is quite natural, in particular in the geophysical context of fluid motions on very thin domains (typically the atmosphere and the oceans) where "reduced" models are frequently used, involving only two space variables [163, 187, 394], as will be discussed in section 4.6.

### Step 1: mollification

We first prove the following approximation result:

**Proposition 4.5.1.** *Let  $Q = [0, 1] \times D \times \mathcal{A}$  where  $D = \mathbb{T}^d$  and the label space is  $\mathcal{A} = \mathbb{T}$ . Let  $(c, m = cv)$  be a given pair in the dual Banach space  $E'$ , where  $E = C^0(Q; \mathbb{R} \times \mathbb{R}^d)$ , such that*

$$c \geq 0, \quad \int_a c = 1, \quad \partial_t c + \nabla \cdot (cv) = 0, \quad K(c, m) = \frac{1}{2} \int |v|^2 c < +\infty.$$

*Then, we can find a sequence  $(c_n, m_n = c_n v_n)$ , made of smooth functions on  $Q$ , valued in  $\mathbb{R} \times \mathbb{R}^d$ , such that the following hold:*

- $(c_n, m_n) \rightharpoonup (c, m)$ , for the the weak-\* convergence of measures;
- $c_n \geq \frac{1}{n}$  and  $\int_{\mathcal{A}} c_n(t, x, a) da = 1$ ;
- $\partial_t c_n + \nabla \cdot m_n = 0$ ,
- $K(c_n, m_n) \leq K(c, m) + o(1)$  as  $n \rightarrow +\infty$ .

*Proof.* The proof will consist in first extending the time variable  $t$  to  $\mathbb{R}$ , while shrinking the temporal interval  $t \in [0, 1]$  to  $t \in [\varepsilon, 1 - \varepsilon]$ , where  $\varepsilon = 1/n$ ,  $n \geq 2$ , and finally performing a suitable mollification by convolution in all variables  $(t, x, a)$ . Every step will keep the action arbitrarily close to  $K(c, m)$  while both the continuity equation and the incompressibility condition will be preserved.

*Extension and retraction* We first extend and retract  $(c, m)$  to  $\mathbb{R} \times D \times \mathbb{T}$ , i.e. to all  $t \in \mathbb{R}$ , by setting for all  $(x, a) \in D \times \mathbb{T}$ ,

$$c_\varepsilon(t, x, a) = c\left(\frac{t - \varepsilon}{1 - \varepsilon}, x, a\right) \quad \forall t \in [\varepsilon, 1 - \varepsilon],$$

$$\begin{aligned}
c_\varepsilon(t, x, a) &= c_0(x, a), \quad \forall t < \varepsilon, \quad c(t, x, a) = c_1(x, a), \quad \forall t > 1 - \varepsilon, \\
m_\varepsilon(t, x, a) &= 0, \quad \forall t \in \mathbb{R} \setminus [\varepsilon, 1 - \varepsilon], \\
m_\varepsilon(t, x, a) &= \frac{1}{1 - \varepsilon} m\left(\frac{t - \varepsilon}{1 - \varepsilon}, x, a\right) \quad \forall t \in [\varepsilon, 1 - \varepsilon].
\end{aligned}$$

By doing so, we keep for  $(c_\varepsilon, m_\varepsilon)$  the main properties of  $(c, m)$  namely the nonnegativity of  $c$ , the continuity equation (extended to  $\mathbb{R} \times D \times \mathbb{T}$ ), the incompressibility condition and the time boundary conditions. In addition,  $K(c_\varepsilon, m_\varepsilon)$  differs from  $K(c, m)$  only by  $O(\varepsilon)$ .

*Positivity of  $c_\varepsilon$  and convolution.* We first perform a convex interpolation by substituting for  $(c_\varepsilon, m_\varepsilon)$  the new pair  $(\varepsilon + (1 - \varepsilon)c_\varepsilon, (1 - \varepsilon)m_\varepsilon)$ , which maxes  $c_\varepsilon \geq \varepsilon > 0$ , without affecting the continuity equation and the incompressibility condition, while  $K(c_\varepsilon, m_\varepsilon)$  is reduced since  $K$  is convex and  $K(1, 0) = 0$ . To keep notations simple, we still denote by  $(c_\varepsilon, m_\varepsilon)$  the result of this second step. Finally we perform the convolution  $(c_\varepsilon, m_\varepsilon)(t, x, a)$  in *all* variables  $(t, x, a)$  by a mollifier  $\zeta_\varepsilon(t)\gamma_\varepsilon(x, a)$  where  $\gamma_\varepsilon$  is a periodic positive mollifier on  $\mathbb{T}^d \times \mathbb{T} = \mathbb{T}^{d+1}$  and  $\zeta_\varepsilon$  is a compactly supported nonnegative mollifier with support in  $[-\varepsilon, \varepsilon]$ . Again, this convex operation affects neither the continuity equation nor the incompressibility condition and diminishes  $K(c_\varepsilon, m_\varepsilon)$  (by convexity of  $K$ ).

Let us emphasize that, at each step, we have only performed small, controllable, modifications of  $(c, m)$  in the weak-\* sense of measures, which completes the Proof.  $\square$

## Step 2: Construction of a classical incompressible flow with one more space dimension

Now we take  $(c_n, m_n = c_n v_n)$ , for some fixed  $n$  big enough, as in the previous section, and we temporarily denote it by  $(c, m = cv)$  to make notations lighter. We now consider the new spatial domain  $D \times \mathbb{T}$  where  $D = \mathbb{T}^d$ , whose variable will be denoted by  $(x, z) \in D \times \mathbb{T} = \mathbb{T}^{d+1}$ . The new vertical coordinate  $z \in \mathbb{T}$  is going to substitute, in a non-trivial way, for the label variable  $a \in \mathbb{T}$ .

To pass from the label  $a \in \mathbb{T}$  to the vertical variable  $z \in \mathbb{T}$  representing the “extra dimension”, we consider the monotone rearrangement map  $R(t, x, \cdot) : \mathbb{T} \rightarrow \mathbb{T}$  sending  $c(t, x, a)da$  to the 1D Lebesgue measure on  $\mathbb{T}$ . More precisely, we implicitly define the unique smooth function  $z \in \mathbb{R} \mapsto R(t, x, z) \in \mathbb{R}$ , such that  $\partial_z R > 0$ ,  $R(t, x, z) - z$  is  $\mathbb{T}$ -periodic in  $z$  with zero mean and,

$$\int_{\mathbb{T}} f(R(t, x, z))dz = \int_{\mathbb{T}} f(a)c(t, x, a)da,$$

for all bounded Borel  $\mathbb{T}$ -periodic function  $f$  and for all  $(t, x) \in [0, 1] \times D$ . We then define a smooth time-dependent divergence-free vector field

$$(t, x, z) \in [0, 1] \times D \times \mathbb{T} \rightarrow (u(t, x, z), w(t, x, z)) \in \mathbb{R}^d \times \mathbb{R}$$

by setting first

$$u(t, x, z) = v(t, x, R(t, x, z)), \quad v = \frac{m}{c},$$

and then defining  $w$  to be, for each fixed  $(t, x)$  the unique  $\mathbb{T}$ -periodic function  $z \in \mathbb{T} \rightarrow w(t, x, z)$ , with zero mean, such that

$$\partial_z w(t, x, z) = -\nabla_x \cdot u(t, x, z).$$

which exactly means that  $(u, w)$  is divergence-free on  $D \times \mathbb{T}$ . Next, we introduce the volume-preserving flow  $(\xi_t, \eta_t)$  generated on  $D \times \mathbb{T}$  by  $(u, w)$  through:

$$\partial_t \xi = u(t, \xi, \eta) \quad \partial_t \eta = w(t, \xi, \eta).$$

By construction of  $R$ ,

$$\int_{t,x,a} f(t, x, a) c(t, x, a) = \int_{t,x,z} f(t, x, R(t, x, z)), \quad \forall f \in C^0(Q).$$

Since  $(\xi_t, \eta_t)$  is a volume-preserving diffeomorphism, this can be also written

$$\int_{t,x,a} f(t, x, a) c(t, x, a) = \int_{t,x,z} f(t, \xi_t(x, z), \tilde{R}(t, x, z))$$

where

$$\tilde{R}(t, x, z) = R(t, \xi_t(x, z), \eta_t(x, z)).$$

Similarly, by definition of  $R$  and  $u$ ,

$$\begin{aligned} \int_{t,x,a} f(t, x, a) m(t, x, a) &= \int_{t,x,z} f(t, x, R(t, x, z)) v(t, x, R(t, x, z)) \\ &= \int_{t,x,z} f(t, x, R(t, x, z)) u(t, x, z) = \int_{t,x,z} f(t, \xi_t(x, z), \tilde{R}(t, x, z)) u(t, \xi_t(x, z), \eta_t(x, z)) \\ &= \int_{t,x,z} f(t, \xi_t(x, z), \tilde{R}(t, x, z)) \frac{d}{dt} \xi_t(x, z). \end{aligned}$$

Now, let us use that  $(c, m) = (c_n, m_n)$  satisfies the continuity equation so that, for all sufficiently smooth function  $f(t, x, a)$ ,

$$\int_{t,x,a} \partial_t f c + \nabla_x f \cdot m = BT_n(f) \sim BT(f), \quad n \rightarrow \infty,$$

where

$$BT_n(f) = \int_{x,a} f(T, x, a) c_n(1, x, a) - f(0, x, a) c_n(0, x, a),$$

$$BT(f) = \int_a f(1, x, a) c_1(x, a) - f(0, x, a) c_0(x, a).$$

Using the new expression of  $c$  in terms of  $\xi$  and  $R$ , we get

$$\begin{aligned} BT_n(f) &= \int_{t,x,z} (\partial_t f)(t, \xi_t(x, z), \tilde{R}(t, x, z)) + (\nabla_x f)(t, \xi_t(x, z), \tilde{R}(t, x, z)) \frac{d}{dt} \xi_t(x, z) \\ &= \int_{t,x,z} \frac{d}{dt} [f(t, \xi_t(x, z), \tilde{R}(t, x, z))] - (\partial_a f)(t, \xi_t(x, z), \tilde{R}(t, x, z)) \partial_t \tilde{R}(t, x, z) \\ &= \int_{x,z} [f(T, \xi_T(x, z), \tilde{R}(T, x, z)) - f(0, x, \tilde{R}(0, x, z))] \\ &\quad - \int_{t,x,z} (\partial_a f)(t, \xi_t(x, z), \tilde{R}(t, x, z)) \partial_t \tilde{R}(t, x, z). \end{aligned}$$

In particular, whenever  $f$  vanishes at  $t = 0$  and  $t = T$ , we get

$$0 = \int_{t,x,z} (\partial_a f)(t, \xi_t(x, z), \tilde{R}(t, x, z)) \partial_t \tilde{R}(t, x, z),$$

The right-hand side can also be written, using the definition of  $\tilde{R}$

$$\int_{t,x,z} (\partial_a f)(t, \xi_t(x, z), R(t, \xi_t(x, z), \eta_t(x, z))) (D_t R)(t, \xi_t(x, z), \eta_t(x, z)),$$

(where  $D_t R$  is a short notation for  $(\partial_t + u \cdot \nabla_x + w \partial_z)R$ ) which is nothing but

$$\int_{t,x,z} (\partial_a f)(t, x, R(t, x, z)) D_t R(t, x, z)$$

(since  $(\xi_t, \eta_t)$  is a volume-preserving diffeomorphism).

Introducing  $g(t, x, z) = f(t, x, R(t, x, z))$ , so that

$$\partial_z g(t, x, z) = (\partial_a f)(t, x, R(t, x, z)) \partial_z R(t, x, z),$$

we deduce

$$\int_{t,x,z} \partial_z g(t, x, z) \frac{D_t R(t, x, z)}{\partial_z R(t, x, z)} = 0,$$

which is possible only if  $D_t R(t, x, z) = \partial_z R(t, x, z) \beta(t, x)$  for some function  $\beta(t, x)$ . In other words

$$(\partial_t + u \cdot \nabla_x + (w - \beta) \partial_z) R = 0.$$

Since  $w(t, x, z)$  is  $\mathbb{T}$ -periodic in  $z$  with zero mean we deduce that  $\beta(t, x) = 0$  and get:

$$(\partial_t + u \cdot \nabla_x + w \partial_z) R = 0.$$

This means that  $R(t, \xi_t(x, z), \eta_t(x, z)) = R(0, x, z)$  and widely simplifies the formulae we have obtained for  $(c, m)$ . Indeed, we may now write

$$\int_{t,x,a} f(t, x, a) c(t, x, a) = \int_{t,x,z} f(t, \xi_t(x, z), R(0, x, z))$$

$$\int_{t,x,a} f(t, x, a) m(t, x, a) = \int_{t,x,z} f(t, \xi_t(x, z), R(0, x, z)) \frac{d}{dt} \xi_t(x, z),$$

Finally, denoting  $(R, \xi)$  by  $(R_n, \xi^n)$ , in order to remind their dependence on  $n$ , we have obtained the following behavior for the time-boundary term

$$BT_n(f) = \int_{x,z} f(1, \xi_1^n(x, z), R_n(0, x, z)) - f(0, x, R_n(0, x, z))$$

$$\sim BT(f) = \int_a f(1, x, a) c(1, x, a) - f(0, x, a) c_0(x, a),$$

for all  $f$ , as  $n \rightarrow \infty$ .

### Step 3: matching of the time-boundary data

At this stage, we limit ourself to the case when the time-boundary data  $(c_0, c_1)$  are of special form

$$c_1(1, x, a) = \delta(x - X_1(a)), \quad c_0(x, a) = \delta(x - X_0(a))$$

where  $a \in \mathbb{T} \rightarrow X_0(a) \in D$  and  $a \in \mathbb{T} \rightarrow X_1(a) \in D$  are two given Lebesgue-measure preserving maps such that, for each  $\varepsilon$ , there is a smooth map  $h_\varepsilon : D \rightarrow D$  with

$$\int_{\mathbb{T}} |X_1(a) - h_\varepsilon(X_0(a))|^2 da \leq \varepsilon^2.$$

(Notice that the domain of definition  $\mathbb{T}$  and the range  $D = \mathbb{T}^d$  of these maps may be of different dimension, so that  $X_0$  and  $X_1$  cannot be expected to be smooth.) Let us now introduce smooth approximation for  $X_0$  and  $X_1$ , respectively denoted by  $X_0^\varepsilon$  and  $X_1^\varepsilon$ , so that

$$\int_{\mathbb{T}} |X_0^\varepsilon(a) - X_0(a)|^2 da \leq \varepsilon^2, \quad \int_{\mathbb{T}} |X_1^\varepsilon(a) - X_1(a)|^2 da \leq \varepsilon^2.$$

By choosing successively  $f(t, x, a) = (1-t)|x - X_0^\varepsilon(a)|^2$  and  $f(t, x, a) = t|x - X_1^\varepsilon(a)|^2$  in the asymptotic formula we have just obtained, namely

$$\begin{aligned} & \lim_n \int_{x,z} f(1, \xi_1^n(x, z), R_n(0, x, z)) - f(0, x, R_n(0, x, z)) \\ &= \int_{x,a} f(1, x, a)c(1, x, a) - f(0, x, a)c_0(x, a), \end{aligned}$$

we get

$$\begin{aligned} \lim_n \int_{x,z} |\xi_1^n(x, z) - X_1^\varepsilon(R_n(0, x, z))|^2 &= \int_{[0,1]} |X_1^\varepsilon(a) - X_1(a)|^2 da \leq \varepsilon^2, \\ \lim_n \int_{x,z} |x - X_0^\varepsilon(R_n(0, x, z))|^2 &= \int_{[0,1]} |X_0^\varepsilon(a) - X_0(a)|^2 da \leq \varepsilon^2. \end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} & \sqrt{\int_{x,z} |x - X_0^\varepsilon(R_n(0, x, z))|^2} - \sqrt{\int_{x,z} |x - X_0(R_n(0, x, z))|^2} \\ & \leq \sqrt{\int_{x,z} |X_0^\varepsilon(R_n(0, x, z)) - X_0(R_n(0, x, z))|^2} \\ & = \sqrt{\int_{\mathbb{T}} |X_0^\varepsilon(a) - X_0(a)|^2 da} \leq \varepsilon \end{aligned}$$

(by construction of  $R_n$ ). Similarly, we get

$$\sqrt{\int_{x,z} |\xi_1^n(x, z) - X_1^\varepsilon(R_n(0, x, z))|^2} \leq \sqrt{\int_{x,z} |\xi_1^n(x, z) - X_1(R_n(0, x, z))|^2} + \varepsilon.$$

So, we can pass to the limit in  $\varepsilon$  and get

$$\int_{x,z} |\xi_1^n(x, z) - X_1(R_n(0, x, z))|^2 \rightarrow 0,$$

$$\int_{x,z} |x - X_0(R_n(0, x, z))|^2 \rightarrow 0.$$

At this stage, we limit ourself to the case when  $X_0$  is one-to-one (this looks strange since  $X_0$  maps  $\mathbb{T}$  to  $D = \mathbb{T}^d$ , but is perfectly plausible: this just means that  $X_0$  is a measure preserving Borel isomorphism between  $\mathbb{T}$  equipped with the 1D Lebesgue measure and  $D = \mathbb{T}^d$  equipped with the  $d$ -dimensional Lebesgue measure (cf. [411]). Thus we may consider  $h = X_1 \circ X_0^{-1}$  as a volume preserving map of  $D = \mathbb{T}^d$ , which, for every  $\varepsilon > 0$  admits some approximation by a smooth map  $h_\varepsilon : D \rightarrow D$  with respect to the  $L^2(D; \mathbb{R}^d)$  norm:

$$\int_D |X_1 \circ X_0^{-1}(x) - h_\varepsilon(x)|^2 dx \leq \varepsilon^2.$$

which also means

$$\int_{\mathbb{T}} |X_1(a) - h_\varepsilon(X_0(a))|^2 da \leq \varepsilon^2.$$

Thus,

$$\sqrt{\int_{x,z} |\xi_1^n(x, z) - X_1(R_n(0, x, z))|^2} - \sqrt{\int_{x,z} |\xi_1^n(x, z) - h_\varepsilon(X_0(R_n(0, x, z)))|^2}$$

$$\leq \sqrt{\int_{x,z} |X_1(R_n(0, x, z)) - h_\varepsilon(X_0(R_n(0, x, z)))|^2} = \sqrt{\int_a |X_1(a) - h_\varepsilon(X_0(a))|^2} \leq \varepsilon.$$

Using that

$$\int_{x,z} |h_\varepsilon(x) - h_\varepsilon(X_0(R_n(0, x, z)))|^2 \leq \text{Lip}(h_\varepsilon)^2 \int_{x,z} |x - X_0(R_n(0, x, z))|^2 \rightarrow 0,$$

we have obtained

$$\limsup_n \int_{x,z} |\xi_1^n(x, z) - h_\varepsilon(x)|^2 \leq \varepsilon^2,$$

which can also be written

$$\limsup_n \int_{a,z} |\xi_1^n(X_0(a), z) - h_\varepsilon(X_0(a))|^2 \leq \varepsilon^2,$$

By passing to the limit in  $\varepsilon$ , we have finally obtained:

**Proposition 4.5.2.**

$$\lim \int_{a,z} |\xi_1^n(X_0(a), z) - X_1(a)|^2 = 0. \quad (4.5.1)$$

### Step 5: rescaling the vertical direction

In this last and very simple step, we just rescale the vertical variable by substituting  $\mathbb{R}/\varepsilon\mathbb{Z}$  for  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . Accordingly, we define

$$\tilde{u}(t, x, z) = u(t, x, z/\varepsilon), \quad \tilde{w}(t, x, z) = \varepsilon w(t, x, z/\varepsilon),$$

where,  $\tilde{u}(t, x, z)$  and  $\tilde{w}(t, x, z)$  are now  $\varepsilon\mathbb{T}$ -periodic in  $z$ . and we introduce the corresponding flow  $\tilde{\xi}, \tilde{\eta}$  as above. The action of this classical volume-preserving flow can be easily estimated as follows:

$$\begin{aligned} \frac{1}{2} \int |\partial_t \xi|^2 + \int |\partial_t \eta|^2 &= \frac{1}{2} \int |\tilde{u}|^2 + |\tilde{w}|^2 \\ &\sim \frac{1}{2} \int |u|^2 + \varepsilon^2 \int |w|^2 \\ &\leq K(c, m) + o(1), \end{aligned}$$

while the previous estimates on the time-boundary conditions, as well as the continuity and incompressibility equations, continue to hold by straightforward computations, which completes the proof of our variant of Theorem 4.4.1 using one extra space dimension.

## 4.6 Hydrostatic solutions to the Euler equations

In this section, we want to relate, following [100], the concept of *generalized* solution to the Euler equations on a two dimensional domain  $D$  to the concept of *classical* solution to the so-called "hydrostatic approximation", somewhat in the same spirit as in the previous section.

More precisely, let us consider a "classical" solution  $(v(t, x), p(t, x))$  of the Euler equations, in a very thin three-dimensional domain such as  $D_\varepsilon = D \times \mathbb{T}_\varepsilon$ , where  $D$ , for simplicity is just  $D = \mathbb{T}^2$  and  $\mathbb{T}_\varepsilon$  is just the 1D torus with period  $\varepsilon$ :  $\mathbb{T}_\varepsilon = \mathbb{R}/\varepsilon\mathbb{Z}$ . Let us rescale the vertical coordinate  $x^3$  and the third component  $v^3$  of the velocity field:  $(x^3, v^3) \rightarrow (\varepsilon x^3, \varepsilon v^3)$ . After this rescaling we get, on the rescaled 3D domain  $D \times [0, 1]$ , no longer the Euler equations but a rescaled version of them, namely

$$I_\varepsilon D_t v + \nabla p = 0, \quad D_t = \partial_t + v \cdot \nabla, \quad \nabla \cdot v = 0,$$

where  $I_\varepsilon$  denotes the diagonal matrix  $I_\varepsilon = \text{diag}(1, 1, \varepsilon^2)$ . Notice that the operators  $D_t$  and  $\nabla \cdot$  are unchanged and  $\varepsilon$  only features in  $I_\varepsilon$ . It is very customary in geosciences to neglect  $\varepsilon$  by substituting  $I_0 = \text{diag}(1, 1, 0)$  for  $I_\varepsilon$ . This is the so-called hydrostatic approximation, for which the pressure does not depend on the vertical coordinate  $x^3$ :

$$I_0 D_t v + \nabla p = 0, \quad D_t = \partial_t + v \cdot \nabla, \quad \nabla \cdot v = 0.$$

This approximation of the 3D Euler equations in a thin domain is very commonly used in ocean-atmosphere computational models [163, 187, 394]. As an evolution equation, the hydrostatic limit of the Euler equations is much more singular than the original Euler equations: it is ill-posed, in some sense, on any linear Sobolev space, but well-posed on some adequate functional convex cone [92, 95, 282, 355]. (See

also [127, 254, 287, 330, 146] for closely related problems.) Of course, all smooth solutions of the 2D Euler equations on the 2D domain  $D$  are particular solutions of this hydrostatic limit, but there are many other solutions that are genuinely three dimensional.

Let us consider a smooth solution  $(v(t, x), p(t, x))$  of this hydrostatic limit of the Euler equations on the 3D domain  $D \times \mathbb{T}$ . We denote by  $g_t(x)$  the volume-preserving flow in  $D \times \mathbb{T}$  generated by

$$\frac{d}{dt}g_t(x) = v(t, g_t(x)), \quad g_0(x) = x, \quad x \in D \times \mathbb{T}.$$

Let us now consider an arbitrarily chosen one-to-one Borel map  $\bar{X}_0 : D \rightarrow D \times \mathbb{T}$  that transports the 2D Lebesgue measure on  $D$  to the 3D Lebesgue measure on  $D \times \mathbb{T}$ , i.e.

$$\int_D f(\bar{X}_0(a))da = \int_{D \times \mathbb{T}} f(x)dx, \quad \forall f \in C^0(D \times \mathbb{T}).$$

(Such maps do exist but cannot be smooth. See [411] for more details. Many examples can be easily obtained just by using binary notations and  $\{0, 1\}^{\mathbb{N}}$  as an intermediate space between  $D$  and  $D \times \mathbb{T}$ .)

Next, we define  $\bar{X}_t(a) = g_t(\bar{X}_0(a)) \in D \times \mathbb{T}$ , for all  $a \in D$ . Let us now denote  $X_t(a) = (\bar{X}_t^1(a), \bar{X}_t^2(a)) \in D$  the two first components of  $\bar{X}_t(a)$ . This defines a time-dependent family of maps  $D \rightarrow D$  that preserves the 2D Lebesgue measure on  $D$ . Indeed, if we consider a continuous function  $f$  on  $\mathbb{T}^2$ , we can trivially lift it as a continuous function  $F$  on  $\mathbb{T}^3$  by setting  $F(x^1, x^2, x^3) = f(x^1, x^2)$  and we get

$$\begin{aligned} \int_D f(X_t(a))da &= \int_D F(\bar{X}_t(a))da = \int_{D \times \mathbb{T}} F(x^1, x^2, x^3)dx^1 dx^2 dx^3 \\ &= \int_{D \times \mathbb{T}} f(x^1, x^2)dx^1 dx^2 dx^3 = \int_D f(x^1, x^2)dx^1 dx^2, \end{aligned}$$

which is enough to show that  $X_t$  preserves the 2D Lebesgue measure on  $D$ . Meanwhile, since  $(v, p)$  is solution of the hydrostatic limit of the Euler equations, we get for  $X$ :

$$\frac{d^2}{dt^2}X_t(a) + (\nabla p)(t, X_t(a)) = 0$$

where  $\nabla$  denotes the two-dimensional gradient on the two-dimensional domain  $D$ . (We again have used that  $p(t, x)$  does not depend on  $x^3$  and, therefore, can be seen as a time-dependent function on the two-dimensional domain  $D$ .) So, we have obtained that  $(X_t(a), p(t, x))$  is a solution of the Euler equations on the 2D domain  $D$ , in a generalized sense (already discussed in section 2.4), although they are not solutions of the 2D Euler equations in the classical sense.

Even more provocative is the perspective of 1D solutions to the Euler equations. Indeed, in the classical setting, there are only trivial solutions of the Euler equations, because of the divergence-free condition. Indeed, on the 1D torus  $\mathbb{T}$ , the only possible solutions are constant velocity fields  $v$ . However, there are many non-trivial 1D solutions to the Euler equations with the generalized definition we have just used. Once again, such solutions can be obtained by rescaling a thin 2D domain and by passing to the hydrostatic limit in the 2D Euler equations, by dimension reduction, exactly as we did from three to two dimensions.



## 4.7 Explicit solutions to the OIT problem

Let us finish this chapter devoted to the OIT problem by providing very few examples of explicit solutions. So far, we have systematically made the assumption  $D = \mathbb{T}^d$  and we limited ourself to the time normalized time interval  $[0, 1]$  for simplicity. However, it is easier to provide explicit examples on domains with boundary such as the unit cube or the unit disk and on more general time intervals  $[0, T]$ . The simplest non trivial explicit 1D generalized solution to the Euler equations, in the sense of the OIT, known to us, can be written as follows. We take  $D = [-1, 1]$  equipped with the normalized 2D Lebesgue measure  $dx$ . We set  $T = \pi$  and define, for  $\tilde{a} = (a, \omega) \in D \times [0, 1]$  and  $(t, x) \in [0, T] \times D$ ,

$$X_t(\tilde{a}) = X_t(a, \omega) = a \cos t + \sqrt{1 - a^2} \sin t \cos(2\pi\omega), \quad p(t, x) = p(x) = x^2/2.$$

One can check (easily) that

$$\frac{d^2}{dt^2} X_t(\tilde{a}) = -X_t(\tilde{a}) = -p'(X_t(\tilde{a}))$$

and (not so easily but crucially) that  $X_t$  transports the Lebesgue measure on  $D \times [0, 1]$  to the Lebesgue measure on  $D$ . At  $T = \pi$ , we have  $X_0(\tilde{a}) = X_0(a, \omega) = a$  and  $X_T(\tilde{a}) = X_T(a, \omega) = -a$ , while  $p''(x) = 1$ . Then, the corresponding measures  $(c, q)$  defined by

$$c(t, x, \tilde{a}) = \delta(x - X_t(a, \omega)), \quad q(t, x, \tilde{a}) = \partial_t X_t(a, \omega) \delta(x - X_t(a, \omega)), \quad \tilde{a} = (a, \omega),$$

can be shown, thanks to the 1D Poincaré inequality, to be an optimal solution for the IOT problem set on  $[0, T] \times D$  with boundary data

$$c_0(x, \tilde{a}) = \delta(x - a), \quad c_T(x, \tilde{a}) = \delta(x + a), \quad \tilde{a} = (a, \omega).$$

A closely related generalized solution can be defined in 2D on the unit disk  $D$  (with normalized Lebesgue measure). The formulae are very similar. (Actually the previous 1D solution can be interpreted just as the projection from the unit disk to  $[-1, 1]$  of this one.) We define

$$\tilde{a} = (a, \omega) = (a^1, a^2, \omega) \in D \times [0, 1]$$

$$X_t(\tilde{a}) = X_t(a, \omega) = a \cos t + \sqrt{1 - |a|^2} \sin t \exp(2\pi i \omega), \quad p(t, x) = |x|^2/2.$$

with an abusive complex notation and, again, set

$$c(t, x, \tilde{a}) = \delta(x - X_t(a, \omega)), \quad q(t, x, \tilde{a}) = \partial_t X_t(a, \omega) \delta(x - X_t(a, \omega)), \quad \tilde{a} = (a, \omega).$$

Observe that we have  $D_x^2 p(t, x) = Id$ ,  $X_0(a, \omega) = a$ ,  $X_T(a, \omega) = -a$ , if we choose  $T = \pi$ . Once again, this provides a generalized solution to the Euler equations and  $(c, q)$  can be shown to be optimal for the OIT on  $[0, T] \times D$  with data

$$c_0(x, \tilde{a}) = \delta(x - a), \quad c_T(x, \tilde{a}) = \delta(x + a), \quad \tilde{a} = (a, \omega) \in D \times [0, 1].$$

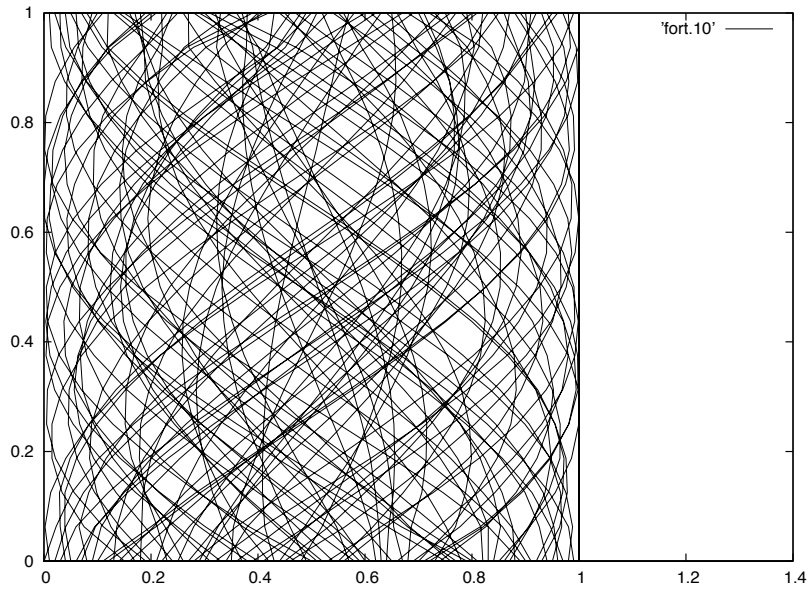
This OIT amounts to transferring all particles from their initial position to the opposite one on the unit disk  $D$ , during the time interval  $[0, \pi]$ , in an incompressible fashion inside  $D$ . Of course the obtained motion is not at all conventional: every

“particle” issued from  $x$  in the unit disk get split according to the “microscopical” (or “hidden”) variable  $\omega$  and follow a continuum of different trajectories parameterized by  $\omega \in [0, 1]$ , with equal probability, and eventually reaches its destination  $-x$  at time  $T = \pi$ . This strange motion looks much more conventional, once lifted as a 3D incompressible motion by adding a vertical coordinate  $x^3$  along a small interval of length  $\varepsilon$ , and projecting back to the 2D basis. This is just another example of hydrostatic limit of the 3D Euler equation. The multiplicity of trajectories observed on the 2D domain  $D$  just correspond to the projection of three dimensional trajectories in  $D \times [0, \varepsilon]$ . Accordingly, the “hidden” variable  $\omega$  is just keeping record (in a non-trivial way) of the missing vertical coordinate  $x^3$ .

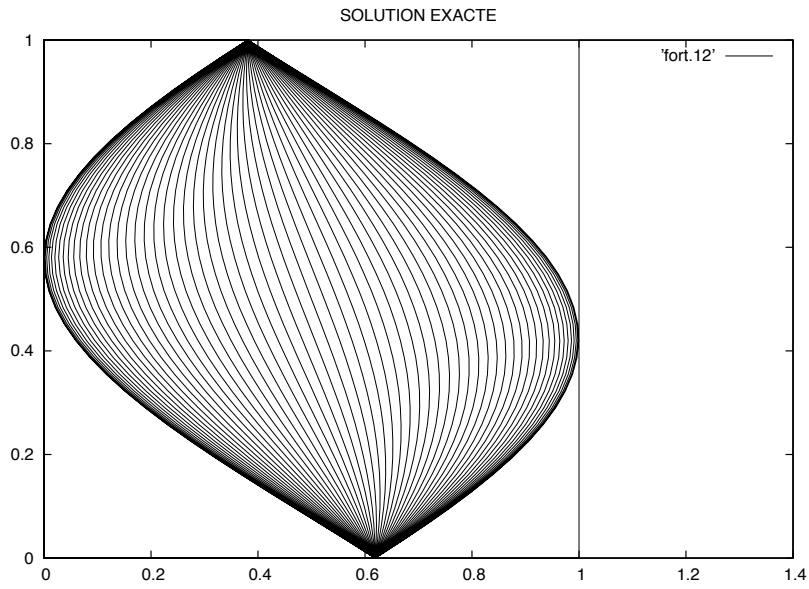
It is interesting to notice, that in the 2D case, there are two other solutions  $X^+$  and  $X^-$  to the very same OIT problem, namely

$$X_t^+(a, \omega) = a \exp(it), \quad X_t^-(a, \omega) = a \exp(-it), \quad p(t, x) = |x|^2/2,$$

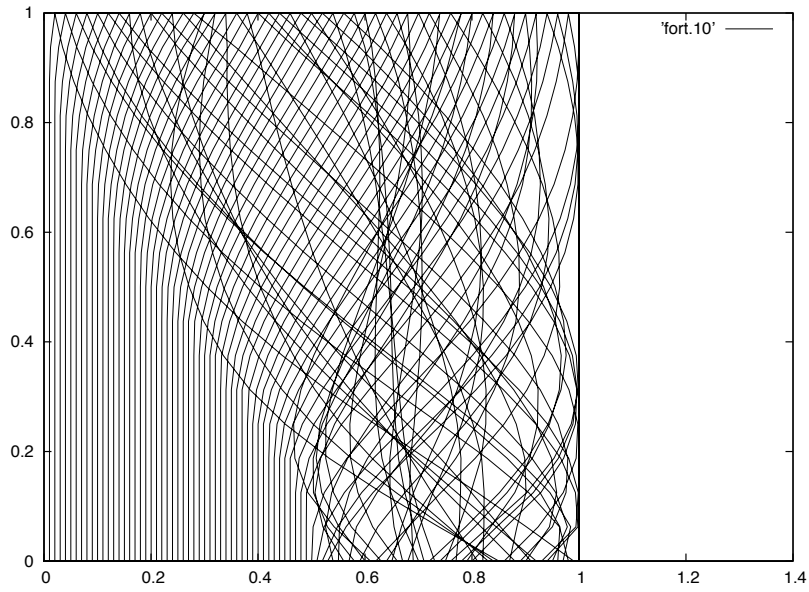
with an obvious complex notation. They actually do not depend on the “micro” variable  $\omega$  and correspond to two classical solutions of the 2D Euler equations with (stationary) velocity fields  $v^+(x) = (-x^2, x^1)$ ,  $v^-(x) = (x^2, -x^1)$ . Geometrically, they correspond to simple rigid rotations of the unique disk. We further point out that these three different solutions to the same IOT problem share the same pressure field, which is fully consistent with Theorem 4.2.1. Surprisingly enough, there is a very rich family of other solutions to the same OIT problem, obtained by M. Bernot, A. Figalli and F. Santambrogio [56]. In particular, our generalized solution can be “decomposed” as the average of two more “fundamental” generalized solutions of the Euler equations (which was very surprizing to us).



Exact 1D generalized solution to the Euler equations.  
(Horizontal axis:  $x \in [-1, 1]$ , vertical axis:  $t \in [0, \pi]$ .)



Exact 1D generalized solution to the Euler equations.  
Only a few selected trajectories are drawn  
(Horizontal axis:  $x \in [-1, 1]$ , vertical axis:  $t \in [0, \pi]$ .)



Another 1D generalized solution to the Euler equations.  
(Horizontal axis:  $x \in [0, 1]$ , vertical axis:  $t \in [0, 1]$ .)

# Chapter 5

## Solutions of various initial value problems by convex minimization

Least square methods are quite common in the important field of data assimilation (which is of key importance for weather prediction, cf., among many others, [25, 68, 170, 351]...). Solving initial value problems by convex minimization is an old idea going back to the least square method for linear equations. For nonlinear systems of PDEs, in particular for parabolic equations and various gradient flows, there has been many contributions, including Brezis-Ekeland, Ghoussoub, Mielke-Stefanelli, Visintin [134, 258, 369, 453] etc... In a recent work [110], we have introduced a different approach, essentially based on the concept of weak, distributional solutions, that works for systems of hyperbolic conservation laws with a convex entropy, including the Euler equations of fluid mechanics, and the simple Burgers equation without viscosity. This has been further extended by Vorotnikov [458] to a large class of Fluid Mechanics models.

