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Non autonomous maximal regularity for the fractional evolution equations

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Abstract

We consider the problem of maximal regularity for the semilinear non-autonomous fractional equations

$$B^\alpha u(t) + A(t)u(t) = F(t,u),\ t\text{-a.e.}$$

Here, B^α denotes the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ w.r.t. time and the time dependent operators $A(t)$ are associated with (time dependent) sesquilinear forms on a Hilbert space $\mathcal{H}$. We prove maximal L^p-regularity results and other regularity properties for the solutions of the above equation under minimal regularity assumptions on the forms and the inhomogeneous term F.
keywords: Fractional equations, maximal regularity, non-autonomous evolution equations.

1 Introduction

The present paper deals with maximal L^p-regularity for non-autonomous evolution fractional equations in the setting of Hilbert spaces. Before explaining our results we introduce some notations and assumptions.
Let $(\mathcal{H}, (\cdot,\cdot), \|\cdot\|)$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. We consider another Hilbert space $\mathcal{V}$ which is densely and continuously embedded into $\mathcal{H}$. We denote by $\mathcal{V}'$ the (anti-) dual space of $\mathcal{V}$ so that

$$\mathcal{V} \hookrightarrow_d \mathcal{H} \hookrightarrow_d \mathcal{V}'.$$

We denote by $\langle , \rangle$ the duality $\mathcal{V}$-$\mathcal{V}'$ and note that $\langle \psi, v\rangle = (\psi, v)$ if $\psi, v \in \mathcal{H}$. Noting here that $\mathcal{V}, \mathcal{H}, \mathcal{V}'$ are all UMD-Banach spaces (we refer the reader to [20] for the definition and more details about this space type).
Given $\tau \in (0,\infty)$ and consider a family of sesquilinear forms

$$\mathfrak{a} : [0,\tau] \times \mathcal{V} \times \mathcal{V} \to \mathbb{C}$$

such that

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- [H1]: $D(\mathfrak{a}(t)) = \mathcal{V}$ (constant form domain),
- [H2]: $|\mathfrak{a}(t,u,v)| \leq M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ (uniform boundedness),
- [H3]: $\Re\mathfrak{a}(t,u,u) + \nu\|u\|^2 \geq \delta\|u\|^2_{\mathcal{V}}$ $(\forall u \in \mathcal{V})$ for some $\delta > 0$ and some $\nu \in \mathbb{R}$ (uniform quasi-coercivity).

Here and throughout this paper, $\|\cdot\|_{\mathcal{V}}$ denotes the norm of $\mathcal{V}$.

To each form $\mathfrak{a}(t)$ we can associate two operators $A(t)$ and $\mathcal{A}(t)$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. Recall that $u \in \mathcal{H}$ is in the domain $D(A(t))$ if there exists $h \in \mathcal{H}$ such that for all $v \in \mathcal{V}$: $\mathfrak{a}(t,u,v) = (h,v)$. We then set $A(t)u := h$. The operator $\mathcal{A}(t)$ is a bounded operator from $\mathcal{V}$ into $\mathcal{V}'$ such that $\mathcal{A}(t)u = \mathfrak{a}(t,u,\cdot)$. The operator $A(t)$ is the part of $\mathcal{A}(t)$ on $\mathcal{H}$. It is a classical fact that $-A(t)$ and $-\mathcal{A}(t)$ are sectorial operators and both generators of holomorphic semigroups $(e^{-rA(t)})_{r\geq 0}$ and $(e^{-r\mathcal{A}(t)})_{r\geq 0}$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. The semigroup $e^{-rA(t)}$ is the restriction of $e^{-r\mathcal{A}(t)}$ to $\mathcal{H}$. In addition, $e^{-rA(t)}$ induces a holomorphic semigroup on $\mathcal{V}$ (see, e.g., Ouhabaz [30, Chapter 1]).
We define $k_\alpha \in L^1_{loc}(\mathbb{R}^+), \alpha \in (0,1)$ by $k_\alpha(t) = \frac{1}{\Gamma(1-\alpha)}t^{-\alpha}$, where Γ is the gamma function. It is easy to see that $l_\alpha(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$ satisfies $k_\alpha * l_\alpha = 1$ in $\mathbb{R}^+$, $k_\alpha * l_\alpha$ stands the convolution on the halfline, i.e. $k_\alpha * l_\alpha(t) = \int_0^t k_\alpha(t-s)l_\alpha(t)(s)\,ds$.
Given a function f defined on $[0,\tau]$ with values either in $\mathcal{H}$ or in $\mathcal{V}'$. In this paper we study the abstract problem

$$\frac{d}{dt}[k_\alpha * (u - u_0)](t) + A(t)u(t) = f(t),\ t\text{-a.e.} \tag{1.1}$$

Here, $\frac{d}{dt}[k_\alpha * (u-u_0)]$ is Riemann-Liouville fractional operator of order α and u_0 plays the role of the initial value for u. This is an abstract non-autonomous fractional equation and our aim is to prove well-posedness and maximal L^p-regularity for $p \in (1,\infty)$ in $\mathcal{V}'$ and in $\mathcal{H}$.
For the autonomous case (i.e. $A(t) = A(0)$ for all $t \in [0,\tau]$), the problem (1.1) was the subject of treatment of many authors, see for instance [14], [43], [44], [6] and [38].

Definition 1.1. *We say that the problem (1.1) has maximal L^p-regularity in $\mathcal{H}$(resp. $\mathcal{V}'$) if for every $f \in L^p(0,\tau,\mathcal{H})$(resp. $L^p(0,\tau,\mathcal{V}')$), there exists a unique $u \in L^p(0,\tau,\mathcal{V})$ with $u(t) \in D(A(t))$(resp. $\mathcal{V}$) for a.e. $t \in [0,\tau]$ such that $\mathcal{A}(.)u \in L^p(0,\tau,\mathcal{H})$(resp. $L^p(0,\tau,\mathcal{V}')$) and u is a solution of* (1.1) *in the L^p-sense.*

Note that if (1.1) has maximal L^p-regularity in $\mathcal{H}$ then the terms $\frac{d}{dt}[k_\alpha*(u-u_0)](.), A(.)u(.)$ and f lie in the space $L^p(0,\tau,\mathcal{H})$, which is the reason for the terminology "maximal regularity".
A well known result [41][Theorem 3.1] proved maximal L^2-regularity in the space $\mathcal{V}'$. That is for every $f \in L^2(0,\tau;\mathcal{V}')$ and $u_0 \in \mathcal{H}$ there exists a unique $u \in H^\alpha(0,\tau;\mathcal{V}') \cap L^2(0,\tau;\mathcal{V})$ which solves the equation (1.1). In this result only measurability of $t \to \mathfrak{a}(t,.,.)$ with respect to the time variable is required to have a solution $u \in L^2(0,\tau;\mathcal{V})$ and the proof is based on the form method. However, considering boundary valued problems one is

interested in strong solutions, i.e., $A(.)u \in L^2(0,\tau;\mathcal{H})$ and not only in $L^2(0,\tau;\mathcal{V}')$ (note that $\mathcal{H} \hookrightarrow \mathcal{V}'$ by the canonical identification). So the central problem can be formulated as follows.

Problem 1.2. *Under which conditions on the forms* $\mathfrak{a}(.)$ *the equation* (1.1) *has maximal* L^p*-regularity in* $\mathcal{H}$.

Noting that in the parabolic case (i.e. for the equation $u'(t) + A(t)u(t) = f(t)$) this problem is called Lions' problem on maximal regularity, we refer to [2] for an account on recent development on this topic.
Our result in $\mathcal{V}'$ (see Theorem 3.14) is to extend the results in [41] in two directions. The first direction is to deal with maximal L^p-regularity, for all $p \in (1,\infty)$. The second direction is to assume less regularity on initial data u_0. In this paper we give an answer to the Problem 1.2. In order to achieve this we shall use in a crucial way the results of Monniaux and Prüss [34] for dore-venni type theorem for noncummuting operators. We remark that $L_p(L_q)$-maximal regularity for non autonomous time fractional diffusion equations in $\mathbb{R}^d$ have been proved recently in [9], [11] by PDE methods. For L_p-estimates for fractional equations in divergence form with measurable coefficients we refere the reader to [10]. We refer also to [7] where the authors applied the approach of [41] to establish the existence of strong solutions for non autonomous time fractional diffusion equations.
Our main results can be summarized as follows (see Theorems 3.10, 3.16 and 4.2 for more general and precise statements). Suppose that for some $\beta \in [0,1], K > 0, \epsilon > 0$

$$|\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \leq K|t-s|^{\frac{\alpha\beta}{2}+\epsilon}\|u\|_{\mathcal{V}}\|v\|_{[\mathcal{H},\mathcal{V}]_\beta}, t,s \in [0,\tau], u,v \in \mathcal{V}.$$

Then the equation (1.1) has maximal L^p-regularity in $\mathcal{H}$. Moreover for $p = 2, \gamma \in [0,1]$ we have

$$\int_0^\tau \frac{\|u(t)\|^2_{[\mathcal{H},\mathcal{V}]_\gamma}}{t^{(2-\gamma)\alpha}}\, dt < \infty$$

and $u(t) \in [\mathcal{H},\mathcal{V}]_{2\alpha-1}$ for all $t \in [0,\tau]$. Assume in addition that $t \to F(t,x), x \in \mathcal{H}$ satisfies $F(.,0) \in L^p(0,\tau;\mathcal{H})$ and the following continuity property: for any $\epsilon > 0$ there exists a constant $N_{\epsilon,p} > 0$ such that

$$\|F(.,u) - F(.,v)\|^p_{L^p(0,\tau;\mathcal{H})} \leq \epsilon\|u-v\|^p_{MR(\alpha,p,\tau)} + N_{\epsilon,p}\|u-v\|^p_{L^p(0,\tau;\mathcal{H})}, \tag{1.2}$$

for any $u,v \in MR(\alpha,p,\tau)$. Here, $\|u\|_{MR(\alpha,p,\tau)} = \|\frac{d}{dt}[k_\alpha*(u-u_0)](.)\|_{L^p(0,\tau;\mathcal{H})} + \|A(.)u\|_{L^p(0,\tau;\mathcal{H})}$. Then there exists a unique $u \in MR(\alpha,p,\tau)$ be the solution to the semilinear equation

$$\frac{d}{dt}[k_\alpha * (u-u_0)](.) + A(.)u = F(.,u).$$

This work is structured as follows. In the second section we work towards an time fractional operator, we prove some results and preparatory lemma. Section (3) we prove maximal L^p-regularity and some other results for the solution to the Problem (1.1). We

prove our results for the semilinear equation in section (4). In section (6), we show that the embedding $MR(\alpha,p,\tau)$ into $L^p(0,\tau;\mathcal{V})\cap C([0,\tau];\mathcal{H})$ is compact whenever $\mathcal{V}$ is compactly embedded in $\mathcal{H}$. This is important for our application to quasilinear problems given in Section.
We illustrate our abstract results by four applications. One of them concerns the heat equation with non-autonomous Robin-boundary-conditions

$$\partial_\mu u(t)+\gamma(t,.)u=0.$$

on a bounded Lipschitz domain Ω. Here ∂_μ denotes the normal derivative. Under appropriate assumptions on γ we prove maximal regularity, i.e., that the solution is in $MR(\alpha,p,\tau)$. This is of great importance if non-linear problems are considered.
Notation. We denote by $\mathcal{L}(E,F)$ (or $\mathcal{L}(E)$) the space of bounded linear operators from E to F (from E to E). The spaces $L^p(a,b;E)$ and $W^{1,p}(a,b;E)$ denote respectively the Lebesgue and Sobolev spaces of function on (a,b) with values in E. Recall that the norms of $\mathcal{H}$ and $\mathcal{V}$ are denoted by $\|\cdot\|$ and $\|\cdot\|_{\mathcal{V}}$. The scalar product of $\mathcal{H}$ is $(\cdot,\cdot)$.
We denote by C, C' or c... all inessential constants. Their values may change from line to line.
Finally, by $(E,F)_{\theta,p},[E,F]_\theta,\theta\in(0,1),p\in(1,\infty)$ we denote the real and complex interpolation spaces respectively between E and F.

2 Time fractional operator

In this section we prove several properties and estimates for the time fractional operator which will play an important role in the proof of our main results.
We define the differentiation operator on $X:=L^2(0,\tau;\mathcal{H})$ by

$$D(B)=H_0^1(0,\tau;\mathcal{H}):=\{u\in H^1(0,\tau;\mathcal{H});u(0)=0\},B:D(B)\to X,Bu=u',\|u\|_{D(B)}=\|u'\|_X$$

and consider B as a closed operator in X.

Theorem 2.1 ([35](Theorem 3.1))**.** *We have*

- $\mathbb{R}^-\cup\{0\}\subset\rho(B)$ *and for all* $\lambda\geq 0,\|(\lambda+B)^{-1}\|_{\mathcal{L}(X)}\leq\frac{C_0}{1+|\lambda|}$.
- $\forall s\in\mathbb{R},B^{is}\in\mathcal{L}(X)$, *moreover* $s\to B^{is}$ *is a strongly continuous group in* $\mathcal{L}(X)$ *with* $\|B^{is}\|_{\mathcal{L}(X)}\leq C_1(1+s^2)e^{\frac{\pi}{2}|s|}$.

We note that B is a positive operator, then by Balakrishnan formula we have for $\alpha\in(0,1)$ and $f\in D(B^\alpha)$

$$B^\alpha f:=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}B\int_0^\infty\xi^{\alpha-1}(\xi+B)^{-1}fd\xi,$$

where $(\xi+B)^{-1}f(t)=\int_0^t e^{-(t-s)\xi}f(s)\,ds$, $t\in[0,\tau],\xi\in[0,\infty)$.

Proposition 2.2. *We have*

- $D(B^\alpha) = [X, D(B)]_\alpha$ *for all* $\alpha \in (0,1)$ *and for* $t \in (0,\tau)$

$$(B^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{1}{(t-s)^\alpha} f(s)\, ds, f \in D(B^\alpha)$$

  is the Riemann-Liouville fractional derivative operator.

