The Unruh state for massless fermions on Kerr spacetime and its Hadamard property

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Abstract. We give a rigorous definition of the Unruh state in the setting of massless Dirac fields on slowly rotating Kerr spacetimes. In the black hole exterior region, we show that it is asymptotically thermal at Hawking temperature on the past event horizon. Furthermore, we demonstrate that in the union of the exterior and interior regions, the Unruh state is pure and Hadamard. The main ingredients are the Häfner–Nicolas scattering theory, new microlocal estimates for characteristic Cauchy problems and criteria on the level of square-integrable solutions.

1. Introduction and summary

1.1. **Introduction.** One of the major open problems in mathematical Quantum Field Theory on curved spacetimes is to determine the final quantum state arising from the collapse into a black hole and to describe its thermodynamical properties.

In the last decade, valuable insight has been gained especially from simplified models in which the black hole is eternal and non-rotating.

Most notably, years after Unruh's proposal for a distinguished state on Schwarzschild spacetime [59] and subsequent developments, including e.g. works by Candelas [4] and Dimock–Kay [17, 18], a rigorous definition was eventually provided by Dappiaggi–Moretti–Pinamonti [12]. The same authors gave also more clue to the physical relevance of the Unruh state by proving that it satisfies the *Hadamard condition* on the union of the exterior and the interior region, thus ruling out infinite accumulation of energy at the event horizon. The remarkable fact is that imposing the Hadamard condition singles out a state at the *Hawking temperature* at the event horizon.

The Hadamard condition was also shown recently for the Unruh state on Reissner–Nordström–de Sitter spacetime by Hollands–Wald–Zahn [40], who used it as a reference state to demonstrate the quantum instability of the Cauchy horizon in that setting.

The essential feature of e.g. Schwarzschild spacetime that makes it more tractable than the case of rotating black holes is the existence of a Killing vector field which is time-like in the whole exterior region. Schwarzschild spacetime has also the special structure of a *static bifurcate Killing horizon*, which makes it also possible to consider a different distinguished state, the *Hartle–Hawking–Israel* state [32, 41], conjectured in the '70s to be well-defined on the whole Kruskal–Szekeres extension. While its uniqueness is known since the work of Kay–Wald [42], its rigorous construction and the proof of its Hadamard property was established relatively recently by Sanders [55],

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followed by a generalization to the stationary case by Gérard [26]. Although believed to be too idealized to describe the final black hole collapse state accurately, the Hartle–Hawking–Israel state is an important theoretical model nevertheless in view of its high level of symmetry and its connection to black hole thermodynamics [62].

In the physically more realistic rotating case, however, there is no rigorous result so far that gives the existence of a distinguished Hadamard state. The absence of a global time-like Killing vector field causes severe difficulties both on the conceptual and technical level, which are expressed in the following non-existence theorems:

(1) The Kay–Wald theorem asserts the non-existence of a state which is invariant under the flow of the Killing field $v_{\mathscr{H}}$ that generates the horizon, under the assumption that a certain *superradiance* property holds true [42]. The latter is conjectured to be verified in the case of bosonic fields. (Note that on the other hand, there is no superradiance for fermions in this sense.)

(2) A theorem due to Pinamonti–Sanders–Verch asserts that a thermal state associated with a complete Killing vector field v cannot be Hadamard if there is a point at which v is space-like [50]. This result holds true in a broad setting including bosonic and fermionic non-interacting fields.

Focusing our attention on the exterior Kerr spacetime $(\mathrm{M}_{\mathrm{I}}, g)$ for the moment, the closest analogues of the time-like Killing vector field ∂_t on Schwarzschild are the two Killing vector fields $v_{\mathscr{H}} = \partial_t + \Omega_{\mathscr{H}}\partial_\varphi$ (the generator of the past horizon $\mathscr{H}^-$; the constant $\Omega_{\mathscr{H}}$ is the angular velocity of $\mathscr{H}^-$) and $v_{\mathscr{I}} = \partial_t$ (the generator of past null infinity $\mathscr{I}^-$), each of which is time-like only in a subregion of the exterior. While the result (2) (and in all likelihood (1) as well in the bosonic case) implies bad properties of any state in the exterior region that is thermal with respect to either $v_{\mathscr{H}}$ or $v_{\mathscr{I}}$, we propose instead to split the solution space into two parts, and use $v_{\mathscr{H}}$ for solutions coming from $\mathscr{H}^-$ and $v_{\mathscr{I}}$ for those coming from $\mathscr{I}^-$ in the sense of scattering theory.

This is consistent with a formal definition in terms of mode expansions, proposed by Ottewill–Winstanley in the case of scalar fields [49] and Casals–Dolan–Nolan–Ottewill–Winstanley for fermions [6], cf. [7] for electromagnetic fields. The approach via mode expansions is advantageous for practical computations. However, making the definition of the state rigorous and proving the Hadamard condition requires a sufficiently precise scattering theory (which distinguishes between solutions coming from $\mathscr{H}^-$ and solutions coming from $\mathscr{I}^-$), combined with high frequency estimates for solutions in terms of their asymptotic data. On top of that, the fact that none of the Killing vector fields is everywhere time-like makes it impossible to use the arguments employed in [12] to show the Hadamard condition.

1.2. **Main result.** In the present paper, we provide the first rigorous result of existence of a distinguished Hadamard state on a rotating black hole spacetime. More precisely, we give a precise definition and prove the Hadamard property of the *Unruh state* (or more pedantically, of the *past Unruh state*) in the case of massless Dirac fields on Kerr spacetime.

We consider the four-dimensional spacetime $(\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}, g)$ which is the union of the Kerr black hole exterior and interior spacetimes with rotation parameter $a > 0$ and view it as a subregion of the Kerr–Kruskal extension (M, g) (see Figure 1).

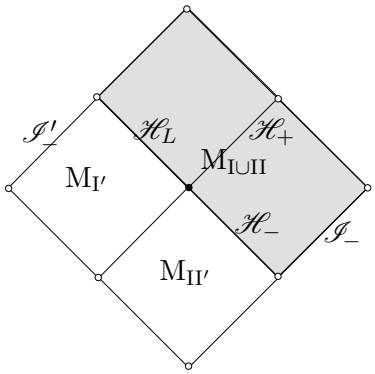

FIGURE 1. The Carter–Penrose diagram of the spacetime $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$ (represented by the shaded region) embedded in the larger spacetime M. Scattering in the exterior region M_{I} refers to data at the past horizon $\mathscr{H}_- \subset \mathscr{H}_L$ and at past null infinity $\mathscr{I}_-$. Scattering in $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$ refers to data at the long horizon $\mathscr{H}_L$ and at $\mathscr{I}_-$. Scattering in M requires an extra piece of data at $\mathscr{I}'_-$.

The Dirac operator $\not{D}$ acting on smooth sections of the canonical spinor bundle $\mathcal{S}$ over M is the differential operator defined as

$$\not{D} = g^{\mu\nu}\gamma(e_\mu)\nabla^{\mathcal{S}}_{e_\nu},$$

see Sections 2 and 5 for details. In the massless case considered here, it is well known that the whole analysis can be reduced to the *Weyl equation* $\mathbb{D}\phi = 0$, which accounts for half of the degrees of freedom, see Section 2.

The main reason for us to consider massless fermions is that in this case the scattering theory in the exterior region M_{I} is sufficiently well developed thanks to results by Häfner–Nicolas [34], and in contrast to bosons, there are no extra difficulties due to superradiance. In particular there is a notion of scattering data at $\mathscr{H}_-$ and $\mathscr{I}_-$ of square-integrable solutions. Furthermore, for sufficiently regular solutions these data agree with traces at $\mathscr{H}_-$ and $\mathscr{I}_-$.

The traces at $\mathscr{H}_-$ and $\mathscr{I}_-$ extend to solutions in the Kerr-Kruskal spacetime M as traces at the left long horizon $\mathscr{H}_L$ (see Figure 1) and at the two conformal infinities $\mathscr{I}_-$ and $\mathscr{I}'_-$. Using these traces it is possible to define a *pure state* ω_{M} in the whole of M, which we call the *Unruh state*.

As explained in Subsect. 4.1, to construct a state it suffices to specify a pair of positive semidefinite operators C^+ and C^- acting on square-integrable solutions of $\mathbb{D}\phi = 0$ and satisfying $C^+ + C^- = \mathbf{1}$ (these are the so-called *covariances*). We demonstrate that the *Hadamard condition* can be then formulated as the requirement that for each solution ϕ, the wavefront set of $(C^\pm)^{\frac{1}{2}}\phi$ is confined to only one component of the characteristic set of $\mathbb{D}$, namely

$$\mathrm{WF}((C^\pm)^{\frac{1}{2}}\phi) \subset \mathcal{N}^\pm, \tag{1.1}$$

where the two components $\mathcal{N}^+$ and $\mathcal{N}^-$ are defined in (3.8) and the definition of the wavefront set is recalled in Appendix E. The main difficulty is how to control a high-frequency condition such as (1.1) when $(C^\pm)^{\frac{1}{2}}\phi$ is defined through its asymptotic data.

Our main result is the following theorem. Below, $\mathbf{1}_{\mathbb{R}^\pm}$ is the characteristic function of the positive, resp. negative half-line, $(U, V, \theta, \varphi^\#)$ are the Kruskal–Boyer–Lindquist coordinates, and $(t^*, r, \theta, \varphi^*)$ are the Kerr-star coordinates in the exterior region, see Section 5.

Theorem 1.1 (cf. Theorems 8.2 and 8.3)**.** *Let* (M, g) *be the Kerr–Kruskal spacetime, and let* $\mathbb{D}$ *be the Weyl operator on* (M, g)*. Then there exists* $0 < a_0 \leq 1$ *such that if* $|a|M^{-1} < a_0$ *then the following holds:*

(1) *there exists a unique state* ω_{M} *for* $\mathbb{D}$*, called the* Unruh state*, with data* $\mathbf{1}_{\mathbb{R}^\pm}(-\mathrm{i}^{-1}\partial_U)$ *at the horizon* $\mathscr{H}_L$ *and* $\mathbf{1}_{\mathbb{R}^\pm}(\mathrm{i}^{-1}\partial_{t^*})$*, resp.* $\mathbf{1}_{\mathbb{R}^\mp}(\mathrm{i}^{-1}\partial_{t^*})$ *at null infinity* $\mathscr{I}_-$*, resp.* $\mathscr{I}'_-$*;*
(2) ω_{M} *and its restriction* $\omega_{\mathrm{M_{I\cup II}}}$ *to* $\mathrm{M_{I\cup II}}$ *are pure states;*
(3) *the restriction* $\omega_{\mathrm{M_I}}$ *of* ω_{M} *to* $\mathrm{M_I}$ *is thermal with respect to* $v_{\mathscr{H}}$ *at the past horizon* $\mathscr{H}_-$ *with temperature equal to the Hawking temperature* $(2\pi)^{-1}\kappa_+$*;*
(4) $\omega_{\mathrm{M_{I\cup II}}}$ *is a Hadamard state.*

The property that the state ω_{M} is *pure* is equivalent to its covariances $C^\pm_{\mathrm{M}}$ being projections, and since $\mathbf{1}_{\mathbb{R}^\pm}(\mathrm{i}^{-1}\partial_U)$ and $\mathbf{1}_{\mathbb{R}^\pm}(\mathrm{i}^{-1}\partial_{t^*})$ are projections, this is a direct consequence of the fact that the scattering data maps that we construct are unitary. The same arguments apply to $\omega_{\mathrm{M_{I\cup II}}}$.

The Hawking temperature at $\mathscr{H}_-$ arises exactly as anticipated. Namely, the restriction of $\mathbf{1}_{\mathbb{R}^\mp}(\mathrm{i}^{-1}\partial_U)$ to $\mathscr{H}_- = \mathscr{H}_L \cap \{U > 0\}$ is a function of $v_{\mathscr{H}}$ (using that $v_{\mathscr{H}}$ equals $-\kappa_+ U\partial_U$ on $\mathscr{H}_-$), and this function is precisely the fermionic thermal distribution $\chi^\pm_{\mathscr{H}_-}(s) = (1 + \mathrm{e}^{\mp 2\pi s/\kappa_+})^{-1}$. This can be understood through an elementary computation presented in Appendix D.

The difficult part is unquestionably the Hadamard property of $\omega_{\mathrm{M_{I\cup II}}}$. Part of the proof relies on wavefront set estimates for solutions in terms of their characteristic data. This is a refinement of the strategy used in works including [45, 11, 12, 2, 28], and it applies to points lying on bicharacteristics that intersect $\mathscr{H}$ or $\mathscr{I}_-$ (with some extra care if the intersection lies on the crossing sphere). In contrast to the Schwarzschild case, however, it is essential to deal with bicharacteristics that do not have this property and that cannot be dealt with by similar arguments.

As a way around that difficulty, we first show that in the exterior region $\mathrm{M_I}$, the Unruh state has covariances of the form:

$$C^\pm_{\mathrm{M_I}} = \mathrm{P}_{\mathscr{H}_-} \circ \chi^\pm_{\mathscr{H}_-}(\mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}}) + \mathrm{P}_{\mathscr{I}_-} \circ \chi^\pm_{\mathscr{I}_-}(\mathrm{i}^{-1}\mathcal{L}_{\mathscr{I}}), \tag{1.2}$$

where $\mathrm{P}_{\mathscr{H}_-}$ and $\mathrm{P}_{\mathscr{I}_-}$ are projections defined using scattering theory, and $\chi^\pm_{\mathscr{I}_-}(s) = \mathbf{1}_{\mathbb{R}^\pm}(s)$. The operator $\mathcal{L}_{\mathscr{H}/\mathscr{I}}$ is the *spinorial Lie derivative* of $v_{\mathscr{H}/\mathscr{I}}$, and in (1.2) we interpret $\mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}/\mathscr{I}}$ as a self-adjoint operator acting on the Hilbert space of square integrable solutions of $\mathbb{D}\phi = 0$.

The form (1.2) allows us to prove wave front set estimates for $(C^\pm_{\mathrm{M_I}})^{\frac{1}{2}}\phi$ in the region where $v_{\mathscr{H}/\mathscr{I}}$ is time-like. The use of the projections $\mathrm{P}_{\mathscr{H}_-/\mathscr{I}_-}$ is also needed to separate solutions coming from $\mathscr{H}_-$ and from $\mathscr{I}_-$ when using arguments based on wavefront sets of traces. All this requires a careful examination of the relationship between scattering theory and characteristic Cauchy problems.

Next, we show that in the slowly rotating case, i.e. for sufficiently small a, all bicharacteristics in $\mathrm{M_I}$ that do not meet $\mathscr{H}_-$ nor $\mathscr{I}_-$ necessarily cross a region where $v_{\mathscr{H}}$ and $v_{\mathscr{I}}$ are time-like. Therefore, the estimates in this good region can be propagated to the problematic points.

Finally, in the remaining part of the larger spacetime $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$, we propagate estimates from M_{I} and from the analogous time-reflected block $\mathrm{M}_{\mathrm{I}'}$. Then, an extra argument near the crossing sphere allows us to conclude the proof of Theorem 1.1.

We emphasize that while the results in Theorem 1.1 are formulated for the Weyl equation, they can be immediately translated to *massless Dirac* fields, see for example the discussion in [27, Sect. 17.15].

1.3. **Physical interpretation and Hawking effect.** The Unruh state is conjectured to be the final state emerging from the collapse of a star into a black hole. This is the physical context of the celebrated *Hawking effect* [36]. Mathematically rigorous works on this process exist in simplified models where the boundary of the star is a perfect mirror, see Bachelot [1] for the Klein–Gordon equation in the spherically symmetric setting and Häfner [33] for the Dirac equation on rotating black holes, including also the charged and massive case (cf. Drouot [22] for Klein–Gordon fields on Schwarzschild-de Sitter spacetime). The final state arising in this process can be interpreted as a state on the Kerr exterior spacetime, and its asymptotic data was shown in [33, Thm. 1.1] to be exactly that of the Unruh state $\omega_{\mathrm{M}_{\mathrm{I}}}$ considered in the present work. Therefore, the Unruh state *is* the conjectured final state arising from the collapse in this model. Furthermore, since we now know that the final state is Hadamard, it would be interesting to see if the arguments of Fredenhagen–Haag [23] about the derivation of the Hawking radiation could be generalized for fermions to the rotating case.

1.4. **Bibliographical remarks and related settings.** The problem of constructing Hadamard states from asymptotic data was considered in various settings by Hollands [38], Moretti [44, 45], Dappiaggi–Pinamonti–Moretti [11, 12], Dappiaggi–Siemssen [13], Benini–Dappiaggi–Murro [2], Gérard–Wrochna [28] using characteristic data methods, by Gérard–Wrochna [29] using scattering theory and time-dependent pseudo-differential calculus, and by Vasy–Wrochna [60] using radial propagation estimates. This includes the already mentioned result [11] on the Unruh state on Schwarzschild spacetime, which uses decay bounds for scalar waves due to Dafermos–Rodnianski [9] (cf. [3] for a variant of [11] for the Schwarzschild-de Sitter metric), and [40] on Reissner–Nordström–de Sitter, which uses estimates by Hintz–Vasy [37].

The results of the present paper strongly rely on the scattering theory for massless Dirac fields in the Kerr exterior region M_{I} due to Häfner–Nicolas [34], see also [35] for a concise presentation. The fact that asymptotic data in the sense of scattering theory coincides with traces at $\mathscr{H}_-$ and conformal traces at $\mathscr{I}_-$ is what allows us to combine the formalism with wavefront set estimates from characteristic data.

For *massive Dirac* fields, the scattering theory [34] was extended by Daudé, see [14, Ch. IV]. In that setting, the scattering data on $\mathscr{H}_-$ can still be interpreted as traces, but this is no longer true for the scattering data at spatial infinity, since the massive Dirac equation is no more conformally invariant. In this situation it is possible to define the Unruh state in M_{I}, $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$ or even M as in the present paper, but its Hadamard property in $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$ or M_{I} is unknown.