More recently, we figured out how the method also applies to some parabolic problems, one of them being the quadratic porous medium equations. This case is so simple and the analysis is so straightforward that we have decided to describe it as our first example, although the strategy was first defined for the Euler equations of incompressible fluids.

In addition, let us mention that the convex optimization problems obtained by this method can be seen as some generalized variational mean-field games à la Lasry-Lions [315] (see also [1, 147]), with the peculiarity that they usually involve matrix-valued rather than scalar density fields, which is, to the best of our knowledge, still unusual in the theory of MFGs.

### 5.1 The porous medium equation with quadratic non linearity

The porous media equations with quadratic non linearity (QPME, in brief), set on the periodic cube  $\mathbb{T}^d$  (for simplicity), reads

$$\partial_t u = \Delta u^2/2, \quad u = u(t, x) \in \mathbb{R}, \quad t \geq 0, \quad x \in \mathbb{T}^d,$$

where  $u$  is, a priori, a nonnegative function that can be interpreted as a "density" function for some fluid moving in a porous medium.

N.B. From a statistical mechanics viewpoint, this equation, set on the entire euclidean space  $\mathbb{R}^d$ , can be obtained, as, more or less, in [333], as the macroscopic limit of the properly rescaled very simple (deterministic) system of  $N$  interacting particles:

$$\frac{dX_k}{dt} = \epsilon^{-1} \sum_{j=1, N} (X_k - X_j) \exp\left(-\frac{|X_k - X_j|^2}{\epsilon}\right),$$

$$u(t, x) \sim \frac{1}{N} \sum_{j=1, N} \delta(x - X_j(t)), \quad 1/N \ll \epsilon^d \ll 1.$$

This equation admits a Ljapunov (or "entropy") functional, namely

$$\int_{\mathbb{T}^d} u^2(t, x) dx,$$

for which we get, at least formally

$$\frac{d}{dt} \int_{\mathbb{T}^d} u^2(t, x) dx = - \int_{\mathbb{T}^d} u(t, x) |\nabla u|^2(t, x) dx,$$

We start with the rather absurd problem of minimizing, on a given finite time interval  $[0, T]$ , the time integral of the "entropy"

$$\int_Q u^2(t, x) dx dt, \quad Q = [0, T] \times \mathbb{T}^d,$$

among all weak (i.e. distributional) solutions in  $L^2([0, T] \times \mathbb{T}^d)$  of the QPME

$$\partial_t u = \Delta u^2/2, \quad u = u(t, x) \geq 0, \quad t \geq 0, \quad x \in \mathbb{T}^d,$$

with a prescribed initial condition  $u_0 \geq 0$ , given, for simplicity, in  $L^\infty(\mathbb{T}^d)$ . A priori this problem is absurd since it is well known since the 80s that the Cauchy problem is uniquely solvable, for nonnegative distributional solutions, in  $L^1(\mathbb{R}^d)$  [133], and that all  $L^p$  spaces (in particular  $L^2$ ) are preserved by the corresponding semi-group of (nonnegative) solutions. Therefore, once  $u_0$  is prescribed, there is a unique nonnegative admissible solution and the minimization problem looks trivial. However, we do not require that the weak solutions are nonnegative, which makes the problem more uncertain.

Anyway, this strange minimization problem admits a saddle point formulation which reads

$$I(u_0) = \inf_u \sup_\phi \int_Q (u^2 - 2\partial_t \phi u - \Delta \phi u^2 + 2u_0 \partial_t \phi),$$

where the only constraints are:

- i) for test function  $\phi$  to be smooth and vanish at  $t = T$ ;
- ii) for function  $u$  to be square integrable on  $Q$ . By reversing the inf and the sup, we get a (non trivial!) relaxed problem

$$J(u_0) = \sup_\phi \inf_u \int_Q (u^2 - 2\partial_t \phi u - \Delta \phi u^2 + 2u_0 \partial_t \phi).$$

At this level, we may just claim that  $I(u_0) \geq J(u_0)$  and there may be a "duality gap" since the problem we started from is not formulated as a convex problem. The relaxed problem is very simple. Indeed, it is enough to perform the minimization in  $u$  pointwise in  $(t, x)$ , since there is no more constraint on  $u$ :

$$J(u_0) = \sup_{\phi} \inf_u \int_Q (u^2 - 2\partial_t \phi u - \Delta \phi u^2 + 2u_0 \partial_t \phi) =$$

$$\sup_{\phi} \int_Q \left( -\frac{(\partial_t \phi)^2}{1 - \Delta \phi} + 2u_0 \partial_t \phi \right), \quad \Delta \phi \leq 1, \quad \phi(T, \cdot) = 0.$$

Notice that the optimal value of  $u$ , for a given point  $(t, x)$ , is given by

$$u = \frac{\partial_t \phi(t, x)}{1 - \Delta \phi(t, x)},$$

under the condition that  $\Delta \phi(t, x) < 1$  (otherwise the infimum in  $u$  is  $-\infty$ , unless both  $\Delta \phi(t, x) = 1$  and  $\partial_t \phi(t, x) = 0$  hold true simultaneously.).

Setting  $q = \partial_t \phi$ ,  $\sigma = 1 - \Delta \phi$ , we get an alternative formulation:

$$J(u_0) = \sup_{\sigma, q} \int_Q \left( -\frac{q^2}{\sigma} + 2u_0 q \right), \quad \partial_t \sigma + \Delta q = 0, \quad \sigma \geq 0, \quad \sigma(T, \cdot) = 1.$$

*Remark.* This optimization problem is strongly reminiscent of the optimal transport problem (with quadratic cost), in its temporal (also known as Benamou-Brenier) formulation. Furthermore, in the 1D case, it is identical (up to the time-boundary conditions) to the optimization problem introduced by Huesmann and Trevisan in [295]. In their paper, the authors obtain a "Benamou-Brenier" formulation of the so-called martingale optimal transport problem (a very popular subject in the last years, initially motivated by financial mathematics, that will not be covered in this book [44, 43, 259]) and they already point out a connection with the 1D porous medium equation.

### Analysis of the relaxed concave optimization problem

Let us now perform a rough analysis of our relaxed concave optimization problem, using what is already known about the QPME. To make our reasoning easier, we limit ourself to the easy case when  $u_0$  is smooth and positive on  $\mathbb{T}^d$ . We want to prove

**Theorem 5.1.1.** *Any smooth positive solution  $(t, x) \in Q = [0, T] \times \mathbb{T}^d \rightarrow u(t, x)$  of the quadratic porous medium equation QPME*

$$\partial_t u = \Delta u^2 / 2$$

can be recovered as

$$u = \frac{\partial_t \phi}{1 - \Delta \phi},$$

where  $\phi$  solves the concave optimization problem

$$J(u_0) = \sup_{\phi} \int_Q \left( -\frac{(\partial_t \phi)^2}{1 - \Delta \phi} + 2u_0 \partial_t \phi \right), \quad \Delta \phi \leq 1, \quad \phi(T, \cdot) = 0,$$

and satisfies

$$1 - \Delta \phi \geq (t/T)^{d/(d+2)}.$$



*Proof.* By standard parabolic regularity theory, the unique nonnegative weak solution  $u(t, x)$  with smooth positive initial condition  $u_0$  is a smooth and positive function of  $(t, x) \in Q = [0, T] \times \mathbb{T}^d$ . It is known [449] that all (nonnegative) solutions  $u = u(t, x)$  of the QPME satisfy the Aronson-Bénilan estimate

$$\Delta u \geq -\kappa/t,$$

where  $\kappa = d/(d+2)$  just depends on  $d$ . Let us try to find a solution  $\phi$  to the concave optimization problem just by solving the final value problem

$$\partial_t \phi = (1 - \Delta \phi)u, \quad \phi(T, \cdot) = 0,$$

i.e., in terms of  $\alpha = 1 - \Delta \phi$ ,

$$\partial_t \alpha + \Delta(\alpha u) = 0, \quad \alpha(T, \cdot) = 1.$$

We claim that  $\alpha(t, x) \geq (t/T)^\kappa$  follows from the Aronson-Bénilan estimate. Indeed, since  $u$  is smooth, we can write

$$\partial_t \alpha + \Delta(\alpha u) = \partial_t \alpha + u \Delta \alpha + 2 \nabla \alpha \cdot \nabla u + \alpha \Delta u = 0$$

and, using both the maximum principle and the Aronson-Bénilan estimate, we get for  $A(t) = \inf_{x \in \mathbb{T}^d} \alpha(t, x)$  the differential inequality

$$A'(t) \leq \kappa A(t)/t.$$

So,  $\log A(T) - \log A(t) \leq \kappa(\log T - \log t)$ , and therefore  $A(t) \geq (t/T)^\kappa$  (since  $A(T) = 1$ ). This estimate shows that the function  $\alpha = 1 - \Delta \phi$  stays positive on  $]0, T] \times \mathbb{T}^d$ . Let us now finally show that  $\phi$  is optimal for the concave maximization problem. For that purpose, let us just evaluate

$$j = \int_Q \left( -\frac{(\partial_t \phi)^2}{1 - \Delta \phi} + 2u_0 \partial_t \phi \right).$$

which, by definition of  $J(u_0)$ , is certainly bounded from above by  $J(u_0)$ . Since  $u$  solves the QPME with initial condition  $u_0$ , we have

$$\int_Q (2\partial_t \phi u + \Delta \phi u^2 - 2\partial_t \phi u_0) = 0.$$

Thus, since  $\phi$  solves  $\partial_t \phi = (1 - \Delta \phi)u$ ,

$$j = \int_Q \left( -\frac{(\partial_t \phi)^2}{1 - \Delta \phi} + 2u \partial_t \phi + \Delta \phi u^2 \right) = \int_Q u^2$$

which shows that  $\phi$  is optimal since, by construction,

$$J(u_0) \geq j = \int_Q u^2 \geq I(u_0) \geq J(u_0).$$

*End of Proof.*

## Various comments

1) Through additional technical work, this proof should extend to *all* initial conditions in  $L^2(\mathbb{R}^d)$ . The theory should also apply to the case of the entire euclidean space  $\mathbb{R}^d$  and to the famous "Barenblatt profiles", that have compact support and saturate the Aronson-Bénilan estimate [449].

2) Notice that, strictly speaking, we have not shown the uniqueness of a maximizer for the concave maximization problem.

3) Our formulation in terms of convex optimization might be a useful way of getting new regularity results for the QPME. This problem is of current interest since new regularity results have been obtained:

a) in [257] by Gess, Sauer and Tadmor, for the porous medium equation, through quite unusual methods in the elliptic setting such as "average lemmas" coming from kinetic theory [276];

b) in [274] by Goldman and Otto, for the quadratic optimal transport problem in its temporal "Benamou-Brenier" formulation, which looks very similar to the relaxed concave optimization problem we have just obtained for the QPME.

## 5.2 The viscous Hamilton-Jacobi equation and the Schrödinger problem

The analysis performed for the porous medium equation also applies to the viscous quadratic Hamilton-Jacobi equation

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = \frac{\epsilon}{2} \Delta \phi,$$

with initial condition  $\phi_0$  where  $\epsilon > 0$  is the viscosity coefficient. (Let us just mention the paramount importance of the vanishing viscosity limit of this equation and the related theory of "viscosity solutions" [184]. See Appendix 11.) We set  $Q = [0, T] \times D$ , with  $D = \mathbb{T}^d$  for simplicity, and assume the initial condition  $B_0$  to be the gradient of a periodic function  $\phi_0$  of zero mean on  $D$ . This scalar equation can be written in divergence form by introducing the vector field  $B = \nabla \phi$ , which leads to the IVP

$$\partial_t B + \nabla \left( \frac{|B|^2 - \epsilon \nabla \cdot B}{2} \right) = 0, \quad B(0, \cdot) = B_0 = \nabla \phi_0.$$

Then, we want to minimize  $\int_Q |B|^2$  among all *weak* solutions  $B$  of the IVP with initial condition  $B_0$ . Using Lagrange multipliers, we get the saddle-point problem

$$\inf_B \sup_A \int_Q \frac{|B|^2}{2} - \partial_t A \cdot (B - B_0) - \nabla \cdot A \frac{|B|^2}{2} - \frac{\epsilon}{2} \nabla(\nabla \cdot A) \cdot B$$

where the vector field  $A = A(t, x) \in \mathbb{R}^d$  is just subject to  $A(T, \cdot) = 0$ . (Notice that we do not have to enforce that  $B$  is a gradient, since it automatically follows from the weak formulation.) The dual problem is just obtained by exchanging the sup and the inf and can be very easily computed (since there is no constraint on  $B$ ). We get

$$\sup_A \int_Q - \frac{|\partial_t A + \epsilon \nabla(\nabla \cdot A)/2|^2}{2(1 - \nabla \cdot A)} + \partial_t A \cdot B_0,$$

where  $A$  is subject to  $A(T, \cdot) = 0$  and inequality  $\nabla \cdot A \leq 1$ . This dual problem can be nicely formulated in terms of

$$\rho(t, x) = 1 - \nabla \cdot A(t, x) \geq 0, \quad q(t, x) = \partial_t A(t, x) \in \mathbb{R}^d,$$

More precisely:

**Proposition 5.2.1.** *The dual problem generated by the viscous Hamilton-Jacobi equation reads*

$$\sup_{\rho, q} \int_Q -\frac{|q - \epsilon \nabla \rho / 2|^2}{2\rho} + q \cdot B_0,$$

where the fields  $\rho \geq 0$ ,  $q \in \mathbb{R}^d$  are constrained by

$$\partial_t \rho + \nabla \cdot q = 0, \quad \rho(T, \cdot) = 1.$$

In addition, there is no duality gap in the saddle-point formulation.

Before proving that there is no duality gap, let us make several observations.

### Connection with the Schrödinger problem

The optimization problem we have derived from the viscous Hamilton-Jacobi equation can be written in a slightly different way by noticing first that

$$\begin{aligned} & \int_Q -\frac{|q - \epsilon \nabla \rho / 2|^2}{2\rho} + \int_Q \frac{|q|^2 + |\epsilon \nabla \rho / 2|^2}{2\rho} = \int_Q \epsilon q \cdot \frac{\nabla \rho}{\rho} = \int_Q -\epsilon \log \rho \nabla \cdot q \\ & = \int_Q \epsilon \log \rho \partial_t \rho = \int_Q \epsilon \partial_t (\rho \log \rho - \rho) = \int_Q \epsilon \partial_t (\rho \log \rho) = - \int_D \epsilon (\rho \log \rho)(t = 0, \cdot) \end{aligned}$$

(using that  $\rho(T, \cdot) = 1$ ) and, next, that

$$\int_Q q \cdot B_0 = \int_Q -\nabla \cdot q \phi_0$$

(since  $B_0 = \nabla \phi_0$ )

$$= \int_Q \partial_t \rho \phi_0 = \int_D (1 - \rho(t = 0, \cdot)) \phi_0 = \int_D -\rho(t = 0, \cdot) \phi_0$$

(using that  $\rho(T, \cdot) = 1$  and that  $\phi_0$  has zero mean). So, the maximization problem now reads

$$\sup_{\rho, q} \int_Q -\frac{|q|^2 + |\epsilon \nabla \rho / 2|^2}{2\rho} + \int_D -\rho(t = 0, \cdot) \phi_0 - \epsilon (\rho \log \rho)(t = 0, \cdot),$$

where  $(\rho, q)$  are constrained by

$$\partial_t \rho + \nabla \cdot q = 0, \quad \rho(T, \cdot) = 1.$$

At this stage, we have obtained a variant (with a different time-boundary term) of the famous Schrödinger problem [418], intensively studied in the recent years, in particular after Ch. Léonard [320], as a natural "entropic regularization" of the optimal transport problem (with quadratic cost) [48, 49, 399], with a stochastic interpretation in terms of brownian clouds. In that framework, the regularization term is the well-known "Fisher information"

$$\rho \rightarrow \int \frac{|\nabla \rho|^2}{2\rho}$$

which plays an important role in various fields (information theory, statistics, functional analysis, quantum mechanics...).

## Connection with the Schrödinger equation

Not so surprisingly, the Schrödinger problem (1931) is closely related to the Schrödinger equation (1925). Indeed the solutions of the Schrödinger equation, written in the hydrodynamical formulation due to Madelung (1926) [349], exactly correspond to the critical points  $(\rho, q)$  of the following action -featuring a crucial change of sign-

$$\int \frac{|q(t, x)|^2 - |\nabla \rho(t, x)|^2}{2\rho(t, x)} dx dt$$

under space-time compactly supported perturbations and constraint

$$\partial_t \rho + \nabla \cdot q = 0,$$

one of the optimality equation being

$$q = \rho \nabla \theta,$$

for some scalar potential  $\theta = \theta(t, x) \in \mathbb{R}$ . (See [455] for more details.) Then, the wave function  $\psi = \psi(t, x)$  solution of the Schrödinger equation is simply recovered by polar factorization through the Madelung transform (1926) [349] as

$$\psi(t, x) = \sqrt{\rho(t, x)} e^{i\theta(t, x)} \in \mathbb{C}.$$

Notice that there is a degeneracy of this transform when the wave function vanishes, which makes the Madelung formulation of the Schrödinger equation not entirely satisfactory [126, 149].

## No duality gap in the saddle-point formulation

To conclude this section, let us check that there is no duality gap between the inf-sup and the sup-inf in the saddle-point formulation, namely let us prove that

$$\sup_A \inf_B = \inf_B \sup_A \inf_B \int_Q \frac{|B|^2}{2} - \partial_t A \cdot (B - B_0) - \nabla \cdot A \frac{|B|^2}{2} - \frac{\epsilon}{2} \nabla(\nabla \cdot A) \cdot B.$$

For simplicity, we assume the initial condition  $\phi_0$  to be smooth so that the viscous Hamilton-Jacobi equation admits a unique smooth solution that we denote  $\phi^s = \phi^s(t, x)$  on the compact set  $Q = [0, T] \times D$ , where  $D = \mathbb{T}^d$ , and we set  $B^s(t, x) = \nabla \phi^s(t, x)$  so that

$$B^s(0, x) = B_0(x) = \nabla \phi_0(x).$$

(The superscript  $s$  means "solution".) The proof is very elementary and even simpler than in the case of the porous medium equation discussed in the previous section. By definition, we first get

$$\frac{1}{2} \int_Q |\nabla \phi^s|^2 = \frac{1}{2} \int_Q |B^s|^2 \geq \inf \sup.$$

Next, we notice that a good guess for the optimal solution  $(\rho, q)$  of the dual problem is obtained by minimizing in  $B$  in the saddle-point problem. This leads to solving the backward linear PDE in  $A$ :

$$(1 - \nabla \cdot A) B^s = \partial_t A + \frac{\epsilon}{2} \nabla(\nabla \cdot A)$$

with final condition  $A(T, \cdot) = 0$ , where we have input  $B^s$  for  $B$ . We get, after taking the divergence of the equation, the backward transport-diffusion equation

$$\nabla \cdot (\rho B^s) = -\partial_t \rho - \frac{\epsilon}{2} \Delta \rho.$$

for  $\rho(t, x) = 1 - \nabla \cdot A(t, x)$ , with final condition  $\rho(T, x) = 1$ . This standard PDE admits a unique smooth positive solution  $\rho^s(t, x)$ , since the field  $B^s$  is smooth. The previous equation now reads:

$$\rho^s B^s = \partial_t A - \frac{\epsilon}{2} \nabla \rho^s$$

so that

$$A(t, x) = - \int_t^T (\rho^s B^s + \frac{\epsilon}{2} \nabla \rho^s)(\tau, x) d\tau$$

since  $A(T, \cdot) = 0$ . Next, we define

$$q^s(t, x) = \partial_t A(t, x) = \rho^s B^s(t, x) + \frac{\epsilon}{2} \nabla \rho^s(t, x)$$

We have

$$-\partial_t \rho^s = \nabla \cdot (\rho^s B^s) + \frac{\epsilon}{2} \Delta \rho^s = \nabla \cdot q^s,$$

so that the continuity equation is satisfied which makes  $(\rho^s, q^s)$  an admissible solution for the dual problem:

$$\sup \inf = \sup_{\rho, q} \int_Q -\frac{|q - \epsilon \nabla \rho / 2|^2}{2\rho} + q \cdot B_0.$$

Thus

$$\begin{aligned} \sup \inf &\geq \int_Q -\frac{|q^s - \epsilon \nabla \rho^s / 2|^2}{2\rho^s} + q^s \cdot B_0 \\ &= \int_Q -\frac{\rho^s |B^s|^2}{2} + q^s \cdot B_0 \end{aligned}$$

(using the definition of  $q^s$ )

$$= \int_Q -\frac{\rho^s |B^s|^2}{2} + \partial_t \rho^s \phi_0$$

(using the continuity equation and that  $B_0 = \nabla \phi_0$ )

$$= \int_Q -\frac{\rho^s |B^s|^2}{2} + \int_D (1 - \rho^s(0, \cdot)) \phi_0$$

(using that  $\rho^s(T, \cdot) = 1$  and that  $\phi_0$  does not depend on  $t$ ).

$$= \int_Q -\frac{\rho^s |\nabla \phi^s|^2}{2} + \int_D (1 - \rho^s(0, \cdot)) \phi_0.$$

Now, we use both the transport-diffusion equation for  $\rho^s$  and the viscous Hamilton-Jacobi equation for  $\phi^s$ , to get

$$\partial_t((1 - \rho^s)\phi^s) = (\nabla \cdot (\rho^s \nabla \phi^s) + \frac{\epsilon}{2} \Delta \rho^s)\phi^s - (1 - \rho^s) \left( \frac{|\nabla \phi^s|^2}{2} - \frac{\epsilon}{2} \Delta \phi^s \right)$$

and deduce (using integration by part)

$$\frac{d}{dt} \int_D (1 - \rho^s) \phi^s = \int_D (-\rho^s - 1) \frac{|\nabla \phi^s|^2}{2}.$$

So

$$\int_D (\rho^s(0, \cdot) - 1) \phi_0 = \int_Q (-\rho^s - 1) \frac{|\nabla \phi^s|^2}{2}$$

(by integration in  $t \in [0, T]$ , using that  $\rho^s(T, \cdot) = 1$ ). Since we had just obtained

$$\sup \inf \geq \int_Q -\frac{\rho^s |\nabla \phi^s|^2}{2} + \int_D (1 - \rho^s(0, \cdot)) \phi_0,$$

we finally get

$$\sup \inf \geq \int_Q \frac{|\nabla \phi^s|^2}{2}$$

and conclude that indeed there is no duality gap since we already know

$$\int_Q \frac{|\nabla \phi^s|^2}{2} \geq \inf \sup \geq \sup \inf.$$

### 5.3 The Navier-Stokes equations

Now, we want to minimize  $\int_Q |v|^2$  on the time-space domain  $Q = [0, T] \times D$ ,  $D = \mathbb{T}^d$ , among all *weak* solutions  $v = v(t, x) \in \mathbb{R}^d$ , of the Navier-Stokes equations of incompressible fluids with initial condition  $v_0$ :

$$\partial_t v + \nabla \cdot (v \otimes v) + \nabla p = \epsilon \Delta v, \quad \nabla \cdot v = 0,$$

(where, as usual,  $\nabla p$  can be eliminated thanks to the divergence-free condition  $\nabla \cdot v = 0$ ), which can be also written as

$$\partial_t v + \nabla \cdot (v \otimes v) + \nabla p = \epsilon \nabla \cdot \left( \frac{\nabla v + \nabla v^t}{2} \right), \quad \nabla \cdot v = 0.$$

This problem can be immediately written as a saddle-point problem:

$$\inf_v \sup_{A, h} \int_Q \frac{1}{2} (|v|^2 - (\nabla A + \nabla A^t) : v \otimes v) - \partial_t A \cdot (v - v_0) - \epsilon v \cdot (\nabla \cdot \left( \frac{\nabla A + \nabla A^t}{2} \right)) - v \cdot \nabla h,$$

where  $A = A(t, x) \in \mathbb{R}^d$  is a divergence-free vector field such that  $A(T, \cdot) = 0$  and  $h = h(t, x) \in \mathbb{R}$  is a Lagrange multiplier for the divergence-free condition on  $v$ . We get a dual problem by exchanging sup and inf.

**Proposition 5.3.1.** *The dual problem generated by the Navier-Stokes equations can be written as a kind of generalized Schrödinger problem:*

$$\sup_{M, q} \int_Q q \cdot v_0 - \frac{(q - \epsilon \nabla \cdot M) \cdot M^{-1} \cdot (q - \epsilon \nabla \cdot M)}{2}$$

where the symmetric matrix-valued field  $M = M(t, x) \geq 0$  and the vector field  $q = q(t, x) \in \mathbb{R}^d$  are subject to

$$\partial_t M + \mathcal{L}q = 0, \quad M(T, \cdot) = I_d,$$

where  $\mathcal{L}$  is the constant coefficient first-order pseudo-differential operator

$$\mathcal{L}q = \nabla q + \nabla q^T - 2D^2 \Delta^{-1} \nabla \cdot q.$$

*Proof.*

After exchanging the sup and the inf, the minimization in  $v$  is very easy and leads to

$$\inf_v \sup_{A,h} \int_Q \frac{1}{2} (I_d - \nabla A - \nabla A^t)^{-1} \cdot \left( \partial_t A + \nabla h + \epsilon \nabla \cdot \left( \frac{\nabla A + \nabla A^t}{2} \right) \right) + \partial_t A \cdot v_0,$$

where  $A$  is subject to  $I_d - \nabla A - \nabla A^t \geq 0$  in the sense of symmetric matrices. We now introduce

$$M = I_d - \nabla A - \nabla A^t, \quad q = \partial_t A + \nabla h.$$

Since  $A$  is divergence free, we have

$$\Delta h = \nabla \cdot q,$$

and therefore

$$\partial_t A = q - \nabla \Delta^{-1} \nabla \cdot q.$$

So, we get the compatibility condition between  $M$  and  $q$  that allows us to recover  $A$  and  $h$  from them:

$$\partial_t M + \nabla q + \nabla q^T - 2D^2 \Delta^{-1} \nabla \cdot q = 0, \quad M(T, \cdot) = I_d,$$

which completes the proof.

*Remarks.*

1) The generalized Schrödinger problem generated by the NS equations features a matrix-valued version of the Fisher information

$$(\nabla \cdot M) \cdot M^{-1} \cdot (\nabla \cdot M), \quad M = M^T \geq 0,$$

very roughly similar to the Einstein-Hilbert Lagrangian, which reads, in 4 space-time dimension, up to a null Lagrangian [211],

$$(\Gamma_{ij}^m g^{ij} \Gamma_{km}^k - \Gamma_{ik}^m g^{ij} \Gamma_{jm}^k) \sqrt{-\det g}$$

for which  $g$  is a Lorentzian metric and  $\Gamma$  is its Levi-Civita connection:

$$\Gamma_{jk}^i = g^{im} (g_{km,j} + g_{jm,k} - g_{kj,m}) / 2.$$

2) The generalized Schrödinger problem derived from the Navier-Stokes equations looks very similar to the "Brödinger problem" (or rather "Bredinger") introduced by Arnaudon, Cruzeiro, Léonard, Zambrini [20, 34], in particular in its recent interpretation by Baradat and Monsaingeon [35]. This problem can be seen as the "entropic regularization" of the incompressible optimal problem already extensively discussed in this book in connection with the Euler equations of incompressible fluids.

## 5.4 The quantum diffusion equation

Just to indicate, without any further analysis, a highly non trivial example of a parabolic system for which the initial value problem could be fruitfully addressed

in terms of convex optimization, let us mention the so-called quantum diffusion equation ([263, 299] written as a system in weak form according to [263] sect. 1.8):

$$\text{QDE : } \quad \partial_t u + \Delta^2 u - D^2 : \frac{g \otimes g}{u} = 0, \quad g = \nabla u,$$

where  $u : (t, x) \in Q = [0, T] \times \mathbb{T}^d \rightarrow u(t, x) \geq 0$ , for which

$$\int_{\mathbb{T}^d} \frac{|g(t, x)|^2}{2u(t, x)} dx$$

is a Ljapunov function, or an "entropy" in Otto's framework of gradient flows for transportation metrics [263].

We start by minimizing the time integral over  $[0, T]$  of the entropy among all weak solutions of QDE with given initial condition  $u_0$ , which leads to the saddle point problem:

$$\begin{aligned} I(u_0) = & \inf_{(u \geq 0, g)} \sup_{(\phi, P)} - \int_{\mathbb{T}^d} u_0(x) \phi(0, x) dx \\ & + \int_Q \left( \frac{|g|^2}{2u} - \partial_t \phi u + \Delta^2 \phi u - D^2 \phi : \frac{g \otimes g}{u} - P \cdot g - u \nabla \cdot P \right) (t, x) dx dt, \end{aligned}$$

(where  $P = P(t, x) \in \mathbb{R}^d$  is a Lagrange multiplier for constraint  $g = \nabla u$ ). Reversing the inf and the sup leads to the desired relaxed concave maximization problem. By minimizing in  $g$  (pointwise in  $(t, x)$  since there is no constraint on  $g$ ), we first get

$$\begin{aligned} J(u_0) = & \sup_{(\phi, P)} \inf_{u \geq 0} - \int_{\mathbb{T}^d} u_0(x) \phi(0, x) dx \\ & + \int_Q u(t, x) \left( -\frac{1}{2} (\mathbb{I}_d - 2D^2 \phi)^{-1} : P \otimes P - \partial_t \phi + \Delta^2 \phi - \nabla \cdot P \right) (t, x) dx dt, \end{aligned}$$

where  $\mathbb{I}_d$  is the identity matrix and  $\phi : (t, x) \in Q = [0, T] \times \mathbb{T}^d \rightarrow \phi(t, x) \in \mathbb{R}$  is subject to  $D^2 \phi \leq \mathbb{I}_d$  and  $\phi(T, \cdot) = 0$ . Then, after minimizing, again pointwise, in  $u \geq 0$ , we finally obtain:

$$J(u_0) = \sup_{(\phi, P)} - \int_{\mathbb{T}^d} u_0(x) \phi(0, x) dx,$$

where,  $\phi$  is subject, again, to  $D^2 \phi \leq \mathbb{I}_d$  and  $\phi(T, \cdot) = 0$  and also to the pointwise inequality:

$$\partial_t \phi - \Delta^2 \phi + 1/2 (I_d - 2D^2 \phi)^{-1} : (P \otimes P) + \nabla \cdot P \leq 0,$$

for some unknown vector field  $P : (t, x) \in [0, T] \times \mathbb{T}^d \rightarrow P(t, x) \in \mathbb{R}^d$ .

## 5.5 Entropic conservation laws

A system of first-order conservation laws read

$$\partial_t U + \nabla \cdot (F(U)) = 0, \quad U = U(t, x) \in \mathcal{W} \subset \mathbb{R}^m, \quad t \in \mathbb{R}, \quad x \in D,$$



where we assume  $D = \mathbb{T}^d$  for simplicity. Such a system is called entropic [193] if the given function  $F$  (usually called the "flux function") enjoys the symmetry property

$$\sum_{\beta=1}^m \partial_{\beta} \mathcal{E}(W) \partial_{\alpha} F^{i\beta}(W) = \partial_{\alpha} Q^i(W), \quad \forall W \in \mathcal{W},$$

for some pair of functions  $(\mathcal{E}, Q) : \mathcal{W} \rightarrow \mathbb{R}^{1+d}$ , where  $\mathcal{W}$  is an open convex subset of  $\mathbb{R}^m$  and  $\mathcal{E}$  (usually called "entropy") is strictly convex over  $\mathcal{W}$ . This structural condition implies that, whenever  $U = U(t, x)$  is a smooth solution of the system, we get the additional conservation law

$$\partial_t(\mathcal{E}(U)) + \nabla \cdot (Q(U)) = 0.$$

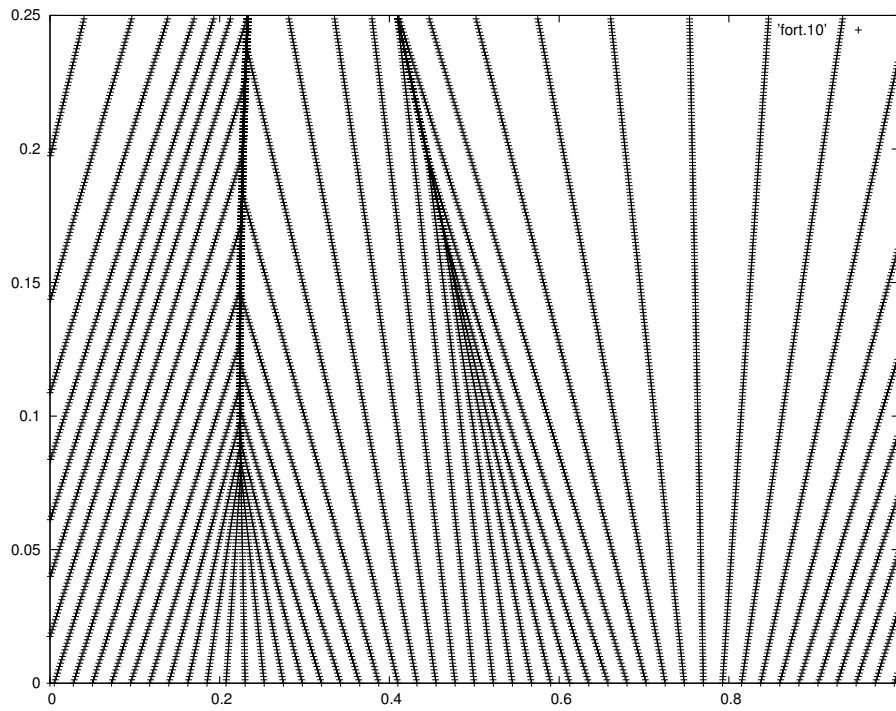
Indeed, in coordinates (with implicit summation on repeated indices),

$$\begin{aligned} -\partial_t(\mathcal{E}(U)) &= \partial_{\alpha} \mathcal{E}(U) \partial_i(\mathcal{F}^{i\alpha}(U)) = \partial_{\alpha} \mathcal{E}(U) \partial_{\beta} \mathcal{F}^{i\alpha}(U) \partial_i U^{\beta} \\ &= \partial_{\beta} \mathcal{Q}^i(U) \partial_i U^{\beta} = \partial_i(\mathcal{Q}^i(U)). \end{aligned}$$

Of course the simplest example is the so-called "inviscid Burgers" equation, where  $U = u(t, x)$  is a real-valued function of a single space variable  $x$  with the simplest nonlinear flux function  $F = u^2/2$ :

$$\partial_t u + \partial_x(u^2/2) = 0.$$

It is well established [193] that, in most situations, such systems admit smooth solutions that blow up (in Lipschitz norm) after a finite time, phenomenon known as "shock formation", by reference to compressible gas dynamics.



Inviscid Burgers equation :  $\partial_t u + \partial_x (u^2/2) = 0$ ,  $u = u(t, x)$ ,  $x \in \mathbb{R}/\mathbb{Z}$ ,  $t \geq 0$ .  
 Formation of two shock waves. (Vertical axis:  $t \in [0, 1/4]$ . horizontal axis:  $x \in \mathbb{T}$ .)

**A canonical example:  
the Euler equations of isothermal compressible fluids.**

They simply read

$$\partial_t \rho + \nabla \cdot q = 0, \quad \partial_t q + \nabla \cdot \left( \frac{q \otimes q}{\rho} \right) + \nabla \rho = 0,$$

and fit into the general framework just by defining

$$U = (\rho, q) \in \mathcal{W} = ]0, +\infty[ \times \mathbb{R}^3, \quad F = \left( q, \frac{q \otimes q}{\rho} + I_3 \rho \right), \quad \mathcal{E} = -\frac{|q|^2}{2\rho} - \rho \log \rho$$

**The least square approach?**

Given  $U_0$  on  $\mathbb{T}^d$  and  $T > 0$ , if  $F(U)$  is linear in  $U$ , the least square method can be used for the IVP and clearly leads to a (degenerate) convex problem

$$\inf_{U(t=0, \cdot) = U_0} \int_{[0, T] \times \mathbb{T}^d} |\partial_t U + \nabla \cdot (F(U))|^2$$

(see [61] in the scalar case with non constant coefficients) but this is no longer true for nonlinear systems.

Alternately, we are going to use the convex optimization method based on weak solutions, that we have already presented for several parabolic equations, for instance the quadratic porous medium equation.

**Minimization approach to the initial value problem**

Given  $U_0$  on  $D = \mathbb{T}^d$  and  $T > 0$ , we minimize the time integral over  $[0, T]$  of the entropy among all weak solutions  $U$  of the IVP:

$$I(U_0) = \inf_U \int_0^T \int_D \mathcal{E}(U), \quad U = U(t, x) \in \mathcal{W} \subset \mathbb{R}^m \text{ subject to}$$

$$\int_0^T \int_D \partial_t A \cdot U + \nabla A \cdot F(U) + \int_D A(0, \cdot) \cdot U_0 = 0$$

for all smooth  $A = A(t, x) \in \mathbb{R}^m$  with  $A(T, \cdot) = 0$ . The problem is not trivial since there may be many weak solutions starting from  $U_0$  which are not entropy-preserving (by "convex integration" à la De Lellis-Székelyhidi) [196, 197, 198]. We get the resulting saddle-point problem

$$\begin{aligned} \inf_U \sup_A \int_0^T \int_D \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U) \\ - \int_D A(0, \cdot) \cdot U_0 \end{aligned}$$

where  $A = A(t, x) \in \mathbb{R}^m$  is smooth with  $A(T, \cdot) = 0$ . Here  $U_0$  is the initial condition and  $T$  the final time.