- B^α *is a sectorial operator of angle* $\theta_b = \alpha\frac{\pi}{2}$ *and has a bounded imaginary power with*

$$\|(B^\alpha)^{is}\|_{\mathcal{L}(X)} \leq C_1(1+\alpha^2 s^2)e^{\frac{\alpha\pi}{2}|s|},\ s \in \mathbb{R}.$$

- *For* $\lambda \in \Sigma_{\pi-\theta_b}, f \in L^2(0,\tau;\mathcal{H})$

$$(\lambda + B^\alpha)^{-1} f(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s)\, ds,$$

  here $E_{\alpha,\alpha}$ *is the Mittag-Leffler function defined by*

$$E_{\alpha,\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \alpha)}, z \in \mathbb{C}.$$

- *For all* $u \in D(B^\alpha)$

$$\|u\|_{L^2(0,\tau;\mathcal{H},\frac{dt}{t^{2\alpha}})} \leq 2^\alpha \|u\|_{D(B^\alpha)},$$

  where $\|u\|^2_{L^2(0,\tau;\mathcal{H},\frac{dt}{t^{2\alpha}})} = \int_0^\tau \|u(t)\|^2 \frac{dt}{t^{2\alpha}}$.

Remark 2.3. • *Let* $0 \leq s \leq \tau$. *If we define the differentiation operator on* $X_s := L^2(s,\tau;\mathcal{H})$ *by*

$$D(B_s) = \{u \in H^1(s,\tau;\mathcal{H}) : u(s) = 0\}, B_s : D(B_s) \to X, B_s u = u'.$$

  Then B_s^α *is defined in* X_s *by*

$$(B_s^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_s^t \frac{1}{(t-r)^\alpha} f(r)\, dr.$$

  Note that $B_s^\alpha f = B^\alpha(1_{(s,\tau)} f)$ *but* $B_s^\alpha f \neq 1_{(s,\tau)} B^\alpha f$ *in general. Indeed, let* $f = H(.)$ *be the Heaviside function (*$H(t) = 1$ *if* $t \geq 0$ *and* 0 *otherwise). We have* $(B_s^\alpha f)(t) = B^\alpha(1_{(s,\tau)}H)(t) = B^\alpha H(.-s)(t) = \frac{1}{\Gamma(1-\alpha)}(t-s)^{-\alpha}H(t-s)$. *But* $(B^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$. *Hence, if* $s > 0$ *we obtain* $B_s^\alpha f \neq 1_{(s,\tau)} B^\alpha f$.

- $(B^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds$, *and so*

$$\|B^{-\alpha}\|_{\mathcal{L}(X)} \leq \frac{\tau^\alpha}{\Gamma(\alpha+1)}.$$

Proof. Since B is a positive and maximal accretive operator (see [32] Example p. 252) with dense domain and has bounded imaginary power, it follows by ([29], Theorem 4.2.6) that $D(B^\alpha) = [X, D(B)]_\alpha$ for all $\alpha \in (0,1)$.
Let $t \in (0,\tau)$. Then for $f \in D(B^\alpha)$

$$\begin{aligned}(B^\alpha f)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\int_0^\infty \xi^{\alpha-1}\Big((\xi+B)^{-1}f\Big)(t)d\xi \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\int_0^\infty \xi^{\alpha-1}\int_0^t e^{-(t-s)\xi}f(s)\,ds\,d\xi \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\int_0^\infty \xi^{\alpha-1}e^{-(t-s)\xi}\,d\xi f(s)\,ds \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{1}{(t-s)^\alpha}f(s)\,ds.\end{aligned}$$

Now, since B is closed and maximal accretive operator, by [32][Theorem 24] B^α is regularly accretive with index $\leq \tan\frac{\alpha\pi}{2}$. The latter means that

$$|\Im(B^\alpha x, x)_X| \leq C\Re(B^\alpha x, x)_X, x \in D(B^\alpha), C \leq \tan\frac{\alpha\pi}{2}.$$

Thus, B^α is a sectorial operator of angle $\alpha\frac{\pi}{2}$.
Let $\lambda \in \Sigma_{\pi-\theta}$ and $g \in D(B^\alpha)$ and set $f = (\lambda + B^\alpha)g$. By the Laplace transform we have $\widehat{f}(s) = \lambda\widehat{g}(s) + \widehat{(B^\alpha g)}(s) = \lambda\widehat{g}(s) + s^\alpha\widehat{g} = (\lambda + s^\alpha)\widehat{g}(s)$, here the hat indicates the Laplace transform. Then $\widehat{g}(s) = \frac{1}{\lambda+s^\alpha}\hat{f}(s)$. Therefore $g(t) = \Big(L^{-1}(\frac{1}{\lambda+s^\alpha})1_{\mathbb{R}^+} * f\Big)(t)$, here L^{-1} denotes the inverse Laplace transform. Since $L^{-1}(\frac{1}{\lambda+s^\alpha})(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$, then the third claim follows immediately.
Define the spaces

$$F_0 := L^2(0,\tau;\mathcal{H}, s^2ds),\ \ F_1 := X.$$

Then for every $\alpha \in (0,1)$,

$$[F_0, F_1]_\alpha = L^2(0,\tau;\mathcal{H}, s^{2(1-\alpha)}ds),$$

(see [37], p. 130). For $u \in D(B)$ we define

$$T(u)(s) := \frac{u(s)}{s}.$$

Then, $T : D(B) \to F_1$ is bounded. Indeed, by the Hardy's inequality

$$\begin{aligned}\|T(u)\|_{F_1}^2 &= \int_0^\tau \frac{(\|\int_0^s u'(l)dl\|)^2}{s^2}ds \\ &\leq \int_0^t \Big(\frac{1}{s}\int_0^s \|u'(l)\|dl\Big)^2 ds \\ &\leq 4\int_0^\tau \|u'(l)\|^2 dl \\ &\leq 4\|u\|_{D(B)}^2.\end{aligned}$$

It follows immediately from the definition that $T : L^2(0,\tau;\mathcal{H}) \to F_0$ is bounded with

$$\|T(u)\|_{F_0}^2 = \|u\|_X^2.$$

Therefore, by interpolation

$$T : [X, D(B)]_\alpha \to [F_0, F_1]_\alpha$$

is a bounded operator with

$$\|u\|_{L^2(0,\tau;\mathcal{H},\frac{ds}{s^{2\alpha}})} = \|Tu\|_{L^2(0,\tau;\mathcal{H},s^{2(1-\alpha)}ds)} \leq 2^\alpha \|u\|_{D(B^\alpha)}. \tag{2.1}$$

□

Lemma 2.4. *We have*

$$D(B^\alpha) = \begin{cases} H_0^\alpha(0,\tau;\mathcal{H}) := \{u \in H^\alpha(0,\tau;\mathcal{H}); u(0) = 0\}, \alpha \in (\frac{1}{2}, 1] \\ \{u \in H^{\frac{1}{2}}(0,\tau;\mathcal{H}); \int_0^\tau \frac{\|u(t)\|^2}{t}\, dt < \infty\}, \alpha = \frac{1}{2} \\ H^\alpha(0,\tau;\mathcal{H}), \alpha \in (0, \frac{1}{2}). \end{cases} \tag{2.2}$$

In addition, $\|u\|_{D(B^\alpha)} = \|u\|_{H^\alpha(0,\tau;\mathcal{H})}$ *if* $\alpha \neq \frac{1}{2}$ *and*

$$\|u\|_{D(B^{\frac{1}{2}})} = \|u\|_{H^{\frac{1}{2}}(0,\tau;\mathcal{H})} + \Big(\int_0^\tau \frac{\|u(t)\|^2}{t}\, dt\Big)^{\frac{1}{2}}.$$

Remark 2.5. *The space* $H^\alpha(0,\tau;\mathcal{H})$ *endowed with norm*

$$\|u\|_{H^\alpha(0,\tau;\mathcal{H})}^2 := \|u\|_X^2 + \int_0^\tau \int_0^\tau \frac{\|u(t) - u(s)\|^2}{|t-s|^{2\alpha+1}}\, ds\, dt.$$

Proof. Since by Lemma 2.2 $D(B^\alpha) = [X, H_0^1(0,\tau;\mathcal{H})]_\alpha$, then the result follows immediately by [[33], p. 68]. □

For the adjoint operator $B^{*\alpha}$ we have

Proposition 2.6. *For* $f \in D(B^{*\alpha}), \alpha \in (0,1)$ *we get*

$$(B^{*\alpha} f)(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\tau \frac{1}{(s-t)^\alpha} f(s)\, ds.$$

Moreover, we obtain that for $\alpha \in (0, \frac{1}{2}), D(B^{*\alpha}) = D(B^\alpha)$ *but* $D(B^{*\alpha}) \neq D(B^\alpha)$ *for* $\alpha \in [\frac{1}{2}, 1)$.

Proof. By integration by parts we can show easily that

$$D(B^*) = H_\tau^1(0,\tau;\mathcal{H}) := \{u \in H^1(0,\tau;\mathcal{H}); u(\tau) = 0\}, B^* : D(B^*) \to X, B^* u = -u'.$$

Then by Balakrishnan formula we have

$$(B^{*\alpha} f)(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\tau \frac{1}{(s-t)^\alpha} f(s)\, ds, f \in D(B^{*\alpha}).$$

Moreover,

$$D(B^{*\alpha}) = [X, H^1_\tau(0,\tau;\mathcal{H})]_\alpha = \begin{cases} H^\alpha_\tau(0,\tau;\mathcal{H}) = \{u \in H^\alpha(0,\tau;\mathcal{H}); u(\tau) = 0\}, \alpha \in (\frac{1}{2}, 1] \\ \{u \in H^{\frac{1}{2}}(0,\tau;\mathcal{H}); \int_0^\tau \frac{\|u(t)\|^2}{\tau - t}\, dt < \infty\}, \alpha = \frac{1}{2} \\ H^\alpha(0,\tau;\mathcal{H}), \alpha \in (0, \frac{1}{2}). \end{cases} \tag{2.3}$$

This gives $D(B^{*\alpha}) = D(B^\alpha), \alpha \in (0, \frac{1}{2})$. Now, let $\alpha \in [\frac{1}{2}, 1), u \in D(B^{*\alpha})$. Since u need not satisfy the boundary condition on $D(B^\alpha)$ (even if $u \in D(B^*)$), hence $D(B^{*\alpha}) \neq D(B^\alpha)$. □

Lemma 2.7. *Let $u \in D(B^\alpha)$ with $\alpha \in (\frac{1}{2}, 1)$. Then for all $s \in (0, \tau]$ we have*

$$\int_0^s \|u(s) - u(s-l)\|^2 l^{-2\alpha}\, dl \leq C_{s,\alpha} \|u\|_{D(B^\alpha)}.$$

Proof. Let $0 < s \leq \tau$. For $l \in [0, s]$ we set $v(l) = u(s) - u(s-l)$. Then by Proposition 2.2 we get

$$\int_0^s \|v(l)\|^2 l^{-2\alpha}\, dl \leq C_\alpha \|v\|_{D(B^\alpha)} = C_\alpha \|v\|_{H^\alpha(0,s;\mathcal{H})}.$$

In light of the definition of the norm in $H^\alpha(0,\tau;\mathcal{H})$ and the fact that $H^\alpha(0,s;\mathcal{H}) \hookrightarrow L^\infty(0,s;\mathcal{H})$ (see Lemma 2.9) we find

$$\begin{aligned} \|v\|_{H^\alpha(0,s;\mathcal{H})} &\leq C_s(\|u(s)\| + \|u\|_{H^\alpha(0,\tau;\mathcal{H})}) \\ &\leq C'_{s,\alpha} \|u\|_{H^\alpha(0,\tau;\mathcal{H})}. \end{aligned}$$

This finishes the proof of the lemma. □

Lemma 2.8. *Let $0 < \alpha_2 < \alpha_1 < 1$. Then for all $u \in D(B^{\alpha_1}), \epsilon > 0$ there exists a $K(\epsilon) > 0$ such that*

$$\|u\|_{D(B^{\alpha_2})} \leq \epsilon \|u\|_{D(B^{\alpha_1})} + K(\epsilon) \|u\|_X.$$

Proof. The reiteration theorem for the real method [[37], 1.10.3, Theorem 2] or property of power of positive operator [29][Theorem 4.3.11] shows that

$$D(B^{\alpha_2}) = [\mathcal{H}, D(B)]_{\alpha_2} = [\mathcal{H}, [\mathcal{H}, B]_{\alpha_1}]_{\frac{\alpha_2}{\alpha_1}} = [\mathcal{H}, D(B^{\alpha_1})]_{\frac{\alpha_2}{\alpha_1}}. \tag{2.4}$$

Let $u \in D(B^{\alpha_1})$. Then the interpolation inequality (see [29][Corollary 2.1.8]), the Holder inequality and (2.4) gives

$$\begin{aligned} \|u\|_{D(B^{\alpha_2})} &\leq \|u\|_X^{1-\frac{\alpha_2}{\alpha_1}} \|u\|_{D(B^{\alpha_1})}^{\frac{\alpha_2}{\alpha_1}} \\ &\leq \epsilon \|u\|_{D(B^{\alpha_1})} + K(\epsilon, \alpha_1, \alpha_2) \|u\|_X. \end{aligned}$$

Where $K(\epsilon, \alpha_1, \alpha_2) = C(\alpha_1, \alpha_2) \epsilon^{\frac{\alpha_2 - \alpha_1}{\alpha_2}}$ and $C(\alpha_1, \alpha_2)$ is a positive constant depending only on α_1 and α_2. □

Let $p \in (1, \infty), v \in L^p(0,\tau;\mathcal{H})$ and set $C_\alpha v = l_\alpha * v$.

Lemma 2.9. *We have*

- $C_\alpha \in \mathcal{L}(L^p(0,\tau;\mathcal{H}), C^{\alpha-\frac{1}{p}}([0,\tau];\mathcal{H}))$ *for* $\alpha \in (\frac{1}{p},1)$ *and for* $\alpha \in (0,\frac{1}{p})$ *we get* $C_\alpha \in \mathcal{L}(L^p(0,\tau;\mathcal{H}), L^{\frac{p}{1-\alpha p}}(0,\tau;\mathcal{H}))$.

- $D(B^\alpha) \hookrightarrow C^{\alpha-\frac{1}{2}}([0,\tau];\mathcal{H})$ *for* $\alpha \in (\frac{1}{2},1)$ *and for* $\alpha \in (0,\frac{1}{2})$, $D(B^\alpha) \hookrightarrow L^{\frac{2}{1-2\alpha}}(0,\tau;\mathcal{H})$.
- *For all* $u \in L^p(0,\tau;\mathcal{H})$ *such that* $B^\alpha u \in L^p(0,\tau;\mathcal{H})$, *we have for* $t \in (0,\tau]$
$$\int_0^t \|u(s)\|^p\,ds \le K(\alpha,\tau,p)\int_0^t (t-s)^{\alpha-1}\int_0^s \|B^\alpha u(r)\|^p\,dr\,ds,$$
where $K(\alpha,\tau,p) = \frac{\tau^{\alpha\frac{p^2}{p-1}}}{\alpha^{p-1}\Gamma(\alpha)^p}$.

