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► **To cite this version:**

Christophe Profeta. The area under a spectrally positive stable excursion and other related processes. 2020. hal-02928052v1

HAL Id: hal-02928052

<https://hal.science/hal-02928052v1>

Preprint submitted on 2 Sep 2020 (v1), last revised 27 Apr 2021 (v2)

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THE AREA UNDER A SPECTRALLY POSITIVE STABLE EXCURSION AND OTHER RELATED PROCESSES

CHRISTOPHE PROFETA

ABSTRACT. We study the distribution of the area under the normalized excursion of a spectrally positive stable Lévy process L , as well as the area under its meander, and under L conditioned to stay positive. Our results involve a special case of Wright's function, which may be seen as a generalization of the classic Airy function appearing in similar Brownian's areas.

1. INTRODUCTION

Let L be an α -stable Lévy process without negative jumps, with $\alpha > 1$. We assume that L is normalized to have characteristic exponent :

$$\ln \left(\mathbb{E} \left[e^{i\lambda L_1} \right] \right) = (i\lambda)^\alpha e^{-i\pi\alpha \operatorname{sgn}(\lambda)}, \quad \lambda \in \mathbb{R}.$$

It is well-known, see for instance [23, Prop. 3.4.1], that the distribution of the area under L also follows a stable distribution :

$$\int_0^1 L_u du \stackrel{(\text{law})}{=} (1 + \alpha)^{-\frac{1}{\alpha}} L_1. \tag{1.1}$$

The purpose of this note is to study the distribution of the area under three related processes : the normalized excursion of L , the meander of L , and L conditioned to stay positive. These distributions have already been extensively studied in the Brownian case, see in particular the survey by Janson [8] or the paper by Perman & Wellner [21]. For all these Brownian areas, one observes the occurrence of the classic Airy function Ai . We shall prove that for spectrally positive stable Lévy processes, the role of Ai is played by the following M-Wright's function

$$\Phi_\alpha(x) = \frac{1}{\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \Gamma\left(\frac{1+n}{1+\alpha}\right) \sin\left(\pi \frac{1+n}{\alpha+1}\right) (1+\alpha)^{\frac{n-\alpha}{1+\alpha}} x^n.$$

The function Φ_α is known to be related to a time-fractional diffusion equation, see [19] and the references within. In particular, it admits the integral representation :

$$\Phi_\alpha(x) = \frac{1}{\pi} \int_0^{+\infty} e^{-\sin(\frac{\pi\alpha}{2}) \frac{z^{1+\alpha}}{1+\alpha}} \cos\left(\cos\left(\frac{\pi\alpha}{2}\right) \frac{z^{1+\alpha}}{1+\alpha} - zx\right) dz$$

2010 *Mathematics Subject Classification.* 60G52 - 60G18 - 60E10.

Key words and phrases. Stable processes - Area - Normalized excursion - Meander.

from which we immediately see that $\Phi_2(x) = \text{Ai}(x)$. Another function of interest will be its cosine counterpart :

$$\Psi_\alpha(x) = \frac{1}{\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \Gamma\left(\frac{1+n}{1+\alpha}\right) \cos\left(\pi \frac{1+n}{\alpha+1}\right) (1+\alpha)^{\frac{n-\alpha}{1+\alpha}} x^n.$$

We now state the main results of this paper.

1.1. The area under a normalized excursion. Let $L^{(\text{ex})}$ denote a normalized excursion of L on the segment $[0, 1]$ and set

$$\mathcal{A}_{\text{ex}} = \int_0^1 L_t^{(\text{ex})} dt.$$

Theorem 1. *The double Laplace transform of \mathcal{A}_{ex} is given by :*

$$\int_0^{+\infty} (1 - e^{-\lambda t}) \mathbb{E} \left[e^{-t^{1+\frac{1}{\alpha}} \mathcal{A}_{\text{ex}}} \right] t^{-(1+\frac{1}{\alpha})} dt = \alpha \Gamma\left(1 - \frac{1}{\alpha}\right) \left(\frac{\Phi'_\alpha(0)}{\Phi_\alpha(0)} - \frac{\Phi'_\alpha(\lambda)}{\Phi_\alpha(\lambda)} \right).$$

In the Brownian case, i.e. when $\alpha = 2$, the distribution of \mathcal{A}_{ex} is nowadays designated in the literature as the Airy distribution. We refer the reader to Louchard [16, 17] for a study of the Brownian excursion area via the Feynman-Kac formula, to Takács [24] for an approach via random walks and to Flajolet & Louchard [6] for a study of Airy distribution via its moments.

In the physics literature, this distribution has also appeared in the study of fluctuating interfaces, see [18]. We finally mention that more recently, some authors have investigated the area under a normalized Bessel excursion and shown its relation with the cooling of atoms [1, 12].

As is the case for the Airy distribution, we may deduce from Theorem 1 a recurrence relation for the moments of \mathcal{A}_{ex} . To this end, let us define a sequence $(B_{n,k}, 1 \leq n, 1 \leq k \leq n)$ by¹

$$B_{n,1} = \frac{(2-\alpha)_{n-1}}{(n+1)(n+2)}, \quad n \geq 1,$$

and for $k \geq 1$,

$$B_{n,k+1} = \frac{1}{k+1} \sum_{l=k}^{n-1} \binom{n}{l} B_{n-l,1} \times B_{l,k}, \quad n \geq 1, \quad (1.2)$$

This sequence corresponds to the values of the exponential partial Bell polynomials taken on the sequence $(B_{n,1}, n \geq 1)$, see Comtet [5, Section 3.3]. We then set $c_0^{(\alpha)} = 1$ and for $p \geq 1$,

$$c_p^{(\alpha)} = \frac{1}{(2p)! \sqrt{\pi}} \left(\frac{2}{\alpha}\right)^p \sum_{k=1}^{2p} B_{2p,k} \Gamma\left(p+k+\frac{1}{2}\right) (2(\alpha-1))^k. \quad (1.3)$$

The coefficients $(c_p^{(\alpha)}, p \geq 0)$ are the ones appearing in the asymptotic expansion of the function Φ_α , see Proposition 10 in the Appendix.

¹For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $(x)_n = x(x+1)\dots(x+n-1)$ denotes the usual Pochhammer symbol, with the convention $(x)_0 = 1$.

Corollary 2. *Let us set, for $n \geq 1$,*

$$\Omega_n = \frac{1}{n!} \frac{\mathbb{E}[\mathcal{A}_{\text{ex}}^n]}{\alpha \Gamma\left(1 - \frac{1}{\alpha}\right)} \Gamma\left(\frac{(n-1)(\alpha+1)}{\alpha} + 1\right).$$

Then, the sequence (Ω_n) follows the recurrence relation :

$$\Omega_n = c_{n-1}^{(\alpha)} \frac{(2n-1)(\alpha+1) - 2}{2\alpha} - \sum_{k=1}^{n-1} \Omega_k c_{n-k}^{(\alpha)}. \quad (1.4)$$

As consequence, we have the asymptotics² :

$$\left(\mathbb{E}[\mathcal{A}_{\text{ex}}^n]\right)^{\frac{1}{n}} \underset{n \rightarrow +\infty}{\asymp} n^{1-\frac{1}{\alpha}} \quad \text{and} \quad \ln \mathbb{P}(\mathcal{A}_{\text{ex}} > x) \underset{x \rightarrow +\infty}{\asymp} -x^{\frac{\alpha}{\alpha-1}}.$$

The first moments are given by :

$$\mathbb{E}[\mathcal{A}_{\text{ex}}] = \frac{\alpha-1}{2} \Gamma\left(1 - \frac{1}{\alpha}\right) \quad \text{and} \quad \mathbb{E}[\mathcal{A}_{\text{ex}}^2] = \frac{\Gamma\left(1 - \frac{1}{\alpha}\right) (\alpha-1) (2\alpha+1)}{12\Gamma\left(1 + \frac{1}{\alpha}\right)}.$$

Note that in the Brownian case, since $B_{n,1} = 0$ as soon as $n \geq 2$, the coefficients $c_n^{(2)}$ simplify to

$$c_n^{(2)} = \frac{1}{(2n)! \sqrt{\pi}} \left(\frac{1}{3}\right)^{2n} \Gamma\left(3n + \frac{1}{2}\right)$$

and we recover a classic recurrence formula for the moments of Airy distribution, see [17]. In this case, a complete asymptotic expansion for $\mathbb{P}(\mathcal{A}_{\text{ex}} > x)$ was computed by Janson & Louchard in [9].

1.2. The area under a stable meander. Let $L^{(\text{me})}$ denotes the meander of L on the segment $[0, 1]$ and set

$$\mathcal{A}_{\text{me}} = \int_0^1 L_t^{(\text{me})} dt.$$

Theorem 3. *The double Laplace transform of \mathcal{A}_{me} is given by :*

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{E}\left[e^{-t^{1+\frac{1}{\alpha}} \mathcal{A}_{\text{me}}}\right] t^{-\frac{1}{\alpha}} dt = \pi \Gamma\left(1 - \frac{1}{\alpha}\right) \frac{\Psi'_\alpha(\lambda) \Phi_\alpha(\lambda) - \Phi'_\alpha(\lambda) \Psi_\alpha(\lambda)}{\Phi_\alpha(\lambda)}. \quad (1.5)$$

In particular,

$$\mathbb{E}[\mathcal{A}_{\text{me}}] = \Gamma\left(1 - \frac{1}{\alpha}\right) \frac{\alpha+1}{2\alpha}$$

and there is the asymptotics, for $1 < \alpha < 2$:

$$\mathbb{P}(\mathcal{A}_{\text{me}} > x) \underset{x \rightarrow +\infty}{\sim} \frac{(\alpha-1)\Gamma(1+\alpha)\Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(2-\alpha)\Gamma\left(2 + \alpha - \frac{1}{\alpha}\right)} x^{-\alpha}.$$

²We write $a_p \asymp b_p$ as $p \rightarrow +\infty$ to state that there exist two constants $0 < \kappa_1 \leq \kappa_2 < +\infty$ such that $\kappa_1 a_p \leq b_p \leq \kappa_2 a_p$ for p large enough.