## Reversing infimum and supremum

This leads to a *concave* maximization problem in  $A$ , namely

$$\begin{aligned} J(U_0) &= \sup_{A(T, \cdot)=0} \inf_U \int_0^T \int_D \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U) - \int_D A(0, \cdot) \cdot U_0 \\ &= \sup_{A(T, \cdot)=0} \int_0^T \int_D -G(\partial_t A, \nabla A) - \int_D A(0, \cdot) \cdot U_0 \end{aligned}$$

where  $G$  is defined by

$$G(E, B) = \sup_{V \in \mathcal{WC}\mathbb{R}^m} E \cdot V + B \cdot F(V) - \mathcal{E}(V), \quad (E, B) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}.$$

Notice that  $G$  is automatically convex (but presumably degenerate!). Thus we have obtained a (possibly degenerate) space-time elliptic system in  $A$ , which is reminiscent of those appearing in optimal transport theory (as will be discussed later on). Here is the paradox! How a convex optimization problem could be compatible with a well-posed evolution problem? For instance, if  $G$  were just a square, we would get

$$\sup_A \int_0^T \int_D -|\partial_t A|^2 - |\nabla A|^2 - \int_D A(0, \cdot) \cdot U_0$$

which would correspond to an ill-posed equation for  $A$ :

$$\partial_{tt}^2 A + \Delta A = 0.$$

The answer to the paradox is that, in our construction,  $G$  is very likely to be convex *degenerate* which is presumably still compatible with the solution of a well-posed initial value problem.

## Examples and interpretation in terms of matrix-valued variational mean-field games

Let us look more carefully at explicit examples of hyperbolic conservation laws, such as the Burgers equation (without viscosity) and the much more challenging Euler equations. In the elementary example of the Burgers equation, the maximization problem in  $A$  simply reads

$$\sup_A \int_{[0, T] \times \mathbb{T}} -\frac{(\partial_t A)^2}{2(1 - \partial_x A)} - \int_{\mathbb{T}} A(0, \cdot) u_0.$$

with  $A = A(t, x) \in \mathbb{R}$  subject to  $A(T, \cdot) = 0$ ,  $\partial_x A \leq 1$ . Introducing

$$\rho = 1 - \partial_x A \geq 0, \quad q = \partial_t A,$$

we get:

$$\sup_{(\rho, q)} \left\{ \int_{[0, T] \times \mathbb{T}} -\frac{q^2}{2\rho} - q u_0 \mid \partial_t \rho + \partial_x q = 0, \quad \rho(T, \cdot) = 1 \right\}.$$

This problem can be interpreted, in our opinion, as the "ballistic" version (à la Ghoussoub [42]) of the optimal transport problem with quadratic cost and, as well, as a rather trivial example of mean-field game (MFG) à la Lasry-Lions [315]. (See also

[1, 147, 446] (without noise nor interaction) of variational type. So we may expect more interesting connections with MFG, while addressing more complex equations than the inviscid Burgers equation.

Also notice that the resulting problem

$$\sup_{(\rho, q)} \left\{ \int_{[0, T] \times \mathbb{T}} -\frac{q^2}{2\rho} - qu_0 \mid \partial_t \rho + \partial_x q = 0, \rho(T, \cdot) = 1 \right\}$$

is so close to an optimal transport problem (in its so-called Benamou-Brenier formulation) that, at the computational level, it differs from it just by two lines of (fortran) code, when using the algorithm designed in [47].

Let us now move to the more sophisticated case of the isothermal Euler equations:

$$\partial_t \rho + \nabla \cdot q = 0, \quad \partial_t q + \nabla \cdot \left( \frac{q \otimes q}{\rho} \right) + \nabla \rho = 0.$$

We easily get the convex optimization problem

$$\int_{[0, T] \times D} \exp(u) \exp\left(\frac{1}{2} Q \cdot M^{-1} \cdot Q\right) + \int_D \sigma_0 \rho_0 + w_0 \cdot q_0,$$

among all fields  $u = u(t, x) \in \mathbb{R}$ ,  $Q = Q(t, x) \in \mathbb{R}^d$ ,  $M = M(t, x) = M^t(t, x) \in \mathbb{R}^{d \times d}$ ,  $M \geq 0$ , of form:

$$u = \partial_t \sigma + \partial^i w_i, \quad Q_i = \partial_t w_i + \partial_i \sigma, \quad M_{ij} = \delta_{ij} - \partial_i w_j - \partial_j w_i,$$

where  $\sigma$  and  $w$  must vanish at  $t = T$ . This optimization problem can be interpreted as a *generalized* (variational deterministic) mean-field game involving fields of non-negative symmetric matrices instead of density fields. Also observe that the *linear* wave equation, written as a first order system in  $(\sigma, w)$  with right-hand side  $(u, Q)$ ,

$$\partial_t \sigma + \partial^i w_i = u, \quad \partial_t w_i + \partial_i \sigma = Q_i$$

directly features, without any linearization, in this optimization problem which has been derived from the *nonlinear* (isothermal) Euler equations,

Finally, let us discuss the Euler equations of incompressible fluids that can be seen as a singular limit of the compressible case (as well known [308, 314, 365]):

$$\partial_t q + \nabla \cdot (q \otimes q) = -\nabla p, \quad \nabla \cdot q = 0,$$

where  $q$  is prescribed at  $t = 0$  and  $p$  is now a Lagrange multiplier for constraint  $\nabla \cdot q = 0$ . We get again a generalized MFG for measures valued in the cone of semi-definite symmetric matrices.

$$\sup_{(M, Q)} - \int_{[0, T] \times D} q_0 \cdot Q + \frac{1}{2} Q \cdot M^{-1} \cdot Q,$$

where now  $Q$  is a vector field (not necessarily divergence-free) and  $M = M^t \geq 0$  is a field of semi-definite symmetric matrices subject to

$$M_{ij}(T, \cdot) = \delta_{ij}, \quad \partial_t M_{ij} = \partial_j Q_i + \partial_i Q_j + 2\partial_i \partial_j (-\Delta)^{-1} \partial_k Q^k.$$

So, we see that our convex optimization method to solve IVP is a natural way to obtain non trivial matrix-valued generalizations of the concept of (variational) MFG.

## Main results for entropic conservation laws

**Theorem 5.5.1.** *If  $U$  is a smooth solution to the IVP and  $T$  is not too large, so that*

$$\forall t, x, \forall V \in \mathcal{W}, \mathcal{E}''(V) - (T - t)F'''(V) \cdot \nabla(\mathcal{E}'(U(t, x))) > 0,$$

*in the sense of symmetric matrices, then  $U$  can be recovered from the concave maximization problem which admits  $A(t, x) = (t - T)\mathcal{E}'(U(t, x))$  as solution.*

Notice that the smallness condition requires, in particular,

$$\mathcal{E}''(V) - TF'''(V) \cdot \nabla(\mathcal{E}'(U_0(x))) > 0, \quad \forall x, \forall V \in \mathcal{W},$$

and definitely restricts the choice of  $T$  with respect to  $U_0$ . This is clearly a drawback of the theory. So we could worry about the generic apparition of shock waves and give up any hope to be able to solve the initial value problem for arbitrarily large values of  $T$ . Observe, however, that the smallness condition gets less restrictive as  $t$  approaches  $T$  and even allows a blow-up of  $\partial_i(\partial_\alpha \mathcal{E}(U(t, x)))$  of order  $(T - t)^{-1}$ . As a matter of fact, in the very special and elementary case of the "inviscid" Burgers equation with initial condition  $u_0$ , the smallness condition simply reads

$$1 + (T - t)\partial_x u(t, x) > 0, \quad \forall t \in [0, T], x \in \mathbb{T}$$

and turns out to be equivalent to:

$$1 + Tu'_0(x) > 0, \quad \forall x, \in \mathbb{T}$$

This exactly means that  $T$  is smaller than

$$T^* = \inf_{x \in \mathcal{T}} \frac{1}{\max\{-u'_0(x), 0\}} \in ]0, +\infty],$$

which is exactly the first time when a shock forms. So, at least in this very elementary case, *all* smooth solutions can be recovered from the maximization problem without any restriction.

## Proof of the Theorem

Since  $U$  is supposed to be a smooth solution of the system of conservation laws, we have

$$\partial_t U^\alpha + \partial_\beta \mathcal{F}^{i\alpha}(U) \partial_i U^\beta = 0.$$

Thus  $W$  defined by

$$W_\alpha(t, x) = (t - T)\partial_\alpha \mathcal{E}(U(t, x)), \quad \alpha \in \{1, \dots, m\},$$

solves

$$\partial_t W_\gamma - \partial_\gamma \mathcal{E}(U) = (t - T)\partial_{\alpha\gamma}^2 \mathcal{E}(U) \partial_t U^\alpha = -(t - T)\partial_{\alpha\gamma}^2 \mathcal{E}(U) \partial_\beta \mathcal{F}^{i\alpha}(U) \partial_i U^\beta$$

which is equal, thanks to the structural symmetry property, to

$$\begin{aligned} -(t - T)\partial_{\alpha\beta}^2 \mathcal{E}(U) \partial_\gamma \mathcal{F}^{i\alpha}(U) \partial_i U^\beta &= -(t - T)\partial_i(\partial_\alpha \mathcal{E}(U)) \partial_\gamma \mathcal{F}^{i\alpha}(U) \\ &= -\partial_i W_\alpha \partial_\gamma \mathcal{F}^{i\alpha}(U). \end{aligned}$$

Thus, we have obtained

$$\partial_t W_\gamma + \partial_i W_\alpha \partial_\gamma \mathcal{F}^{i\alpha}(U) - \partial_\gamma \mathcal{E}(U) = 0,$$

which precisely means that, at each point  $(t, x)$ ,  $V = U(t, x)$  satisfies the first order optimality condition in the definition of  $G(\partial_t W(t, x), DW(t, x))$  through

$$G(\partial_t W(t, x), DW(t, x)) = \sup_{V \in \mathcal{W}} \partial_t W_\gamma(t, x) V^\gamma + \partial_i W_\alpha(t, x) \mathcal{F}^{i\alpha}(V) - \mathcal{E}(V).$$

Meanwhile, the smallness condition tells us, by definition of  $W$ , that

$$\partial_{\beta\gamma}^2 \mathcal{E}(V) - \partial_i W_\alpha(t, x) \partial_{\beta\gamma}^2 \mathcal{F}^{i\alpha}(V)$$

is a positive definite matrix for all  $(t, x, V)$ , which means that, for each fixed  $(t, x)$ ,

$$V \in \mathcal{W} \rightarrow \partial_i W_\alpha(t, x) \mathcal{F}^{i\alpha}(V) - \mathcal{E}(V)$$

is a concave function. So the first order optimality condition we have already obtained for  $V = U(t, x)$  is enough to deduce that

$$G(\partial_t W, DW) = \partial_t W_\gamma U^\gamma + \partial_i W_\alpha \mathcal{F}^{i\alpha}(U) - \mathcal{E}(U).$$

Thus, integrating on  $Q = [0, T] \times D$ , where  $D = \mathbb{T}^d$ , and using that  $U$  is solution of the system of conservation laws, we get

$$\begin{aligned} \int_Q G(\partial_t W, DW) + \mathcal{E}(U) &= \int_Q \partial_t W_\gamma U^\gamma + \partial_i W_\alpha \mathcal{F}^{i\alpha}(U) = \\ &= \int_D W_\gamma(T, \cdot) U^\gamma(T, \cdot) - W_\gamma(0, \cdot) U^\gamma(0, \cdot) = \int_D -W_\gamma(0, \cdot) U_0 \end{aligned}$$

since  $U_0$  is the initial condition and, by definition,  $W(T, \cdot) = 0$ . By definition, the optimal value  $J(U_0)$  of the maximization problem is larger than

$$\int_Q -G(\partial_t W, DW) - \int_D -W_\gamma(0, \cdot) U_0^\gamma.$$

Thus, we have obtained

$$J(U_0) \geq \int_Q \mathcal{E}(U).$$

But, by definition,  $I(U_0)$  is certainly smaller than  $\int_Q \mathcal{E}(U)$  (since  $U$  solves the system of conservation laws) and is also larger than  $J(U_0)$ . (Indeed  $\inf \sup \geq \sup \inf$  is always true.) We conclude that  $I(U_0) = J(U_0)$  which shows that there is no duality gap and that  $W$  is optimal for the maximization problem. This completes the proof.

### The special case of the inviscid Burgers equation

In the very elementary case of the Burgers equation, all entropy solutions (in the sense of Kruzhkov, see [193] for this concept of solutions) can be recovered, for arbitrarily large  $T$ , but in some unusual way. More precisely

**Theorem 5.5.2.** *If  $u$  is a Kruzhkov solution of the inviscid Burgers equation on some fixed time interval  $T$  with initial condition  $u_0$ , then the relaxed convex optimisation problem enables us to recover not necessarily the Kruzhkov solution itself but rather the unique solution  $u^T(t, x)$  of the inviscid Burgers equation enjoying the following properties:*

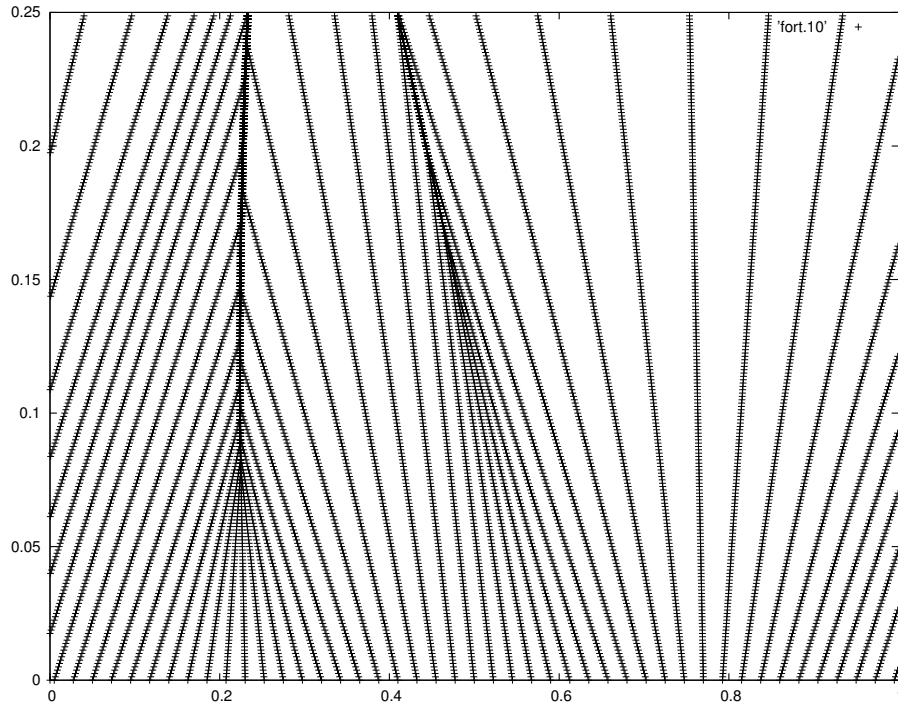
- 1)  $u^T$  and  $u$  coincide at the final time  $T$ ;
- 2)  $u^T$  is shock free up to time  $t = T$  (not included).

*In general, the initial value of  $u^T$  differs from  $u_0$ , unless no shock have formed before  $T$ .*

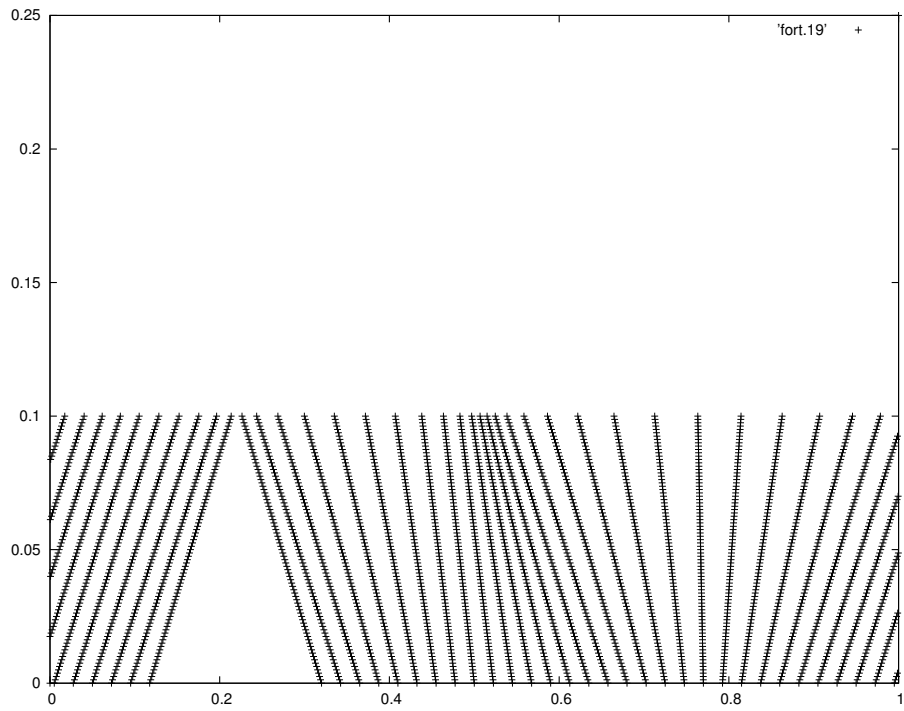
A proof can be found in [110] and will not be reproduced here.

So, our method is able to recover the right Kruzhkov entropy but only at the final given time  $T$ , as soon as shock have formed before  $T$ . This result is also a new answer to the paradox discussed earlier. Something is left from the degenerate space-time ellipticity of the convex minimization problem in the sense that the smoothest possible solution of the inviscid Burgers equation compatible with the right final solution is selected, just by substituting for the given initial condition  $u_0$  another one, namely  $u^T(0, \cdot)$ .

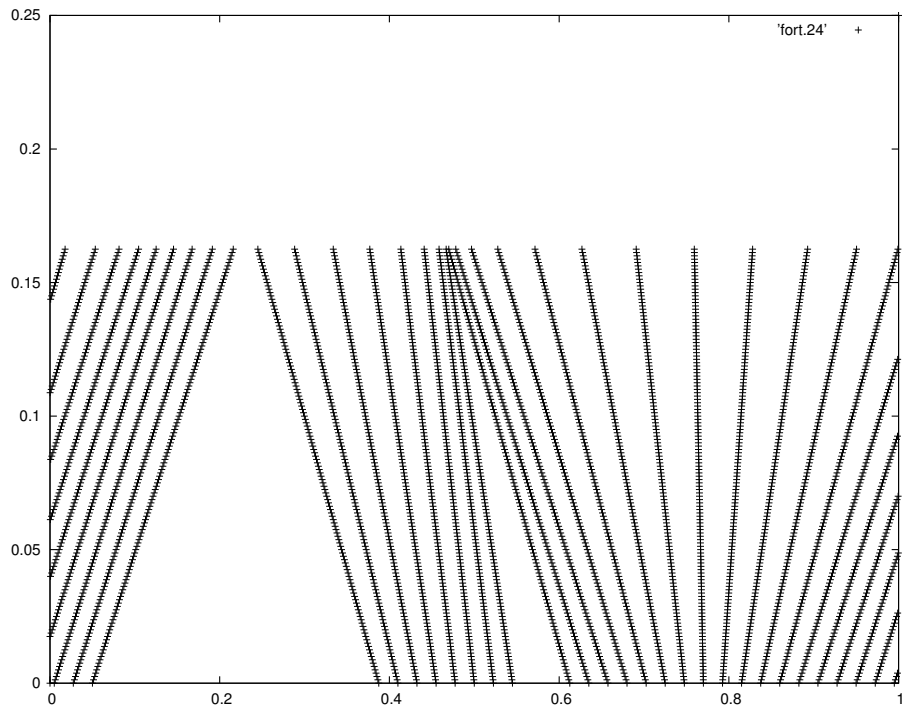




Inviscid Burgers equation :  $\partial_t u + \partial_x (u^2/2) = 0$ ,  $u = u(t, x)$ ,  $x \in \mathbb{R}/\mathbb{Z}$ ,  $t \geq 0$ .  
 Formation of two shock waves. (Vertical axis:  $t \in [0, 1/4]$ . horizontal axis:  $x \in \mathbb{T}$ .)



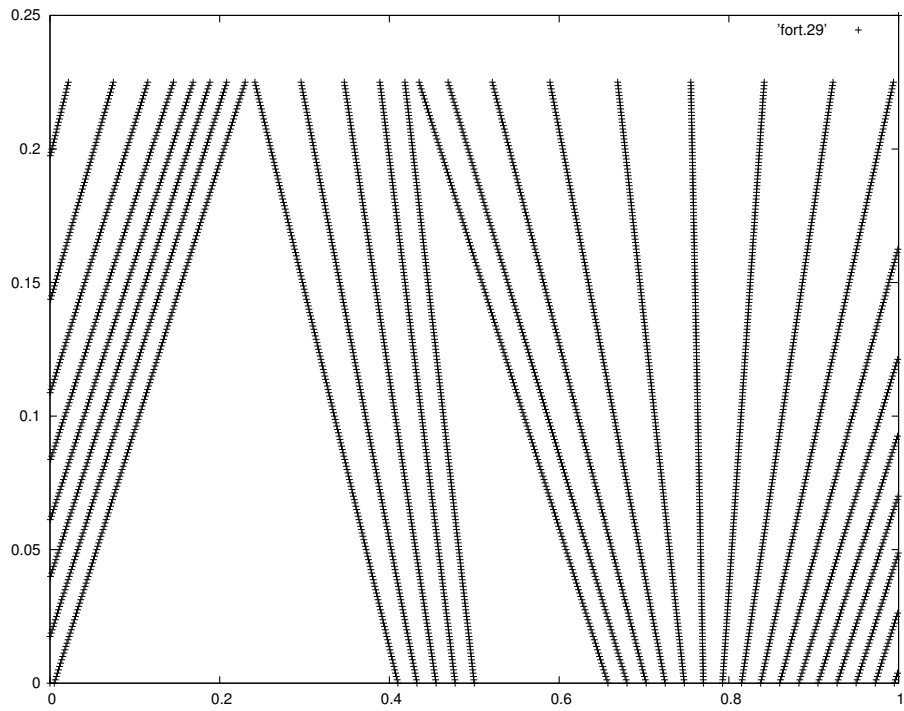
Inviscid Burgers equation :  $\partial_t u + \partial_x (u^2/2) = 0$ ,  $u = u(t, x)$ ,  $x \in \mathbb{R}/\mathbb{Z}$ ,  $t \geq 0$ .  
 Recovery of the solution at time  $T=0.1$  by convex optimization.  
 Observe the formation of a vacuum zone as the first shock has formed.



Inviscid Burgers equation :  $\partial_t u + \partial_x (u^2/2) = 0$ ,  $u = u(t, x)$ ,  $x \in \mathbb{R}/\mathbb{Z}$ ,  $t \geq 0$ .

Recovery of the solution at time  $T=0.16$  by convex optimization.

Observe the formation of a second vacuum zone as the second shock has formed.



Inviscid Burgers equation :  $\partial_t u + \partial_x (u^2/2) = 0$ ,  $u = u(t, x)$ ,  $x \in \mathbb{R}/\mathbb{Z}$ ,  $t \geq 0$ .  
 Recovery of the solution at time  $T=0.225$  by convex optimization.  
 Observe the extension of both vacuum zones.



# Chapter 6

## Convex formulations of first order systems of conservation laws

### 6.1 A short review of first order systems of conservation laws

First order systems of conservation laws read:

$$\partial_t u + \sum_{i=1}^d \partial_{x_i} (Q_i(u)) = 0,$$

or, in short, using the nabla notation,

$$\partial_t u + \nabla \cdot (Q(u)) = 0,$$

where  $u = u(t, x) \in \mathbb{R}^m$  depends on  $t \geq 0$ ,  $x \in \mathbb{R}^d$ , and  $\cdot$  denotes the inner product in  $\mathbb{R}^d$ . The  $Q_i$  (for  $i = 1, \dots, d$ ) are given smooth functions from  $\mathbb{R}^m$  into itself. The system is called hyperbolic when, for each  $\tau \in \mathbb{R}^d$  and each  $U \in \mathbb{R}^m$ , the  $m \times m$  matrix  $\sum_{i=1, d} \tau_i Q'_i(U)$  can be put in diagonal form with real eigenvalues. There is no general theory to solve globally in time the initial value problem for such systems of PDEs. (See [70, 193, 273, 322, 350, 421] for a general introduction to the field.) In general, smooth solutions are known to exist for short times but are expected to become discontinuous in finite time. Therefore, it is usual to consider discontinuous weak solutions, satisfying additional "entropy conditions", to address the initial value problem. Some special situations are far better understood. First, for some very special (but nevertheless very important in Physics and Geometry) systems (enjoying "linear degeneracy" or "null conditions"), smooth solutions may be global (shock free), at least for "small" initial data (see [306, 329, 432], for instance). This includes the famous result on the stability of the Minkowski space in General Relativity by Klainerman and Christodoulou [172]. Next, in one space dimension  $d = 1$ , for a large class of systems, existence and uniqueness of global weak entropy solutions have been proven by Bianchini et Bressan for initial data of sufficiently small total variation [63, 128]. Still, in one space dimension, for a limited class of systems (typically for  $m = 2$ ), existence of global weak entropy solutions have been obtained for large initial data by "compensated compactness" arguments

[440, 206, 335]. Finally, there is a very comprehensive theory in the much simpler case of a single "scalar" conservation laws, i.e. when  $m = 1$ . Kruzhkov [314] showed that such a scalar conservation law has a unique "entropy solution"  $u \in L^\infty$  for each given initial condition  $u_0 \in L^\infty$ . (If the derivative  $Q'$  is further assumed to be bounded, then we can substitute  $L^1_{loc}$  for  $L^\infty$  in this statement.) An entropy (or Kruzhkov) solution is an  $L^\infty$  function that satisfies the following distributional inequality

$$\partial_t C(u) + \nabla_x \cdot (Q^C(u)) \leq 0,$$

for all Lipschitz convex function  $C : \mathbb{R} \rightarrow \mathbb{R}$ , where the derivative of  $Q^C$  is defined by  $(Q^C)' = C'Q'$  (the initial condition  $u_0$  being prescribed by continuity at  $t = 0$ , in  $L^1_{loc}$ , namely:

$$\lim_{t \rightarrow 0} \int_B |u(t, x) - u_0(x)| dx = 0,$$

for all compact subset  $B$  of  $\mathbb{R}^d$ ). Beyond their existence and uniqueness, the Kruzhkov solutions enjoy many interesting properties. Each entropy solution  $u(t, \cdot)$ , with initial condition  $u_0$ , continuously depends on  $t \geq 0$  in  $L^1_{loc}$  and can be written  $T(t)u_0$ , where  $(T(t), t \geq 0)$  is a family of order preserving operators:

$$T(t)u_0 \geq T(t)\tilde{u}_0, \quad \forall t \geq 0,$$

whenever  $u_0 \geq \tilde{u}_0$ . Since constants are trivial entropy solutions to a scalar conservation law, it follows that if  $u_0$  takes its values in some fixed compact interval, so does  $u(t, \cdot)$  for all  $t \geq 0$ . Next, two solutions  $u$  and  $\tilde{u}$ , with  $u_0 - \tilde{u}_0 \in L^1$ , are  $L^1$  stable with respect to their initial conditions:

$$\int |u(t, x) - \tilde{u}(t, x)| dx \leq \int |u_0(x) - \tilde{u}_0(x)| dx,$$

for all  $t \geq 0$ . As a consequence, the total variation  $TV(u(t, \cdot))$  of a Kruzhkov solution  $u$  at time  $t \geq 0$  cannot be larger than the total variation of its initial condition  $u_0$ . This easily comes from the translation invariance of the scalar conservation law and from one of the most classical definitions of the total variation of a function  $v$ , namely:

$$TV(v) = \sup_{\eta \in \mathbb{R}^d, \eta \neq 0} \int \frac{|v(x + \eta) - v(x)|}{|\eta|} dx,$$

where  $|\cdot|$  denotes the Euclidean norm on both  $\mathbb{R}$  and  $\mathbb{R}^d$ . As a matter of fact, the space  $L^1$  plays a key role in Kruzhkov's theory. Indeed, there is no  $L^p$  stability with respect to initial conditions in any  $p > 1$ . Typically, for  $p > 1$ , the Sobolev norm  $\|u(t, \cdot)\|_{W^{1,p}}$  of a Kruzhkov solution blows up in finite time. This fact has induced a great amount of pessimism about the possibility of a unified theory of global solutions for general multidimensional systems of hyperbolic conservation laws. Indeed, simple linear systems, such as the wave equation (written as a first order system) or the Maxwell equations, are not well posed in any  $L^p$  but for  $p = 2$  [125]. However, as we are going to see that  $L^2$  turns out to be a perfectly suitable space for entropy solutions to multidimensional scalar conservation laws, provided a different formulation is used, based on a combination of level-set, kinetic and transport-collapse approximations, in the spirit of previous works by Giga, Miyakawa, Osher, Tsai and the author [79, 81, 82, 97, 267, 445]. As a matter of fact, this new formulation is really due to Panov [391] and was just rediscovered, in a different style, by the

author in [101]. (See [395].) Let us also mention the more recent approach of Serre and Vasseur where the space  $L^2$  can also be used for conservation laws, from a quite different angle [425]. Finally let us emphasise that this new formulation à la Panov is entirely convex, and provides a remarkable example of "hidden convexity" in nonlinear PDEs.

## 6.2 Panov formulation of scalar conservation laws

### The main result

N.B. For notational simplicity, we limit ourself to initial conditions  $u_0$  that can be written as

$$u_0(x) = \int_0^1 1\{Y_0(a, x) < 1/2\} da,$$

for some "level set function"  $Y_0$  enjoying the following properties

$$Y_0(0, x) = 0, \quad Y_0(1, x) = 1, \quad \partial_a Y_0(a, x) > 0.$$

(As a matter of fact, this way we may recover all  $u_0$  with a range compactly supported in  $]0, 1[$ , and, therefore all  $u_0$  in  $L^\infty(\mathbb{T}^d)$ , up to a trivial rescaling of the "flux function"  $Q$ .)

**Theorem 6.2.1.** *Let  $Y_0(a, x)$  be any  $L^\infty$  function of  $x \in \mathbb{T}^d$  and  $a \in [0, 1]$  such that*

$$Y_0(0, x) = 0, \quad Y_0(1, x) = 1, \quad \partial_a Y_0(a, x) > 0.$$

*Let, for all  $y \in [0, 1]$ ,*

$$u_0(x, y) = \int_0^1 1\{Y_0(a, x) < y\} da,$$

*Then, the unique Kruzhkov solution to the scalar conservation law*

$$\partial_t u + \nabla \cdot (Q(u)) = 0,$$

*with initial condition  $u_0(x, y)$  can be written*

$$u(t, x) = \int_0^1 1\{Y(t, a, x) < y\} da,$$

*where  $Y$  solves the subdifferential inclusion in  $L^2(\mathbb{T}^d \times [0, 1])$ :*

$$0 \in \partial_t Y + q(a) \cdot \nabla_x Y + \partial K[Y],$$

*with  $q = Q'$ ,  $K[Y] = 0$  if  $\partial_a Y \geq 0$ , and  $K[Y] = +\infty$  otherwise.*

Let us be more explicit for the definition of this subdifferential inclusion.

**Definition 6.2.2.** *We say that  $Y$  is a solution to*

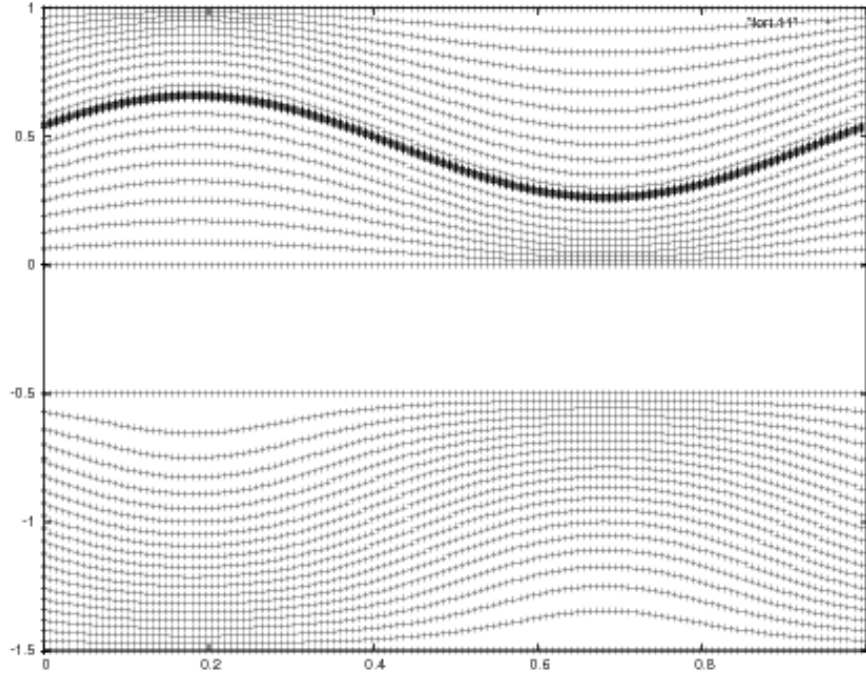
$$0 \in \partial_t Y + q(a) \cdot \nabla_x Y + \partial K[Y], \quad \text{if :}$$

- 1)  $t \rightarrow Y(t, \cdot, \cdot) \in L^2(\mathbb{T}^d \times [0, 1])$  is continuous and satisfies  $\partial_a Y \geq 0$ ,
- 2)  $Y$  satisfies, in the sense of distribution,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d \times [0, 1]} |Y - Z|^2(t, a, x) dadx \\ & + \int_{\mathbb{T}^d \times [0, 1]} (Y - Z)(t, a, x) (\partial_t Z + q(a) \cdot \nabla_x Z)(t, a, x) dadx \leq 0, \end{aligned}$$

*for each smooth function  $Z(t, a, x)$  such that  $\partial_a Z \geq 0$ .*

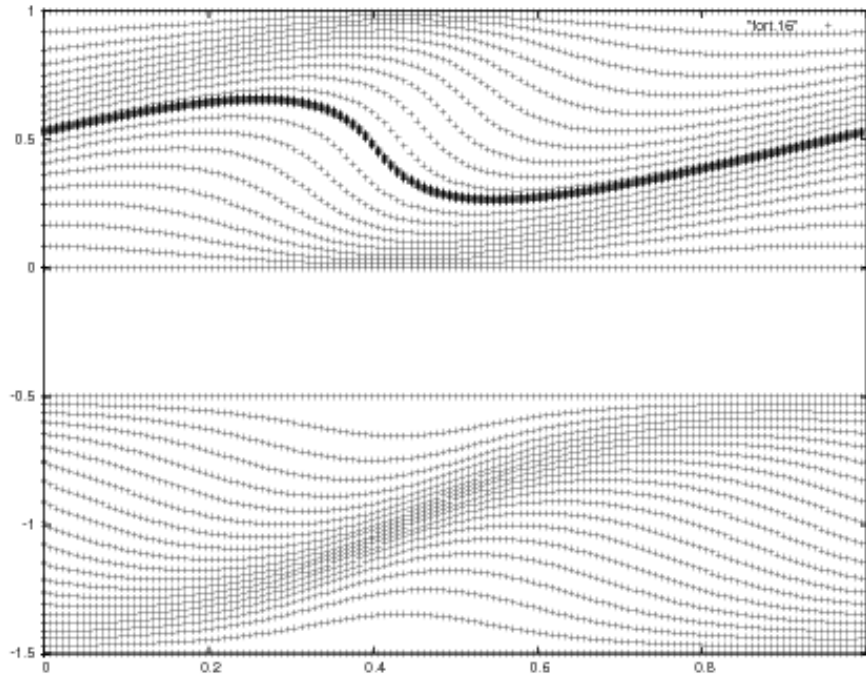




Inviscid Burgers equation :  $\partial_t u + \partial_x(u^2/2) = 0$ ,  $x \in \mathbb{R}/\mathbb{Z}$ ,  $t \geq 0$ .

Top: drawing of a set of initial data  $x \rightarrow u_0(x, y)$  increasing from 0 to 1 in  $y \in [0, 1]$  (N.B. value  $y = 1/2$  is emphasized).

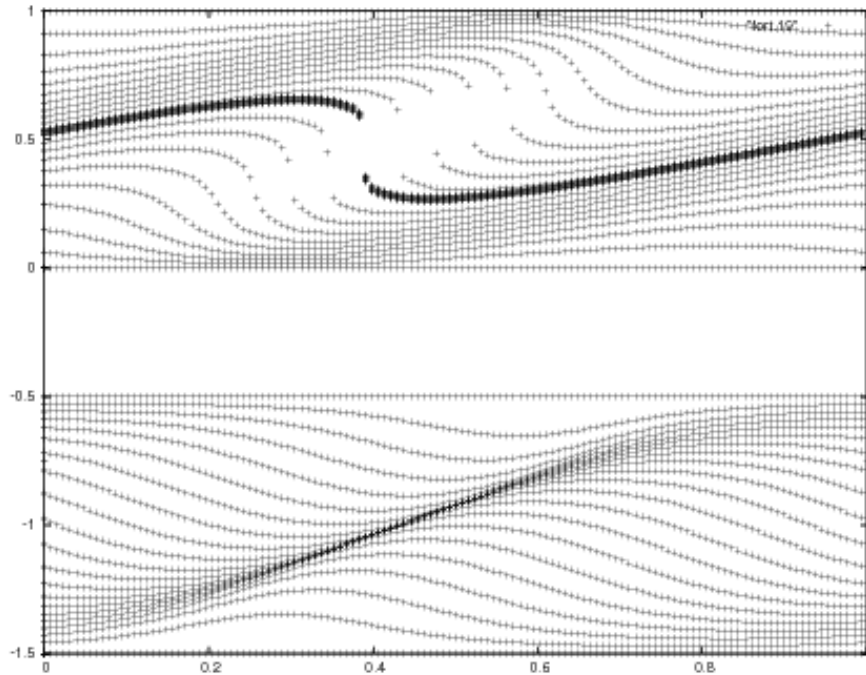
Bottom: drawing of the pseudo-inverse  $Y_0(a, x)$ ,  $a \in [0, 1]$  so that:  
 $y = Y_0(u_0(x, y), x)$ ,  $a = u_0(x, Y_0(a, x))$ .