Proof. Let $\alpha \in (\frac{1}{p},1), u \in L^p(0,\tau;\mathcal{H}), t \in [0,\tau]$. The Hölder's inequality implies that
$$\|(C_\alpha v)(t)\| \le \|l_\alpha\|_{L^q(0,t)}\|v\|_{L^p(0,t;\mathcal{H})} \le \frac{1}{(q(\alpha-\frac{1}{p}))^{\frac{1}{q}}\Gamma(\alpha)} t^{\alpha-\frac{1}{p}}\|v\|_{L^p(0,\tau;\mathcal{H})},$$
where $q = \frac{p}{p-1}$.
Let $0 \le s \le t \le \tau$. We have
$$\begin{aligned}
\|(C_\alpha v)(t) - (C_\alpha v)(s)\| &= \|(l_\alpha * v)(t) - (l_\alpha * v)(s)\| \\
&\le \frac{1}{\Gamma(\alpha)}\|\int_s^t (t-r)^{\alpha-1}v(r)\,dr + \int_0^s \Big((t-r)^{\alpha-1} - (s-r)^{\alpha-1}\Big)v(r)\,dr\| \\
&\le \frac{1}{(q(\alpha-\frac{1}{p}))^{\frac{1}{q}}\Gamma(\alpha)}(t-s)^{\alpha-\frac{1}{p}}\|v\|_{L^p(s,t;\mathcal{H})} \\
&+ \frac{1}{\Gamma(\alpha)}\Big(\int_0^s \Big((t-r)^{\alpha-1} - (s-r)^{\alpha-1}\Big)^q dr\Big)^{\frac{1}{q}}\|v\|_{L^p(0,s;\mathcal{H})}.
\end{aligned}$$
By using the inequality
$$(a-b)^p \le a^p - b^p, a \ge b \ge 0, p \ge 1,$$
we get
$$\begin{aligned}
\|(C_\alpha v)(t) - (C_\alpha v)(s)\| &\le \frac{1}{(q(\alpha-\frac{1}{p}))^{\frac{1}{q}}\Gamma(\alpha)}(t-s)^{\alpha-\frac{1}{p}}\|v\|_{L^p(s,t;\mathcal{H})} \\
&+ \frac{1}{\Gamma(\alpha)}\Big(\int_0^s (t-r)^{q(\alpha-1)} - (s-r)^{q(\alpha-1)}\,dr\Big)^{\frac{1}{q}}\|v\|_{L^p(0,s;\mathcal{H})} \\
&= \frac{1}{(q(\alpha-\frac{1}{p}))^{\frac{1}{q}}\Gamma(\alpha)}(t-s)^{\alpha-\frac{1}{p}}\|v\|_{L^p(s,t;\mathcal{H})} \\
&+ \frac{1}{(q(\alpha-\frac{1}{p}))^{\frac{1}{q}}\Gamma(\alpha)}\Big((t-s)^{q(\alpha-1)+1} + (t^{q(\alpha-1)+1} - s^{q(\alpha-1)+1})\Big)^{\frac{1}{q}}\|v\|_{L^p(0,s;\mathcal{H})} \\
&\le \frac{2}{(q(\alpha-\frac{1}{p}))^{\frac{1}{q}}\Gamma(\alpha)}(t-s)^{\alpha-\frac{1}{p}}\|v\|_{L^p(0,\tau;\mathcal{H})}.
\end{aligned}$$

Let $v \in L^p(0,\tau;\mathcal{H})$ and $\alpha \in (0,\frac{1}{p})$. We obtain for $t \in (0,\tau)$, $\|(C_\alpha v)(t)\| \leq \big(l_\alpha * \|v(.)\|\big)$. Applying [19][Theorem 4] we get

$$\|C_\alpha v\|_{L^{\frac{p}{1-\alpha p}}(0,\tau;\mathcal{H})} \leq C(\alpha,p)\|v\|_{L^p(0,\tau;\mathcal{H})}$$

which finishes the proof of the second assertion.

Let $u \in L^p(0,\tau;\mathcal{H})$ such that $B^\alpha u \in L^p(0,\tau;\mathcal{H})$. Set $v = B^\alpha u = (k_\alpha * u)'$. Then $u = l_\alpha * v = C_\alpha v$. Therefore the second assertion follows immediately by the first assertion. For $t \in (0,\tau]$ we obtain

$$\begin{aligned}
\|u(t)\| = \|(C_\alpha v)(t)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|B^\alpha u(s)\|\,ds \\
&\leq \frac{1}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1}\Big)^{\frac{1}{q}}\Big(\int_0^t (t-s)^{\alpha-1}\|B^\alpha u(s)\|^p\,ds\Big)^{\frac{1}{p}} \\
&\leq \frac{\tau^{\frac{\alpha}{q}}}{\Gamma(\alpha)\alpha^{\frac{1}{q}}}\Big(\int_0^t (t-s)^{\alpha-1}\|B^\alpha u(s)\|^p\,ds\Big)^{\frac{1}{p}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^t \|u(s)\|^p\,ds &\leq K(\alpha,\tau,p)\int_0^t\int_0^s (s-r)^{\alpha-1}\|B^\alpha u(r)\|^p\,dr\,ds \\
&= \frac{K(\alpha,\tau,p)}{\alpha}\int_0^t (t-r)^{\alpha}\|B^\alpha u(r)\|^p\,dr \\
&= K(\alpha,\tau,p)\int_0^t\int_r^t (t-s)^{\alpha-1}\,ds\,\|B^\alpha u(r)\|^p\,dr \\
&= K(\alpha,\tau,p)\int_0^t (t-s)^{\alpha-1}\int_0^s \|B^\alpha u(r)\|^p\,dr\,ds.
\end{aligned}$$

This finishes the proof. □

3 Maximal regularity for non-autonomous fractional equation

In this section we prove our main result on maximal regularity for the linear equation. We begin by proving most of the arguments which we will need for the proofs.

We consider the non-autonomous fractional equation

$$B^\alpha(u-u_0)(t) + A(t)u(t) = f(t),\ t\text{-a.e.} \tag{3.1}$$

Note that (3.1) is equivalent to the integro-differential equation $u - u_0 + l_\alpha * (A(.)u) = l_\alpha * f$. In the sequel we assume that $\nu = 0$.

Definition 3.1. *Let $\alpha \in (0,1), p \in (1,\infty)$. We say that the fractional evolution equation* (3.1) *has the maximal L^p-regularity if for all $f \in L^p(0,\tau;\mathcal{H})$ and $u_0 \in \mathcal{H}_{\alpha,p} \subseteq \mathcal{H}$ there exists a unique solution $u(t) \in D(A(t))$, $t-$a.e such that $A(.)u \in L^p(0,\tau;\mathcal{H})$ and $B^\alpha(u - u_0) \in L^p(0,\tau;\mathcal{H})$.*

Maximal regularity may fail even for ordinary non-autonomous fractional equation, letting $\mathcal{H} = \mathbb{R}$.

Example 3.2. *Consider $\phi(t) = t^p, p \in (-\frac{1}{2}, 0)$. Then $\phi \in L^2(0,\frac{1}{2})$. Let $\alpha \in (p+1,1)$. For $t \in (0,\frac{1}{2}]$ we define $a(t) := \frac{-\Gamma(p+1)}{\Gamma(p-\alpha+1)}(t^{-\alpha} - 1) > \frac{-\Gamma(p+1)}{\Gamma(p-\alpha+1)}(\sqrt{2} - 1) > 0$. Then $a \in L^q(0,\frac{1}{2})$ for all $q \in [1,\frac{1}{\alpha})$. Consider now the ordinary non-autonomous fractional equation*

$$B^\alpha u(t) + a(t)u(t) = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}\phi(t).$$

It easy to check that $u(t) = t^p$ is the solution for this equation. But

$$B^\alpha u(t) = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha} \notin L^1(0,\frac{1}{2}).$$

Notice however, this example is not a counterexample to the questions we raise, since our standing hypothesis [H2] is not satisfied here.

Proposition 3.3. *The solution of the Problem* (3.1) *is unique.*

Proof. Assume that there are two solutions u_1, u_2 to the problem (3.1). Obviously, $v = u_1 - u_2$ satisfies in X

$$B^\alpha v + A(.)v = 0.$$

Since B^α is regularly accretive we have $\Re(A(.)v,v) \leq \Re(B^\alpha v,v)_X + \Re(A(.)v,v) = 0$. Which gives $\delta\|v\|_{L^2(0,\tau;\mathcal{V})} = 0$. Thus, $u_1 = u_2$. □

We assume the following conditions for some $\beta \in [0,1], \epsilon > 0, K > 0$

$$|\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \leq K|t-s|^{\frac{\alpha\beta}{2}+\epsilon}\|u\|_{\mathcal{V}}\|v\|_{[\mathcal{H},\mathcal{V}]_\beta}, u,v \in \mathcal{V} \text{ and } t,s \in [0,\tau]. \tag{3.2}$$

Note that the assumption (3.2) is equivalent to $\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},([\mathcal{H},\mathcal{V}]_\beta)')} \leq K|t - s|^{\frac{\alpha\beta}{2}+\epsilon}, t,s \in [0,\tau]$.

Lemma 3.4. *1- For all $0 < \beta < \frac{1}{2}, s \in [0,\tau]$ we have*

$$D(A(s)^\beta) = [\mathcal{H}, D(A(s))]_\beta = [\mathcal{H},\mathcal{V}]_{2\beta} \text{ with equivalent norms.}$$

As consequence $D(A(s)^{\alpha-\frac{1}{2}}) = [\mathcal{H},\mathcal{V}]_{2\alpha-1}$ for all $\frac{1}{2} < \alpha < 1$.

2- For all $s \in \mathbb{R}, t \in [0,\tau], \mathcal{A}(t)^{is} \in \mathcal{L}(Y), Y = \mathcal{V}', \mathcal{H}$, moreover $s \to \mathcal{A}(t)^{is}$ is a strongly continuous group in $\mathcal{L}(Y)$ with

$$\|\mathcal{A}(t)^{is}\|_{\mathcal{L}(Y)} \leq e^{\frac{\pi}{2}|s|}.$$

Proof. For the proof of the first assertion see [[32],Theorem 3.1]. For the second assertion we refer to [29][Theorem 4.3.5]. □

It is known that $-A(t)$ is sectorial operator and generates a bounded holomorphic semigroup on $\mathcal{H}$. The same is true for $-\mathcal{A}(t)$ on $\mathcal{V}'$. From [30] (Theorem 1.52 and Theorem 1.55), we have the following lemma which point out that the constants involved in the estimates are uniform with respect to t

Lemma 3.5. *For any $t \in [0,\tau]$, the operators $-A(t)$ and $-\mathcal{A}(t)$ generate strongly continuous analytic semigroups of angle $\gamma = \frac{\pi}{2} - arctan(\frac{M}{\delta})$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. In addition, there exist constant C_θ, independent of t, such that*

$$\|(z + A(t))^{-1}\|_{\mathcal{L}(Y)} \leq \frac{C_\theta}{1+|z|} \text{ for all } z \in \Sigma_{\pi-\theta} \text{ with fixed } \theta < \gamma.$$

Here, $Y = \mathcal{H}, \mathcal{V}$ or $\mathcal{V}'$.
All of the previous estimates holds for the adjoint operator $A(t)^*$.

Lemma 3.6. *We have for all $t \in [0,\tau]$, $\|(\lambda + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{H},[\mathcal{H},\mathcal{V}]_\beta)} \leq \frac{C_{\beta,\theta}}{1+|\lambda|^{1-\frac{\beta}{2}}}, \lambda \in \Sigma_{\pi-\theta}$. This estimate is also hold for the adjoint operator and we obtain $\|(\lambda+\mathcal{A}(t)^*)^{-1}\|_{\mathcal{L}(\mathcal{H},[\mathcal{H},\mathcal{V}]_\beta)} \leq \frac{C_{\beta,\theta}}{1+|\lambda|^{1-\frac{\beta}{2}}}, \lambda \in \Sigma_{\pi-\theta}$.*

Proof. Let $v \in \mathcal{H}$. We get

$$\begin{aligned}
\delta\|(\lambda + \mathcal{A}(t))^{-1}v\|_{\mathcal{V}}^2 &\leq \Re\Big(A(t)(\lambda + \mathcal{A}(t))^{-1}v, (\lambda + \mathcal{A}(t))^{-1}v\Big) \\
&\leq \|A(t)(\lambda + A(t))^{-1}v\|\|(\lambda + \mathcal{A}(t))^{-1}v\| \\
&\leq \frac{1 + C_{\beta,\theta}}{1+|\lambda|}\|v\|^2.
\end{aligned}$$

Therefore

$$\|(\lambda + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \leq \frac{C_\theta}{\sqrt{1+|\lambda|}}.$$

By interpolation we obtain

$$\begin{aligned}
\|(\lambda + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{H},[\mathcal{H},\mathcal{V}]_\beta)} &\leq C_\beta\|(\lambda + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{H})}^{1-\beta}\|(\lambda + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})}^{\beta} \\
&\leq \frac{C_{\beta,\theta}}{(1+|\lambda|)^{1-\frac{\beta}{2}}} \\
&\leq \frac{2^{\frac{\beta}{2}}C_{\beta,\theta}}{1+|\lambda|^{1-\frac{\beta}{2}}},
\end{aligned}$$

where in the last inequality we use $(1 + x^{1-\frac{\beta}{2}}) \leq 2^{\frac{\beta}{2}}(1+x)^{1-\frac{\beta}{2}}, x \geq 0$. □

Lemma 3.7. *We have for all $0 \leq s,t \leq \tau$ and $\lambda \in \Sigma_{\pi-\theta}$*

$$\|\mathcal{A}(t)(\lambda + \mathcal{A}(t))^{-1}(\mathcal{A}(t)^{-1} - \mathcal{A}(s)^{-1})\|_{\mathcal{L}(\mathcal{H})} \leq C\frac{|t-s|^{\frac{\alpha\beta}{2}+\epsilon}}{1+|\lambda|^{1-\frac{\beta}{2}}}.$$