The rigorous definition and Hadamard property of the Unruh state on rotating black holes for bosonic, e.g. *scalar* fields is still a major open problem. Useful insight is provided by the work of Ottewill–Winstanley [49], which uses mode analysis. In the case of the wave equation on the Kerr metric, a scattering theory was developed by Dafermos–Rodnianski–Shlapentokh-Rothman [10], and similar results at fixed angular momentum

were obtained for the Klein-Gordon equation on the Kerr–De Sitter metric by Gérard–Georgescu–Häfner [25]. What makes however the case of bosons more difficult than fermions is that there is no obvious Hilbert space that is equally convenient from the point of view of energy estimates and quantization. It is also not fully understood whether superradiance is an obstruction for the existence of the Unruh state.

1.5. **Structure of the paper.** In Sections 2–5 we introduce the setup in detail and prove several preliminary results. This includes a Green's formula for square-integrable solutions and partially null hypersurfaces, see Proposition 3.4, and a proof of the criterion (1.1) for the Hadamard condition, see Theorem 4.3. The purpose of Sections 6–7 is to summarize the scattering results from [34], translate them in a more covariant way to the setting of Weyl fields in $\mathrm{M_I}$ and then extend them to a scattering theory on M. An important role is played by different choice of tetrads, specially adapted either to scattering or to formulas involving traces at $\mathscr{H}_-$ and $\mathscr{I}_-$. The main results are proved in Section 8.

Many auxiliary results are collected in the appendices. This includes the proof of global hyperbolicity and the construction of Cauchy hypersurfaces for the Kerr–Kruskal spacetime M and the $\mathrm{M_I}$ and $\mathrm{M_{I\cup II}}$ regions in Appendix C.

2. Dirac and Weyl equations on Lorentzian manifolds

2.1. **Notation.** Let (M, g) be an oriented and time oriented Lorentzian manifold of dimension 4. We denote by $\Omega_g \in \bigwedge^4 M$ the volume form associated to g and by $d\mathrm{vol}_g = |\Omega_g|$ the volume density.

If $\mathcal{S} \xrightarrow{\pi} M$ is a vector bundle, we denote by $C^\infty(M;\mathcal{S})$ the space of its smooth sections. In the case when $\mathcal{S}$ is a complex vector bundle, we denote its anti-dual bundle by $\mathcal{S}^*$. The complex conjugate bundle (obtained by considering the fibers as complex vector spaces with the opposite complex structure) is denoted by $\overline{\mathcal{S}}$.

If $\mathcal{X}$ is a complex vector space, we denote by $L_{\mathrm{h}}(\mathcal{X}, \mathcal{X}^*)$, resp. $L_{\mathrm{a}}(\mathcal{X}, \mathcal{X}^*)$, the space of Hermitian, resp. anti-Hermitian forms on $\mathcal{X}$. Generally, if $\beta \in L(\mathcal{X}, \mathcal{X}^*)$, i.e. if β is a sesquilinear form, we denote by $\overline{\psi_1} \cdot \beta\psi_2$ its evaluation on $\psi_1, \psi_2 \in \mathcal{X}$. If instead $\beta \in C^\infty(M; End(\mathcal{S}, \mathcal{S}^*))$, the same notation is used for the fiberwise evaluation of $\psi_1, \psi_2 \in C^\infty(M;\mathcal{S})$.

2.2. **Spinor bundles.** In what follows we give a brief introduction to spinors and Weyl equations on four-dimensional spacetimes. We closely follow the exposition in [27, Sect. 17], which is based mainly on works by Trautman (i.e., [56] and references therein), and which uses the formalism of *spinor bundles*. This formalism is closely related to the broadly used approach via *spin structures* (see [57] for a comparison) and it has the advantage of being somewhat more direct from the analyst's point of view.

We follow the presentation in [56]. A *spinor bundle* is a vector bundle $\mathcal{S} \xrightarrow{\pi} M$ with the following objects:

(1) a linear map $\gamma : C^\infty(M; TM) \to C^\infty(M; End(\mathcal{S}))$ such that

$$\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = 2X\cdot gY\mathbf{1}, \quad X, Y \in C^\infty(M; TM), \tag{2.1}$$

and for each $x \in M$, γ_x induces a faithful irreducible representation of the Clifford algebra $\mathrm{Cl}(T_xM, g_x)$ in $\mathcal{S}_x$;

(2) a section $\beta \in C^\infty(M; End(\mathcal{S}, \mathcal{S}^*))$ such that β_x is Hermitian non-degenerate for each $x \in M$ and

$$\begin{aligned} &i) \quad \gamma(X)^*\beta = -\beta\gamma(X), \quad \forall X \in C^\infty(M; TM), \\ &ii) \quad \mathrm{i}\beta\gamma(e) > 0, \text{ for } e \text{ a time-like, future directed vector field on } M; \end{aligned} \tag{2.2}$$

(3) a section $\kappa \in C^\infty(M; End(\mathcal{S}, \overline{\mathcal{S}}))$ such that

$$\kappa\gamma(X) = \gamma(X)\kappa, \quad \kappa^2 = \mathbf{1}; \tag{2.3}$$

(4) a connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$, called a *spin connection*, such that:

$$\begin{aligned} &i) \quad \nabla^{\mathcal{S}}_X(\gamma(Y)\psi) = \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla^{\mathcal{S}}_X\psi, \\ &ii) \quad X(\overline{\psi}\cdot\beta\psi) = \overline{\nabla^{\mathcal{S}}_X\psi}\cdot\beta\psi + \overline{\psi}\cdot\beta\nabla^{\mathcal{S}}_X\psi, \\ &iii) \quad \kappa\nabla^{\mathcal{S}}_X\psi = \nabla^{\mathcal{S}}_X\kappa\psi, \end{aligned} \tag{2.4}$$

for all $X, Y \in C^\infty(M; TM)$ and $\psi \in C^\infty(M; \mathcal{S})$, where ∇ is the Levi-Civita connection on (M, g).

It is well known that the rank of $\mathcal{S}$ is necessarily equal to 4 and that if (2.2) holds for some time-like future directed vector field e, then it holds for all such vector fields.

Remark 2.1. A linear map γ as in (1) is called a *Clifford representation*. A section β as in (2) is called a *positive energy Hermitian form* (for the Clifford representation γ), while a section κ as in (3) is called a *charge conjugation* (for the Clifford representation γ).

It is well-known, see e.g. [56], that given a Clifford representation γ, the space of positive energy Hermitian forms is a principal bundle over M with structure group $\mathbb{R}^{+\times}$, while the space of charge conjugations is a principal bundle over M with structure group $\mathbb{S}^1$.

The properties listed in (2.4) are usually summarized by saying that γ, β, κ are *covariantly constant* w.r.t. the connection $\nabla^{\mathcal{S}}$.

2.2.1. *Spinor bundles from global frames.* Suppose that (M, g) has a global oriented and time oriented orthonormal frame $(e_0, \dots, e_3)$. This will be in particular the case for all the Kerr spacetimes considered in later sections.

Let $(\mathbb{R}^{1,3}, \eta)$ be the Minkowski space with canonical basis $(u_0, \dots, u_3)$ and $\mathrm{Cliff}(1,3)$ the associated Clifford algebra. We fix a faithful and irreducible representation $\rho_0 : \mathrm{Cliff}(1,3) \to M_4(\mathbb{C})$. We denote by $\gamma_i \in M_4(\mathbb{C})$ the gamma matrices associated to u_i for $0 \leq i \leq 3$, and fix a Hermitian sesquilinear form β_0 on $\mathbb{C}^4$, and a complex conjugation κ_0 such that

$$\gamma_i^*\beta_0 = -\beta_0\gamma_i, \quad \mathrm{i}\beta\gamma_0 > 0, \quad \gamma_i\kappa_0 = \kappa_0\gamma_i.$$

Then we can construct a spinor bundle $\mathcal{S} \xrightarrow{\pi} M$ by setting:

$$\begin{aligned} &i) \quad \mathcal{S} = M \times \mathbb{C}^4, \\ &ii) \quad \gamma(e_i) = \gamma_i, \ \beta(x) = \beta_0, \ \kappa(x) = \kappa_0, \\ &iii) \quad \nabla^{\mathcal{S}}_{e_i}\psi = e_i\cdot\nabla\psi + \sigma_i\psi \text{ where} \\ &\qquad \sigma_i = \tfrac{1}{4}\Gamma^k_{ij}\gamma_k\gamma^j, \quad \gamma^k = \eta^{kl}\gamma_l, \quad \nabla_{e_i}e_j =: \Gamma^k_{ij}e_k. \end{aligned}$$

2.3. **Dirac operators.** If $\mathcal{S} \xrightarrow{\pi} M$ is a spinor bundle and $m \in C^\infty(M;\mathbb{R})$, the *Dirac operator* $\not{D}$ acting on $C^\infty(M;\mathcal{S})$ is the differential operator defined as:

$$\not{D} = g^{\mu\nu}\gamma(e_\mu)\nabla^{\mathcal{S}}_{e_\nu} + m.$$

where $(e_0,\dots,e_3)$ is a local frame of TM. We assume throughout the paper that $m \equiv 0$, in which case $\not{D}$ is called the *massless* Dirac operator.

2.4. **Weyl spinors.** If $(e_0,\dots,e_3)$ is an oriented orthonormal local frame of TM one defines the volume element $\eta = \gamma(e_0)\cdots\gamma(e_3)$, which is independent on the choice of such a local frame. The section $\eta \in C^\infty(M;End(\mathcal{S}))$ satisfies:

$$\eta^2 = -\mathbf{1}, \quad \eta\gamma(X) = -\gamma(X)\eta. \tag{2.5}$$

Using (2.5) and (2.4) together with the Clifford relations (2.1) we get:

$$\eta^*\beta = \beta\eta, \quad \nabla^{\mathcal{S}}_X(\eta\psi) = \eta\nabla^{\mathcal{S}}_X\psi. \tag{2.6}$$

It follows that $\mathcal{S}_x = \mathcal{W}_{\mathrm{e},x}\oplus\mathcal{W}_{\mathrm{o},x}$, where $\mathcal{W}_{\mathrm{e},x} = \mathrm{Ker}(\mathrm{i}\eta(x)-\mathbf{1})$, $\mathcal{W}_{\mathrm{o},x} = \mathrm{Ker}(\mathrm{i}\eta(x)+\mathbf{1})$, and the bundle $\mathcal{S}$ splits as $\mathcal{W}_{\mathrm{e}}\oplus\mathcal{W}_{\mathrm{o}}$, where $\mathcal{W}_{\mathrm{e/o}}$ is the bundle of *even/odd Weyl spinors*. One can check that

$$\overline{\psi}_1\cdot\beta\kappa\psi_2 = -\overline{\psi}_2\cdot\beta\psi_1, \quad \psi_i \in C^\infty(M;\mathcal{S}), \tag{2.7}$$

see e.g. [27, Sect. 17.5]. If $\mathcal{W}$ is a vector bundle, we denote by $\mathcal{W}^\#$ its dual bundle, so that $\mathcal{W}^* = \overline{\mathcal{W}^\#}$ in the complex case. Note that κ maps sections of $\mathcal{W}_{\mathrm{o}}$ to sections of $\overline{\mathcal{W}_{\mathrm{e}}}$. Setting

$$\mathbb{S} := \mathcal{W}^*_{\mathrm{e}},$$

one identifies $\mathcal{S}$ with $\mathbb{S}^*\oplus\mathbb{S}^\#$ by the map

$$C^\infty(M;\mathcal{S}) \ni \psi \mapsto \psi_{\mathrm{e}}\oplus\kappa\psi_{\mathrm{o}} =: \chi\oplus\phi \in C^\infty(M;\mathbb{S}^*)\oplus C^\infty(M;\mathbb{S}^\#).$$

From (2.7) one sees that $\mathbb{S}$ is equipped with the symplectic form

$$\epsilon := \frac{1}{\sqrt{2}}(\beta\kappa)^{-1} \in C^\infty(M;End(\mathbb{S},\mathbb{S}^\#)).$$

One can also identify T_xM with $L_{\mathrm{a}}(\mathbb{S}^*,\mathbb{S})$, as real vector spaces by the map:

$$T_xM \ni v \mapsto \beta_x\circ\gamma_x(v) \in L_{\mathrm{a}}(\mathcal{W}_{\mathrm{e},x},\mathcal{W}^*_{\mathrm{e},x}).$$

This map is injective since the representation in (2.1) is faithful and bijective since both spaces have dimension 4. By complexification we obtain an isomorphism

$$\tau_x : \mathbb{C}T_xM \ni z \mapsto \beta_x\gamma_x(z) \in L_{\mathrm{a}}(\mathcal{W}_{\mathrm{e},x},\mathcal{W}^*_{\mathrm{e},x}) \sim \mathbb{S}_x\otimes\overline{\mathbb{S}}_x,$$

and hence an isomorphism

$$\tau : C^\infty(M;\mathbb{C}TM) \ni v \mapsto \beta\gamma(v) \in C^\infty(M;\mathbb{S}\otimes\overline{\mathbb{S}}) \tag{2.8}$$

If we extend g to $\mathbb{C}TM$ as a *bilinear* (not sesquilinear) form, one can show that

$$\tau^\#\circ(\epsilon\otimes\overline{\epsilon})\circ\tau = g.$$

2.5. **Null tetrads and associated frames.** A *normalized null tetrad* is a global frame $(l, n, m, \overline{m})$ of $\mathbb{C}TM$ such that:

$$\begin{aligned} &l, n \text{ are real}, \quad l\cdot gl = n\cdot gn = 0, \quad l\cdot gn = -1 \\ &m\cdot gm = l\cdot gm = n\cdot gm = 0, \quad m\cdot g\overline{m} = 1. \end{aligned} \tag{2.9}$$

It follows that if

$$u_0 = \frac{1}{\sqrt{2}}(l+n), \quad u_1 = \frac{1}{\sqrt{2}}(l-n), \quad u_2 = \sqrt{2}\,\mathrm{Re}\, m, \quad u_3 = \sqrt{2}\,\mathrm{Im}\, m,$$

then $(u_0, \ldots, u_3)$ is a global orthonormal frame for (M, g) and we associate to a spinor bundle as in 2.2.1.

2.5.1. *Time orientation.* A normalized null tetrad induces a time orientation of (M, g) by declaring that the time-like vector $\frac{1}{\sqrt{2}}(l+n)$ is future directed. Note that for any smooth functions α, β with $\alpha, \beta \geq 0, \alpha + \beta > 0$, $\alpha l + \beta n$ is also future directed, in particular l, n are future directed null vectors.

2.5.2. *Spin frames.* Using the isomorphism τ in (2.8) one associates to a normalized null tetrad a global frame $(o, \imath)$ of $\mathbb{S}$ such that:

$$\begin{aligned} &\mathrm{i}\tau(l) = o \otimes \overline{o}, \quad \mathrm{i}\tau(n) = \imath \otimes \bar{\imath}, \\ &\mathrm{i}\tau(m) = o \otimes \bar{\imath}, \quad \mathrm{i}\tau(\overline{m}) = \imath \otimes \overline{o}, \\ &o\cdot\epsilon\imath = 1. \end{aligned} \tag{2.10}$$

If $s_1, s_2 \in \mathbb{S}$ we denote by $|s_1)(s_2| \in L(\mathbb{S}^*, \mathbb{S})$ the map $w \mapsto (s_2|w)s_1$. Using this notation we have:

$$\mathrm{i}\Gamma(l) = |o)(o|, \quad \mathrm{i}\Gamma(n) = |\imath)(\imath|, \quad \mathrm{i}\Gamma(m) = |o)(\imath|, \quad \mathrm{i}\Gamma(\overline{m}) = |\imath)(o|, \tag{2.11}$$

where we have set:

$$\Gamma(X) = \beta\gamma(X) \in C^\infty(M, End(\mathbb{S}^*, \mathbb{S})), \quad X \in C^\infty(M; TM). \tag{2.12}$$

2.6. **Dirac and Weyl equations.** Let $\not{D}$ be the massless Dirac operator acting on $C^\infty(M; \mathcal{S})$. From (2.5), (2.6) we obtain that $\eta \circ \not{D} = -\not{D} \circ \eta$, hence after identifying $\mathcal{S}$ with $\mathcal{W}_\mathrm{e} \oplus \mathcal{W}_\mathrm{o}$ we can write $\not{D}$ as:

$$\not{D} = \begin{pmatrix} 0 & \not{D}_\mathrm{o} \\ \not{D}_\mathrm{e} & 0 \end{pmatrix},$$

for $\not{D}_{\mathrm{e/o}} = \not{D}_{|C^\infty(M;\mathcal{W}_{\mathrm{e/o}})}$. Using Subsect. 2.4 we define the *Weyl operator*:

$$\mathbb{D} := (\beta\not{D})_{|C^\infty(M;\mathcal{W}_\mathrm{e})} : C^\infty(M; \mathbb{S}^*) \to C^\infty(M; \mathbb{S}).$$

We can write

$$\mathbb{D} = g^{\mu\nu}\Gamma(e_\mu)\nabla^{\mathcal{S}}_{e_\nu}. \tag{2.13}$$

The *Weyl equation* is

$$\mathbb{D}\phi = 0. \tag{2.14}$$

Remark 2.2. In the physics literature, it is customary to study the Weyl anti-neutrino equation. Here we rather stick to the Weyl neutrino equation (2.14) and we use the notation ϕ (usually reserved for anti-neutrinos) for its solutions.

2.7. **Conformal transformations.** Let $\hat{g} = c^2 g$ be a metric conformal to g. Denoting with hats the objects attached to $\hat{g}$ we have in particular $\hat{\gamma}(X) = c\gamma(X)$ for $X \in C^\infty(M;TM)$. To fix the spin connection $\hat{\nabla}^{\mathcal{S}}$ one has to fix the Hermitian form $\hat{\beta}$ and the charge conjugation $\hat{\kappa}$. It is natural to choose $\hat{\kappa} = \kappa$, but several choices of $\hat{\beta}$ are possible. A choice which is natural when one considers Weyl spinors is to set

$$\hat{\beta} = c^{-1}\beta,$$

so that the isomorphism τ in (2.8) is unchanged, i.e. $\hat{\tau} = \tau$. A straightforward computation shows then that

$$\hat{\nabla}^{\mathcal{S}}_X = \nabla^{\mathcal{S}}_X + \frac{1}{2}c^{-1}\gamma(X)\gamma(\nabla c) - c^{-1}X\cdot dc\mathbf{1}.$$

Consequently, if $\hat{\not{D}}, \hat{\mathbb{D}}$ are the associated Dirac and Weyl operators, they are related by:

$$\hat{\not{D}} = c^{-2}\not{D}c, \quad \hat{\mathbb{D}} = c^{-3}\mathbb{D}c. \tag{2.15}$$

2.8. **Lie derivative of spinors.** The notion of Lie derivative of spinors was introduced by Kosman [43]. We recall its definition below.

