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Spectral stability of the critical front in the extended Fisher-KPP equation

Montie Avery* Louis Garénaux†

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Abstract

We revisit the existence and stability of the critical front in the extended Fisher-KPP equation, refining earlier results of Rottschäfer and Wayne [28] which establish stability of fronts without identifying a precise decay rate. We verify that the front is marginally spectrally stable: while the essential spectrum touches the imaginary axis at the origin, there are no unstable eigenvalues and no eigenvalue (or *resonance*) embedded in the essential spectrum at the origin. Together with the recent work of Avery and Scheel [3], this implies nonlinear stability of the critical front with sharp $t^{-3/2}$ decay rate, as previously obtained in the classical Fisher-KPP equation. The main challenges are to regularize the singular perturbation in the extended Fisher-KPP equation and to track eigenvalues near the essential spectrum, and we overcome these difficulties with functional analytic methods.

1 Introduction

1.1 Background and main results

We consider the extended Fisher-KPP equation

$$u_t = -\delta^2 u_{xxxx} + u_{xx} + f(u), \quad (1.1)$$

for small values of the parameter δ . Notice that the sign of $\delta \in \mathbb{R}$ will play no role in what follows. Here we assume f is of Fisher-KPP type: $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, and $f''(u) < 0$ for all $u \in (0, 1)$; see Section 1.2 for comments on this last assumption. Equation (1.1) may be derived as an amplitude equation near certain co-dimension 2 bifurcations in reaction-diffusion systems [27], and for $\delta = 0$, it reduces to the classical Fisher-KPP equation, which is a fundamental model for front propagation into unstable states. The extended Fisher-KPP equation is of interest in the physics literature as a concrete model in which one can study the transition from monotone to pattern-forming invasion fronts [8]. In both the classical and extended Fisher-KPP equations, much of the interesting dynamics are driven by traveling fronts, which solve

$$0 = -\delta^2 q'''' + q'' + cq' + f(q), \quad q(-\infty) = 1, \quad q(\infty) = 0. \quad (1.2)$$

In the classical Fisher-KPP equation, $\delta = 0$, the front equation (1.2) admits monotone solutions for all speeds $c \geq c_*(0) = 2\sqrt{f'(0)}$; see [32]. Using comparison principles [1, 18, 25, 24] or probabilistic methods [5, 6] it can be shown that the *critical front* with speed $c_*(0)$ governs the dynamics for large classes of initial data, in the sense that solutions typically converge to the critical front, with some higher order corrections to the position. Such results are not available for the extended Fisher-KPP equation due to the lack of a comparison principle.

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Much work has also been done on the stability of fronts in the Fisher-KPP equation with respect to sufficiently localized perturbations. For *supercritical fronts*, with speeds $c > c_*(0)$, stability with an exponential decay rate $e^{-\Lambda t}$ can be achieved by using exponential weights to push the essential spectrum entirely into the left half-plane and suppress the translational eigenvalue at the origin. Stability of the critical front is more difficult, since with the optimal choice of exponential weight, the essential spectrum still touches the imaginary axis due to the presence of absolute spectrum [29] at the origin. Local stability of the critical front was established by Kirchgässner [23] and later refined in [9, 15, 10, 3]. The sharpest results [15, 10, 3] establish that sufficiently localized perturbations to the critical front decay with algebraic rate $t^{-3/2}$. This result was first obtained by Gallay [15] using renormalization group theory, and it was later observed in [30] that this improved decay rate, compared to the diffusive decay rate $t^{-1/2}$, can be explained by the lack of an eigenvalue embedded in the essential spectrum at the origin, a perspective directly exploited to obtain nonlinear stability in [10, 3].

One typically expects that the linearization about a traveling wave in a translation invariant system will have an eigenvalue at the origin, with the derivative of the wave giving the eigenfunction. However, in the classical Fisher-KPP equation, the critical front $q_*(\cdot; 0)$ has weak exponential decay

$$q_*(x; 0) \sim x e^{-\eta_*(0)x}, \quad \text{as } x \rightarrow \infty,$$

where $\eta_*(0) = c_*(0)/2 = \sqrt{f'(0)}$. Since one uses exponential weight to stabilize the essential spectrum, the derivative of the front is unbounded in this weighted space, with $e^{\eta_*(0)x} q'_*(x; 0) \sim \eta_*(0)x$ as $x \rightarrow \infty$. Hence, in the appropriately weighted space, there is not an eigenvalue at the origin even corresponding to a *bounded* eigenfunction. Such an embedded eigenvalue to a bounded eigenfunction is sometimes called a *resonance*. This generalized notion of an eigenvalue is the appropriate, dynamically relevant approach to considering eigenvalues embedded in the essential spectrum — for instance, resonances correspond to zeros of the Evans function, once the Evans function is extended into the essential spectrum with the gap lemma [22, 16]. The lack of a zero of the Evans function at the origin was central to the proof of stability of the critical front via pointwise semigroup methods in [10].

For δ small, the extended Fisher-KPP equation is a singular perturbation of the classical Fisher-KPP equation. The presence of fourth order dissipation, however small, immediately breaks the comparison principle, rendering stability questions more difficult. Using geometric singular perturbation theory, Rottschäfer and Wayne showed [28] that the picture for existence of fronts in the classical Fisher-KPP equation extends to δ small but nonzero: there is a critical speed $c_*(\delta)$ such that there exist monotone front solutions to (1.2) if and only if $c \geq c_*(\delta)$. The authors also proved asymptotic stability of all of these fronts in the extended Fisher-KPP equation against sufficiently localized perturbations. However, their approach, using control of certain energy functionals, does not identify a precise decay rate for the perturbations. We also mention that the existence of both invasion fronts and fronts connecting two stable states in fourth order parabolic equations, including the extended Fisher-KPP equation with δ not necessarily small, was established in [4] using topological arguments.

Here, we revisit the stability of the critical front in the extended Fisher-KPP equation, putting this equation in the general framework of stability of pulled fronts recently presented in [3]. There, Scheel and the first author obtain the $t^{-3/2}$ decay rate for perturbations to critical fronts in general higher order parabolic equations, under assumptions on the existence and spectral stability of critical fronts. The main difficulty in verifying these assumptions for the extended Fisher-KPP equation is to simultaneously handle the singular nature of the perturbation while also tracking potential eigenvalues embedded in the essential spectrum. The main contributions of the present paper are as follows:

1. We verify the spectral stability assumptions of [3] for (1.2), thereby obtaining the $t^{-3/2}$ decay rate as well as an asymptotic description of the perturbation as immediate corollaries.
2. Our approach illustrates conceptually that these spectral stability assumptions as well as the existence of critical fronts are robust within the class of equations considered in [3]: these features persist under small perturbations to the equation considered, even in our singularly perturbed setting.
3. We do not rely on geometric singular perturbation theory or the machinery of normally hyperbolic invariant manifolds, instead further developing self-contained, functional analytic approaches to singular perturbations and the study of eigenvalues embedded in the essential spectrum previously explored in

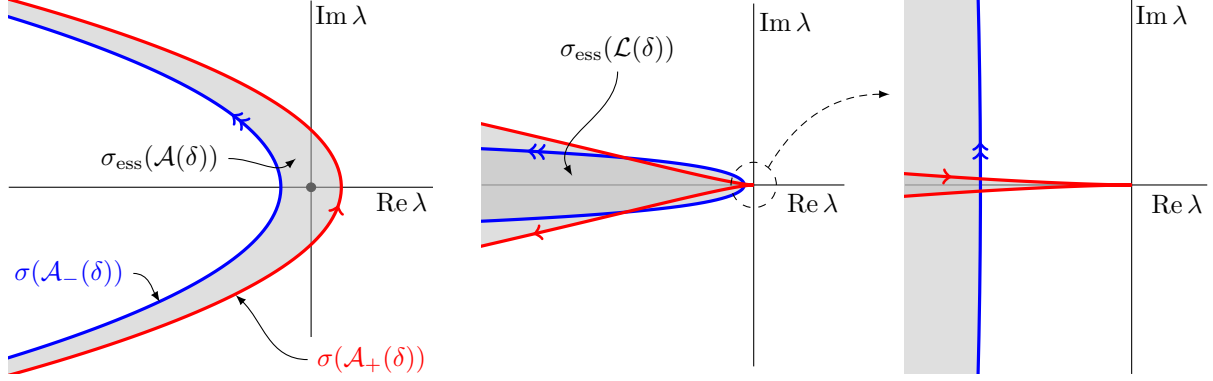


Figure 1: Left: the essential spectrum of the unweighted operator $\mathcal{A}(\delta)$ is bounded by the Fredholm borders in blue and red. Middle and right: overview and zoom near the origin of the essential spectrum of the weighted operator $\mathcal{L}(\delta)$.

[17] and [26]. We also reprove the existence of the critical front using a far-field/core decomposition relying only on functional analysis.

The threshold speed $c_*(\delta)$ identified in [28] is the *linear spreading speed* for the extended Fisher-KPP equation, in the sense that solutions with compactly supported initial data to the linearization of (1.2) about $u \equiv 0$ in a co-moving frame,

$$u_t = -\delta^2 u_{xxxx} + u_{xx} + cu_x + f'(0)u$$

generically grow pointwise for $c < c_*(\delta)$ and decay exponentially pointwise for $c > c_*(\delta)$. In this case, the linear spreading speed is determined by the presence of a simple pinched double root of the *dispersion relation*

$$d_c^+(\lambda, \nu) = -\delta^2 \nu^4 + \nu^2 + c\nu + f'(0) - \lambda, \quad (1.3)$$

at $(\lambda, \nu) = (0, -\eta_*(\delta))$ for $c = c_*(\delta)$; see Lemma 1.1 for details, and [20] for background on linear spreading speed theory. We first reprove the existence of the critical front using a functional analytic approach to regularize the perturbation.

Theorem 1 (Existence of the critical front). *For δ sufficiently small and for $c = c_*(\delta)$, there exists a smooth traveling front q_* solving (1.2), such that*

$$q_*(x; \delta) = (\mu(\delta) + x)e^{-\eta_*(\delta)x} + O(e^{-(\eta_*(\delta)+\eta)x}), \quad x \rightarrow \infty$$

for some $\eta > 0$. Moreover, $q_*(\cdot; \delta) \rightarrow q_*(\cdot; 0)$ uniformly in space as $\delta \rightarrow 0$.

We then turn to the stability of this critical front under a reasonable class of perturbations. Writing the solution to (1.1) as $u(t, x) = q_*(x - c_*(\delta)t; \delta) + v(t, x - c_*(\delta)t)$, the perturbation v solves

$$v_t = \mathcal{A}(\delta)v + f(q_* + v) - f(q_*) - f'(q_*)v, \quad (1.4)$$

where $\mathcal{A}(\delta) : H^4(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is the linearization about the critical front, defined through

$$\mathcal{A}(\delta) := -\delta^2 \partial_x^4 + \partial_x^2 + c_* \partial_x + f'(q_*(x; \delta)). \quad (1.5)$$

This linear operator has unstable essential spectrum at $+\infty$, due to the fact that $f'(0) > 0$; see Figure 1. Hence, the critical front $q_*(\cdot; \delta)$ is unstable to perturbations $v_0 \in H^4(\mathbb{R})$, and we thereby restrict ourselves to exponentially localized perturbations. For each δ small we define a smooth positive exponential weight $\omega_*(\cdot; \delta)$ satisfying

$$\omega_*(x; \delta) = \begin{cases} e^{\eta_*(\delta)x}, & x \geq 1, \\ 1, & x \leq -1, \end{cases} \quad (1.6)$$

and assume that the perturbation has the form $v(t, \cdot) = p(t, \cdot)\omega_*^{-1}$, with $p(t, \cdot) \in H^4(\mathbb{R})$. We sometimes suppress the dependence of ω_* on δ if it is clear from context. The linear dynamics for p are then driven by $\mathcal{L}(\delta) : H^4(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, given by

$$\mathcal{L}(\delta) := \omega_* \mathcal{A}(\delta) \omega_*^{-1} = -\delta^2 \partial_x^4 + \delta^2 a_3 \partial_x^3 + (1 + \delta^2 a_2) \partial_x^2 + a_1 \partial_x + a_0,$$

where the coefficients $a_i(x; \delta)$ converge to limits $a_i^\pm(\delta)$ exponentially quickly when $x \rightarrow \pm\infty$, and are defined using the local notation $\varpi(x) := 1/\omega_*(x)$ by the following expressions:

$$a_3 = -4 \frac{\varpi'}{\varpi}, \quad a_2 = -6 \frac{\varpi''}{\varpi}, \quad a_1 = c_* + 2 \frac{\varpi'}{\varpi} - 4\delta^2 \frac{\varpi'''}{\varpi}, \quad a_0 = f'(q_*) + c_* \frac{\varpi'}{\varpi} + \frac{\varpi''}{\varpi} - \delta^2 \frac{\varpi''''}{\varpi}. \quad (1.7)$$

We note that $\varpi^{(k)}(x)/\varpi(x) = (-\eta_*)^k$ for $x \geq 1$.