Proof. Let $0 \leq s, t \leq \tau$ and $\lambda \in \Sigma_{\pi-\theta}$. We get by Lemma 3.6

$$\begin{aligned}
&\|\mathcal{A}(t)(\lambda+\mathcal{A}(t))^{-1}(\mathcal{A}(t)^{-1}-\mathcal{A}(s)^{-1})\|_{\mathcal{L}(\mathcal{H})} \\
&= \|(\lambda+\mathcal{A}(t))^{-1}(\mathcal{A}(t)-\mathcal{A}(s))\mathcal{A}(s)^{-1}\|_{\mathcal{L}(\mathcal{H})} \\
&= \sup_{\|u\|=\|v\|=1} |\mathfrak{a}(t,\mathcal{A}(s)^{-1}u,(\bar{\lambda}+\mathcal{A}(t)^*)^{-1}v)-\mathfrak{a}(s,\mathcal{A}(s)^{-1}u,(\bar{\lambda}+\mathcal{A}(t)^*)^{-1}v)| \\
&\leq K|t-s|^{\frac{\alpha\beta}{2}+\epsilon} \sup_{\|u\|=\|v\|=1} \|\mathcal{A}(s)^{-1}u\|_{\mathcal{V}}\|(\bar{\lambda}+\mathcal{A}(t)^*)^{-1}v\|_{[\mathcal{H},\mathcal{V}]_\beta} \\
&\leq C\frac{|t-s|^{\frac{\alpha\beta}{2}+\epsilon}}{1+|\lambda|^{1-\frac{\beta}{2}}}.
\end{aligned}$$

□

Let $r_{\lambda,\alpha}, \alpha \in (0,1)$ be the solution of the Volterra equation

$$r_{\lambda,\alpha}(t)+\lambda\int_0^t l_\alpha(t-s)r_{\lambda,\alpha}(s)\,ds = l_\alpha(t), t \geq 0, \lambda \in \Sigma_{\pi-\theta_r}, \theta_r > \frac{\alpha\pi}{2}.$$

By Laplace transform we get $\widehat{r_{\lambda,\alpha}}(z) = \frac{1}{z^\alpha+\lambda}$. Then by [[36], 2.15] we obtain

$$r_{\lambda,\alpha}(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha), t \geq 0.$$

By a simple computation we can see also that $r_{\lambda,\alpha}$ is solution of the equation $B^\alpha r_{\lambda,\alpha} + \lambda r_{\lambda,\alpha} = \delta_0$, where δ_0 is the Dirac measure at 0.
Therefore $r_{\lambda,\alpha}(t) = \Big((B^\alpha+\lambda)^{-1}\delta_0\Big)(t) = \Big(s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^\alpha) * \delta_0\Big)(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha), t \geq 0.$

Lemma 3.8. *We have* $\int_0^\tau |t^{\frac{\alpha\beta}{2}+\epsilon}r_{\lambda,\alpha}(t)|\,dt \leq \frac{C}{|\lambda|^{1+\frac{\beta}{2}+\epsilon}}$.

Proof. We get

$$\begin{aligned}
&\int_0^\tau |t^{\frac{\alpha\beta}{2}+\epsilon}r_{\lambda,\alpha}(t)|\,dt \\
&= \int_0^\tau |t^{\frac{\alpha\beta}{2}+\epsilon}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)|\,dt \\
&\leq C\int_0^\tau t^{\frac{\alpha\beta}{2}+\alpha+\epsilon-1}\frac{1}{(|\lambda|t^\alpha)^{1+\frac{\beta}{2}+\epsilon}}\,dt \\
&\leq \frac{C\tau^{\epsilon(1-\alpha)}}{|\lambda|^{1+\frac{\beta}{2}+\epsilon}},
\end{aligned}$$

where in the first inequality we have used (see [39][Theorem 1.4])

$$|E_{\alpha,\alpha}(z)| \leq \frac{C}{|z|^{1+\gamma}}, 0 \leq \gamma \leq 1, |arg(z)| > \frac{\alpha\pi}{2}.$$

□

Let $p \in (1, \infty)$. Define the space $Tr^p_\alpha = \mathcal{H}$ if $\alpha < \frac{1}{p}$ and $Tr^p_\alpha = \{0\}$ for $\alpha \geq \frac{1}{p}$ and for $u_0 \in Tr^p_\alpha$

$$MR(\alpha, p, \tau) := \{u \in D(A(t)),\, t{-}a.e : A(.)u, B^\alpha(u - u_0) \in L^p(0, \tau; \mathcal{H})\}$$

endowed with norm

$$\|u\|_{MR(\alpha,p,\tau)} = \|B^\alpha(u - u_0)\|_{L^p(0,\tau;\mathcal{H})} + \|A(.)u\|_{L^p(0,\tau;\mathcal{H})}.$$

From [[34], Theorem 2, Corollary 3] we have the following theorem for the evolutionary integral equation

Theorem 3.9. *Let $a \in L^1_{loc}(\mathbb{R}^+)$ a nontrivial scalar kernel of subexponential growth. The latter means that*

$$\int_0^\infty |a(t)|e^{-\varepsilon t}\, dt, \text{ for each } \varepsilon > 0.$$

Assume that

(a) $|arg\ \widehat{a}(\lambda)| \leq \theta_b, |\widehat{a}'(\lambda)/\widehat{a}(\lambda)| \leq k$ *for all* $\lambda \in \Sigma_{\frac{\pi}{2}}$.

Here $\theta_b \in (0, \pi)$ and $k > 0$ are constants.
Consider the following evolutionary integral equation:

$$u(t) + \int_0^t a(t - s)[L(t)u(t) - f(s)]\, ds = 0, t \in [0, \tau]. \tag{3.3}$$

Let r_λ be the solution of the Volterra equation

$$r_\lambda(t) + \lambda \int_0^t a(t - s)r_\lambda(s)\, ds = a(t), t \geq 0, \lambda \in \Sigma_{\pi-\theta_b}.$$

Assume that for each $t \in [0, \tau], L(t)$ is a sectorial operator with angles θ_A and admit bounded imaginary powers on a UMD-Banach space Y such that

$$\|(\lambda{+}L(t))^{-1}\|_{\mathcal{L}(Y)} \leq \frac{M_A}{1 + |\lambda|}; \|L(t)^{is}\|_{\mathcal{L}(Y)} \leq K_s e^{s\theta_A}; t \in [0, \tau]; \lambda \in \Sigma_{\pi-\theta_A}; s \in \mathbb{R}; M_A, K_s \geq 0. \tag{3.4}$$

Suppose in addition that $\theta_A + \theta_b < \pi$ and

$$\|L(t)(\lambda + L(t))^{-1}(L(t)^{-1} - L(s)^{-1})\|_{\mathcal{L}(Y)} \leq \frac{M_1|t - s|^\gamma}{1 + |\lambda|^{1-d}} \text{ for all } t, s \in [0, \tau], \lambda \in \Sigma_{\pi-\theta_A}, \tag{3.5}$$

there exist constants $\rho > d \geq 0$ and $M_2 > 0$ such that

$$\int_0^\tau t^\gamma |r_\lambda(t)|\, dt \leq \frac{M_2}{|\lambda|^{1+\rho}}, \lambda \in \Sigma_{\pi-\theta_b}. \tag{3.6}$$

Then for every $f \in L^p(0, \tau; Y)$, (3.3) admits a unique solution $u \in L^p(0, \tau; Y)$ such that $L(.)u \in L^p(0, \tau; Y)$.

The following theorem is our main result in this section

Theorem 3.10. *Assume that* (3.2). *Then for all* $u_0 \in Tr^p_\alpha$ *the fractional evolution equation* (3.1) *has maximal* L^p*-regularity.*

Remark 3.11. *Noting that by the closed graph theorem it follows that there is* $N > 0$ *independent of* u_0, f *such that*

$$\|u\|_{MR(\alpha,p,\tau)} \leq N(\|u_0\|_{Tr^p_\alpha} + \|f\|_{L^p(0,\tau;\mathcal{H})}).$$

Proof. First we consider the case $u_0 = 0$. We remark that (3.1) is equivalent to the evolutionary integral equation (3.3) with $L(t) = A(t), a(t) = l_\alpha(t)$. We would like to apply Theorem 3.9, so we need to verify that the assumptions of this theorem are satisfied. Let us first observe that the operator $-A(t)$ is a sectorial operator with angle $\theta < \frac{\pi}{2}$ and admit bounded imaginary powers on $\mathcal{H}$ and in $\mathcal{V}'$ and the assumptions (3.4) are verified by Lemmas 3.4 and 3.5. Let us turn our attention to the assumptions (3.5), (3.6). Indeed, the condition (3.5) is verified by Lemma 3.7 for $\gamma = \frac{\alpha\beta}{2} + \epsilon, d = \frac{\beta}{2}$ and the condition (3.6) is satisfied by Lemma 3.8 for $\rho = \frac{\beta}{2} + \epsilon$. The condition (a) in Theorem 3.9 is obvious. Therefore the fractional evolution equation (3.1) has maximal L^p-regularity.
Now let $u_0 \in Tr^p_\alpha$, $g = f + k_\alpha u_0$ and u be the solution of (3.1) for initial data 0 and the inhomogeneous term g. It is clear that $t \to k_\alpha(t)u_0 \in L^p(0,\tau;\mathcal{H})$ and u is the solution of the integro-differential equation

$$u = -l_\alpha * A(.)u + l_\alpha * g = -l_\alpha * A(.)u + l_\alpha * f + (l_\alpha * k_\alpha)u_0 = -l_\alpha * A(.)u + l_\alpha * f + u_0.$$

Then u is the unique solution to the Problem (3.1) for $u_0 \in Tr^p_\alpha$ and the source term f. This finishes the proof. □

Proposition 3.12. *For all* $f \in L^2(0,\tau;\mathcal{H})$ *there exists a unique* $v \in H^1_0(0,\tau;\mathcal{H}) \cap L^2(0,\tau;\mathcal{V})$ *be the solution to the following problem*

$$v'(t) + \frac{1}{\Gamma(\alpha)} A(t) \frac{d}{dt} \int_0^t \frac{1}{(t-s)^{1-\alpha}} v(s)\, ds = f(t),\ t - a.e. \tag{3.7}$$

Moreover, there exists $C > 0$ *such that*

$$\|v\|_{D(B)} + \|A(.)B^{1-\alpha}v\| \leq C\|f\|_{L^2(0,\tau;\mathcal{H})}.$$

Remark 3.13.

- *We note that*

$$(B^{1-\alpha}v)(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t \frac{1}{(t-s)^{1-\alpha}} v(s)\, ds$$

and since $v \in H^1_0(0,\tau;\mathcal{H})$, *then by a simple computation we find*

$$(B^{1-\alpha}v)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} v'(s)\, ds$$

which is the Caputo fractional derivative.

Proof. Let $u \in D(B^\alpha) \cap L^2(0,\tau;\mathcal{V})$ be the unique solution to the problem $B^\alpha u + A(.)u = f$ in X. Set $v = k_\alpha * u$. Then $v \in H_0^1(0,\tau;\mathcal{H}) \cap L^2(0,\tau;\mathcal{V})$ and we have $v' = B^\alpha u = -A(.)u + f$. Now since $v = k_\alpha * u$, one has $l_\alpha * v = l_\alpha * k_\alpha * u = 1_{\mathbb{R}^+} * u$. Thus $B^{1-\alpha} v = u$ and so $v' + A(.)B^{1-\alpha} v = f$ in X. □

For maximal L^p-regularity in $\mathcal{V}'$ of the fractional evolution equation (3.1) we have the following theorem

Theorem 3.14. *Assume that* $|\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \leq C|t-s|^\epsilon \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}$ *fore some* $\epsilon > 0$ *and all* $t, s \in [0,\tau], u, v \in \mathcal{V}$. *Then for all* $p \in (1,\infty)$, $f \in L^p(0,\tau;\mathcal{V}')$, $u_0 \in \mathcal{V}'$ *for* $\alpha \in (0, \frac{1}{p})$ *and* $u_0 \in (\mathcal{V}', \mathcal{V})_{1-\frac{1}{p},p}$ *for* $\alpha \in [\frac{1}{p}, 1)$ *there exists a unique* $u \in L^p(0,\tau;\mathcal{V})$ *be the unique weak-solution to the Problem* (3.1).

Remark 3.15.

- *We note that for* $p = 2$ *we have* $(\mathcal{V}', \mathcal{V})_{\frac{1}{2},2} = \mathcal{H}$. *For* $p = 2$ *the result was proving in [41][Theorem 3.1] with less regularity in time for the forms* ($t \to \mathfrak{a}(t)$ *measurable) and* $u_0 \in \mathcal{H}$ *for all* $\alpha \in (0,1)$.
- *In the case* $\alpha \in (\frac{1}{p}, 1), u_0$ *play the role of initial data of* u. *Indeed, since* $u(.) - u_0 = l_\alpha * (-\mathcal{A}(.)u + f) = C_\alpha(-\mathcal{A}(.)u + f)$, *where* C_α *is the operator defined in Lemma 2.9, then if we follows the proof of Lemma 2.9 we can show that* $u(.) - u_0 \in C([0,\tau];\mathcal{V}'), u(t) - u_0 \to 0$ *as* $t \to 0$. *Thus,* $u(0) = u_0$.

Proof. First we prove the result for $u_0 = 0$. As in the proof of Theorem 3.10 we apply Theorem 3.9, so we need to verify that the assumptions of this theorem are satisfied. We shall verify only the conditions (3.5), (3.6), the others are obvious.
Let $0 \leq s, t \leq \tau$ and $\lambda \in \Sigma_{\pi-\theta}$. We get by Lemma 3.6

$$\begin{aligned}
&\|\mathcal{A}(t)(\lambda + \mathcal{A}(t))^{-1}(\mathcal{A}(t)^{-1} - \mathcal{A}(s)^{-1})\|_{\mathcal{L}(\mathcal{V}')} \\
&= \|(\lambda + \mathcal{A}(t))^{-1}(\mathcal{A}(t) - \mathcal{A}(s))\mathcal{A}(s)^{-1}\|_{\mathcal{L}(\mathcal{V}')} \\
&= \sup_{\|u\|_{\mathcal{V}'} = \|v\|_{\mathcal{V}} = 1} |\mathfrak{a}(t, \mathcal{A}(s)^{-1}u, (\bar{\lambda} + \mathcal{A}(t)^*)^{-1}v) - \mathfrak{a}(s, \mathcal{A}(s)^{-1}u, (\bar{\lambda} + \mathcal{A}(t)^*)^{-1}v)| \\
&\leq C|t-s|^\epsilon \sup_{\|u\|_{\mathcal{V}'} = \|v\|_{\mathcal{V}} = 1} \|\mathcal{A}(s)^{-1}u\|_{\mathcal{V}} \|(\bar{\lambda} + \mathcal{A}(t)^*)^{-1}v\|_{\mathcal{V}} \\
&\leq C\frac{|t-s|^\epsilon}{1+|\lambda|}.
\end{aligned}$$

So the condition (3.5) is satisfied for $d = 0$ and $\gamma = \epsilon$. We now use Lemma 3.8 in the case $\beta = 0$ to get the assumption (3.6). Therefore there exists a unique solution to the Problem (3.1) with $u_0 = 0$.