In the Brownian case, this distribution was studied for instance by Takács [25], see also Perman & Wellner [21]. Note that here, since the process is not pinned down at $t = 1$, the presence of positive jumps yields a polynomial decay of the tail of \mathcal{A}_{me} , which is different from the case $\alpha = 2$, see Janson & Louchard [9].

Remark 4. Again, when $\alpha = 2$, Theorem 3 simplifies. Indeed, observe that in this case, Ψ_2 is a solution of the following differential equation

$$\Psi_2''(\lambda) = \lambda\Psi_2(\lambda) - \frac{1}{\pi}$$

hence the derivative of the numerator on the right-hand side of (1.5) equals

$$\left(\Psi_2'(\lambda)\text{Ai}(\lambda) - \text{Ai}'(\lambda)\Psi_2(\lambda) \right)' = -\frac{1}{\pi}\text{Ai}(\lambda).$$

By integration, this implies that

$$\Psi_2'(\lambda)\text{Ai}(\lambda) - \text{Ai}'(\lambda)\Psi_2(\lambda) = \frac{1}{3\pi} - \frac{1}{\pi} \int_0^\lambda \text{Ai}(x)dx = \frac{1}{\pi} \int_\lambda^{+\infty} \text{Ai}(x)dx$$

which agrees for instance with [21, Corollary 3.2].

1.3. The area under L conditioned to stay positive. Let L^\uparrow be the process L started from 0 and conditioned to stay positive. We set

$$\mathcal{A}^\uparrow = \int_0^1 L_t^\uparrow dt.$$

Theorem 5. *The double Laplace transform of \mathcal{A}^\uparrow is given by :*

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{E} \left[e^{-t^{1+\frac{1}{\alpha}} \mathcal{A}^\uparrow} \right] dt = \pi\lambda \frac{\Phi_\alpha'(\lambda) \Psi_\alpha(\lambda) - \Psi_\alpha'(\lambda) \Phi_\alpha(\lambda)}{\Phi_\alpha(\lambda)} - \frac{\Phi_\alpha'(\lambda)}{\Phi_\alpha(\lambda)}. \quad (1.6)$$

In particular, there is the asymptotics, for $1 < \alpha < 2$:

$$\mathbb{P}(\mathcal{A}^\uparrow > x) \underset{x \rightarrow +\infty}{\sim} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\frac{1}{\alpha})\Gamma(2-\alpha)} x^{1-\alpha}.$$

In the Brownian case, \mathcal{A}^\uparrow corresponds, up to a $\sqrt{2}$ factor, to the integral on $[0, 1]$ of a three-dimensional Bessel process started from 0. As for the meander, Formula (1.6) simplifies when $\alpha = 2$, and the right-hand side equals

$$-\lambda \frac{\int_\lambda^{+\infty} \text{Ai}(x)dx}{\text{Ai}(\lambda)} - \frac{\text{Ai}'(\lambda)}{\text{Ai}(\lambda)} = \frac{-\lambda \int_\lambda^{+\infty} \text{Ai}(x)dx + \int_\lambda^{+\infty} x\text{Ai}(x)dx}{\text{Ai}(\lambda)} = \frac{\int_0^{+\infty} x\text{Ai}(x+\lambda)dx}{\text{Ai}(\lambda)}$$

which agrees with [3, p.440].

1.4. Outline of the paper. The remainder of the paper is divided as follows. Section 2 provides some notation as well as a key proposition which is of independent interest. Section 3, 4 and 5 give the proofs of the main Theorems, respectively on the normalized excursion, the meander, and L conditioned to stay positive. Finally, Section 6 is an appendix on the Wright's function Φ_α , where we compute its asymptotic expansion at infinity.

2. PRELIMINARIES

2.1. **Notations.** We start by recalling the definition of the considered processes, for which we mainly refer to Chaumont [4]. We assume that L is defined on the Skorokhod space of càdlàg processes. We denote by \mathbb{P}_x its law when $L_0 = x$, with the convention that $\mathbb{P} = \mathbb{P}_0$, and by $(\mathcal{F}_t, t \geq 0)$ its natural filtration. Define, for $z \in \mathbb{R}$,

$$T_z = \inf\{t \geq 0, L_t = z\}.$$

- (1) We denote by $\mathbb{P}_{x,y}^t$ the law of the bridge of L of length t , going from x to y .
- (2) We denote by \mathbb{P}_x^\uparrow the law of L started at $x > 0$ and conditioned to stay positive. It is classically given by the h -transform :

$$\forall \Lambda_t \in \mathcal{F}_t, \quad \mathbb{P}_x^\uparrow(\Lambda_t) = \frac{1}{x} \mathbb{E}_x [L_t 1_{\Lambda_t} 1_{\{T_0 > t\}}]. \quad (2.1)$$

This law admits a weak limit in the Skorokhod sense as $x \downarrow 0$ which we shall denote \mathbb{P}_0^\uparrow .

- (3) We denote by $\mathbb{P}^{(me)}$ the law of the meander of L , which is given by the limit

$$\forall \Lambda_t \in \mathcal{F}_t, \quad \mathbb{P}^{(me)}(\Lambda_t) = \lim_{x \downarrow 0} \mathbb{P}_x(\Lambda_t | T_0 > t).$$

- (4) Finally, we recall that, since L is spectrally positive, the law of the stable excursion of length t is equivalent to the bridge of L conditioned to stay positive, and starting and ending at 0. From Lemma 4 in [4], this law may be for instance defined by

$$\forall s < t \quad \Lambda_s \in \mathcal{F}_s, \quad \mathbb{P}_{0,0}^{\uparrow t}(\Lambda_s) = \lim_{x \downarrow 0} \mathbb{P}_x(\Lambda_s | T_0 = t). \quad (2.2)$$

2.2. **The key proposition.** The proofs of Theorems 1, 3 and 5 will rely heavily on the following proposition, which gives the joint Laplace transform of the pair $(T_0, \int_0^{T_0} L_s ds)$.

Proposition 6. *For $z, \lambda \geq 0$ and $\mu > 0$ we have :*

$$\mathbb{E}_z \left[e^{-\lambda T_0 - \mu \int_0^{T_0} L_s ds} \right] = \frac{\Phi_\alpha \left(\mu^{\frac{1}{1+\alpha}} \left(z + \frac{\lambda}{\mu} \right) \right)}{\Phi_\alpha \left(\mu^{-\frac{\alpha}{1+\alpha}} \lambda \right)}.$$

Proof. We first assume that $\lambda = 0$. The law of $\int_0^{T_0} L_s ds$ has been studied by Letemplier & Simon in [15]. In particular, they obtain the Mellin transform :

$$\mathbb{E}_1 \left[\left(\int_0^{T_0} L_s ds \right)^\nu \right] = (\alpha + 1)^\nu \frac{\Gamma(\frac{\alpha}{\alpha+1}) \Gamma(1 - (\alpha + 1)\nu)}{\Gamma(\frac{\alpha}{\alpha+1} - \nu) \Gamma(1 - \nu)}, \quad \nu < \frac{1}{1 + \alpha}.$$

Replacing ν by $-\nu$ and using the definition of the Gamma function, we obtain :

$$\int_0^{+\infty} \mu^{\nu-1} \mathbb{E}_1 \left[e^{-\mu \int_0^{T_0} L_s ds} \right] d\mu = (\alpha + 1)^\nu \frac{\Gamma(\nu) \Gamma(\frac{\alpha}{\alpha+1}) \Gamma(1 + (\alpha + 1)\nu)}{\Gamma(\frac{\alpha}{\alpha+1} + \nu) \Gamma(\nu + 1)}.$$

We now invert this Mellin transform following Janson [10] :

$$\mathbb{E}_1 \left[e^{-\mu \int_0^{T_0} L_s ds} \right] = 1 + \Gamma \left(\frac{\alpha}{1 + \alpha} \right) \sum_{n=1}^{+\infty} \frac{(-1)^n (1 + \alpha)^{\frac{n}{1+\alpha}}}{n! \Gamma \left(\frac{\alpha-n}{\alpha+1} \right)} \mu^{\frac{n}{1+\alpha}}$$

hence, by scaling, we thus obtain

$$\mathbb{E}_z \left[e^{-\mu \int_0^{T_0} L_s ds} \right] = \Gamma \left(\frac{\alpha}{1 + \alpha} \right) (1 + \alpha)^{\frac{\alpha}{1+\alpha}} \Phi_\alpha \left(z \mu^{\frac{1}{1+\alpha}} \right) = \frac{\Phi_\alpha \left(z \mu^{\frac{1}{1+\alpha}} \right)}{\Phi_\alpha(0)} \quad (2.3)$$

which is Proposition 6 when $\lambda = 0$. We now deal with the general case. Let $x > y > 0$. Applying the Markov property, we deduce from the absence of negative jumps that

$$\mathbb{E}_x \left[e^{-\mu \int_0^{T_0} L_s ds} \right] = \mathbb{E}_x \left[e^{-\mu \int_0^{T_y} L_s ds} \right] \times \mathbb{E}_y \left[e^{-\mu \int_0^{T_0} L_s ds} \right]. \quad (2.4)$$