For such a linear operator, the essential spectrum is delimited by the two Fredholm borders, which are defined using the asymptotic dispersion relations.

Lemma 1.1 (Linear spreading speed). *There exists $\delta_0 > 0$ such that for $\delta \in (-\delta_0, \delta_0)$, there exists a critical speed $c_* = c_*(\delta)$, and an exponent $\eta = \eta_*(\delta) > 0$ for the critical weight such that the right dispersion relation (1.3) satisfies the following properties.*

(i) *Simple pinched double root: for λ, ν near $0 \in \mathbb{C}$:*

$$d_{c_*}^+(\lambda, -\eta_* + \nu) = \nu^2 \sqrt{1 - 12\delta^2 f'(0)} - \lambda + O(\nu^3), \quad (1.8)$$

with $\sqrt{1 - 12\delta^2 f'(0)} > 0$.

(ii) *Minimal critical spectrum: If $d_{c_*}^+(i\kappa, -\eta_* + i\kappa) = 0$ for some $\kappa, k \in \mathbb{R}$, then $\kappa = k = 0$.*

(iii) *No unstable essential spectrum: If $d_{c_*}^+(\lambda, -\eta_* + ik) = 0$ for some $k \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, then $\text{Re } \lambda \leq 0$.*

We prove this lemma in Section 2.1. The left dispersion relation, given by

$$d_{c_*}^-(\lambda, \nu) = -\delta^2 \nu^4 + \nu^2 + c_*(\delta)\nu + f'(1) - \lambda$$

is also relevant for determining the essential spectrum of $\mathcal{L}(\delta)$; intuitively, it determines what happens to perturbations in the wake of the front. Since the invading state $u \equiv 1$ is locally stable, with $f'(1) < 0$, one immediately obtains the following lemma.

Lemma 1.2 (Stability on the left). *If $d_{c_*}^-(\lambda, ik) = 0$ for some $k \in \mathbb{R}$, then $\text{Re } \lambda < 0$.*

These two lemmas together with Palmer's theorem imply that the essential spectrum of $\mathcal{L}(\delta)$ is marginally stable, touching the imaginary axis only at the origin [21, 14]; see Figure 1. Our main result is the following theorem, which states that $\mathcal{L}(\delta)$ has no unstable eigenvalues and no resonance at $\lambda = 0$.

Theorem 2 (Spectral stability). *There exists a $\delta_0 > 0$ such that for all $\delta \in (-\delta_0, \delta_0)$ the operator $\mathcal{L}(\delta)$ has no eigenvalues λ with $\text{Re } \lambda \geq 0$, and there does not exist a bounded solution to $\mathcal{L}(\delta)u = 0$.*

Together, Lemma 1.1, Lemma 1.2, and Theorem 2 imply that for δ small, the critical front in the extended Fisher-KPP equation satisfies the spectral stability assumptions of [3], and so as a corollary we obtain two results on nonlinear stability of the critical front. To state these, we first define for $r \in \mathbb{R}$ a smooth positive one-sided algebraic weight ρ_r which satisfies

$$\rho_r(x) = \begin{cases} 1, & x \leq -1, \\ (1 + x^2)^{r/2}, & x \geq 1. \end{cases}$$

Corollary 1.3 (Nonlinear stability). *Let $r > \frac{3}{2}$. There exist constants $\varepsilon > 0$ and $C > 0$ such that if $\|\omega_* \rho_r v_0\|_{H^1} < \varepsilon$, then*

$$\|\omega_* \rho_{-r} v(t, \cdot)\|_{H^1} \leq \frac{C\varepsilon}{(1+t)^{3/2}},$$

where v is the solution to (1.4) with initial data v_0 .

Corollary 1.4 (Nonlinear stability with asymptotics). *Let $r > \frac{5}{2}$. There exist constants $\varepsilon > 0$ and $C > 0$ such that if $\|\omega_* \rho_r v_0\|_{H^1} < \varepsilon$, then there exists a real number $\alpha_* = \alpha_*(\omega_* \rho_r v_0)$, depending smoothly on $\omega_* \rho_r v_0$ in $H^1(\mathbb{R})$, such that for $t > 1$,*

$$\|\rho_{-r} \omega_*(v(t, \cdot) - \alpha_* t^{-3/2} q'_*(\cdot; \delta))\|_{H^1} \leq \frac{C\varepsilon}{(1+t)^2},$$

where v is the solution to (1.4) with initial data v_0 .

1.2 Remarks

Assumptions on f . Since we prove our results by perturbing from the classical Fisher-KPP equation, our results hold for any smooth nonlinearity f which satisfies $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, and for which existence and spectral stability of the critical front hold for the classical Fisher-KPP equation with this reaction term. In particular, this is implied by the assumption $f''(u) < 0$ for $u \in (0, 1)$ [32, Theorem 5.5], which we state in the introduction. This can be weakened, for instance, to the assumption that $0 < f(u) \leq f'(0)u$ for $u \in (0, 1)$; see e.g. [1].

General approach – preconditioning. Our approach to regularizing the singular perturbation is based on preconditioning with an appropriately chosen operator. To illustrate the main idea, briefly consider the eigenvalue problem for the unweighted linearization, $(\mathcal{A}(\delta) - \lambda)u = 0$. Applying $(1 - \delta^2 \partial_x^2)^{-1}$ to $\mathcal{A}(\delta) - \lambda$, we obtain

$$\begin{aligned} (1 - \delta^2 \partial_x^2)^{-1}(\mathcal{A}(\delta) - \lambda) &= (1 - \delta^2 \partial_x^2)^{-1}[(1 - \delta^2 \partial_x^2) \partial_x^2 + c_*(\delta) \partial_x + f'(q_*) - \lambda] \\ &= \partial_x^2 + (1 - \delta^2 \partial_x^2)^{-1}(c_*(\delta) \partial_x + f'(q_*) - \lambda) \\ &= \partial_x^2 + c_*(\delta) \partial_x + f'(q_*(\cdot; \delta)) - \lambda + T(\delta)(c_*(\delta) \partial_x + f'(q_*(\cdot; \delta)) - \lambda), \end{aligned}$$

where $T(\delta) = (1 - \delta^2 \partial_x^2)^{-1} - 1$. Once we prove that the terms involving $T(\delta)$ are continuous in δ , the eigenvalue problem becomes essentially a regular perturbation of the classical Fisher-KPP linearization, at $\delta = 0$. We prove the necessary estimates on the preconditioners using direct Fourier analysis in Section 2.3. This approach is inspired by that used to construct oblique stripe solutions in a quenched Swift-Hohenberg equation in [17].

Stability to less localized perturbations. We note that under the spectral stability conditions we prove here, in addition to Corollaries 1.3 and 1.4, one also immediately obtains from the results of [3] stability under less localized perturbations, with a prescribed decay rate which is slower than $t^{-3/2}$. See [3, Theorems 3 and 4] for details.

Geometric vs. functional analytic point of view. We remark here that one should also be able to prove the spectral stability results obtained here using geometric dynamical systems methods, in particular geometric singular perturbation theory in the sense of Fenichel [13] together with the gap lemma [16, 22], which is used to extend the Evans function into the essential spectrum. An attractive feature of our approach here is that it is quite self contained, ultimately relying mostly on basic Fredholm theory and Fourier analysis. We also remark that in principle the functional analytic methods could be adapted, together with the approach to linear stability through obtaining resolvent estimates via far-field/core decompositions in [3], to problems in stability of critical fronts in nonlocal equations, since these methods do not rely as heavily on the presence of an underlying phase space. Some of the relevant Fredholm theory for nonlocal operators has been developed in [12, 11].

Natural range for δ . In this paper, we have restricted to small δ . However, we believe that similar results should hold true for larger values of this parameter. While the existence of fronts is established in [4] for all speeds $c > 0$ and $\delta \in \mathbb{R}$, we do not have access to explicit decay at $+\infty$ for this fronts, which seems necessary to establish precise stability. Monotonicity of the front would imply such a precise decay by use of Ikehara's theorem [7]. An important value is $\bar{\delta} = 1/\sqrt{12f''(0)}$, at which the dispersion relation admits a triple root, and the essential spectrum of the linearized operator becomes tangent to the imaginary axis. Stability at or above this value of δ is therefore fundamentally outside the scope of [3].

Supercritical and subcritical fronts. If we consider a supercritical front, traveling with speed $c > c_*(\delta)$ and constructed in [28], one can simplify the argument of Theorem 2 to prove that the linearization about such a front has no unstable point spectrum. For these fronts, one can use an exponential weight to push

the essential spectrum entirely into the left half plane, and thereby with the analogue of Theorem 2 obtain stability of supercritical fronts with an exponential decay rate using standard semigroup methods (see e.g. [19]). Subcritical fronts, with $c < c_*(\delta)$, have unstable absolute spectrum, meaning in particular that the essential spectrum of the linearization about any of these fronts is unstable in any exponentially weighted space. A modified version of our proof of Theorem 1 should also give existence of these supercritical and subcritical fronts using functional analytic methods, although we do not give the details here.

Outline. The remainder of this paper is organized as follows. In Section 2, we compute some preliminary information needed for our analysis (the linear spreading speed in (1.2) and the cokernel of $\mathcal{L}(0)$) and prove some necessary estimates on our preconditioner. In Section 3, we use our preconditioning and a far-field/core decomposition to prove Theorem 1, establishing existence of the critical front. In Section 4, we define a functional analytic analogue of the Evans function near $\lambda = 0$, and use it together with knowledge of the spectrum of $\mathcal{L}(0)$ to prove that $\mathcal{L}(\delta)$ has no resonance at the origin or unstable eigenvalues for δ small. In Section 5, we complete the proof of Theorem 2 by showing that there are also no unstable eigenvalues away from the origin.

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2 Preliminaries

2.1 Exponential weights

In addition to the critical weight (1.6) which we use to shift the essential spectrum out of the right half plane, we will further need exponential weights to recover Fredholm properties of $\mathcal{L}(\delta)$ and related operators for our far-field/core analysis. For $\eta_{\pm} \in \mathbb{R}$, we define a smooth positive weight function ω_{η_-, η_+} satisfying

$$\omega_{\eta_-, \eta_+} = \begin{cases} e^{\eta_- x}, & x \leq -1, \\ e^{\eta_+ x}, & x \geq 1. \end{cases}$$

If $\eta_- = 0$ and $\eta_+ = \eta$, then we write $\omega_{\eta_-, \eta_+} = \omega_{\eta}$. If $\eta_- = \eta_+ = \eta$, we choose $\omega_{\eta, \eta}(x) = e^{\eta x}$.

Given an integer m , we define the exponentially weighted Sobolev space $H_{\eta_-, \eta_+}^m(\mathbb{R})$ through the norm

$$\|f\|_{H_{\eta_-, \eta_+}^m} = \|\omega_{\eta_-, \eta_+} f\|_{H^m}.$$

We note that for $\eta > 0$ we have $H_{0, \eta}^m(\mathbb{R}) = H^m(\mathbb{R}) \cap H_{\eta, \eta}^m(\mathbb{R})$ as well as the following equivalence of norms

$$\|f\|_{H_{0, \eta}^m} \sim \|f\|_{H^m} + \|f\|_{H_{\eta, \eta}^m}. \quad (2.1)$$

This characterization of the one-sided weighted spaces is useful in obtaining estimates on operators defined by Fourier multipliers on these spaces, and we make use of this below in Section 2.3.