Inviscid Burgers equation :  $\partial_t u + \partial_x(u^2/2) = 0$ ,  $x \in \mathbb{R}/\mathbb{Z}$ ,  $t \geq 0$ .

Top: drawing of the solution (before formation of shocks)  $x \rightarrow u(t, x, y)$  at  $t = 0.5$  (N.B. value  $y = 1/2$  is emphasized).

Bottom: drawing of the pseudo-inverse  $Y(t, a, x)$ ,  $a \in [0, 1]$  so that:  
 $y = Y(t, u(t, x, y), x)$ ,  $a = u(t, x, Y(t, a, x))$ .



Inviscid Burgers equation :  $\partial_t u + \partial_x(u^2/2) = 0$ ,  $x \in \mathbb{R}/\mathbb{Z}$ ,  $t \geq 0$ .

Top: drawing of the solution (after formation of shocks)  $x \rightarrow u(t, x, y)$  at  $t = 1$  (N.B. value  $y = 1/2$  is emphasized).

Bottom: drawing of the pseudo-inverse  $Y(t, a, x)$ ,  $a \in [0, 1]$  so that:  
 $y = Y(t, u(t, x, y), x)$ ,  $a = u(t, x, Y(t, a, x))$ .

## Remark

As shown by Perepelitsa in [395],  $Y \rightarrow F'(a) \cdot \nabla_x Y + \partial\Phi[Y]$  actually is a maximal monotone in the classical sense of [129] and generates a semi-group of contractions in  $L^2$ . It is rather astonishing that scalar conservation laws can be reduced to the rather conventional theory of maximal monotone operators in  $L^2$ . Indeed, in the 80s, scalar conservation laws were frequently presented as one of the most striking applications of the more advanced theory of maximal operators...in  $L^1$ !

## Idea of the proof

We follow the presentation of [101] rather than the earlier work of Panov [391]. (We refer to [395] for a more detailed comparison of [391] and [101].)

The main idea is to consider, instead of a single initial condition  $u_0(x)$  for the scalar conservation law

$$\partial_t u + \nabla \cdot (Q(u)) = 0,$$

a one-parameter family of initial conditions  $u_0(x, y)$ . We make the crucial assumption that this family is monotonically increasing with respect to the parameter  $y$ . By the standard comparison principle for scalar conservation laws, the corresponding Kruzhkov solutions  $u(t, x, y)$  are also monotone with respect to  $y$ . Assume, for a while, that  $u(t, x, y)$  is *a priori* smooth and strictly increasing in  $y$ . Thus, we can write

$$u(t, x, Y(t, a, x)) = a, \quad Y(t, x, u(t, x, y)) = y$$

where  $Y(t, a, x)$  is smooth and strictly increasing in  $a \in [0, 1]$ . Then, a straightforward calculation shows that  $Y$  must solve the simple linear equation

$$\partial_t Y + q(a) \cdot \nabla_x Y = 0$$

(which admits  $Y(t, a, x) = Y(t = 0, a, x - tq(a))$  as exact solution). This is just a rephrasing of the celebrated "method of characteristics". Unfortunately, this linear equation is not able to preserve the monotonicity condition  $\partial_a Y \geq 0$  in the large. However, by properly correcting it, namely by adding the subdifferential term  $\partial K$ , it is possible to enforce  $\partial_a Y \geq 0$ , and, this way, to recover the correct Kruzhkov entropy solutions. More precisely, as  $Y$  solves the subdifferential inclusion stated above, then

$$u(t, x, y) = \int_0^1 1\{Y(t, a, x) < y\} da$$

will be shown to be, for each fixed value  $y$ , the right entropy solution with initial conditions  $x \rightarrow u_0(x, y)$ .

Observe that this approach is strongly related to both the kinetic formulation and the level set method for scalar conservation laws. Let us recall that the kinetic approach amounts to lift a non-linear scalar conservation law by averaging out a linear advection equation involving a hidden extra variable. This idea (that has obvious roots in the kinetic theory of Maxwell and Boltzmann) was introduced for scalar conservation laws in parallel by Giga-Miyakawa and the author [79, 81, 82, 267]. Its time continuous counter-part is nothing but the celebrated "kinetic formulation" of Lions, Perthame and Tadmor [336] which, with the crucial help of the so-called "averaging lemma" [276], provided the first regularity results (in suitable fractional Sobolev spaces) for multidimensional scalar conservation laws,

(under suitable nonlinearity conditions). (See also related results [112, 194, 256, 335].) Concerning the "level set method", its application to scalar conservation laws by Tsai, Giga and Osher [445] can be interpreted as a parabolic approximation of our subdifferential inclusion, as will be discussed below.

### Elements of a proof

We follow the constructive proof of [101] based on the analysis of the time-discrete scheme known as the "transport-collapse method" [82]. (This time-discrete scheme is somewhat related to the important family of "projection methods" in Computational Fluid Dynamics [168, 169, 218, 400, 443]. We will show that, as the time step goes to zero, the approximate solutions we are going to construct both converge to solutions in the Kruzhkov sense and solutions in the subdifferential sense. We assume that  $Y_0(a, x) \in [0, 1]$  (which is consistent with the statement of Theorem 6.2.1). We fix a time step  $h > 0$  and approximate  $Y(nh, a, x)$  by  $Y_n(a, x)$ , for each positive integer  $n$ . To get  $Y_n$  from  $Y_{n-1}$ , we perform two steps, making the following induction assumptions:

$$\partial_a Y_{n-1} \geq 0, \quad Y_{n-1} \in [0, 1],$$

which are consistent with our assumptions on  $Y_0$ .

### Predictor step

The first "predictor" step amounts to solve the linear equation

$$\partial_t Y + q(a) \cdot \nabla_x Y = 0,$$

for  $nh - h < t < nh$ , with  $Y_{n-1}$  as initial condition at  $t = nh - h$ . We exactly get at time  $t = nh$  the predicted value:

$$Y_n^*(a, x) = Y_{n-1}(a, x - h q(a))$$

Thanks to the induction assumption, we still have  $Y_n^* \in [0, 1]$ , however, although  $\partial_a Y_{n-1}$  is nonnegative, the same may not be true for  $\partial_a Y_n^*$ . This is why, we need a "corrector step".

### Corrector step

In the second step, we 'rearrange'  $Y^*$  in increasing order with respect to  $a \in [0, 1]$ , for each fixed  $x$ , and get the corrected function  $Y_n$ . Let us recall some elementary facts about rearrangements (see [327] and some applications in [138, 279]):

**Lemma 6.2.3.** *Let:  $a \in [0, 1] \rightarrow X(a) \in \mathbb{R}$  an  $L^\infty$  function. Then, there is unique  $L^\infty$  function  $Y : [0, 1] \rightarrow \mathbb{R}$ , such that  $Y' \geq 0$  and:*

$$\int_0^1 H(y - Y(a)) da = \int_0^1 H(y - X(a)) da, \quad \forall y \in \mathbb{R}.$$

*We say that  $Y$  is the rearrangement of  $X$ . In addition, for all  $Z \in L^\infty$  such that  $Z' \geq 0$ , the following rearrangement inequality:*

$$\int_0^1 |Y(a) - Z(a)|^p da \leq \int_0^1 |X(a) - Z(a)|^p da.$$

*holds true for all  $p \geq 1$ .*

So, we define  $Y_n(a, x)$  to be, for each fixed  $x$ , the rearrangement of  $Y_n^*(a, x)$  in  $a \in [0, 1]$ :

$$\partial_a Y_n \geq 0, \quad \int_0^1 H(y - Y_n(a, x)) da = \int_0^1 H(y - Y_n^*(a, x)) da, \quad \forall y \in \mathbb{R}.$$

Equivalently, we may define the auxiliary function:

$$u_n(x, y) = \int_0^1 H(y - Y_n^*(a, x)) da, \quad \forall y \in \mathbb{R},$$

i.e.

$$u_n(x, y) = \int_0^1 H(y - h Y_{n-1}(a, x - h q(a))) da,$$

and set:

$$Y_n(a, x) = \int_0^\infty H(a - u_n(x, y)) dy.$$

At this point,  $Y_n$  is entirely determined by  $Y_{n-1}$ . Notice that, from the very definition of the rearrangement step,  $u_n$ , by definition, can be equivalently written:

$$u_n(x, y) = \int_0^1 H(y - Y_n(a, x)) da.$$

Also notice that, for all function  $Z(a, x)$  such that  $\partial_a Z \geq 0$ , and all  $p \geq 1$ :

$$\int |Y_n(a, x) - Z(a, x)|^p da dx \leq \int |Y_n^*(a, x) - Z(a, x)|^p da dx$$

follows from the rearrangement inequality. Finally, we see that  $\partial_a Y_n \geq 0$  is automatically satisfied (this was the purpose of the rearrangement step) as well as  $Y_n \in [0, 1]$  (since the convex hull of the range of  $Y_n^*$  has been preserved by the rearrangement step). So, the induction assumption is enforced at step  $n$  and the scheme is well defined.

## Remark

Observe that, for any fixed  $x$ ,  $u_n(x, y)$ , as a function of  $y$ , is the (generalized) inverse of  $Y_n(a, x)$ , viewed as a function of  $a$ , in the sense of Lemma 6.2.3. Also notice that the level sets  $\{(a, y); y \geq Y_n(a, x)\}$  and  $\{(a, y); a \leq u_n(x, y)\}$  coincide.

## The transport-collapse scheme revisited

The time-discrete scheme can be entirely recast in terms of the auxiliary function  $u_n$  defined as above. Indeed, introducing

$$ju_n(x, y, a) = H(u_n(x, y) - a),$$

we can rewrite the "predictor-corrector" steps in terms of  $u_n$  and  $ju_n$  as simply as:

$$u_n(x, y) = \int_0^1 ju_{n-1}(x - h q(a), y, a) da,$$

which exactly define the "transport-collapse" (TC) approximation to the scalar conservation law, or, equivalently, its "kinetic" approximation, according to [79, 81, 82, 267].

## Convergence to the Kruzhkov solution

We are now going to prove that, on one hand,  $Y_n(a, x)$  converges to  $Y(t, a, x)$  as  $nh \rightarrow t$ , and, on the other hand,  $u_n(x, y)$  converges to  $u(t, x, y)$ , where  $Y$  and  $u$  are respectively the unique solution to the subdifferential inclusion

$$0 \in \partial_t Y + q(a) \cdot \nabla_x Y + \partial K[Y],$$

with initial condition  $Y_0(a, x)$  and the unique Kruzhkov solution to the scalar conservation law with initial condition (where  $y$  is just a parameter)

$$u_0(x, y) = \int_0^1 H(y - Y_0(a, x)) da. \quad (6.2.1)$$

We take for granted the convergence analysis of the TC method [79, 80, 81, 82, 267] and obtain that, as  $nh \rightarrow t$ ,

$$\int |u_n(x, y) - u(t, x, y)| dy dx \rightarrow 0,$$

where  $u$  is the unique Kruzhkov solution with initial value  $u_0$ . More precisely, if we extend the time discrete approximations  $u_n(x, y)$  to all  $t \in [0, T]$  by linear interpolation in time:

$$u^h(t, x, y) = u_{n+1}(x, y) \frac{t - nh}{h} + u_n(x, y) \frac{nh + h - t}{h},$$

then  $u^h - u$  converges to 0 in the space  $C^0([0, T], L^1(\mathbb{T}^d \times \mathbb{R}))$  as  $h \rightarrow 0$ . It is now natural to introduce the level-set function  $Y$  defined from the Kruzhkov solution by

$$Y(t, a, x) = \int_0^\infty H(a - u(t, x, y)) dy.$$

(Notice that, at this point, we do not know that  $Y$  is a solution to the subdifferential inclusion.) Let us interpolate the  $Y_n$  by

$$Y^h(t, a, x) = Y_{n+1}(a, x) \frac{t - nh}{h} + Y_n(a, x) \frac{nh + h - t}{h},$$

for all  $t \in [nh, nh + h]$  and  $n \geq 0$ . Next, we crucially use the "co-area formula" (or in other words Lebesgue's "horizontal" integration by level sets) to get

$$\int |Y(t, a, x) - Y_n(a, x)| da dx = \int |u(t, x, y) - u_n(x, y)| dy dx.$$

Thus:

$$\sup_{t \in [0, T]} \|Y(t, \cdot) - Y^h(t, \cdot)\|_{L^1} \leq \sup_{t \in [0, T]} \|u(t, \cdot) - u^h(t, \cdot)\|_{L^1} \rightarrow 0,$$

and we conclude that the approximate solution  $Y^h$  must converge to  $Y$  in  $C^0([0, T], L^1([0, 1] \times \mathbb{T}^d))$  as  $h \rightarrow 0$ . Notice that, since the  $Y^h$  are uniformly bounded in  $L^\infty$ , the convergence also holds true in  $C^0([0, T], L^2([0, 1] \times \mathbb{T}^d))$ .

We are finally left with proving that  $Y$  is the solution to the subdifferential inclusion with initial condition  $Y_0$  in the sense of Definition 6.2.2.

## Consistency of the transport-collapse scheme

Let us check that the TC scheme is consistent with the subdifferential formulation in the precise sense of Definition 6.2.2. For each smooth function  $Z(t, a, x)$  with  $\partial_a Z \geq 0$  and  $p \geq 1$ , we have

$$\begin{aligned} & \int |Y_{n+1}(a, x) - Z(nh + h, a, x)|^p dadx \\ & \leq \int |Y_{n+1}^*(a, x) - Z(nh + h, a, x)|^p dadx \end{aligned}$$

(because of the rearrangement step, which is non expansive in any  $L^p$ )

$$= \int |Y_n(a, x - h q(a)) - Z(nh + h, a, x)|^p dadx$$

(by definition of the predictor step)

$$\begin{aligned} & = \int |Y_n(a, x) - Z(nh + h, a, x + h q(a))|^p dadx \\ & = \int |Y_n - Z(nh, \cdot)|^p dadx + h \Gamma + o(h) \end{aligned}$$

where:

$$\Gamma = p \int (Y_n - Z(nh, \cdot)) |Y_n - Z(nh, \cdot)|^{p-2} \{-\partial_t Z(nh, \cdot) - q \cdot \nabla_x Z(nh, \cdot)\} dadx$$

(by Taylor expanding  $Z$  about  $(nh, a, x)$ ). Since the approximate solution provided by the TC scheme has a unique limit  $Y$ , as shown in the previous section, this limit must satisfy:

$$\frac{d}{dt} \int |Y - Z|^p dadx \leq p \int (Y - Z) |Y - Z|^{p-2} (-\partial_t Z - q(a) \cdot \nabla_x Z) dadx,$$

in the distributional sense in  $t$ . In particular, for  $p = 2$ , we exactly recover the differential inequality of Definition 6.2.2. We conclude that the approximate solutions generated by the TCM scheme do converge to the solutions of the subdifferential inclusion in the sense of Definition 6.2.2, which completes the proof of Theorem 6.2.1.

## Viscous approximations

A natural regularization for our subdifferential inclusion amounts to substituting a barrier function for the convex cone  $K$  in  $L^2([0, 1] \times \mathbb{T}^d)$  of all functions  $Y$  such that  $\partial_a Y \geq 0$ . Typically, we introduce a convex function  $\phi : \mathbb{R} \rightarrow ]-\infty, +\infty]$  such that  $\phi(\tau) = +\infty$  if  $\tau < 0$ , we define, for all  $Y \in K$ ,

$$\Phi[Y] = \int \phi(\partial_a Y) dadx,$$

and set  $\Phi[Y] = +\infty$  if  $Y$  does not belong to  $K$ . Typical examples are:

$$\phi(\tau) = -\log(\tau), \quad \phi(\tau) = \tau \log(\tau), \quad \phi(\tau) = \frac{1}{\tau}, \quad \forall \tau > 0.$$



Then, we considered the perturbed subdifferential inclusion

$$0 \in \partial_t Y + q(a) \cdot \nabla_x Y - q_0(a) + \varepsilon \partial \Phi[Y],$$

for  $\varepsilon > 0$ . The general theory of maximal monotone operators guarantees the convergence of the corresponding solutions as  $\varepsilon \rightarrow 0$ . It is not difficult (at least formally) to identify the corresponding perturbation to our scalar conservation

$$\partial_t u + \nabla \cdot (Q(u)) = 0.$$

Indeed, assuming  $\phi(\tau)$  to be smooth for  $\tau > 0$ , we get, for each smooth function  $Y$  such that  $\partial_a Y > 0$ :

$$\partial \Phi(Y) = -\partial_a(\phi'(\partial_a Y)).$$

Thus, any smooth solution  $Y$  of the perturbed subdifferential inclusion satisfying  $\partial_a Y > 0$ , solves the following parabolic equation:

$$\partial_t Y + q(a) \cdot \nabla_x Y = \varepsilon \partial_a(\phi'(\partial_a Y)).$$

Introducing, the function  $u(t, x, y)$  implicitly defined by

$$u(t, x, Y(t, a, x)) = a,$$

we get (by differentiating with respect to  $a$ ,  $t$  and  $x$ ):

$$(\partial_y u)(t, x, Y(t, a, x)) \partial_a Y(t, a, x) = 1,$$

$$(\partial_t u)(t, x, y) + (\partial_y u)(t, x, y) \partial_t Y = 0,$$

$$(\nabla_x u)(t, x, y) + (\partial_y u)(t, x, y) \nabla_x Y = 0.$$

Then, we get

$$-\partial_t u - q(u) \cdot \nabla_x u - q_0(u) \partial_y u = \varepsilon \partial_y(\phi'(\frac{1}{\partial_y u})).$$

In particular, in the case  $\phi(\tau) = -\log \tau$ , we obtain

$$\partial_t u + q(u) \cdot \nabla_x u = \varepsilon \partial_{yy}^2 u,$$

with viscosity only in the  $y$  variable. This includes viscous effects not on the space variable  $x$  but rather on the "level-set parameter"  $y \in \mathbb{R}$ . This unusual type of regularization has already been used and analyzed in the level-set framework developed by Giga, Giga, Osher and Tsai for scalar conservation laws [266, 445].

## Related equations

A similar method can be applied to some special systems of conservation laws. A typical example (which was crucial for our understanding) is the 'Born-Infeld-Chaplygin' system considered in [97], and the related concept of 'order-preserving strings'. This system reads:

$$\partial_t(hv) + \partial_y(hv^2 - hb^2) - \partial_x(hb) = 0,$$

$$\partial_t h + \partial_y(hv) = 0, \quad \partial_t(hb) - \partial_x(hv) = 0,$$

where  $h, b, v$  are real valued functions of time  $t$  and two space variables  $x, y$ . In [97] this system is related to the following subdifferential system:

$$0 \in \partial_t Y - \partial_x W + \partial K[Y], \quad \partial_t W = \partial_x Y,$$

where  $(Y, W)$  are real valued functions of  $(t, a, x)$  and  $K[Y]$  is still 0 or  $+\infty$  according to whether  $\partial_a Y \geq 0$  is true or not. The (formal) correspondence between is obtained by setting:

$$\begin{aligned} h(t, x, Y(t, a, x)) \partial_a Y(t, a, x) &= 1, \\ v(t, x, Y(t, a, x)) &= \partial_t Y(t, a, x), \quad b(t, x, Y(t, a, x)) = \partial_x Y(t, a, x). \end{aligned}$$

Unfortunately, this system is very special (its smooth solutions are easily integrable). In our opinion, it is very unlikely that  $L^2$  formulations can be found for general hyperbolic conservation laws as easily as in the multidimensional scalar case.

### More details on the subdifferential inclusion

Let us examine few additional properties of the subdifferential inclusion

$$0 \in \partial_t Y + q(a) \cdot \nabla_x Y + \partial K[Y],$$

obtained from the "transport-collapse" approximation scheme. First, we observe that, in the TC scheme,

- 1) the predictor step (a simple translation in the  $x$  variable by  $h q(a)$ ) is isometric in all  $L^p$  spaces,
- 2) the corrector step (an increasing rearrangement in the  $a$  variable) is non-expansive in all  $L^p$ .

Thus the scheme is non-expansive in all  $L^p([0, 1] \times \mathbb{T}^d)$

Since the scheme is also invariant under translations in the  $x$  variable, we get the following a priori estimate:

$$\|\nabla_x Y_n\|_{L^p} \leq \|\nabla_x Y_0\|_{L^p}.$$

Moreover, if we compare two solutions of the scheme  $Y_n$  and  $\tilde{Y}_n = Y_{n+1}$  obtained with initial condition  $\tilde{Y}_0 = Y_1$ , we deduce:

$$\begin{aligned} \int |Y_{n+1}(a, x) - Y_n(a, x)|^p da dx &\leq \int |Y_1(a, x) - Y_0(a, x)|^p da dx \\ &\leq \int |Y_1^*(a, x) - Y_0(a, x)|^p da dx = \int |Y_0(a, x - h q(a)) - Y_0(a, x)|^p da dx. \end{aligned}$$

So we get a second a priori estimate:

$$\|Y_{n+1} - Y_n\|_{L^p} \leq \|q\|_{L^\infty} \|\nabla_x Y_0\|_{L^p} h.$$

We conclude that the solutions  $Y$  to the subdifferential inclusion obtained from the TC scheme satisfy the a priori bounds:

$$\begin{aligned} \|\nabla_x Y(t, \cdot)\|_{L^p} &\leq \|\nabla_x Y_0\|_{L^p}, \\ \|\partial_t Y(t, \cdot)\|_{L^p} &\leq \|q_0\|_{L^p} + \|q\|_{L^\infty} \|\nabla_x Y_0\|_{L^p}. \end{aligned}$$

## $L^p$ and Monge-Kantorovich stability properties

As just mentioned, the solutions of the subdifferential inclusion enjoy the  $L^p$  stability property with respect to their initial conditions, not only for  $p = 2$  but also for all  $p \geq 1$ . The case  $p = 1$  is of particular interest. Indeed, let us consider two solutions  $Y$  and  $\tilde{Y}$  of the subdifferential inclusion and the corresponding Kruzhkov solutions  $u$  and  $\tilde{u}$ , as in the proof of Theorem 6.2.1. Using the co-area formula we find, for all  $t \geq 0$ ,

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{T}^d} |u(t, x, y) - \tilde{u}(t, x, y)| dx dy = \\
&= \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{T}^d} |H(u(t, x, y) - a) - H(\tilde{u}(t, x, y) - a)| da dx dy \\
&= \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{T}^d} |H(y - Y(t, a, x)) - H(y - \tilde{Y}(t, a, x))| da dx dy \\
&= \int_0^1 \int_{\mathbb{T}^d} |Y(t, a, x) - \tilde{Y}(t, a, x)| dx da \leq \int_0^1 \int_{\mathbb{T}^d} |Y_0(a, x) - \tilde{Y}_0(a, x)| dx da \\
&= \int_{\mathbb{R}} \int_{\mathbb{T}^d} |u_0(x, y) - \tilde{u}_0(x, y)| dx dy.
\end{aligned}$$

Thus, Kruzhkov's  $L^1$  stability property is nothing but a *very* incomplete output of the much stronger  $L^p$  stability property enjoyed by the subdifferential inclusion!

As a matter of fact, it is possible to translate the  $L^p$  stability of the level set function  $Y$  in terms of the Kruzhkov solution  $u$  by using Monge-Kantorovich (MK) distances. Let us first recall that for two probability measures  $\mu$  and  $\nu$  compactly supported on  $\mathbb{R}^D$ , their  $p$  MK distance can be defined (see [451] for instance), for  $p \geq 1$ , by:

$$\delta_p^p(\mu, \nu) = \sup \int \phi(x) d\mu(x) + \int \psi(y) d\nu(y),$$

where the supremum is taken over all pair of continuous functions  $\phi$  and  $\psi$  such that:

$$\phi(x) + \psi(y) \leq |x - y|^p, \quad \forall x, y \in \mathbb{R}^D.$$

In dimension  $D = 1$ , this definition reduces to:

$$\delta_p(\mu, \nu) = \|Y - Z\|_{L^p},$$

where  $Y$  and  $Z$  are respectively the "generalized inverse" of  $u$  and  $v$  defined on  $\mathbb{R}$  by:

$$u(y) = \mu([-\infty, y]), \quad v(y) = \nu([-\infty, y]), \quad \forall y \in \mathbb{R}.$$

Next, observe that, for each  $x \in \mathbb{T}^d$ , the  $y$  derivative of the Kruzhkov solution  $u(t, x, y)$ , can be seen as a probability measure compactly supported on  $\mathbb{R}$ . (Indeed,  $\partial_y u \geq 0$ ,  $u = 0$  near  $y = -\infty$  and  $u = 1$  near  $y = +\infty$ .) Then, the  $L^p$  stability property simply reads:

$$\int_{\mathbb{T}^d} \delta_p^p(\partial_y u(t, \cdot, x), \partial_y \tilde{u}(t, \cdot, x)) dx \leq \int_{\mathbb{T}^d} \delta_p^p(\partial_y u_0(\cdot, x), \partial_y \tilde{u}_0(\cdot, x)) dx.$$

Let us refer to [72] and [153] for recent occurrences of MK distances in the field of scalar conservation laws.

## Uniqueness theory

Let us consider a solution  $Y$  to the subdifferential inclusion in the sense of Definition 6.2.2. By definition  $Y(t, \cdot)$  depends continuously of  $t \in [0, T]$  in  $L^2$ . From definition (6.2.2), using  $Z = 0$  as a test function, we see that:

$$\frac{d}{dt} \|Y(t, \cdot)\|_{L^2}^2 \leq 2 \int Y(t, a, x) q_0(a) \, dadx \leq \|Y(t, \cdot)\|_{L^2}^2 + \|q\|_{L^2}^2,$$

which implies that the  $L^2$  norm  $Y(t, \cdot)$  stays uniformly bounded on any finite interval  $[0, T]$ . Thus,  $T > 0$  being fixed, we can mollify  $Y$  and get, for each  $\epsilon \in ]0, 1]$  a smooth function  $Y_\epsilon(t, a, x)$ , still increasing in  $a$ , so that:

$$\sup_{t \in [0, T]} \|Y(t, \cdot) - Y_\epsilon(t, \cdot)\|_{L^2} \leq \epsilon.$$

Let us now consider an initial condition  $Z_0$  such that  $\nabla_x Z_0$  belongs to  $L^2$ . We know that there exist a solution  $Z$  to the subdifferential inclusion, still in the sense of Definition 6.2.2. obtained by TC approximation, for which both  $\partial_t Z(t, \cdot)$  and  $\nabla_x Z(t, \cdot)$  stay uniformly bounded in  $L^2$  for all  $t \in [0, T]$ . This function  $Z$  has enough regularity to be used as a test function when expressing that  $Y$  is a solution in the sense of Definition 6.2.2. So, for each smooth nonnegative function  $\theta(t)$ , compactly supported in  $]0, T[$ , we get from Definition 6.2.2

$$\int \{\theta'(t)|Y - Z|^2 + 2\theta(t)(Y - Z)(q_0(a) - \partial_t Z - q(a) \cdot \nabla_x Z)\} dadxdt \geq 0.$$

Substituting  $Y_\epsilon$  for  $Y$ , we get

$$\int \{\theta'(t)|Y_\epsilon - Z|^2 + 2\theta(t)(Y_\epsilon - Z)(q_0(a) - \partial_t Z - q(a) \cdot \nabla_x Z)\} dadxdt \geq -C\epsilon,$$

where  $C$  is a constant depending on  $\theta$ ,  $Z$ ,  $q_0$  and  $q$  only. Since  $Z$  is also a solution, using  $Y_\epsilon$  as a test function, we get from Definition 6.2.2:

$$\int \{\theta'(t)|Z - Y_\epsilon|^2 + 2\theta(t)(Z - Y_\epsilon)(q_0(a) - \partial_t Y_\epsilon - q(a) \cdot \nabla_x Y_\epsilon)\} dadxdt \geq 0.$$

Adding up these two inequalities, we deduce:

$$\int \{2\theta'(t)|Y_\epsilon - Z|^2 + 2\theta(t)(Y_\epsilon - Z)(\partial_t(Y_\epsilon - Z) + q(a) \cdot \nabla_x(Y_\epsilon - Z))\} dadxdt \geq -C\epsilon.$$

Integrating by part in  $t \in [0, T]$  and  $x \in \mathbb{T}^d$ , we simply get:

$$\int \theta'(t)|Y_\epsilon - Z|^2 dadxdt \geq -C\epsilon.$$

Letting  $\epsilon \rightarrow 0$ , we deduce:

$$\frac{d}{dt} \int |Y - Z|^2 dadx \leq 0.$$

We conclude, at this point, that:

$$\|Y(t, \cdot) - Z(t, \cdot)\|_{L^2} \leq \|Y_0 - Z_0\|_{L^2}, \quad \forall t \in [0, T]$$

This immediately implies the uniqueness of  $Y$ . Indeed, any other solution  $\tilde{Y}$  with initial condition  $Y_0$  must also satisfy:

$$\|\tilde{Y}(t, \cdot) - Z(t, \cdot)\|_{L^2} \leq \|Y_0 - Z_0\|_{L^2}.$$

Thus, by the triangle inequality:

$$\|\tilde{Y}(t, \cdot) - Y(t, \cdot)\|_{L^2} \leq 2\|Y_0 - Z_0\|_{L^2}.$$

Since  $Z_0$  is any function such that  $\nabla_x Z_0$  belongs to  $L^2$ , we can make  $\|Y_0 - Z_0\|_{L^2}$  arbitrarily small and conclude that  $\tilde{Y} = Y$ , which completes the proof of uniqueness.

### 6.3 Entropic systems of conservation law

We consider general systems of conservative laws of form:

$$\partial_t U^\alpha + \partial_i(\mathcal{F}^{i\alpha}(U)) = 0, \quad \alpha = 1, \dots, m,$$

(with implicit summation on repeated indices) where  $U = U(t, x) \in \mathcal{W} \subset \mathbb{R}^m$ ,  $t \geq 0$ ,  $x \in \mathbb{R}^d$ ,  $\partial_t = \frac{\partial}{\partial t}$ ,  $\partial_i = \frac{\partial}{\partial x_i}$ ,  $\mathcal{W}$  is a smooth convex subset of  $\mathbb{R}^m$  and the "flux function"  $\mathcal{F} : \mathcal{W} \rightarrow \mathbb{R}^{d \times m}$  is smooth with some suitable control near  $\partial\mathcal{W}$ . Once again, we can go back to Euler to start the theory, with his equations of compressible fluids which read, in the isothermal case,

$$\partial_t \rho + \nabla \cdot q = 0, \quad \partial_t q + \nabla \cdot \left( \frac{q \otimes q}{\rho} \right) + \nabla \rho = 0$$

( $\rho > 0$  and  $q \in \mathbb{R}^d$  respectively denoting the density and the momentum of the fluid), which fits to the general framework by setting

$$U = (\rho, q) \in \mathcal{W} = ]0, +\infty[ \times \mathbb{R}^d, \quad \mathcal{F}(U) = \left( q, \frac{q \otimes q}{\rho} + \rho I_d \right).$$

From now on, we limit ourself to the subclass of "entropic system of conservation laws" (ESCL):

**Definition 6.3.1.** *We call ESCL a system of conservation laws for which the flux function  $\mathcal{F}$  satisfies the additional symmetry condition*

$$\forall i \in \{1, \dots, n\}, \quad \forall \beta, \gamma \in \{1, \dots, m\}, \quad \partial_{\alpha\beta}^2 \mathcal{E} \partial_\gamma \mathcal{F}^{i\alpha} = \partial_{\alpha\gamma}^2 \mathcal{E} \partial_\beta \mathcal{F}^{i\alpha},$$

for some smooth function, called "entropy"  $\mathcal{E} : \mathcal{W} \rightarrow \mathbb{R}$ , strictly convex in the sense that the symmetric matrix  $(\partial_{\alpha\beta}^2 \mathcal{E})$  is everywhere definite positive on  $\mathcal{W}$ .

This property looks strange, at first glance, but is essentially equivalent to the "conservation of entropy" in the sense that every  $C^1$  solution of the ESCL satisfies the additional conservation law

$$\partial_t(\mathcal{E}(U)) + \partial_i(\mathcal{Q}^i(U)) = 0,$$

where the "entropy flux function"  $\mathcal{Q} : \mathcal{W} \rightarrow \mathbb{R}^d$  can be explicitly computed from  $\mathcal{F}$  and  $\mathcal{E}$ .

[Indeed, the symmetry condition with respect to  $\mathcal{E}$  is equivalent to

$$\partial_\gamma(\partial_\alpha \mathcal{E} \partial_\beta \mathcal{F}^{i\alpha}) = \partial_\beta(\partial_\alpha \mathcal{E} \partial_\gamma \mathcal{F}^{i\alpha}),$$

which means that  $\partial_\alpha \mathcal{E} \partial_\beta \mathcal{F}^{i\alpha}$  is the gradient of some function  $Q^i : \mathcal{W} \rightarrow \mathbb{R}$ , i.e.  $\partial_\alpha \mathcal{E} \partial_\beta \mathcal{F}^{i\alpha} = \partial_\beta Q^i$ . Therefore, for any solution  $U$  of class  $C^1$ ,

$$\begin{aligned} -\partial_t(\mathcal{E}(U)) &= \partial_\alpha \mathcal{E}(U) \partial_i(\mathcal{F}^{i\alpha}(U)) = \partial_\alpha \mathcal{E}(U) \partial_\beta \mathcal{F}^{i\alpha}(U) \partial_i U^\beta \\ &= \partial_\beta Q^i(U) \partial_i U^\beta = \partial_i(Q^i(U)), \end{aligned}$$

which implies the conservation of entropy.]

The class of ESCL contains many examples from Continuum Mechanics, Physics and Geometry (Euler equations of compressible fluids, Elastodynamics, Electromagnetism, Magneto-Hydrodynamics, Extremal surfaces in Lorentzian spaces, etc...) Of course the simplest nonlinear example of ESCL is the Burgers equation (without viscosity)

$$\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0, \quad u \in \mathbb{R},$$

where  $\mathcal{F}(u) = u^2/2$  and for which a possible choice of entropy is  $\mathcal{E}(u) = u^2/2$ , with  $\mathcal{Q}(u) = u^3/3$ .

More general is the class of scalar conservation laws when  $m = 1$ ,  $\mathcal{W} = \mathbb{R}$ , for which the symmetry condition is trivially satisfied and any convex function  $\mathcal{E}$  can play the role of an entropy. We have already seen that this subclass enjoys a "hidden convexity" property, through the Panov formulation, as discussed in section 6.2.1.

The example of Euler's equations is richer. For instance, in the isothermal case, we find as a strictly convex entropy

$$\mathcal{E}(U) = \frac{|q|^2}{2\rho} + \rho(\log \rho - 1), \quad U = (\rho, q).$$

### Few results on the ESCL

In order to get general results without too much technicalities in our proofs, we make some simplifying assumptions, which are not necessarily satisfied by our basic examples (inviscid Burgers and Euler equations). So, we assume:

- i)  $\mathcal{W} = \mathbb{R}^m$ ;
- ii) all derivatives of  $F$  are bounded;
- iii) there is a constant  $r \in ]0, 1]$  such that, for all points in  $\mathcal{W} = \mathbb{R}^m$ , the spectrum of matrix  $\partial_{\alpha\beta}^2 \mathcal{E}$  is contained in  $[r, 1/r]$ ,

and we consider only solutions  $U = U(t, x)$  that are  $\mathbb{Z}^d$ -periodic in  $x$  (in other words,  $x \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ ).

A first structural property is the possibility of writing any ESCL in symmetric form.

**Theorem 6.3.2.** *For any solution  $U = U(t, x)$  of class  $C^1$  on  $[0, T] \times \mathbb{T}^d$ , the ESCL can be written in non-conservative form*

$$A_{\alpha\beta}^0(t, x) \partial_t U^\beta(t, x) + A_{\alpha\gamma}^j(t, x) \partial_j U^\gamma(t, x) = 0,$$

where  $A^0, A^j, j = 1, \dots, m$ , are fields of symmetric  $m \times m$  matrices,  $A^0$  being definite positive.

This "symmetric" writing is important because it leads to a local existence and uniqueness result:

**Theorem 6.3.3.** *For any initial condition  $U_0$  in  $H^s(\mathbb{T}^d)$ , with  $s - d/2 > 1$ , there is a time  $T > 0$  (depending on  $U_0$ ) and a unique solution  $U = U(t, x)$ , of class  $H^s$ , to the ESCL with initial condition  $U_0: U(0, \cdot) = U_0$ .*

Observe that the exponent  $s - d/2 > 1$  corresponds to the continuous injection of the Sobolev space  $H^s(\mathbb{T}^d)$  in  $C^1(\mathbb{T}^d)$ . Next, we address the link between classical and weak solutions.

**Definition 6.3.4.** *We call weak solution of the ESCL with initial condition  $U_0$ , on a given time interval  $[0, T]$ , any function  $U \in L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^m)$  such that*

$$\int_{[0, T] \times \mathbb{T}^d} \partial_t W_\alpha U^\alpha + \partial_i W_\alpha \mathcal{F}^{i\alpha}(U) + \int_{\mathbb{T}^d} W_\alpha(0, \cdot) U_0^\alpha = 0,$$

for all smooth function  $(t, x) \in [0, T] \times \mathbb{T}^d \rightarrow W = W(t, x) \in \mathbb{R}^m$ , such that  $W(T, \cdot) = 0$ .

(The choice of  $L^p$  with  $p = 2$  is not essential and just related to the simplifying assumptions we have made. For concrete applications,  $p$  is subject to change.)

**Theorem 6.3.5.** *Let  $U$  be a solution of the ESCL, de classe  $C^1$  on  $[0, T] \times \mathbb{T}^d$  with initial condition  $U_0$ . Then,  $U$  is the unique weak solution with initial condition  $U_0$ , such that*

$$\int_{\mathbb{T}^d} \mathcal{E}(U(t, x)) dx \leq \int_{\mathbb{T}^d} \mathcal{E}(U_0(x)) dx,$$

for a.e.  $t \in [0, T]$ .