For any $X \in C^\infty(M;TM)$, one sets:

$$\mathcal{L}_X\psi = \nabla^{\mathcal{S}}_X\psi + \frac{1}{8}((\nabla_a X)_b - (\nabla_b X)_a)\gamma^a\gamma^b\psi, \quad \psi \in C^\infty(M,S). \tag{2.16}$$

Note the sign change w.r.t. [43, (I.16)], coming from the convention for the Clifford algebra in [43, (I.1)].

One also defines the Lie derivative of γ by:

$$\mathcal{L}_X(\gamma(v)\psi) =: (\mathcal{L}_X\gamma)(v)\psi + \gamma(\mathcal{L}_X v)\psi + \gamma(v)\mathcal{L}_X\psi, \quad \psi \in C^\infty(M;\mathcal{S}),$$

for all $v \in C^\infty(M;TM)$. Properties of $\mathcal{L}_X$ which are relevant for us are discussed in Appendix A.

3. Weyl equation on globally hyperbolic spacetimes

If $v \in C^\infty_{\mathrm{c}}(M;\mathbb{S})$ and $\phi \in C^\infty(M;\mathbb{S}^*)$ we set

$$(\phi|v)_M = \overline{(v|\phi)_M} := \int_M \overline{\phi(x)}\cdot v(x)\, d\mathrm{vol}_g, \tag{3.1}$$

and extend this notation to other natural cases, like for example $v \in \mathcal{E}'(M;\mathbb{S})$ and $\phi \in C^\infty(M;\mathbb{S}^*)$, etc.

3.1. **Space-compact solutions.** From now on we assume that (M,g) is a globally hyperbolic spacetime (of dimension 4) with a spin structure and denote by $\mathbb{D}$ the associated Weyl operator.

3.1.1. *Space-compact solutions.* Let $\mathrm{Sol}_{\mathrm{sc}}(M)$ be the space of smooth space-compact solutions of $\mathbb{D}\phi = 0$, $\phi \in C^\infty(M;\mathbb{S}^*)$ (ϕ *space-compact* means that the intersection of $\operatorname{supp}\phi$ with a space-like Cauchy surface is compact). The *current* $J(\phi_1,\phi_2) \in C^\infty(M;T^*M)$ defined by

$$J(\phi_1,\phi_2)\cdot X := \overline{\phi}_1\cdot\Gamma(X)\phi_2, \quad X \in C^\infty(M;TM),\ \phi_i \in \mathrm{Sol}_{\mathrm{sc}}(M), \tag{3.2}$$

satisfies

$$\nabla^a J_a(\phi_1,\phi_2) = 0, \quad \phi_i \in \mathrm{Sol}_{\mathrm{sc}}(M). \tag{3.3}$$

Since $\nabla^a J_a \Omega_g = d(g^{-1}J \lrcorner \Omega_g)$, we deduce from Stokes' formula that

$$\int_{\partial U} i^*(g^{-1}J(\phi_1,\phi_2)\lrcorner\Omega_g) = 0,\ \phi_i \in \mathrm{Sol}_{\mathrm{sc}}(M), \tag{3.4}$$

if U is any open set whose boundary ∂U is a union of smooth hypersurfaces, such that $\operatorname{supp} J(\phi_1,\phi_2) \cap \partial U$ is compact.

Let $S \subset M$ be a smooth hypersurface and let $i : S \to M$ be the canonical injection. To express

$$\int_S i^*(g^{-1}J\lrcorner\Omega_g)$$

for J a 1-form on M, we choose a vector field $l = l^a$ transverse to S, future pointing and a 1-form $\nu = \nu_a dx^a$ on M such that $TS = \operatorname{Ker}\nu$, normalized so that $\nu \cdot l = 1$. Splitting $g^{-1}J$ as

$$g^{-1}J = (\nu \cdot g^{-1}J)l + R, \text{ where } R \text{ is tangent to } S,$$

we obtain:

$$i^*(g^{-1}J\lrcorner\Omega_g) = (\nu\cdot g^{-1}J) i^*(l\lrcorner\Omega_g). \tag{3.5}$$

3.1.2. *Characteristic manifold.* The *principal symbol* of $\mathbb{D}$ is the section $\sigma_{\mathbb{D}} \in C^\infty(T^*M\setminus o; End(\mathbb{S}^*,\mathbb{S}))$ given by

$$\sigma_{\mathbb{D}}(x,\xi) = \Gamma(g^{-1}(x)\xi), \quad (x,\xi) \in T^*M \setminus o. \tag{3.6}$$

Lemma 3.1. *The Weyl operator $\mathbb{D}$ is pre-normally hyperbolic, i.e., there exists a differential operator $\mathbb{D}'$ such that $(\sigma_{\mathbb{D}} \circ \sigma_{\mathbb{D}'})(x,\xi) = (\xi \cdot g^{-1}(x)\xi)\mathbf{1}$.*

Proof. The Dirac operator $\not{D}$ is pre-normally hyperbolic because

$$\sigma_{\not{D}}^2 = (\xi \cdot g^{-1}(x)\xi)\mathbf{1}, \tag{3.7}$$

as can be checked using $\sigma_{\not{D}} = \gamma(g^{-1}(x)\xi)$ and the Clifford relations. After identification of $\mathcal{S}$ with $\mathcal{W}_{\mathrm{e}} \oplus \mathcal{W}_{\mathrm{o}}$, $\sigma_{\not{D}}$ is anti-diagonal, therefore we can conclude from (3.7) that $\not{D}_{\mathrm{o}}$ and $\not{D}_{\mathrm{e}}$ are pre-normally hyperbolic. It follows that $\mathbb{D}$ is pre-normally hyperbolic. □

The *characteristic manifold* of $\mathbb{D}$ is defined as

$$\mathrm{Char}(M) = \{(x,\xi) \in T^*M \setminus o \,:\, \sigma_{\mathbb{D}}(x,\xi) \text{ is not invertible}\}.$$

By Lemma 3.1,

$$\mathrm{Char}(M) = \{(x,\xi) \in T^*M \setminus o \,:\, \xi \cdot g^{-1}(x)\xi = 0\} =: \mathcal{N}.$$

Its two connected components are

$$\mathcal{N}^\pm := \mathcal{N} \cap \{(x,\xi) \in T^*M \setminus o \,:\, \pm v\cdot\xi > 0\ \forall v \in T_xM \text{ future directed time-like}\}. \tag{3.8}$$

3.1.3. *Propagators and Cauchy problem.* An adaptation of an argument due to Dimock [19] to the case of general pre-normally hyperbolic operators (see [46, Thm. 1]) gives the existence and uniqueness of *retarded* and *advanced propagators*, $\mathbb{G}_{\mathrm{ret}}$ and $\mathbb{G}_{\mathrm{adv}}$. Recall that $\mathbb{G}_{\mathrm{ret/adv}}$ is by definition a two-sided inverse of $\mathbb{D}$ (on test sections) such that

$$\operatorname{supp}\mathbb{G}_{\mathrm{ret/adv}} v \subset J_\pm(\operatorname{supp} v), \quad v \in C_{\mathrm{c}}^\infty(M;\mathbb{S}),$$

where $J_\pm(K)$ stands for the causal future/past of $K \subset M$. The *Pauli-Jordan* or *causal propagator* is the difference

$$\mathbb{G} = \mathbb{G}_{\mathrm{ret}} - \mathbb{G}_{\mathrm{adv}}.$$

We have

$$(v_1|\mathbb{G}v_2)_M = -(\mathbb{G}v_1|v_2)_M,\ v_i \in C_{\mathrm{c}}^\infty(M;\mathbb{S}), \tag{3.9}$$

i.e. $\mathbb{G}^* = -\mathbb{G}$ for the pairing $(\cdot|\cdot)_M$ defined in (3.1).

If S is a space-like Cauchy surface, the Cauchy problem

$$\begin{cases} \mathbb{D}\phi = 0, \\ r_S\phi = \varphi \in C_c^\infty(S; \mathbb{S}_S^*), \end{cases} \tag{3.10}$$

where $\mathbb{S}_S^*$ is the restriction of $\mathbb{S}^*$ to S and $r_S\phi = \phi_{|S}$, has a unique solution $\phi =: \mathbb{U}_S\varphi \in \mathrm{Sol}_{\mathrm{sc}}(M)$ (see e.g. [46, Thm. 2]).

For all $\phi \in \mathrm{Sol}_{\mathrm{sc}}(M)$ one has:

$$\phi(x) = -\int_S \mathbb{G}(x,y)\Gamma(g^{-1}\nu)(y)\phi(y) i_l^*(d\mathrm{vol}_g)(y), \tag{3.11}$$

where l and ν are as in (3.5). Choosing $l = n, \nu = -gn$, where n the future directed vector field normal to S, this can be rewritten as

$$\phi(x) = -\int_S \mathbb{G}(x,y)\Gamma(n(y))\phi(y)\, d\mathrm{vol}_h(y), \tag{3.12}$$

where h is the induced Riemannian metric on S.

If S is given by $\{f = 0\}$ for some function f with $df \neq 0$ on S and if we can complete f near S with coordinates $y^1, \dots, y^{n-1}$ such that $df \wedge dy^1 \cdots \wedge dy^{n-1}$ is direct, ∂_f future pointing, we take $l = \partial_f$, $\nu = df$ in (3.5) and obtain:

$$\phi(x) = -\int_S \mathbb{G}(x,y)\Gamma(\nabla f)(y)\phi(y)|g|^{\frac{1}{2}}\, dy^1 \dots dy^{n-1}. \tag{3.13}$$

3.2. L^2 solutions.

3.2.1. *Hilbertian scalar product.* Let now $S \subset M$ be any smooth Cauchy surface (with the convention that a Cauchy surface does not need to be space-like). We set

$$\begin{aligned} (\phi_1|\phi_2)_{\mathbb{D}} &:= \mathrm{i}\int_S i^*(g^{-1}J(\phi_1,\phi_2)\lrcorner\Omega_g) \\ &= \mathrm{i}\int_S \overline{\phi}_1 \cdot \Gamma(g^{-1}\nu)\phi_2 \;\; i_l^* d\mathrm{vol}_g, \end{aligned} \tag{3.14}$$

where $i_l^* d\mathrm{vol}_g = |i^*(l\lrcorner\Omega_g)|$.

From (3.4) we see that the r.h.s. in (3.14) is independent on the choice of the Cauchy surface S.

If S is space-like, we obtain as in (3.12):

$$(\phi_1|\phi_2)_{\mathbb{D}} = \mathrm{i}\int_S \overline{\phi_1} \cdot \Gamma(n)\phi_2\, d\mathrm{vol}_h. \tag{3.15}$$

By (2.2) $\mathrm{i}\Gamma(n)$ is positive definite, which shows that $(\cdot|\cdot)_{\mathbb{D}}$ is a Hilbertian scalar product on $\mathrm{Sol}_{\mathrm{sc}}(M)$.

If S is given by $\{f = 0\}$ for some function f as in (3.13) we obtain:

$$(\phi_1|\phi_2)_{\mathbb{D}} = \mathrm{i}\int_S \overline{\phi}_1 \cdot \Gamma(\nabla f)\phi_2 |g|^{\frac{1}{2}}\, dy^1 \dots dy^{n-1}.$$

Definition 3.2. The Hilbert space $\mathrm{Sol}_{\mathrm{L}^2}(M)$, called the *space of L^2 solutions*, is the completion of $\mathrm{Sol}_{\mathrm{sc}}(M)$ for the scalar product $(\cdot|\cdot)_{\mathbb{D}}$.

Note that $\mathrm{Sol}_{\mathrm{L}^2}(M) \subset L^2_{\mathrm{loc}}(M; \mathbb{S})$ continuously, so elements of $\mathrm{Sol}_{\mathrm{L}^2}(M)$ are distributional solutions of $\mathbb{D}\phi = 0$.

3.2.2. *Conformal transformations.* Let us denote by $\hat{M}$ the spacetime M equipped with the metric $\hat{g} = c^2 g$. From Subsect. 2.7 and (3.15) we obtain that

$$\mathrm{Sol}_{\mathrm{L}^2}(M) \ni \phi \mapsto \hat{\phi} = c^{-1}\phi \in \mathrm{Sol}_{\mathrm{L}^2}(\hat{M}) \tag{3.16}$$

is unitary.

3.2.3. *Equivalent Hilbert spaces.* We now recall other Hilbert spaces unitarily equivalent to $\mathrm{Sol}_{\mathrm{L}^2}(M)$, see e.g. [27, Sect. 17.14]. Let Σ be a smooth space-like Cauchy surface and $\mathbb{S}_\Sigma$, r_Σ defined in 3.1.3.

Proposition 3.3. *The following maps are unitary*

$$\Big(\tfrac{C_{\mathrm{c}}^\infty(M;\mathbb{S})}{\mathbb{D}C_{\mathrm{c}}^\infty(M;\mathbb{S}^*)}, \mathrm{i}\mathbb{G}\Big) \xrightarrow{\ \mathbb{G}\ } \big(\mathrm{Sol}_{\mathrm{sc}}(M), (\cdot|\cdot)_{\mathbb{D}}\big) \xrightarrow{\ r_\Sigma\ } \big(C_{\mathrm{c}}^\infty(\Sigma;\mathbb{S}_\Sigma^*), \nu_\Sigma\big), \tag{3.17}$$

where

$$\overline{\varphi}_1 \cdot \nu_\Sigma \varphi_2 = \mathrm{i}\int_\Sigma \overline{\varphi}_1 \cdot \Gamma(n)\varphi_2 \, d\mathrm{vol}_h, \ \varphi_i \in C_{\mathrm{c}}^\infty(\Sigma;\mathbb{S}_\Sigma^*).$$

As a consequence of Proposition 3.3 we have the identity

$$(\phi_1|\phi_2)_{\mathbb{D}} = (v_1|\mathrm{i}\mathbb{G}v_2)_M = (\mathrm{i}\mathbb{G}v_1|v_2)_M, \ \text{for } \phi_i = \mathbb{G}v_i, \ v_i \in C_{\mathrm{c}}^\infty(M;\mathbb{S}), \tag{3.18}$$

which extends to

$$(v|\phi)_M = (\mathrm{i}\mathbb{G}v|\phi)_{\mathbb{D}}, \ v \in C_{\mathrm{c}}^\infty(M;\mathbb{S}), \ \phi \in \mathrm{Sol}_{\mathrm{L}^2}(M). \tag{3.19}$$

3.3. **Traces of L^2 solutions on hypersurfaces.** Let $S \subset M$ be a piecewise smooth hypersurface, equipped with a piecewise smooth density dm. If $u \in \mathcal{E}'(S;\mathbb{S}_S)$ we denote by $\delta_S \otimes u \in \mathcal{D}'(M;\mathbb{S})$ the distribution defined by

$$(v|\delta_S \otimes u)_M := \int_S \overline{v}_{|S} \cdot u \, dm, \ v \in C_{\mathrm{c}}^\infty(M). \tag{3.20}$$

If S is a smooth space-like Cauchy hypersurface and $\phi \in \mathrm{Sol}_{\mathrm{sc}}(M)$, we can rewrite (3.12) or (3.13) as

$$\phi = -\mathbb{G}(\delta_S \otimes \Gamma_S r_S \phi), \ \phi \in \mathrm{Sol}_{\mathrm{sc}}(M), \tag{3.21}$$

where S is equipped with the appropriate density and $\Gamma_S : S \to End(\mathbb{S}_S^*, \mathbb{S}_S)$ denotes the factor $\Gamma(n)$ or $\Gamma(\nabla f)$ in (3.12) or (3.13).

Note that (3.21) extends to $\phi \in \mathrm{Sol}_{\mathrm{L}^2}(M)$. In fact the trace $r_S\phi$ of ϕ on S is well defined using that $\mathrm{WF}(\phi) \subset \mathcal{N}$.

In this subsection we will show how to extend (3.21) for $\phi \in \mathrm{Sol}_{\mathrm{L}^2}(M)$ to cases where S is not necessarily smooth nor everywhere space-like. We recall that $J(K)$ for $K \subset M$ is the causal shadow of K.