2.2 Essential spectrum

The boundaries of the essential spectrum of \mathcal{L} are determined by the essential spectrum of the limiting operators \mathcal{L}_{\pm} , obtained by sending $x \rightarrow \pm\infty$ [21, 14]. From the construction of c_* , η_* (see the proof of Lemma 1.1 below), we have at $+\infty$:

$$\mathcal{L}_+(\delta) = -\delta^2 \partial_x^4 + 4\eta_* \delta^2 \partial_x^3 + (1 - 6\delta^2 \eta_*^2) \partial_x^2. \quad (2.2)$$

The spectrum of this constant coefficient operator is, via Fourier transform, readily seen to be marginally stable, see the red curves of Figure 1. Notice that $\eta = \eta_*$ is the only reasonable value for which \mathcal{L}_+ has a non positive zeroth order term. Any other uniformly bounded $\eta(\delta)$ will lead to spectral instability for $\mathcal{L}(\delta)$. At $-\infty$, there is no contribution from ω_* , hence

$$\mathcal{L}_- = \mathcal{A}_- = -\delta^2 \partial_x^4 + \partial_x^2 + c_* \partial_x + f'(1)$$

has a stable spectrum, with spectral gap $f'(1) < 0$, by Lemma 1.2.

Proof of Lemma 1.1. We first look for $c_*, \eta_* > 0$ which satisfy (1.8). The polynomial $\nu \mapsto d_c(\lambda, \nu)$ at $\lambda = 0$ admits $-\eta$ as a double root if and only if

$$\begin{cases} 0 = d_c^+(0, -\eta) = -\delta^2 \eta^4 + \eta^2 - c\eta + f'(0), \\ 0 = \partial_\nu d_c^+(0, -\eta) = 4\delta^2 \eta^3 - 2\eta + c. \end{cases}$$

We remove c from the first equation by using the second one, and find a quadratic equation satisfied by η^2 , which has roots $\pm\eta_1, \pm\eta_2$ where

$$\eta_1 := \frac{1}{|\delta|\sqrt{6}} \sqrt{1 + \sqrt{1 - 12\delta^2 f'(0)}} \sim \frac{1}{|\delta|\sqrt{3}}, \quad \eta_2 := \frac{1}{|\delta|\sqrt{6}} \sqrt{1 - \sqrt{1 - 12\delta^2 f'(0)}} \sim \sqrt{f'(0)}. \quad (2.3)$$

where the asymptotics hold for $\delta \rightarrow 0$. The choice $\eta_* = \eta_2$ and $c_* = 2\eta_* - 4\delta^2 \eta_*^3$ leads to

$$c_*(\delta) = 2\sqrt{f'(0)} - \delta^2 f'(0)^{3/2} + O(\delta^4).$$

The other double roots do not determine linear spreading speeds, as they are not *pinched*; see [20] for details. We now fix $\delta_0 = 1/\sqrt{12f'(0)}$. Then for $|\delta| < \delta_0$, and using the expression of $\eta_* = \eta_2$, we obtain:

$$\frac{\partial_\nu^2 d_c^+(0, -\eta_*)}{2!} = 1 - 6\delta^2 \eta_*^2 = \sqrt{1 - 12\delta^2 f'(0)} > 0.$$

Hence $(\lambda, \nu) = (0, -\eta_*)$ is a simple double root and (1.8) is proved. Such an expansion together with the lack of unstable essential spectrum ensures that this root is pinched; see [20, Lemma 4.4]. Alternatively, Lemma 4.2 below directly proves that the root is pinched.

We now check the two remaining conditions in Lemma 1.1. We equate the polynomial $P(X) = d_*^+(\lambda, X)$ with its Taylor series centered at the double root $X = -\eta_*$ to obtain

$$\operatorname{Re} d_*^+(\lambda, -\eta_* + ik) = -\operatorname{Re} \lambda + \operatorname{Re} \left(\sum_{j=0}^4 (ik)^j \frac{P^{(j)}(-\eta_*)}{j!} \right) = -\operatorname{Re} \lambda - \delta^2 k^4 - (1 - 6\delta^2 \eta_*^2) k^2 \leq 0 \quad (2.4)$$

if $\operatorname{Re} \lambda > 0$, since from (2.3), we have $1 - 6\delta^2 \eta_*^2 = \sqrt{1 - 12\delta^2 f'(0)} \geq 0$. This proves hypothesis (iii). Furthermore, the inequality in (2.4) is an equality if and only if $k = 0$ and $\lambda = 0$, for which we have $d_*^+(0, -\eta_*) = 0$. Hence, hypothesis (ii) is proved. \square

2.3 Preconditioner estimates

Here we prove the estimates we will need on our preconditioner $(1 - \delta^2 \partial_x^2)^{-1}$, by directly examining its Fourier symbol.

Lemma 2.1. *Fix $\eta > 0$ sufficiently small, and fix an integer m . Then there exist constants $\delta_0 > 0$ and $C = C(\delta_0, \eta)$ such that if $|\delta| < \delta_0$,*

$$\|(1 - \delta^2 \partial_x^2)^{-1}\|_{L_{0,\eta}^2 \rightarrow L_{0,\eta}^2} \leq C, \quad (2.5)$$

$$\|(1 - \delta^2 \partial_x^2)^{-1}\|_{H_{0,\eta}^m \rightarrow H_{0,\eta}^{m+1}} \leq \frac{C}{|\delta|}. \quad (2.6)$$

Proof. By (2.1), it suffices to prove the estimates separately for L^2 and for $L_{\eta,\eta}^2$ with $\eta > 0$ small. Since multiplication by e^η is an isomorphism from $L_{\eta,\eta}^2(\mathbb{R})$ to $L^2(\mathbb{R})$, to prove estimates for $(1 - \delta^2 \partial_x^2)^{-1}$ on $L_{\eta,\eta}^2$, it suffices to consider the inverse of the conjugate operator

$$e^\eta (1 - \delta^2 \partial_x^2) e^{-\eta} = 1 - \delta^2 (\partial_x - \eta)^2$$

acting on $L^2(\mathbb{R})$. This is the advantage of using (2.1) to separate estimates on $L_{0,\eta}^2(\mathbb{R})$ into estimates on $L^2(\mathbb{R})$ and $L_{\eta,\eta}^2(\mathbb{R})$: the conjugate operator arising from studying $(1 - \delta^2 \partial_x^2)$ on $L_{\eta,\eta}^2(\mathbb{R})$ has constant coefficients since the weight is a fixed exponential function, and so we can directly estimate its inverse using the Fourier transform.

Fix $\eta \geq 0$. By Plancherel's theorem,

$$\|(1 - \delta^2(\partial_x - \eta)^2)^{-1}f\|_{L^2} = \left\| \frac{1}{1 - \delta^2(i \cdot - \eta)^2} \hat{f}(\cdot) \right\|_{L^2} \leq \sup_{k \in \mathbb{R}} \left| \frac{1}{1 - \delta^2(ik - \eta)^2} \right| \|\hat{f}\|_{L^2}.$$

Let $\delta_0 = 1/(\sqrt{2}\eta)$, so that $\delta_0^2\eta^2 = 1/2$, and hence if $|\delta| < \delta_0$,

$$1 + \delta^2(k^2 - \eta^2) = 1 - \delta^2\eta^2 + \delta^2k^2 \geq \frac{1}{2} + \delta^2k^2. \quad (2.7)$$

Then for any δ with $|\delta| < \delta_0$, we have

$$\left| \frac{1}{1 - \delta^2(ik - \eta)^2} \right|^2 = \frac{1}{(1 + \delta^2(k^2 - \eta^2))^2 + 4k^2\delta^4\eta^2} \leq \frac{1}{(1 + \delta^2(k^2 - \eta^2))^2} \leq \frac{1}{\frac{1}{2} + \delta^2k^2} \leq C,$$

with C depending only on δ_0 and η , and so

$$\|(1 - \delta^2\partial_x^2)^{-1}\|_{L^2_{\eta,\eta} \rightarrow L^2_{\eta,\eta}} \leq C.$$

Since this holds for any fixed $0 \leq \eta < 1$, in particular also for $\eta = 0$, we obtain (2.5) by combining these estimates with (2.1).

Now we prove (2.6), again by obtaining bounds on the Fourier symbol of the inverse of the conjugate operator for $\eta > 0$ and $\eta = 0$. By Plancherel's theorem, for any fixed $0 \leq \eta < 1$, we have

$$\|(1 - \delta^2(\partial_x^2 - \eta))^{-1}f\|_{H^{m+1}} = \left\| \frac{1}{1 - \delta^2(i \cdot - \eta)^2} \langle \cdot \rangle^{m+1} \hat{f}(\cdot) \right\| \leq \sup_{k \in \mathbb{R}} \left| \frac{1}{1 - \delta^2(ik - \eta)^2} \langle k \rangle \right| \|\hat{f}\|_{H^m}.$$

Again, let $\delta_0 = 1/(\sqrt{2}\eta)$. Then, by (2.7), we have

$$\left| \frac{1}{1 - \delta^2(ik - \eta)^2} \langle k \rangle \right|^2 = \frac{\delta^2 + \delta^2k^2}{(1 + \delta^2(k^2 - \eta^2))^2 + 4k^2\delta^4\eta^2} \frac{1}{\delta^2} \leq \frac{\delta^2 + \delta^2k^2}{(\frac{1}{2} + \delta^2k^2)^2} \frac{1}{\delta^2} \leq \frac{C}{\delta^2},$$

from which we obtain

$$\|(1 - \delta^2(\partial_x^2 - \eta))^{-1}f\|_{H^{m+1}} \leq \frac{C}{|\delta|} \|f\|_{L^2}.$$

Since this holds for $\eta \geq 0$, we obtain (2.6) from the equivalence of norms (2.1). \square

We now state and prove the estimates we will need on the difference between the preconditioner and the identity, $T(\delta) = (1 - \delta^2\partial_x^2)^{-1} - 1$.

Lemma 2.2. *Fix $\eta > 0$ sufficiently small. There exists a constant δ_0 such that the mapping $\delta \mapsto T(\delta)$ is continuous from $(-\delta_0, \delta_0)$ to $\mathcal{B}(H_{0,\eta}^1, L_{0,\eta}^2)$, the space of bounded linear operators from $H_{0,\eta}^1(\mathbb{R})$ to $L_{0,\eta}^2(\mathbb{R})$ with the operator norm topology.*

Proof. As in the proof of Lemma 2.1, it suffices to establish continuity in δ of the conjugate operator $T_\eta(\delta) := (1 - \delta^2(\partial_x - \eta)^2)^{-1} - 1$ on $L^2(\mathbb{R})$ for $\eta \geq 0$ sufficiently small. For δ nonzero, we write

$$1 - \delta^2(\partial_x - \eta)^2 = \delta^2 \left(\frac{1}{\delta^2} - (\partial_x - \eta)^2 \right).$$

By standard spectral theory, we therefore see that $T_\eta(\delta)$ is continuous in δ provided δ is nonzero and $1/\delta^2$ is in the resolvent set of the operator $(\partial_x - \eta)^2$. Computing the spectrum of this operator with the Fourier transform, one readily finds that there exists a δ_1 depending on η such that this continuity holds for $0 < \delta < \delta_1$.