In this statement, called "strong-weak uniqueness", the condition that the entropy of the weak solution is always bounded from above by the entropy of the initial condition plays a crucial role.

(As a matter of fact, the method of "convex integration" applied by De Lellis, Székelyhidi and their co-authors to several ESCL of importance, show they are an infinite number of weak solutions for generic initial data!)

### Proof of Theorem 6.3.2

Let  $U$  be solution of the ESCL, of class  $C^1$  on  $[0, T] \times \mathbb{T}^d$ . Since we have

$$\partial_t (\mathcal{E}_{,\alpha}(U)) = \mathcal{E}_{,\alpha\beta}(U) \partial_t U^\beta = \mathcal{E}_{,\alpha\beta}(U) \mathcal{F}_{,\gamma}^{j\beta}(U) \partial_j U^\gamma$$

(where partial derivatives are temporarily denoted by comma), it is enough to set

$$\begin{aligned} A_{\alpha\beta}^0(t, x) &= \mathcal{E}_{,\alpha\beta}(U(t, x)) \\ A_{\alpha\gamma}^j(t, x) &= \mathcal{E}_{,\alpha\beta}(U(t, x)) \mathcal{F}_{,\gamma}^{j\beta}(U(t, x)) \\ &= \mathcal{E}_{,\gamma\beta}(U(t, x)) \mathcal{F}_{,\alpha}^{j\beta}(U(t, x)) \end{aligned}$$

(because of the symmetry condition that characterizes the ESCL, on top of the convexity of  $\mathcal{E}$ ). This completes the proof.

### Elements of proof for Theorem 6.3.3

This result is standard in the field of conservation laws [193, 350]. The starting point is a stability result in the space  $L^2(\mathbb{T}^d)$ , and more generally in Sobolev spaces  $H^s(\mathbb{T}^d)$ , of the *linear* system with variable coefficients:

$$A_{\alpha\beta}^0(t, x)\partial_t U^\beta(t, x) + A_{\alpha\gamma}^j(t, x)\partial_j U^\gamma(t, x) = M_{\alpha\gamma}(t, x)U^\gamma(t, x)$$

where the  $M$ ,  $A^0$ ,  $A^j$ ,  $j = 1, \dots, m$ , are *given* fields of  $m \times m$  symmetric matrices, definite positive in the case of the  $A^0$ . Once this result is established, the nonlinear system where the  $A^k$  depend on  $U$ , via:

$$A_{\alpha\beta}^0(t, x) = \mathcal{E}_{,\alpha\beta}(U(t, x))$$

$$A_{\alpha\gamma}^j(t, x) = \mathcal{E}_{,\gamma\beta}(U(t, x))\mathcal{F}_{,\alpha}^{j\beta}(U(t, x)),$$

can be analyzed by some fixed-point argument, through a careful control of the various nonlinearities by the  $C^1(\mathbb{T}^d)$  norm of  $U$ , which, itself, can be controlled by the Sobolev  $H^s(\mathbb{T}^d)$  norm of  $U$ , as soon as  $s - d/2 > 1$ . The complete proof is too technical to be reproduced here and we limit ourself to a sketch of proof of the  $L^2$  stability of the linear system with variable coefficients mentioned above.

**Proposition 6.3.6.** *Assume that there exist constants  $r \in ]0, 1]$  and  $\kappa \in \mathbb{R}$  such that, at each point  $(t, x)$ , the symmetric matrices  $A^0$  and*

$$C = \partial_t A^0 - \partial_j A^j + M + M^T$$

*have their spectrum uniformly contained respectively in  $[r, 1/r]$  and  $]-\infty, \kappa]$ . Then the linear system*

$$A_{\alpha\beta}^0(t, x)\partial_t U^\beta(t, x) + A_{\alpha\gamma}^j(t, x)\partial_j U^\gamma(t, x) = M_{\alpha\gamma}(t, x)U^\gamma(t, x)$$

*is  $L^2(\mathbb{T}^d)$  stable:*

$$\|U(t, \cdot)\|_{L^2(\mathbb{T}^d)} \leq \|U(s, \cdot)\|_{L^2(\mathbb{T}^d)} \exp(\kappa|t - s|/r^2), \quad \forall t, s \in \mathbb{R}.$$

By multiplying the linear system by  $U^\alpha$ , we get

$$\partial_t (U^\alpha A_{\alpha\beta}^0 U^\beta) - \partial_j (U^\alpha A_{\alpha\beta}^j U^\beta) = U^\alpha C_{\alpha\beta} U^\beta.$$

Thus, by integrating in  $x \in \mathbb{T}^d$ , we obtain

$$\frac{d}{dt} \int_{\mathbb{T}^d} U^\alpha A_{\alpha\beta}^0 U^\beta = \int_{\mathbb{T}^d} U^\alpha C_{\alpha\beta} U^\beta.$$

By assumption, we deduce

$$\left| \frac{d}{dt} \int_{\mathbb{T}^d} U^\alpha A_{\alpha\beta}^0 U^\beta \right| \leq \kappa/r \int_{\mathbb{T}^d} U^\alpha A_{\alpha\beta}^0 U^\beta$$

and, therefore,

$$\int_{\mathbb{T}^d} U^\alpha(t, \cdot) A_{\alpha\beta}^0(t, \cdot) U^\beta(t, \cdot) \leq \exp(\kappa|t - s|/r) \int_{\mathbb{T}^d} U^\alpha(s, \cdot) A_{\alpha\beta}^0(s, \cdot) U^\beta(s, \cdot).$$

Finally:

$$\|U(t, \cdot)\|_{L^2(\mathbb{T}^d)} \leq \|U(s, \cdot)\|_{L^2(\mathbb{T}^d)} \exp(\kappa|t - s|/r), \quad \forall t, s \in \mathbb{R}.$$

N.B. With additional work, one get similar estimates for all  $H^s$  norm for  $s \in \mathbb{N}$  and, once  $s - d/2 > 1$ , we may control the  $C^1$  norm of  $U$  (which is crucial for the fixed-point argument, when addressing nonlinear systems).



### Proof of Theorem 6.3.5

Let  $(t, x) \in [0, T] \times \mathbb{T}^d \rightarrow U(t, x) \in \mathbb{R}^m$  be a weak solution of the ESCL in the sense of Definition 6.3.4 and let  $(t, x) \in [0, T] \times \mathbb{T}^d \rightarrow V(t, x) \in \mathbb{R}^m$  be a smooth function. Let us introduce

$$\eta(u, v) = \mathcal{E}(u) - \mathcal{E}(v) - \mathcal{E}_{,\alpha}(v)(u^\alpha - v^\alpha) \quad \forall u, v \in \mathbb{R}^m,$$

$$\zeta^{i\alpha}(u, v) = \mathcal{F}^{i\alpha}(u) - \mathcal{F}^{i\alpha}(v) - \mathcal{F}^{i\alpha}_{,\gamma}(v)(u^\gamma - v^\gamma) \quad \forall u, v \in \mathbb{R}^m, \quad i \in \{1, \dots, d\}, \quad \alpha \in \{1, \dots, m\}.$$

From the assumptions made on  $\mathcal{E}$  and  $\mathcal{F}$ , we easily get

$$r|u - v|^2 \leq \eta(u, v) \leq |u - v|^2/r, \quad |\zeta(u, v)| \leq C\eta(u, v)$$

(where  $C$  is a constant depending on the sup norm of the second derivatives of  $\mathcal{F}$ ), so that

$$\int_{\mathbb{T}^d} \eta(U(t, x), V(t, x)) dx$$

controls

$$\|U(t, \cdot) - V(t, \cdot)\|_{L^2}^2.$$

Let us perform the following calculations in the sense of distributions on  $]0, T[ \times \mathbb{T}^d$ :

$$\begin{aligned} & \partial_t (\mathcal{E}(V) + \mathcal{E}_{,\alpha}(V)(U^\alpha - V^\alpha)) \\ &= \mathcal{E}_{,\alpha}(V) \partial_t V^\alpha + \mathcal{E}_{,\alpha\beta}(V) \partial_t V^\beta (U^\alpha - V^\alpha) + \mathcal{E}_{,\alpha}(V) (-\partial_i (\mathcal{F}^{i\alpha}(U)) - \partial_t V^\alpha) \end{aligned}$$

(using that  $U$  is a weak solution which gives a rigorous meaning to  $\mathcal{E}_{,\alpha}(V) \partial_i (\mathcal{F}^{i\alpha}(U))$  in the sense of distributions)

$$\begin{aligned} &= \mathcal{E}_{,\alpha\beta}(V) (R^\beta[V] - \mathcal{F}^{i\beta}_{,\gamma}(V) \partial_i V^\gamma) (U^\alpha - V^\alpha) \\ &\quad - \partial_i (\mathcal{E}_{,\alpha}(V) \mathcal{F}^{i\alpha}(U)) + \mathcal{E}_{,\alpha\gamma}(V) \partial_i V^\gamma \mathcal{F}^{i\alpha}(U) \end{aligned}$$

[where we have introduced the "redisual"

$$R^\beta[V] = \partial_t V^\beta + \partial_i (\mathcal{F}^{i\beta}(V)) = \partial_t V^\beta + \mathcal{F}^{i\beta}_{,\gamma}(V) \partial_i V^\gamma$$

which makes  $V \rightarrow R[V]$  a nonlinear operator which vanishes as soon as  $V$  is a  $C^1$  solution of the ESCL, which will be used a little later]

$$\begin{aligned} &= \mathcal{E}_{,\alpha\beta}(V) (U^\alpha - V^\alpha) R^\beta[V] - \mathcal{E}_{,\gamma\beta}(V) \mathcal{F}^{i\beta}_{,\alpha}(V) \partial_i V^\gamma (U^\alpha - V^\alpha) \\ &\quad - \partial_i (\mathcal{E}_{,\alpha}(V) \mathcal{F}^{i\alpha}(U)) + \mathcal{E}_{,\beta\gamma}(V) \partial_i V^\gamma \mathcal{F}^{i\beta}(U) \end{aligned}$$

(where we have crucially used the symmetry property of  $\mathcal{F}$  with respect to  $\mathcal{E}$  and also replaced mute index  $\alpha$  by  $\beta$  in the very last term)

$$= \mathcal{E}_{,\alpha\beta}(V) (U^\alpha - V^\alpha) R^\beta[V] + \mathcal{E}_{,\gamma\beta}(V) \partial_i V^\gamma (\zeta^{i\beta}(U, V) + \mathcal{F}^{i\beta}(V)) - \partial_i (\mathcal{E}_{,\alpha}(V) \mathcal{F}^{i\alpha}(U))$$

(where we have used the definition of  $\zeta$ ). Note that, by definition of  $\mathcal{Q}$ ,

$$\begin{aligned} \mathcal{E}_{,\gamma\beta}(V) \partial_i V^\gamma \mathcal{F}^{i\beta}(V) &= \partial_i (\mathcal{E}_{,\beta}(V) \mathcal{F}^{i\beta}(V)) - \mathcal{F}^{i\beta}_{,\gamma}(V) \mathcal{E}_{,\beta}(V) \partial_i V^\gamma \\ &= \partial_i (\mathcal{E}_{,\beta}(V) \mathcal{F}^{i\beta}(V) - \mathcal{Q}^i(V)). \end{aligned}$$

So, we have obtained, still in the sense of distributions on  $]0, T[ \times \mathbb{T}^d$ ,

$$\begin{aligned} & \partial_t (\mathcal{E}(V) + \mathcal{E}_{,\alpha}(V)(U^\alpha - V^\alpha)) \\ &= \mathcal{E}_{,\alpha\beta}(V)(U^\alpha - V^\alpha)R^\beta[V] + \mathcal{E}_{,\gamma\beta}(V)\partial_i V^\gamma \zeta^{i\beta}(U, V) - \partial_i (\mathcal{Q}^i(V)). \end{aligned}$$

Since  $U$  is a weak solution in the sense of definition 6.3.4, one can write this equation in integral form while incorporating the initial condition  $U_0$ . By doing so, we get for every test function  $\psi(t, x) = \chi(t) \otimes 1$  with  $\chi \in C^\infty(\mathbb{R})$  supported in  $] - \infty, T[$ ,

$$\begin{aligned} & - \int_0^T \chi'(t) \int_{\mathbb{T}^d} (\mathcal{E}(V) + \mathcal{E}_{,\alpha}(V)(U^\alpha - V^\alpha))(t, x) dx dt \\ & - \chi(0) \int_{\mathbb{T}^d} (\mathcal{E}(V(0, x)) + \mathcal{E}_{,\alpha}(V(0, x))(U_0^\alpha(x) - V^\alpha(0, x))) dx \\ &= \int_0^T \chi(t) \int_{\mathbb{T}^d} (\mathcal{E}_{,\alpha\beta}(V)(U^\alpha - V^\alpha)R^\beta[V] + \mathcal{E}_{,\gamma\beta}(V)\partial_i V^\gamma \zeta^{i\beta}(U, V))(t, x) dx dt. \end{aligned}$$

At this stage, we incorporate the term  $\mathcal{E}(U)$  in the left-hand side in order to exhibit

$$\eta(U, V) = \mathcal{E}(U) - \mathcal{E}(V) - \mathcal{E}_{,\alpha}(V)(U^\alpha - V^\alpha).$$

We find (after changing all signs)

$$\begin{aligned} & - \int_0^T \chi'(t) \int_{\mathbb{T}^d} \eta(U, V)(t, x) dx dt = - \int_0^T \chi'(t) \int_{\mathbb{T}^d} \mathcal{E}(U)(t, x) dx dt \\ & - \chi(0) \int_{\mathbb{T}^d} \eta(U_0(x), V(0, x)) dx + \chi(0) \int_{\mathbb{T}^d} \mathcal{E}(U_0(x)) dx \\ & - \int_0^T \chi(t) \int_{\mathbb{T}^d} \mathcal{E}_{,\alpha\beta}(V)(U^\alpha - V^\alpha)R^\beta[V](t, x) dx dt \\ & - \int_0^T \chi(t) \int_{\mathbb{T}^d} \mathcal{E}_{,\gamma\beta}(V)\partial_i V^\gamma \zeta^{i\beta}(U, V)(t, x) dx dt. \end{aligned}$$

Using the assumptions made on  $\mathcal{E}$  and  $\mathcal{F}$ , and assuming now on that  $\chi \geq 0$ , we easily dominate the very last term by

$$c \int_0^T \chi(t) \lambda(t) \int_{\mathbb{T}^d} \eta(U, V)(t, x) dx dt,$$

where we denote by  $\lambda(t)$  the Lipschitz constant in  $x \in \mathbb{T}^d$  of  $V(t, \cdot)$  and by  $c$  a generic constant depending only on functions  $\mathcal{E}$  et  $\mathcal{F}$ . Denoting temporarily

$$\begin{aligned} \theta(t) &= \int_{\mathbb{T}^d} \eta(U, V)(t, x) dx, & h(t) &= \int_{\mathbb{T}^d} \mathcal{E}(U(t, x)) dx, \\ \theta_0 &= \int_{\mathbb{T}^d} \eta(U_0(x), V(0, x)) dx, & h_0 &= \int_{\mathbb{T}^d} \mathcal{E}(U_0(x)) dx, \\ \rho(t) &= \int_{\mathbb{T}^d} (\mathcal{E}_{,\alpha\beta}(V)(U^\alpha - V^\alpha)R^\beta[V])(t, x) dx \end{aligned}$$

we have so obtained

$$-\int_0^T \chi'(t)\theta(t)dt \leq -\int_0^T \chi'(t)h(t)dt + \chi(0)(\theta_0 - h_0) - \int_0^T \chi(t)\rho(t)dt + c \int_0^T \chi(t)\lambda(t)\theta(t)dt.$$

Almost every  $\tau \in [0, T[$  is a Lebesgue point of functions  $\theta$  and  $h$ . In such a point, that we fix for a while, we take  $\epsilon > 0$  small enough so that  $\tau + \epsilon < T$  and we take  $\chi \in C_c^\infty(\mathbb{R})$  so that:

i) for  $t \in [-1, \tau - \epsilon]$ ,  $\chi(t) = 1$  ;

ii) for  $t > \tau + \epsilon$ ,  $\chi(t) = 0$  ;

iii) for  $t \in [\tau - \epsilon, \tau + \epsilon]$ ,  $\chi(t)$  is non increasing. Through the limit  $\epsilon \downarrow 0$ , we get

$$\theta(\tau) \leq h(\tau) + \theta_0 - h_0 - \int_0^\tau \rho(t)dt + c \int_0^\tau \lambda(t)\theta(t)dt.$$

At this point, we crucially use the assumption

$$\int_{\mathbb{T}^d} \mathcal{E}(U)(\tau, x)dx \leq \int_{\mathbb{T}^d} \mathcal{E}(U_0(x))dx$$

which holds true for a.e.  $\tau \in [0, T]$ , i.e.  $h(\tau) \leq h_0$ . We deduce that for a.e.  $\tau \in [0, T]$ ,

$$\theta(\tau) \leq \theta_0 - \int_0^\tau \rho(t)dt + c \int_0^\tau \lambda(t)\theta(t)dt$$

and, using the Gronwall lemma, we have obtained:

**Proposition 6.3.7.** *For a.e.  $t \in [0, T]$ ,*

$$\theta(t) \leq \theta_0 \exp\left(c \int_0^t \lambda(s)ds\right) - \int_0^t \rho(s) \exp\left(c \int_s^t \lambda(\sigma)d\sigma\right)ds.$$

where  $\lambda(t)$  is the Lipschitz constant in  $x \in \mathbb{T}^d$  of  $V(t, \cdot)$ ,  $c$  is a constant depending only on functions  $\mathcal{E}$ ,  $\mathcal{F}$ , and

$$\theta(t) = \int_{\mathbb{T}^d} \eta(U, V)(t, x)dx, \quad \theta_0 = \int_{\mathbb{T}^d} \eta(U_0(x), V(0, x))dx,$$

$$\rho(t) = \int_{\mathbb{T}^d} (\mathcal{E}_{,\alpha\beta}(V)(U^\alpha - V^\alpha)R^\beta[V])(t, x)dx.$$

Assuming that  $V$  is a smooth solution of the ESCL with initial condition  $U_0$ , we automatically get  $R[V] = 0$ , since

$$R^\beta[V] = \partial_t V^\beta + \partial_i(\mathcal{F}^{i\beta}(V)),$$

and  $\theta_0 = 0$ . Thus

$$\int_{\mathbb{T}^d} \eta(U, V)(t, x)dx = 0,$$

for a.e.  $t \in [0, T]$ . Since this quantity dominates, up to a multiplicative positive constant, the squared  $L^2$  norm of  $U(t, \cdot) - V(t, \cdot)$ , we conclude that  $U = V$  which shows the uniqueness of  $V$  among all weak solutions with initial condition  $U_0$  that keep their entropy at time  $t$  below the entropy of  $U_0$ , for a.e.  $t$ . This completes the proof of Theorem 6.3.5.

## 6.4 A convex concept of "dissipative solutions"

During the proof of Theorem 6.3.5, we have established Proposition 6.3.7 which suggest a new concept of generalized solutions for the ESCL. This idea goes back to the works of Dafermos and DiPerna in the 80s. (See [193, 206, 207].) Lions made this concept more explicit in the special case of the Euler equations of incompressible fluids [331], and introduced the wording of "dissipative solutions", that we will conserve in this book, although the word "dissipative solution" is used in different contexts by several authors. Strictly speaking, the Euler equations of incompressible fluids do not belong to the ESCL class. However they are just a limit case and the concept easily goes through. The main observation is that the inequality obtained in Proposition 6.3.7 is *convex* with respect to solution  $U$ . Indeed,  $\eta(U, V)$  is convex in  $U$  by definition, and, in the right-hand side, only feature linear terms in  $U$ . This is a very fruitful property which easily provides some weak compactness. More precisely, let us introduce the space  $C_w^0([0, T], L^2(\mathbb{T}^d; \mathbb{R}^m))$  of all functions

$$U : t \in [0, T] \rightarrow U(t, \cdot) \in L^2(\mathbb{T}^d; \mathbb{R}^m)$$

which are continuous in  $t$  with respect to the *weak* topology of  $L^2(\mathbb{T}^d; \mathbb{R}^m)$ , i.e. such that, for each function  $\psi \in L^2(\mathbb{T}^d; \mathbb{R}^m)$ ,

$$t \in [0, T] \rightarrow \int_{\mathbb{T}^d} U^\alpha(t, x) \psi_\alpha(x) dx$$

is continuous.

**Definition 6.4.1.** *We say that  $U \in C_w^0([0, T], L^2(\mathbb{T}^d; \mathbb{R}^m))$  is a "dissipative solution" of the ESCL with initial condition  $U_0$  if  $U(0, \cdot) = U_0$  and the inequality established in Proposition 6.3.7 holds true for all smooth function  $V$ .*

Then, it is immediate to check:

**Proposition 6.4.2.** *Given  $U_0 \in L^2(\mathbb{T}^d; \mathbb{R}^m)$ , the set of all dissipative solutions of the ESCL with initial condition  $U_0$ :*

*i) is convex (if not empty!)*

*ii) has a single element as soon as the ESCL admits a smooth solution  $U$  with initial value  $U_0$  and this element is precisely  $U$ .*

This result is far from being satisfactory. However, it turns out that:

- i) it is usually possible (although sometimes quite technical) to get an existence proof through suitable approximations enjoying the same type of weak compactness, and for *arbitrarily* long time interval, which is usually impossible for smooth solutions;
- ii) the concept is very useful to show that the ESCL can be rigorously derived from a more fundamental model by passing to the limit with suitable small parameters.

Let us quote the example of the Euler equations of incompressible fluids that can be derived from the Navier-Stokes equations [331] or from the Boltzmann equation [413]. More generally, relative entropy methods have been used in many problems of asymptotic analysis. Let us just quote few examples [57, 94, 122, 217, 238, 265, 270, 316, 403, 420, 457]. In such cases, the relative entropy approach has been a useful alternative to compactness methods such as Young's measures, currents or varifolds, compensated compactness, averaging lemma, semi-classical or microlocal defect measures (see [27, 78, 132, 210, 253, 276, 332, 334, 379, 407, 427, 434, 440, 441, 442]...)



# Chapter 7

## Hidden convexity in some models of Convection

Convection is one of the most important phenomena in natural sciences (oceanography, volcanism, continental drift, terrestrial magnetism, etc...) [163, 187, 394] and also in daily life (weather, heating and boiling!). It describes in particular the way that incompressible fluids move under the differential action of gravity caused by their inhomogeneity which, itself, results of difference of mass, temperature, salinity, etc...Typically fluid parcels try to rearrange themselves in order to reach more stable states (typically, heavy parcels at bottom and light ones at top), which creates motion and, therefore, generates new instabilities and so on. In this chapter, we discuss some crude convection models derived from the Euler or Navier-Stokes equations of incompressible fluids including additional terms describing buoyancy and Coriolis forces in some suitable asymptotic regimes of physical interest. Some of these models will be shown to exhibit some hidden convexity, in close relationship with the concept, well known in optimal transport theory, of rearrangement of maps as maps with convex potential, as we have already seen in this book on Section 3.2.

### 7.1 A caricatural model of climate change

Let  $D$  be a smooth bounded domain  $D \subset \mathbb{R}^3$  (or, alternately, the torus  $\mathbb{T}^3$ ) in which moves an incompressible fluid of velocity  $v(t, x)$  at  $x \in D$ ,  $t \geq 0$ , subject to the Navier-Stokes-Boussinesq (NSB) equations

$$\begin{aligned}(\partial_t v + v \cdot \nabla)v - \nu \Delta v + \nabla p &= y, \\ (\partial_t + v \cdot \nabla)y &= \epsilon G(\epsilon t, x)\end{aligned}$$

with  $\nabla \cdot v = 0$  and  $v = 0$  along  $\partial D$ .

The field  $y = y(t, x) \in \mathbb{R}^3$  is a "generalized buoyancy", vector-valued, force, with a small, slowly evolving, source term, where  $G$  is a given smooth function with bounded derivatives.

We can see these equations as a caricatural model of climate change: we look for the long time impact of a small, slowly evolving, source term of amplitude  $\epsilon$  on long time scales of order  $\epsilon^{-1}$ .

By substituting  $(t, v, p, y)$  for  $(\epsilon t, \epsilon v, p, y)$  in the NSB equations, we get the following rescaled RNSB equations

$$(\text{RNSB}) \quad y = \nabla p + \epsilon^2(\partial_t v + (v \cdot \nabla)v) - \epsilon \nu \Delta v, \quad \nabla \cdot v = 0, \quad \partial_t y + (v \cdot \nabla)y = G(t, x).$$

We call "hydrostatic Boussinesq" HB equations, the formal limit obtained for  $\epsilon = 0$ :

$$y = \nabla p, \quad \nabla \cdot v = 0, \quad \partial_t y + (v \cdot \nabla)y = G(t, x).$$

**Remark 1**

In the concrete convection model considered in [113], there is no  $x_2$  dependence and  $G_1 = 0$ . Then the force field  $y$  is vector-valued and combines both Coriolis (in the  $x_1$  direction) and buoyancy (in the  $x_3$  direction) effects. The  $\epsilon \rightarrow 0$  limit is, then, related to the Hoskins "x-z" semi-geostrophic equations [190, 294]. (See also [9, 46, 188, 189, 340]...)

**Remark 2**

From the PDE viewpoint, global existence of weak solutions in 3D follows from Leray [321] and Diperna-Lions [209] (see also [383]).

**Remark 3**

For any suitable test function  $f$  we have INDEPENDENTLY of  $\epsilon, v$  the following key property

$$\frac{d}{dt} \int_D f(y(t, x)) dx = \int_D (\nabla f)(y(t, x)) \cdot G(t, x) dx$$

This is valid even for the Leray weak solutions, thanks to DiPerna-Lions' theory on ODEs [209].

**Remark 4**

When both the source term and the initial force are gradients and the fluid initially is at rest

$$G = G(x) = \nabla g(x), \quad y(0, x) = \nabla p_0(x), \quad v(0, x) = 0,$$

then the rescaled NSB system has a trivial but interesting "convection-free" solution, independently of  $\epsilon$ , namely

$$v(t, x) = 0, \quad y(t, x) = \nabla p(t, x), \quad p(t, x) = p_0(x) + tg(x).$$

Of course, these solutions are also trivial solutions to the HB system.

## 7.2 Hidden convexity in the Hydrostatic-Boussinesq system

The Hydrostatic Boussinesq system

$$(HB) \quad y = \nabla p, \quad \nabla \cdot v = 0, \quad \partial_t y + (v \cdot \nabla)y = G(t, x),$$

we have formally obtained by setting  $\epsilon$  to zero in the rescaled Navier-Stokes-Boussinesq equations looks strange since there is no evolution equation for  $v$ . However, we have a constraint for  $y$ , namely to be a gradient. Thus, we can recover  $v$  as a kind of Lagrange multiplier of this constraint. Indeed, notice first that,

$$(v \cdot \nabla)y = (D_x^2 p \cdot v)$$

and  $v = \nabla \times A$ , for some divergence-free vector potential  $A = A(t, x) \in \mathbb{R}^3$ , at least when  $d = 3$ . Then, taking the curl of the evolution equation in the HB system, we get

$$\nabla \times (D_x^2 p(t, x) \cdot \nabla \times A) = \nabla \times G.$$

At each fixed time  $t$ , knowing  $p$ , this is just a linear "magnetostatic" system in  $A$ , which is elliptic whenever  $p$  is convex in the strong sense

$$(SCC) \quad c Id < D_x^2 p(t, x) < c^{-1} Id, \quad \forall x,$$

for some constant  $c > 0$  that may depend on  $t$ . This strongly suggests that the HB system is well-posed, under this strong convexity assumption, which, presumably, is sustainable, at least on short time intervals. It is a typical example of hidden convexity! This intuition is indeed correct and was proven by Loeper (for a specific choice of  $G$ , but his method goes through the general case of a smooth function  $G$  with bounded derivatives), using a Monge-Ampère reformulation of the system [340]. The proof has been obtained by Loeper only in the case of a periodic domain, such as  $D = \mathbb{T}^3$ . This periodic setting requires a little bit of care: the pressure  $p(t, x)$  should be understood as the sum of  $|x|^2/2$  and a  $\mathbb{Z}^3$ -periodic function  $p'(t, x)$ , the strong convexity condition meaning

$$c Id < Id + D_x^2 p'(t, x) < c^{-1} Id, \quad \forall x,$$

for some constant  $c > 0$ . Accordingly,  $y(t, x) - x = \nabla p'(t, x)$  is also a  $\mathbb{Z}^3$ -periodic, vector-valued function, just as  $v(t, x)$ . Notice that this condition implies that the Legendre-Fenchel transform of  $p$ , defined as usual by

$$p^*(t, y) = \sup_{x \in \mathbb{R}^d} x \cdot y - p(t, x),$$

also satisfies

$$c Id < D_x^2 p^*(t, y) < c^{-1} Id, \quad \forall y.$$

As a consequence, both  $x \rightarrow \nabla p(t, x)$  and  $y \rightarrow \nabla p^*(t, y)$  define global orientation-preserving diffeomorphisms of  $\mathbb{R}^3$ .

## Derivation of the HB model under strong convexity condition

The strong convexity condition (SCC) is sufficient to get a rigorous derivation of the HB equations from the RNSB equations as  $\epsilon$  goes to zero, at least in the case of a periodic domain.

**Theorem 7.2.1.** *Let  $D = \mathbb{T}^3$ . Assume  $G$  to be smooth with bounded derivatives up to second order. Let  $(y^\epsilon, v^\epsilon, p^\epsilon)$  be a Leray-type solution to the RNSB equations. Let  $(y = \nabla p, v)$  be a smooth solution to the HB equations on a given finite time interval  $[0, T]$ . Assume that the strong convexity condition (SCC) is satisfied up to time  $T$ . Then, the  $L^2$  distance between  $y^\epsilon$  and  $y$  stays uniformly of order  $\sqrt{\epsilon}$  as  $\epsilon$  goes to zero, uniformly in  $t \in [0, T]$ , provided it does at  $t = 0$  and the initial velocity  $v^\epsilon(t = 0, x)$  stays uniformly bounded in  $L^2$ .*

Let us just tell a brief idea of the proof. (See [106] for a detailed proof.) A natural but very faulty idea would be to compare  $y^\epsilon$  and  $y$  directly in  $L^2$  (or more generally Sobolev) norm and try to get Gronwall-type differential inequalities for it.



This method completely fails, due to the presence of an irreducible term of size  $\epsilon^{-1}$ . The right idea is to consider the "relative entropy"

$$\int_D \left\{ K(t, y^\epsilon(t, x), y(t, x)) + \frac{\epsilon^2}{2} |v^\epsilon - v|^2 \right\} dx$$

where

$$K(t, y', y) = p^*(t, y') - p^*(t, y) - \nabla p^*(t, y) \cdot (y' - y) \sim |y - y'|^2,$$

where  $p^*$  is the Legendre-Fenchel transform of  $p$ . Then we can get a Gronwall estimate to deduce that the relative entropy, which is small at time  $t$ , cannot grow more than exponentially in time with a rate that depends on the smoothness of  $p^*$ . This is enough to get convergence as  $\epsilon$  goes to zero.

*Remark.*

Notice the remarkable feature of this "relative entropy" with respect to the previous relative entropies discussed earlier in this book. Instead of a universal convex function which is expanded about all possible limit solutions as we have seen so far in the previous sections, here the convex function reads

$$(v, y) \rightarrow p^*(t, y) + \frac{\epsilon^2}{2} |v|^2,$$

is not at all universal and involves the limit solution  $p^*$  itself!

### Breakdown of convexity and concept of "entropy" solutions

Unfortunately, we cannot expect the strong condition (SCC) to be sustainable for large times. This can be seen immediately with the trivial solutions already mentioned, namely:

$$v(t, x) = 0, \quad y(t, x) = \nabla p(t, x), \quad p(t, x) = p_0(x) + tg(x)$$

Indeed, it is sufficient to have a source term  $G = \nabla g$ , with  $D^2g(x) \leq -cId$  for some positive constant  $c$ , to fail the strong convexity condition in finite time. However, these trivial solutions, of both the HB and the RNSB system, can be expected to be dynamically very unstable solutions of the RNSB equations, especially as  $\epsilon$  gets smaller and smaller. This is why, it seems reasonable to look for solutions of the HB system which keep the convexity condition, at least in the large sense

$$D^2p(t, x) \geq 0.$$

In the framework of semi-geostrophic equations [190, 294], this condition is called the Cullen-Purser condition [190]. By analogy with the theory of hyperbolic conservation laws we rather call this convexity condition "entropy condition".

The main point now is that any "entropy solution"  $y(t, x) = \nabla p(t, x)$ , square integrable at each time  $t$ , can be entirely recovered by the knowledge of all "observables"

$$f \rightarrow \int_D f(y(t, x)) dx,$$

for all continuous function  $f$  with at most quadratic growth at infinity. This is a direct consequence of the optimal transport theorem 3.2.1. Now, we have already obtained an evolution equation for all these observables, namely

$$\frac{d}{dt} \int_D f(y(t, x)) dx = \int_D (\nabla f)(y(t, x)) \cdot G(t, x) dx$$

which is valid for the RNSB equations independently of both  $v$  and  $\epsilon$ . This suggests the following concept of "entropy" solution for the HB system:

**Definition 7.2.2.** *We say that  $(t \rightarrow y(t, \cdot)) \in C^0(\mathbb{R}_+, L^2(D, \mathbb{R}^3))$  is an entropy to the HB system*

$$(HB) \quad y = \nabla p, \quad \nabla \cdot v = 0, \quad \partial_t y + (v \cdot \nabla)y = G(t, x),$$

if, for each time  $t$ ,  $y = \nabla p$  is a map with convex potential  $p$  and if

$$\frac{d}{dt} \int_D f(y(t, x)) dx = \int_D (\nabla f)(y(t, x)) \cdot G(t, x) dx,$$

for all  $C^1$  function  $f$  with  $\sup_y (1 + |y|)^{-1} |\nabla f(y)| < \infty$ .

### Global existence of "entropy" solutions for the HB system

The global existence of entropy conditions is an easy consequence of the convergence of the following time-discrete scheme with time step  $\tau > 0$ , where we approximate  $y(t = n\tau, x)$  by  $y_{n,\tau}(x)$ , for  $n = 0, 1, 2, \dots$ , as follows:

i) we first perform a predictor step:  $\tilde{y}_{n+1,\tau}(x) = y_n(x) + \tau G(x)$ .

ii) then, the corrector step amounts to perform a rearrangement as a map with convex potential:  $y_{n+1,\tau} = (\tilde{y}_{n+1,\tau})^\# = \nabla p_{n+1,\tau}$  where  $p_{n+1,\tau}$  is convex (in the large sense of  $D^2 p_{n+1,\tau} \geq 0$ ).

Observe that the last step is possible thanks to the optimal transport theory we have discussed earlier in this book. It is indeed enough to apply Theorem 3.2.1 to get  $\nabla p_{n+1}$  as the unique gradient of a convex function that transports the Lebesgue measure on  $D$  to its image by map  $x \rightarrow \tilde{y}_{n+1,\tau}(x)$ .

**Theorem 7.2.3.** *As  $\tau \rightarrow 0$ , the time-discrete scheme has converging subsequences. Each limit  $y$  belongs to the space  $C^0(\mathbb{R}_+, L^2(D, \mathbb{R}^d))$ , admits a convex potential:  $y(t, \cdot) = \nabla p(t, \cdot)$  for each  $t \geq 0$  and satisfies*

$$\frac{d}{dt} \int_D f(y(t, x)) dx = \int_D (\nabla f)(y(t, x)) \cdot G(t, x) dx$$

for all smooth function  $f$  such that  $\sup_y (1 + |y|)^{-1} |\nabla f(y)| < \infty$ . This exactly means that  $y$  is a global entropy solution to the HB equations in the sense of Definition 7.2.2.

The proof is rather easy and can be found in [106]. Let us just check the consistency of the scheme, in the special case  $G = G(x)$ . Given a smooth function  $f$ , we get

$$\int_D f(y_{n+1,\tau}(x)) dx = \int_D f(\tilde{y}_{n+1}(x)) dx$$

(because  $y_{n+1,\tau}$  is a rearrangement of  $\tilde{y}_{n+1}$ )

$$= \int_D f(y_{n,\tau}(x) + \tau G(x)) dx$$

(by definition of predictor  $\tilde{y}_{n+1,\tau}$ )

$$= \int_D f(y_{n,\tau}(x))dx + \tau \int_D (\nabla f)(y_{n,\tau}(x)) \cdot G(x)dx + O(\tau^2),$$

which, indeed, means that the time-discrete scheme is consistent.

## 7.3 The 1D time-discrete rearrangement scheme

Remarkably enough, the rearrangement scheme we have just introduced still makes perfect sense in one space dimension, although it has been derived from a model of incompressible fluids requiring at least 2 space dimensions. We should not be surprised by this paradoxical phenomenon after all the time we have devoted to the generalized formulations of the Euler equations in the first part of this book (cf. section 2.4)!

As a matter of fact, it is quite interesting to look at the 1D case. First, because the analysis of convergence can be very much improved thanks to the theory of scalar conservation laws already discussed in this book. Second, because the discrete scheme makes sense as a crude model of 1D, "column", convection. Finally and unexpectedly, it also admits interesting interpretations in the field of social sciences.

### Rearrangement in increasing order

Before revisiting the time-discrete scheme in 1D, let us recall the well-known fact of Analysis (see [327] for example). Any  $L^2$  real-valued function

$$x \in [0, 1] \rightarrow z(x),$$

admits a unique rearrangement in increasing order, i.e. a unique non decreasing  $L^2$  function  $z^\sharp$  such that,

$$\int_{[0,1]} f(z^\sharp(x))dx = \int_{[0,1]} f(z(x))dx$$

for all continuous function  $f$  with at most quadratic growth.