Furthermore, by translation, we also have :

$$\mathbb{E}_x \left[e^{-\mu \int_0^{T_y} L_s ds} \right] = \mathbb{E}_{x-y} \left[e^{-\mu y T_0 - \mu \int_0^{T_0} L_s ds} \right].$$

Finally, setting $x - y = z > 0$ and $\lambda = \mu y > 0$, and plugging (2.3) in (2.4) yields

$$\mathbb{E}_z \left[e^{-\lambda T_0 - \mu \int_0^{T_0} L_s ds} \right] = \frac{\mathbb{E}_x \left[e^{-\mu \int_0^{T_0} L_s ds} \right]}{\mathbb{E}_y \left[e^{-\mu \int_0^{T_0} L_s ds} \right]} = \frac{\Phi_\alpha \left(\mu^{\frac{1}{1+\alpha}} \left(z + \frac{\lambda}{\mu} \right) \right)}{\Phi_\alpha \left(\mu^{-\frac{\alpha}{1+\alpha}} \lambda \right)}$$

which ends the proof of Proposition 6. \square

Remark 7. Proposition 6 was proven in the Brownian case by Lefebvre [14], by applying the Feynman-Kac formula. A generalization where T_0 is replaced by the exit time from an interval was obtained by Lachal [13] by a similar method.

3. THE AREA UNDER A SPECTRALLY POSITIVE STABLE EXCURSION

3.1. Proof of Theorem 1 : the double Laplace transform. Notice first that by monotone convergence, the absolute continuity formula (2.2) remains true at $s = t$:

$$\mathbb{E}_{0,0}^{\uparrow t} \left[e^{-\int_0^t L_u du} \right] = \lim_{x \downarrow 0} \mathbb{E}_x \left[e^{-\int_0^t L_u du} | T_0 = t \right]. \quad (3.1)$$

Next, starting from Proposition 6, we may write

$$\frac{\Phi_\alpha(z)}{\Phi_\alpha(0)} - \frac{\Phi_\alpha(z + \lambda)}{\Phi_\alpha(\lambda)} = \int_0^{+\infty} (1 - e^{-\lambda t}) \mathbb{E}_z \left[e^{-\int_0^t L_u du} | T_0 = t \right] \mathbb{P}_z(T_0 \in dt). \quad (3.2)$$

Recall now from Sato [22, Theorem 46.3] that since L has no negative jumps, T_0 is a positive stable random variable of index $1/\alpha$, i.e. for $z > 0$:

$$\mathbb{P}_z(T_0 \in dt)/dt = \frac{1}{\pi} \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n!} \sin(n\pi/\alpha) \Gamma(1 + n/\alpha) z^n t^{-1-n/\alpha}. \quad (3.3)$$

Dividing (3.2) by z and letting $z \downarrow 0$, we deduce from (3.1) and (3.3) that

$$\frac{\Phi'_\alpha(0)}{\Phi_\alpha(0)} - \frac{\Phi'_\alpha(\lambda)}{\Phi_\alpha(\lambda)} = \frac{\sin(\pi/\alpha)}{\pi} \Gamma\left(1 + \frac{1}{\alpha}\right) \int_0^{+\infty} (1 - e^{-\lambda t}) \mathbb{E}_{0,0}^{\uparrow t} \left[e^{-\int_0^t L_u du} \right] t^{-1-\frac{1}{\alpha}} dt.$$

Theorem 1 now follows by the complement formula for the Gamma function and the scaling property. \square

3.2. Proof of Corollary 1 : study of the positive moments. To get information on the moments, we shall work with a slight modification of the formula of Theorem 1. Indeed, using the decomposition

$$\begin{aligned} & \alpha \Gamma\left(1 - \frac{1}{\alpha}\right) \left(\frac{\Phi'_\alpha(0)}{\Phi_\alpha(0)} - \frac{\Phi'_\alpha(\lambda)}{\Phi_\alpha(\lambda)} \right) \\ &= \int_0^{+\infty} \mathbb{E} \left[e^{-t^{1+\frac{1}{\alpha}} \mathcal{A}_{\text{ex}}} - 1 \right] \frac{dt}{t^{1+\frac{1}{\alpha}}} + \int_0^{+\infty} e^{-\lambda t} \mathbb{E} \left[1 - e^{-t^{1+\frac{1}{\alpha}} \mathcal{A}_{\text{ex}}} \right] \frac{dt}{t^{1+\frac{1}{\alpha}}} - \Gamma\left(-\frac{1}{\alpha}\right) \lambda^{1/\alpha} \end{aligned}$$

we deduce the alternative formula

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{E} \left[1 - e^{-t^{1+\frac{1}{\alpha}} \mathcal{A}_{\text{ex}}} \right] t^{-(1+\frac{1}{\alpha})} dt = \Gamma\left(-\frac{1}{\alpha}\right) \left(\frac{\Phi'(\lambda)}{\Phi(\lambda)} + \lambda^{1/\alpha} \right). \quad (3.4)$$

Setting $x = \lambda^{-(1+1/\alpha)}$ and making the change of variable $s = \lambda t$ then yields

$$\int_0^{+\infty} e^{-s} \mathbb{E} \left[1 - e^{-x s^{1+\frac{1}{\alpha}} \mathcal{A}_{\text{ex}}} \right] s^{-(1+\frac{1}{\alpha})} ds = \Gamma\left(-\frac{1}{\alpha}\right) \left(x^{\frac{1}{1+\alpha}} \frac{\Phi'\left(x^{-\frac{\alpha}{1+\alpha}}\right)}{\Phi\left(x^{-\frac{\alpha}{1+\alpha}}\right)} + 1 \right)$$

and it remains to compute the asymptotic expansion of both sides as $x \rightarrow 0$. Using the asymptotic expansions of Φ_α and Φ'_α given in Proposition 10 in the Appendix, we get

$$\sum_{n=1}^{+\infty} (-1)^n x^n \Omega_n \underset{x \rightarrow 0}{\sim} \frac{\sum_{p=0}^{+\infty} (-1)^{p+1} d_p^{(\alpha)} x^p}{\sum_{p=0}^{+\infty} (-1)^p c_p^{(\alpha)} x^p} + 1$$

where we recall that $(c_p^{(\alpha)})$ is given by (1.3) and $(d_p^{(\alpha)})$ is given by

$$d_0^{(\alpha)} = 1 \quad \text{and} \quad d_p^{(\alpha)} = c_p^{(\alpha)} - c_{p-1}^{(\alpha)} \left(\frac{(2p-1)(\alpha+1)-2}{2\alpha} \right) \quad \text{for } p \geq 1.$$

By identification, we thus obtain the announced recurrence relation :

$$\Omega_n = c_{n-1}^{(\alpha)} \frac{(2n-1)(\alpha+1)-2}{2\alpha} - \sum_{j=1}^{n-1} \Omega_j c_{n-j}^{(\alpha)}.$$

3.3. Proof of Corollary 1 : asymptotics. To study the asymptotics of (Ω_n) , observe first that by definition, this sequence is positive, and so is the sequence $(c_n^{(\alpha)})$. Therefore, we deduce from the recurrence relation (1.4) and from Corollary 11 in the Appendix that there exists a finite constant κ_1 such that for n large enough

$$\Omega_n^{\frac{1}{n}} \leq \left(c_{n-1}^{(\alpha)} \frac{(2n-1)(\alpha+1)-2}{2\alpha} \right)^{\frac{1}{n}} \leq \kappa_1 n.$$

Stirling's formula now implies the upper bound for $\mathbb{E}[\mathcal{A}_{\text{ex}}^n]^{\frac{1}{n}}$, which may be written

$$\limsup_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{A}_{\text{ex}}^n]^{\frac{1}{n}}}{n^{1-\frac{1}{\alpha}}} \leq \kappa_2 \quad (3.5)$$

for some finite constant $\kappa_2 > 0$. To obtain the tail decay of \mathcal{A}_{ex} , we shall rely on Kasahara's Tauberian theorem of exponential type. Indeed, applying [11, Theorem 4], we deduce from (3.5), that there exists a constant $0 < \kappa_3 < +\infty$ such that

$$\limsup_{x \rightarrow +\infty} \frac{1}{x} \ln \mathbb{P}(\mathcal{A}_{\text{ex}} > x^{1-\frac{1}{\alpha}}) \leq -\kappa_3$$

which gives the announced upper bound.