We consider now the case $u_0 \neq 0$. For the case $\alpha \in (0, \frac{1}{p})$ we proceed analogously to the proof of Theorem 3.10. Let $\alpha \in [\frac{1}{p}, 1), v$ satisfies $B^\alpha v(t) + \mathcal{A}(t)v(t) = f(t)$ and w be the solution to the problem $w'(t) + \mathcal{A}(t)w(t) = 0, w(0) = u_0, u_0 \in (\mathcal{V}', \mathcal{V})_{1-\frac{1}{p},p}$. Then by [3][Theorem 2.2] one has $w \in W^{1,p}(0,\tau;\mathcal{V}') \cap L^p(0,\tau;\mathcal{V})$. Set $u = w + v$. Hence,

$u \in L^p(0,\tau;\mathcal{V})$ such that $B^\alpha(u-u_0) \in L^p(0,\tau;\mathcal{V}')$ and u is the solution to $B^\alpha(u-u_0)(t) + \mathcal{A}(t)u(t) = f(t) + (B^\alpha(w-u_0)(t) + \mathcal{A}(t)w(t)), u(0) = u_0$. Then for $f = g - (B^\alpha(w-u_0) + \mathcal{A}(.)w), g \in L^p(0,\tau;\mathcal{V}')$, we have $u \in L^p(0,\tau;\mathcal{V})$ is the unique solution to $B^\alpha(u-u_0) + \mathcal{A}(.)u = g, u(0) = u_0$. □

We remark that for $u \in H_0^\alpha(0,\tau;\mathcal{V}') \cap L^2(0,\tau;\mathcal{V}), \alpha \in (0,1)$ we have by [41][Corollary 3.1]

$$\int_0^\tau \frac{\|u(t)\|^2}{t^\alpha} dt < \infty.$$

Moreover, if $\alpha > \frac{1}{2}$ we get by [40][Proposition 2.5] that $u \in C([0,\tau];[\mathcal{V}',\mathcal{V}]_{\alpha-\frac{1}{2}})$.
For the space $MR(\alpha,2,\tau)$ we have the following theorem.

Theorem 3.16.

- *In case $p=2$ with $u_0 = 0$ we have for all $\alpha \in (0,1), \gamma \in [0,1], u \in MR(\alpha,2,\tau)$*

$$\int_0^\tau \frac{\|u(t)\|^2_{[\mathcal{H},\mathcal{V}]_\gamma}}{t^{(2-\gamma)\alpha}} dt \le C_{\alpha,\gamma} \|u\|^2_{MR(\alpha,2,\tau)}.$$

- *For all $s \in [0,\tau], \alpha \in (\frac{1}{2},1)$ the operator*

$$\begin{aligned} TR_s : MR(\alpha,2,\tau) &\longrightarrow [\mathcal{H};\mathcal{V}]_{2\alpha-1} \\ u &\longrightarrow u(s) \end{aligned}$$

is well defined and bounded.

Proof. First we prove the first assertion for $\gamma = 1$. Let $u \in MR(\alpha,2,\tau)$. Then by the uniform quasi-coercivity of the forms and Proposition 2.2 we get for $u \in MR(\alpha,2,\tau)$

$$\begin{aligned} \int_0^\tau \frac{\|u(t)\|^2_{\mathcal{V}}}{t^\alpha} dt &\le \frac{1}{\delta} \int_0^\tau \Re(A(t)u(t),u(t)) \frac{dt}{t^\alpha} \\ &\le \frac{1}{2\delta^2} \|Au\|^2_{L^2(0,\tau;\mathcal{H})} + \frac{1}{2} \int_0^\tau \frac{\|u(t)\|^2}{t^{2\alpha}} dt \\ &\le \frac{1}{2\delta^2} \|Au\|^2_{L^2(0,\tau;\mathcal{H})} + 2^{2\alpha-1} \int_0^\tau \|B^\alpha u(t)\|^2 dt \\ &\le C(\alpha,\delta) \|u\|^2_{MR(\alpha,2,\tau)}, \end{aligned}$$

where $C(\alpha,\delta) = \frac{1}{2}\Big(\frac{1}{\delta^2} + 2^{2\alpha}\Big)$.
By interpolation we have for $\gamma \in [0,1]$

$$\begin{aligned} &\int_0^\tau \frac{\|u(t)\|^2_{[\mathcal{H},\mathcal{V}]_\gamma}}{t^{(2-\gamma)\alpha}} dt \\ &\le \int_0^\tau \frac{\|u(t)\|^{2(1-\gamma)}}{t^{2(1-\gamma)\alpha}} \frac{\|u(t)\|^{2\gamma}_{\mathcal{V}}}{t^{\gamma\alpha}} dt \\ &\le C_\gamma \int_0^\tau \frac{\|u(t)\|^2}{t^{2\alpha}} dt + C_\gamma \int_0^\tau \frac{\|u(t)\|^2_{\mathcal{V}}}{t^\alpha} dt \\ &\le C(\alpha,\delta,\gamma) \|u\|^2_{MR(\alpha,2,\tau)}. \end{aligned}$$

This finishes the proof of the first assertion.
Noting that by Lemma 3.4 we have $[\mathcal{H},\mathcal{V}]_{2\alpha-1}=[\mathcal{H},D(A(s))]_{\alpha-\frac{1}{2}}$ for all $s\in[0,\tau]$. Let $u\in MR(\alpha,2,\tau)$ and $s\in]0,\tau]$ we get

$$\begin{aligned}
&\int_0^s \|A(s)e^{-lA(s)}u(s)\|^2 l^{2(1-\alpha)}\,dl\\
&\le 2\int_0^s \|A(s)e^{-lA(s)}(u(s)-u(s-l))\|^2 l^{2(1-\alpha)}\,dl\\
&+4\int_0^s \|\left[A(s)e^{-lA(s)}-A(s-l)e^{-lA(s-l)}\right]u(s-l)\|^2 l^{2(1-\alpha)}\,dl\\
&+4\int_0^s \|e^{-lA(s-l)}A(s-l)u(s-l)\|^2 l^{2(1-\alpha)}\,dl\\
&:=I_1(s)+I_2(s)+I_3(s).
\end{aligned}$$

The analyticity of the semigroup and Lemma 2.7 gives

$$I_1(s)\le M\int_0^s \|u(s)-u(s-l)\|^2 l^{-2\alpha}\,dl\le M_\alpha\|u\|_{D(B^\alpha)}.$$

Choose a contour Γ in the positive half-plane and write by the holomorphic functional calculus for the sectorial operators $A(s),A(s-l)$

$$A(s)e^{-lA(s)}-A(s-l)e^{-lA(s-l)}=\frac{1}{2\pi i}\int_\Gamma \lambda e^{-t\lambda}(\lambda I-\mathcal{A}(s))^{-1}\Big(\mathcal{A}(s)-\mathcal{A}(s-l)\Big)(\lambda I-\mathcal{A}(s-l))^{-1}\,d\lambda.$$

Therefore, by using Lemmas 3.6 and 3.5 we obtain

$$\begin{aligned}
&\|\left[A(s)e^{-lA(s)}-A(s-l)e^{-lA(s-l)}\right]u(s-l)\|\\
&\le C\int_0^\infty |\lambda|e^{-t|\cos\gamma||\lambda|}\|(\lambda I-\mathcal{A}(s))^{-1}\|_{\mathcal{L}([\mathcal{H},\mathcal{V}]'_\beta,\mathcal{H})}\\
&\times\|(\lambda I-\mathcal{A}(s-l))^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})}\,d|\lambda|\|\mathcal{A}(s)-\mathcal{A}(s-l)\|_{\mathcal{L}(\mathcal{V},[\mathcal{H},\mathcal{V}]'_\beta)}\|u(s-l)\|\\
&\le C\frac{\|\mathcal{A}(s)-\mathcal{A}(s-l)\|_{\mathcal{L}(\mathcal{V},[\mathcal{H},\mathcal{V}]'_\beta)}}{l^{\frac{\beta+1}{2}}}\|u(s-l)\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_2(s)&\le C\int_0^s \frac{\|\mathcal{A}(s)-\mathcal{A}(s-l)\|^2_{\mathcal{L}(\mathcal{V},[\mathcal{H},\mathcal{V}]'_\beta)}}{l^{\beta+2\alpha-1}}\,dl\|u\|^2_{L^\infty(0,\tau;\mathcal{H})}\\
&\le C_{\alpha,\beta,s}\|u\|^2_{D(B^\alpha)}.
\end{aligned}$$

For I_3 one has

$$I_3(s)\le Ms^{2(1-\alpha)}\int_0^s \|A(s-l)u(s-l)\|^2\,dl\le Ms^{2(1-\alpha)}\|A(.)u\|^2_X.$$

Therefore, by the definition of the interpolation space (see [29][Proposition 5.1.1]) we have

$$\|u(s)\|^2_{[\mathcal{H},\mathcal{V}]_{2\alpha-1}}=\|u(s)\|^2_{D(A(s)^{\alpha-\frac{1}{2}})}\le C_s\Big(\|u(s)\|^2+\int_0^s \|A(s)e^{-lA(s)}u(s)\|^2 l^{2(1-\alpha)}\,dl\Big)\le C_{s,\alpha}\|u\|^2_{MR(\alpha,2,\tau)},$$

which prove the desired result.
For the case $s=0$, we estimate $\int_0^\tau \|A(s)e^{-lA(s)}u(s)\|^2 l^{2(1-\alpha)}\,dl$ and to do this we proceed similarly as before. □

4 Semilinear equation

Let $F(t,u) : (0,\tau) \times \mathcal{H} \to \mathcal{H}$ and $F_0(t) = F(t,0)$. Assume that $F_0 \in L^p(0,\tau;\mathcal{H})$ and $F(.,u)$ satisfies the following continuity property: for any $\epsilon > 0$ there exists a constant $N_{\epsilon,p} > 0$ such that

$$\|F(.,u) - F(.,v)\|^p_{L^p(0,\tau;\mathcal{H})} \leq \epsilon \|u-v\|^p_{MR(\alpha,p,\tau)} + N_{\epsilon,p}\|u-v\|^p_{L^p(0,\tau;\mathcal{H})}, \tag{4.1}$$

for any $u, v \in MR(\alpha,p,\tau)$.

Example 4.1. *1- If we assume that $\|F(t,x) - F(t,y)\| \leq K\|x-y\|_{\mathcal{V}}, K > 0, x, y \in \mathcal{V}, t \in (0,\tau)$ then the conditions (4.1) is satisfied. Indeed, let $u, v \in MR(\alpha,p,\tau)$ one has*

$$\begin{aligned}
\|F(.,u) - F(.,v)\|^p_{L^p(0,\tau;\mathcal{H})} &\leq K^p \|u-v\|^p_{L^p(0,\tau;\mathcal{V})} \\
&\leq \frac{K^p}{\delta^{\frac{p}{2}}} \int_0^\tau \Big(\delta \|u(t)-v(t)\|^2_{\mathcal{V}}\Big)^{\frac{p}{2}} dt \\
&\leq \frac{K^p}{\delta^{\frac{p}{2}}} \int_0^\tau \Big(\Re\Big(A(t)(u(t)-v(t)), u(t)-v(t)\Big)\Big)^{\frac{p}{2}} dt \\
&\leq \frac{K^p}{\delta^{\frac{p}{2}}} \int_0^\tau \|A(t)(u(t)-v(t))\|^{\frac{p}{2}} \|u(t)-v(t)\|^{\frac{p}{2}} dt \\
&\leq \epsilon \|A(.)(u-v)\|^p_{L^p(0,\tau;\mathcal{H})} + N_{\epsilon,p}\|u-v\|^p_{L^p(0,\tau;\mathcal{H})},
\end{aligned}$$

where $N_{\epsilon,p} = \frac{K^{2p}}{\delta^p \epsilon}$.

2- Let $F(t,u) = B^{\alpha_1}u, u \in MR(\alpha,2,\tau), 0 < \alpha_1 < \alpha$. Then the Lemma 2.8 implies

$$\|F(.,u)\|_X = \|B^{\alpha_1}u\|_X \leq \epsilon \|u\|_{D(B^\alpha)} + K(\alpha,\alpha_1,\epsilon)\|u\|_X.$$

Hence, the condition (4.1) is satisfied.

In the following theorem we assume the same hypothesis with Theorem 3.10 for the forms $(\mathfrak{a}(t))_{t\in[0,\tau]}$

Theorem 4.2. *Let $u_0 \in Tr^p_\alpha$. Then the equation*

$$B^\alpha(u-u_0) + A(.)u = F(.,u) \tag{4.2}$$

admits a unique solution $u \in MR(\alpha,p,\tau)$.
Moreover, there is $C_{\alpha,p} > 0$ independent of u_0, F_0 such that

$$\|u\|_{MR(\alpha,p,\tau)} \leq C_{\alpha,p}(\|F_0\|_{L^p(0,\tau;\mathcal{H})} + \|u_0\|_{Tr^p_\alpha}).$$

Proof. For $v \in MR(\alpha,p,\tau)$ consider the linear equation

$$B^\alpha u + A(.)u = F(.,v).$$

By Theorem 3.10, this equation has a unique solution $u \in MR(\alpha, p, \tau)$.
We define

$$\begin{aligned} S : MR(\alpha, p, \tau) &\to MR(\alpha, p, \tau) \\ v &\mapsto u. \end{aligned}$$

For $v_1, v_2 \in MR(\alpha, p, \tau)$ we have by Theorem 3.10 and Lemma 2.9

$$\begin{aligned} \|Sv_1 - Sv_2\|^p_{MR(\alpha,p,\tau)} &\leq N\|F(., v_1) - F(., v_2)\|^p_{L^p(0,\tau;\mathcal{H})} \\ &\leq N\epsilon\|v_1 - v_2\|^p_{MR(\alpha,p,\tau)} + NN_{\epsilon,p}\|v_1 - v_2\|^p_{L^p(0,\tau;\mathcal{H})} \\ &\leq N\epsilon\|v_1 - v_2\|^p_{MR(\alpha,p,\tau)} + NN_{\epsilon,p}\int_0^\tau (\tau - s)^{\alpha-1}\int_0^s \|B^\alpha(v_1 - v_2)(r)\|^p \, dr \, ds \\ &\leq N\epsilon\|v_1 - v_2\|^p_{MR(\alpha,p,\tau)} + NN_{\epsilon,p}\int_0^\tau (\tau - s)^{\alpha-1}\|v_1 - v_2\|^p_{MR(\alpha,p,s)} \, ds. \end{aligned}$$