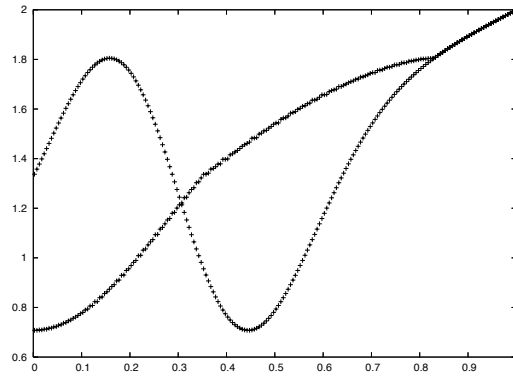
Notice that in the discrete case when

$$z(x) = Z_j, \quad j/N < x < (j+1)/N, \quad j = 0, \dots, N-1,$$

then  $z^\sharp(x) = Z_j^\sharp$  where  $(Z_1^\sharp, \dots, Z_N^\sharp)$  is just  $(Z_1, \dots, Z_N)$  sorted in increasing order. (Of course, this result is just a special occurrence of the optimal transport theorem 3.2.1.)

## A function and its rearrangement in increasing order

$N = 200$  grid points in  $x$



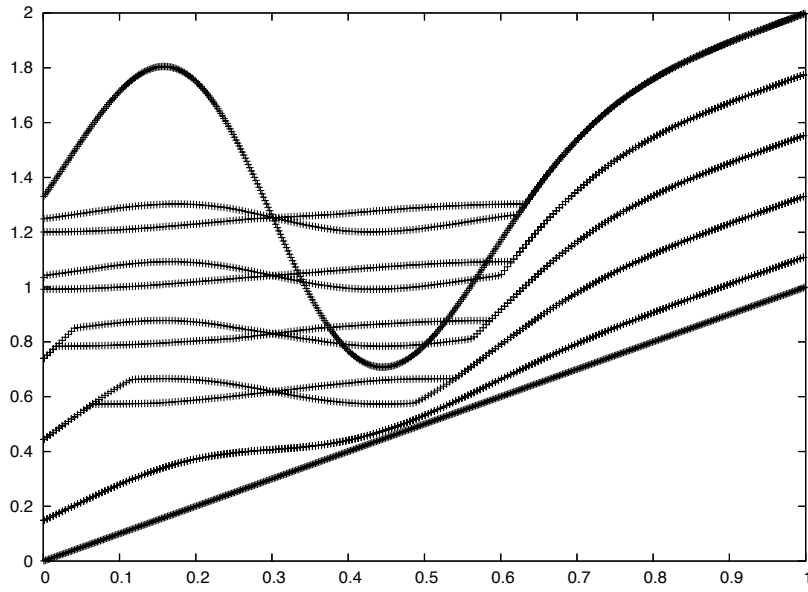
### The 1D rearrangement-scheme as a very crude model of column convection

We consider a vertical column  $x \in [0, 1]$  and denote by  $y(t, x)$  the temperature field along the column. We assume the existence of a steady source of heat along the column:  $G = G(x)$ . The convection model is described through the following time-discrete scheme with time step  $\tau > 0$ , and two sub-steps:

-predictor (heating):  $\tilde{y}_{n+1,\tau}(x) = y_{n,\tau}(x) + \tau G(x)$

-corrector ("instantaneous" convection):  $y_{n+1,\tau} = (\tilde{y}_{n+1,\tau})^\sharp$

so that the temperature profile stays monotonically increasing at *each* time step. (This actually corresponds to a succession of stable equilibria with a boost of heating at each time step.) We see that we exactly recover, in its 1D version, the time-discrete scheme introduced in the previous section in several space dimensions.



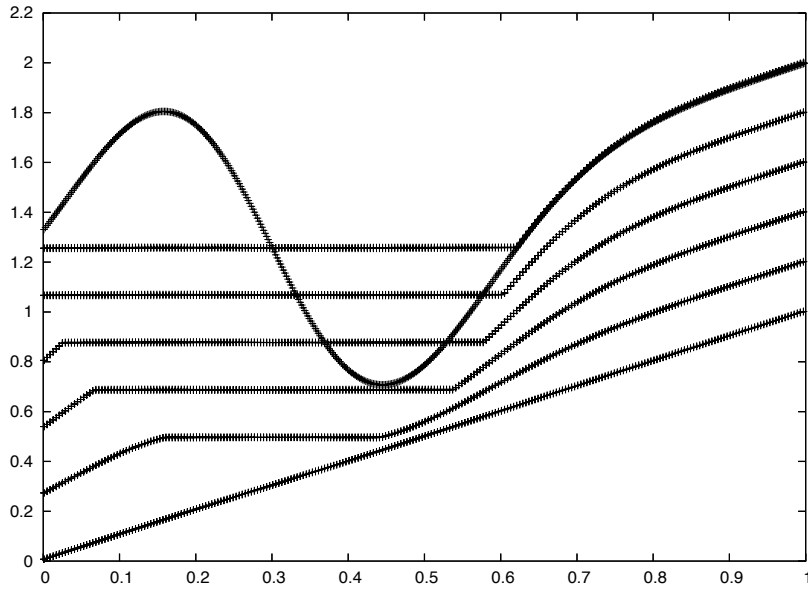
Column convection.

Heat profiles at different times with a rough time step

Data:  $G(x) = 1 + \exp(-25(x - 0.2)^2) - \exp(-20(x - 0.4)^2)$

$t, x \in [0, 1]$   $\tau = 0.1$  (= 10 time steps) 500 grid points in  $x$ ,

$y=y(t,x)$  versus  $x$  drawn every 2 time steps (predictor and corrector).



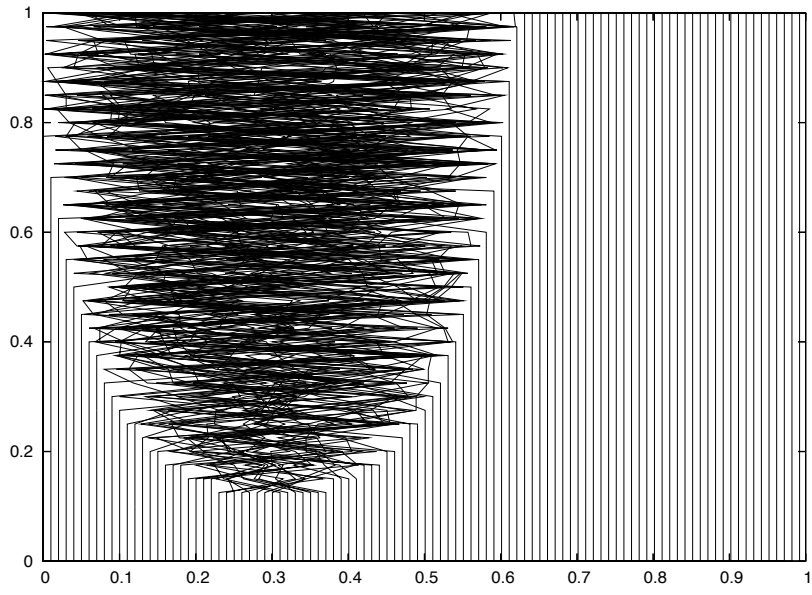
Column convection.

Heat profiles at different times with a fine time step

Data:  $G(x) = 1 + \exp(-25(x - 0.2)^2) - \exp(-20(x - 0.4)^2)$

$t, x \in [0, 1]$   $\tau = 0.005$  (= 200 time steps) 500 grid points in  $x$ ,

$y=y(t,x)$  versus  $x$  drawn every 40 time steps (predictor and corrector).



Column convection.  
Drawing of the temperature mixing zone.



## Convergence analysis

Using the classical theory of maximal monotone operators [129], it is fairly easy to prove

**Theorem 7.3.1.** *As  $\tau \rightarrow 0$ , the time-discrete scheme has a unique limit  $y = y(t, x)$ , monotonically increasing in  $x$ , characterized as the unique solution in  $C^0(\mathbb{R}_+, L^2(D, \mathbb{R}^d))$  of the subdifferential inclusion:*

$$G(x) \in \partial_t y + \partial C[y], \quad y(t = 0, \cdot) = y_0,$$

where  $C[y] = 0$  or  $+\infty$ , according to whether or not  $y$  is a non decreasing function of  $x$ .

In addition, the cumulative function  $u(t, s) = \int_0^1 1\{y(t, x) < s\} dx$ , which is the "pseudo-inverse" function of  $y$ , is an entropy solution to the scalar conservation law

$$\partial_t u + \partial_s(g(u)) = 0, \quad g(v) = \int_0^v G(w) dw.$$

The second statement of this theorem is not a surprise. Indeed, the scheme we have described is nothing but the "transport-collapse" method [82, 80], that we have already used in the framework of Panov's formulation of multidimensional scalar conservation laws. (See section 6.2.)

## Qualitative features

Scalar conservation laws such as  $\partial_t u + \partial_s(g(u)) = 0$ , are known to produce in finite time solutions  $s \rightarrow u(t, s)$  with discontinuities, known as "shock waves". For the temperature field  $x \rightarrow y(t, x)$ , this means the formation of a plateau, which corresponds to a zone where the temperature field is homogenized. In the canonical example  $G = G(x) = 1 - x$ , corresponding to the famous "inviscid" Burgers equation  $\partial_t u + \partial_s(u - u^2/2) = 0$ , it can be shown that, for all initial conditions, a single plateau forms for large  $t$ , which corresponds to a perfectly homogenized temperature. For functions like  $G(x) = 1 - \cos(3\pi x)$ , the long-time behavior is more complex, featuring a central plateau surrounded by two tails, one cold at bottom and one hot at top.

## 7.4 Related models in social sciences

### A model of competition by rank

For  $N$  agents (factories, researchers, universities...) in competition, we denote by  $X_{n,\tau}(\alpha)$  the cumulated production of agent  $\alpha = 1, \dots, N$  at time  $n\tau$ ,  $n \in \mathbb{N}$ , where  $\tau > 0$  is the time step, and by  $\sigma_{n,\tau}(\alpha)$  the rank of agent  $\alpha$  at time  $n\tau$ , in reverse order so that  $\sigma_{n,\tau}(\alpha) = N$  (resp.  $= 1$ ) for the agent  $\alpha$  with highest (resp. lowest) production at time  $n$  and  $\sigma_{n,\tau}$  can be seen as an element of the symmetric group  $\mathfrak{S}_N$ .

Then, the model assumes the existence of a bounded function  $G$  defined on  $[0, 1]$  such that

$$X_{n+1,\tau}(\alpha) = X_{n,\tau}(\alpha) + \tau G(N^{-1}\sigma_{n,\tau}(\alpha))$$

which means that the production between two different times depends only on the ranking.

For example  $G(u) = 1 - u$  describes an equalitarian behaviour where the top people slow down their production while the bottom people catch up as fast as possible. A choice like  $G(u) = 1 - \cos(3\pi u)$  seems more realistic: bottom people are discouraged while top people get even more competitive:

$$G(0) = 0, \quad G(1/3) = 2, \quad G(2/3) = 0, \quad G(1) = 2.$$

We observe that the corresponding sorted sequence  $Y_{n,\tau} = X_{n,\tau}^\sharp$  satisfies:

$$Y_{n+1,\tau} = (Y_{n,\tau} + \tau G)^\sharp,$$

which is just a space-discrete version of the rearrangement-scheme discussed in the previous sub-section.

### Tax on capital according to rank

We denote by  $Z_n(\alpha) \geq 0$  the capital for year  $n$  of each tax-payer  $\alpha \in \{1, \dots, N\}$ . We introduce  $\sigma_n(\alpha) \in \{1, \dots, N\}$  the (reverse) rank of the capital of taxpayer  $\alpha$  at year  $n$ . We assume

$$Z_{n+1}(\alpha) = Z_n(\alpha) \exp(r\tau) \exp(-F(N^{-1}\sigma_n)\tau)$$

where  $\tau$  is the time step,  $r$  is the capital growth, which we assume, very crudely, to be the same for each tax-payer, while the taxation rate depends only on the rank through a given real bounded function  $F$  defined on  $[0, 1]$ .

Thus we recover for  $X_{n,\tau} = \log Z_n$  exactly the same scheme we had in the previous model, namely,

$$X_{n+1,\tau}(\alpha) = X_{n,\tau}(\alpha) + \tau G(N^{-1}\sigma_{n,\tau}(\alpha))$$

just by setting

$$G(u) = r - F(u), \quad \forall u \in [0, 1].$$

The social science interpretation is that, depending on the choice of  $G$ , different policies may be enforced. For instance, an equalitarian policy can be obtained by homogenizing the capital of the different taxpayers (with a final discrepancy of order  $O(\tau)$ ) which will hold true provided that  $G$  satisfies the condition

$$g(u) = \int_0^u G(v)dv > g(0) = g(1), \quad \forall u \in ]0, 1[,$$

which corresponds to the formation of a single shock wave. For different choices of function  $G$ , several shock waves may form, leading to a segmentation of the taxpayers in different homogenized classes.



# Chapter 8

## Augmentation of conservation laws with polyconvex entropy

This chapter closely follows the papers [98, 124, 215] by Xianglong Duan, Wenan Yong and the author. We discuss two examples: the nonlinear theory of Electromagnetism designed in 1934 by Max Born et Leopold Infeld [73]; the theory of time-like extremal surfaces in the Minkowski space, at least of those which can be written as graphs. In terms of applications, both examples are well known in High Energy Physics (String Theory and "Dirichlet-branes") [401]. In both cases, we get system of first order conservation laws with non-convex entropy. So, we cannot directly apply the concepts of relative entropy and dissipative solutions already discussed in this book (section 6). However, it turns out that, in each case, the entropy is a "polyconvex" function, in the sense that it is a convex function of some nonlinear combination of the unknowns (cf. [30]). For instance, in the second case, the unknowns are matrix-valued and the entropy is a convex function of the minors of the corresponding matrices. Then, the basic idea amounts to find extra-conservation laws for these extra-variables and trying to get an enlarged system of conservation laws, with the hope there is a convex entropy for the augmented system. To the best of our knowledge, this idea has been first successfully applied by Qin to a large class of models in non-linear Elasticity [404]. In the two examples covered in this chapter, there is an additional remarkable property. Indeed, we can rewrite the augmented systems in the amazingly simple non-conservative form:

$$\partial_t U_\alpha + A_\alpha^{i\beta\gamma} U_\gamma \partial_i U_\beta = 0,$$

(with implicit summation on repeated indices), where  $U = U(t, x) \in \mathbb{R}^m$ ,  $x \in \mathbb{R}^d$ , and the coefficients  $A_\alpha^{i\beta\gamma}$  are constant. So these systems look like non-trivial generalizations of the famous inviscid version of the Burgers equation, namely:

$$\partial_t u + u \partial_x u = 0.$$

In addition, for each fixed  $i = 1, \dots, d$ ,  $\gamma = 1, \dots, m$ , the  $A_\alpha^{i\beta\gamma}$  form a symmetric  $m \times m$  matrix in  $\alpha, \beta$ . This is enough, with any further effort, to guarantee [193, 350] that the initial value problem (IVP) is locally well-posed in all Sobolev spaces  $H^s(\mathbb{R}^d)$  with continuous injection in  $C^1$ , i.e. for all  $s > 1 + d/2$ .

## 8.1 The Born-Infeld equations

In 1934, Max Born and Leopold Infeld introduced a non-linear correction of the classical Maxwell model. This amounts to finding critical points (with respect to compactly supported perturbations)

$$(t, x) \in \mathbb{R}^{1+3} \rightarrow (E, B)(t, x) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

of the following action

$$A_\lambda[E, B] = \int \int (1 - \sqrt{1 + \lambda^{-2}(B^2 - E^2) - \lambda^{-4}(B \cdot E)^2}) \, dxdt$$

where  $\lambda > 0$  is a physical constant (the "absolute field"), under constraints

$$\nabla \cdot B = 0, \quad \partial_t B + \nabla \times E = 0.$$

In the "low-field" limit  $\lambda \rightarrow \infty$ , the classical Maxwell model is recovered

$$\lambda^2 A_\lambda[E, B] \sim \frac{1}{2} \int (E^2 - B^2) \, dxdt$$

leading to the famous (homogeneous) Maxwell equations

$$\partial_t B + \nabla \times E = 0, \quad \partial_t E = \nabla \times B, \quad \nabla \cdot B = \nabla \cdot E = 0.$$

Originally designed for Quantum-Electrodynamics (without real success [239]), the Born-Infeld model has attracted since a lot of very different fields (from String Theory [401] to Quantum Electrodynamics, Fluid Dynamics and Numerical Analysis [71, 203, 290, 302, 415, 444]!

### The electrostatic case

The electrostatic case is consistently obtained by canceling the magnetic field  $B$ :

$$A_\lambda[E, 0] = \int \int (1 - \sqrt{1 - \lambda^{-2}E^2}) \, dxdt$$

under constraint

$$\nabla \times E = 0.$$

So, the constant  $\lambda > 0$  just appears as the maximal possible electrostatic field in the theory (just like 1 is the maximal possible velocity in Special Relativity). This was Max Born's original idea (inspired by earlier ideas of Gustav Mie).

### Remark: a more general and geometric definition

For a general  $1 + d$  dimensional Lorentzian manifold with metric  $g_{ij}dx^i dx^j$  the BI model involves a closed 2-form  $\mathcal{B} = B_{ij}dx^i \wedge dx^j$  and the Born-Infeld Action now reads

$$A_\lambda[g, \mathcal{B}] = \int (\sqrt{-\det g} - \sqrt{-\det(g + \lambda \mathcal{B})}).$$

Notice that this Action is "fully covariant", i.e. invariant as  $g$  and  $\mathcal{B}$  are deformed by any space-time diffeomorphism. (Indeed, there is an exact compensation between

the determinant and the modifications brought to  $g_{ij}dx^i dx^j$  and  $B_{ij}dx^i \wedge dx^j$  by any diffeomorphism

$$x = (x^0, \dots, x^d) \in \mathbb{R}^{1+d} \rightarrow \Phi(x) \in \mathbb{R}^{1+d}.$$

Of course, in the special case  $d = 3$ ,  $g = \text{diag}(-1, 1, 1, 1)$ , one may recover (through an elementary but instructive calculation, involving elementary linear algebra and properties of  $4 \times 4$  skew symmetric matrices) the previous formulae introduced in 1934 in the special case of the standard 1+3 Minkowski space.

### **Remark: high-field limit of the Born-Infeld model and Magnetohydrodynamics**

The original Born-Infeld model

$$A_\lambda[E, B] = \int \int (1 - \sqrt{1 + \lambda^{-2}(B^2 - E^2) - \lambda^{-4}(B \cdot E)^2}) \, dxdt$$

$$\nabla \cdot B = 0, \quad \partial_t B + \nabla \times E = 0$$

admits an interesting "high-field" limit obtained as  $\lambda \rightarrow 0$ , namely, at least formally,

$$\lambda A_\lambda[E, B] \sim - \int \int \sqrt{B^2 - E^2} \, dxdt$$

under the additional pointwise constraint  $E \cdot B = 0$ . This pointwise constraint  $E \cdot B = 0$  is equivalent to  $E = B \times v$  for some new field  $v = v(t, x)$ . This leads to

$$\lambda A_\lambda[E, B] \sim - \int \int \sqrt{B^2(1 - v^2) + (B \cdot v)^2} \, dxdt$$

with differential constraints

$$\nabla \cdot B = 0, \quad \partial_t B + \nabla \times (B \times v) = 0$$

which can be interpreted as the "induction equation" in ideal Magnetohydrodynamics [22, 59, 255, 300, 374], where  $B$  and  $v$  may be seen respectively as the magnetic field and the velocity field of a charged fluid.

### **The Born-Infeld equations in Hamiltonian form**

After normalization  $\lambda = 1$ , written in Hamiltonian form, the Born-Infeld equations read

$$\partial_t B + \nabla \times \left( \frac{B \times (D \times B) + D}{\sqrt{1 + D^2 + B^2 + (D \times B)^2}} \right) = 0, \quad \nabla \cdot B = 0,$$

$$\partial_t D + \nabla \times \left( \frac{D \times (D \times B) - B}{\sqrt{1 + D^2 + B^2 + (D \times B)^2}} \right) = 0, \quad \nabla \cdot D = 0.$$

As shown by Speck [432], using Klainerman's null forms, global smooth solutions to the initial value problem have been proven to uniquely exist for small localized initial conditions. We are going to follow a very different way to analyse the Born-Infeld equations, by augmenting the system and finding a suitable convex "entropy function".

## The energy-momentum conservation laws

By Noether's theorem, since the Born-Infeld Action is manifestly invariant under time and space translations in the Minkowski space  $\mathbb{R}^{1+3}$ , we expect four extra conservation laws. There calculation is elementary but not completely obvious:

$$\partial_t Q + \nabla \cdot \left( \frac{Q \otimes Q - B \otimes B - D \otimes D}{h} \right) = \nabla \left( \frac{1}{h} \right), \quad \partial_t h + \nabla \cdot Q = 0$$

for the energy and momentum fields

$$h = \sqrt{1 + D^2 + B^2 + (D \times B)^2}, \quad Q = D \times B.$$

## The augmented Born-Infeld system

Following [98] we define the 10 by 10 *augmented* Born-Infeld system (ABI) as the original BI system augmented by the 4 energy-momentum conservation laws

$$\begin{aligned} \partial_t B + \nabla \times \left( \frac{B \times Q + D}{h} \right) &= \partial_t D + \nabla \times \left( \frac{D \times Q - B}{h} \right) = 0 \\ \partial_t Q + \nabla \cdot \left( \frac{Q \otimes Q - B \otimes B - D \otimes D}{h} \right) &= \nabla \left( \frac{1}{h} \right), \quad \partial_t h + \nabla \cdot Q = 0 \end{aligned}$$

while disregarding the original algebraic constraints

$$h = \sqrt{1 + D^2 + B^2 + (D \times B)^2}, \quad Q = D \times B,$$

which define a 6 dimensional algebraic submanifold in the space  $(h, Q, D, B) \in \mathbb{R}^{10}$  that we call the "BI manifold".

## The ABI system in non-conservative variables

Here, our analysis follows [124] rather than [98]. Indeed, the augmented BI system looks even simpler in so-called "non-conservative variables"

$$b = B/h, \quad d = D/h, \quad v = Q/h, \quad \tau = 1/h$$

Namely

$$\begin{aligned} \partial_t b + (v \cdot \nabla) b - (b \cdot \nabla) v + \tau \nabla \times d &= 0 \\ \partial_t d + (v \cdot \nabla) d - (d \cdot \nabla) v - \tau \nabla \times b &= 0 \\ \partial_t v + (v \cdot \nabla) v - (b \cdot \nabla) b - (d \cdot \nabla) d - \tau \nabla \tau &= 0 \\ \partial_t \tau + (v \cdot \nabla) \tau - \tau \nabla \cdot v &= 0 \end{aligned}$$

This turns out to be just a symmetric system with purely quadratic non-linearities! In some sense, a generalization of the inviscid Burgers equation, of form

$$\partial_t U_\alpha + A_\alpha^{i\beta\gamma} U_\gamma \partial_i U_\beta = 0,$$

written "in coordinates" (with implicit summation on repeated indices), where  $U = U(t, x) \in \mathbb{R}^{10}$  and, for each fixed indices  $i = 1, \dots, 3$  and  $\gamma = 1, \dots, 10$ , the  $10 \times 10$  matrices  $(A_\alpha^{i\beta\gamma})$  are symmetric in  $\alpha, \beta$ . Also observe that there is no limitation of range for the variables  $U = (n, d, v, \tau)$  in the space  $\mathbb{R}^{10}$ . (In particular it makes

sense to consider negative or null values of  $\tau$ , which is not possible in the conservative formulation of the ABI system since  $\rho = 1/\tau$ . This is a remarkable advantage of the non-conservative version! Of course, we don't make any comment on the possible physical meaning of considering negative values of  $\tau$ ! Concerning the BI manifold, its expression in terms of non-conservative variables is even simpler. We get the following algebraic (quadratic) 6-dimensional submanifold of  $\mathbb{R}^{10}$ :

$$\text{NCBIM } \tau^2 + b^2 + d^2 + v^2 = 1, \quad \tau v = d \times b.$$

(Notice that we may consider both positive and negative values of  $\tau$  in this definition!)

So, we obtain, essentially for free, the following result

**Theorem 8.1.1.** *The non-conservative augmented Born-Infeld (NCABI) system is locally well-posed in any Sobolev space  $H^s(\mathbb{R}^3)$  continuously imbedded in  $C^1$  (namely, for any  $s > 5/2$ ). In addition the non-conservative Born-Infeld manifold is preserved under evolution.*

Because of the preservation of the manifold, we have immediately, without any further analysis, obtained the local well-posedness of the original Born-Infeld equations. Of course, the analysis provided by Speck [432] is much more sophisticated and leads to a *global* existence and uniqueness result of smooth solutions to the expense of assuming initial conditions to be small and localized, which is in no way needed in our cruder analysis. An interesting open question is the possible global existence of smooth solutions not only for the original BI system but also for its augmented version.

**Remark: reduced versions of the NCABI system:  
motion of strings and photons**

It is perfectly consistent to assume  $\tau = 0, d = 0$  in the non-conservative augmented BI (NCABI) system. We then get a reduced system which describes a continuum of vibrating strings

$$\partial_t b + (v \cdot \nabla) b - (b \cdot \nabla) v = 0, \quad \partial_t v + (v \cdot \nabla) v - (b \cdot \nabla) b = 0$$

The corresponding BI manifold  $b^2 + v^2 = 1, v \cdot b = 0$  corresponds to relativistic strings, like in "classical" String Theory (i.e. without quantization). We may further consistently assume  $b = 0$  in the NCABI and get  $\partial_t v + (v \cdot \nabla) v = 0$  with reduced BI-manifold  $v^2 = 1$  which describes the motion of (classical) massless particles moving at the speed of light (e.g. photons).

**First appearance of convexity in the augmented Born-Infeld system**

Let us now go back to the  $10 \times 10$  augmented ABI system in conservative form. Surprisingly enough, the augmented system, as shown in [98], admits an extra conservation law, namely

$$\partial_t \eta + \nabla \cdot \omega = 0, \quad \eta = \frac{1 + D^2 + B^2 + Q^2}{h}, \quad \omega = \omega(h, Q, D, B)$$

where  $\eta$  is a strictly *convex* function and the "entropy flux"  $\omega$  can be explicitly computed. This makes the ABI system an example of entropic system of conservation laws (ESCL), for which we can use all the concepts of "relative entropy method" and "dissipative solutions" we discussed in section 6.4.



### Remark: Galilean invariance of the augmented Born-Infeld system

The ABI system looks pretty much like classical MHD equations and enjoys an astonishing *classical* Galilean invariance, under the transform

$$(t, x) \rightarrow (t, x + W t), \quad (h, Q, D, B) \rightarrow (h, Q - hU, D, B)$$

for any constant speed  $W \in \mathbb{R}^3$ ! This looks contradictory with the definite Lorentzian origin of the Born-Infeld system. However, there is no contradiction since those Galilean transforms are *incompatible* with the Born-Infeld manifold, where  $Q$  is algebraically slaved by  $B$  and  $D$  through  $Q = D \times B$ ! Moreover, we conjecture that this amazing property characterizes the Born-Infeld model among all alternative Electromagnetic theories, including ...Maxwell's one!

### Second appearance of convexity in the augmented Born-Infeld system

The  $10 \times 10$  ABI (augmented Born-Infeld) system is *linearly degenerate* (in the sense of Lax [193]) and enjoy an interesting stability under weak-\* convergence. More precisely:

**Theorem 8.1.2.** *Each weak-\* limit of uniformly bounded sequences in  $L^\infty$  of smooth solutions depending on one space variable of the ABI system are still solutions of the ABI system.*

This follows from a straightforward application of the Murat-Tartar 'div-curl' lemma [379, 440]. This suggests that the convex hull of the BI manifold might be a natural completed configuration space for the Born-Infeld theory. However, this is not so clear, as pointed out to the author by Felix Otto, since one has to take into account the differential constraints  $\nabla \cdot D = \nabla \cdot B = 0$ . Anyway, as shown in [98], the convex hull has full dimension in  $\mathbb{R}^{10}$  and has been explicitly computed by Serre [424] and is defined by the single inequality

$$h \geq \sqrt{1 + D^2 + B^2 + Q^2 + 2\sqrt{|P - D \times B|^2 + (B \cdot P)^2 + (D \cdot P)^2}}$$

Moreover Müller and Palombaro [378], using convex integration theory, have proven that the differential constraints  $\nabla \cdot D = \nabla \cdot B = 0$  are not an obstruction to the conjecture.

On the convexified BI manifold, defined by Serre's inequality, we have the following properties:

- 1) The electromagnetic field  $(D, B)$  and the 'density and momentum' fields  $(h, Q)$  can be chosen *independently* of each other at initial time, provided they satisfy Serre's inequality
- 2) The augmented BI system can be interpreted (in MHD style) as the coupling of an electromagnetic field with a fluid

$$\begin{aligned} \partial_t B + \nabla \times \left( \frac{B \times Q + D}{h} \right) &= \partial_t D + \nabla \times \left( \frac{D \times Q - B}{h} \right) = 0 \\ \partial_t Q + \nabla \cdot \left( \frac{Q \otimes Q - B \otimes B - D \otimes D}{h} \right) &= \nabla \cdot \left( \frac{1}{h} \right), \quad \partial_t h + \nabla \cdot Q = 0. \end{aligned}$$

(while the original Born-Infeld model is purely electromagnetic, without any interaction with matter).

3) 'Matter' may exist without electromagnetic field, in the case when  $B = D = 0$ , which leads to the so-called "Chaplygin gas" [423] (which has been advocated as a possible model for "dark energy" or "vacuum energy") with an unusual speed of sound  $c$ , namely  $c = 1/h$ ,

$$\partial_t Q + \nabla \cdot \left( \frac{Q \otimes Q}{h} \right) = \nabla \left( \frac{1}{h} \right), \quad \partial_t h + \nabla \cdot Q = 0$$

4) 'Moderate' Galilean transforms are allowed

$$(t, x) \rightarrow (t, x + U t), \quad (h, Q, D, B) \rightarrow (h, Q - hU, D, B)$$

(which is impossible on the original BI manifold). As a matter of fact, this seems to be a general feature of Special Relativity under weak completion (cf. "subrelativistic" conditions, as discussed in [45, 99]).

## 8.2 Extremal time-like surfaces in the Minkowski space

Let us now consider a second example of an augmented system with convex entropy derived from a system of conservation laws with a polyconvex entropy. This section narrowly follows the paper [215] by Xianglong Duan.

In the  $(1+n+m)$ -dimensional Minkowski space  $\mathbb{R}^{1+(n+m)}$ , let  $X(t, x)$  be a time-like  $(1+n)$ -dimensional surface (called  $n$ -brane in String Theory [401]), namely,

$$(t, x) \in \bar{\Omega} \subset \mathbb{R} \times \mathbb{R}^n \rightarrow X(t, x) = (X^0(t, x), \dots, X^{n+m}(t, x)) \in \mathbb{R}^{1+(n+m)},$$

where  $\Omega$  is a bounded open set. This surface is called an extremal surface if  $X$  is a critical point, with respect to compactly supported perturbations in the open set  $\Omega$ , of the following area functional (which corresponds to the Nambu-Goto action in the case  $n = 1$ )

$$- \iint_{\Omega} \sqrt{-\det(G_{\mu\nu})} \, , \quad G_{\mu\nu} = \eta_{MN} \partial_{\mu} X^M \partial_{\nu} X^N \, ,$$

where  $M, N = 0, 1, \dots, n+m$ ,  $\mu, \nu = 0, 1, \dots, n$ , and  $\eta = (-1, 1, \dots, 1)$  denotes the Minkowski metric, while  $G$  is the induced metric on the  $(1+n)$ -surface by  $\eta$ . Here  $\partial_0 = \partial_t$  and we use the convention of implicit summations on repeated indices.

Through the least-action principle, the Euler-Lagrange equations gives the well-known equations of extremal surfaces,

$$\partial_{\mu} \left( \sqrt{-G} G^{\mu\nu} \partial_{\nu} X^M \right) = 0, \quad M = 0, 1, \dots, n+m,$$

where  $G^{\mu\nu}$  is the inverse of  $G_{\mu\nu}$  and  $G = \det(G_{\mu\nu})$ .

Now, let us concentrate on the special case where the extremal surfaces are graphs of the form

$$X^0 = t, \quad X^i = x^i, \quad i = 1, \dots, n, \quad X^{n+\alpha} = X^{n+\alpha}(t, x), \quad \alpha = 1, \dots, m.$$

By using notation

$$V_\alpha = \partial_t X^{n+\alpha}, \quad F_{\alpha i} = \partial_i X^{n+\alpha}, \quad \alpha = 1, \dots, m, \quad i = 1, \dots, n.$$

$$D_\alpha = \frac{\sqrt{\det(I_n + F^T F)} (I_m + F F^T)^{-1}_{\alpha\beta} V_\beta}{\sqrt{1 - V^T (I_m + F F^T)^{-1} V}}$$

we find that the extremal surface equation is now equivalent to the following system for the matrix-valued function  $F = (F_{\alpha i})_{q \times p}$  and a vector valued function  $D = (D_\alpha)_{\alpha=1,2,\dots,q}$ ,

$$\partial_t F_{\alpha i} + \partial_i \left( \frac{D_\alpha + F_{\alpha j} P_j}{h} \right) = 0, \quad \partial_t D_\alpha + \partial_i \left( \frac{D_\alpha P_i + \xi'(F)_{\alpha i}}{h} \right) = 0,$$

$$\partial_j F_{\alpha i} = \partial_i F_{\alpha j}, \quad P_i = F_{\alpha i} D_\alpha, \quad h = \sqrt{D^2 + P^2 + \xi(F)}, \quad 1 \leq i, j \leq p, \quad 1 \leq \alpha \leq q,$$

where

$$\xi(F) = \det(I + F^T F), \quad \xi'(F)_{\alpha i} = \frac{1}{2} \frac{\partial \xi(F)}{\partial F_{\alpha i}} = \xi(F) (I + F^T F)^{-1}_{ij} F_{\alpha j}.$$

As we have seen for the Born-Infeld equations, there are extra conservation laws for the "energy" density  $h$  and the "momentum" vector  $P$  as defined above, namely,

$$\partial_t h + \nabla \cdot P = 0, \quad \partial_t P_i + \partial_j \left( \frac{P_i P_j}{h} - \frac{\xi(F) (I + F^T F)^{-1}_{ij}}{h} \right) = 0.$$

Viewing  $h$  and  $P$  as independent variables, the new system admits a polyconvex entropy (which means that the entropy can be written as a convex function of the minors of  $F$ ). Here, for  $1 \leq k \leq r$ , and any ordered sequences  $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_k \leq m$  and  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ , let  $A = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$ ,  $I = \{i_1, i_2, \dots, i_k\}$ , the minor of  $F$  with respect to the rows  $\alpha_1, \alpha_2, \dots, \alpha_k$  and columns  $i_1, i_2, \dots, i_k$  is defined as

$$[F]_{A,I} = \det \left( (F_{\alpha_p i_q})_{p,q=1,\dots,k} \right).$$

Now, by viewing these minors  $[F]_{A,I}$  as new independent variables, we can further enlarge this system. As for the Born-Infeld equations, the augmented system is hyperbolic with a convex entropy, linearly degenerate and preserves the algebraic constraints that have been given up in the process of augmenting the system.

### The augmented system

Now let us consider the energy density  $h$ , the vector field  $P$  and the minors  $[F]_{A,I}$  as independent variables. As shown by Xianglong Duan [215], the original system can be augmented to the following system of conservation laws. More precisely, for  $h > 0$ ,  $D = (D_\alpha)_{\alpha=1,2,\dots,m}$ ,  $P = (P_i)_{i=1,2,\dots,n}$ ,  $M_{A,I}$  with  $A \subseteq \{1, 2, \dots, m\}$ ,  $I \subseteq \{1, 2, \dots, n\}$ ,  $1 \leq |A| = |I| \leq r = \min\{m, n\}$ , the augmented system reads

$$\begin{aligned} \partial_t h + \nabla \cdot P &= 0 \\ \partial_t D_\alpha + \partial_i \left( \frac{D_\alpha P_i}{h} \right) + \sum_{\substack{A,I,i \\ \alpha \in A, i \in I}} (-1)^{O_A(\alpha) + O_I(i)} \partial_i \left( \frac{M_{A,I} M_{A \setminus \{\alpha\}, I \setminus \{i\}}}{h} \right) &= 0 \end{aligned}$$

$$\begin{aligned} \partial_t P_i + \sum_{\substack{A,I,j \\ j \in I, i \notin I \setminus \{j\}}} (-1)^{O_I(j) + O_{I \setminus \{j\}}(i)} \partial_j \left( \frac{M_{A, (I \setminus \{j\}) \cup \{i\}} M_{A,I}}{h} \right) \\ + \partial_j \left( \frac{P_i P_j}{h} \right) - \partial_i \left( \frac{1 + \sum_{A,I} M_{A,I}^2}{h} \right) = 0 \quad (8.2.1) \end{aligned}$$

$$\begin{aligned} \partial_t M_{A,I} + \sum_{\substack{i,j \\ i \in I, j \notin I \setminus \{i\}}} (-1)^{O_{I \setminus \{i\}}(j) + O_I(i)} \partial_i \left( \frac{M_{A, (I \setminus \{i\}) \cup \{j\}} P_j}{h} \right) \\ + \sum_{\substack{\alpha, i \\ \alpha \in A, i \in I}} (-1)^{O_A(\alpha) + O_I(i)} \partial_i \left( \frac{M_{A \setminus \{\alpha\}, I \setminus \{i\}} D_\alpha}{h} \right) = 0 \quad (8.2.2) \\ \sum_{i \in I} (-1)^{O_I(i)} \partial_i \left( M_{A', I \setminus \{i\}} \right) = 0, \quad 2 \leq |I| = |A'| + 1 \leq r + 1. \end{aligned}$$

Here  $O_A(\alpha)$  denotes the integer such that  $\alpha$  is the  $O_A(\alpha)$ th smallest element in  $A \cup \{\alpha\}$ . All the sum are taken in the convention that  $A \subseteq \{1, \dots, m\}$ ,  $I \subseteq \{1, \dots, n\}$ ,  $1 \leq \alpha \leq m$ ,  $1 \leq i, j \leq n$ .