It does not seem easy to obtain a lower bound for Ω_n from the recurrence relation (1.4). Instead, we shall rather study directly the tail of the survival function of \mathcal{A}_{ex} . To do so, we recall the following absolute continuity formula for the normalized excursion, see [4, Formula (11)]:

$$\forall s < 1, \quad \Lambda_s \in \mathcal{F}_s, \quad \mathbb{P}_{0,0}^{\uparrow,1}(\Lambda_s) = c \mathbb{E}_0^{\uparrow} \left[1_{\Lambda_s} \frac{j_{1-s}^*(L_s)}{L_s} \right]$$

where c is a normalization constant and, from Monrad & Silverstein [20, Formula (3.25)], j_{1-s}^* is a measurable function which admits the asymptotics :

$$j_{1-s}^*(x) \underset{x \rightarrow +\infty}{\sim} \gamma(1-s)^{-1/\alpha} x^{\frac{\alpha}{2(\alpha-1)}} e^{-\eta(1-s)^{-\frac{1}{\alpha-1}} x^{\frac{\alpha}{\alpha-1}}}$$

where γ and η are two positive constants. Using this absolute continuity formula, we first deduce that :

$$\begin{aligned} \mathbb{P}(\mathcal{A}_{\text{ex}} > x) &\geq \mathbb{P}_{0,0}^{\uparrow,1} \left(\int_{1/4}^{3/4} L_u^{\uparrow} du > x \right) \\ &= c \mathbb{E}_0^{\uparrow} \left[\frac{j_{1/4}^*(L_{3/4}^{\uparrow})}{L_{3/4}^{\uparrow}} 1_{\{\int_{1/4}^{3/4} L_u^{\uparrow} du > x\}} \right] \geq c \mathbb{E}_0^{\uparrow} \left[\frac{j_{1/4}^*(L_{3/4}^{\uparrow})}{L_{3/4}^{\uparrow}} 1_{\left\{ \inf_{\frac{1}{4} \leq u \leq \frac{3}{4}} L_u^{\uparrow} > 2x \right\}} \right]. \end{aligned}$$

Applying the Markov property, we further obtain

$$\begin{aligned} \mathbb{P}(\mathcal{A}_{\text{ex}} > x) &\geq c \mathbb{E}_0^\uparrow \left[\mathbb{E}_{L_{1/4}^\uparrow}^\uparrow \left[\frac{j_{1/4}^*(L_{1/2}^\uparrow)}{L_{1/2}^\uparrow} 1_{\left\{ \inf_{0 \leq u \leq \frac{1}{2}} L_u^\uparrow > 2x \right\}} \right] 1_{\{L_{1/4}^\uparrow > 2x\}} \right] \\ &= c \mathbb{E}_0^\uparrow \left[\frac{1}{L_{1/4}^\uparrow} \mathbb{E}_{L_{1/4}^\uparrow}^\uparrow \left[j_{1/4}^*(L_{1/2}) 1_{\left\{ \inf_{0 \leq u \leq \frac{1}{2}} L_u > 2x \right\}} \right] 1_{\{L_{1/4}^\uparrow > 2x\}} \right], \end{aligned}$$

where we used in the last equality the absolute continuity formula (2.1) for $\mathbb{P}_{L_{1/4}^\uparrow}^\uparrow$. Now, since $L_{1/2} \geq \inf_{0 \leq u \leq \frac{1}{2}} L_u > 2x$, we deduce from the asymptotics of $j_{1/4}^*$ that for x large enough there exist two positive constants $\tilde{\gamma}$ and $\tilde{\eta}$ such that

$$\begin{aligned} \mathbb{P}(\mathcal{A}_{\text{ex}} > x) &\geq \tilde{\gamma} e^{-\tilde{\eta} x^{\frac{\alpha}{\alpha-1}}} \mathbb{E}_0^\uparrow \left[\frac{1}{L_{1/4}^\uparrow} \mathbb{P}_{L_{1/4}^\uparrow} \left(\inf_{0 \leq u \leq \frac{1}{2}} L_u > 2x \right) 1_{\{L_{1/4}^\uparrow > 2x\}} \right] \\ &\geq \tilde{\gamma} e^{-\tilde{\eta} x^{\frac{\alpha}{\alpha-1}}} \mathbb{E}_0^\uparrow \left[\frac{1}{L_{1/4}^\uparrow} \mathbb{P}_{L_{1/4}^\uparrow} \left(\inf_{0 \leq u \leq \frac{1}{2}} L_u > 2x, L_{1/2} < 4x \right) 1_{\{4x > L_{1/4}^\uparrow > 3x\}} \right] \\ &\geq \tilde{\gamma} e^{-\tilde{\eta} x^{\frac{\alpha}{\alpha-1}}} \frac{1}{4x} \mathbb{E}_0^\uparrow \left[\mathbb{P}_0 \left(\inf_{0 \leq u \leq \frac{1}{2}} L_u > -x, L_{1/2} < 0 \right) 1_{\{4x > L_{1/4}^\uparrow > 3x\}} \right] \\ &= \tilde{\gamma} e^{-\tilde{\eta} x^{\frac{\alpha}{\alpha-1}}} \frac{1}{4x} \mathbb{P}_0 \left(\inf_{0 \leq u \leq \frac{1}{2}} L_u > -x, L_{1/2} < 0 \right) \mathbb{P}_0^\uparrow \left(4x > L_{1/4}^\uparrow > 3x \right) \end{aligned}$$

where, in the third inequality, we have used the independent increments property of L . The lower bound now follows by taking the logarithm on each side, and using the limits

$$\mathbb{P}_0 \left(\inf_{0 \leq u \leq \frac{1}{2}} L_u > -x, L_{1/2} < 0 \right) \xrightarrow{x \rightarrow +\infty} \mathbb{P}_0(L_{1/2} < 0) = \frac{1}{\alpha}$$

and, see Chaumont [4, p.12],

$$\mathbb{P}_0^\uparrow \left(4x > L_{1/4}^\uparrow > 3x \right) \underset{x \rightarrow +\infty}{\sim} \kappa x^{-1-\alpha}$$

for some $\kappa > 0$. The lower bound for $\mathbb{E}[\mathcal{A}_{\text{ex}}^n]^\frac{1}{n}$ follows then as before from Kasahara's Tauberian theorem of exponential type [11, Theorem 4]. \square

3.4. Study of some (fractional) negative moments. As was observed by Flajolet & Louchard [6] for the classic Airy distribution, one may also compute by recurrence some specific negative moments. These moments are related to the asymptotic expansion of Φ_α at $\lambda = 0$, which is given by its very definition as a series.

Corollary 8. *Let us set for $n \geq 1$:*

$$\Delta_n = \mathbb{E} \left[\mathcal{A}_{\text{ex}}^{\frac{1-\alpha n}{\alpha+1}} \right] \Gamma \left(\frac{\alpha n - 1}{\alpha + 1} \right) (1 + \alpha)^{-\frac{n+1}{1+\alpha}}.$$

Then, the sequence (Δ_n) follows the recurrence relation

$$\Delta_n + \sum_{k=1}^{n-1} \binom{n}{k} \Delta_{n-k} \frac{\Gamma \left(\frac{\alpha}{\alpha+1} \right)}{\Gamma \left(\frac{\alpha-k}{\alpha+1} \right)} = \left(1 + \frac{1}{\alpha} \right) \Gamma \left(-\frac{1}{\alpha} \right) \left(\frac{\Gamma \left(\frac{\alpha}{\alpha+1} \right)}{\Gamma \left(\frac{\alpha-1-n}{\alpha+1} \right)} - \frac{\Gamma^2 \left(\frac{\alpha}{\alpha+1} \right)}{\Gamma \left(\frac{\alpha-1}{\alpha+1} \right) \Gamma \left(\frac{\alpha-n}{\alpha+1} \right)} \right).$$

In particular, the first value is given by :

$$\mathbb{E} \left[\mathcal{A}_{\text{ex}}^{\frac{1-\alpha}{\alpha+1}} \right] = (1 + \alpha)^{\frac{3+\alpha}{1+\alpha}} \frac{\Gamma \left(-\frac{1}{\alpha} \right)}{\Gamma \left(\frac{\alpha-1}{\alpha+1} \right)} \left(\frac{\Gamma \left(\frac{\alpha}{\alpha+1} \right)}{\Gamma \left(\frac{\alpha-2}{\alpha+1} \right)} - \frac{\Gamma^2 \left(\frac{\alpha}{\alpha+1} \right)}{\Gamma^2 \left(\frac{\alpha-1}{\alpha+1} \right)} \right).$$

Proof. Starting from Theorem 1 and multiplying both sides by Φ_α , we obtain

$$\begin{aligned} \frac{\alpha}{\alpha+1} \Phi_\alpha(\lambda) \mathbb{E} \left[\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n!} \lambda^n \mathcal{A}_{\text{ex}}^{\frac{1-\alpha n}{\alpha+1}} \Gamma \left(\frac{\alpha n - 1}{\alpha + 1} \right) \right] \\ = \Gamma \left(-\frac{1}{\alpha} \right) \left(\Phi'_\alpha(\lambda) + (1 + \alpha)^{\frac{1}{1+\alpha}} \frac{\Gamma \left(\frac{\alpha}{\alpha+1} \right)}{\Gamma \left(\frac{\alpha-1}{\alpha+1} \right)} \Phi_\alpha(\lambda) \right). \end{aligned}$$

The result now follows by computing the Cauchy product on the left-hand side, and by identifying both expansions. \square