Proposition 3.4. *Let $S \subset M$ be a piecewise smooth hypersurface and $\mathbb{S}_S$ the restriction of $\mathbb{S}$ to S. Let dm be a piecewise smooth density on S and $\mathrm{i}\Gamma_S : S \to End(\mathbb{S}_S^*, \mathbb{S}_S)$ a piecewise smooth positive Hermitian structure on $\mathbb{S}_S^*$. We denote by $L^2(S;\mathbb{S}_S^*)$ the Hilbert space of L^2 sections of $\mathbb{S}_S^*$ equipped with the scalar product:*

$$(u|u)_S := \mathrm{i}\int_S \overline{u} \cdot \Gamma_S u \, dm.$$

We assume the following:

(1) *$J(K) \cap S$ is compact for all $K \Subset M$.*
(2) *there exists a map $r_S : \mathrm{Sol}_{\mathrm{sc}}(M) \to L^2(S;\mathbb{S}_S^*)$ such that*

$$\phi = -\mathbb{G}(\delta_S \otimes \Gamma_S r_S \phi), \ \forall \phi \in \mathrm{Sol}_{\mathrm{sc}}(M).$$

(3) *the map r_S extends as a bounded operator*

$$T_S : \mathrm{Sol}_{\mathrm{L}^2}(M) \to L^2(S; \mathbb{S}^*_S).$$

Then for each $\chi \in C^\infty_{\mathrm{c}}(M)$ and $\chi_S \in C^\infty_{\mathrm{c}}(S)$ such that $\chi_S \equiv 1$ on $J(\mathrm{supp}\,\chi) \cap S$ one has:

$$\chi\phi = -\chi \mathbb{G}(\delta_s \otimes \Gamma_S \chi_S T_S \phi), \ \forall \phi \in \mathrm{Sol}_{\mathrm{L}^2}(M).$$

Proof. For $v \in C^\infty_{\mathrm{c}}(M; \mathbb{S})$, $u \in L^2(S; \mathbb{S}^*_S)$ and $\chi_S \in C^\infty_{\mathrm{c}}(S)$, we have

$$\begin{aligned}(v|\mathrm{i}\mathbb{G}(\delta_S \otimes \Gamma_S \chi_S u))_M &= (\mathrm{i}\mathbb{G}v|\delta_S \otimes \Gamma_S \chi_S u)_M \\ &= (\mathrm{i} r_S \mathbb{G}v|\chi_S u)_S = (\mathrm{i} T_S \mathbb{G}v|\chi_S u)_S \\ &= (\mathrm{i}\mathbb{G}v|T^*_S \chi_S u)_{\mathbb{D}} = (v|T^*_S \chi_S u)_M.\end{aligned}$$

In the first line we use (3.9), in the second line we use that $\mathbb{G}v \in \mathrm{Sol}_{\mathrm{sc}}(M)$ and in the third line we use that $T^*_S : L^2(S; \mathbb{S}^*_S) \to \mathrm{Sol}_{\mathrm{L}^2}(M) \subset \mathcal{D}'(M; \mathbb{S}^*)$ and the identity (3.19).

It follows that

$$\mathrm{i}\mathbb{G}(\delta_S \otimes \Gamma_S \chi_S u) = T^*_S \chi_S u, \ u \in L^2(S, \mathbb{S}^*_S). \tag{3.22}$$

Let now $\chi \in C^\infty_{\mathrm{c}}(M)$ and $\chi_S \in C^\infty_{\mathrm{c}}(S)$ such that $\chi_S \equiv 1$ on $J(\mathrm{supp}\,\chi) \cap S$. Because of the support of properties of $\mathbb{G}$ we have

$$\begin{aligned}\chi\phi &= -\chi \mathbb{G}(\delta_S \otimes \Gamma_S r_S \phi) \\ &= -\chi \mathbb{G}(\delta_S \otimes \Gamma_S \chi_S r_S \phi) = \mathrm{i}\chi T^*_S \chi_S T_S \phi, \ \phi \in \mathrm{Sol}_{\mathrm{sc}}(M).\end{aligned} \tag{3.23}$$

Observing that $\chi : \mathrm{Sol}_{\mathrm{L}^2}(M) \to L^2(M; \mathbb{S}^*)$ is bounded, the identity (3.23) extends to $\phi \in \mathrm{Sol}_{\mathrm{L}^2}(M)$ by density. □

3.4. **Action of Killing vector fields.** Let X be a complete Killing vector field on (M, g). By Lemmas A.3, A.4, $\mathcal{L}^*_X$ preserves $C^\infty(M; \mathbb{S})$ and $\mathbb{D}C^\infty_{\mathrm{c}}(M; \mathbb{S}^*)$, and $\mathcal{L}_X$ preserves $\mathrm{Sol}_{\mathrm{sc}}(M)$.

Proposition 3.5. *The operator $\mathrm{i}^{-1}\mathcal{L}_X$ with domain $\mathrm{Sol}_{\mathrm{sc}}(M)$ is essentially selfadjoint on the Hilbert space $\mathrm{Sol}_{\mathrm{L}^2}(M)$.*

Proof. Let $\phi_1, \phi_2 \in \mathrm{Sol}_{\mathrm{sc}}(M)$. Since X is Killing, using the properties of the spinorial Lie derivative listed in Appendix A, we obtain

$$X(\overline{\phi}_1 \cdot \Gamma(v)\phi_2) = \overline{\mathcal{L}_X \phi}_1 \cdot \Gamma(v)\phi_2 + \overline{\phi}_1 \cdot \Gamma(v)\mathcal{L}_X \phi_2 + \overline{\phi}_1 \cdot \Gamma(\mathcal{L}_X v)\phi_2,$$

so defining the 1-form $K(\phi_1, \phi_2) \in C^\infty(M; T^*M)$ by

$$K(\phi_1, \phi_2) \cdot v := \overline{\mathcal{L}_X \phi}_1 \cdot \Gamma(v)\phi_2 + \overline{\phi}_1 \cdot \Gamma(v)\mathcal{L}_X \phi_2,$$

we have $K(\phi_1, \phi_2) = \mathcal{L}_X J(\phi_1, \phi_2)$, where $J(\phi_1, \phi_2)$ is the conserved current defined in (3.2).

Since X is Killing we have:

$$(g^{-1}\mathcal{L}_X J) \lrcorner\, \Omega_g = \mathcal{L}_X (g^{-1} J) \lrcorner\, \Omega_g = \mathcal{L}_X (g^{-1} J \lrcorner\, \Omega_g),$$

and by Cartan's formula:

$$\mathcal{L}_X (g^{-1} J \lrcorner\, \Omega_g) = X \lrcorner\, d(g^{-1} J \lrcorner\, \Omega_g) + d(X \lrcorner\, g^{-1} J \lrcorner\, \Omega_g).$$

Since J is conserved we have $d(g^{-1} J \lrcorner\, \Omega_g) = 0$ hence:

$$g^{-1} K \lrcorner\, \Omega_g = d(X \lrcorner\, g^{-1} J \lrcorner\, \Omega_g).$$

and since $i^* d\omega = d i^* \omega$ we get:

$$\int_S i^*(g^{-1} K \lrcorner\, \Omega_g) = \int_S d i^*(X \lrcorner\, g^{-1} J \lrcorner\, \Omega_g) = 0.$$

If S is space-like this yields:

$$\int_\Sigma (\overline{\mathcal{L}_X\phi}_1 \cdot \Gamma(n)\phi_2 + \overline{\phi}_1 \cdot \Gamma(n)\mathcal{L}_X\phi_2)\, d\mathrm{vol}_h = 0,$$

i.e. $\mathrm{i}^{-1}\mathcal{L}_X$ is symmetric on $\mathrm{Sol}_{\mathrm{sc}}(M)$. The one parameter group $\{\mathrm{e}^{s\mathcal{L}_X}\}_{s\in\mathbb{R}}$ preserves $\mathrm{Sol}_{\mathrm{sc}}(M)$, is isometric and strongly continuous on $\mathrm{Sol}_{\mathrm{sc}}(M)$. Therefore, by Nelson's invariant domain theorem, it extends to a unitary group $\{U_s\}_{s\in\mathbb{R}}$ on $\mathrm{Sol}_{\mathrm{L}^2}(M)$, whose generator is the closure of $\mathrm{i}^{-1}\mathcal{L}_X$ on $\mathrm{Sol}_{\mathrm{sc}}(M)$. □

In the sequel we will also denote by $\mathrm{i}^{-1}\mathcal{L}_X$ the corresponding selfadjoint extension on $\mathrm{Sol}_{\mathrm{L}^2}(M)$.

3.5. **Use of null tetrads.** Let $(l, n, m, \overline{m})$ be a normalized null tetrad (see Subsect. 2.5) and $(o, \imath)$ the associated frame of $\mathbb{S}$. For $\phi \in C^\infty(M; \mathbb{S}^*)$ one sets then:

$$\phi_0 = \phi\cdot o, \quad \phi_1 = \phi\cdot \imath, \quad U\phi = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \in C^\infty(M; \mathbb{C}^2), \tag{3.24}$$

so that $\phi = \phi_0 o^* + \phi_1 \imath^*$ if $(o^*, \imath^*)$ is the dual frame of $\mathbb{S}^*$. If the tetrad is chosen such that $l + n$ is normal to a space-like Cauchy surface S , then from (2.11) we obtain

$$(\phi|\phi)_{\mathbb{D}} = \frac{1}{\sqrt{2}} \int_S \left(|\phi_0|^2 + |\phi_1|^2\right) d\mathrm{vol}_h. \tag{3.25}$$

3.5.1. *Action of Killing vector fields.* Let X be a complete Killing vector field, and $\mathrm{i}^{-1}\mathcal{L}_X$ its selfadjoint action on $\mathrm{Sol}_{\mathrm{L}^2}(M)$. We would like to compute $U \circ \mathcal{L}_X \circ U^{-1}$, where U is defined in (3.24). Assume that the null tetrad $(l, n, m, \overline{m})$ is invariant under X. Then by Lemma A.6 we have $\mathcal{L}_X^* o = \mathcal{L}_X^* \imath = 0$. We have $\mathcal{L}_X o^* = \mathcal{L}_X \imath^* = 0$ so

$$\mathcal{L}_X(\phi_0 o^* + \phi_1 \imath^*) = X(\phi_0) o^* + X(\phi_1) \imath^*.$$

Therefore

$$U \circ \mathcal{L}_X \circ U^{-1} = X, \tag{3.26}$$

where on the right we mean the trivial action of the vector field X on $C^\infty(M; \mathbb{C}^2)$.

4. Hadamard states for the Weyl equation

4.1. **The CAR $*$-algebra and states.** We denote by $\mathrm{CAR}(M)$ the CAR $*$-algebra associated to any of the equivalent pre-Hilbert spaces in Proposition 3.3 (see Appendix B for the definition).

We can use any of the three pre-Hilbert spaces in Proposition 3.3 to specify a quasi-free state on $\mathrm{CAR}(M)$. If one considers the first one as the starting point (which is possibly the most natural choice), then one can show that to define a quasi-free state ω it suffices to specify a pair of *spacetime covariances* (or *two-point functions* if one speaks of the associated Schwartz kernels), i.e. a pair of operators $\mathbb{\Lambda}^\pm$ satisfying:

$$\begin{aligned} &i) \quad \mathbb{\Lambda}^\pm \colon C_{\mathrm{c}}^\infty(M; \mathbb{S}) \to \mathcal{D}'(M; \mathbb{S}^*) \text{ is linear continuous,} \\ &ii) \quad \mathbb{\Lambda}^\pm \geq 0, \\ &iii) \quad \mathbb{\Lambda}^+ + \mathbb{\Lambda}^- = \mathrm{i}\mathbb{G}, \\ &iv) \quad \mathbb{D}\mathbb{\Lambda}^\pm = \mathbb{\Lambda}^\pm \mathbb{D} = 0. \end{aligned} \tag{4.1}$$

Alternatively, one can define the state ω by its *solution space covariances*, i.e. operators $C^\pm \in B(\mathrm{Sol}_{\mathrm{L}^2}(M))$ such that

$$C^\pm \geq 0, \quad C^+ + C^- = \mathbf{1}. \tag{4.2}$$

By Proposition 3.3, the two types of covariances are related as follows:

$$\overline{v}\cdot\mathbb{\Lambda}^{\pm}v = (\mathbb{G}v|C^{\pm}\mathbb{G}v)_{\mathbb{D}},\ v\in C_{\mathrm{c}}^{\infty}(M;\mathbb{S}). \tag{4.3}$$

4.2. **Hadamard states.** The definition of Hadamard states for Weyl fields is analogous to the well-known case of Dirac fields, see e.g. [39].

Definition 4.1. A quasi-free state ω on $\mathrm{CAR}(M)$ is a *Hadamard state* if it satisfies (the *Hadamard condition*):

$$\mathrm{WF}(\mathbb{\Lambda}^{\pm})'\subset\mathcal{N}^{\pm}\times\mathcal{N}^{\pm}, \tag{4.4}$$

where $\mathcal{N}^{+}$ and $\mathcal{N}^{-}$ are the two components of the characteristic set defined in (3.8).

Several equivalent definitions of the wave front set $\mathrm{WF}(\phi)$ of a distribution ϕ are recalled in Appendix E. Here we only sketch very briefly the characterization of $\mathrm{WF}(\phi)$ in terms of what we call *generalized oscillatory functions*.

Namely, in the simplest $\mathbb{R}^n$ setting, an *oscillatory function at* $q_0=(x_0,\xi_0)\in T^*\mathbb{R}^n$ is a function (strictly speaking, family of functions) on $\mathbb{R}^n$ of the form

$$w_q^{\lambda}(x)=\chi(x)\mathrm{e}^{\mathrm{i}\lambda(x-y)\cdot\eta},\ \lambda\geq 1,\ q=(y,\eta)\in T^*\mathbb{R}^n,$$

where $\chi\in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ and $\chi(x_0)\neq 0$. This can be extended to the setting of manifolds in the obvious way. Next, we say that v_q^{λ} is a *generalized oscillatory function at* $q_0\in T^*M\setminus o$ if it is of the form $v_q^{\lambda}=A^*w_q^{\lambda}$, where v_q^{λ} is an oscillatory function at q_0 and A is a properly supported pseudo-differential operator of order 0 and elliptic at q_0.

With these definitions, $q_0=(x_0,\xi_0)\notin\mathrm{WF}(\phi)$ iff there exists a generalized oscillatory test function v_q^{λ} at q_0 such that for all $N\in\mathbb{N}$,

$$|(v_q^{\lambda}|\phi)_M|\leq C_N\lambda^{-N},\ \lambda\geq 1,$$

uniformly for q in a neighborhood of (x_0,ξ_0) in $T^*M\setminus o$. If $\mathbb{\Lambda}\colon C_{\mathrm{c}}^{\infty}(M;\mathbb{S})\to\mathcal{D}'(M;\mathbb{S}^*)$ is linear and continuous, then $\mathrm{WF}(\mathbb{\Lambda})$ is by definition the wavefront set of the Schwartz kernel of $\mathbb{\Lambda}$, and $\mathrm{WF}(\mathbb{\Lambda})'$ is defined by the usual convention

$$(x,\xi,y,\eta)\in\mathrm{WF}(\mathbb{\Lambda})'\ \Leftrightarrow\ (x,-\xi,y,-\eta)\in\mathrm{WF}(\mathbb{\Lambda}).$$

The fact that the phase space for Weyl or Dirac fields is a (pre)-Hilbert space has some important consequences for Hadamard states, which we will now explain.

Let us assume that the quasi-free state ω is defined by its solution space covariances $C^{\pm}$, see Subsect. 4.1.

Lemma 4.2. *Suppose that for any $q_0\in\mathcal{N}^{\mp}$ there exists a generalized oscillatory test function v_q^{λ} at q_0 such that if $\phi_q^{\lambda}=\mathbb{G}v_q^{\lambda}$ one has*

$$\|(C^{\pm})^{\frac{1}{2}}\phi_q^{\lambda}\|_{\mathbb{D}}\leq C_N\lambda^{-N},\ \forall N\in\mathbb{N}$$

uniformly for q in a neighborhood of q_0 in $T^*M\setminus o$. Then ω is a Hadamard state.

Proof. By the same argument as for Klein-Gordon fields, see e.g. [52], it suffices to prove that $\mathrm{WF}(\mathbb{\Lambda}^{\pm})'\cap\Delta\subset\mathcal{N}^{\pm}\times\mathcal{N}^{\pm}$, where $\Delta\subset T^*M\times T^*M$ is the diagonal. If $q_0\in\mathcal{N}^{\mp}$ and v_q^{λ} are as in the lemma, we have

$$\overline{v_q}^{\lambda}\cdot\mathbb{\Lambda}^{\pm}v_q^{\lambda}=(\phi_q^{\lambda}|C^{\pm}\phi_q^{\lambda})_{\mathbb{D}}=\|(C^{\pm})^{\frac{1}{2}}\phi_q^{\lambda}\|_{\mathbb{D}}^2\in O(\lambda^{-N}),\ N\in\mathbb{N},$$

so $(q_0,q_0)\notin\mathrm{WF}(\mathbb{\Lambda}^{\pm})'$, which by the remark above implies that ω is a Hadamard state. □

Theorem 4.3. *Suppose that*

$$\mathrm{WF}((C^{\pm})^{\frac{1}{2}}\phi) \subset \mathcal{N}^{\pm} \quad \forall \phi \in \mathrm{Sol}_{\mathrm{L}^2}(M). \tag{4.5}$$

Then the state ω is a Hadamard state.

Proof. Let $q_0 \in \mathcal{N}^{\mp}$ and $N \in \mathbb{N}$. By (4.5) and Lemma E.4, there exists a generalized oscillatory test function v_q^λ at q_0 such that

$$\sup_{\lambda \geq 1} \lambda^N |(v_q^\lambda|((C^{\pm})^{\frac{1}{2}}\phi)_M| < \infty \quad \forall \phi \in \mathrm{Sol}_{\mathrm{L}^2}(M).$$

Applying the uniform boundedness principle to the family of linear forms

$$T_\lambda : \mathrm{Sol}_{\mathrm{L}^2}(M) \ni \phi \mapsto \lambda^N (v_q^\lambda|((C^{\pm})^{\frac{1}{2}}\phi)_M \in \mathbb{C}$$

we obtain that

$$\sup_{\lambda \geq 1, \|\phi\|_{\mathbb{D}}=1} \lambda^N |(v_q^\lambda|((C^{\pm})^{\frac{1}{2}}\phi)_M| < \infty. \tag{4.6}$$

Denoting $\phi_q^\lambda = \mathbb{G} v_q^\lambda$ and using also (3.19) this gives

$$\begin{aligned} \|(C^{\pm})^{\frac{1}{2}}\phi_q^\lambda\|_{\mathbb{D}} &= \sup_{\|\phi\|_{\mathbb{D}}=1} |((C^{\pm})^{\frac{1}{2}}\phi_q^\lambda|\phi)_{\mathbb{D}}| \\ &= \sup_{\|\phi\|_{\mathbb{D}}=1} |(\phi_q^\lambda|(C^{\pm})^{\frac{1}{2}}\phi)_{\mathbb{D}}| = \sup_{\|\phi\|_{\mathbb{D}}=1} |(v_q^\lambda|(C^{\pm})^{\frac{1}{2}}\phi)_M| \in O(\lambda^{-N}) \end{aligned} \tag{4.7}$$

which by Lemma 4.2 implies that ω is a Hadamard state. □

The first, obvious advantage of Theorem 4.3 is that it gives a criterion in terms of solutions rather than bi-solutions. On top of that, the merit is that we now only need to consider square-integrable solutions rather than arbitrary distributional ones. This will become crucial in the next chapters, where the use of scattering theory will force us to work in an L^2 setting.