Set $K_0 := N\epsilon$ and $K_1 := NN_{\epsilon,p}$. Then repeating the above inequality and using the identity

$$\int_0^t (t - s_1)^{\alpha-1}\int_0^{s_1} (s_1 - s_2)^{\alpha-1} ... \int_0^{s_{n-1}} (s_{n-1} - s_n)^{\alpha-1} \, ds_n ... ds_1 = \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha + 1)} t^{n\alpha},$$

we obtain

$$\begin{aligned} \|S^n v_1 - S^n v_2\|^p_{MR(\alpha,p,\tau)} &\leq \sum_0^n \binom{n}{k} K_0^{n-k}(K_1\tau^\alpha)^k \frac{\Gamma(\alpha)^k}{\Gamma(k\alpha + 1)}\|v_1 - v_2\|^p_{MR(\alpha,p,\tau)} \\ &\leq (2K_0)^n \Big[\max_{k=1,..,n}\Big(\frac{\Big(K_0^{-1}\tau^\alpha K_1\Gamma(\alpha)\Big)^k}{\Gamma(k\alpha + 1)}\Big)\Big]\|v_1 - v_2\|^p_{MR(\alpha,p,\tau)}. \end{aligned}$$

For the second inequality we use $\sum_0^n \binom{n}{k} = 2^n$. Note that $\max_{k=1,..,n}\Big(\frac{\Big(K_0^{-1}\tau^\alpha K_1\Gamma(\alpha)\Big)^k}{\Gamma(k\alpha+1)}\Big)$ is bounded for all $n \in \mathbb{N}^*$.
Now, we take $\epsilon < \frac{1}{4N}$, which gives $K_0 < \frac{1}{4}$ and n sufficiently large to get

$$\begin{aligned} \|S^n v_1 - S^n v_2\|^p_{MR(\alpha,p,\tau)} &< \frac{1}{2^n}\Big[\max_{k=1,..,n}\Big(\frac{\Big(K_0^{-1}\tau^\alpha K_1\Gamma(\alpha)\Big)^k}{\Gamma(k\alpha + 1)}\Big)\Big]\|v_1 - v_2\|^p_{MR(\alpha,p,\tau)} \\ &< \|v_1 - v_2\|^p_{MR(\alpha,p,\tau)}. \end{aligned}$$

Then S^n is a contraction on $MR(\alpha, p, \tau)$ and this yields the existence and uniqueness of a solution $u \in MR(\alpha, p, \tau)$ to the equation (4.2). Therefore it only remains to prove the a priori estimate. From the linear equation and (4.1) we have for all $\epsilon > 0$

$$\begin{aligned} \|u\|^p_{MR(\alpha,p,\tau)} &\leq N\|F(., u)\|^p_{L^p(0,\tau;\mathcal{H})} \\ &\leq NC(p)\|F(., u) - F_0(.)\|^p_{L^p(0,\tau;\mathcal{H})} + NC(p)\|F_0(.)\|^p_{L^p(0,\tau;\mathcal{H})} \\ &\leq NC(p)\epsilon\|u\|^p_{MR(\alpha,p,\tau)} + NN_{\epsilon,p}C(p)\|u\|^p_{L^p(0,\tau;\mathcal{H})} + NC(p)\|F_0(.)\|^p_{L^p(0,\tau;\mathcal{H})} \\ &\leq NC(p)\epsilon\|u\|^p_{MR(\alpha,p,\tau)} + NN_{\epsilon,p}C(p)\int_0^\tau (\tau - s)^{\alpha-1}\|u\|^p_{MR(\alpha,p,s)} \, ds + NC(p)\|F_0(.)\|^p_{L^p(0,\tau;\mathcal{H})}. \end{aligned}$$

Here, $C(p)$ is a positive constant depending only on p.
Now, taking $\epsilon = \frac{1}{4NC(p)}$ and applying Gronwall's lemma gives that there exists $C > 0$ such that

$$\|u\|_{MR(\alpha,p,\tau)} \leq C\|F_0\|_{L^p(0,\tau;\mathcal{H})}.$$

□

Corollary 4.3. *Let $C : (0,\tau) \times \mathcal{V} \to \mathcal{H}$ is a linear operator such that $\|C(t)u\| \leq K\|u\|_{\mathcal{V}}, K > 0, u \in \mathcal{V}, t \in (0,\tau)$ and $F(.,.)$ as in the Theorem 4.2. Then for $\lambda \in \mathbb{C}$ the semi-linear equation*

$$B^\alpha(u - u_0) + A(.)u + C(.)u + \lambda u = F(., u) \tag{4.3}$$

admits a unique solution $u \in MR(\alpha, p, \tau)$ for all $u_0 \in Tr^p_\alpha$.
Moreover, there is $C_{\alpha,p} > 0$ independent of u_0, F_0 such that

$$\|u\|_{MR(\alpha,p,\tau)} \leq C_{\alpha,p}\Big(\|F_0\|_{L^p(0,\tau;\mathcal{H})} + \|u\|_{Tr^p_\alpha}\Big).$$

Proof. Set $G(t,u) = F(t,u) - (C(t)u + \lambda u)$. Then it is easy to check that G satisfies the conditions (4.1) and $G(t,0) = F_0(t)$. So the claim follows immediately by Theorem 4.2. □

Remark 4.4. *As consequence of the previous corollary, we can take $\nu = 0$ in [H3] without loss a generality.*

For the non-autonomous evolution equations with two fractional time derivatives we have

Corollary 4.5. *Let $C : (0,\tau) \times \mathcal{V} \to \mathcal{H}$ is a linear operator such that $\|C(t)u\| \leq K\|u\|_{\mathcal{V}}, K > 0, u \in \mathcal{V}, t \in (0,\tau)$ and $F(.,.)$ as in the Theorem 4.2. Then for $\lambda \in \mathbb{C}, 0 < \alpha_1 < \alpha$ the semi-linear equation*

$$B^\alpha(u - u_0) + A(.)u + C(.)u + \lambda B^{\alpha_1}u = F(., u) \tag{4.4}$$

admits a unique solution $u \in MR(\alpha, 2, \tau)$ for all $u_0 \in Tr^2_\alpha$.
Moreover, there is $C_\alpha > 0$ independent of u_0, F_0 such that

$$\|u\|_{MR(\alpha,2,\tau)} \leq C_\alpha\Big(\|F_0\|_{L^2(0,\tau;\mathcal{H})} + \|u\|_{Tr^2_\alpha}\Big).$$

Proof. Set $G(t,u) = F(t,u) - (C(t)u + \lambda B^{\alpha_1}u)$. Then it is easy to check that G satisfies the conditions (4.1) and $G(t,0) = F_0(t)$. So the claim follows immediately by Theorem 4.2. □

In the following proposition we prove that maximal L^p-regularity holds without requiring any regularity in time for the operators (or the forms).

Proposition 4.6. *Assume that*

$$t \to \mathfrak{a}(t,x,y), x,y \in \mathcal{V} \text{ is measurable and} |\mathfrak{a}(t,x,y) - \mathfrak{a}(s,x,y)| \leq C\|x\|_{\mathcal{V}}\|y\|, s,t \in [0,\tau]. \tag{4.5}$$

Then the equation

$$B^{\alpha}u + A(.)u = f \tag{4.6}$$

admits a unique solution $u \in MR(\alpha,p,\tau)$ *for all* $f \in L^p(0,\tau;\mathcal{H}), p \in (1,\infty)$. *Moreover, there is* $C_\alpha > 0$ *independent of* f *such that*

$$\|u\|_{MR(\alpha,p,\tau)} \leq C_\alpha \|f\|_{L^p(0,\tau;\mathcal{H})}$$

and in the case $p = 2$ *we get for* $\beta \in [0,1]$ *that* $u \in H^{\alpha\beta}(0,\tau;A(0)^{1-\beta}) \cap L^2(0,\tau;D(A(0)^{1-\beta}), \frac{dt}{t^{2\alpha\beta}})$.

Remark 4.7. *The condition* (4.5) *is equivalent to* $\mathcal{A}(t) - \mathcal{A}(s) \in \mathcal{L}(\mathcal{V},\mathcal{H}), s,t \in [0,\tau]$ *and we have* $D(A(t)) = D(A(s))$ *with equivalent norms.*

Proof. Set $F(t,u) = f + (A(t) - A(0))u(t), u \in L^p(0,\tau;D(A(0)))$.
We have for $u,v \in L^p(0,\tau;D(A(0)))$

$$\begin{aligned}
\|F(.,u) - F(.,v)\|^p_{L^p(0,\tau;\mathcal{H})} &= \|(A(.) - A(0))(u-v)\|^p_{L^p(0,\tau;\mathcal{H})} = \|\sup_{\|x\|=1}\Big((A(.) - A(0))(u-v), x\Big)\|^p_{L^p(0,\tau)} \\
&= \|\sup_{\|x\|=1}\Big(a(.,u-v,x) - a(0,u-v,x)\Big)\|^p_{L^p(0,\tau)} \\
&\leq C^p\|u-v\|^p_{L^p(0,\tau;\mathcal{V})} \\
&\leq \epsilon\|A(0)(u-v)\|^p_{L^p(0,\tau;\mathcal{H})} + \frac{C^{2p}}{\delta^p\epsilon}\|u-v\|^p_{L^p(0,\tau;\mathcal{H})}.
\end{aligned}$$

Now we proceed exactly as the proof of Theorem 4.2 we prove that there exists a unique $u \in L^p(0,\tau;D(A(0)))$ such that $B^\alpha u \in L^p(0,\tau;\mathcal{H})$ be the solution to the semi-linear equation

$$B^\alpha u + A(0)u = F(.,u).$$

Hence, u is the unique solution to the following problem

$$B^\alpha u(t) + A(t)u = f(t), \text{ t-a.e}$$

and we have the apriori estimate

$$\|B^\alpha u\|_{L^p(0,\tau;\mathcal{H})} + \|A(0)u\|_{L^p(0,\tau;\mathcal{H})} \leq C_\alpha\|f\|_{L^p(0,\tau;\mathcal{H})}.$$

Define the operator

$$E_z u = B^{\alpha z}A(0)^{1-z}u, z \in \mathbb{C}, u \in D(B^\alpha) \cap L^2(0,\tau;D(A(0))).$$

We have for $z = is, s \in \mathbb{R}$

$$\begin{aligned}
\|E_{is}u\|_X = \|B^{is\alpha}A(0)^{1-is}u\|_X &\leq \|B^{is\alpha}\|_{\mathcal{L}(X)}\|A(0)^{-is}\|_{\mathcal{L}(X)}\|A(0)u\|_X \\
&\leq C_{\alpha,s}\|u\|_{D(B^\alpha)\cap L^2(0,\tau;D(A(0)))}.
\end{aligned}$$

Similarly for $z = 1 + is$ we have

$$\|E_{1+is}u\|_X = \|B^{(1+is)\alpha}A(0)^{-is}u\|_X = \|A(0)^{-is}B^{is\alpha}B^{\alpha}u\|_X \leq C_{\alpha,s}\|u\|_{D(B^{\alpha})\cap L^2(0,\tau;D(A(0)))}.$$

We apply Stein's complex interpolation theorem to obtain that

$$E_{\beta} = B^{\alpha\beta}A(0)^{1-\beta} \in \mathcal{L}\Big(D(B^{\alpha}) \cap L^2(0,\tau;D(A(0))), X\Big), \beta \in [0,1].$$

Hence,

$$\|u\|_{H^{\alpha\beta}(0,\tau;A(0)^{1-\beta})} = \|B^{\alpha\beta}A(0)^{1-\beta}u\|_X \leq C_{\alpha,\beta}\|u\|_{D(B^{\alpha})\cap L^2(0,\tau;D(A(0)))} \leq C_{\alpha,\beta}\|f\|_{L^2(0,\tau;\mathcal{H})}.$$

By using Lemma 2.2 we get $H^{\alpha\beta}(0,\tau;A(0)^{1-\beta}) \hookrightarrow L^2(0,\tau;D(A(0)^{1-\beta}),\frac{dt}{t^{2\alpha\beta}})$. This finishes the proof. □

5 Time dependent perturbations-Maximal regularity

Let $\mathfrak{a}(t), A(t), \mathcal{V}$ and $\mathcal{H}$ be as above and suppose again that the standard assumptions [H1]-[H3] are satisfied with $\nu = 0$. Let $(C(t))_{t\in[0,\tau]}$ be a family of bounded invertible operators on $\mathcal{H}$. We assume that there exist constants $\delta_1 > 0$ and $M_1 > 0$ independent of t such that

$$\Re(C(t)^{-1}u, u) \geq \delta_1\|u\|^2_{\mathcal{H}} \quad \forall u \in \mathcal{H}, \tag{5.1}$$

and

$$\|C(t)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq M_1. \tag{5.2}$$

We assume moreover that for all $s, t \in [0;\tau]$

$$\|C(t) - C(s)\|_{\mathcal{L}(\mathcal{H})} \leq L|t-s|^{\epsilon},\ L, \epsilon > 0.$$

As a consequence of (5.1) and (5.2) the numerical range of $C(t)^{-1}$ is contained in a sector of angle w_1 for some $w_1 \in [0, \frac{\pi}{2})$, independent of t. Note that (5.1) implies that

$$\|C(t)^{-1}u\| \geq \delta\|u\|$$

and hence

$$\|C(t)\|_{\mathcal{L}(\mathcal{H})} \leq \frac{1}{\delta}. \tag{5.3}$$

We denote by w_0 the common angle of the numerical range of forms $\mathfrak{a}(t), t \in [0,\tau]$. We assume in addition that

$$w_0 + w_1 < \frac{\pi}{2}, \tag{5.4}$$

and the operator $C(t)A(t)$ is accretive for all $t \in [0,\tau]$, i.e.

$$\Re(C(t)A(t)u, u) \geq C\|u\|^2, u \in D(A(t)), C > 0.$$

Note that $w_0 \leq \arctan(\frac{M}{\delta})$ and $w_1 \leq \arctan(\frac{M_1}{\delta_1})$.
From [34] we have the following Dore-venni theorem type of non-commuting operators on UMD-Banach space

Theorem 5.1. *Let A and B be two sectorial operators with angles θ_A, θ_B and admit a bounded imaginary powers on a UMD-Banach space Y. Suppose that $0 \in \rho(A)$ and $\theta_A + \theta_B < \pi$. Suppose in addition that for some $0 \leq \alpha < \beta \leq 1$ and $c \geq 0$ the Labbas-Terreni commutator estimate*

$$I(\lambda, \sigma) = \|A(\lambda + A)^{-1}\Big(A^{-1}(\sigma + B)^{-1} - (\sigma + B)^{-1}A^{-1}\Big)\|_{\mathcal{L}(Y)} \leq \frac{c}{(1 + |\lambda|^{1-\alpha})|\sigma|^{1+\beta}}$$

holds for all λ and σ with $|\arg(\lambda)| < \pi - \theta_A$ and $|\arg(\sigma)| < \pi - \theta_B$. Then there exists a c_0 such that $c_0 + A + B$ with domain $D(A) \cap D(B)$ is invertible and there is a constant $C > 0$ such that

$$\|(\lambda + A + B)^{-1}\|_{\mathcal{L}(Y)} \leq \frac{C}{\lambda}, \text{ for all } \lambda > c_0.$$

Remark 5.2. *We have the same result if*

$$I(\lambda, \sigma) \leq \frac{c}{(1 + |\lambda|^{1-\alpha})|\sigma|^{1+\beta}} + \frac{c'}{(1 + |\lambda|^{1-\alpha_1})|\sigma|^{1+\beta_1}}$$

for $0 \leq \alpha < \beta \leq 1$ and $0 \leq \alpha_1 < \beta_1 \leq 1, c \geq 0, c' \geq 0$ with $|\arg(\lambda)| < \pi - \theta_A$ and $|\arg(\sigma)| < \pi - \theta_B$.