4. THE AREA UNDER A SPECTRALLY POSITIVE STABLE MEANDER

4.1. Proof of Theorem 3 : the double Laplace transform. We shall work here with a Laplace-Fourier transform to avoid integrability problems. Applying the Markov property and using Proposition 6, we have for $z > 0$:

$$\begin{aligned} & \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_z \left[e^{-i \int_0^t L_u du} \mathbf{1}_{\{T_0 > t\}} \right] dt \\ &= \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_z \left[e^{-i \int_0^t L_u du} \right] dt - \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_z \left[e^{-i \int_0^t L_u du} \mathbf{1}_{\{T_0 \leq t\}} \right] dt \\ &= \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_z \left[e^{-i \int_0^t L_u du} \right] dt - \mathbb{E}_z \left[e^{-\lambda T_0 - i \int_0^{T_0} L_u du} \right] \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0 \left[e^{-i \int_0^t L_u du} \right] dt \\ &= \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0 \left[e^{-izt - i \int_0^t L_u du} \right] dt - \frac{\Phi_\alpha \left(i^{\frac{1}{1+\alpha}} \left(z + \frac{\lambda}{i} \right) \right)}{\Phi_\alpha \left(i^{-\frac{\alpha}{1+\alpha}} \lambda \right)} \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0 \left[e^{-i \int_0^t L_u du} \right] dt. \end{aligned}$$

We now divide this equality by z and let $z \downarrow 0$. Using (3.3), the scaling property and the definition of the meander, the left-hand side converges towards

$$\frac{1}{z} \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_z \left[e^{-i \int_0^t L_u du} \mathbf{1}_{\{T_0 > t\}} \right] \mathbb{P}_z(T_0 > t) dt \xrightarrow{z \downarrow 0} \frac{1}{\Gamma(1 - \frac{1}{\alpha})} \int_0^{+\infty} e^{-\lambda t} \mathbb{E} \left[e^{-it^{1+1/\alpha} \mathcal{A}_{\text{me}}} \right] t^{-1/\alpha} dt$$

while the right-hand side yields

$$- \int_0^{+\infty} ite^{-\lambda t} \mathbb{E}_0 \left[e^{-i \int_0^t L_u du} \right] dt - i \frac{1}{1+\alpha} \frac{\Phi'_\alpha \left(i^{-\frac{\alpha}{1+\alpha}} \lambda \right)}{\Phi_\alpha \left(i^{-\frac{\alpha}{1+\alpha}} \lambda \right)} \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0 \left[e^{-i \int_0^t L_u du} \right] dt. \quad (4.1)$$

Let us set, to simplify the notation :

$$F_\alpha(\lambda) = \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0 \left[e^{-i \int_0^t L_u du} \right] dt. \quad (4.2)$$

From (1.1), we deduce that for $\lambda \in \mathbb{C}$:

$$\begin{aligned} F_\alpha(\lambda) &= \int_0^{+\infty} e^{-\lambda t} \exp \left(e^{\frac{i\pi}{2}\alpha} \frac{t^{1+\alpha}}{1+\alpha} \right) dt \\ &= \frac{1}{1+\alpha} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \lambda^n \Gamma \left(\frac{n+1}{\alpha+1} \right) e^{\frac{i\pi(2-\alpha)}{2} \frac{n+1}{\alpha+1}} (1+\alpha)^{\frac{n+1}{\alpha+1}} \\ &= \pi i^{-\frac{\alpha}{1+\alpha}} \left(\Psi_\alpha \left(i^{-\frac{\alpha}{1+\alpha}} \lambda \right) + i \Phi_\alpha \left(i^{-\frac{\alpha}{1+\alpha}} \lambda \right) \right) \end{aligned} \quad (4.3)$$

which shows that Φ_α and Ψ_α are closely related to the integral of L . Notice also that

$$- \int_0^{+\infty} ite^{-\lambda t} \mathbb{E}_0 \left[e^{-i \int_0^t L_u du} \right] dt = i F'_\alpha(\lambda). \quad (4.4)$$

Plugging (4.3) and (4.4) in (4.1) finally yields the formula

$$\begin{aligned} &\int_0^{+\infty} e^{-\lambda t} \mathbb{E} \left[e^{-it^{1+\frac{1}{\alpha}} \mathcal{A}_{me}} \right] t^{-\frac{1}{\alpha}} dt \\ &= \Gamma \left(1 - \frac{1}{\alpha} \right) \frac{i^{\frac{1-\alpha}{1+\alpha}} \Psi'_\alpha \left(i^{-\frac{\alpha}{1+\alpha}} \lambda \right) \Phi_\alpha \left(i^{-\frac{\alpha}{1+\alpha}} \lambda \right) - \Phi'_\alpha \left(i^{-\frac{\alpha}{1+\alpha}} \lambda \right) \Psi_\alpha \left(i^{-\frac{\alpha}{1+\alpha}} \lambda \right)}{\Phi_\alpha \left(i^{-\frac{\alpha}{1+\alpha}} \lambda \right)} \end{aligned}$$

and the announced result follows by analytical continuation.

4.2. Proof of Theorem 3 : first moment and asymptotics. To simplify the following computation, we set

$$H_\alpha(\lambda) = \pi \frac{\Psi'_\alpha(\lambda) \Phi_\alpha(\lambda) - \Phi'_\alpha(\lambda) \Psi_\alpha(\lambda)}{\Phi_\alpha(\lambda)}.$$

Differentiating the formula

$$\int_0^{+\infty} e^{-t} \mathbb{E} \left[e^{-\left(\frac{t}{\lambda}\right)^{1+\frac{1}{\alpha}} \mathcal{A}_{me}} \right] t^{-\frac{1}{\alpha}} dt = \Gamma \left(1 - \frac{1}{\alpha} \right) \lambda^{1-\frac{1}{\alpha}} H_\alpha(\lambda)$$

we deduce that

$$(\alpha+1) \int_0^{+\infty} e^{-t} \mathbb{E} \left[\mathcal{A}_{me} e^{-\left(\frac{t}{\lambda}\right)^{1+\frac{1}{\alpha}} \mathcal{A}_{me}} \right] t dt = \Gamma \left(1 - \frac{1}{\alpha} \right) \left((\alpha-1) \lambda^2 H_\alpha(\lambda) + \alpha \lambda^3 H'_\alpha(\lambda) \right)$$

and the value $\mathbb{E}[\mathcal{A}_{me}]$ will follow by letting $\lambda \rightarrow +\infty$ and applying the monotone convergence theorem on the left-hand side. Since the asymptotics of Φ_α and Φ'_α are given in Proposition

10 in the Appendix, it only remains to study those of Ψ_α . To this end, we observe from (4.3) that

$$\Psi_\alpha(\lambda) = \frac{i^{\frac{\alpha}{1+\alpha}}}{\pi} F_\alpha(\lambda i^{\frac{\alpha}{1+\alpha}}) - i\Phi_\alpha(\lambda).$$

Applying Watson's lemma, we deduce from the definition (4.2) of F_α that

$$F_\alpha(\lambda i^{\frac{\alpha}{1+\alpha}}) \underset{\lambda \rightarrow +\infty}{\sim} i^{-\frac{\alpha}{1+\alpha}} \sum_{n=0}^{+\infty} \frac{1}{n!} \frac{\Gamma(1 + (1+\alpha)n)}{(1+\alpha)^n} \lambda^{-n(1+\alpha)-1}$$

which implies, since Φ_α decreases exponentially fast, the asymptotic expansion :

$$\Psi_\alpha(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} \frac{1}{\pi} \sum_{n=0}^{+\infty} \frac{1}{n!} \frac{\Gamma(1 + (1+\alpha)n)}{(1+\alpha)^n} \lambda^{-n(1+\alpha)-1}.$$

Therefore, going back to the definition of H_α , we deduce that

$$H_\alpha(\lambda) \underset{\lambda \rightarrow +\infty}{=} \left(\lambda^{\frac{1}{\alpha}-1} - \frac{\alpha+1}{2\alpha} \lambda^{-2} + \Gamma(1+\alpha) \lambda^{-2-\alpha+\frac{1}{\alpha}} \right) + o(\lambda^{-2-\alpha+\frac{1}{\alpha}}). \quad (4.5)$$

As a consequence, we obtain the limit

$$\lim_{\lambda \rightarrow +\infty} (\alpha-1) \lambda^2 H_\alpha(\lambda) + \alpha \lambda^3 H'_\alpha(\lambda) = \frac{(\alpha+1)^2}{2\alpha}$$

from which we deduce the value of the first moment

$$\mathbb{E}[\mathcal{A}_{\text{me}}] = \Gamma\left(1 - \frac{1}{\alpha}\right) \frac{\alpha+1}{2\alpha}.$$

Next, to compute the asymptotics of $\mathbb{P}(\mathcal{A}_{\text{me}} > x)$, we shall work with Mellin transforms, which is a convenient tool when dealing with stable processes. Using Formula (1.5) and applying the Fubini-Tonelli theorem, we have for $\nu \in (0, 1 - \frac{1}{\alpha})$,

$$\Gamma\left(1 - \frac{1}{\alpha}\right) \int_0^{+\infty} \lambda^{\nu-1} H_\alpha(\lambda) d\lambda = \frac{\alpha\Gamma(\nu)}{1+\alpha} \Gamma\left(\frac{\alpha-1-\alpha\nu}{\alpha+1}\right) \mathbb{E}\left[(\mathcal{A}_{\text{me}})^{\frac{1-\alpha+\alpha\nu}{1+\alpha}}\right]. \quad (4.6)$$