We now establish continuity at $\delta = 0$ via direct estimates on the Fourier multiplier.

$$\hat{T}_\eta(\delta, k) = \frac{\delta^2(ik - \eta)^2}{1 - \delta^2(ik - \eta)^2}.$$

Since we are proving continuity of $T_\eta(\delta)$ from H^1 to L^2 , we gain a helpful factor of $\langle k \rangle$ — that is, it suffices to estimate $|\hat{T}_\eta(\delta, k)|/\langle k \rangle$. By (2.7), for $|\delta| < \delta_0 := \min\{\delta_1, 1/\sqrt{2}\eta\}$ we have

$$\begin{aligned} \left| \frac{\hat{T}_\eta(\delta, k)}{\langle k \rangle} \right| &= \left| \frac{\delta^2(ik - \eta)^2}{1 - \delta^2(ik - \eta)^2} \frac{1}{\langle k \rangle} \right| = \frac{\delta^2 \sqrt{(\eta^2 - k^2)^2 + 4k^2\eta^2}}{\sqrt{(1 - \delta^2(\eta^2 - k^2))^2 + 4\delta^4 k^2 \eta^2}} \frac{|\delta|}{(\delta^2 + \delta^2 k^2)^{1/2}} \\ &\leq |\delta| \left(\frac{\delta^4 \eta^4 + \delta^4 k^4 + 2\delta^4 k^2 \eta^2}{\left[\left(\frac{1}{2} + \delta^2 k^2 \right)^2 + 4\delta^4 k^2 \eta^2 \right] (\delta^2 + \delta^2 k^2)} \right)^{1/2}, \end{aligned}$$

using (2.7) in the denominator. We now split the factor in the parenthesis, first estimating

$$\frac{\delta^4 \eta^4 + 2\delta^4 k^2 \eta^2}{\left[\left(\frac{1}{2} + \delta^2 k^2 \right)^2 + 4\delta^4 k^2 \eta^2 \right] (\delta^2 + \delta^2 k^2)} \leq \frac{\delta^4 \eta^4 + 2\delta^4 k^2 \eta^2}{\frac{1}{4}(\delta^2 + \delta^2 k^2)} = \frac{\delta^2 \eta^4 + 2\delta^2 k^2 \eta^2}{\frac{1}{4}(1 + k^2)} \leq C,$$

where C depends only on δ_0 and η . For the remaining term, we have

$$\frac{\delta^4 k^4}{\left[\left(\frac{1}{2} + \delta^2 k^2 \right)^2 + 4\delta^4 k^2 \eta^2 \right] (\delta^2 + \delta^2 k^2)} \leq \frac{\delta^4 k^4}{\left(\frac{1}{2} + \delta^2 k^2 \right)^2 (\delta^2 k^2)} = \frac{\delta^2 k^2}{\left(\frac{1}{2} + \delta^2 k^2 \right)^2} \leq C,$$

again with constant C only depending on η and δ_0 . From this estimate on the Fourier symbol together with Plancherel's theorem, we obtain

$$\|T_\eta(\delta)\|_{H^1 \rightarrow L^2} \leq C|\delta|,$$

for $|\delta| < \delta_0$, and so in particular $\delta \mapsto T_\eta(\delta)$ is continuous at $\delta = 0$, which completes the proof of the lemma. \square

2.4 Fredholm properties at $\delta = 0$

We will further need the Fredholm properties of $\mathcal{L}(0)$, which is the linearization in the weighted space of the classical FKPP problem $\delta = 0$. The classical Fisher-KPP front, at $\delta = 0$, may be constructed via simple phase plane methods (see [32]), and we denote this front by q_0 . In the following two lemmas, we describe the kernel, the cokernel and the range of $\mathcal{L}(0)$. They will both be needed for the existence of the critical front $q_*(\cdot; \delta)$ in Section 3, and for the control of small eigenvalues in Section 4.

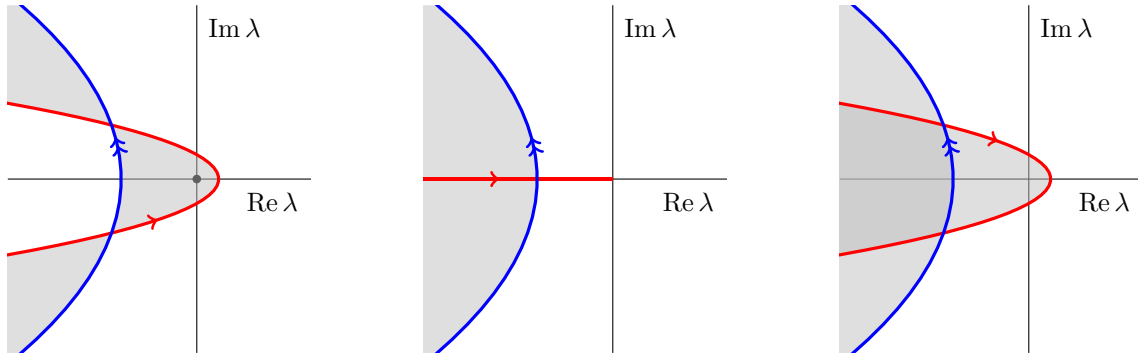


Figure 2: Essential spectrum of $\mathcal{L}_\eta(0)$ for $\eta < 0$ at left, $\eta = 0$ at middle, and $\eta > 0$ at right. For $\eta > 0$, the positive Fredholm border has reverse orientation so that $\text{fred}(\mathcal{L}_\eta) = -1$, while $\text{fred}(\mathcal{L}_\eta) = 1$ for $\eta < 0$.

Lemma 2.3. *For $\eta > 0$, the operator $\mathcal{L}(0) : H_{0,\eta}^2(\mathbb{R}) \rightarrow L_{0,\eta}^2(\mathbb{R})$ is Fredholm with index -1 , with trivial kernel and with cokernel spanned by $\varphi(x) = (\omega_*(x; 0))^{-1} e^{c_*(0)x} q_0'(x)$.*

Proof. Recall that the asymptotic operators are given by $\mathcal{L}_+(0) = \partial_x^2$ and $\mathcal{L}_-(0) = \partial_x^2 + c_*(0)\partial_x + f'(1)$. For $\eta > 0$, define the conjugate operator:

$$\mathcal{L}_\eta(0) = \omega_{0,\eta} \mathcal{L}(0) \omega_{0,\eta}^{-1} : H^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}),$$

with asymptotic operators $\mathcal{L}_{\eta,+} = (\partial_x - \eta)^2$ and $\mathcal{L}_{\eta,-} = \mathcal{L}_-$. Since the multiplication $\omega_{0,\eta} : L_{0,\eta}^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is an isomorphism, the Fredholm indexes satisfy

$$\text{fred } \mathcal{L}_\eta(0) = 0 + \text{fred } \mathcal{L}(0) + 0 = \text{fred } \mathcal{L}(0).$$

Then the conjugate operator is defined on a unweighted space, and its Fredholm borders are the two oriented curves $\sigma(\mathcal{L}_{\eta,+}) = \{-k^2 + i2\eta k + \eta^2 : k \in \mathbb{R}\}$ and $\sigma(\mathcal{L}_{\eta,-}) = \{-k^2 + ic_*(0)k + f'(1) : k \in \mathbb{R}\}$, which are away from $0 \in \mathbb{C}$; see Figure 2. This ensures that $\mathcal{L}_\eta(0)$ is Fredholm, we now compute its index $\text{fred}(\mathcal{L}_\eta(0) - \lambda)$ at $\lambda = 0$. For λ to the right of the essential spectrum, we use Palmer's theorem to compute the Fredholm index from the Morse indices (see e.g. [21, 14]),

$$\text{fred}(\mathcal{L}_\eta(0) - \lambda) = \dim E_-^u(\lambda) - \dim E_+^u(\lambda) = 2 - 2 = 0.$$

where E_\pm^u are the unstable eigenspaces at $\pm\infty$. To prove that this spaces share the same dimension, one can take $|\lambda|$ large enough and use a standard normalization; see [10, proof of Lemma 3.1]. Then the index decreases to -1 when λ crosses $\sigma(\mathcal{L}_{\eta,+})$, since the latter curve has reverse orientation, see [21]. Hence at $\lambda = 0$, we have shown that $\text{fred } \mathcal{L}(0) = \text{fred } \mathcal{L}_\eta(0) = -1$.

To compute the kernel, we note that

$$u \in \ker \mathcal{L}(0) \quad \text{if and only if} \quad u \in H_{0,\eta}^2(\mathbb{R}) \text{ and } \mathcal{A}(0)\omega_*^{-1}u = 0, \quad (2.8)$$

with $\mathcal{A}(0) = \partial_x^2 + c_*(0)\partial_x + f'(q_0(x))$. Studying the asymptotic growth of the ODE $\mathcal{A}(0)u = 0$, one can construct a basis of solutions $\{q'_0, \phi\}$, with exponential behavior at $-\infty$: $\phi(x) \sim \exp((-\sqrt{f'(0)} - \alpha)x)$ and $q'_0(x) \sim \exp((-\sqrt{f'(0)} + \alpha)x)$ with $\alpha = \sqrt{f'(0) - f'(1)} > \sqrt{f'(0)} > 0$. See [10, proof of Lemma 2.2] for a similar construction. Furthermore, the derivative of the front has weak exponential decay at $+\infty$: $q'_0(x) \sim x\omega_*(x)^{-1}$. Hence, neither ϕ nor q'_0 are sufficiently localized to satisfy the right hand condition in (2.8), so that $\ker \mathcal{L}(0) = \{0\}$.

Finally, it is easily computed that $\tilde{\mathcal{A}}(0) := \exp(\frac{c_*}{2}\cdot)\mathcal{A}(0)\exp(-\frac{c_*}{2}\cdot)$ is self-adjoint, so that for $v \in H_{0,-\eta}^2(\mathbb{R})$ and $u \in H_{0,\eta}^2(\mathbb{R})$:

$$\langle u, \mathcal{L}^*v \rangle = \langle \mathcal{L}u, v \rangle = \langle \mathcal{A}(\omega_*^{-1}u), \omega_*v \rangle = \langle \tilde{\mathcal{A}}(e^{\frac{c_*}{2}\cdot}\omega_*^{-1}u), e^{-\frac{c_*}{2}\cdot}\omega_*v \rangle = \langle e^{c_*\cdot}\omega_*^{-1}u, \mathcal{A}(e^{-c_*\cdot}\omega_*v) \rangle,$$

which ensures that $v \in \ker(\mathcal{L}(0)^*)$ if and only if $v \in H_{0,-\eta}^2(\mathbb{R})$ and $\mathcal{A}(0)e^{-c_*\cdot}\omega_*v = 0$. For $x \rightarrow -\infty$, $\omega_*(x)^{-1}e^{c_*x}\phi(x) \sim \exp((\sqrt{f'(0)} - \alpha)x)$ is not bounded, hence $\ker(\mathcal{L}(0)^*) = \text{Span}(\omega_*^{-1}e^{c_*\cdot}q'_0)$. \square

Lemma 2.4. *For $\eta > 0$ small enough, the range of $\mathcal{L}(0) : H_{0,\eta}^2(\mathbb{R}) \rightarrow L_{0,\eta}^2(\mathbb{R})$ is*

$$\text{im}(\mathcal{L}(0)) = \{u \in L_{0,\eta}^2(\mathbb{R}) : \langle u, \varphi \rangle = 0\},$$

where φ is defined in the above Lemma 2.3. We let $P : L_{0,\eta}^2(\mathbb{R}) \longrightarrow \text{im}(\mathcal{L}(0))$ denote the orthogonal projection onto $\text{im } \mathcal{L}(0)$ with respect to the $L_{0,\eta}^2(\mathbb{R})$ -inner product.