5. The Weyl equation on Kerr spacetime

5.1. **Boyer-Lindquist blocks.** We now recall the relevant facts on Kerr black hole geometry.

We fix $a \in \mathbb{R}, M > 0$ with $0 < |a| < M$, i.e. we consider the *slowly rotating* Kerr case. One sets

$$\begin{aligned} &\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2\theta, \\ &\sigma^2 = (r^2+a^2)^2 - a^2\Delta \sin^2\theta = (r^2+a^2)\rho^2 + 2a^2 Mr \sin^2\theta, \end{aligned}$$

and $r_\pm = M \pm \sqrt{M^2 - a^2}$ for the two roots of Δ as a function of r.

The Boyer-Lindquist blocks are the manifolds $(\mathrm{M_I}, g)$, $(\mathrm{M_{II}}, g)$, where

$$\begin{aligned} \mathrm{M_I} &= \mathbb{R}_t \times]r_+, +\infty[_r \times \mathbb{S}^2_{\theta,\varphi}, \\ \mathrm{M_{II}} &= \mathbb{R}_t \times]r_-, r_+[_r \times \mathbb{S}^2_{\theta,\varphi}, \end{aligned}$$

$\theta \in [0,\pi]$, $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ are the spherical coordinates on $\mathbb{S}^2$, and

$$g = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4aMr\sin^2\theta}{\rho^2}\, dt d\varphi + \frac{\rho^2}{\Delta}\, dr^2 + \rho^2 d\theta^2 + \frac{\sigma^2}{\rho^2}\sin^2\theta\, d\varphi^2.$$

(We will not need to consider the third Boyer-Lindquist block M_{III} corresponding to $r \in]-\infty, r_-[$). The global coordinates (t, r, θ, φ) on $\mathrm{M_I}, \mathrm{M_{II}}$ are called *Boyer-Lindquist coordinates*. We have $\det g = -\rho^4 \sin^2\theta$ and the inverse metric g^{-1} has components:

$$\begin{aligned} g^{tt} &= -\frac{\sigma^2}{\Delta\rho^2}, \quad g^{rr} = \frac{\Delta}{\rho^2}, \\ g^{t\varphi} &= -\frac{2aMr}{\Delta\rho^2}, \quad g^{\varphi\varphi} = \frac{\Delta - a^2\sin^2\theta}{\Delta\rho^2\sin^2\theta}, \quad g^{\theta\theta} = \frac{1}{\rho^2}, \end{aligned} \tag{5.1}$$

all other being equal to 0. Since $g^{tt} < 0$ in $\mathrm{M_I}$, the vector field ∇t is time-like and one fixes the time orientation of $(\mathrm{M_I}, g)$ by declaring $-\nabla t$ to be future directed. By convention the time orientation of $\mathrm{M_{II}}$ is the one inherited from its embedding into K^*, see 5.3.2 below.

The following fact appears to be folklore knowledge. For lack of a reference that show the precise statement, a proof is given in Proposition C.8.

Proposition 5.1. $(\mathrm{M_I}, g)$ *is globally hyperbolic.*

5.2. **Time reversal.** When considering the gluing of various blocks into a larger spacetime it is useful to introduce the following notation.

If (M, g) is a spacetime we denote by (M', g) the same Lorentzian manifold with the opposite time orientation. So, $\mathrm{id} : (M, g) \to (M', g)$ is an isometric involution reversing the time orientation. If (M, g) admits a spinor bundle $\mathcal{S} \xrightarrow{\pi} M$, then so does (M', g). We will generally decorate with primes the objects associated to (M', g).

Consequences of the time reversal on the level of classical and quantized Weyl fields are discussed in Section B.2 in the appendix.

5.3. **The** K^* **and** ${}^*\mathrm{K}$ **spacetimes.** We now recall the *Kerr-star* and *star-Kerr* coordinates, which allow to glue together $\mathrm{M_I}$ and $\mathrm{M_{II}}$ along parts of $\{r = r_+\}$. As pre-announced, we will remove from K^* and ${}^*\mathrm{K}$ the parts corresponding to the third Boyer-Lindquist block $\mathrm{M_{III}}$.

5.3.1. *The* K^* *spacetime.* We set

$$\mathrm{K}^* = \mathbb{R}_{t^*} \times]r_-, +\infty[_r \times \mathbb{S}^2_{\theta,\varphi^*},$$

equipped with the metric

$$g = g_{tt}dt^{*2} + 2g_{t\varphi}dt^*d\varphi^* + g_{\varphi\varphi}d\varphi^{*2} + 2dt^*dr - 2a\sin^2\theta d\varphi^* dr + \rho^2 d\theta^2.$$

The global coordinates $(t^*, r, \theta, \varphi^*)$ are called *Kerr-star coordinates*.

We will denote by $\partial_{r^*}, \partial_{\theta^*}$, the coordinate vector fields $\partial_r, \partial_\theta$ in Kerr-star coordinates. We time orient K^* by declaring that the null vector $-\partial_{r^*}$ is future directed. We have $\det g = -\rho^4 \sin^2\theta$. The inverse metric g^{-1} has components:

$$\begin{aligned} g^{t^*t^*} &= \frac{a^2\sin^2\theta}{\rho^2}, \quad g^{t^*r^*} = \frac{r^2+a^2}{\rho^2}, \quad g^{r^*r^*} = \frac{\Delta}{\rho^2}, \\ g^{t^*\varphi^*} &= \frac{a}{\rho^2}, \quad g^{r^*\varphi^*} = \frac{a}{\rho^2}, \\ g^{\varphi^*\varphi^*} &= \frac{1}{\rho^2\sin^2\theta}, \quad g^{\theta^*\theta^*} = \frac{1}{\rho^2}, \end{aligned} \tag{5.2}$$

all other being equal to 0.

5.3.2. *Embedding* $\mathrm{M_I}$ *and* $\mathrm{M_{II}}$ *into* K^*. Let $x(r)$ and $\Lambda(r)$ for $r \in]r_-, r_+[\cup]r_+, +\infty[$ be such that

$$\frac{dx}{dr} = \frac{r^2+a^2}{\Delta}, \quad \frac{d\Lambda}{dr} = \frac{a}{\Delta}. \tag{5.3}$$

We choose, see [48, Lem. 3.4.2]:

$$\begin{aligned} x(r) &= r + \tfrac{1}{2\kappa_+}\ln(|r-r_+|) + \tfrac{1}{2\kappa_-}\ln(|r-r_-|), \\ \Lambda(r) &= \tfrac{a}{r_+-r_-}\ln(\frac{|r-r_+|}{|r-r_-|}), \end{aligned} \tag{5.4}$$

where

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2(r_\pm^2+a^2)}, \tag{5.5}$$

so that $\kappa_- < 0 < \kappa_+$ and $\kappa_+\kappa_-^{-1} = -r_-r_+^{-1}$.

We define the map $j^* : \mathrm{M_I} \cup \mathrm{M_{II}} \to \mathrm{K}^*$ by:

$$t^* \circ j^* = t + x(r), \quad r \circ j^* = r, \quad \theta \circ j^* = \theta, \quad \varphi^* \circ j^* = \varphi + \Lambda(r),$$

which allows to identify isometrically $(\mathrm{M_I}, g)$ resp. $(\mathrm{M_{II}}, g)$ with $(\mathrm{M_I^*}, g)$, resp. $(\mathrm{M_{II}^*}, g)$, where

$$\mathrm{M_I^*} = \mathrm{K}^* \cap \{r_+ < r\}, \quad \mathrm{M_{II}^*} = \mathrm{K}^* \cap \{r_- < r < r_+\}.$$

Using j^* one can glue $\mathrm{M_I}$ with $\mathrm{M_{II}}$ inside K^* along the *future horizon*:

$$\mathscr{H}_+ = \mathbb{R}_{t^*} \times \{r = r_+\} \times \mathbb{S}^2_{\theta,\varphi}.$$

The embedding of $\mathrm{M_I}$ and $\mathrm{M_{II}}$ into K^* respects the time orientation. For coherence of notation we will use the following definition.

Definition 5.2. We set

$$\mathrm{M_{I\cup II}} := (j^*)^{-1}(\mathrm{K}^*) = \mathrm{M_I} \cup \mathrm{M_{II}} \cup \mathscr{H}_+,$$

with the spacetime structure inherited from K^*.

The following fact will be checked in Appendix C, see Proposition C.11.

Proposition 5.3. $(\mathrm{M_{I\cup II}}, g)$ *is globally hyperbolic.*

5.3.3. *The* ${}^*\mathrm{K}$ *spacetime.* Similarly we set

$${}^*\mathrm{K} = \mathbb{R}_{{}^*t} \times]r_-, +\infty[_r \times \mathbb{S}^2_{\theta,{}^*\varphi},$$

equipped with the metric

$$g = g_{tt}d{}^*t^2 + 2g_{t\varphi}d{}^*t d{}^*\varphi + g_{\varphi\varphi}d{}^*\varphi^2 - 2d{}^*t dr + 2a\sin^2\theta d{}^*\varphi dr + \rho^2 d\theta^2.$$

The global coordinates $({}^*t, r, \theta, {}^*\varphi)$ are called *star-Kerr coordinates* and as before ∂_{*r}, $\partial_{*\theta}$ denote the coordinate vector fields ∂_r, ∂_θ in star-Kerr coordinates. We time orient ${}^*\mathrm{K}$ by declaring that the null vector ∂_{*r} is future directed. We have $\det g = -\rho^4 \sin^2\theta$.

5.3.4. *Embedding* $\mathrm{M_I}$ *and* $\mathrm{M_{II}}$ *into* ${}^*\mathrm{K}$. Again, the map ${}^*j : \mathrm{M_I} \cup \mathrm{M_{II}} \to \mathrm{K}^*$ defined by:

$$ {}^*t \circ {}^*j = t - x(r), \quad r \circ {}^*j = r, \quad \theta \circ {}^*j = \theta, \quad {}^*\varphi \circ {}^*j = \varphi - \Lambda(r), $$

allows to identify isometrically $(\mathrm{M_I}, g)$ resp. $(\mathrm{M_{II}}, g)$ with $({}^*\mathrm{M_I}, g)$, resp. $({}^*\mathrm{M_{II}}, g)$, where

$$ {}^*\mathrm{M_I} = {}^*\mathrm{K} \cap \{r_+ < r\}, \quad {}^*\mathrm{M_{II}} = {}^*\mathrm{K} \cap \{r_- < r < r_+\}. $$

Using ${}^*j$ one can glue $\mathrm{M_I}$ with $\mathrm{M_{II}}$ inside ${}^*\mathrm{K}$ along the *past horizon*:

$$ \mathscr{H}_- = \mathbb{R}_{{}^*t} \times \{r = r_+\} \times \mathbb{S}^2_{\theta, {}^*\varphi}. \tag{5.6} $$

The embedding of $\mathrm{M_I}$ (resp. $\mathrm{M_{II}}$) into ${}^*\mathrm{K}$ respects (resp. reverses) the time orientation, see [48, Lem. 3.1.3].

In the original Boyer–Lindquist coordinates, the future and past horizons $\mathscr{H}_+$ and $\mathscr{H}_-$ are reached at infinite values of t.

5.4. **Conformal extension of** $\mathrm{M_I}$**.** The Penrose conformal extension of $\mathrm{M_I}$ is obtained by setting $\hat{g} = w^2 g$, where $w = r^{-1} \in \left]0, r_+^{-1}\right[$. The metric $\hat{g}$ expressed in coordinates $(t^*, w, \theta, \varphi^*)$ equals , see e.g. [34, (8.36)]:

$$ \begin{aligned} \hat{g} = &-\Big(w^2 - \frac{2Mw^3}{1 + a^2 w^2 \cos^2\theta}\Big) dt^{*2} - \frac{4Maw^3 \sin^2\theta}{1 + a^2 w^2 \cos^2\theta} dt^* d\varphi^* \\ &+ \Big(1 + a^2 w^2 + \frac{2Ma^2 w^3 \sin^2\theta}{1 + a^2 w^2 \cos^2\theta}\Big) \sin^2\theta d\varphi^{*2} \\ &+ \big(1 + a^2 w^2 \cos^2\theta\big) d\theta^2 - 2dt^* dw + 2a \sin^2\theta d\varphi^* dw. \end{aligned} $$

We have $\det \hat{g} = -(1 + a^2 w^2 \cos^2\theta)^2 \sin^2\theta$, so $\hat{g}$ extends as a smooth, non-degenerate metric to $w \in \left]-\infty, r_+^{-1}\right[$. Clearly we can fix $\epsilon_0 > 0$ small enough so that $\hat{g}$ is Lorentzian for $w \in \left]-\epsilon_0, r_+^{-1}\right[$. We define the conformal extension of $\mathrm{M_I}$ as

$$ \hat{\mathrm{M}}_\mathrm{I} := \mathbb{R}_{t^*} \times \left]-\epsilon_0, r_+^{-1}\right]_w \times \mathbb{S}^2_{\theta, \varphi^*}. $$

Past null infinity is then the null hypersurface

$$ \mathscr{I}_- = \mathbb{R}_{t^*} \times \{w = 0\} \times \mathbb{S}^2_{\theta, \varphi^*}. \tag{5.7} $$

In an analogous way, extending $\hat{g}$ expressed in coordinates $({}^*t, w, \theta, {}^*\varphi)$ to

$$ \mathbb{R}_{{}^*t} \times \left]-\epsilon_0, r_+^{-1}\right[_w \times \mathbb{S}^2_{\theta {}^*\varphi} $$

allows to define *future null infinity* $\mathscr{I}_+ = \mathbb{R}_{{}^*t} \times \{w = 0\} \times \mathbb{S}^2_{\theta {}^*\varphi}$, see Figure 2 for the conformal diagram.

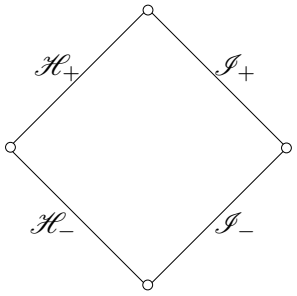

FIGURE 2. The conformal extensions of $\mathrm{M_I}$.

The time orientations of the conformal extensions of $\mathrm{M_I}$ are inherited from the time orientation of $\mathrm{M_I}$.

5.5. **The Kerr-Kruskal extension.** Recall that $\kappa_\pm$ was defined in (5.5). The *Kerr-Kruskal extension* M is the manifold (M, g) with:

$$\mathrm{M} = \mathbb{R}_U \times \mathbb{R}_V \times \mathbb{S}^2_{\theta,\varphi^\#}.$$

The global coordinates $(U, V, \theta, \varphi^\#)$ on M are called *KBL coordinates*. One defines r as a smooth function on M by the equation:

$$\frac{r - r_+}{UV} = G(r), \text{ for } G(r) = \mathrm{e}^{-2\kappa_+ r}(r - r_-)^{r_-/r_+},$$

see [48, Lem. 3.4.9]. M is then equipped with the metric [48, Prop. 3.5.3]:

$$\begin{aligned} g = {} & \frac{G^2(r)a^2\sin^2\theta}{4\kappa_+^2\rho^2}\frac{(r-r_-)(r+r_+)}{(r^2+a^2)(r_+^2+a^2)}\Big(\frac{\rho^2}{r^2+a^2}+\frac{\rho_+^2}{r_+^2+a^2}\Big)(U^2dV^2+V^2dU^2) \\ & + \frac{G(r)(r-r_-)}{2\kappa_+^2\rho^2}\Big(\frac{\rho^4}{(r^2+a^2)^2}+\frac{\rho_+^4}{(r_+^2+a^2)^2}\Big)dUdV \\ & + \frac{G(r)a\sin^2\theta}{\kappa_+^2\rho^2(r_+^2+a^2)}\big(\rho_+^2(r-r_-)+(r^2+a^2)(r+r_+)\big)(UdV-VdU)d\varphi^\sharp \\ & + \rho^2 d\theta^2 + \Big(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\rho^2}\Big)\sin^2\theta d\varphi^{\sharp 2}, \end{aligned} \tag{5.8}$$

where $\rho_+ = \rho(r_+, \theta)$. Again, the following result is proved in Proposition C.12.

Proposition 5.4. (M, g) *is globally hyperbolic.*

5.5.1. *Null tetrad on* M. One sets:

$$\begin{aligned} \mathfrak{l} &= \frac{\mathrm{e}^{-\kappa_+ r}(r-r_-)^{M/r_+}}{(r-r_-)\sqrt{2\rho^2}}\left(\frac{2\kappa_+(r^2+a^2)}{G(r)}\partial_V - \frac{a(r+r_+)}{r_+^2+a^2}U\partial_{\varphi^\#}\right), \\ \mathfrak{n} &= \frac{\mathrm{e}^{-\kappa_+ r}(r-r_-)^{M/r_+}}{(r-r_-)\sqrt{2\rho^2}}\left(-\frac{2\kappa_+(r^2+a^2)}{G(r)}\partial_U - \frac{a(r+r_+)}{r_+^2+a^2}V\partial_{\varphi^\#}\right), \\ m &= \frac{1}{\sqrt{2}p}\left(\mathrm{i}a\sin\theta\partial_t + \partial_\theta + \frac{\mathrm{i}}{\sin\theta}\partial_\varphi\right), \end{aligned} \tag{5.9}$$

for $p = r + \mathrm{i}a\cos\theta$. Then $(\mathfrak{l}, \mathfrak{n}, m, \overline{m})$ is a global null tetrad on M. We time orient (M, g) by saying that $\mathfrak{l} + \mathfrak{n}$ is future directed.

The submanifolds $\mathscr{H}_R := \{U = 0\}$, $\mathscr{H}_L := \{V = 0\}$ are called the *long horizons*, and they intersect at the *crossing sphere* $S(r_+) := \{U = V = 0\}$.