Integrating twice by parts to remove the singularities at $\nu = 1 - \frac{1}{\alpha}$ and $\nu = 2$, we further obtain

$$\begin{aligned} & \Gamma\left(1 - \frac{1}{\alpha}\right) \int_0^{+\infty} \lambda^{\nu-2} ((\alpha-1)2\lambda H_\alpha(\lambda) + (4\alpha-1)\lambda^2 H'_\alpha + \alpha\lambda^3 H''_\alpha(\lambda)) d\lambda \\ &= (2-\nu)\Gamma(\nu)\Gamma\left(\frac{\alpha(2-\nu)}{\alpha+1}\right) (1-\alpha+\alpha\nu) \int_0^{+\infty} \lambda^{\nu-2} \lambda^{\frac{1}{\alpha}} \mathbb{P}\left(\mathcal{A}_{\text{me}} > \lambda^{1+\frac{1}{\alpha}}\right) d\lambda \end{aligned}$$

where, as $\lambda \rightarrow +\infty$, the integrand on the left-hand side is equivalent to

$$(\alpha-1)2\lambda H_\alpha(\lambda) + (4\alpha-1)\lambda^2 H'_\alpha + \alpha\lambda^3 H''_\alpha(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} \Gamma(1+\alpha)(1+\alpha)^2(\alpha-1)\lambda^{-1-\alpha+\frac{1}{\alpha}}.$$

The Mellin transform on the left-hand side thus admits a simple pole at $\nu = 2 + \alpha - \frac{1}{\alpha}$, hence so does the Mellin transform on the right-hand side. Applying the converse mapping theorem, we deduce that, as $\lambda \rightarrow +\infty$,

$$\Gamma\left(1 - \frac{1}{\alpha}\right) \Gamma(1 + \alpha) \lambda^{-1 - \alpha + \frac{1}{\alpha}} \underset{\lambda \rightarrow +\infty}{\sim} -\Gamma\left(2 + \alpha - \frac{1}{\alpha}\right) \Gamma(1 - \alpha) \lambda^{\frac{1}{\alpha}} \mathbb{P}\left(\mathcal{A}_{\text{me}} > \lambda^{1 + \frac{1}{\alpha}}\right)$$

which yields the announced asymptotics.

5. THE AREA UNDER L CONDITIONED TO STAY POSITIVE.

5.1. Proof of Theorem 5 : the double Laplace transform. We proceed as for the meander. Applying the Markov property, we first have

$$\begin{aligned} & \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_z \left[L_t e^{-i \int_0^t L_u du} 1_{\{T_0 > t\}} \right] dt \\ &= \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_z \left[L_t e^{-i \int_0^t L_u du} \right] dt - \frac{\Phi_\alpha\left(i^{-\frac{1}{1+\alpha}}\left(z + \frac{\lambda}{i}\right)\right)}{\Phi_\alpha\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right)} \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0 \left[L_t e^{-i \int_0^t L_u du} \right] dt. \end{aligned}$$

Dividing both sides by z and letting $z \downarrow 0$ then yields by definition of \mathbb{P}_0^\uparrow

$$\begin{aligned} & \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0^\uparrow \left[e^{-i \int_0^t L_u du} \right] dt \\ &= \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0 \left[(1 - itL_t) e^{-i \int_0^t L_u du} \right] dt - i^{-\frac{1}{1+\alpha}} \frac{\Phi'_\alpha\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right)}{\Phi_\alpha\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right)} \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0 \left[L_t e^{-i \int_0^t L_u du} \right] dt. \end{aligned}$$

Observe next that, integrating by parts the definition (4.2) of F_α , we also have

$$F_\alpha(\lambda) = \frac{1}{\lambda} - \frac{i}{\lambda} \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0 \left[L_t e^{-i \int_0^t L_u du} \right] dt.$$

As a consequence, we deduce from (4.3) that

$$\begin{aligned} & \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0^\uparrow \left[e^{-i \int_0^t L_u du} \right] dt \\ &= -\lambda F'_\alpha(\lambda) - i^{-\frac{\alpha}{1+\alpha}} \frac{\Phi'_\alpha\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right)}{\Phi_\alpha\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right)} (1 - \lambda F_\alpha(\lambda)) \\ &= \pi i^{-\frac{2\alpha}{1+\alpha}} \lambda \frac{\Phi'_\alpha\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right) \Psi_\alpha\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right) - \Psi'_\alpha\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right) \Phi\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right)}{\Phi\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right)} - i^{-\frac{\alpha}{1+\alpha}} \frac{\Phi'\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right)}{\Phi\left(i^{-\frac{\alpha}{1+\alpha}}\lambda\right)} \end{aligned}$$

and the result follows as before by analytic continuation.

5.2. Proof of Theorem 5 : asymptotics. The asymptotics of the tail of \mathcal{A}^\dagger is easy to obtain as we can work with Laplace transforms and use repeatedly Karamata's Tauberian theorem. Indeed, since we are dealing with monotone integrands, there is the asymptotics for $\alpha < 2$:

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{E} \left[1 - e^{-t^{1+\frac{1}{\alpha}} \mathcal{A}^\dagger} \right] dt = \frac{1}{\lambda} + \lambda H_\alpha(\lambda) + \frac{\Phi'_\alpha(\lambda)}{\Phi_\alpha(\lambda)} \underset{\lambda \rightarrow +\infty}{\sim} \frac{\Gamma(1+\alpha)}{\lambda^{1+\alpha-\frac{1}{\alpha}}}$$

which implies that

$$\mathbb{E} \left[1 - e^{-t^{1+\frac{1}{\alpha}} \mathcal{A}^\dagger} \right] \underset{t \rightarrow 0}{\sim} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\frac{1}{\alpha})} t^{\alpha-\frac{1}{\alpha}}$$

which in turn implies that

$$\mathbb{P}(\mathcal{A}^\dagger > x) \underset{x \rightarrow +\infty}{\sim} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\frac{1}{\alpha}) \Gamma(2-\alpha)} x^{1-\alpha}.$$

6. APPENDIX ON M-WRIGHT'S FUNCTIONS

We gather and prove in this section several useful formulae for the M-Wright's function Φ_α and its derivative.

6.1. The integral representation. Following the notation of [7], the classic Wright's function is defined by :

$$\phi(\rho, \beta; z) = \sum_{k=0}^{+\infty} \frac{z^k}{k! \Gamma(\rho k + \beta)}, \quad \rho > -1, \beta \in \mathbb{C}.$$

A special case of this function is obtained when $\rho = \beta - 1 = -\frac{1}{1+\alpha}$, for which, applying the compensation formula for the Gamma function, we obtain :

$$\begin{aligned} \Phi_\alpha(x) &= \frac{1}{\pi} (1+\alpha)^{-\frac{\alpha}{1+\alpha}} \phi \left(-\frac{1}{1+\alpha}, \frac{\alpha}{1+\alpha}; -x(1+\alpha)^{\frac{1}{1+\alpha}} \right) \\ &= \frac{1}{\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \Gamma \left(\frac{1+n}{1+\alpha} \right) \sin \left(\pi \frac{1+n}{\alpha+1} \right) (1+\alpha)^{\frac{n-\alpha}{1+\alpha}} x^n \end{aligned}$$

which is nowadays referred in the literature as an M-Wright's function. This function is also closely related to the probability density of the positive stable distribution of parameter $\frac{1}{\alpha+1}$, see Sato [22, p.88]. To get an integral expression, recall the formulae, since $\alpha > 1$:

$$\int_0^{\infty e^{\pm i\pi/(1+\alpha)}} z^n e^{\frac{z^{1+\alpha}}{1+\alpha}} dz = e^{\pm i\pi \frac{(n+1)}{1+\alpha}} (1+\alpha)^{\frac{n-\alpha}{\alpha+1}} \Gamma \left(\frac{n+1}{\alpha+1} \right)$$

which implies

$$\begin{aligned} \Phi_\alpha(x) &= \frac{1}{2\pi i} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n \left(\int_0^{\infty e^{i\pi/(1+\alpha)}} z^n e^{\frac{z^{1+\alpha}}{1+\alpha}} dz - \int_0^{\infty e^{-i\pi/(1+\alpha)}} z^n e^{\frac{z^{1+\alpha}}{1+\alpha}} dz \right) \\ &= \frac{1}{2\pi i} \int_{\infty e^{-i\pi/(1+\alpha)}}^{\infty e^{i\pi/(1+\alpha)}} e^{\frac{z^{1+\alpha}}{1+\alpha} - zx} dz. \end{aligned}$$

Applying Cauchy's integral theorem, we may deform the path of integration to pass by the imaginary axis, and thus obtain the integral representation :

$$\Phi_\alpha(x) = \frac{1}{\pi} \int_0^{+\infty} e^{-\sin(\frac{\pi\alpha}{2}) \frac{z^{1+\alpha}}{1+\alpha}} \cos\left(\cos\left(\frac{\pi\alpha}{2}\right) \frac{z^{1+\alpha}}{1+\alpha} - zx\right) dz. \quad (6.1)$$

6.2. Asymptotic expansion of Φ_α and Φ'_α . We now study the asymptotics of $\Phi_\alpha(x)$ as $x \rightarrow +\infty$. A general (theoretical) asymptotic expansion for ϕ was computed by Wright [26] (see also [7, Theorem 2.1.3]), but it seems difficult to extract from his formula an explicit expression for the coefficients. This will be our objective here.