Proof. Assume that $u \in \text{im}(\mathcal{L}(0))$, so that $u = \mathcal{L}(0)\tilde{u}$ with $\tilde{u} \in H_{0,\eta}^2(\mathbb{R})$. Then $\langle u, \varphi \rangle = \langle \tilde{u}, \mathcal{L}(0)^*\varphi \rangle = 0$. To prove the reverse inclusion, write $u \in L_{0,\eta}^2(\mathbb{R})$ as

$$u = Pu + (1 - P)u.$$

From Lemma 2.3, $\mathcal{L}(0)$ is Fredholm, hence its range is closed and P is well defined. Furthermore, $\text{fred } \mathcal{L}(0) = -1$ and $\ker \mathcal{L}(0) = \{0\}$, so that $1 - P$ has a one dimensional range:

$$(1 - P)u = \alpha(u)\psi,$$

with $\psi \in L_{0,\eta}^2(\mathbb{R})$ fixed, and $\alpha : H_{0,\eta}^2(\mathbb{R}) \longrightarrow \mathbb{R}$ linear. Assuming that $\langle u, \varphi \rangle = 0$, we obtain

$$0 = \langle Pu, \varphi \rangle + \langle (1 - P)u, \varphi \rangle = \langle \tilde{u}, \mathcal{L}(0)^*\varphi \rangle + \alpha(u)\langle \psi, \varphi \rangle = \alpha(u)\langle \psi, \varphi \rangle, \quad (2.9)$$

for some $\tilde{u} \in H_{0,\eta}^2(\mathbb{R})$. Hence either $\alpha(u) = 0$ or $\langle \psi, \varphi \rangle = 0$. If $\langle \psi, \varphi \rangle = 0$, then for all $v \in H_{0,\eta}^2(\mathbb{R})$, we would have $\langle v, \varphi \rangle = \langle \tilde{v}, \mathcal{L}(0)^*\varphi \rangle + \alpha(v)\langle \psi, \varphi \rangle = 0$, which is to say that $\varphi = 0$ and is a contradiction. Hence from (2.9) we conclude $\alpha(u) = 0$, so that $u = Pu \in \text{im}(\mathcal{L}(0))$. \square

3 Existence of the critical front – proof of Theorem 1

Our approach is to capture the weak exponential decay at $+\infty$ implied by the pinched double root by solving (1.2) with an ansatz

$$q(x; \delta) = \chi_-(x) + w(x) + \chi_+(x)(\mu + x)e^{-\eta_*(\delta)x}, \quad (3.1)$$

where χ_+ is a smooth positive cutoff function satisfying

$$\chi_+(x) = \begin{cases} 1, & x \geq 3, \\ 0, & x \leq 2, \end{cases} \quad (3.2)$$

and $\chi_-(x) = \chi_+(-x)$. For brevity, we denote by $\psi(\mu, \delta)$ the function

$$\psi(x; \mu, \delta) = (\mu + x)e^{-\eta_*(\delta)x}.$$

We will require w to be exponentially localized, with a decay rate faster than $e^{-\eta_*(\delta)x}$ — this localized piece is the *core* of the solution, while χ_- and $\chi_+\psi$ capture the *far-field* behavior. Similar far-field/core decompositions have been used to construct heteroclinic solutions to pattern-forming systems in [2, 17]. Inserting the ansatz (3.1) into the traveling wave equation (1.2), we get an equation

$$\mathcal{A}_+(\delta)(\chi_- + w + \chi_+\psi(\mu, \delta)) + N(\chi_- + w + \chi_+\psi(\mu, \delta)) = 0, \quad (3.3)$$

where $\mathcal{A}_+(\delta) = -\delta^2\partial_x^4 + \partial_x^2 + c_*(\delta)\partial_x + f'(0)$, and $N(q) = f(q) - f'(0)q$. Since we want to require w to decay faster than the front itself, we first let $v = \omega_*w$, so that (3.3) becomes

$$0 = F(v; \mu, \delta) := \mathcal{S}(\delta)v + \omega_*\mathcal{A}_+(\delta)(\chi_- + \chi_+\psi) + \omega_*N(\chi_- + \omega_*^{-1}v + \chi_+\psi),$$

where $\mathcal{S}(\delta) = \omega_*\mathcal{A}_+(\delta)\omega_*^{-1}$ is the conjugate operator

$$\mathcal{S}(\delta) = -\delta^2\partial_x^4 + \delta^2a_3(x; \delta)\partial_x^3 + (1 + \delta^2a_2(x; \delta))\partial_x^2 + a_1(x; \delta)\partial_x + \tilde{a}_0(x; \delta), \quad (3.4)$$

where the coefficients a_i are given in (1.7) for $i = 1, 2$ or 3 while

$$\tilde{a}_0 = f'(0) + \omega_* (c_*\partial_x + \partial_x^2 - \delta^2\partial_x^4) \omega_*^{-1},$$

since we are linearizing about the unstable state $u \equiv 0$ rather than the front itself, which we are in the process of constructing.

Since $\omega_*(x; \delta) = 1$ on the support of χ_- and $\omega_*(x; \delta) = e^{\eta_*(\delta)x}$ on the support of χ_+ , we simplify F to

$$F(v; \mu, \delta) = \mathcal{S}(\delta)v + \mathcal{A}_+(\delta)\chi_- + \mathcal{S}(\delta)[(\mu + \cdot)\chi_+] + \omega_*N(\chi_- + \omega_*^{-1}v + \chi_+\psi).$$

Then, we extract from N terms that are linear in v , together with residual terms that are v -independent. We write

$$\omega_*N(\chi_- + \omega_*^{-1}v + \chi_+\psi) = \mathcal{N}(v; \mu, \delta) + Q(\mu, \delta)v + R(\mu, \delta)$$

where

$$\mathcal{N}(v; \mu, \delta) = \omega_* [f(\chi_+ + \omega_*^{-1}v + \chi_+\psi) - f(\chi_- + \chi_+\psi) - f'(\chi_- + \chi_+\psi)\omega_*^{-1}v], \quad (3.5)$$

and

$$Q(\mu, \delta)v = (f'(\chi_- + \chi_+\psi) - f'(0))v, \quad R(\mu, \delta) = \omega_*[f(\chi_- + \chi_+\psi) - f'(0)(\chi_- + \chi_+\psi)].$$

Altogether, F decomposes as the sum of a linear term, a residual term, and a nonlinear term:

$$F(v; \mu, \delta) = [\mathcal{S}(\delta) + Q(\mu, \delta)]v + \mathcal{R}(\mu, \delta) + \mathcal{N}(v; \mu, \delta), \quad (3.6)$$

where $\mathcal{N}(v; \mu, \delta)$ is given by (3.5), and

$$\mathcal{R}(\mu, \delta) = R(\mu, \delta) + \mathcal{A}_+(\delta)\chi_- + \mathcal{S}(\delta)[(\mu + \cdot)\chi_+].$$

At $\delta = 0$, the equation $F(v; \mu, 0) = 0$ is the traveling wave equation for the Fisher-KPP equation, and so we have a solution $F(v_0; \mu_0, 0) = 0$ where

$$v_0 = \omega_*(\cdot; 0)q_0 - \chi_- - \chi_+\omega_*(\cdot; 0)\psi(\mu_0, 0),$$

and q_0 is the translate of the critical Fisher-KPP front for which

$$q_0(x) = (\mu_0 + x)e^{-\eta_*(0)x} + O(x^2e^{-2\eta_*(0)x}), \quad x \rightarrow \infty,$$

so that v_0 is exponentially localized (see e.g. [15] for asymptotics of the critical Fisher-KPP front).

To regularize the singular perturbation and enforce exponential localization of v , we consider

$$G(v; \mu, \delta) = (1 - \delta^2\partial_x^2)^{-1}F(v; \mu, \delta),$$

as a nonlinear function $G : H_{0,\eta}^2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \rightarrow L_{0,\eta}^2(\mathbb{R})$, for $\eta > 0$ sufficiently small.

Lemma 3.1. *Fix $\eta > 0$ sufficiently small. There exists $\delta_0 > 0$ such that $(v, \mu, \delta) \mapsto G(v, \mu, \delta) : H_{0,\eta}^2(\mathbb{R}) \times \mathbb{R} \times (-\delta_0, \delta_0) \rightarrow L_{0,\eta}^2(\mathbb{R})$ is well-defined, smooth in v , and continuous in μ and δ . Moreover, $\partial_v G$ and $\partial_\mu G$ are continuous in δ .*

Proof. We use (3.6) to write G as

$$G(v; \mu, \delta) = (1 - \delta^2\partial_x^2)^{-1}\mathcal{S}(\delta)v + (1 - \delta^2\partial_x^2)^{-1}[Q(\mu, \delta)v + \mathcal{R}(\mu, \delta) + \mathcal{N}(v; \mu, \delta)]. \quad (3.7)$$

Using the fact that f is smooth and that $H_{0,\eta}^2(\mathbb{R})$ is a Banach algebra, one readily finds by Taylor expanding f where it appears in \mathcal{N} and \mathcal{R} that if $v \in H_{0,\eta}^2(\mathbb{R})$, then

$$\|Q(\mu, \delta)v + R(\mu, \delta) + \mathcal{N}(v; \mu, \delta)\|_{L_{0,\eta}^2} < \infty.$$

The remaining terms $A_+(\delta)\chi_-$ and $\mathcal{S}(\delta)[(\mu + \cdot)\chi_+]$ in $\mathcal{R}(\mu, \delta)$ are strongly localized by the choice of the far-field ansatz: $\chi_-(x)$ is identically zero for x large, and for x large every term in $\mathcal{S}(\delta)$ has at least two derivatives in it, so $\mathcal{S}(\delta)(\mu + \cdot) \equiv 0$ on the support of χ_+ , and the only terms that remain are compactly supported commutator terms. Hence we also obtain $\|\mathcal{R}(\mu, \delta)\|_{L_{0,\eta}^2} < \infty$.

Together with (2.5) of Lemma (2.1), this implies that the second term of (3.7) is in $L_{0,\eta}^2(\mathbb{R})$, and so to check that G is well-defined, it only remains to estimate the first term in (3.7). For this term, we use the specific form of $\mathcal{S}(\delta)$, given in (3.4), to write

$$(1 - \delta^2\partial_x^2)^{-1}\mathcal{S}(\delta) = \partial_x^2 + \delta^2(1 - \delta^2\partial_x^2)^{-1}[a_3\partial_x^3 + a_2\partial_x^2] + (1 - \delta^2\partial_x^2)^{-1}(a_1\partial_x + \tilde{a}_0). \quad (3.8)$$

Since a_3 and a_2 are smooth, constant outside of fixed compact set, and bounded uniformly in δ , we have

$$\|a_3\partial_x^3 + a_2\partial_x^2\|_{H_{0,\eta}^2 \rightarrow H_{0,\eta}^{-1}} \leq C.$$

Combining this with estimate (2.6) of Lemma 2.1, we obtain

$$\|\delta^2(1 - \delta^2\partial_x^2)^{-1}(a_3\partial_x^3 + a_2\partial_x^2)\|_{H_{0,\eta}^2 \rightarrow L_{0,\eta}^2} \leq C|\delta|. \quad (3.9)$$

The other terms in (3.8) are readily seen to be uniformly bounded in δ as operators from $H_{0,\eta}^2(\mathbb{R})$ to $L_{0,\eta}^2(\mathbb{R})$ for δ sufficiently small, from which we conclude that G is well-defined.

Since f is smooth, smoothness in v follows readily from the fact that $H_{0,\eta}^2(\mathbb{R})$ is a Banach algebra whose norm controls the L^∞ norm. Smoothness in μ is also readily attainable from smoothness of f and the exponential localization of our ansatz. The preconditioner plays little role in these arguments — when treating the residual terms or the nonlinearity, we do not need to use the preconditioner at all to obtain smoothness in v and μ .

The residual terms as well as the nonlinearity are also readily seen to be continuous in δ . The main subtlety is to handle the term $(1 - \delta^2 \partial_x^2)^{-1} \mathcal{S}(\delta)$, which we write as

$$(1 - \delta^2)^{-1} \mathcal{S}(\delta) = (\partial_x^2 + a_1 \partial_x + \tilde{a}_0) + \delta^2 (1 - \delta^2 \partial_x^2)^{-1} (a_3 \partial_x^3 + a_2 \partial_x^2) + T(\delta) (a_1 \partial_x + \tilde{a}_0),$$

where $T(\delta) = (1 - \delta^2 \partial_x^2)^{-1} - 1$. The operator $\partial_x^2 + a_1(x, \delta) \partial_x + \tilde{a}_0(x, \delta)$ is continuous in δ from $H_{0,\eta}^2$ to $L_{0,\eta}^2$, since the coefficients are smooth and uniformly bounded in δ . The second term is continuous in δ by (3.9), and the last term is continuous in δ by Lemma 2.2. Continuity in δ of $\partial_v G$ and $\partial_\mu G$ proceeds analogously. \square

With the appropriate regularity of G in hand, we now aim to solve near $(v_0, \mu_0, 0)$ using the implicit function theorem. The linearization about this solution in v is given by

$$\partial_v G(v_0; \mu_0, 0) = \mathcal{S}(0) + Q(\mu_0, 0) + \partial_v \mathcal{N}(v_0; \mu_0, 0) = \mathcal{S}(0) + f'(q_0) - f'(0) = \mathcal{L}(0).$$

From Lemma 2.3, $\partial_v G(v_0; \mu_0, 0)$ is Fredholm with index -1 , so that the joint linearization $\partial_{(v,\mu)} G(v_0; \mu_0, 0)$ is Fredholm index 0 by the Fredholm bordering lemma [31, Lemma 4.4]. We show that in fact the joint linearization has full range, and hence is invertible.