The following changes of coordinates allow to embed isometrically $\mathrm{M_I}, \mathrm{M_{II}}, \mathrm{M'_I}, \mathrm{M'_{II}}$ into M, respecting the time orientations:

$$\begin{aligned} & U = \mathrm{e}^{-\kappa_+ {}^*t},\ V = \mathrm{e}^{\kappa_+ t^*}, \text{ on } \mathrm{M_I}, \\ & U = -\mathrm{e}^{-\kappa_+ {}^*t},\ V = \mathrm{e}^{\kappa_+ t^*} \text{ on } \mathrm{M_{II}}, \\ & U = -\mathrm{e}^{-\kappa_+ {}^*t},\ V = -\mathrm{e}^{\kappa_+ t^*} \text{ on } \mathrm{M'_I}, \\ & U = \mathrm{e}^{-\kappa_+ {}^*t},\ V = -\mathrm{e}^{\kappa_+ t^*} \text{ on } \mathrm{M'_{II}}, \\ & \varphi^\sharp = \tfrac{1}{2}(\varphi^* + {}^*\varphi - \tfrac{a}{r_+^2+a^2}(t^* + {}^*t)) = \varphi - \tfrac{a}{r_+^2+a^2}t. \end{aligned} \tag{5.10}$$

We recall that if M is a spacetime, M' stands for the Lorentzian manifold M with the reversed time orientation. By slightly altering this rule we denote by

$$\begin{aligned} & \mathrm{M_I} = \{U > 0,\ V > 0\}, \quad \mathrm{M_{II}} = \{U < 0,\ V > 0\}, \\ & \mathrm{M_{I'}} = \{U < 0,\ V < 0\}, \quad \mathrm{M_{II'}} = \{U > 0,\ V < 0\}, \end{aligned}$$

the four quadrants of $\mathrm{M} \setminus (\mathscr{H}_L \cup \mathscr{H}_R)$.

5.5.2. *The wedge reflection.* The map

$$R: \begin{array}{l} \mathrm{M} \to \mathrm{M} \\ (U, V, \theta, \varphi^{\#}) \mapsto (-U, -V, \theta, \varphi^{\#}) \end{array} \tag{5.11}$$

is called the *wedge reflection*. It preserves the orientation, it reverses the time orientation and it gives an identification of M'_{I} and $\mathrm{M}_{\mathrm{I}'}$ (resp. of $\mathrm{M}'_{\mathrm{II}}$ and $\mathrm{M}_{\mathrm{II}'}$) as spacetimes.

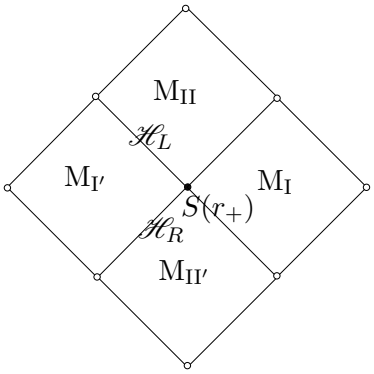

FIGURE 3. The Kerr–Kruskal spacetime M.

5.6. **The Weyl equation on the Kerr–Kruskal spacetime.** From the global null tetrad $(\mathfrak{l}, \mathfrak{n}, m, \overline{m})$ defined in (5.9), we obtain a global orthonormal frame as in Subsect. 2.5 and hence a spinor bundle as in 2.2.1. We denote by $\mathbb{D}$ the associated Weyl operator and by $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M})$ the space of L^2 solutions as in Subsect. 3.2.

The restriction of $\mathbb{D}$ on M_{I}, $\mathrm{M}_{\mathrm{I}'}$ etc will be denoted by the same letter, and the corresponding spaces of solutions by $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}_{\mathrm{I}})$, $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}_{\mathrm{I}'})$, etc.

5.6.1. *Killing vector fields.* An important role is played by the two Killing vector fields:

$$v_{\mathscr{I}} = \partial_t = \kappa_+(-U\partial_U + V\partial_V) - \Omega_{\mathscr{H}}\partial_{\varphi^{\#}},$$
$$v_{\mathscr{H}} = \partial_t + \Omega_{\mathscr{H}}\partial_\varphi = \kappa_+(-U\partial_U + V\partial_V),$$

where $\Omega_{\mathscr{H}} = \frac{a}{r_+^2 + a^2}$ is the *angular velocity* of the horizon. The vector field $v_{\mathscr{H}}$ is tangent to $\mathscr{H}_-$ (in ${}^*\mathrm{K}$), while $v_{\mathscr{I}}$ is tangent to $\mathscr{I}_-$ (in the conformal extension $\hat{\mathrm{M}}_{\mathrm{I}}$ of M_{I}).

The self-adjoint generators of their unitary actions on $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}_{\mathrm{I}})$ will be denoted by $\mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}}$ and $\mathrm{i}^{-1}\mathcal{L}_{\mathscr{I}}$ (so that $\mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}} = \mathrm{i}^{-1}\mathcal{L}_{v_{\mathscr{H}}}$ and $\mathrm{i}^{-1}\mathcal{L}_{\mathscr{I}} = \mathrm{i}^{-1}\mathcal{L}_{v_{\mathscr{I}}}$ in the notation of Prop. 3.5), and we also set $\mathrm{i}^{-1}\mathcal{L}_{\varphi} := \mathrm{i}^{-1}\mathcal{L}_{\partial_\varphi}$.

6. Evolutionary form of the Weyl equation in M_{I}

6.1. **The HN reduction.** Following [34], we now explain how the Weyl equation $\mathbb{D}\phi = 0$ can be reduced to an equation of the form $\partial_t \Psi - \mathrm{i}H\Psi = 0$ with H a t-independent differential operator.

In [34] two related null tetrads are used over $\mathrm{M_I}$. The basic null tetrad is given by:

$$\begin{aligned} l &= \frac{1}{\sqrt{2\Delta\rho^2}}\left((r^2+a^2)\partial_t + \Delta\partial_r + a\partial_\varphi\right), \\ n &= \frac{1}{\sqrt{2\Delta\rho^2}}\left((r^2+a^2)\partial_t - \Delta\partial_r + a\partial_\varphi\right), \\ m &= \frac{1}{\sqrt{2}p}\left(\mathrm{i}a\sin\theta\partial_t + \partial_\theta + \frac{\mathrm{i}}{\sin\theta}\partial_\varphi\right), \end{aligned} \tag{6.1}$$

for $p = r + \mathrm{i}a\cos\theta$. As in Subsect. 3.5 we set

$$\phi_0 = \phi\cdot o, \quad \phi_1 = \phi\cdot\imath.$$

Another null tetrad $(\mathbf{l},\mathbf{n},\mathbf{m},\overline{\mathbf{m}})$ adapted to the foliation of $\mathrm{M_I}$ by the hypersurfaces $\Sigma_s = \{t = s\}$ is used for the scattering theory arguments, with the property that

$$\mathbf{l} + \mathbf{n} = 2\mathbf{t},$$

where $\mathbf{t}$ is the future directed unit normal vector field to this foliation. Concretely we have, see [34, 2.5.1]:

$$\begin{aligned} \mathbf{l} &= \frac{\sigma}{\sqrt{2\Delta\rho^2}}\left(\partial_t + \frac{2aMr}{\sigma^2}\partial_\varphi\right) + \sqrt{\frac{\Delta}{2\rho^2}}\partial_r, \\ \mathbf{n} &= \frac{\sigma}{\sqrt{2\Delta\rho^2}}\left(\partial_t + \frac{2aMr}{\sigma^2}\partial_\varphi\right) - \sqrt{\frac{\Delta}{2\rho^2}}\partial_r, \\ \mathbf{m} &= \frac{1}{\sqrt{2\rho^2}}\left(\partial_\theta + \frac{\mathrm{i}\rho^2}{\sigma\sin\theta}\partial_\varphi\right). \end{aligned} \tag{6.2}$$

If $(\mathbf{o},\mathbf{i})$ is the spinor basis associated to $(\mathbf{l},\mathbf{n},\mathbf{m},\overline{\mathbf{m}})$, one sets then:

$$\Psi := \left(\frac{\Delta\rho^2\sigma^2}{(r^2+a^2)^2}\right)^{\frac{1}{4}}\begin{pmatrix}\phi\cdot\mathbf{o}\\ \phi\cdot\mathbf{i}\end{pmatrix} = \begin{pmatrix}\Psi_0\\ \Psi_1\end{pmatrix} =: \mathcal{V}\phi, \tag{6.3}$$

and uses the coordinates (t,x,θ,φ) where $x(r)$ is defined in (5.3). Let us define the matrix $\mathbf{U}\in\mathbf{M_2}(\mathbb{C})$ corresponding to the above change of spinor basis by

$$\begin{pmatrix}\phi\cdot\mathbf{o}\\ \phi\cdot\mathbf{i}\end{pmatrix} = \mathbf{U}\begin{pmatrix}\phi\cdot o\\ \phi\cdot\imath\end{pmatrix}. \tag{6.4}$$

We refer the reader to [34, (2.50)] for the concrete expression of $\mathbf{U}$. We have:

$$\begin{pmatrix}\Psi_0\\ \Psi_1\end{pmatrix} = \left(\frac{\Delta\rho^2\sigma^2}{(r^2+a^2)^2}\right)^{\frac{1}{4}}\mathbf{U}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}. \tag{6.5}$$

Finally, one sets

$$D = \mathcal{V}\mathbb{D}\mathcal{V}^{-1},$$

and

$$(\Psi|\Psi)_D = \frac{1}{\sqrt{2}}\int_\Sigma \left(|\Psi_0|^2 + |\Psi_1|^2\right) dx d^2\omega. \tag{6.6}$$

The equation $D\Psi = 0$ can be rewritten as

$$\partial_t\Psi - \mathrm{i}H\Psi = 0,$$

which we will call the *reduced Weyl equation*. A concrete expression for the t-independent differential operator H can be found in [34, (2.56)] (the operator H is denoted by $\not{D}_K$ therein).

We denote by $Sol_{L^2}(\mathrm{M_I})$ the closure of the space $Sol_{\mathrm{sc}}(\mathrm{M_I})$ of space-compact solutions of $D\Psi = 0$ for the scalar product $(\cdot|\cdot)_D$ defined in (6.6). Using (3.25) and the identity

$$d\mathrm{vol}_\Sigma = \sqrt{\frac{\sigma^2\rho^2\Delta}{r^2+a^2}}\, dx d^2\omega,$$

we obtain that:

$$\mathcal{V} : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}) \xrightarrow{\sim} Sol_{L^2}(\mathrm{M_I}) \text{ is unitary.} \tag{6.7}$$

6.1.1. *Cauchy evolution.* Let $\mathcal{H}$ be the Hilbert space associated to $(\Psi|\Psi)_D$. We set

$$\rho_\Sigma : Sol_{L^2}(\mathrm{M_I}) \ni \psi \mapsto \psi(0) \in \mathcal{H}$$

and denote by $\Psi = U_\Sigma f$ the unique solution of the Cauchy problem

$$\begin{cases} D\Psi = 0, \\ \rho_\Sigma \Psi = f \in \mathcal{H}. \end{cases}$$

6.1.2. *Killing vector fields.* The Killing vector fields $v_{\mathscr{H}_-/\mathscr{I}_-}$ preserve $(\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}})$ and $v_{\mathscr{H}/\mathscr{I}} \left(\frac{\Delta\rho^2\sigma^2}{(r^2+a^2)^2}\right)^{\frac{1}{4}} = 0$, hence by (3.26) we have, in the sense of unitary equivalence of selfadjoint operators:

$$\mathcal{V} \circ \mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}_-/\mathscr{I}_-} \circ \mathcal{V}^{-1} = \mathrm{i}^{-1} v_{\mathscr{H}/\mathscr{I}}, \tag{6.8}$$

where we also denote by $v_{\mathscr{H}/\mathscr{I}}$ the first order differential operators associated to $v_{\mathscr{H}_-/\mathscr{I}_-}$, acting on $Sol_{L^2}(\mathrm{M_I})$.

Furthermore, we have:

$$\mathrm{i}^{-1} v_{\mathscr{H}} \circ U_\Sigma = U_\Sigma \circ (H + \Omega_{\mathscr{H}} \mathrm{i}^{-1}\partial_\varphi), \quad \mathrm{i}^{-1} v_{\mathscr{I}} \circ U_\Sigma = U_\Sigma \circ H. \tag{6.9}$$

6.2. Decomposition of $Sol_{L^2}(\mathrm{M_I})$.

6.2.1. *Asymptotic velocity.* The first important result of [34] is the existence of the *asymptotic velocity* for solutions of the reduced Weyl equation. For the sake of brevity we will consider only the past asymptotics $t \to -\infty$. The following result is proved in [34, Thm. 2.1] (see also [35] for an introductory account).

Theorem 6.1. *There exists a selfadjoint operator $v \in B(\mathcal{H})$, called the* past asymptotic velocity *such that:*

$$\chi(v) = \mathrm{s}\text{-}\lim_{t\to-\infty} \mathrm{e}^{-\mathrm{i}tH} \chi\left(\frac{x}{t}\right) \mathrm{e}^{\mathrm{i}tH}, \quad \forall \chi \in C_{\mathrm{c}}^\infty(\mathbb{R}).$$

The spectrum of v is $\mathrm{sp}(v) = \{-1, 1\}$, *and*

$$[v, H] = [v, \partial_\varphi] = 0. \tag{6.10}$$

Definition 6.2. We set

$$\pi_{\mathscr{H}_-} := \mathbf{1}_{\{1\}}(v), \quad \pi_{\mathscr{I}_-} := \mathbf{1}_{\{-1\}}(v).$$

The orthogonal decomposition $\mathbf{1} = \pi_{\mathscr{H}_-} + \pi_{\mathscr{I}_-}$ allows to split an initial data $f \in \mathcal{H}$ as $f_{\mathscr{H}_-} + f_{\mathscr{I}_-}$ with $\mathrm{e}^{\mathrm{i}tH} f_{\mathscr{H}_-}$ resp. $\mathrm{e}^{\mathrm{i}tH} f_{\mathscr{I}_-}$ moving towards $\mathscr{H}_-$, resp. $\mathscr{I}_-$ when $t \to -\infty$.

It follows that we can decompose $Sol_{L^2}(\mathrm{M_I})$ as an orthogonal direct sum:

$$Sol_{L^2}(\mathrm{M_I}) = Sol_{L^2,\mathscr{H}_-}(\mathrm{M_I}) \oplus Sol_{L^2,\mathscr{I}_-}(\mathrm{M_I}),$$

where

$$Sol_{L^2,\mathscr{H}_-/\mathscr{I}_-}(\mathrm{M_I}) := U_\Sigma \circ \pi_{\mathscr{H}_-/\mathscr{I}_-} \mathcal{H}$$

correspond to solutions going entirely through $\mathscr{H}_-$, resp. $\mathscr{I}_-$ in the past. We will denote by

$$\Pi_{\mathscr{H}_-/\mathscr{I}_-} = U_\Sigma \circ \pi_{\mathscr{H}_-/\mathscr{I}_-} \circ \rho_\Sigma \tag{6.11}$$

the orthogonal projections on $Sol_{L^2,\mathscr{H}_-/\mathscr{I}_-}(\mathrm{M_I})$.

6.2.2. *Killing vector fields.* By (6.10), we know that $\pi_{\mathscr{H}_-/\mathscr{I}_-}$ commute with H and ∂_φ hence by (6.9) $\mathrm{i}^{-1} v_{\mathscr{H}}$, resp. $\mathrm{i}^{-1} v_{\mathscr{I}}$ preserves $Sol_{L^2,\mathscr{H}_-}(\mathrm{M_I})$ resp. $Sol_{L^2,\mathscr{I}_-}(\mathrm{M_I})$.

6.3. **Traces at infinities.** Another important fact proved in [34] is the existence of traces of solutions on $\mathscr{H}_-$ and $\mathscr{I}_-$ and their relationship with the spaces $Sol_{L^2,\mathscr{H}_-/\mathscr{I}_-}(\mathrm{M_I})$.

6.3.1. *Traces on $\mathscr{H}_-$.* For $\Psi \in Sol_{\mathrm{sc}}(\mathrm{M_I})$, the trace

$$T_{\mathscr{H}_-}\Psi := \Psi_{1|\mathscr{H}_-} \in C^\infty(\mathscr{H}_-; \mathbb{C})$$

is well defined, see [34, Subsect. 8.2]. The following result is shown in [34, Thm. 8.2].

Proposition 6.3. *$T_{\mathscr{H}_-}$ uniquely extends as a bounded operator*

$$T_{\mathscr{H}_-} : Sol_{L^2}(\mathrm{M_I}) \to L^2(\mathscr{H}_-, \mathrm{dvol}_{\mathscr{H}_-}),$$

where $\mathscr{H}_-$ is identified with $\mathbb{R}_{*t} \times \mathbb{S}^2_{\theta,*\varphi}$, see (5.6) and $\mathrm{dvol}_{\mathscr{H}_-} = \sin\theta d^*t d\theta d^*\varphi$. One has:

$$\begin{aligned} &\mathrm{Ker}\, T_{\mathscr{H}_-} = Sol_{L^2,\mathscr{I}_-}(\mathrm{M_I}), \quad \mathrm{Ran}\, T_{\mathscr{H}_-} = L^2(\mathscr{H}_-, \mathrm{dvol}_{\mathscr{H}_-}), \\ &(\Psi|\Psi)_D = \tfrac{1}{\sqrt{2}} \textstyle\int_{\mathscr{H}_-} |T_{\mathscr{H}_-}\Psi|^2 \, \mathrm{dvol}_{\mathscr{H}_-}, \quad \Psi \in Sol_{L^2,\mathscr{H}_-}(\mathrm{M_I}). \end{aligned} \tag{6.12}$$

Note that at $\mathscr{H}_-$, only the Ψ_1 component is relevant for the trace, whereas at $\mathscr{I}_-$ only the Ψ_0 component is relevant.

6.3.2. *Killing vector field on $\mathscr{H}_-$.* The Killing vector field $v_{\mathscr{H}}$ is null and tangent to $\mathscr{H}$. On $\mathscr{H}_-$ it equals $\partial_{*t} + \Omega_{\mathscr{H}} \partial_{*\varphi}$, resp. $-\kappa_+ U \partial_U$ in star-Kerr, resp. KBL coordinates.