Theorem 5.3. *Assume that* (3.2). *Then the fractional evolution equation*

$$B^\alpha u + C(.)A(.)u = f \tag{5.5}$$

has maximal L^2-regularity.

Remark 5.4. *If τ is small enough, the regularity assumption on $C(t)$ can be weakened considerably. Indeed, the continuity is sufficient and in order to avoid the inadequate expense of the size of descriptions, we do not prove this result.*

Proof. Define the operator $\mathbb{A}$ in X by $(\mathbb{A}u)(t) = C(t)A(t)u(t)$ with domain

$$D(\mathbb{A}) = \{u \in X, u(t) \in D(A(t)) \text{ a.e } t \text{ and } A(.)u \in L^2(0, \tau; \mathcal{H})\}.$$

We would like to apply the Theorem 5.3, so we need to verify that the assumptions of this theorem are satisfied. Then we have to prove the operator $\mathcal{L} = B^\alpha + \mathbb{A}$ (which is the sum of non-commuting operators) is bounded invertible on X. By using [1][Proposition 3.2] it follows that $C(t)A(t)$ is a sectorial operator of angle $\theta_{\mathbb{A}} = w_0 + w_1$ on $\mathcal{H}$ for all $t \in [0, \tau]$ such that

$$\|(\lambda + C(t)A(t))^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{1 + |\lambda|}, C > 0, \lambda \in \Sigma_{\pi - \theta_{\mathbb{A}}}.$$

Hence, $\mathbb{A}$ is a sectorial operator on X. Now, since $C(t)A(t)$ is an accretive operator we obtain by [29][Theorem 4.3.5] that $C(t)A(t)$ has a bounded imaginary power for all $t \in [0, \tau]$ and

$$\|\Big(C(t)A(t)\Big)^{is}\|_{\mathcal{L}(\mathcal{H})} \leq e^{\frac{\pi}{2}|s|}, s \in \mathbb{R}.$$

Thus,

$$\|(\mathbb{A})^{is}\|_{\mathcal{L}(X)} \leq e^{\frac{\pi}{2}|s|}, s \in \mathbb{R}.$$

We remark that by Proposition 2.2, B^α is a sectorial operator of angle $\theta_b = \frac{\alpha\pi}{2}$ and has a bounded imaginary such that for all $\epsilon > 0$

$$\|(B^\alpha)^{is}\|_{\mathcal{L}(\mathcal{H})} \leq C(1+\alpha^2 s^2)e^{\frac{\alpha\pi}{2}|s|} \leq C_\epsilon e^{(\frac{\alpha\pi}{2}+\epsilon)|s|}, s \in \mathbb{R}.$$

Therefore it remains only to prove the Labbas-Terreni commutator estimate. Indeed, let λ and σ with $|\arg(\lambda)| < \pi - \theta_{\mathbb{A}}$ and $|\arg(\sigma)| < \pi(1-\frac{\alpha}{2})$. Then

$$\begin{aligned}
&I(\lambda,\sigma)^2\\
&= \sup_{f\in X, \|f\|_X=1} \|\mathbb{A}(\lambda+\mathbb{A})^{-1}\Big(\mathbb{A}^{-1}(\sigma+B^\alpha)^{-1} - (\sigma+B^\alpha)^{-1}\mathbb{A}^{-1}\Big)f\|_X^2\\
&= \sup_{f\in X, \|f\|_X=1} \int_0^\tau \|C(t)A(t)(\lambda+C(t)A(t))^{-1}\Big[\Big(C(t)A(t)\Big)^{-1}(\sigma+B^\alpha)^{-1}f(t)\\
&- (\sigma+B^\alpha)^{-1}\Big(C(.)A(.)\Big)^{-1}f(t)\Big]\|^2\,dt.
\end{aligned}$$

Since

$$(\sigma+B^\alpha)^{-1}f(t) = \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\sigma(t-s)^\alpha)f(s)\,ds,$$

we get

$$\begin{aligned}
&\Big(C(t)A(t)\Big)^{-1}(\sigma+B^\alpha)^{-1}f(t) - (\sigma+B^\alpha)^{-1}\Big(C(.)A(.)\Big)^{-1}f(t)\\
&= \Big(C(t)A(t)\Big)^{-1}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\sigma(t-s)^\alpha)f(s)\,ds\\
&- \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\sigma(t-s)^\alpha)\Big(C(s)A(s)\Big)^{-1}f(s)\,ds\\
&= \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\sigma(t-s)^\alpha)\Big[\Big(C(t)A(t)\Big)^{-1} - \Big(C(s)A(s)\Big)^{-1}\Big]f(s)\,ds.
\end{aligned}$$

This gives

$$\begin{aligned}
&I(\lambda,\sigma)^2\\
&= \sup_{f\in X, \|f\|_X=1} \int_0^\tau \|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\sigma(t-s)^\alpha)C(t)A(t)(\lambda+C(t)A(t))^{-1}\\
&\Big[\Big(C(t)A(t)\Big)^{-1} - \Big(C(s)A(s)\Big)^{-1}\Big]f(s)\,ds\|^2\,dt.
\end{aligned}$$

We write

$$\begin{aligned}
&\Big(C(t)A(t)\Big)^{-1} - \Big(C(s)A(s)\Big)^{-1}\\
&= [\Big(C(t)A(t)\Big)^{-1} - \Big(C(s)A(t)\Big)^{-1}] + [\Big(C(s)A(t)\Big)^{-1} - \Big(C(s)A(s)\Big)^{-1}]\\
&= A(t)^{-1}[C(t)^{-1} - C(s)^{-1}] + [A(t)^{-1} - A(s)^{-1}]C(s)^{-1}.
\end{aligned}$$

This gives

$$
\begin{aligned}
&C(t)A(t)(\lambda+C(t)A(t))^{-1}\Big[\Big(C(t)A(t)\Big)^{-1}-\Big(C(s)A(s)\Big)^{-1}\Big]f(s)\\
&=C(t)A(t)(\lambda+C(t)A(t))^{-1}A(t)^{-1}[C(t)^{-1}-C(s)^{-1}]f(s)\\
&+C(t)A(t)(\lambda+C(t)A(t))^{-1}[A(t)^{-1}-A(s)^{-1}]C(s)^{-1}f(s)\\
&=(\lambda+C(t)A(t))^{-1}[C(s)-C(t)]C(s)^{-1}f(s)+(\lambda+C(t)A(t))^{-1}C(t)[\mathcal{A}(s)-\mathcal{A}(s)]\mathcal{A}(s)^{-1}C(s)^{-1}f(s)\\
&:=K_1(t,\lambda)+K_2(t,\lambda).
\end{aligned}
$$

For the first term on the RHS we obtain

$$
\begin{aligned}
\|K_1(t,\lambda)\| &\leq \|(\lambda+C(t)A(t))^{-1}\|_{\mathcal{L}(\mathcal{H})}\|C(s)-C(t)\|_{\mathcal{L}(\mathcal{H})}\|C(s)^{-1}\|_{\mathcal{L}(\mathcal{H})}\|f(s)\|\\
&\leq \frac{C|t-s|^{\epsilon}}{1+|\lambda|}\|f(s)\|.
\end{aligned}
$$

By [1][Proposition 3.2, Proposition 3.4] we have

$$
\|(\lambda+C(t)A(t))^{-1}C(t)\|_{\mathcal{L}(([\mathcal{H},\mathcal{V}]_\beta)',\mathcal{H})}\leq \frac{C}{(1+|\lambda|^{1-\frac{\beta}{2}})}, C>0, \lambda\in\Sigma_{\pi-\theta_{\mathbb{A}}}.
$$

Thus,

$$
\begin{aligned}
\|K_2(t,\lambda)\| &\leq \|(\lambda+C(t)A(t))^{-1}C(t)\|_{\mathcal{L}(([\mathcal{H},\mathcal{V}]_\beta)',\mathcal{H})}\|\mathcal{A}(s)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},([\mathcal{H},\mathcal{V}]_\beta)'}\|\mathcal{A}(s)^{-1}C(s)^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})}\|f(s)\|\\
&\leq \frac{C|t-s|^{\frac{\alpha\beta}{2}+\epsilon}}{1+|\lambda|^{1-\frac{\beta}{2}}}\|f(s)\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I(\lambda,\sigma)^2 &\leq \frac{C_1}{(1+|\lambda|)^2}\sup_{f\in X,\|f\|_X=1}\int_0^\tau\Big(\int_0^t(t-s)^{\alpha+\epsilon-1}|E_{\alpha,\alpha}(-\sigma(t-s)^\alpha)|\|f(s)\|\,ds\Big)^2\,dt\\
&+\frac{C_2}{(1+|\lambda|^{1-\frac{\beta}{2}})^2}\sup_{f\in X,\|f\|_X=1}\int_0^\tau\Big(\int_0^t(t-s)^{\frac{\alpha\beta}{2}+\alpha+\epsilon-1}|E_{\alpha,\alpha}(-\sigma(t-s)^\alpha)|\|f(s)\|\,ds\Big)^2\,dt.
\end{aligned}
$$

In light of Lemma 3.8 and Young's inequality for convolution we have

$$
\begin{aligned}
I(\lambda,\sigma) &\leq \frac{C_1}{1+|\lambda|}\int_0^\tau t^{\alpha+\epsilon-1}|E_{\alpha,\alpha}(-\sigma t^\alpha)|\,dt\sup_{f\in X,\|f\|_X=1}\|f\|_X\\
&+\frac{C_2}{1+|\lambda|^{1-\frac{\beta}{2}}}\int_0^\tau t^{\frac{\alpha\beta}{2}+\alpha+\epsilon-1}|E_{\alpha,\alpha}(-\sigma t^\alpha)|\,dt\sup_{f\in X,\|f\|_X=1}\|f\|_X\\
&\leq \frac{C_1}{(1+|\lambda|)|\sigma|^{1+\epsilon}}+\frac{C_2}{(1+|\lambda|^{1-\frac{\beta}{2}})|\sigma|^{1+\frac{\beta}{2}+\epsilon}}.
\end{aligned}
$$

This prove the Labbas-Terreni commutator estimate. Therefore, there exists a $c_0>0$ such that $c_0+\mathcal{L}$ is bounded invertible on X. Now, by applying the fixed point argument we can easily prove that $\mathcal{L}$ is bounded invertible on X. Then, (5.5) has maximal L^2-regularity. □

6 An Aubin-Lions lemma for $MR(\alpha, p, \tau)$

The following theorem is due to Simon [28]. Let X be a Banach space and let $\tau > 0, p \in (1, \infty)$.

Theorem 6.1. *A subset $F \subset L^p(0, \tau; X)$ is relatively compact in $L^p(0, \tau; X)$ for $1 \leq p < \infty$ or in $C([0, \tau], X)$ for $p = \infty$ if and only if*

1- $\{\int_s^t f\, dr : f \in F\}$ is relatively compact in X for all $0 < s < t < \tau$ and

2- $\|f(. + h) - f\|_{L^p(0,\tau-h;X)} \to 0$ as $h \to 0$ uniformly for $f \in F$.

Theorem 6.2. *Suppose that $\mathcal{V}$ is compactly embedded in $\mathcal{H}$. Then $MR(\alpha, p, \tau)$ is compactly embedded in $L^p(0, \tau; \mathcal{V}) \cap C([0, \tau]; \mathcal{H})$ for all $p \in (1, \infty)$ and $\alpha \in (\frac{1}{p}, 1)$.*

Proof. Let us first prove that $MR(\alpha, p, \tau) \hookrightarrow_c C([0, \tau]; \mathcal{H})$, if $\mathcal{V} \hookrightarrow_c \mathcal{H}$. To do this we show the conditions of Theorem (6.1). Let F be the unit ball of $MR(\alpha, p, \tau)$. The first condition is easy to check since F is a subset of the unit ball of $L^p(0, \tau; \mathcal{V})$. For the second let $0 < h < \tau$ and $u \in F$, then by Lemma 2.9 we have for $\alpha \in (\frac{1}{p}, 1)$

$$\begin{aligned}
\|u(. + h) - u\|_{L^\infty(0,\tau-h;\mathcal{H})} &= \|(l_\alpha * v)(. + h) - l_\alpha * v\|_{L^\infty(0,\tau-h;\mathcal{H})} \\
&= \|(C_\alpha v)(. + h) - C_\alpha v\|_{L^\infty(0,\tau-h;\mathcal{H})} \\
&\leq K(\alpha, \tau, p) h^{\alpha - \frac{1}{p}} \|v\|^p_{L^p(0,\tau;\mathcal{H})},
\end{aligned}$$

where $v = B^\alpha(u - u_0)$.
Let $u \in MR(\alpha, p, \tau)$. Then

$$\begin{aligned}
\|u\|^p_{L^p(0,\tau;\mathcal{V})} &\leq \frac{1}{\delta^{\frac{p}{2}}} \int_0^\tau (\delta \|u(t)\|^2_{\mathcal{V}})^{\frac{p}{2}}\, dt \\
&\leq \frac{1}{\delta^{\frac{p}{2}}} \int_0^\tau (\Re\mathfrak{a}(t, u, u))^{\frac{p}{2}}\, dt \\
&\leq \frac{1}{\delta^{\frac{p}{2}}} \int_0^\tau \|A(t)u(t)\|^{\frac{p}{2}} \|u(t)\|^{\frac{p}{2}}\, dt \\
&\leq \frac{1}{\delta^{\frac{p}{2}}} \|A(.)u\|^{\frac{p}{2}}_{L^p(0,\tau;\mathcal{H})} \|u\|^{\frac{p}{2}}_{L^p(0,\tau;\mathcal{H})}. \qquad (6.1)
\end{aligned}$$

Let $(u_n)_{n\in\mathbb{N}} \subset MR(\alpha, p, \tau)$ be a bounded sequence. Thus there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ which is Cauchy in $L^p(0, \tau; \mathcal{H})$. Finally by (6.1) and the boundedness of $(Au_n)_{n\in\mathbb{N}}$ in $L^p(0, \tau; \mathcal{H})$ we obtain that this subsequence is also Cauchy in $L^p(0, \tau; \mathcal{V})$. Hence, $MR(\alpha, p, \tau)$ is compactly embedded in $L^p(0, \tau; \mathcal{V})$. □

7 Applications

This section is devoted to application of our results on existence and maximal regularity of the sections to concrete evolution equations. We show how they can be applied to both linear and non-linear evolution equations.