Lemma 3.2. *The joint linearization $\partial_{(v,\mu)} G(v_0; \mu_0, 0) : H_{0,\eta}^2(\mathbb{R}) \times \mathbb{R} \rightarrow L_{0,\eta}^2(\mathbb{R})$ is invertible.*

Proof. To show that $\partial_{(v,\mu)} G(v_0; \mu_0, 0)$ is invertible, we show that $\partial_\mu G(v_0; \mu_0, 0)$ is linearly independent from the range of $\mathcal{L}(0)$. From Lemma 2.4, it is enough to obtain $\langle \partial_\mu G(v_0; \mu_0, 0), \varphi \rangle \neq 0$. After a short computation, one finds

$$\partial_\mu G(v_0; \mu_0, 0) = \mathcal{S}(0) \chi_+ + (f'(q_0) - f'(0)) \chi_+ = \mathcal{L}(0) \chi_+.$$

We compute $\langle \mathcal{L}(0) \chi_+, \varphi \rangle$ via integration by parts, with the goal being to move $\mathcal{L}(0)$ onto the other side of the inner product as its adjoint and exploit the fact that $\mathcal{L}(0)^* \varphi = 0$. However, we must be careful since φ and χ_+ are not localized at ∞ , and in fact there is one boundary term from integration by parts which does not vanish. We see this by writing

$$\int_{\mathbb{R}} \chi_+'' \varphi dx = - \int_{\mathbb{R}} \chi_+' \varphi' dx = \int_{\mathbb{R}} \chi_+ \varphi'' - [\chi_+' \varphi']_{-\infty}^{\infty} = \langle \chi_+, \varphi'' \rangle - \varphi'(\infty) = \langle \chi_+, \varphi'' \rangle + \eta_*(0),$$

where we have observed from Lemma 2.3 that $\varphi'(\infty) = -\eta_*(0)$. Recalling that $\mathcal{L}(0) = \partial_x^2 + f'(q_*)$ for $x \geq 1$, we obtain

$$\langle \mathcal{L}(0) \chi_+, \varphi \rangle = \langle \chi_+, \mathcal{L}(0)^* \varphi \rangle + \eta_*(0) = \eta_*(0) = \frac{c_*(0)}{2} > 0,$$

which concludes the proof. \square

Proof of Theorem 1. Since $G(v_0; \mu_0, 0) = 0$, G is smooth in v and μ and continuous in δ near $(v_0; \mu_0, 0)$, $\partial_{(v,\mu)} G(v_0; \mu_0, 0)$ is invertible, and $\partial_{(v,\mu)} G(v; \mu, \delta)$ is continuous in δ , the implicit function theorem implies that for δ small, there exist $v(\delta) \in H_{0,\eta}^2(\mathbb{R})$ and $\mu(\delta) \in \mathbb{R}$ depending continuously on δ near $\delta = 0$ such that $G(v(\delta); \mu(\delta), \delta) = 0$. By construction of G , this implies that

$$q_*(x; \delta) := \chi_-(x) + \omega_*(x; \delta)^{-1} v(x; \delta) + \chi_+(x) (\mu(\delta) + x) e^{-\eta_*(\delta)x}$$

solves (1.2). The claim that $q_*(\cdot, \delta) \rightarrow q_*(\cdot, 0) = q_0$ uniformly in space follows from the form of this ansatz, together with the fact that $H_{0,\eta}^2(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$. \square

4 Small eigenvalues

Having established existence of the critical front, we are now ready to study the point spectrum of the linearization about the front. Here we show that there is no eigenvalue in a neighborhood of the origin, and in particular no resonance embedded in the essential spectrum at the origin. For this, we follow [26]: apply a Lyapunov-Schmidt reduction to construct a scalar function which vanishes at the eigenvalues, in a similar manner to the Evans function.

Throughout this section, we set $\Omega(\delta) := \{0\} \cup (\mathbb{C} \setminus \sigma_{\text{ess}}(\mathcal{L}_\delta))$, and restrict to $\lambda \in \Omega(\delta)$. Then λ is off the negative real axis, so that the principal value of $\gamma := \sqrt{\lambda}$ is defined by $\text{Re } \gamma \geq 0$.

Proposition 4.1. *There exists $\delta_0, \gamma_0 > 0$ and a function $E : (-\delta_0, \delta_0) \times B(0, \gamma_0) \rightarrow \mathbb{C}$, continuous in δ and analytic in γ such that for all $\gamma \in \Omega(\delta)$, the eigenvalue problem*

$$(\mathcal{L}(\delta) - \gamma^2)u = 0 \tag{4.1}$$

admits a bounded solution u if and only if $E(\delta, \gamma) = 0$. Furthermore, $E(0, 0) \neq 0$. In particular, there exists $\gamma_1, \delta_1 > 0$ such that for all $\delta \in (-\delta_1, \delta_1)$, $\mathcal{L}(\delta)$ has no eigenvalues on $B(0, \gamma_1^2) \cap \Omega(\delta_1)$.

For any fixed $\delta \neq 0$, notice that (4.1) is a linear, non degenerate ODE with smooth coefficients, so that any solution u is smooth. Furthermore, such a solution admits exponential expansions at $\pm\infty$ (see the proof of Lemma 4.2 hereafter), so that when γ^2 is to the right of the essential spectrum, u is bounded if and only if it lies in $H^4(\mathbb{R})$, which is to say it is an eigenfunction. We will therefore consider bounded solutions from this point forward: for γ^2 to the right of the essential spectrum, they correspond with eigenfunctions, while at $\gamma = 0$ they capture resonances of $\mathcal{L}(\delta)$.

We first show that a bounded solution of (4.1) decomposes into two parts: a uniformly localized part, and a slowly decaying part, whose rate is γ -close to 0.

Lemma 4.2. *Near $(\delta, \gamma) = (0, 0)$, the roots of the polynomial $\nu \mapsto d_{c_*}^+(\gamma^2, -\eta_* + \nu)$ satisfy:*

$$\nu_1 = -\frac{1}{|\delta|} + \mathcal{O}(1), \quad \nu_2 = -\gamma + \mathcal{O}(\delta\gamma + \gamma^2), \quad \nu_3 = \gamma + \mathcal{O}(\delta\gamma + \gamma^2), \quad \nu_4 = \frac{1}{|\delta|} + \mathcal{O}(1),$$

where each \mathcal{O} is taken as δ and γ goes to 0.

In particular, there exists $\delta_0 > 0, \gamma_0 > 0$ and $\eta > 0$ such that for all $\delta \in (-\delta_0, \delta_0)$, and $\gamma \in B(0, \gamma_0)$ with $\gamma^2 \in \Omega(\delta)$, a bounded and smooth solution u of (4.1) decompose as

$$u(x) = w(x) + \beta\chi_+(x)e^{\nu_2 x}, \tag{4.2}$$

where $w \in H_{0,\eta}^2(\mathbb{R})$ and $\beta \in \mathbb{C}$. In this decomposition, χ_+ is the cutoff function (3.2).

Proof. The claimed expansions of the four roots is purely technical and is postponed to the end of the proof. Rewrite (4.1) as a first order ODE in \mathbb{R}^4 :

$$\partial_x U = M(x; \delta, \gamma).$$

where $U = (u, u', u'', u^{(3)})^T$. The matrix M converges towards $M_{\pm}(\delta, \gamma)$ when $x \rightarrow \pm\infty$, with an exponential rate which is independent of δ and γ . The eigenvalues of this asymptotic matrices M_{\pm} are the roots of the dispersion relations $d_{c_*}^{\pm}(\gamma^2, -\eta_* + \cdot)$. It is standard that with such a convergence rate, these eigenvalues determine the behavior of U at $\pm\infty$; see for example [10, proof of Lemma 2.2].

More precisely, the behavior at $+\infty$ is the following. For $\gamma \neq 0$, the four roots are distinct, so that the exponential behavior is ensured: $U(x) \sim \sum_{i=1}^4 c_i(U)e^{\nu_i x}$ when $x \rightarrow +\infty$, with $c_i(U, \delta, \gamma)$ are vectors that does not depend on x . As $\gamma^2 \notin \sigma_{\text{ess}}(\mathcal{L}(\delta))$, the two small roots satisfy $\text{Re } \nu_2(\delta, \gamma) < 0 < \text{Re } \nu_3(\delta, \gamma)$, so that a bounded U has exactly the claimed form. At $\gamma = 0$, the two small roots merge to form a Jordan block. The proof in the above reference adapts, and we have the following expansion: $U(x) \sim c_1(U)e^{\nu_1 x} + c_2(U) + c_3(U)x + c_4(U)e^{\nu_4 x}$ when $x \rightarrow +\infty$. Once again the claimed decomposition is satisfied.

At $-\infty$, the four roots of $d_{c_*}^-(\gamma^2, -\eta_* + \cdot)$ are distinct, and bounded away from 0 with spectral gap uniform in (δ, γ) . Then the expansion $U(x) \sim \sum_{i=1}^4 c_i^-(U)e^{\nu_i^- x}$ holds at $x \rightarrow -\infty$, so that any bounded U lies in $H^2(\mathbb{R}_-)$. Hence the claimed decomposition holds. For an alternative argument not relying on the dynamical systems view of exponential expansions, see Remark 4.7.

We now establish the expansions of the roots by applying the implicit function theorem to $d_{c_*}^+$. From the choice of η_* (see also (2.2)) we have

$$g_0(\delta, \gamma, \nu) := d_{c_*}^+(\gamma^2, -\eta_* + \nu) = -\delta^2 \nu^4 + 4\eta_* \delta^2 \nu^3 + (1 - 6\delta^2 \eta_*^2) \nu^2 - \gamma^2.$$

To avoid any δ singularity, we get rid of the δ^2 in the dominant term by changing variables $\mu := \nu|\delta|$:

$$g_1(\delta, \gamma, \mu) := \delta^2 g_0(\gamma, \delta, \frac{\mu}{|\delta|}) = -\mu^4 + 4\eta_* |\delta| \mu^3 + (1 - 6\delta^2 \eta_*^2) \mu^2 - \gamma^2 \delta^2.$$

At $(\delta, \gamma) = (0, 0)$, this reduces to $g_1(0, 0, \mu) = -\mu^2(\mu - 1)(\mu + 1)$. Applying implicit function theorem to the simple root -1 , we construct a root $\mu_1(\delta, \gamma)$ for $g_1(\delta, \gamma, \cdot)$ whose derivatives can be computed iteratively by differentiating the relation $g_1(\delta, \gamma, \mu_1(\delta, \gamma)) = 0$. One can show by induction that any pure derivative in γ is null: $\partial_\gamma^k \mu_1(0, 0) = 0$ for $k \in \mathbb{N}^*$. This ensures that the Taylor expansion has the form

$$\mu_1(\delta, \gamma) = -1 - \delta \frac{\partial_\delta g_1(0, 0, -1)}{\partial_\mu g_1(0, 0, -1)} - \gamma \frac{\partial_\gamma g_1(0, 0, -1)}{\partial_\mu g_1(0, 0, -1)} + O(\delta^2 + \delta\gamma) = -1 + O(\delta).$$

Coming back to the original variable, we define $\nu_1(\delta, \gamma) = \mu_1(\delta, \gamma)/|\delta|$, which satisfies the claimed expansion. The same steps can be applied to define $\mu_4(\delta, \gamma) = 1 + O(\delta)$, which in turn leads to $\nu_4(\delta, \gamma)$ as claimed.

To unfold the double root at $\mu = 0$, we change variables once again to $\nu = \gamma\sigma$:

$$g_2(\delta, \gamma, \sigma) = \frac{g_0(\delta, \gamma, \gamma\sigma)}{\gamma^2} = -\delta^2\gamma^2\sigma^4 + 4\eta_*\delta^2\gamma\sigma^3 + (1 - 6\delta^2\eta_*^2)\sigma^2 - 1.$$

At $(\delta, \gamma) = (0, 0)$, this reduces to $g_2(0, 0, \sigma) = (\sigma - 1)(\sigma + 1)$. Applying the implicit function theorem once again gives rise to

$$\sigma_2(\delta, \gamma) = -1 + O(\delta + \gamma), \quad \sigma_3(\delta, \gamma) = 1 + O(\delta + \gamma),$$

which in turns leads to the claimed estimates on $\nu_2(\delta, \gamma) = \gamma\sigma_2(\delta, \gamma)$ and $\nu_3(\delta, \gamma) = \gamma\sigma_3(\delta, \gamma)$. \square

As in the existence of the critical front, our problem is singular at $\delta = 0$. Hence, we apply the same preconditioner: when δ is small, (4.1) is equivalent to

$$(1 - \delta^2\partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)u = 0.$$

We now use the decomposition of Lemma 4.2 to separate out the localized part of our problem from the far-field behavior, which will allow us to make use of the Fredholm properties on weighted spaces of Section 2.4. In the following, for $\delta \in (-\delta_0, \delta_0)$ and $\gamma \in B(0, \gamma_0)$ we let

$$A(\delta, \gamma, \eta) = \{w + \beta\chi_+ e^{\nu_2(\delta, \gamma)} : w \in H_{0, \eta}^2(\mathbb{R}), \beta \in \mathbb{R}\},$$

denote the set where the ansatz obtained above holds.