Following [215], it can be first checked that the augmented system reduces to the original system under the algebraic constraints which were given up in order to enlarge the system, namely

$$P_i = F_{\alpha i} D_\alpha, \quad h = \sqrt{D^2 + P^2 + \xi(F)}, \quad M_{A,I} = [F]_{A,I}.$$

The following result is obtained in [215]:

**Proposition 8.2.1.** *The augmented system written above admits an additional conservation law for the convex entropy*

$$S(h, D, P, M) = \frac{1 + D^2 + P^2 + \sum_{A,I} M_{A,I}^2}{2h},$$

namely:

$$\begin{aligned} \partial_t S + \nabla \cdot \left( \frac{SP}{h} \right) + \sum_{\substack{A,I,i \\ \alpha \in A, i \in I}} (-1)^{O_A(\alpha) + O_I(i)} \partial_i \left( \frac{D_\alpha M_{A \setminus \{\alpha\}, I \setminus \{i\}} M_{A,I}}{h^2} \right) \\ + \sum_{\substack{A,I,j \\ j \in I, i \notin I \setminus \{j\}}} (-1)^{O_I(j) + O_{I \setminus \{j\}}(i)} \partial_j \left( \frac{P_i M_{A, (I \setminus \{j\}) \cup \{i\}} M_{A,I}}{h^2} \right) \\ - \partial_j \left( \frac{P_j (1 + M_{A,I}^2)}{h^2} \right) = 0. \end{aligned}$$

### Non-conservative form

The non-conservative form of the augmented system has a very simple structure, as shown by Xianglong Duan [215]:

**Theorem 8.2.2.** *In the case of graphs, the equations of extremal time-like surfaces of dimension  $1+n$  in the Minkowski space of dimension  $1+n+m$  can be translated into a first order symmetric hyperbolic system of PDEs, which admits the very simple form*

$$\partial_t W + \sum_{j=1}^n A_j(W) \partial_{x_j} W = 0, \quad W : (t, x) \in \mathbb{R}^{1+n} \rightarrow W(t, x) \in \mathbb{R}^{n+m+\binom{m+n}{n}},$$

where each  $A_j(W)$  are suitable  $(n+m+\binom{m+n}{n}) \times (n+m+\binom{m+n}{n})$  symmetric matrix depending linearly on  $W$ . Accordingly, this system is automatically well-posed, locally in time, in the Sobolev space  $W^{s,2}$  as soon as  $s > n/2 + 1$ .

The structure of the resulting equations is reminiscent of the celebrated prototype of all nonlinear hyperbolic PDEs, the so-called inviscid Burgers equation  $\partial_t u + u \partial_x u = 0$ , where  $u$  and  $x$  are both just valued in  $\mathbb{R}$ , with the simplest possible nonlinearity. Of course, to get such a simple structure, the relation to be found between  $X$  (valued in  $\mathbb{R}^{1+n+m}$ ) and  $W$  (valued in  $\mathbb{R}^{n+m+\binom{m+n}{n}}$ ) is very involved [215]. More precisely, it can be shown that the case of extremal surfaces corresponds to a special subset of solutions of the augmented system for which  $W$  lives in a suitable algebraic sub-manifold of  $\mathbb{R}^{n+m+\binom{m+n}{n}}$ , which is preserved by the dynamics of the augmented system.

As for the augmented Born-Infeld equations, the strategy of proof follows the concept of system of conservation laws with “polyconvex” entropy in the sense of Dafermos [193]. The first step is to lift the original system of conservation laws to a (much) larger one which enjoys a convex entropy rather than a polyconvex one. This strategy has been successfully applied in many situations, such as nonlinear Elastodynamics [404], nonlinear Electromagnetism [98, 124, 422], just to quote few examples. Let us add that the calculations provided in [215] crucially rely on the classical Cauchy-Binet formula.

# Chapter 9

## Convex entropic formulation of some degenerate parabolic systems

As we have already seen in Chapter 6, entropy methods are very useful to address system of first order conservation laws. In the present chapter, we extend this approach to some parabolic equations, the prototype being the linear heat equation. We will address more sophisticated examples, coming from Continuum Mechanics, such as the Muskat system (also known as the incompressible porous media equation), or Geometry, such as mean curvature flows of various co-dimensions. (Mean curvature flows is an enormous subject in Geometric Analysis and Computation. Let us just mention very few related works [6, 62, 78, 136, 156, 298, 317, 347, 364]). All the examples we are going to cover can be derived, through a simple asymptotic method, from suitable systems of first order conservation laws with a convex entropy so that we will be able to transfer convex entropic formulations straightforwardly from the hyperbolic level to the parabolic level. Our tool to derive parabolic systems from systems of first order conservation laws is extremely simple, although not usual in the literature for evolution PDEs, to the best of our knowledge. It amounts to performing a quadratic change of time near  $t = 0$  and, then, neglecting the higher order terms. Let us explain this idea through the very simple prototype of dynamical systems with a convex potential.

### 9.1 From dynamical systems to gradient flows by quadratic change of time

Let us first apply the quadratic change of time (QCT) method to the simple dynamical system

$$\frac{d^2 X}{dt^2} = -(\nabla\varphi)(X),$$

by setting

$$X(t) = Y(\theta), \quad \theta = t^2/2 \quad \theta' = \frac{d\theta}{dt} = t.$$

This leads to

$$\frac{dX}{dt} = \theta' \frac{dY}{d\theta}, \quad -(\nabla\varphi)(Y(\theta)) = \frac{d}{dt}(\theta' \frac{dY}{d\theta}) = \theta'' \frac{dY}{d\theta} + (\theta')^2 \frac{d^2 Y}{d\theta^2}$$

and thus

$$\frac{dY}{d\theta} + 2\theta \frac{d^2 Y}{d\theta^2} = -(\nabla\varphi)(Y).$$

For large  $\theta$ , we get the purely inertial motion governed by:

$$\frac{d^2Y}{d\theta^2} = 0,$$

while, for small  $\theta$ , we rather get the so-called "gradient flow" regime with:

$$\frac{dY}{d\theta} = -(\nabla\varphi)(Y).$$

*Remark :*

The quadratic rescaling  $\theta = t^2/2$  perfectly fits with Galileo's experiment: a rigid ball descends a rigid ramp of constant slope, with zero initial velocity and constant acceleration  $G$ , reaching position  $X(t) = x_0 + Gt^2/2 = x_0 + G\theta = Y(\theta)$  at time  $t$ . So,  $Y$  is just a linear function of the rescaled time  $\theta$ !

$$\frac{dY}{d\theta} + 2\theta\frac{d^2Y}{d\theta^2} = G$$

but also *simultaneously*

$$\frac{dY}{d\theta} = G, \quad \frac{d^2Y}{d\theta^2} = 0,$$

i.e. *both* the gradient flow and the inertial regimes.

*End of remark.*



The Galileo experiment.

Small bells are set up along the ramp according to a parabolic spacing (1, 4, 9, 16, 25...) so that, when falling down, the ball rings the bells periodically in time.



For the original dynamical system,

$$\frac{d^2 X}{dt^2} = -\nabla\varphi(X),$$

we get the usual conservation of energy

$$\frac{d}{dt} \left[ \frac{1}{2} \left| \frac{dX}{dt} \right|^2 + \varphi(X) \right] = 0$$

For the time-rescaled version  $Y(\theta) = X(t)$ ,  $\theta = t^2/2$ , we find

$$\frac{d}{d\theta} [\varphi(Y)] + \theta \frac{d}{d\theta} \left| \frac{dY}{d\theta} \right|^2 = - \left| \frac{dY}{d\theta} \right|^2$$

In the asymptotic regime when  $\theta$  is very small, we recover the gradient flow

$$\frac{dY}{d\theta} = -\nabla\varphi(Y)$$

and the classical "energy – dissipation" relation

$$\frac{d}{d\theta} [\varphi(Y)] = - \left| \frac{dY}{d\theta} \right|^2.$$

We may compare, for short times,  $X$  solution of the original equation, with zero initial velocity, to  $Y$  solution of the gradient flow

$$\frac{d^2 X}{dt^2} = -\nabla\varphi(X), \quad X'(0) = 0, \quad \frac{dY}{d\theta} = -\nabla\varphi(Y), \quad Y(0) = X(0).$$

Under strong convexity and smoothness assumptions on  $\varphi$ , Assuming the spectrum of the symmetric matrix  $D^2\varphi(x)$  to be contained in a fixed interval  $[r, 1/r]$ , uniformly in  $x$ , for some constant  $r > 0$ , we may easily prove, through a standard Gronwall estimate,

$$|X(t) - Y(t^2/2)|^2 + \left| \frac{dX}{dt}(t) - t \frac{dY}{d\theta}(t^2/2) \right|^2 \leq t^4 \exp(t^2 c) c.$$

by monitoring the "relative energy"

$$\frac{1}{2} \left| \frac{dX}{dt} - t \frac{dY}{d\theta} \right|^2 + \varphi(X) - \varphi(Y) - \nabla\varphi(Y) \cdot (X - Y),$$

which is just obtained (as a "relative entropy") by subtracting from the energy of  $X$  what we obtain by expanding linearly the energy in  $X$  about  $Y$ . Notice that constant  $c$  depends only on  $r$  and on  $Y$ .

## 9.2 From the Euler equations to the heat equation by quadratic change of time

Let us now get back, as a leitmotiv, to the Euler equations, this time for *compressible* fluids. They read, as written by Euler (i.e. without thermodynamics nor energy equation; they are frequently called "isentropic Euler equations"):

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) = -\nabla p$$

where  $(\rho, p, v) \in \mathbb{R}^{1+1+3}$  are the density, pressure and velocity fields of a fluid and  $p$  is assumed to be a given function of  $\rho$ . Let us now perform the quadratic change of time (QCT)

$$\tilde{\rho}(t, x) = \rho(\theta, x), \quad \tilde{v}(t, x) = \theta' v(\theta, x), \quad \theta = \theta(t) = t^2/2 \quad \theta' = \frac{d\theta}{dt} = t$$

(so that  $\tilde{v}(t, x)dt = v(\theta, x)d\theta$ ). We get:

$$\begin{aligned} \partial_t \tilde{\rho} + \nabla \cdot (\tilde{\rho} \tilde{v}) &= 0 \quad \rightarrow \quad \theta' \partial_\theta \rho + \theta' \nabla \cdot (\rho v) = 0 \\ \partial_t (\tilde{\rho} \tilde{v}) + \nabla \cdot (\tilde{\rho} \tilde{v} \otimes \tilde{v}) &= -\nabla p(\tilde{\rho}) \quad \rightarrow \\ \theta'' \rho v + (\theta')^2 \partial_\theta (\rho v) + (\theta')^2 \nabla \cdot (\rho v \otimes v) &= -\nabla p(\rho) \\ \rightarrow \quad \rho v + 2\theta \partial_\theta (\rho v) + 2\theta \nabla \cdot (\rho v \otimes v) &= -\nabla p(\rho) \end{aligned}$$

So, after the quadratic change of time, the Euler equations become

$$\partial_\theta \rho + \nabla \cdot (\rho v) = 0, \quad \rho v + 2\theta [\partial_\theta (\rho v) + \nabla \cdot (\rho v \otimes v)] = -\nabla p(\rho)$$

Notice that the continuity equation has stayed unchanged. (Actually, this was the main purpose of the different rescaling of variables  $\rho$  and  $v$ .) The new system of evolution PDEs is no longer "autonomous": it depends *explicitly* on the new time variable  $\theta$ , actually in a very simple, linear, way. So we may consider two asymptotic regimes, according to the size of  $\theta$ . For very large  $\theta$ , we just obtain the so-called "pressureless Euler" equations:

$$\partial_\theta \rho + \nabla \cdot (\rho v) = 0, \quad \partial_\theta (\rho v) + \nabla \cdot (\rho v \otimes v) = 0,$$

which is just a degenerate (but tricky!) version of the Euler equations. We are much more interested in the second regime when  $\theta$  is very small. Then, we obtain the so-called "porous media equation"

$$\partial_\theta \rho + \nabla \cdot (\rho v) = 0, \quad \rho v = -\nabla p,$$

or, in short,

$$\partial_\theta \rho = \Delta(p(\rho)),$$

including the heat equation in the special ("isothermal") case  $p(\rho) = \rho$ . So, the quadratic change of time has clearly introduced a change of type in the equations, since we have moved from the hyperbolic, first order, setting of the Euler equations to the parabolic, second order in space, setting of the heat and the porous medium equations.

### 9.3 Inhomogeneous incompressible Euler and Muskat equations

Another example where we can fruitfully derive degenerate parabolic equations out of entropic systems of conservation laws come from Fluid Mechanics. This the Muskat system, also known as incompressible porous media equation. We start with the Euler equations, set on  $\mathbb{T}^d$  for simplicity, of an incompressible *inhomogeneous*

fluid subject to the action of an external potential  $\Phi$  and we use the Boussinesq approximation:

$$\begin{aligned}\partial_t \rho + \nabla \cdot (\rho v) &= 0, \quad \nabla \cdot v = 0, \\ \bar{\rho}(\partial_t v + \nabla \cdot (v \otimes v)) + \nabla p &= -\rho \nabla \Phi, \quad \bar{\rho} = cst.\end{aligned}$$

Notice that the density field  $\rho$  is advected by the velocity field  $v$  in the sense that

$$(\partial_t \rho + v \cdot \nabla) \rho = 0,$$

which is a consequence of *both* the continuity equation *and* the divergence-free condition on  $v$ .

*Remark.*

In geophysical Fluid Mechanics [394], the Boussinesq approximation, which is still widely used because it substantially simplifies numerical computations, amounts to neglecting the variation of the density in the acceleration term and substituting for it the constant  $\bar{\rho}$  which should be considered as an average density (and, accordingly,  $\rho$  should be thought as the density minus its average rather than the density itself, which does not affect the equations since adding a constant to  $\rho$  does not modify them, thanks to the pressure term and the divergence-free condition). (See [187, 394].) Anyway, this model is fully consistent with the least action principle without requiring any approximation, provided the action is defined by

$$\mathcal{A} = \int \int \left( \frac{1}{2} \bar{\rho} |v(t, x)|^2 - \rho(t, x) \Phi(x) \right) dx dt$$

subject to constraints:

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \nabla \cdot v = 0.$$

Indeed, introducing two Lagrange multipliers  $\theta = \theta(t, x) \in \mathbb{R}$  and  $q = q(t, x) \in \mathbb{R}$  for the constraints, we form the Lagrangian

$$\mathcal{L} = \int \int \left( \frac{1}{2} |v(t, x)|^2 - \rho(t, x) \Phi(x) - \partial_t \theta \rho - \nabla \theta \cdot \rho v - \nabla q \cdot v \right) dx dt$$

(where we have set  $\bar{\rho} = 1$  for notational simplicity) and get, by successively varying  $v$  and  $\rho$ ,

$$v = \rho \nabla \theta + \nabla q, \quad \partial_t \theta + v \cdot \nabla \theta + \Phi = 0,$$

which leads back to

$$\partial_t v + \nabla \cdot (v \otimes v) + \nabla p = -\rho \nabla \Phi,$$

after elementary calculations, where  $p$  is related to  $q$  through:

$$p = \frac{1}{2} |v|^2 - v \cdot \nabla q.$$

[Strictly speaking this derivation is incomplete as  $d > 3$  (which does not matter from a mechanical viewpoint) since the "Clebsch" decomposition  $v = \rho \nabla \theta + \nabla q$  is too restrictive to describe a divergence-free vector field as  $d > 3$ . Then, additional Lagrange multipliers must be added in the action principle.]

*End of remark.*

From now on, we simplify notations by setting  $\bar{\rho} = 1$  and define the "Euler-Boussinesq" equations as

$$(EB) : \quad \partial_t v + \nabla \cdot (v \otimes v) + \nabla p = -\rho \nabla \Phi, \quad \partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \nabla \cdot v = 0.$$

Observe the (formal) conservation of energy:

$$\frac{d}{dt} \int_{\mathbb{T}^d} \left( \frac{1}{2} |v(t, x)|^2 + \rho(t, x) \Phi(x) \right) dx = 0.$$

Also notice that for any suitable function  $\Psi$  we get the extra conservation

$$\frac{d}{dt} \int_{\mathbb{T}^d} \Psi(\rho(t, x)) dx = 0.$$

So, we may as well rewrite the conservation of energy as

$$\frac{d}{dt} \int_{\mathbb{T}^d} \{ |v(t, x)|^2 + (\rho(t, x) + \Phi(x))^2 \} dx = 0.$$

(just by taking  $\Psi(r) = r^2$ ).

### From Euler to Muskat by quadratic change of time

Let us again use the quadratic change of time method, applied to the Euler-Boussinesq (EB) system:

$$\begin{aligned} t \rightarrow \theta = t^2/2, \quad \text{new } \rho(\theta, x) &= \text{old } \rho(\sqrt{2\theta}, x), \\ \text{new } v(\theta, x) &= \frac{1}{\sqrt{2\theta}} \text{old } v(\sqrt{2\theta}, x), \end{aligned}$$

After this change, the Euler-Boussinesq system becomes

$$\begin{aligned} \partial_\theta \rho + \nabla \cdot (\rho v) &= 0, \quad \nabla \cdot v = 0, \\ v + 2\theta(\partial_\theta v + \nabla \cdot (v \otimes v)) + \nabla p &= -\rho \nabla \Phi, \end{aligned}$$

For small  $\theta$  we just find, as asymptotic equations, the Muskat equations

$$\partial_\theta \rho + \nabla \cdot (\rho v) = 0, \quad \nabla \cdot v = 0, \quad v + \nabla p = -\rho \nabla \Phi.$$

### Relative energy estimate for the Euler-Boussinesq equations

**Proposition 9.3.1.** *If  $(v, \rho)$  is a weak solution of Euler-Boussinesq with decreasing energy. Then, for all smooth fields  $(\tilde{v}, \tilde{\rho})$  such that  $\nabla \cdot \tilde{v} = 0$ , we get the "relative energy" differential inequality*

$$\begin{aligned} \frac{d}{dt} \{ \|v - \tilde{v}\|_{L^2(\mathbb{T}^d)}^2 + \|\rho - \tilde{\rho}\|_{L^2(\mathbb{T}^d)}^2 \} &\leq 2 \int_{\mathbb{T}^d} \mathcal{L} + \mathcal{Q}, \\ \mathcal{L} &= (\tilde{v} - v) \cdot \tilde{E}_1 + (\tilde{\rho} - \rho) \tilde{E}_2 \\ \mathcal{Q} &= (\tilde{\rho} - \rho)(\tilde{v} - v) \cdot \nabla(\Phi + \tilde{\rho}) - (\tilde{v} - v) \otimes (\tilde{v} - v) \cdot (\nabla \tilde{v} + \nabla \tilde{v}^T), \\ \tilde{E}_1 &= \partial_t \tilde{v} + \nabla \cdot (\tilde{v} \otimes \tilde{v}) + \tilde{\rho} \nabla \Phi, \quad \tilde{E}_2 = \partial_t \tilde{\rho} + \partial_j (\rho v^j), \end{aligned}$$

At this point, we have just mimicked what Lions did for the homogeneous Euler equations in [331]. Then, still following Lions, we may deduce from the relative energy estimate a good concept of "dissipative" solutions to the Euler-Boussinesq system and easily get global existence and "weak-strong" stability (and uniqueness) results for such solutions.

## Dissipative solutions" for the Muskat system

From the "relative energy" estimate obtained for the Euler-Boussinesq system, we almost immediately get a corresponding new concept of "dissipative solution" for the Muskat system just by using, again, the quadratic change of time method. The result is therefore just a definition:

**Definition 9.3.2.** *We say that  $(\rho, v) \in (C^0(L_w^2) \times L^2)([0, T] \times \mathbb{T}^d)$  is a dissipative solution to the Muskat system if: i)  $\nabla \cdot v = 0$ ,  
ii)  $\forall(\tilde{\rho}, \tilde{v}) \in (W^{1,\infty} \times L^2)([0, T] \times \mathbb{T}^d)$  s.t.  $\nabla \cdot \tilde{v} = 0$ ,*

$$\forall t \in [0, T], \int_{\mathbb{T}^d} (\tilde{\rho} - \rho)(t, \cdot)^2 \leq e^{t\tilde{r}} \int_{\mathbb{T}^d} (\tilde{\rho} - \rho)(0, \cdot)^2$$

$$- \int_0^t e^{(t-s)\tilde{r}} \int_{\mathbb{T}^d} \{2(v - \tilde{v}) \cdot \tilde{E}_1 + 2(\rho - \tilde{\rho})\tilde{E}_2$$

$$+ |\tilde{v} - v|^2 + |\tilde{v} - v - (\tilde{\rho} - \rho)\nabla(\Phi + \tilde{\rho})|^2\}(s, x) dx ds,$$

$$\tilde{E}_1 = \tilde{v} + \tilde{\rho}\nabla\Phi, \quad \tilde{E}_2 = \partial_t\tilde{\rho} + \tilde{v} \cdot \nabla\tilde{\rho}, \quad \tilde{r} = \|\nabla(\Phi + \tilde{\rho})\|_{L^\infty}.$$

## 9.4 Quadratic change of time for mean-curvature flows

We are now going to get some mean curvature flows from hyperbolic equations (typically geometric wave equations) through the quadratic change of time method. This has been developed for the curve-shortening flow (which is the mean-curvature flow in dimension 1, i.e. in co-dimension  $d - 1$ ), with Xianglong Duan [115]. Here, we focus on the substantially simpler case of mean curvature flow for graphs, with co-dimension one. In this section, we narrowly follow [111].

**Theorem 9.4.1.** *Through the quadratic change of time method, the nonlinear wave equation, which describes graphs of extremal area in the Minkowski space  $\mathbb{R}^{1+d}$ ,*

$$\partial_t\left(\frac{\partial_t\phi}{R}\right) = \nabla \cdot \left(\frac{\nabla\phi}{R}\right), \quad R = \sqrt{1 - \partial_t\phi^2 + |\nabla\phi|^2}$$

*generates two twin evolution PDEs. The first one is the "arctangential" heat equation  $\partial_t D = \Delta(\arctan D)$ , while the second one is just the well known mean curvature flow for graphs*

$$\partial_t\phi = \sqrt{1 + |\nabla\phi|^2} \nabla \cdot \left(\frac{\nabla\phi}{\sqrt{1 + |\nabla\phi|^2}}\right).$$

**Remark: interpretation of the arctangential heat equation in optimal transport terms:**

The arctangential flow  $\partial_t D = \lambda\Delta(\arctan(D\lambda^{-1}))$  (where we have input the scaling parameter  $\lambda > 0$ ) can be easily written in optimal transport style (à la Otto) [387, 390]

$$\partial_t D = \nabla \cdot (D \nabla(\mathcal{F}'(D))).$$

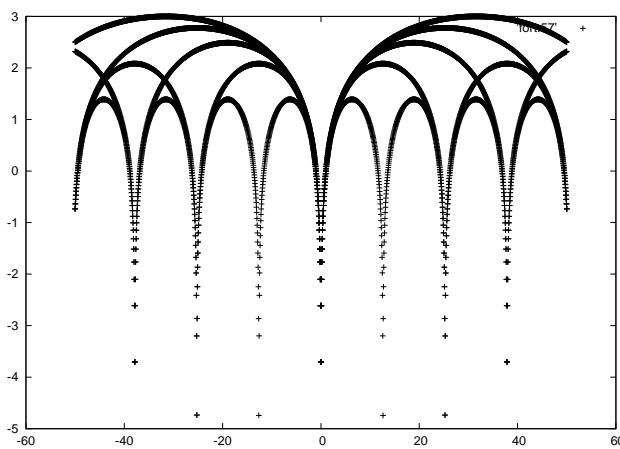
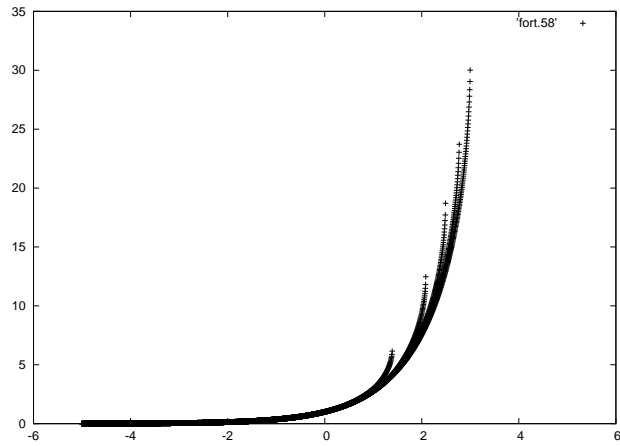
Indeed, it is enough to set

$$\mathcal{F}(D) = D \log \left( \frac{D}{\sqrt{1 + D^2 \lambda^{-2}}} \right) - \lambda \arctan(D \lambda^{-1}).$$

Notice that function  $\mathcal{F}$  is nothing but the Legendre transform of

$$u \rightarrow \lambda \arcsin(\lambda^{-1} e^u)$$

(extended by  $+\infty$  for  $u > \log \lambda$ ), which can be seen, interestingly enough, as a “catastrophic” version of the usual exponential. (N.B. In addition, the inverse of this “catastrophic” exponential  $u \rightarrow \lambda \arcsin(\lambda^{-1} \exp(u))$ , which can be symmetrized and periodized as  $v \rightarrow \frac{1}{2} \log(\lambda^2 \sin^2(v \lambda^{-1}))$ , also plays a crucial role in the recent theory of “unbalanced optimal transport” [167, 312, 328].)



The “catastrophic” exponential function, drawn for different values of parameter  $\lambda$ , and its inverse (after symmetrization and periodization):  $v \rightarrow \frac{1}{2} \log(\lambda^2 \sin^2(v\lambda^{-1}))$

### Proof of Theorem 9.4.1

We want to derive from the nonlinear wave equation (studied by Lindblad in [329])

$$\partial_t \left( \frac{\partial_t \phi}{R} \right) = \nabla \cdot \left( \frac{\nabla \phi}{R} \right), \quad R = \sqrt{1 - \partial_t \phi^2 + |\nabla \phi|^2},$$

at once, both the arctangential heat flow

$$\partial_t D = \Delta(\arctan D)$$

the mean curvature flow for graphs

$$\partial_t \phi = \sqrt{1 + |\nabla \phi|^2} \nabla \cdot \left( \frac{\nabla \phi}{\sqrt{1 + |\nabla \phi|^2}} \right).$$

*Proof/First step.*

Here we proceed as we did for the Born-Infeld equations, by introducing a suitable augmented system revealing the hidden convexity structure of the wave equation. More precisely:

**Theorem 9.4.2.** *As  $\phi(t, x)$  solves the equation of extremal surfaces in Minkowski's space, then*

$$(D, B, P) = \frac{1}{\sqrt{1 - \partial_t \phi^2 + |\nabla \phi|^2}} (\partial_t \phi, \nabla \phi, -\partial_t \phi \nabla \phi)$$

*solves the "entropic" system of conservation laws:*

$$\partial_t B + \nabla \cdot \left( \frac{P \cdot B - D}{h} \right) = 0, \quad \partial_t D + \nabla \cdot \left( \frac{PD - B}{h} \right) = 0,$$

$$\partial_t P + \nabla \cdot \left( \frac{P \otimes P + B \otimes B}{h} \right) = \nabla \cdot \left( \frac{1 + B^2}{h} \right),$$

*with*

$$h = h(D, B, P) = \sqrt{1 + D^2 + B^2 + P^2}$$

*as convex "entropy", which is a strictly convex function of  $(D, B, P)$  and obeys an extra conservation law.*

Let us postpone the proof of this result for a moment and continue the proof of Theorem 9.4.1.

*Proof of Theorem 9.4.1. /Second step.*

We apply the quadratic change of time method  $t \rightarrow \theta = t^2/2$  in two different ways. A first possible rescaling is

$$\begin{aligned} \mathcal{B}(\theta, x) &= B(\sqrt{2\theta}, x), \\ \mathcal{D}(\theta, x) &= \frac{D(\sqrt{2\theta}, x)}{\sqrt{2\theta}}, \quad \mathcal{P}(\theta, x) = \frac{P(\sqrt{2\theta}, x)}{\sqrt{2\theta}}, \end{aligned}$$

requiring initial condition  $D = P = 0$  at  $t = 0$ , which corresponds to  $\partial_t \phi(0, x) = 0$  in terms of the solution  $\phi$  to the nonlinear wave equation.

In a somewhat dual way, a second natural change is

$$\mathcal{D}(\theta, x) = D(\sqrt{2\theta}, x),$$



$$\mathcal{B}(\theta, x) = \frac{B(\sqrt{2\theta}, x)}{\sqrt{2\theta}}, \quad \mathcal{P}(\theta, x) = \frac{P(\sqrt{2\theta}, x)}{\sqrt{2\theta}},$$

requiring initial condition  $B = P = 0$  at  $t = 0$ , which corresponds to  $\nabla\phi = 0$  at  $t = 0$  in terms of  $\phi$ .

After performing the change of time  $t \rightarrow \theta = t^2/2$ , we get, in the 1st case, the non autonomous system:

$$\begin{aligned} \partial_\theta \mathcal{B} &= \nabla \left( \mathcal{D} - \frac{\mathcal{P} \cdot \mathcal{B}}{\mathcal{H}} \right), \quad \mathcal{H} = \sqrt{1 + \mathcal{B}^2 + 2\theta(\mathcal{D}^2 + \mathcal{P}^2)}, \\ \mathcal{D} - \nabla \cdot \left( \frac{\mathcal{B}}{\mathcal{H}} \right) &= -2\theta \left( \partial_\theta \mathcal{D} + \nabla \cdot \left( \frac{\mathcal{P}\mathcal{D}}{\mathcal{H}} \right) \right), \\ \mathcal{P} + \nabla \cdot \left( \frac{\mathcal{B} \otimes \mathcal{B}}{\mathcal{H}} \right) - \nabla \left( \frac{1 + \mathcal{B}^2}{\mathcal{H}} \right) & \\ &= -2\theta \left( \partial_\theta \mathcal{P} + \nabla \cdot \left( \frac{\mathcal{P} \otimes \mathcal{P}}{\mathcal{H}} \right) \right), \end{aligned}$$

Neglecting the red terms leads to the mean curvature flow (for graphs), written as an augmented system, in form:

$$\begin{aligned} \partial_\theta \mathcal{B} &= \nabla \left( \mathcal{D} - \frac{\mathcal{P} \cdot \mathcal{B}}{\mathcal{H}} \right), \quad \mathcal{H} = \sqrt{1 + \mathcal{B}^2} \\ \mathcal{D} &= \nabla \cdot \left( \frac{\mathcal{B}}{\mathcal{H}} \right), \quad \mathcal{P} + \nabla \cdot \left( \frac{\mathcal{B} \otimes \mathcal{B}}{\mathcal{H}} \right) = \nabla \left( \frac{1 + \mathcal{B}^2}{\mathcal{H}} \right). \end{aligned}$$

Symmetrically, the second rescaling leads to the arctangential heat equation and, then, the twin gradient flow structures easily follow.

*End of proof.*

### Proof of Theorem 9.4.2

*First step* : Hamiltonian form of the minimal surface equations.

The non linear wave equation

$$\partial_t \left( \frac{\partial_t \phi}{R} \right) = \nabla \cdot \left( \frac{\nabla \phi}{R} \right), \quad R = \sqrt{1 - \partial_t \phi^2 + |\nabla \phi|^2},$$

is easily obtained by finding critical points  $\phi$  of the Minkowski area of the graph  $(t, x) \rightarrow (t, x, \phi(t, x))$ , namely

$$- \int \int \sqrt{1 - \partial_t \phi^2 + \partial^k \phi \partial_k \phi} \, dt dx,$$

under space-time compactly supported perturbations. For the sequel, it is crucial to use the Hamiltonian form of the nonlinear wave equation. For that purpose, we introduce the fields

$$E(t, x) = \partial_t \phi(t, x), \quad B_i(t, x) = \partial_i \phi(t, x),$$

which are linked by the differential constraint  $\partial_t B_i = \partial_i E$ . Introducing the Lagrangian function

$$L(E, B) = -\sqrt{1 - E^2 + B_k B^k},$$

we look at critical points  $(E, B)$  of

$$\int \int L(E(t, x), B(t, x)) dt dx$$

under space-time compactly supported perturbations, subject to the differential constraints. In other words, we look for saddle-points  $(E, B, \psi)$  of

$$\int \int (L(E(t, x), B(t, x)) + \partial_t \psi^i B_i(t, x) - \partial_i \psi^i E(t, x)) dt dx$$

where  $\psi = \psi(t, x) \in \mathbb{R}^d$  is a Lagrange multiplier for the differential constraint. Independently of the specific definition of  $L$ , we may introduce the Hamiltonian  $H$  as the partial Legendre-Fenchel transform of the Lagrangian  $L(E, B)$  with respect to  $E$ ,

$$H(D, B) = \sup_{E \in \mathbb{R}} DE - L(E, B)$$

and the corresponding "dual" field

$$D(t, x) = \left( \frac{\partial L}{\partial E} \right) (E(t, x), B(t, x)).$$

Then, we get, by standard differential calculus, the Hamiltonian formulation

$$\partial_t B_i = \partial_i \left( \frac{\partial H}{\partial D} (D, B) \right), \quad \partial_t D = \partial_i \left( \frac{\partial H}{\partial B_i} (D, B) \right),$$

and, as a consequence, an extra conservation law involving  $H$

$$\partial_t (H(D, B)) + \partial_i (P^i(D, B)) = 0, \quad P^i(D, B) = \left( \frac{\partial H}{\partial D} \frac{\partial H}{\partial B_i} \right) (D, B).$$

In the case of the nonlinear wave equation we get, explicitly,

$$H(D, B) = \sqrt{(1 + B_k B^k)(1 + D^2)}$$

and, after elementary calculations, deduce

**Proposition 9.4.3.** *The nonlinear wave equation*

$$\partial_t \left( \frac{\partial_t \phi}{R} \right) = \nabla \cdot \left( \frac{\nabla \phi}{R} \right), \quad R = \sqrt{1 - \partial_t \phi^2 + |\nabla \phi|^2},$$

can be written in Hamiltonian form

$$\partial_t B_i = \partial_i \left( \sqrt{\frac{1 + B_k B^k}{1 + D^2}} D \right), \quad \partial_t D = \partial_i \left( \sqrt{\frac{1 + D^2}{1 + B_k B^k}} B^i \right), \quad (9.4.1)$$

with the extra-conservation law

$$\partial_t H + \partial_i P^i = 0, \quad H = \sqrt{(1 + B_k B^k)(1 + D^2)}, \quad P^i = -DB^i.$$

In addition,  $(D, B)$  are related to  $\phi$  by

$$B_i = \partial_i \phi, \quad D = \frac{\partial_t \phi}{\sqrt{1 - \partial_t \phi^2 + \partial^k \phi \partial_k \phi}}.$$

*Second*]; *step* Construction of an augmented system with convex entropy.  
 Since the Hamiltonian

$$H(D, B) = \sqrt{(1 + B_k B^k)(1 + D^2)}$$

is, unfortunately, not a convex function of  $(D, B)$ , and, therefore the hamiltonian form of the nonlinear wave equation (9.4.1) does not belong to our favorite class of systems of entropic system of conservation laws with a convex entropy. However, there is also an extra conservation law for  $P = -DB$ , namely

$$\partial_t P + \nabla \cdot \left( \frac{P \otimes P + B \otimes B}{h} \right) = \nabla \cdot \left( \frac{1 + B^2}{h} \right),$$

where  $h = h(D, B, P) = \sqrt{1 + D^2 + B_k B^k + P_k P^k}$  is nothing but  $H(D, B)$ , written as a function of  $(D, B, P)$ . We can *add* this new conservation laws to the one we have previously obtained for  $(D, B)$ , namely

$$\partial_t B + \nabla \cdot \left( \frac{P \cdot B - D}{h} \right) = 0, \quad \partial_t D + \nabla \cdot \left( \frac{PD - B}{h} \right) = 0$$

(where we input the new variable  $h$ ). This allows us, ignoring the algebraic constraint  $P = -DB$ , to consider  $(D, B, P)$ , as a solution of an *augmented* system of conservation laws which turns out to enjoy an extra conservation law for the strictly convex "entropy"

$$h(D, B, P) = \sqrt{1 + D^2 + B_k B^k + P_k P^k} .$$

The detailed calculations are provided in the appendix of reference [111].

# Chapter 10

## A dissipative least action principle and its stochastic interpretation

The purpose of this chapter is first to introduce a *modified* least action principle that can include *energy dissipation* and, afterwards, to provide a stochastic interpretation of this modification in terms of large deviations (which will be done in the final section), at least in a special case strongly related to both the Euler equations of incompressible fluids and the gravitational Vlasov-Poisson system that describes Newtonian gravitation. The Vlasov-Poisson system is also of paramount importance in Plasma Physics. Let us quote a few various contributions on Vlasov-Poisson equations [37, 40, 214, 281, 283, 288, 319, 341, 370, 382, 396] somewhat related to our book. As usual in this book, convexity plays a crucial role in this chapter.

There are examples, typically in infinite dimension (but not necessarily), of *formally* hamiltonian systems which *do not* necessarily *preserve* the energy because of some hidden dissipative mechanism:

i) the (inviscid) Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R} \rightarrow u(t, x) \in \mathbb{R};$$

ii) the Euler equations of incompressible fluids: at least at the physical level, it is often believed that the energy could dissipate according to Kolmogorov's "K41" theory of turbulence [244].

Let us start the discussion with a special example of finite dimensional dynamical systems for which a dissipative version of the least action principle can be designed.