If we also denote by $\mathrm{i}^{-1} v_{\mathscr{H}}$ its selfadjoint realization on $L^2(\mathscr{H}_-, \mathrm{dvol}_{\mathscr{H}_-})$ we have:

$$T_{\mathscr{H}_-} \circ (\mathrm{i}^{-1} v_{\mathscr{H}}) = (\mathrm{i}^{-1} v_{\mathscr{H}}) \circ T_{\mathscr{H}_-} \text{ on } Sol_{L^2,\mathscr{H}_-}(\mathrm{M_I}).$$

6.3.3. *Traces on $\mathscr{I}_-$.* For $\Psi \in Sol_{\mathrm{sc}}(\mathrm{M_I})$, the trace

$$T_{\mathscr{I}_-}\Psi := \Psi_{0|\mathscr{I}_-} \in C^\infty(\mathscr{I}_-; \mathbb{C})$$

is well defined, see [34, Subsect. 8.3]. The following result is shown in [34, Thm. 8.3].

Proposition 6.4. *$T_{\mathscr{I}_-}$ uniquely extends as a bounded operator*

$$T_{\mathscr{I}_-} : Sol_{L^2}(\mathrm{M_I}) \to L^2(\mathscr{I}_-, \mathrm{dvol}_{\mathscr{I}_-}),$$

where $\mathscr{I}_-$ is identified with $\mathbb{R}_{t^*} \times \mathbb{S}^2_{\theta,\varphi^*}$ and $\mathrm{dvol}_{\mathscr{I}_-} = \sin\theta \, dt^* d\theta d\varphi^*$. One has

$$\begin{aligned} &\mathrm{Ker}\, T_{\mathscr{I}_-} = Sol_{L^2,\mathscr{H}_-}(\mathrm{M_I}), \quad \mathrm{Ran}\, T_{\mathscr{I}_-} = L^2(\mathscr{I}_-, \mathrm{dvol}_{\mathscr{I}_-}) \\ &(\Psi|\Psi)_D = \tfrac{1}{\sqrt{2}} \textstyle\int_{\mathscr{I}_-} |T_{\mathscr{I}_-}\Psi|^2 \, \mathrm{dvol}_{\mathscr{I}_-}, \quad \Psi \in Sol_{L^2,\mathscr{I}_-}(\mathrm{M_I}). \end{aligned} \tag{6.13}$$

6.3.4. *Killing vector field on $\mathscr{I}_-$.* The Killing vector field $v_{\mathscr{I}}$ is null and tangent to $\mathscr{I}$. On $\mathscr{I}_-$ it equals ∂_{t^*} in Kerr-star coordinates. If we also denote by $\mathrm{i}^{-1}v_{\mathscr{I}}$ its selfadjoint realization on $L^2(\mathscr{I}_-, d\mathrm{vol}_{\mathscr{I}_-})$, we have:

$$T_{\mathscr{I}_-} \circ (\mathrm{i}^{-1}v_{\mathscr{I}}) = (\mathrm{i}^{-1}v_{\mathscr{I}}) \circ T_{\mathscr{I}_-} \text{ on } Sol_{L^2,\mathscr{I}_-}(\mathrm{M_I}).$$

We summarize this subsection in the following theorem.

Theorem 6.5 ([34])**.** *$T = T_{\mathscr{H}_-} \oplus T_{\mathscr{I}_-}$ from $Sol_{L^2}(\mathrm{M_I}) = Sol_{L^2,\mathscr{H}_-}(\mathrm{M_I}) \oplus Sol_{L^2,\mathscr{I}_-}(\mathrm{M_I})$ to $L^2(\mathscr{H}_-, d\mathrm{vol}_{\mathscr{H}_-}) \oplus L^2(\mathscr{I}_-, d\mathrm{vol}_{\mathscr{I}_-})$ is unitary with*

$$T\,\mathrm{i}^{-1}v_{\mathscr{H}}\Pi_{\mathscr{H}_-} = (\mathrm{i}^{-1}v_{\mathscr{H}} \oplus 0)T, \quad T\,\mathrm{i}^{-1}v_{\mathscr{I}}\Pi_{\mathscr{I}_-} = (0 \oplus \mathrm{i}^{-1}v_{\mathscr{I}})T.$$

7. Traces on horizons and at infinity

In Section 6, scattering theory was formulated for *vectors* $\Psi \in Sol_{L^2}(\mathrm{M_I})$. In this section we first re-express these results as the existence of traces on the horizon and infinity for *spinors* $\phi \in \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$. We then extend the traces at the horizon and at infinity to $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M})$.

7.1. **Decomposition of $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$.** The map $\mathcal{V} : \phi \mapsto \Psi$ in (6.3) allows to construct the corresponding orthogonal decomposition of $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$:

$$\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}) = \mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}_-}(\mathrm{M_I}) \oplus \mathrm{Sol}_{\mathrm{L}^2,\mathscr{I}_-}(\mathrm{M_I}), \tag{7.1}$$

where

$$\mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}_-/\mathscr{I}_-}(\mathrm{M_I}) := \mathcal{V}Sol_{L^2,\mathscr{H}_-/\mathscr{I}_-}(\mathrm{M_I}).$$

We denote by

$$\mathrm{P}_{\mathscr{H}_-} = \mathcal{V}\Pi_{\mathscr{H}_-}\mathcal{V}^{-1}, \quad \mathrm{P}_{\mathscr{I}_-} = \mathcal{V}\Pi_{\mathscr{I}_-}\mathcal{V}^{-1}, \tag{7.2}$$

the orthogonal projections on $\mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}_-/\mathscr{I}_-}(\mathrm{M_I})$. By 6.2.2, the subspaces $\mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}_-/\mathscr{I}_-}(\mathrm{M_I})$ are invariant under $\mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}}$, $\mathrm{i}^{-1}\mathcal{L}_{\mathscr{I}}$.

7.2. **Traces on $\mathscr{H}_-$.** For $\phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I})$ we set

$$\mathrm{T}_{\mathscr{H}_-}\phi = \phi_{|\mathscr{H}_-} \in C^\infty(\mathscr{H}_-; \mathbb{C}^2), \tag{7.3}$$

which is clearly well defined. We denote by $\mathrm{L}^2(\mathscr{H}_-)$ the completion of $C^\infty_{\mathrm{c}}(\mathscr{H}_-; \mathbb{C}^2)$ for the (degenerate) scalar product

$$(\phi|\phi)_{\mathscr{H}_-} = -\mathrm{i}\int_{\mathscr{H}_-} \overline{\phi}\cdot\Gamma(\nabla V)\phi|g|^{\frac{1}{2}}\, dU d\theta d\varphi^\#, \tag{7.4}$$

where we recall that $(U, V, \theta, \varphi^\#)$ are the KBL coordinates, see Subsect. 5.5, in terms of which $\mathscr{H}_- = \{0\}_V \times]0, +\infty[_U \times \mathbb{S}^2_{\theta,\varphi^\#}$.

We will see in the proof of Prop. 7.1 below that $(\cdot|\cdot)_{\mathscr{H}_-}$ is positive semidefinite.

Proposition 7.1. *$\mathrm{T}_{\mathscr{H}_-}$ uniquely extends to a bounded operator $\mathrm{T}_{\mathscr{H}_-} : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}) \to \mathrm{L}^2(\mathscr{H}_-)$ with:*

$$\mathrm{Ker}\,\mathrm{T}_{\mathscr{H}_-} = \mathrm{Sol}_{\mathrm{L}^2,\mathscr{I}_-}(\mathrm{M_I}), \quad \mathrm{Ran}\,\mathrm{T}_{\mathscr{H}_-} = \mathrm{L}^2(\mathscr{H}_-), \tag{7.5}$$

$$\begin{aligned}(\phi|\phi)_{\mathbb{D}} &= -\mathrm{i}\int_{\mathscr{H}_-} \mathrm{T}_{\mathscr{H}_-}\overline{\phi}\cdot\Gamma(\nabla V)\mathrm{T}_{\mathscr{H}_-}\phi|g|^{\frac{1}{2}}\, dU d\theta d\varphi^\# \\ &= \mathrm{i}\int_{\mathscr{H}_-} i^*(g^{-1}J(\phi,\phi)\lrcorner\Omega_g), \quad \phi \in \mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}_-}(\mathrm{M_I}).\end{aligned} \tag{7.6}$$

Proof. If $\Psi = \mathcal{V}\phi \in Sol_{L^2,\mathscr{H}_-}(\mathrm{M_I})$ for $\phi \in \mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}_-}(\mathrm{M_I})$, we have by (6.7), (6.12) after passing to KBL coordinates:

$$\|\phi\|^2_{\mathbb{D}} = \|\Psi\|^2_D = \kappa_+^{-1}\frac{1}{\sqrt{2}}\int_{\mathbb{R}^+_U\times\mathbb{S}^2_{\theta,\varphi^\#}} |T_{\mathscr{H}_-}\Psi|^2 U^{-1}\sin\theta\, dU d\theta d\varphi^\#, \tag{7.7}$$

and hence by (7.1) we have:

$$\kappa_+^{-1}\frac{1}{\sqrt{2}}\int_{\mathbb{R}^+_U\times\mathbb{S}^2_{\theta,\varphi^\#}} |T_{\mathscr{H}_-}\Psi|^2 U^{-1}\sin\theta\, dU d\theta d\varphi^\# \leq \|\phi\|^2_{\mathbb{D}},\quad \phi\in\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}). \tag{7.8}$$

We would like to reexpress the l.h.s. above in terms of ϕ, for $\phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I})$ i.e. $\Psi \in Sol_{\mathrm{sc}}(\mathrm{M_I})$. One associates to the KBL coordinates the new null tetrad $(\mathfrak{l},\mathfrak{n},m,\overline{m})$ given by:

$$\mathfrak{l} = \frac{U}{\sqrt{\Delta}}\mathrm{e}^{-\kappa_+ r}(r-r_-)^{M/r_+}l,\quad \mathfrak{n} = \frac{V}{\sqrt{\Delta}}\mathrm{e}^{-\kappa_+ r}(r-r_-)^{M/r_+}n, \tag{7.9}$$

which has the advantage of extending smoothly to $\mathscr{H}_+\cup\mathscr{H}_-$. The associated spinor basis $(\mathfrak{o},\mathfrak{i})$ is:

$$\mathfrak{o} = \left(\frac{U}{\sqrt{\Delta}}\mathrm{e}^{-\kappa_+ r}(r-r_-)^{M/r_+}\right)^{\frac{1}{2}} o,\quad \mathfrak{i} = \left(\frac{V}{\sqrt{\Delta}}\mathrm{e}^{-\kappa_+ r}(r-r_-)^{M/r_+}\right)^{\frac{1}{2}} \imath. \tag{7.10}$$

If we set

$$\mathfrak{f}_0 = \phi\cdot\mathfrak{o},\quad \mathfrak{f}_1 = \phi\cdot\mathfrak{i}, \tag{7.11}$$

it is shown in [34, Corr. 8.1] that for $\Psi \in Sol_{\mathrm{sc}}(\mathrm{M_I})$ one has:

$$(T_{\mathscr{H}_-}\Psi)(U,\theta,\varphi^\#) = \Psi_1(U,0,\theta,\varphi^\#) = C_1\sqrt{\overline{p}_+(\theta)U}\,\mathfrak{f}_1(U,0,\theta,\varphi^\#), \tag{7.12}$$

where:

$$C_1 = \mathrm{e}^{-\kappa_+ r_+/2}(r_+-r_-)^{M/2r_+},\quad p_+(\theta) = r_+ + \mathrm{i}a\cos\theta. \tag{7.13}$$

Therefore we have for $\Psi \in Sol_{\mathrm{sc}}(\mathrm{M_I})$:

$$\begin{aligned}&\kappa_+^{-1}\frac{1}{\sqrt{2}}\int_{\mathbb{R}^+_U\times\mathbb{S}^2_{\theta,\varphi^\#}} |T_{\mathscr{H}_-}\Psi|^2 U^{-1}\sin\theta\, dU\theta d\varphi^\#\\ &= \kappa_+^{-1}\frac{1}{\sqrt{2}}C_1^2\int_{\mathbb{R}^+_U\times\mathbb{S}^2_{\theta,\varphi^\#}} |p_+|(\theta)|\mathfrak{f}_1|^2\sin\theta\, dU d\theta d\varphi^\#\end{aligned} \tag{7.14}$$

On the other hand over $\mathscr{H}_-$ we have $\mathfrak{n} = \frac{C}{|p_+(\theta)|}\partial_U$ (see [34, Subsect. 8.2.2]), for

$$C = -\kappa_+(r_+^2+a^2)\mathrm{e}^{\kappa_+ r_+}(r_+-r_-)^{-M/r_+}.$$

Since the matrix of $\mathrm{i}\Gamma(\mathfrak{n})$ in the spinor basis $(\mathfrak{o},\mathfrak{i})$ equals $\begin{pmatrix}0&0\\0&1\end{pmatrix}$, this implies that

$$|\mathfrak{f}_1|^2 = C|p_+(\theta)|^{-1}\overline{\phi}\cdot\mathrm{i}\Gamma(\partial_U)\phi,$$

and hence:

$$\begin{aligned}&\kappa_+^{-1}\frac{1}{\sqrt{2}}C_1^2\int_{\mathbb{R}^+_U\times\mathbb{S}^2_{\theta,\varphi^\#}} |p_+|(\theta)|\mathfrak{f}_1|^2\sin\theta\, dU d\theta d\varphi^\#\\ &= -\mathrm{i}\sqrt{2}(r_+^2+a^2)\int_{\mathbb{R}^+_U\times\mathbb{S}^2_{\theta,\varphi^\#}} \overline{\phi}\cdot\Gamma(\partial_U)\phi\sin\theta\, dU d\theta d\varphi^\#.\end{aligned} \tag{7.15}$$

Finally over $\mathscr{H}_-$ we have, see [34, (8.21)],

$$g = 2g_{UV}dUdV + g_{VV}dV^2 + 2g_{V\varphi^\#}dVd\varphi^\# + g_{\theta\theta}d\theta^2 + g_{\varphi^\#\varphi^\#}(d\varphi^\#)^2,$$

where $g_{\theta\theta} = \rho^2$, $g_{\varphi^\#\varphi^\#} = \rho^{-2}\sigma^2 \sin^2\theta$. It follows that:

$$|g|^{\frac{1}{2}} = |g_{UV}|(r_+^2 + a^2)\sin\theta, \quad \nabla V = (g_{UV})^{-1}\partial_U. \tag{7.16}$$

Therefore we have:

$$\begin{aligned} &\kappa_+^{-1}\frac{1}{\sqrt{2}}C_1^2 \int_{\mathbb{R}^+_U \times \mathbb{S}^2_{\theta,\varphi^\#}} |p_+|(\theta)|\mathfrak{f}_1|^2 \sin\theta \, dU d\theta d\varphi^\# \\ &= -\mathrm{i}\int_{\mathbb{R}^+_U \times \mathbb{S}^2_{\theta,\varphi^\#}} \overline{\phi}\cdot\Gamma(\nabla V)\phi|g|^{\frac{1}{2}}\, dU d\theta d\varphi^\#, \end{aligned} \tag{7.17}$$

for $\phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I})$. From (7.14) and (7.8) we obtain:

$$-\mathrm{i}\int_{\mathbb{R}^+_U \times \mathbb{S}^2_{\theta,\varphi^\#}} \overline{\phi}\cdot\Gamma(\nabla V)\phi|g|^{\frac{1}{2}}\, dU d\theta d\varphi^\# \leq \|\phi\|^2_{\mathbb{D}}, \quad \phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I}), \tag{7.18}$$

Thus, $\mathrm{T}_{\mathscr{H}_-}$ uniquely extends as a bounded operator on $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$. From (7.7) we obtain that $\mathrm{Ker}\,\mathrm{T}_{\mathscr{H}_-} = \mathrm{Sol}_{\mathrm{L}^2,\mathscr{I}_-}(\mathrm{M_I})$ and

$$(\phi|\phi)_{\mathbb{D}} = -\mathrm{i}\int_{\mathscr{H}_-} \mathrm{T}_{\mathscr{H}_-}\overline{\phi}\cdot\Gamma(\nabla V)\mathrm{T}_{\mathscr{H}_-}\phi|g|^{\frac{1}{2}}\, dU d\theta d\varphi^\#, \quad \phi \in \mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}_-}(\mathrm{M_I}).$$

It remains to check that $\mathrm{Ran}\,\mathrm{T}_{\mathscr{H}_-}$ is the closure of $C^\infty_{\mathrm{c}}(\mathscr{H}_-;\mathbb{C}^2)$ for the scalar product $(\cdot|\cdot)_{\mathscr{H}_-}$ defined in (7.4). This follows from the fact that $\mathrm{Ran}\,T_{\mathscr{H}_-}$, where $T_{\mathscr{H}_-}$ was introduced in 6.3.1, is equal to $L^2(\mathscr{H}_-, d\mathrm{vol}_{\mathscr{H}_-})$, which has $C^\infty_{\mathrm{c}}(\mathscr{H}_-)$ as a dense subspace. This completes the proof of (7.5) and of the first identity of (7.6).

On $\mathscr{H}_-$ we have $\nabla V = g^{-1}_{UV}\partial_U = -\lambda\mathfrak{n}$ for $\lambda > 0$. By the time orientation of M, (see Subsect. 5.5) $\mathfrak{l} + \mathfrak{n}$ is future directed, hence by 2.5.1 $\mathfrak{n}$ is also future directed. It follows that $-\nabla V$ is future directed on $\mathscr{H}_-$. We then apply (3.14) to obtain the second identity of (7.6). □

7.2.1. *Killing vector field on $\mathscr{H}_-$.* As in 6.3.2, we have since $v_{\mathscr{H}}$ is tangent to $\mathscr{H}_-$:

$$\mathrm{T}_{\mathscr{H}_-}\circ\mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}}\phi = \mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}}\circ\mathrm{T}_{\mathscr{H}_-}\phi, \ \phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I}). \tag{7.19}$$

We would like to express $\mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}}\circ\mathrm{T}_{\mathscr{H}_-}\phi$ using the decomposition (7.11). Let us set:

$$\mathcal{S}_{\mathscr{H}_-} : \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I}) \ni \phi \mapsto \phi\cdot\mathfrak{i}_{|\mathscr{H}_-} \in C^\infty(\mathscr{H}_-;\mathbb{C}), \tag{7.20}$$

so that

$$\mathcal{S}_{\mathscr{H}_-}\phi = \mathrm{T}_{\mathscr{H}_-}\phi\cdot\mathfrak{i}.$$

This new trace operator is particularly useful because thanks to the choice of tetrad (7.9), (7.10), it leads to simpler expressions.