Lemma 4.3. *There exist positive constants δ_0, γ_0 and η such that if $\delta \in (-\delta_0, \delta_0)$, $\gamma \in B(0, \gamma_0)$ and $u \in A(\delta, \gamma, \eta)$, then $(1 - \delta^2\partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)u \in L_{0, \eta}^2(\mathbb{R})$.*

Proof. First, $(1 - \delta^2\partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)w$ belongs to $L_{0, \eta}^2(\mathbb{R})$ by the choice of the preconditioner, using the same regularization effect we observed in (3.8). Then, as χ_+ is smooth, vanishes on $(-\infty, 2)$ and is constant on $(3, +\infty)$, it only remains to show that $(\mathcal{L}(\delta) - \gamma^2)e^{\nu_2 \cdot} \in L_{0, \eta}^2(\mathbb{R}_+)$. For $x \geq 1$, almost all coefficients of \mathcal{L} are constants, see (1.7), hence we compute

$$(\mathcal{L}(\delta) - \gamma^2)e^{\nu_2 x} = (\mathcal{L}(\delta) - \mathcal{L}_+(\delta))e^{\nu_2 x} + (\mathcal{L}_+(\delta) - \gamma^2)e^{\nu_2 x} = (f'(q_*(x; \delta)) - f'(0))e^{\nu_2 x} + P(\nu_2, \delta, \gamma)e^{\nu_2 x},$$

where the polynomial $P(X, \delta, \gamma)$ is the symbol defined by: $\mathcal{L}_+(\delta) - \gamma^2 = P(\partial_x, \delta, \gamma)$, and \mathcal{L}_+ is the asymptotic operator (2.2). From the definition of $\nu_2(\delta, \gamma)$, $P(X, \delta, \gamma)$ vanishes at $X = \nu_2(\delta, \gamma)$, hence for $x \geq 1$:

$$(\mathcal{L}(\delta) - \gamma^2)e^{\nu_2 x} = (f'(q_*(x)) - f'(0))e^{\nu_2 x} = f''(0)q_*(x; \delta)e^{\nu_2 x} + O\left(e^{\nu_2 x} q_*(x; \delta)^2\right).$$

The right hand side belongs to $L_{0, \eta}^2(\mathbb{R})$ as long as η satisfies

$$-\eta_* + \operatorname{Re} \nu_2(\delta, \gamma) < -\eta. \quad (4.3)$$

We can take a smaller γ_0 than in Lemma 4.2, so that $\sup_{\delta, \gamma} \{-\eta_*(\delta) + \operatorname{Re} \nu_2(\delta, \gamma)\} < 0$, which then allows to fix $\eta > 0$ so that (4.3) is satisfied for all $\delta \in (-\delta_0, \delta_0)$ and $\gamma \in B(0, \gamma_0)$. This concludes the proof. \square

We can now use Lemma 2.4 to decompose our problem into a part which belongs to $\text{im } \mathcal{L}(0)$ and a complementary part. Recall that $P : L_{0,\eta}^2(\mathbb{R}) \rightarrow \text{im } (\mathcal{L}(0))$ and that φ allows to describe $\text{im } (\mathcal{L}(0))$. Fix $\delta \in (-\delta_0, \delta_0)$, and $\gamma \in B(0, \gamma_0) \cap \Omega(\delta)$. If u is a bounded solution of (4.1) then $(w, \beta) \in H_{0,\eta}^2(\mathbb{R}) \times \mathbb{C}$ defined in Lemma 4.2 solves:

$$\begin{cases} P(1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)(w + \beta h) = 0, \\ \langle (1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)(w + \beta h), \varphi \rangle = 0, \end{cases} \quad (4.4)$$

where $h(x) = \chi_+(x)e^{\nu_2(\gamma)x}$. Reciprocally, if $(w, \beta) \in H_{0,\eta}^2(\mathbb{R}) \times \mathbb{C}$ satisfies (4.4), then $u = w + \beta h$ is bounded and satisfies (4.1). We write the first equation as

$$0 = \mathcal{F}(w, \beta; \gamma, \delta) := P(1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)(w + \beta h). \quad (4.5)$$

and solve it with the implicit function theorem. We will then use the second equation to define $E(\delta, \gamma)$.

Lemma 4.4. *For $\eta > 0$ sufficiently small, the map $\mathcal{F} : H_{0,\eta}^2(\mathbb{R}) \times \mathbb{C} \times B(0, \gamma_0) \times (-\delta_0, \delta_0) \rightarrow L_{0,\eta}^2(\mathbb{R})$ is smooth in w and β , analytic in γ , and continuous in δ . Moreover, $\partial_w \mathcal{F}(w, \beta; \gamma, \delta)$ is continuous in β , γ , and δ .*

Proof. Note that \mathcal{F} is linear in w and β , so smoothness is automatic provided the linear part in w is well defined, which is guaranteed here by Lemma 4.3. For the continuity of $\partial_w \mathcal{F}(w, \beta; \gamma, \delta)$, we write

$$(1 - \delta^2 \partial_x^2)^{-1} \mathcal{L}(\delta) = \partial_x^2 + a_1 \partial_x + a_0 + \delta^2 (1 - \delta^2 \partial_x^2)^{-1} (a_3 \partial_x^3 + a_2 \partial_x^2) + T(\delta)(a_1 \partial_x + a_0).$$

We see by Lemmas 2.1 and 2.2 that $(1 - \delta^2 \partial_x^2)^{-1} \mathcal{L}(\delta)$ is a well-defined family of bounded operators from $H_{0,\eta}^2$ to $L_{0,\eta}^2$, depending continuously on δ . This is of course preserved when we compose with the projection P . We write the other term in the linearization in w as

$$\gamma^2 (1 - \delta^2 \partial_x^2)^{-1} w = \gamma^2 w + \gamma^2 T(\delta) w,$$

which is again continuous in γ and δ as a bounded linear operator from $H_{0,\eta}^2$ to $L_{0,\eta}^2$ by Lemma 2.2. Hence $\partial_w \mathcal{F}$ is continuous in its three last variables. Analyticity of \mathcal{F} in γ follows as in [26, Proposition 5.11]. For the continuity of \mathcal{F} with respect to δ , it only remains to look at the terms associated to h . We rewrite

$$(\mathcal{L}(\delta) - \gamma^2)h = [\mathcal{L}_+(\delta), \chi_+]e^{\nu_2(\delta,\gamma)\cdot} + (\mathcal{L}(\delta) - \mathcal{L}_+(\delta))e^{\nu_2(\delta,\gamma)\cdot},$$

using the fact that $(\mathcal{L}_+(\delta) - \gamma^2)e^{\nu_2(\delta,\gamma)\cdot} = 0$, and where $[\mathcal{L}_+(\delta), \chi_+] = \mathcal{L}_+(\delta)(\chi_+ \cdot) - \chi_+ \mathcal{L}_+(\delta)$ is the commutator between these operators. In this form, we recognize that $[\mathcal{L}_+(\delta), \chi_+]$ and $(\mathcal{L}(\delta) - \mathcal{L}_+(\delta))$ are both differential operators with exponentially localized coefficients, with rate uniform in δ for δ small. By Lemma 4.2, $e^{\nu_2(\delta,\gamma)x}$ is continuous in γ and δ for each fixed x , and the uniform localization of $[\mathcal{L}_+(\delta), \chi_+]$ and $(\mathcal{L}(\delta) - \mathcal{L}_+(\delta))$ guarantees that these terms are continuous in δ in $L_{0,\eta}^2$ for η small. In fact, since h is a smooth function, we see that $\delta \mapsto (\mathcal{L}(\delta) - \gamma^2)h$ is in particular continuous from $(-\delta_1, \delta_1)$ to $H_{0,\eta}^1$. Taking into account the preconditioner, we write

$$(1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)h = (\mathcal{L}(\delta) - \gamma^2)h + T(\delta)(\mathcal{L}(\delta) - \gamma^2)h.$$

By Lemma 2.2, this term is continuous in δ , as desired. \square

Corollary 4.5. *For γ, δ sufficiently small, and for $\beta \in \mathbb{C}$, there is a family of solutions w to (4.5) which have the form*

$$w(\beta; \gamma, \delta) = \beta \tilde{w}(\gamma, \delta). \quad (4.6)$$

Moreover, any solution to (4.5) with γ, δ small has this form.

Proof. We begin with the trivial solution $\mathcal{F}(0, 0; 0, 0) = 0$. The linearization in w about this trivial solution is $\partial_w \mathcal{F}(0, 0; 0, 0) = P\mathcal{L}(0)$, which is invertible by Lemmas 2.3 and 2.4. Together with Lemma 4.4, this implies that we can solve near this trivial solution with the implicit function theorem, obtaining a unique solution $w(\beta; \gamma, \delta)$ in a neighborhood \mathcal{U} of $(0, 0; 0, 0)$. Since (4.5) is linear in w and β , by uniqueness any solution in this neighborhood can be written as

$$w(\beta; \gamma, \delta) = \beta \tilde{w}(\gamma, \delta)$$

for some function $\tilde{w}(\gamma, \delta) \in H_{0,\eta}^2(\mathbb{R})$.

If for some fixed γ, δ small we have another solution (w_0, β_0) to (4.5) which does not a priori have this form, by dividing by a sufficiently large constant $K(\|w_0\|_{H_{0,\eta}^2}, \beta)$ we get another solution which belongs to the neighborhood \mathcal{U} where we have solved with the implicit function theorem, and so we conclude that

$$\frac{w_0}{K(\|w_0\|_{H_{0,\eta}^2}, \beta)} = \frac{\beta}{K(\|w_0\|_{H_{0,\eta}^2}, \beta)} \tilde{w}(\gamma, \delta),$$

and hence the solution (w_0, β_0) in fact has the form (4.6), as claimed. \square

Having solved the first equation in (4.4) with the implicit function theorem, we now insert this solution $w(\beta; \gamma, \delta) = \beta \tilde{w}(\gamma, \delta)$ into the second equation, so that (4.4) has a solution if and only if

$$0 = E(\delta, \gamma) := \langle (1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2) (\tilde{w}(\gamma, \delta) + h), \varphi \rangle. \quad (4.7)$$

Note that we have been able to eliminate the β dependence in this equation, since all terms in this equation are linear in β by Corollary 4.5. Since the projection P played no role in the proof of Lemma 4.4, the same argument shows that E is continuous in both of its arguments.