### 10.1 A special class of Hamiltonian systems

Given an Euclidean space  $H$  (or more generally a Hilbert space) with norm  $\|\cdot\|$  and a potential  $Q : H \rightarrow \mathbb{R}$ ,

$$\frac{1}{2} \|V_t\|^2 + Q[X_t]$$

is the conserved energy (or Hamiltonian) for the dynamical system

$$\frac{dV_t}{dt} = -\nabla Q[X_t], \quad \frac{dX_t}{dt} = V_t, \quad (X_t, V_t) \in H \times H.$$

As well known, its solutions can be obtained from the "least action principle" by looking for critical points of the "action"

$$\int_{t_0}^{t_1} \frac{1}{2} \left\| \frac{dX_t}{dt} \right\|^2 - Q[X_t] \, dt,$$

among all curves  $t \in [t_0, t_1] \rightarrow X_t$  with fixed values at  $t_0$  and  $t_1$ .

We are going to define a special class of hamiltonian systems (in finite dimension), for which a *modified* least action principle can be designed that can include *energy dissipation*. This issue has been already discussed by various authors, Shnirelman and Wolansky, for instance [428, 461]. The systems we are going to discuss are very special but, among them, we will get discrete or approximate versions of the Euler model of incompressible fluids.

Let  $H$  be a Euclidean space and  $S$  a bounded closed subset. Set

$$Q[X] = -\frac{1}{2} \text{dist}^2(X, S) = -\inf_{s \in S} \frac{\|X - s\|^2}{2}$$

and consider the corresponding dynamical system

$$\frac{d^2 X_t}{dt^2} = -\nabla Q[X_t]$$

N.B.:  $Q$  is semi-convex, but not smooth (unless  $S$  is convex).

Indeed:  $Q[X] = -\frac{1}{2} \|X\|^2 + R[X]$ , where  $R[X] = \sup_{s \in S} (\langle X, s \rangle) - \frac{1}{2} \|s\|^2$  is convex.

## 10.2 The main example and the Vlasov-Monge-Ampère system

Let us now describe our main example. Let  $\{A(1), \dots, A(N)\}$  be a cubic lattice of  $N$  points approximating  $D = [-1/2, 1/2]^d \subset \mathbb{R}^d$  as  $N$  tends to infinity. Define

$$H = (\mathbb{R}^d)^N, \quad S = \{(A(\sigma_1), \dots, A(\sigma_N)) \in H, \quad \sigma \in \mathfrak{S}_N\}$$

(where  $\mathfrak{S}_N$  denotes the group of all permutations of the first  $N$  integers, while  $|\cdot|$  and  $\|\cdot\|$  are the euclidean norms respectively on  $\mathbb{R}^d$  and  $\mathbb{R}^{Nd}$ .)

Then, the dynamical system introduced in the previous section reads, after elementary calculations,

$$\beta \frac{d^2 X_t(\alpha)}{dt^2} = X_t(\alpha) - A(\sigma_{opt}(\alpha)), \quad X_t(\alpha) \in \mathbb{R}^d, \quad \alpha = 1, \dots, N \quad (10.2.1)$$

$$\sigma_{opt} = \text{Arginf}_{\sigma \in \mathfrak{S}_N} \sum_{\alpha=1}^N |X_t(\alpha) - A(\sigma(\alpha))|^2 \quad (10.2.2)$$

with  $\beta = 1$ , involving, at each time  $t$ , a discrete optimal transport problem.

This system was introduced, in the case  $\beta = -1$ , in [93], where its hydrodynamic limit to the Euler equations has been established. (Let us mention [48, 86, 247, 363,

393] for related computational methods for fluids.)

Notice that, as  $d = 1$ , this system reduces to

$$\beta \frac{d^2 X_t(\alpha)}{dt^2} = X_t(\alpha) - \frac{1}{2N} \sum_{\alpha' \neq \alpha} \operatorname{sgn}(X_t(\alpha) - X_t(\alpha')).$$

This describes the Newtonian gravitational interaction of  $N$  parallel planes as  $\beta = 1$  (with a global neutralization of the total mass, expressed by the linear term  $X_t$ ).

The continuous version, involving the Monge-Ampère equation, closely related to optimal transport theory, was introduced by B. and Loeper [121, 339], and studied by Cullen, Gangbo, Pisante [188], Ambrosio-Gangbo [12]. We find

$$\partial_t f(t, x, \xi) + \nabla_x \cdot (\xi f(t, x, \xi)) - \nabla_\xi \cdot (\nabla_x \varphi(t, x) f(t, x, \xi)) = 0 \quad (10.2.3)$$

$$\det(\mathbb{I} - \beta D_x^2 \varphi(t, x)) = \int_{\mathbb{R}^d} f(t, x, \xi) d\xi, \quad (t, x, \xi) \in \mathbb{R} \times D \times \mathbb{R}^d. \quad (10.2.4)$$

This fully nonlinear version of the Vlasov-Poisson system is related to Electrodynamics ( $\beta = -1$ ) and Gravitation ( $\beta = 1$ ). The formal limit  $\beta = 0$  reads

$$\partial_t f + \nabla_x \cdot (\xi f) - \nabla_\xi \cdot (\nabla_x p f) = 0, \quad \int_{\mathbb{R}^d} f(t, x, \xi) d\xi = 1,$$

where  $p = p(t, x)$  substitutes for  $\varphi$  as a Lagrange multiplier of constraint  $\int f d\xi = 1$ . It can be understood as a "kinetic formulation" of the Euler equations of homogeneous incompressible fluids (see [84, 91], for this concept and section 2.4 in the present book). Classical solutions  $(v, p)$  to the Euler equations correspond to very special and singular solutions of the kinetic version of form

$$f(t, x, \xi) = \delta(\xi - v(t, x)).$$

### 10.3 A proposal for a modified least action principle

Let us go back to the general case, where  $H$  and  $S$  can be chosen freely, respectively as an Euclidean space and a bounded closed subset. The dynamical system

$$\frac{d^2 X_t}{dt^2} = -\nabla Q[X_t]$$

with  $Q[X] = -\frac{1}{2} \|X\|^2 + R[X]$ , where  $R[X] = \sup_{s \in S} ((X, s) - \frac{1}{2} \|s\|^2)$  is convex, Lipschitz continuous, but not smooth (unless  $S$  is convex), cannot be treated by the usual Cauchy-Lipschitz theory. However the second derivatives of  $R$  are nonnegative bounded measures and we may apply the DiPerna-Lions theory [209] on ODEs with non smooth coefficients, as generalized by Bouchut and Ambrosio to second-order ODEs with "coefficients of bounded variation" [7, 76]. (See also [3, 61, 77, 157, 174, 186, 200, 372, 402] for related topics on ODEs with non smooth coefficients.) In a suitable sense [7], for "almost every initial condition"

$$(X_0, \frac{dX_0}{dt}) \in H \times H,$$

$$\frac{d^2 X_t}{dt^2} = -\nabla Q[X_t] = X_t - \nabla R[X_t]$$

admits a unique global  $C^{1,1}$  solution.

Such a solution is "conservative" and time-reversible. For the system of particles discussed in the previous section, in particular in the framework of 1D-Newtonian gravitation, this corresponds to elastic, non-dissipative collisions.

## Rewriting of the action for "good" curves

There is a subset  $N \subset H$ , which is small in both the Baire category sense and the Lebesgue measure sense (but not empty unless  $S$  is convex), outside of which every point  $X \in H \setminus N$  admits a unique closest point  $\pi[X]$  on  $S$  (cf. related results in [23, 222, 223]) and

$$Q = -\frac{1}{2}\text{dist}^2(\cdot, S)$$

is differentiable at  $X$  with:

$$-\nabla Q[X] = X - \pi[X], \quad Q[X] = -\frac{1}{2}\|X - \pi[X]\|^2 = -\frac{1}{2}\|\nabla Q[X]\|^2.$$

So, the potential can be rewritten as a negative squared gradient.

Thus, for any "good" curve which almost never hits the bad set  $N$ , the action can be written

$$\frac{1}{2} \int_{t_0}^{t_1} \left\| \frac{dX_t}{dt} \right\|^2 + \|\nabla Q[X_t]\|^2 dt$$

which can be rearranged as a perfect square up to a boundary term that does not play any role in the least action principle

$$\frac{1}{2} \int_{t_0}^{t_1} \left\| \frac{dX_t}{dt} + \nabla Q[X_t] \right\|^2 dt - Q[X_{t_1}] + Q[X_{t_0}].$$

## Gradient-flow solutions as special least-action solutions

Due to the very special structure of the action, we find as particular least action solutions any solution to the first-order "gradient-flow equation"

$$\frac{dX_t}{dt} = -\nabla Q[X_t]$$

(somewhat like "instantons" in Yang-Mills theory). However, this is correct only when  $t \rightarrow X_t \in H$  is a "good" curve (i.e. almost never hits the "bad set" where  $Q$  is not differentiable).

## Global dissipative solutions of the gradient-flow

Since  $Q$  is semi-convex, we may use the classical theory of maximal monotone operators (going back to the 70', as in the book by H. Brezis [129]) to solve the initial value problem for the gradient-flow equation.

For each initial condition, there is a unique global solution s.t

$$\frac{d_+ X_t}{dt} = -\overline{\nabla} Q[X_t], \quad \forall t \geq 0, \quad X \in C^0([0, +\infty[, H). \quad (10.3.1)$$

Here,  $\frac{d_+}{dt}$  denotes the right-derivative at  $t$ , and, for each  $X$ ,

$$\overline{\nabla} Q[X] = -X + \overline{\nabla} R[X]$$

where  $\overline{\nabla} R[X]$  is the "relaxed" gradient of the convex function  $R$  at point  $X$ , i.e. the unique  $w \in H$  with lowest norm,  $\|w\|$ , such that

$$R[Z] \geq R[X] + ((w, Z - X)), \quad \forall Z \in H.$$

The relaxed gradient is well defined for every  $X$  and extends the usual gradient to the "bad set"  $N$ . These solutions in the sense of maximal monotone operator theory are in general not conservative solutions (in the sense of Bouchut-Ambrosio) to the original dynamical system. Indeed, they allow velocity jumps and are generally only Lipschitz continuous and not  $C^1$ .

However, they have interesting dissipative features. Indeed, the velocity may jump with an instantaneous loss of kinetic energy.

In the case of one-dimensional gravitating particles, these jumps precisely correspond to sticky collisions [118, 119]. The bad set  $N$  is just the collision set and the relaxed gradient precisely encodes sticky collisions instead of elastic collisions. (Concerning inelastic and sticky collisions, we may refer to [69, 77, 118, 119, 219, 392, 428].)

### The modified action

The conservative solutions, that are only defined for almost every initial condition, manage to hit the bad set only for a negligible amount of time, while the gradient flow solutions enjoy very much staying in it as soon as they enter it.

Our proposal is to pick up the nice dissipative property of the gradient flow solutions and to lift them to the full dynamical system. For that purpose, we introduce the "modified action"

$$\int_{t_0}^{t_1} \left\| \frac{dX_t}{dt} + \bar{\nabla}Q[X_t] \right\|^2 dt \quad (10.3.2)$$

which favors "bad" curves that stay on the "bad set" for a while. Let us recall that  $\bar{\nabla}Q$  denotes the "relaxed" gradient of the semi-convex function

$$Q[X] = -\frac{1}{2} \text{dist}^2(X, S) = -\frac{1}{2} \|X\|^2 + \sup_{s \in S} \left\{ \langle X, s \rangle - \frac{1}{2} \|s\|^2 \right\}. \quad (10.3.3)$$

## 10.4 Stochastic origin of the dissipative least action principle

Using large deviation principles (or alternatively the concept of guiding wave coming from quantum mechanics), we will derive, following [8] and from essentially nothing but noise (namely  $N$  independent Brownian particles without any interaction nor external potential), the dissipative least action principle (10.3.2,10.3.3), for the special system (10.2.1,10.2.2), in the "gravitational" case  $\beta = 1$ . Let us recall that this system is a discretization of the Vlasov-Monge-Ampère system (10.2.3,10.2.4) as well as an approximation of the Euler equations.

The first step of our analysis is very much related to the Schrödinger problem, as analyzed by Christian Léonard [320], and somewhat connected by to recent results by Robert Berman and collaborators, motivated by Kählerian Geometry [50, 52, 53].

### Localization of a Brownian point cloud

Given a point cloud

$$\{A(\alpha) \in \mathbb{R}^d, \alpha = 1, \dots, N\},$$

we consider  $N$  independent Brownian curves issued from this cloud

$$Y_t(\alpha) = A(\alpha) + \sqrt{\epsilon} B_t(\alpha), \quad \alpha = 1, \dots, N.$$



At a fixed time  $T > 0$ , the probability for the moving cloud to reach position  $X = (X(\alpha), \alpha = 1, \dots, N) \in \mathbb{R}^{dN}$  has density

$$\begin{aligned} & \frac{1}{Z} \sum_{\sigma \in \mathfrak{S}_N} \prod_{\alpha=1}^N \exp\left(-\frac{|X(\alpha) - A(\sigma(\alpha))|^2}{2\epsilon T}\right) \\ &= \frac{1}{Z} \sum_{\sigma \in \mathfrak{S}_N} \exp\left(-\frac{\|X - A_\sigma\|^2}{2\epsilon T}\right) \end{aligned}$$

(here  $\mathfrak{S}_N$  denotes the group of all permutations of the first  $N$  integers, while  $|\cdot|$  and  $\|\cdot\|$  are the euclidean norms respectively on  $\mathbb{R}^d$  and  $\mathbb{R}^{Nd}$  and  $Z$  is the normalization factor which is proportional to  $\epsilon^{Nd/2}$ ).

Since

$$-\epsilon \log \frac{1}{Z} \sum_{\sigma \in \mathfrak{S}_N} \exp\left(-\frac{\|X - A_\sigma\|^2}{2\epsilon T}\right) \sim \frac{1}{2T} \inf_{\sigma \in \mathfrak{S}_N} \|X - A_\sigma\|^2$$

as  $\epsilon \rightarrow 0$ , an observer at time  $T$  feels that the particles arrived at  $X_T \in \mathbb{R}^{dN}$ , have travelled along straight lines by "optimal transport"

$$X_t = \left(1 - \frac{t}{T}\right) A_{\sigma_{opt}(T)} + \frac{t}{T} X_T, \quad \sigma_{opt}(T) = \operatorname{Arginf}_{\sigma \in \mathfrak{S}_N} \|X_T - A_\sigma\|^2.$$

This formula implies

$$\frac{dX_t}{dt} = \frac{X_t - A_{\sigma_{opt}(t)}}{t}, \quad \sigma_{opt}(t) = \operatorname{Arginf}_{\sigma \in \mathfrak{S}_N} \|X_t - A_\sigma\|^2.$$

(Indeed, we observe that, for all  $t \in ]0, T[$   $\sigma_{opt}(t)$  is unchanged and equal to  $\sigma_{opt}(T)$ .) The resulting "deterministic" process is, as a matter of fact, just the output of the pure observation of a random process as the level of noise vanishes. This is a good example of order emerging from pure disorder! Of course, this is strongly related to the Schrödinger problem already discussed in this book [320]. It is quite remarkable, as explained in [109], that, from a physical viewpoint, this model is equivalent to the Zeldovich model in Cosmology [465, 426, 245, 116]

### An alternative viewpoint: the pilot wave

We Introduce the heat equation in the space of "clouds"  $X \in \mathbb{R}^{Nd}$

$$\frac{\partial \rho}{\partial t}(t, X) = \frac{\epsilon}{2} \Delta \rho(t, X), \quad \rho(t=0, X) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \delta(X - A_\sigma),$$

where  $\Delta$  is the Laplacian in the very large space  $(\mathbb{R}^d)^N$  and the initial condition has been symmetrized by the symmetric group  $\mathfrak{S}_N$ .

Then, mimicking the idea of "pilot wave" introduced by de Broglie for Quantum Mechanics, we introduce the ODE

$$\frac{dX_t}{dt} = v(t, X_t)$$

where  $v$  is the "pilot" velocity field

$$v(t, X) = -\frac{\epsilon}{2} \nabla_X \log \rho(t, X), \quad t > 0, \quad X \in (\mathbb{R}^d)^N,$$

i.e.

$$\frac{dX_t}{dt} = \frac{1}{2t} \left( X_t - \frac{\sum_{\sigma \in \mathfrak{S}_N} A_\sigma \exp\left(\frac{-\|X_t - A_\sigma\|^2}{2\epsilon t}\right)}{\sum_{\sigma \in \mathfrak{S}_N} \exp\left(\frac{-\|X_t - A_\sigma\|^2}{2\epsilon t}\right)} \right)$$

Notice that, as in de Broglie's theory, the corresponding trajectories are smooth and not at all Brownian curves! [As a matter of fact, a similar calculation also works for the free bosonic Schrödinger equation:

$$(i\partial_t + \kappa\Delta)\psi = 0, \quad \psi(0, X) = \sum_{\sigma \in \mathfrak{S}_N} \exp(-\|X - A_\sigma\|^2/a^2), \quad v = \Im m \nabla \log \psi,$$

where  $\kappa, a > 0$  are suitable constants to be related to the Planck constant. However the analysis becomes much more difficult than for the heat equation and we will not discuss further this very interesting issue.]

Using exponential time  $t = \exp(2\theta)$ , we get

$$\frac{dX_\theta}{d\theta} = X_\theta - \frac{\sum_{\sigma \in \mathfrak{S}_N} A_\sigma \exp\left(\frac{-\|X_\theta - A_\sigma\|^2}{2\epsilon \exp(2\theta)}\right)}{\sum_{\sigma \in \mathfrak{S}_N} \exp\left(\frac{-\|X_\theta - A_\sigma\|^2}{2\epsilon \exp(2\theta)}\right)}.$$

Notice that we may also write (after expanding each square and noticing that  $\|A_\sigma\| = \|A\|$ , for every  $\sigma \in \mathfrak{S}_N$ ):

$$\frac{dX_\theta}{d\theta} = X_\theta - \frac{\sum_{\sigma \in \mathfrak{S}_N} A_\sigma \exp\left(\frac{((X_\theta, A_\sigma))}{\epsilon \exp(2\theta)}\right)}{\sum_{\sigma \in \mathfrak{S}_N} \exp\left(\frac{((X_\theta, A_\sigma))}{\epsilon \exp(2\theta)}\right)} = -\nabla_{\mathbf{X}} Q_\epsilon[\theta, X_\theta],$$

$$Q_\epsilon[\theta, X] = -\frac{\|X\|^2}{2} + \epsilon \exp(2\theta) \log \sum_{\sigma \in \mathfrak{S}_N} \exp\left(\frac{((X, A_\sigma))}{\epsilon \exp(2\theta)}\right), \quad X \in (\mathbb{R}^d)^N.$$

This "potential"  $Q_\epsilon$  is a (time-dependent) semi-convex function. Indeed

$$X \in (\mathbb{R}^d)^N \rightarrow \epsilon \exp(2\theta) \log \sum_{\sigma \in \mathfrak{S}_N} \exp\left(\frac{((X_\theta, A_\sigma))}{\epsilon \exp(2\theta)}\right)$$

is a convex function in  $X$ , with a Lipschitz constant uniformly bounded in  $\epsilon$  and  $\theta$  by  $\|A\|$  and its limit, in sup norm, is just

$$X \rightarrow \sup_{\sigma \in \mathfrak{S}_N} ((X_\theta, A_\sigma)).$$

Thus, the limit in  $\epsilon \rightarrow 0$  of this smooth ODE can be analyzed in the framework of maximal monotone operators [129] and we obtain (10.3.1) the generalized ODE, which should be understood in the sense of maximal monotone operators,

$$\frac{d_+ X_\theta}{d\theta} = -\overline{\nabla} Q[X_\theta],$$

$$Q[X] = -\frac{\|X\|^2}{2} + \sup_{\sigma \in \mathfrak{S}_N} ((X_\theta, A_\sigma)) = \frac{\|A\|^2}{2} - \inf_{\sigma \in \mathfrak{S}_N} \frac{\|X - A_\sigma\|^2}{2},$$

in which features the generalized gradient of the limit potential  $Q$ . So, at this stage, up to the change of time variable  $t = \exp(2\theta)$ , we have fully recovered the dissipative system already discussed in the previous sections. However, in order to get the discrete Vlasov-Monge-Ampère system we are mostly interested in, it is better to keep  $\epsilon > 0$  fixed for a while, and apply the large deviation theory to the "pilot wave" ODE.

## Large deviations of the pilot system

Let us fix  $\epsilon$  for a while and add some noise  $\eta$  to the "guided" trajectories

$$\frac{dX_\theta}{d\theta} = -\nabla Q_\epsilon[\theta, X_\theta] + \sqrt{\eta} \frac{dB_\theta}{d\theta}.$$

Since

$$Q_\epsilon[\theta, X] + \frac{\|X\|^2}{2} = \epsilon \exp(2\theta) \log \sum_{\sigma \in \mathfrak{G}_N} \exp\left(\frac{((X, A_\sigma))}{\epsilon \exp(2\theta)}\right)$$

is a smooth function, Lipschitz continuous in  $X$ , we may apply the standard large deviation theory of Vencel-Freidlin [243] that asserts that the probability to go from a point  $Y_0 \in (\mathbb{R}^d)^N$  at time  $\theta = \theta_0$  to some other point  $Y_1 \in (\mathbb{R}^d)^N$  at time  $\theta = \theta_1$  essentially behaves (in a suitable technical sense), as  $\eta \rightarrow 0$ , as

$$\exp\left(-\frac{\mathcal{A}[\theta_0, \theta_1, Y_0, Y_1]}{\eta}\right), \quad \mathcal{A}[\theta_0, \theta_1, Y_0, Y_1] =$$

$$\inf\{\mathfrak{I}_\epsilon[X; \theta_0, \theta_1]; \quad X \in C^1([\theta_0, \theta_1]; (\mathbb{R}^d)^N), \quad X_{\theta_0} = Y_0, \quad X_{\theta_1} = Y_1\},$$

where  $\mathfrak{I}_\epsilon$  is the so-called good rate function

$$\mathfrak{I}_\epsilon[X; \theta_0, \theta_1] = \frac{1}{2} \int_{\theta_0}^{\theta_1} \left( \left\| \frac{dX_\theta}{d\theta} + \nabla Q_\epsilon[\theta, X_\theta] \right\|^2 \right) d\theta.$$

It also shows that the most likely trajectories converge to minimizers of the good rate function. Finally, we may let  $\epsilon$  go to zero. One can prove, as done in [8], that the good rate function  $\Gamma$ -converges, as  $\epsilon \rightarrow 0$  to

$$\mathfrak{I}[X; \theta_0, \theta_1] = \frac{1}{2} \int_{\theta_0}^{\theta_1} \left( \left\| \frac{dX_\theta}{d\theta} + \bar{\nabla} Q[X_\theta] \right\|^2 \right) d\theta.$$

where

$$Q[X] = -\frac{\|X\|^2}{2} + \sup_{\sigma \in \mathfrak{G}_N} ((X_\theta, A_\sigma)) = \frac{\|A\|^2}{2} - \inf_{\sigma \in \mathfrak{G}_N} \frac{\|X - A_\sigma\|^2}{2},$$

which exactly returns the dissipative least action principle introduced and discussed in the previous sections.

# Chapter 11

## Appendix: Hamilton-Jacobi equations and viscosity solutions

In this book, we have so much emphasized the interest of convex methods that we have entirely omitted the paramount role of Fourier methods in PDEs [26, 293, 442] ! This appendix can be seen as a tribute to Fourier, paradoxically devoted to the Hamilton-Jacobi equation

$$(HJ) \quad \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0,$$

which is a rare example of PDE for which, not only the Fourier analysis, but also the theory of Lebesgue spaces can be entirely ignored, in particular thanks to the remarkable theory of so-called "viscosity solutions", by Crandall, Evans and Lions [184], which relies only on the concept of continuous and semi-continuous functions, without any reference to Lebesgue spaces and, of course, to the Fourier analysis. A typical result is the full understanding of the "Hopf formula" which provides the unique solution  $\phi(t, x)$  of the HJ equation in terms of its initial data  $\phi(0, x)$ , for all  $t \geq 0$  and  $x \in \mathbb{R}^d$ , through:

$$\phi(t, x) = \inf_{\xi \in \mathbb{R}^d} \frac{|\xi - x|^2}{2t} + \phi(0, \xi).$$

This formula is very much related to convex analysis (and more specifically to the Legendre-Fenchel transform). The purpose of this appendix is to explain, following E. Hopf [292], how this beautiful formula can be deduced from the heat equation (and the way Fourier solved it) through the Laplace lemma, which can be seen as an elementary version of the Large Deviation Theory [243].

The basic idea comes from Feynman's interpretation of Quantum Mechanics with his concept of "path integrals". However, let us start at a more conventional level by reminding the well known solution of the heat equation thanks to Gaussian integrals that follow almost instantaneously from its Fourier analysis.

More precisely,

let us introduce the so-called heat semi-group on  $\mathbb{R}^d$

$$(S_\epsilon(t)v)(x) = \int_{\mathbb{R}^d} \exp(-\pi|y|^2) v(x + \sqrt{2\pi\epsilon t} y) dy, \quad t \geq 0, \quad x \in \mathbb{R}^d$$

and recall the well known formula

$$\int_{\mathbb{R}^d} \exp(-\pi|y|^2) dy = 1.$$

A straightforward calculation shows that  $u(t, x) = (S_\epsilon(t)v)(x)$ , which is  $C^\infty$  in  $(t, x) \in ]0, +\infty[ \times \mathbb{R}^d$ , is indeed a classical solution to the heat equation

$$\partial_t u = \epsilon \Delta u / 2,$$

with initial condition  $u(0, \cdot) = v$ .

Remark. Note that the analogous formula

$$\int_{\mathbb{R}^d} \exp(i\pi|y|^2)v(x + \sqrt{2\pi\epsilon t}y)dy, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d$$

provides the general solution to the (free) Schrödinger equation

$$i\partial_t u + \frac{\epsilon}{2}\Delta u = 0,$$

with initial condition  $u(0, \cdot) = v$ .

## Exponential transform and Laplace lemma

As soon as  $v \geq 0$  is not identically null, the solution to the heat equation  $u_\epsilon(t, x) = (S_\epsilon(t)v)(x)$  is strictly positive everywhere for each  $t > 0$  and it makes sense to write it in exponential form

$$u_\epsilon(t, x) = \exp\left(-\frac{\phi_\epsilon(t, x)}{\epsilon}\right).$$

From

$$\partial_t u_\epsilon = \epsilon \Delta u_\epsilon / 2,$$

we easily get

$$\partial_t \phi_\epsilon + \frac{1}{2}|\nabla \phi_\epsilon|^2 = \epsilon \Delta \phi_\epsilon / 2.$$

[Indeed

$$\epsilon du_\epsilon / u_\epsilon = \epsilon d(\log u_\epsilon) = -d\phi_\epsilon$$

and, therefore,

$$\begin{aligned} \epsilon \partial_t u_\epsilon &= -u_\epsilon \partial_t \phi_\epsilon, \\ \epsilon \nabla u_\epsilon &= -u_\epsilon \nabla \phi_\epsilon, \\ \epsilon \Delta u_\epsilon &= \epsilon^{-1} u_\epsilon |\nabla \phi_\epsilon|^2 - u_\epsilon \Delta \phi_\epsilon. \end{aligned}$$

Finally

$$0 = -\epsilon \partial_t u_\epsilon + \epsilon^2 \Delta u_\epsilon / 2 = u_\epsilon |\nabla \phi_\epsilon|^2 / 2 - \epsilon u_\epsilon \Delta \phi_\epsilon / 2 + u_\epsilon \partial_t \phi_\epsilon,$$

after dividing by  $u_\epsilon$ .]

So, we may expect to solve the (fully nonlinear) Hamilton-Jacobi equation

$$\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 = 0,$$

just by passing to the limit  $\epsilon \rightarrow 0$  !

Let us try to pass to the limit in the formula we have already obtained

$$u_\epsilon(t, x) = \int_{\mathbb{R}^d} \exp(-\pi|y|^2)v(x + \sqrt{2\pi\epsilon t}y)dy, \quad t \geq 0, \quad x \in \mathbb{R}^d.$$

It is actually more convenient to let the initial condition depend also on  $\epsilon$  by writing it as

$$v(x) = v_\epsilon(x) = \exp\left(-\frac{\psi(x)}{\epsilon}\right)$$

so that  $\psi(x)$ , which does not depend on  $\epsilon$ , may be seen as the value of  $\phi_\epsilon(t, x)$  at  $t = 0$ . Let us also assume  $\psi$  to be uniformly continuous with

$$\lim_{|x| \rightarrow \infty} \frac{|\psi(x)|}{1 + |x|^2} = 0.$$

So, we have

$$u_\epsilon(t, x) = (2\pi\epsilon t)^{-d/2} \int_{\mathbb{R}^d} \exp\left(-\frac{|\xi - x|^2}{2\epsilon t}\right) v_\epsilon(\xi) d\xi$$

(by performing the change of variable  $y \rightarrow \xi = x + \sqrt{2\pi\epsilon t} y$ )

$$= (2\pi\epsilon t)^{-d/2} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{\epsilon} \left(\frac{|\xi - x|^2}{2t} + \psi(\xi)\right)\right) d\xi.$$

Since

$$u_\epsilon(t, x) = \exp\left(-\frac{\phi_\epsilon(t, x)}{\epsilon}\right),$$

we get

$$\phi_\epsilon(t, x) = -\epsilon \log u_\epsilon(t, x) = \log(2\pi\epsilon t)\epsilon d/2 - \epsilon \log \int_{\mathbb{R}^d} \exp\left(\frac{1}{\epsilon} F(\xi; t, x)\right) d\xi,$$

where

$$F(\xi; t, x) = -\frac{|\xi - x|^2}{2t} - \psi(\xi).$$

Let us now use the Laplace lemma (which can be seen as the starting point of the large deviation theory).

**Lemma 11.0.1.** *Let  $A$  be a non negligible Lebesgue measurable set in  $\mathbb{R}^d$  and let  $F$  be a Lebesgue measurable function such that*

$$0 < \int_A \exp(F(\xi)) d\xi < +\infty.$$

*Then, as  $\epsilon \downarrow 0$ ,*

$$\epsilon \log \left( \int_A \exp\left(\frac{F(\xi)}{\epsilon}\right) d\xi \right) \rightarrow \sup \operatorname{ess}_A F.$$

*Proof.*

We first write  $\epsilon = (1 + R)^{-1}$  so that

$$I = \int_A \exp\left(\frac{F(\xi)}{\epsilon}\right) d\xi = \int_A \exp(F(\xi)) \exp(RF(\xi)) d\xi.$$

Let  $L$  be the essential supremum of  $F$  on  $A$  and define

$$J = \int_A \exp(F(\xi)) d\xi.$$

We have  $0 < J < +\infty$  by assumption which makes its logarithm finite. We first get the obvious upper bound

$$I \leq \exp(RL) \int_A \exp(F(\xi)) d\xi$$

and, therefore,

$$\epsilon \log I = \frac{1}{R+1} \log I \leq \frac{1}{R+1} (RL + \log J) \rightarrow L, \quad \epsilon \downarrow 0.$$

To get a lower bound for  $I$ , let us fix any  $\lambda < L$ . By definition of  $L$ , there is non negligible Lebesgue measurable subset  $B$  of  $A$  such that  $F(\xi) \geq \lambda$  for each  $\xi \in B$ . We have

$$K = \int_B \exp(F(\xi)) d\xi \in ]0, +\infty[.$$

[Indeed,  $K$  is not larger than  $I$  and thus finite. Moreover  $K \geq \exp(\lambda) \int_B d\xi > 0$ .] So

$$I \geq \int_B \exp(F(\xi)) \exp(RF(\xi)) d\xi \geq \exp(R\lambda) \int_B \exp(F(\xi)) d\xi = \exp(R\lambda) K$$

and

$$\epsilon \log I = \frac{1}{R+1} \log I \geq \frac{1}{R+1} (R\lambda + \log K) \rightarrow \lambda, \quad \epsilon \downarrow 0,$$

which completes the proof since  $\lambda$  can be chosen arbitrarily close to  $L$ .

*End of Proof.*

let us now apply the Laplace lemma, for every fixed  $t > 0$  and  $x$ , to the solution  $\phi_\epsilon(t, x)$  of the “viscous” Hamilton-Jacobi equation

$$\partial_t \phi_\epsilon + \frac{1}{2} |\nabla \phi_\epsilon|^2 = \epsilon \Delta \phi_\epsilon / 2$$

with initial condition  $\phi_\epsilon(0, \cdot) = \psi$ , which does not depend on  $\epsilon$ . let us recall that

$$\phi_\epsilon(t, x) = \log(2\pi\epsilon t) \epsilon d / 2 - \epsilon \log \int_{\mathbb{R}^d} \exp\left(\frac{1}{\epsilon} F(\xi; t, x)\right) d\xi,$$

where

$$F(\xi; t, x) = -\frac{|\xi - x|^2}{2t} - \psi(\xi).$$

Since we have assumed

$$\lim_{|\xi| \rightarrow \infty} \frac{|\psi(\xi)|}{1 + |\xi|^2} = 0,$$

we may apply the Laplace lemma with  $A = \mathbb{R}^d$  and  $F(\xi) = F(\xi; t, x)$  (with an abuse of notation,  $(t, x)$  being fixed). Passing to the limit, we get

$$\phi(t, x) = \inf_{\xi \in \mathbb{R}^d} \frac{|\xi - x|^2}{2t} + \psi(\xi)$$

which provides the so-called “Hopf formula” for the Hamilton-Jacobi equation

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0.$$

[An easy way to memorize the Hopf formula is to use the somewhat incorrect but interesting following reasoning:

i) We first write the HJ equation as

$$\sup_{w \in \mathbb{R}^d} \left( \partial_t \phi + w \cdot \nabla \phi - \frac{|w|^2}{2} \right) = 0$$

and we make the (a priori unjustified) ansatz

$$\phi(t, x) = \inf_{w \in \mathbb{R}^d} \Phi(t, x; w)$$

where  $\Phi$  is solution to the underlying constant coefficient linear PDE in  $(t, x)$ , where  $w \in \mathbb{R}^d$  is just a parameter

$$\partial_t \Phi + w \cdot \nabla \Phi - \frac{|w|^2}{2} = 0,$$

with initial condition  $\Phi(0, x; w) = \psi(x)$ . We immediately obtain

$$\Phi(t, x; w) = \Phi(0, x - tw, w) + t \frac{|w|^2}{2} = \psi(x - tw) + t \frac{|w|^2}{2},$$

which leads to

$$\phi(t, x) = \inf_{w \in \mathbb{R}^d} \psi(x - tw) + t \frac{|w|^2}{2} = \inf_{\xi \in \mathbb{R}^d} \frac{|\xi - x|^2}{2t} + \psi(\xi),$$

which is the correct Hopf formula!]

The Hopf formula corresponds to a “vanishing viscosity solution”. Notice that this solution is no longer a smooth function for all  $t > 0$ , unless  $\psi$  is convex and smooth. Indded, for a fixed  $x$ , as  $t$  grows, the infimum can be achieved by several distinct points  $\xi$  which destroys the smoothness of  $\phi$  as a function of  $(t, x)$ , no matter how smooth  $\psi$  can be. This appearance of singularity makes difficult the analysis of the HJ equation.

## The Crandall-Evans-Lions theory of viscosity solutions

It is tempting to go beyond the Hopf formula to treat more general fully nonlinear PDEs such as

$$\partial_t \phi + H(t, x, \nabla \phi) = 0$$

or, even,

$$\partial_t \phi + H(t, x, \nabla \phi, D^2 \phi) = 0,$$

assuming  $(t, x, w, M) \rightarrow H(t, x, w, M) \in \mathbb{R}$

i) to be smooth with respect to  $t \in \mathbb{R}_+$ ,  $x \in \mathbb{R}^d$ ,  $w \in \mathbb{R}^d$  and  $M$ , valued in the set of all symmetric  $d \times d$  matrices;

ii) to satisfy

$$|H(t, x, w, M)| \leq C(1 + |w|^\alpha + |M|^\beta)$$

for suitable constants  $C, \alpha, \beta$ ;

iii) to be non increasing in  $M$



(in the sense that  $H(t, x, w, M) \geq H(t, x, w, \tilde{M})$  whenever  $\tilde{M} - M$  is a nonnegative symmetric matrix).

This is the purpose of the Crandall-Evans-Lions theory of viscosity solutions [184, 185] which started in the 80s with the Hamilton-Jacobi equation

$$\partial_t \phi + H(t, x, \nabla \phi) = 0.$$

(See also [39]. Notice that there are alternative concepts of solutions for the HJ equation [54, 191, 454].) A “viscosity solution” is a priori not supposed to be smooth, but merely continuous (or, at most, Lipschitz continuous, but certainly not  $C^1$ ) and is defined in a particularly original and clever way. Let us consider a smooth test function  $\zeta(t, x)$  and any point  $(t_0, x_0)$  where  $\phi - \zeta$  possibly achieves a minimum (which may be local, as a matter of fact). If  $\phi$  were smooth, we would deduce

$$\partial_t \phi(t_0, x_0) = \partial_t \zeta(t_0, x_0), \quad \nabla \phi(t_0, x_0) = \nabla \zeta(t_0, x_0),$$

which suggests, at point  $(t_0, x_0)$ , to substitute the derivatives of  $\zeta$  for the derivatives of  $\phi$  (which are not well defined). So, we require

$$\partial_t \zeta(t_0, x_0) + H(t_0, x_0, \nabla \zeta(t_0, x_0)) \geq 0.$$

For a (local) maxima, we would require, instead,

$$\partial_t \zeta(t_0, x_0) + H(t_0, x_0, \nabla \zeta(t_0, x_0)) \leq 0.$$

[With a little imagination, we can see this formulation as the  $(\max, +)$  version of the usual formulation of PDEs in the sense of distributions!]

It is quite remarkable that such a formulation does not involve any knowledge on the Lebesgue measure theory and could have been discovered before the Lebesgue integral and without the Fourier transform!

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