Recall the following definition of the coefficients ($c_p^{(\alpha)}$) :

$$c_0^{(\alpha)} = 1 \quad \text{and} \quad c_p^{(\alpha)} = \frac{1}{(2p)!\sqrt{\pi}} \left(\frac{2}{\alpha}\right)^p \sum_{k=1}^{2p} B_{2p,k} \Gamma\left(p+k+\frac{1}{2}\right) (2(\alpha-1))^k, \quad p \geq 1$$

where the sequence ($B_{n,k}$) is defined for any $n \geq 1$ by

$$B_{n,1} = \frac{(2-\alpha)_{n-1}}{(n+1)(n+2)} \quad \text{and} \quad B_{n,k+1} = \frac{1}{k+1} \sum_{l=k}^{n-1} \binom{n}{l} B_{n-l,1} \times B_{l,k} \quad k \geq 1.$$

We start with a simple lemma.

Lemma 9. *For any $k \geq 1$, there is the upper bound :*

$$\forall n \geq 1, \quad \frac{B_{n,k}}{n!} \leq \frac{(B_\infty)^k}{k!} \leq \frac{(1/4)^k}{k!} \quad \text{where} \quad B_\infty = \sum_{n=1}^{+\infty} \frac{B_{n,1}}{n!}. \quad (6.2)$$

Proof. We prove the first inequality by iteration on k . For $k = 1$, this is obviously true since $\frac{B_{n,1}}{n!} \leq B_\infty$. Assume now that (6.2) holds for all the integers up to some k . Then, from the definition (1.2) of the sequence ($B_{n,k}$), we have

$$\frac{B_{n,k+1}}{n!} = \frac{1}{k+1} \sum_{l=k}^{n-1} \frac{B_{n-l,1}}{(n-l)!} \frac{B_{l,k}}{l!} \leq \frac{(B_\infty)^k}{(k+1)!} \sum_{l=k}^{n-1} \frac{B_{n-l,1}}{(n-l)!} \leq \frac{(B_\infty)^{k+1}}{(k+1)!}$$

which proves the first inequality. The second one follows from the fact that $(2-\alpha)_{n-1} \leq (n-1)!$ for all $n \geq 1$, hence, going back to the definition of the sequence ($B_{n,1}$),

$$B_\infty \leq \sum_{n=1}^{+\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}.$$

□

We may now compute the asymptotic expansion of Φ_α and Φ'_α .

Proposition 10. *We have the asymptotic expansions as $x \rightarrow +\infty$:*

$$\Phi_\alpha(x) \underset{x \rightarrow +\infty}{\sim} \frac{1}{\sqrt{2\pi\alpha}} x^{\frac{1-\alpha}{2\alpha}} e^{-\frac{\alpha}{\alpha+1} x^{1+1/\alpha}} \sum_{p=0}^{+\infty} (-1)^p c_p^{(\alpha)} x^{-p(1+1/\alpha)}$$

and

$$\Phi'_\alpha(x) \underset{x \rightarrow +\infty}{\sim} \frac{1}{\sqrt{2\pi\alpha}} x^{\frac{3-\alpha}{2\alpha}} e^{-\frac{\alpha}{\alpha+1}x^{1+1/\alpha}} \sum_{p=0}^{+\infty} (-1)^{p+1} d_p^{(\alpha)} x^{-p(1+1/\alpha)}$$

where $d_0^{(\alpha)} = 1$ and for $p \geq 1$:

$$d_p^{(\alpha)} = c_p^{(\alpha)} - c_{p-1}^{(\alpha)} \left(\frac{(2p-1)(\alpha+1) - 2}{2\alpha} \right).$$

Proof. Since these asymptotics are already known for the Airy function, we shall assume in the following that $\alpha < 2$. We start with the asymptotic expansion of Φ_α . Coming back to the formula (6.1), we may write, using the change of variable $z = x^{\frac{1}{\alpha}}y$:

$$\Phi_\alpha(x) = \frac{x^{\frac{1}{\alpha}}}{2\pi} \int_{-\infty}^{+\infty} e^{ix^{1+\frac{1}{\alpha}} \left(e^{i\frac{\pi\alpha}{2}} \frac{y^{1+\alpha}}{1+\alpha} - y \right)} dy.$$

Applying Cauchy's integral theorem, we first deform the path of integration to pass through the line $\{z \in \mathbb{C}, \Im(z) = -1\}$:

$$\Phi_\alpha(x) = \frac{x^{\frac{1}{\alpha}}}{2\pi} \int_{-\infty}^{+\infty} e^{ix^{1+\frac{1}{\alpha}} \left(i^\alpha \frac{(u-i)^{1+\alpha}}{1+\alpha} - (u-i) \right)} du.$$

Recall next the following Taylor expansion :

$$\frac{(u-i)^{1+\alpha}}{1+\alpha} = \frac{(-i)^{1+\alpha}}{1+\alpha} + u(-i)^\alpha + \frac{u^2}{2}\alpha(-i)^{\alpha-1} + \alpha(\alpha-1)\frac{u^3}{2} \int_0^1 (1-t)^2 (tu-i)^{\alpha-2} dt$$

as well as Euler's integral formula for the hypergeometric function ${}_2F_1$:

$$\int_0^1 (1-t)^2 (tu-i)^{\alpha-2} dt = \frac{1}{3} (-i)^{\alpha-2} {}_2F_1 \left[\begin{matrix} 2-\alpha & 1 \\ 4 \end{matrix}; -iu \right].$$

Setting $\xi = x^{1+1/\alpha}$ and making the change of variable $u\sqrt{\xi} = z$, we further obtain

$$\Phi_\alpha(\xi^{\frac{\alpha}{\alpha+1}}) = \xi^{\frac{1}{1+\alpha} - \frac{1}{2}} e^{-\frac{\alpha}{\alpha+1}\xi} \int_{-\infty}^{+\infty} e^{-\alpha \frac{z^2}{2} + z^2 \alpha(\alpha-1) \varphi\left(\frac{z}{\sqrt{\xi}}\right)} dz \quad (6.3)$$

where

$$\varphi\left(\frac{z}{\sqrt{\xi}}\right) = -\frac{i}{6} \frac{z}{\sqrt{\xi}} {}_2F_1 \left[\begin{matrix} 2-\alpha & 1 \\ 4 \end{matrix}; -i \frac{z}{\sqrt{\xi}} \right].$$

For $|z| < \sqrt{\xi}$, the definition of ${}_2F_1$ as a series yields the alternative expression :

$$\varphi\left(\frac{z}{\sqrt{\xi}}\right) = \sum_{n=1}^{+\infty} \frac{(2-\alpha)_{n-1}}{(2+n)!} \left(-i \frac{z}{\sqrt{\xi}}\right)^n.$$

By definition of the sequence $(B_{n,k}, 1 \leq n, 1 \leq k \leq n)$, we have, still for $|z| < \sqrt{\xi}$:

$$e^{z^2 \alpha(\alpha-1) \varphi\left(\frac{z}{\sqrt{\xi}}\right)} = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \left(\frac{-iz}{\sqrt{\xi}}\right)^n \sum_{k=1}^n (z^2 \alpha(\alpha-1))^k B_{n,k}$$

which may be read as an asymptotic expansion in ξ . It remains now to plug this expansion in (6.3) and integrate term by term to obtain the announced result. However, some care

is needed as the convergence of the expansion is not uniform in z , hence we cannot apply directly [2, Theorem 1.7.5] for instance. Let $N \in \mathbb{N}$. Using the integral definition of the Gamma function and only keeping even terms, we have

$$\begin{aligned} & \left| \xi^{\frac{1}{2} - \frac{1}{1+\alpha}} e^{\frac{\alpha}{\alpha+1}\xi} \Phi_\alpha(\xi^{\frac{\alpha}{\alpha+1}}) - \frac{1}{\sqrt{2\pi\alpha}} \sum_{p=0}^{N-1} (-1)^p c_p^{(\alpha)} \xi^{-p} \right| \\ & \leq \int_{-\infty}^{+\infty} e^{-\alpha \frac{z^2}{2}} \left| e^{z^2 \alpha (\alpha-1) \varphi\left(\frac{z}{\sqrt{\xi}}\right)} - \frac{1}{\sqrt{2\pi\alpha}} - \sum_{p=1}^{N-1} \frac{(-1)^p}{(2p)!} \left(\frac{z}{\sqrt{\xi}}\right)^{2p} \sum_{k=1}^{2p} (z^2 \alpha (\alpha-1))^k B_{2p,k} \right| dz \end{aligned}$$

and we need to prove that this last quantity is smaller than $K \xi^{-N}$ for some constant K independent from ξ . We decompose this last integral according as $|z| < r\sqrt{\xi}$ or $|z| \geq r\sqrt{\xi}$ where $0 < r < 1$ is fixed. On the one hand, when $|z| < r\sqrt{\xi}$, we deduce from Lemma 9 :

$$\begin{aligned} & \int_{-r\sqrt{\xi}}^{r\sqrt{\xi}} e^{-\alpha \frac{z^2}{2}} \left| \sum_{p=N}^{+\infty} \frac{(-1)^p}{(2p)!} \left(\frac{z}{\sqrt{\xi}}\right)^{2p} \sum_{k=1}^{2p} (z^2 \alpha (\alpha-1))^k B_{2p,k} \right| dz \\ & \leq \int_{-r\sqrt{\xi}}^{r\sqrt{\xi}} e^{-\alpha \frac{z^2}{2}} \sum_{p=0}^{+\infty} \frac{1}{(2p+2N)!} \left(\frac{z}{\sqrt{\xi}}\right)^{2p+2N} \sum_{k=1}^{2p+2N} (z^2 \alpha (\alpha-1))^k B_{2p+2N,k} dz \\ & \leq \frac{1}{\xi^N} \int_{-r\sqrt{\xi}}^{r\sqrt{\xi}} e^{-\alpha \frac{z^2}{2}} \sum_{p=0}^{+\infty} z^{2N} r^{2p} \sum_{k=1}^{2p+2N} (z^2 \alpha (\alpha-1))^k \frac{(1/4)^k}{k!} dz \\ & \leq \frac{1}{\xi^N} \sum_{p=0}^{+\infty} r^{2p} \int_{-r\sqrt{\xi}}^{r\sqrt{\xi}} z^{2N} e^{-\alpha \frac{z^2}{2}} e^{\frac{z^2 \alpha (\alpha-1)}{4}} dz \\ & \leq \frac{1}{\xi^N} \frac{1}{1-r^2} \int_{-\infty}^{+\infty} z^{2N} e^{-\frac{z^2}{4} \alpha (3-\alpha)} dz \end{aligned}$$