Lemma 4.6. *The function $E : (-\delta_0, \delta_0) \times B(0, \gamma_0) \rightarrow \mathbb{C}$ is continuous in both arguments, and analytic in γ for fixed δ .*

Proof of Proposition 4.1. It only remains to prove that $E(0, 0) \neq 0$. From (4.7), we see that

$$E(0, 0) = \langle \mathcal{L}(0) (\tilde{w}(0, 0) + \chi_+), \varphi \rangle.$$

Since $\tilde{w}(0, 0) \in H_{0,\eta}^2(\mathbb{R})$, we have

$$\langle \mathcal{L}(0) \tilde{w}(0, 0), \varphi \rangle = \langle \tilde{w}(0, 0), \mathcal{L}(0)^* \varphi \rangle = 0.$$

Hence we obtain

$$E(0, 0) = \langle \mathcal{L}(0) \chi_+, \varphi \rangle = \eta_*(0) \neq 0,$$

by the computation in the proof of Lemma 3.2. \square

Remark 4.7. Rather than using the spatial dynamics approach to exponential expansions outlined in the proof of Lemma 4.2 to show directly that eigenfunctions have the form (4.2), one can instead show that for $\delta \neq 0$, $(\mathcal{L}(\delta) - \gamma^2) : H^4(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is invertible if $E(\delta, \gamma) \neq 0$ and γ^2 is to the right of the essential spectrum of $\mathcal{L}(\delta)$, using an argument adapted from [26]. Indeed, if γ^2 is to the right of the essential spectrum, then $\mathcal{L}(\delta) - \gamma^2$ is Fredholm index 0, and in particular has closed range, so to invert this operator on $L^2(\mathbb{R})$, it suffices to solve $(\mathcal{L}(\delta) - \gamma^2)u = g$ for g in the dense subspace $L_{0,\eta}^2(\mathbb{R})$. The fact that the range of $\mathcal{L}(\delta) - \gamma^2$ is closed then implies $\mathcal{L}(\delta) - \gamma^2$ is surjective, and hence invertible since it is Fredholm of index 0. The open mapping theorem then implies the inverse is bounded, so γ^2 will be in the resolvent set of $\mathcal{L}(\delta)$. To solve $(\mathcal{L}(\delta) - \gamma^2)u = g$ for $g \in L_{0,\eta}^2(\mathbb{R})$, one looks for solutions in the form (4.2), and finds that (w, β) solve the system (4.4) but with $(0, 0)^T$ on the right hand side replaced by $(Pg, \langle g, \phi \rangle)^T$. We can always solve the first equation with the implicit function theorem, and we can solve the second equation precisely when $E(\delta, \gamma) \neq 0$, as claimed. At $\gamma = 0$ we lose Fredholm properties on $L^2(\mathbb{R})$, but the fact that $E(\delta, 0)$ implies there is no solution to $\mathcal{L}(\delta)u = 0$ of the form $u = w + \chi_+$ for w exponentially localized, and this is actually all that is needed in [3] to prove nonlinear stability. One could additionally use a modified far-field/core decomposition at $\gamma = 0$ to prove that all bounded solutions to $\mathcal{L}(\delta)u = 0$ have the form $u = w + \chi_+$.

5 Large and intermediate eigenvalues — proof of Theorem 2

Here, we conclude the study of the point spectrum. We first exclude any large unstable point spectrum, using mostly that the operator is sectorial.

Proposition 5.1. *There exists a compact set $K \subset \mathbb{C}$ such that for all δ small, any eigenvalue λ of $\mathcal{L}(\delta)$ with $\operatorname{Re} \lambda \geq 0$ lies in K . More precisely, an eigenvalue λ satisfies:*

$$\operatorname{Re} \lambda \leq \|b(\cdot; \delta)\|_\infty, \quad |\operatorname{Im} \lambda| \leq \|b(\cdot; \delta)\|_\infty + c_* \sqrt{\|b(\cdot; \delta)\|_\infty - \operatorname{Re} \lambda},$$

where $b(\cdot; \delta) = f'(q_*(\cdot; \delta))$ is uniformly bounded.

Proof. We work with $\delta \in (-\delta_0, \delta_0)$, with δ_0 small enough so that Theorem 1 apply. Assume that $\lambda \in \mathbb{C}$ and $\psi \in H^4(\mathbb{R})$ satisfy $\mathcal{L}(\delta)\psi = \lambda\psi$. Coming back to the unweighted operator $\mathcal{A}(\delta) = \omega_*^{-1}\mathcal{L}(\delta)\omega_*$, defined by (1.5), we obtain

$$\mathcal{A}(\delta)\phi = \lambda\phi \tag{5.1}$$

with $\phi = \omega_*^{-1}\psi \in H^4(\mathbb{R})$. Up to a scalar multiplication, we can assume that $\|\phi\|_{L^2(\mathbb{R})} = 1$. Now we take the Fourier transform in (5.1), multiply by $\widehat{\phi}(\xi)$ and finally integrate to obtain:

$$\int_{\mathbb{R}} (-\delta^2 \xi^4 - \xi^2 + ic_* \xi) |\widehat{\phi}(\xi)|^2 d\xi + \int_{\mathbb{R}} \mathcal{F}(f'(q_*)\phi) \widehat{\phi} d\xi = \lambda, \tag{5.2}$$

where $\mathcal{F}u = \widehat{u}$ denotes the Fourier transform of a function u . Using the facts that $\widehat{\phi} = \overline{\widehat{\phi(-\cdot)}}$, that the Fourier transform preserves the inner product, we can rewrite the 0-th order term as:

$$I_0 := \int_{\mathbb{R}} \mathcal{F}(b\phi) \widehat{\phi} d\xi = \int_{\mathbb{R}} b(x)\phi(x) \overline{\phi(-x)} dx. \tag{5.3}$$

The real and imaginary parts of equation (5.2) give

$$\operatorname{Re} \lambda - \operatorname{Re} I_0 = - \int_{\mathbb{R}} (\delta^2 \xi^4 + \xi^2) |\widehat{\phi}|^2 d\xi, \quad \operatorname{Im} \lambda - \operatorname{Im} I_0 = c_* \int_{\mathbb{R}} \xi |\widehat{\phi}|^2 d\xi.$$

Hence by the Cauchy-Schwartz inequality:

$$0 \leq (\operatorname{Im} \lambda - \operatorname{Im} I_0)^2 \leq c_*^2 \|\widehat{\phi}\|_{L^2}^2 \int_{\mathbb{R}} \xi^2 |\widehat{\phi}|^2 d\xi \leq c_*^2 \int_{\mathbb{R}} (\delta^2 \xi^4 + \xi^2) |\widehat{\phi}|^2 d\xi = c_*^2 (\operatorname{Re} I_0 - \operatorname{Re} \lambda). \tag{5.4}$$

Note that $b(\cdot; \delta)$ is uniformly bounded with respect to $\delta \in (-\delta_0, \delta_0)$, since this holds for $q_*(\cdot; \delta)$ from Theorem 1, and since f' is continuous.

Applying the Cauchy-Schwartz inequality in (5.3) leads to $|I_0| \leq \|b\|_\infty \|\phi\|_{L^2}^2$. Inserting this into (5.4) leads to the claimed bound on $\operatorname{Re} \lambda$. Then, we take the square root of (5.4) to obtain:

$$|\operatorname{Im} \lambda| - |\operatorname{Im} I_0| \leq |\operatorname{Im} \lambda - \operatorname{Im} I_0| \leq c_* \sqrt{\|b\|_\infty - \operatorname{Re} \lambda},$$

which is the claimed bound on $\operatorname{Im} \lambda$. These bounds together with the requirement $\operatorname{Re} \lambda \geq 0$ define a compact set K . \square

We now conclude the proof of Theorem 2 by excluding the possibility of any eigenvalues in the intermediate region; see Figure 3.

Proposition 5.2. *For each $\delta_0 > 0$ sufficiently small, there exists $r(\delta_0) > 0$ with $r(\delta_0) \rightarrow 0$ as $\delta_0 \rightarrow 0$ such that for all δ with $|\delta| < \delta_0$, the operator $\mathcal{L}(\delta)$ has no eigenvalues in $\{\operatorname{Re} \lambda \geq 0\} \setminus B(0, r(\delta_0))$.*

Proof. Suppose to the contrary that there exists a sequence $\delta_n \rightarrow 0$ with corresponding eigenvalues λ_n bounded away from the origin, with $\operatorname{Re} \lambda_n \geq 0$, and with eigenfunctions u_n . We normalize the eigenfunctions so that $\|u_n\|_{H^2} = 1$ for all n . By Proposition 5.1, these eigenvalues all belong to the compact set K . By compactness, we extract a subsequence along which $\lambda_n \rightarrow \lambda_\infty$ for some λ_∞ with $\lambda_\infty \in K$, and $\lambda_\infty \neq 0$, since the sequence was bounded away from the origin. We now show that in this limit, λ_∞ is an eigenvalue for $\mathcal{L}(0)$ with $\operatorname{Re} \lambda \geq 0$, contradicting the spectral stability of this operator.

These eigenfunctions solve $(\mathcal{L}(\delta_n) - \lambda_n)u_n = 0$. We precondition by applying $(1 - \delta_n^2 \partial_x^2)^{-1}$ to both sides of this equation, obtaining

$$[\partial_x^2 + a_1(\cdot, \delta_n) \partial_x + a_0(\cdot, \delta_n) + E_1(\delta_n) + E_2(\delta_n) - \lambda_n + \lambda_n T(\delta_n)] u_n = 0, \tag{5.5}$$

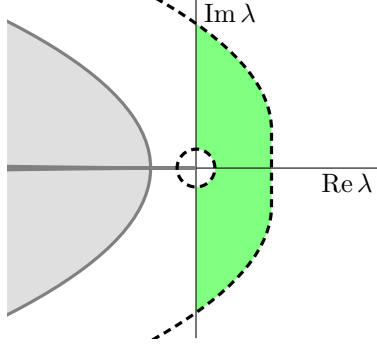


Figure 3: Three regions for the study of the point spectrum. The function $E(\delta, \gamma)$ from Section 4 rules out point spectrum in the dashed ball centered at the origin, together with a potential eigenvalue at the origin. Proposition 5.1 excludes point spectrum to the right of the dashed curve. Finally, the green region to the right contains no point spectrum provided δ is small enough, see Proposition 5.2.

where

$$E_1(\delta_n) = \delta_n^2(1 - \delta_n^2 \partial_x^2)^{-1}(a_3(\cdot, \delta_n) \partial_x^3 + a_2(\cdot, \delta_n) \partial_x^2), \quad E_2(\delta_n) = T(\delta_n)(a_1(\cdot, \delta_n) \partial_x + a_0(\cdot, \delta_n)).$$

We relate this to the KPP linearization $\mathcal{L}(0)$ by rewriting (5.5) as

$$(\mathcal{L}(0) - \lambda_\infty)u_n = -E_1(\delta_n)u_n - E_2(\delta_n)u_n + E_3(\delta_n)u_n + (\lambda_n - \lambda_\infty)u_n + \lambda_n T(\delta_n)u_n =: f_n$$

where

$$E_3(\delta_n) = (a_1(\cdot, 0) - a_1(\cdot, \delta_n)) \partial_x + (a_0(\cdot, 0) - a_0(\cdot, \delta_n)).$$

It follows from Lemma 2.1 and the fact that the coefficients $a_j(\cdot; \delta)$ are uniformly bounded in δ that

$$\|E_1(\delta_n)u_n\|_{L^2} \leq C\delta_n \|u_n\|_{H^2} = C\delta_n.$$

Similarly, by Lemma 2.2 we see that $E_2(\delta_n)u_n \rightarrow 0$ and $\lambda_n T(\delta_n)u_n \rightarrow 0$ in L^2 as $n \rightarrow \infty$, since λ_n and u_n are uniformly bounded in n . Lastly, by the construction of the exponential weights, the fact that $q_*(\cdot; \delta)$ converges uniformly to $q_*(\cdot; 0)$ as $\delta \rightarrow 0$ by Theorem 1, and the assumption that $\|u_n\|_{H^2}$ is uniformly bounded, we see that also $E_3(\delta_n)u_n \rightarrow 0$ in L^2 as $n \rightarrow \infty$. Hence f_n converges to zero in L^2 as $n \rightarrow \infty$.

Since λ_∞ is not in the spectrum of $\mathcal{L}(0)$, we can invert $(\mathcal{L}(0) - \lambda_\infty)$ to write

$$u_n = (\mathcal{L}(0) - \lambda_\infty)^{-1} f_n,$$

from which we observe that $u_n \rightarrow 0$ in $H^2(\mathbb{R})$ as $n \rightarrow \infty$ by boundedness of the resolvent operator. This is a contradiction since we have normalized u_n so that $\|u_n\|_{H^2} = 1$. \square

Proof of Theorem 2. By Proposition 4.1, there exist $\gamma_1, \delta_1 > 0$ so that for all $\delta \in (-\delta_1, \delta_1)$, $\mathcal{L}(\delta)$ has no eigenvalues in $B(0, \gamma_1^2)$, and also has no resonance at $\lambda = 0$. By Proposition 5.2, there exists a $\delta_0 > 0$ so for all $\delta \in (-\delta_0, \delta_0)$, $\mathcal{L}(\delta)$ has no eigenvalues in $\{\text{Re } \lambda \geq 0\} \setminus B(0, \frac{\gamma_1^2}{2})$. Hence for all $\delta \in (-\delta_0, \delta_0)$, $\mathcal{L}(\delta)$ has no eigenvalues in $\{\text{Re } \lambda \geq 0\}$, as desired. \square

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