7.1 Elliptic operators.

Define on $\mathcal{H} = L^2(\mathbb{R}^d)$ the sesquilinear forms

$$\mathfrak{a}(t,u,v) = \sum_{k,j=1}^{d} \int_{\mathbb{R}^d} a_{kj}(t,x)\partial_k u \overline{\partial_j v}\, dx + \sum_{j=1}^{d} \int_{\mathbb{R}^d} b_j(t,x)\partial_j u \overline{v}\, dx + \int_{\mathbb{R}^d} c(t,x) u\overline{v}\, dx, u, v \in H^1(\mathbb{R}^d).$$

We assume that $a_{kj}, b_j, c : [0,\tau] \times \mathbb{R}^d \to \mathbb{C}$ such that:

$$a_{kj}, b_j, c \in L^\infty([0,\tau] \times \mathbb{R}^d) \text{ for } 1 \le k, j \le d,$$

and

$$\Re \sum_{k,j=1}^{d} a_{kj}(t,x)\xi_k \bar{\xi}_j \ge \delta |\xi|^2 \text{ for all } \xi \in \mathbb{C}^d \text{ and a.e. } (t,x) \in [0,\tau] \times \mathbb{R}^d.$$

Here $\delta > 0$ is a constant independent of t.
It easy to check that $\mathfrak{a}(t,.,.)$ is $H^1(\mathbb{R}^d)$-bounded and quasi-coercive. The associated operator with $\mathfrak{a}(t,.,.)$ is elliptic operator given by the formal expression

$$A(t)u = -\sum_{k,j=1}^{d} \partial_j(a_{kj}(t,.)\partial_k u) + \sum_{j=1}^{d} b_j(t,.)\partial_j u + c(t,.)u.$$

In addition to the above assumptions we assume that for $\alpha \in (0,1)$, some constants $K, \epsilon > 0$

$$|a_{kj}(t,x) - a_{kj}(s,x)| \le K|t-s|^{\frac{\alpha}{2}+\epsilon} \text{ for a.e.} x \in \mathbb{R}^d \text{ and all } t, s \in [0,\tau].$$

Let $F(t,u) : (0,\tau) \times \mathcal{H} \to \mathcal{H}$ and $F_0(t) = F(t,0)$. Assume that $F_0 \in L^p(0,\tau;\mathcal{H})$ and $F(.,u)$ satisfies the following continuity property:

$$\|F(t,u) - F(t,v)\|_{L^2(\mathbb{R}^d)} \le K\|u-v\|_{H^1(\mathbb{R}^d)}, K > 0, u, v \in H^1(\mathbb{R}^d), t \in (0,\tau).$$

Therefore, applying Theorem 4.2 and Corollary 4.3 we conclude that for every $u_0 \in \mathcal{H}$ if $\alpha \in (0, \frac{1}{p})$ and $u_0 = 0$ otherwise the problem

$$B^\alpha(u(.) - u_0)(t) - \sum_{k,j=1}^{d} \partial_j(a_{kj}(t,.)\partial_k u(t)) + \sum_{j=1}^{d} b_j(t,.)\partial_j u(t) + c(t,.)u(t) = F(t,u(t)), t-\text{a.e.}$$

has a unique solution $u \in L^p(0,\tau;H^1(\mathbb{R}^d))$ such that $A(.)u \in L^p(0,\tau;\mathcal{H}), B^\alpha(u(.) - u_0) \in L^p(0,\tau;\mathcal{H})$.
The maximal L^p-regularity we proved here holds also in the case of elliptic operators on Lipschitz domains with Dirichlet or Neumann boundary conditions. The arguments are the same. One define the previous forms $\mathfrak{a}(t)$ with domain $\mathcal{V} = H_0^1(\Omega)$ (for Dirichlet boundary conditions) or $\mathcal{V} = H^1(\Omega)$ (for Neumann boundary conditions).

7.2 Non-autonomous Robin boundary conditions

Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. We denote by Tr the classical trace operator. Let $\gamma : [0,\tau] \times \partial\Omega \to \mathbb{R}$ be a bounded measurable function which is Hölder continuous w.r.t. the first variable, i.e.,

$$|\gamma(t,\sigma) - \gamma(s,\sigma)| \leq K|t-s|^{\frac{\alpha}{4}+\varepsilon} \tag{7.1}$$

for some constants $K > 0, \varepsilon > 0$ and for all $t, s \in [0,\tau], \sigma \in \partial\Omega$. We consider the symmetric form

$$\mathfrak{a}(t,u,v) = \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \gamma(t,.)uv \, d\sigma, u, v \in H^1(\Omega). \tag{7.2}$$

The forms $\mathfrak{a}(t)$ is $H^1(\Omega)-$bounded, quasi-coercive and symmetric. The first statement follows readily from the continuity of the trace operator and the boundedness of γ. The second one is a consequence of the inequality

$$\int_{\partial\Omega} |u|^2 d\sigma \leq \varepsilon \|u\|^2_{H^1(\Omega)} + C_\varepsilon \|u\|^2_{L^2(\Omega)}$$

which is valid for all $\varepsilon > 0$ (C_ε is a constant depending on ε). Note that this is a consequence of compactness of the trace as an operator from $H^1(\Omega)$ into $L^2(\partial\Omega, d\sigma)$.
Let $\mathcal{A}(t)$ be the operator associated with $\mathfrak{a}(t,\cdot,\cdot)$. Note that the part $A(t)$ in $H := L^2(\Omega)$ of $\mathcal{A}(t)$ is interpreted as (minus) the Laplacian with time dependent Robin boundary conditions:

$$\partial_\mu v + \gamma(t,.)v = 0 \text{ on } \partial\Omega.$$

Here we use the following weak definition of the normal derivative. Let $v \in H^1(\Omega)$ such that $\Delta v \in L^2(\Omega)$. Let $h \in L^2(\partial\Omega, d\sigma)$. Then $\partial_\mu v = h$ by definition if $\int_\Omega \nabla u \nabla w \, dx + \int_\Omega \Delta u \nabla w \, dx = \int_{\partial\Omega} hw \, d\sigma$ for all $w \in H^1(\Omega)$. Based on this definition, the domain of $A(t)$ is the set

$$D(A(t)) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega),\ \partial_\mu u + \beta_1(t,.)u = 0\}$$

and for $u \in D(A(t))$ the operator is given by $A(t)u := -\Delta u$. Note that for any $\varepsilon > 0$

$$\begin{aligned}
&|\mathfrak{a}(t;u,v) - \mathfrak{a}(s;u,v)| \\
&= |\int_{\partial\Omega} [\gamma(t,.) - \gamma(s,.)]uv \, d\sigma| \\
&\leq \|\gamma(t,.) - \gamma(s,.)\|_{L^\infty(\partial\Omega)} \|u\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)} \|v\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)} \\
&\leq K|t-s|^{\frac{\alpha}{4}+\varepsilon} \|u\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)} \|v\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)},
\end{aligned}$$

where we used the fact that the trace operator is bounded from $H^{\frac{1}{2}+\varepsilon}(\Omega)$ into $L_2(\partial\Omega)$.
Let $m : [0,\tau] \times \Omega \to [r, \frac{1}{r}]$ for some constant $r > 0$, such that for some $\epsilon, L > 0$

$$|m(t,x) - m(s,x)| \leq L|t-s|^\epsilon, t, s \in [0,\tau], x \in \Omega. \tag{7.3}$$

Now, the assumptions (7.1), (7.3) allows us to apply Theorem 5.3 with $\beta = \frac{1}{2} + \varepsilon$ and obtain maximal L^2-regularity for the corresponding evolution equations

$$\begin{cases} B^\alpha u(.)(t) - m(t,.)\Delta u(t) = f(t) \\ \\ \partial_\mu u(t) + \gamma(t,.)u = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{7.4}$$

7.3 Operators with terms of lower order.

Let Ω be a domain of $\mathbb{R}^n$ and let $b_k, m : [0,\tau]\times\Omega \to \mathbb{R}$ be a bounded measurable function for each $k = 1, \cdots, n$.
We define the forms

$$\mathfrak{a}(t,u,v) = \int_\Omega \nabla u.\nabla v\, dx + \sum_{k=1}^n \int_\Omega b_k(t,x)\partial_k uv\, dx + \int_\Omega m(t,x)uv\, dx,$$

with domain $\mathcal{V}$, a closed subset of $H^1(\Omega)$ which contains $H_0^1(\Omega)$. It is clear that

$$|\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \le M_0\|u\|_{\mathcal{V}}\|v\|_2$$

for some constant M_0. This means that the assumption of the Proposition 4.6 is satisfied. Note that the forms $\mathfrak{a}(t)$ are symmetric, then $D(A(t)^{\frac{1}{2}}) = \mathcal{V}$. We apply Proposition 4.6 and obtain that there exists a unique solution $u \in H^\alpha(0,\tau;L^2(\Omega)) \cap H^{\frac{\alpha}{2}}(0,\tau;\mathcal{V})$ for the correspond fractional evolution equation.

7.4 A quasi-linear equation

In this subsection we consider the non-linear evolution equation

$$\begin{cases} B^\alpha u(t) - m(t,x,u(t))\Delta u(t) = f(t) \\ \\ \partial_\mu u(t) + \beta(t,.)u = 0, \quad \text{on}\, \partial\Omega. \end{cases} \tag{7.5}$$

The function m is supposed to be measurable from $[0,\tau]\times\Omega\times\mathbb{R}$ with values in $[\delta, \frac{1}{\delta}]$ for some constant $\delta > 0$ continuous in the first and the last variable. The domain Ω is bounded with Lipschitz boundary and the function β satisfies (7.1).
We have the following result.

Theorem 7.1. *Let $f \in L^2(0,\tau;L^2(\Omega)), \alpha > \frac{1}{2}$. Then there exists a solution $u \in D(B^\alpha) \cap L^2(0,\tau;H^1(\Omega))$ of (7.5).*

Proof. We shall use Schauder's fixed point theorem to prove this result. Let us denote by $\mathcal{H} = L^2(\Omega), \mathcal{V} = H^1(\Omega)$ and the operator $A(t)$ on $\mathcal{H}$ associated with the form $\mathfrak{a}(t,\cdot,\cdot)$ defined by (7.2). Given $v \in L^2(0,\tau;\mathcal{V}) \cap C([0,\tau];\mathcal{H})$ we set for $g \in \mathcal{H}$

$$C_v(t)g = m(t,x,v(t))g.$$

Note that

$$\delta\|g\|^2 \le (C_v(t)g,g)_{\mathcal{H}} \le \frac{1}{\delta}\|g\|^2$$

and $t \to C_v(t)$ is continuous. By Theorem 5.3, for τ small enough there exists a unique $u \in MR(\alpha,2,\tau)$ such that

$$B^\alpha u(t) + C_v(t)A(t)u(t) = f(t).$$

Now we consider the mapping

$$S : L^2(0,\tau;\mathcal{V}) \cap C([0,\tau];\mathcal{H}) \to L^2(0,\tau;V) \cap C([0,\tau];\mathcal{H}),\ Sv = u.$$

Since $V = H^1(\Omega)$ is compactly embedded into H (recall that Ω is bounded and has Lipschitz boundary), we obtain from 6.2 that $MR(\alpha,2,\tau)$ is compactly embedded in $L^2(0,\tau;\mathcal{V}) \cap C([0,\tau];\mathcal{H})$. As a consequence, it is enough to prove continuity of S and then apply Schauder's fixed point theorem to find $u \in MR(\alpha,2,\tau)$ such that $Su = u$.
Now we prove continuity of S. For this, we consider a sequence $(v_n)_n$ which converges to v in $L^2(0,\tau;\mathcal{V}) \cap C([0,\tau];\mathcal{H})$ and let $u_n = S(v_n)$. It is enough to prove that $(v_n)_n$ has a subsequence which converges to Sv. For each $n \in \mathbb{N}$, u_n is the solution of

$$B^\alpha u_n(t) + C_{v_n}(t)A(t)u_n(t) = f(t).$$

the sequence $(u_n)_n$ is bounded in $MR(\alpha,2,\tau)$ and hence by extracting a subsequence we may assume that $(u_n)_n$ converges weakly to some u in $MR(\alpha,2,\tau)$. In particular, $\mathcal{A}(.)u_n \to \mathcal{A}(.)u$ weakly in $L^2(0,\tau;\mathcal{H})$ and u_n converges in norm to u in $L^2(0,\tau;\mathcal{V}) \cap C([0,\tau];\mathcal{H})$. By extracting a subsequence again we can also assume that $v_n(t,x) \to v(t,x)$ and $\nabla v_n(t,x) \to \nabla v(t,x)$ a.e. with respect to t and to x. Hence for $g \in L^2(0,\tau;\mathcal{H})$ we have $C_{v_n}(.)g(.) \to C_v(.)g(.)$ a.e. and also in $L^2(0,\tau;\mathcal{H})$ by the Dominated Convergence Theorem. Thus for all $g \in L^2(0,\tau;\mathcal{H})$ we obtain

$$\begin{aligned}
0 &= \int_0^\tau (B^\alpha u_n(t) + C_{v_n}(t)A(t)u_n(t) - f(t), g(t))\,dt \\
0 &= \int_0^\tau (B^\alpha u_n(t), g(t))\,dt + \int_0^\tau (A(t)u_n(t), C_{v_n}(t)g(t))\,dt + \int_0^\tau (f(t), g(t))\,dt \\
&\to \int_0^\tau (B^\alpha u(t) + C_v(t)A(t)u(t) - f(t), g(t))\,dt \ (n \to \infty).
\end{aligned}$$

Now the particular choice of $g(t) = B^\alpha u(t) + C_v(t)A(t)u(t) - f(t)$ shows that $B^\alpha u(t) + C_v(t)A(t)u(t) = f(t)$. We conclude that $u = Sv$ which is the desired identity. □

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