which is finite since $\alpha \in (1, 2)$. On the other hand, when $|z| \geq r\sqrt{\xi}$, we simply use the triangular inequality and first write :

$$\int_{|z| \geq r\sqrt{\xi}} e^{-\alpha \frac{z^2}{2}} \left| e^{z^2 \alpha (\alpha-1) \varphi\left(\frac{z}{\sqrt{\xi}}\right)} \right| dz \leq \frac{1}{r^{2N} \xi^N} \int_{-\infty}^{+\infty} z^{2N} e^{-\alpha \frac{z^2}{2}} \left| e^{z^2 \alpha (\alpha-1) \varphi\left(\frac{z}{\sqrt{\xi}}\right)} \right| dz.$$

To check that this last integral may be bounded by a constant independent of ξ , we shall simply prove that $\Re(\varphi(u)) \leq 0$ for any $u \in \mathbb{R}$. Indeed, observe that

$$\varphi(u) = -\frac{i u}{2} \int_0^1 (1-t)^2 (t u + i)^{\alpha-2} dt = -\frac{i u}{2} \int_0^1 \frac{(1-t)^2}{(t^2 u^2 + 1)^{\frac{2-\alpha}{2}}} e^{i\theta(\alpha-2)} dt$$

where $\theta = \theta(t, u) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is defined by

$$\cos(\theta) = \frac{1}{\sqrt{1+t^2 u^2}} \quad \text{and} \quad \sin(\theta) = \frac{t u}{\sqrt{1+t^2 u^2}}.$$

Since $\alpha - 2 < 0$, we deduce that the sign of $\sin(\theta(\alpha - 2))$ is the opposite of the sign of u , and thus

$$\Re(\varphi(u)) = \frac{u}{2} \int_0^1 \frac{(1-t)^2}{(t^2 u^2 + 1)^{\frac{2-\alpha}{2}}} \sin(\theta(\alpha - 2)) dt \leq 0.$$

Finally, the remaining term being polynomial, we have using that $\xi > 1$,

$$\begin{aligned} & \int_{|z| \geq r\sqrt{\xi}} e^{-\alpha \frac{z^2}{2}} \left| \frac{1}{\sqrt{2\pi\alpha}} + \sum_{p=1}^{N-1} \frac{(-1)^p}{(2p)!} \left(\frac{z}{\sqrt{\xi}} \right)^{2p} \sum_{k=1}^{2p} (z^2 \alpha (\alpha - 1))^k B_{2p,k} \right| dz \\ & \leq \frac{1}{r^{2N} \xi^N} \int_{-\infty}^{+\infty} z^{2N} e^{-\alpha \frac{z^2}{2}} \left(\frac{1}{\sqrt{2\pi\alpha}} + \sum_{p=1}^{N-1} \frac{1}{(2p)!} \left(\frac{z}{\sqrt{\xi}} \right)^{2p} \sum_{k=1}^{2p} (z^2 \alpha (\alpha - 1))^k B_{2p,k} \right) dz \\ & \leq \frac{1}{r^{2N} \xi^N} \int_{-\infty}^{+\infty} z^{2N} e^{-\alpha \frac{z^2}{2}} \left(\frac{1}{\sqrt{2\pi\alpha}} + \sum_{p=1}^{N-1} \frac{1}{(2p)!} z^{2p} \sum_{k=1}^{2p} (z^2 \alpha (\alpha - 1))^k B_{2p,k} \right) dz \end{aligned}$$

which is also finite and thus ends the proof for Φ_α .

We now study the asymptotic expansion of Φ'_α . Since $\alpha < 2$, we may differentiate under the integral in (6.1) to obtain :

$$\begin{aligned} \Phi'_\alpha(x) &= \int_0^{+\infty} e^{-\sin(\frac{\pi\alpha}{2}) \frac{z^{1+\alpha}}{1+\alpha}} z \sin \left(\cos \left(\frac{\pi\alpha}{2} \right) \frac{z^{1+\alpha}}{1+\alpha} - zx \right) dz \\ &= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} z e^{i e^{i \frac{\pi\alpha}{2}} \frac{z^{1+\alpha}}{1+\alpha} - izx} dz = \frac{1}{2\pi i} x^{\frac{2}{\alpha}} \int_{-\infty}^{+\infty} z e^{ix^{1+\frac{1}{\alpha}} \left(e^{i \frac{\pi\alpha}{2}} \frac{z^{1+\alpha}}{1+\alpha} - z \right)} dz. \end{aligned}$$

Using similar computations to those for Φ_α , we deduce that

$$\Phi'_\alpha(x) \underset{x \rightarrow +\infty}{\sim} \frac{1}{\sqrt{2\pi\alpha}} x^{\frac{3-\alpha}{2\alpha}} e^{-\frac{\alpha}{\alpha+1} x^{1+1/\alpha}} \sum_{p=0}^{+\infty} (-1)^{p+1} d_p^{(\alpha)} x^{-p(1+1/\alpha)}$$

for some coefficients $(d_p^{(\alpha)})$. The formula for these coefficients follows then by integrating the asymptotic expansion of $\left(x^{\frac{\alpha-1}{2\alpha}} e^{\frac{\alpha}{\alpha+1} x^{1+1/\alpha}} \Phi_\alpha(x) \right)'$ and identifying the successive powers. \square

6.3. Asymptotics of the sequence $(c_n^{(\alpha)}, n \geq 0)$.

Corollary 11. *There exists two constants $0 < \kappa_1 < \kappa_2 < +\infty$ such that as $n \rightarrow +\infty$:*

$$\kappa_1 n \leq (c_n^{(\alpha)})^{\frac{1}{n}} \leq \kappa_2 n.$$

Proof. To get the lower bound, we simply observe that $B_{2n,2n} = \left(\frac{1}{6}\right)^{2n}$ and keep only the last term in the sum defining $c_n^{(\alpha)}$:

$$c_n^{(\alpha)} \geq \frac{1}{(2n)! \sqrt{\pi}} \left(\frac{2}{\alpha} \right)^n \left(\frac{1}{6} \right)^n \Gamma \left(3n + \frac{1}{2} \right) (2(\alpha - 1))^{2n}.$$

The result then follows from Stirling's asymptotics. To get the upper bound, we first use Lemma 9 to write

$$c_n^{(\alpha)} \leq \frac{1}{\sqrt{\pi}} \left(\frac{2}{\alpha}\right)^n \sum_{k=1}^{2n} \frac{1}{k!} \Gamma\left(n+k+\frac{1}{2}\right) \left(\frac{\alpha-1}{2}\right)^k.$$

We now decompose this sum according as $k \leq n$ or $k \geq n+1$. For $k \leq n$, we have :

$$\frac{\Gamma\left(n+k+\frac{1}{2}\right)}{k!} = \left(k+\frac{1}{2}\right) \left(k+\frac{3}{2}\right) \dots \left(k+\frac{1}{2}+n-1\right) \leq (2n)^n$$

hence

$$\sum_{k=1}^n \left(\frac{\alpha-1}{2}\right)^k \frac{\Gamma\left(n+k+\frac{1}{2}\right)}{k!} \leq \frac{\alpha-1}{3-\alpha} (2n)^n. \quad (6.4)$$

For $k \geq n+1$, we deduce from Stirling's asymptotics that there exists a constant $\kappa > 0$ such that

$$\begin{aligned} \sum_{k=n+1}^{2n} \left(\frac{\alpha-1}{2}\right)^k \frac{\Gamma\left(n+k+\frac{1}{2}\right)}{k!} &\leq \kappa e^{-n} \sum_{k=n+1}^{2n} \left(\frac{\alpha-1}{2}\right)^k \frac{(n+k+\frac{1}{2})^{n+k}}{k^{k+\frac{1}{2}}} \\ &\leq \kappa e^{-n} \left(3n+\frac{1}{2}\right)^{n-\frac{1}{2}} \sum_{k=n+1}^{2n} \left(\frac{\alpha-1}{2}\right)^k \left(1+\frac{n+\frac{1}{2}}{k}\right)^{k+\frac{1}{2}} \\ &\leq \kappa e^{-n} \left(3n+\frac{1}{2}\right)^{n-\frac{1}{2}} \sum_{k=n+1}^{2n} \left(\frac{\alpha-1}{2}\right)^k 2^{k+\frac{1}{2}} \\ &\leq \kappa e^{-n} \left(3n+\frac{1}{2}\right)^{n-\frac{1}{2}} \sqrt{2n} \end{aligned} \quad (6.5)$$

The result now follows by summing (6.4) and (6.5). \square

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