Definition 7.2. We denote by $L^2(\mathscr{H}_-;\mathbb{C})$ the closure of $C^\infty_{\mathrm{c}}(\mathscr{H}_-;\mathbb{C})$ for the scalar product

$$(\mathfrak{f}_1|\mathfrak{f}_1) = \kappa_+^{-1}\frac{1}{\sqrt{2}}C_1^2\int_{\mathbb{R}^+_U \times \mathbb{S}^2_{\theta,\varphi^\#}} |p_+|(\theta)|\mathfrak{f}_1|^2\sin\theta\, dU d\theta d\varphi^\#.$$

Proposition 7.3. *The map*

$$\mathcal{S}_{\mathscr{H}_-} : \mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}_-}(\mathrm{M_I}) \to L^2(\mathscr{H}_-;\mathbb{C}) \ \textit{is unitary,}$$

and one has:

$$\mathcal{S}_{\mathscr{H}_-}\circ\mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}} = -\mathrm{i}^{-1}\kappa_+(U\partial_U + \frac{1}{2})\circ\mathcal{S}_{\mathscr{H}_-}, \tag{7.21}$$

in the sense of unitary equivalence of selfadjoint operators.

Proof. Since $\phi\cdot\mathfrak{i}=\mathfrak{f}_1$, the identity (7.17) means that $\mathcal{S}_{\mathscr{H}_-}$ is unitary from $\mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}_-}(\mathrm{M_I})$ to $L^2(\mathscr{H}_-;\mathbb{C})$. Next we use that $\mathcal{V}\mathcal{L}_{\mathscr{H}}\phi=v_{\mathscr{H}}\mathcal{V}\phi$ by (6.8). From (7.12) we obtain then that

$$U^{\frac{1}{2}}((\mathcal{L}_{\mathscr{H}}\phi)\cdot\mathfrak{i})_{|\mathscr{H}_-}=-\kappa_+U\partial_U(U^{\frac{1}{2}}\phi\cdot\mathfrak{i}_{|\mathscr{H}_-}),$$

which implies (7.21) since $U^{-\frac{1}{2}}\circ(U\partial_U)\circ U^{\frac{1}{2}}=U\partial_U+\frac{1}{2}$. □

7.3. Traces on $\mathscr{I}_-$.

7.3.1. *Conformal rescaling.* Following [34, Subsect. 8.3] we consider the conformally rescaled metric

$$\hat{g}=c^2g,\quad c=r^{-1}=:w,$$

which can be smoothly extended to the domain $\mathbb{R}_{t^*}\times[0,r_+^{-1}]_w\times\mathbb{S}^2_{\theta,\varphi^*}$, with $\mathscr{I}_-=\mathbb{R}_{t^*}\times\{0\}_w\times\mathbb{S}^2_{\theta,\varphi^*}$. Denoting with hats the objects canonically attached to $(\mathrm{M_I},\hat{g})$ we have by Subsect. 2.7:

$$\hat{\Gamma}(v)=\Gamma(v),\quad\hat{\tau}=\tau,\quad\hat{\epsilon}=c\epsilon,\quad\hat{\mathbb{D}}=c^{-3}\mathbb{D}c. \tag{7.22}$$

In the coordinates (t^*,w,θ,φ^*) we have:

$$l=\frac{1}{\sqrt{2\Delta\rho^2}}\left(2(r^2a^2)\partial_{t^*}-\Delta w^2\partial_w+2a\partial_{\varphi^*}\right),$$

$$n=\frac{\Delta w^2}{\sqrt{2\Delta\rho^2}}\partial_w,$$

$$m=\frac{1}{\sqrt{2}p}\left(\mathrm{i}a\sin\theta\partial_t+\partial_\theta+\frac{\mathrm{i}}{\sin\theta}\partial_\varphi\right).$$

This tetrad extends smoothly up to $\mathscr{I}_-=\{w=0\}$ with

$$l_{|\mathscr{I}_-}=\sqrt{2}\partial_{t^*},\quad\pm n_{|\mathscr{I}_-}=m_{|\mathscr{I}_-}=0,$$

so the tetrad degenerates on $\mathscr{I}_-$. A normalized null tetrad for $(\mathrm{M_I},\hat{g})$ which is non degenerate near $\mathscr{I}_-$ is given by (see [34, 8.3.1]):

$$(\hat{l},\hat{n},\hat{m},\overline{\hat{m}})=(l,c^{-2}n,c^{-1}m,c^{-1}\overline{m}). \tag{7.23}$$

Using (7.22) we see that the associated spinor basis is

$$(\hat{o},\hat{\imath})=(o,c^{-1}\imath). \tag{7.24}$$

7.3.2. *Traces on $\mathscr{I}_-$.* For $\phi\in\mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I})$ we set:

$$\mathrm{T}_{\mathscr{I}_-}\phi:=\hat{\phi}_{|\mathscr{I}_-},\quad\hat{\phi}=c^{-1}\phi\in\mathrm{Sol}_{\mathrm{sc}}(\hat{\mathbb{D}}). \tag{7.25}$$

We denote by $\mathrm{L}^2(\mathscr{I}_-)$ the completion of $C^\infty_{\mathrm{c}}(\mathscr{I}_-;\mathbb{C}^2)$ for the scalar product:

$$(\hat{\phi}|\hat{\phi})_{\mathscr{I}_-}=-\mathrm{i}\int_{\mathscr{I}_-}\overline{\hat{\phi}}\cdot\hat{\Gamma}(\hat{\nabla}w)\hat{\phi}|\hat{g}|^{\frac{1}{2}}\,dt^*d\theta d\varphi^*.$$

As before we will see in the proof of Prop. 7.4 that $(\cdot|\cdot)_{\mathscr{I}_-}$ is positive semidefinite.

Proposition 7.4. *$\mathrm{T}_{\mathscr{I}_-}$ uniquely extends to a bounded operator $\mathrm{T}_{\mathscr{I}_-}:\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})\to\mathrm{L}^2(\mathscr{I}_-)$ with:*

$$\mathrm{Ker}\,\mathrm{T}_{\mathscr{I}_-}=\mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}_-}(\mathrm{M_I}),\quad\mathrm{Ran}\,\mathrm{T}_{\mathscr{I}_-}=\mathrm{L}^2(\mathscr{I}_-), \tag{7.26}$$

$$\begin{aligned}(\phi|\phi)_{\mathbb{D}}&=-\mathrm{i}\textstyle\int_{\mathscr{I}_-}\mathrm{T}_{\mathscr{I}_-}\overline{\phi}\cdot\hat{\Gamma}(\hat{\nabla}w)\mathrm{T}_{\mathscr{I}_-}\phi\sin\theta\,dt^*d\theta d\varphi^*\\&=\mathrm{i}\textstyle\int_{\mathscr{I}_-}i^*(\hat{g}^{-1}\hat{J}(\hat{\phi},\hat{\phi})\lrcorner\Omega_{\hat{g}}),\ \phi\in\mathrm{Sol}_{\mathrm{L}^2,\mathscr{I}_-}(\mathrm{M_I}).\end{aligned} \tag{7.27}$$

Proof. Let $\Psi = \mathcal{V}\phi \in Sol_{L^2,\mathscr{I}_-}(\mathrm{M_I})$ for $\phi \in \mathrm{Sol}_{\mathrm{L}^2,\mathscr{I}_-}(\mathrm{M_I})$. From (6.7) and Prop. 6.4 we have

$$(\phi|\phi)_{\mathbb{D}} = (\Psi|\Psi)_D = \frac{1}{\sqrt{2}} \int_{\mathbb{R}_{t^*}\times\mathbb{S}^2_{\theta,\varphi^*}} |\Psi_0|^2 \sin\theta \, dt^* d\theta d\varphi^*, \tag{7.28}$$

hence by (7.1) we have:

$$\frac{1}{\sqrt{2}} \int_{\mathbb{R}_{t^*}\times\mathbb{S}^2_{\theta,\varphi^*}} |\Psi_0|^2 \sin\theta \, dt^* d\theta d\varphi^* \leq \|\phi\|^2_{\mathbb{D}}, \quad \phi \in \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}). \tag{7.29}$$

As in the proof of Prop. 7.1 we re-express the l.h.s. above in terms of ϕ for $\phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I})$. From the definition (6.5) of Ψ, we obtain:

$$\Psi = \mathbf{U}c \left(\frac{\Delta\rho^2\sigma^2}{(r^2+a^2)^2} \right)^{\frac{1}{4}} \begin{pmatrix} \hat{\phi}\cdot o \\ \hat{\phi}\cdot \imath \end{pmatrix},$$

where we recall that $\hat{\phi} = c^{-1}\phi$. On $\mathscr{I}_-$ the matrix $\mathbf{U}$ introduced in (6.4) is equal to $\mathbf{1}_2$ and $c \left(\frac{\Delta\rho^2\sigma^2}{(r^2+a^2)^2} \right)^{\frac{1}{4}} = 1$, hence

$$\frac{1}{\sqrt{2}} \int_{\mathbb{R}_{t^*}\times\mathbb{S}^2_{\theta,\varphi^*}} |\Psi_0|^2 \sin\theta \, dt^* d\theta d\varphi^* = \frac{1}{\sqrt{2}} \int_{\mathbb{R}_{t^*}\times\mathbb{S}^2_{\theta,\varphi^*}} |\hat{\phi}\cdot o|^2 \sin\theta \, dt^* d\theta d\varphi^*. \tag{7.30}$$

Since $o = \hat{o}$ and the matrix of $\mathrm{i}\hat{\Gamma}(l)$ in the basis $(\hat{o}, \hat{\imath})$ equals $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ we have

$$|\hat{\phi}\cdot o|^2 = |\hat{\phi}\cdot \hat{o}|^2 = \mathrm{i}\overline{\hat{\phi}}\cdot\hat{\Gamma}(l)\hat{\phi} = \mathrm{i}\sqrt{2}\overline{\hat{\phi}}\cdot\hat{\Gamma}(\partial_{t^*})\hat{\phi},$$

hence:

$$\frac{1}{\sqrt{2}} \int_{\mathbb{R}_{t^*}\times\mathbb{S}^2_{\theta,\varphi^*}} |\Psi_0|^2 \sin\theta \, dt^* d\theta d\varphi^* = \mathrm{i} \int_{\mathbb{R}_{t^*}\times\mathbb{S}^2_{\theta,\varphi^*}} \overline{\hat{\phi}}\cdot\hat{\Gamma}(\partial_{t^*})\hat{\phi} \sin\theta \, dt^* d\theta d\varphi^*.$$

Denoting by $\hat{\nabla}$ the gradient w.r.t. $\hat{g}$, we obtain that on $\mathscr{I}_-$, $|\hat{g}| = \sin^2\theta$ and $\hat{\nabla}w = -\partial_{t^*}$. Therefore if we use the coordinates $(t^*, w, \theta, \varphi^*)$ near $\mathscr{I}_-$ we obtain:

$$\frac{1}{\sqrt{2}} \int_{\mathbb{R}_{t^*}\times\mathbb{S}^2_{\theta,\varphi^*}} |\Psi_0|^2 \sin\theta \, dt^* d\theta d\varphi^* = -\mathrm{i} \int_{\mathscr{I}_-} \overline{\hat{\phi}}\cdot\hat{\Gamma}(\hat{\nabla}w)\hat{\phi} \sin\theta \, dt^* d\theta d\varphi^*. \tag{7.31}$$

From (7.29) and (7.31) we obtain:

$$-\mathrm{i} \int_{\mathscr{I}_-} \overline{\hat{\phi}}\cdot\hat{\Gamma}(\hat{\nabla}w)\hat{\phi} \sin\theta \, dt^* d\theta d\varphi^* \leq \|\phi\|^2_{\mathbb{D}}, \quad \phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I}), \tag{7.32}$$

so $\mathrm{T}_{\mathscr{I}_-}$ uniquely extends as a bounded operator on $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$. From (7.28) we obtain that $\mathrm{Ker}\,\mathrm{T}_{\mathscr{I}_-} = \mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}_-}(\mathrm{M_I})$ and

$$(\phi|\phi)_{\mathbb{D}} = -\mathrm{i} \int_{\mathscr{I}_-} \overline{\mathrm{T}_{\mathscr{I}_-}\phi}\cdot\hat{\Gamma}(\hat{\nabla}w)\mathrm{T}_{\mathscr{I}_-}\phi \sin\theta \, dt^* d\theta d\varphi^*, \quad \phi \in \mathrm{Sol}_{\mathrm{L}^2,\mathscr{I}_-}(\mathrm{M_I}).$$

As in Prop. 7.1 we obtain (7.26) and the first identity in (7.27). On $\mathscr{I}_-$ we have $\hat{\nabla}w = -\partial_{t^*}$ and ∂_{t^*} is future directed. Applying (3.14) we obtain the second identity of (7.27). □

7.3.3. *Killing vector field on $\mathscr{I}_-$.* As in 6.3.4 we have

$$\mathrm{T}_{\mathscr{I}_-} \circ \mathrm{i}^{-1}\mathcal{L}_{\mathscr{I}}\phi = \mathrm{i}^{-1}\mathcal{L}_{\mathscr{I}} \circ \mathrm{T}_{\mathscr{I}_-}\phi, \quad \phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I}),$$

since $v_{\mathscr{I}} = \partial_{t^*}$ is tangent to $\mathscr{I}_-$ and the conformal factor $c = r^{-1}$ is invariant under $v_{\mathscr{I}}$. Let us set

$$\mathcal{S}_{\mathscr{I}_-} : \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I}) \ni \phi \mapsto \hat{f}_0 = \hat{\phi}\cdot\hat{o}_{|\mathscr{I}_-} \in C^\infty(\mathscr{I}_-;\mathbb{C}), \tag{7.33}$$

so that

$$\mathcal{S}_{\mathscr{I}_-}\phi = \mathrm{T}_{\mathscr{I}_-}\phi\cdot\hat{o}.$$

Definition 7.5. We denote by $L^2(\mathscr{I}_-;\mathbb{C})$ the closure of $C^\infty_{\mathrm{c}}(\mathscr{I}_-;\mathbb{C})$ for the scalar product

$$(\hat{f}_0|\hat{f}_0) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}_{t^*}\times\mathbb{S}^2_{\theta,\varphi^*}} |\hat{f}_0|^2 \sin\theta\, dt^* d\theta d\varphi^*.$$

Proposition 7.6. *The map*

$$\mathcal{S}_{\mathscr{I}_-} : \mathrm{Sol}_{\mathrm{L}^2,\mathscr{I}_-}(\mathrm{M_I}) \to L^2(\mathscr{I}_-, \sin\theta\, dt^* d\theta d\varphi^*) \text{ is unitary,}$$

and one has:

$$\mathcal{S}_{\mathscr{I}_-} \circ (\mathrm{i}^{-1}\mathcal{L}_{\mathscr{I}}) = \mathrm{i}^{-1}\partial_{t^*} \circ \mathcal{S}_{\mathscr{I}_-}, \tag{7.34}$$

in the sense of unitary equivalence of selfadjoint operators.

Proof. The identities (7.30), (7.31) mean that $\mathcal{S}_{\mathscr{I}_-}$ is unitary. (7.34) follows from the fact that the null tetrad $(\hat{l}, \hat{n}, \hat{m}, \overline{\hat{m}})$ in (7.23) is invariant under $\mathcal{L}_{\mathscr{I}}$, and from the identities in 3.5.1. □

We can summarize the results of Subsect. 7.2 and 7.3 as follows.

Proposition 7.7. (1) *The map* $\mathrm{T}_{\mathrm{M_I}} = \mathrm{T}_{\mathscr{H}_-} \oplus \mathrm{T}_{\mathscr{I}_-}$ *from* $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$ *to* $\mathrm{L}^2(\mathscr{H}_-) \oplus \mathrm{L}^2(\mathscr{I}_-)$ *is unitary.*

(2) *The map* $\mathcal{S}_{\mathrm{M_I}} = \mathcal{S}_{\mathscr{H}_-} \oplus \mathcal{S}_{\mathscr{I}_-}$ *from* $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$ *to* $L^2(\mathscr{H}_-;\mathbb{C}) \oplus L^2(\mathscr{I}_-;\mathbb{C})$ *is unitary.*

(3) *One has*

$$\mathcal{S}_{\mathscr{H}_-} \circ \mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}} = -\mathrm{i}^{-1}\kappa_+(U\partial_U + \tfrac{1}{2}) \circ \mathcal{S}_{\mathscr{H}_-},$$

$$\mathcal{S}_{\mathscr{I}_-} \circ \mathrm{i}^{-1}\mathcal{L}_{\mathscr{I}} = \mathrm{i}^{-1}\partial_{t^*} \circ \mathcal{S}_{\mathscr{I}_-},$$

in the sense of unitary equivalence of selfadjoint operators.

7.4. **Decomposition of** $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M})$**.** By Proposition C.12, $\Sigma_{\mathrm{M}} = \{U = V\}$ is a spacelike Cauchy surface in M. Let us also set

$$\Sigma_{\mathrm{I}} := \Sigma_{\mathrm{M}} \cap \mathrm{M_I} = \{t = 0\} \cap \mathrm{M_I}, \quad \Sigma_{\mathrm{I}'} := \Sigma_{\mathrm{M}} \cap \mathrm{M_{I'}} = \{t = 0\} \cap \mathrm{M_{I'}},$$

so that by (C.18)

$$\Sigma_{\mathrm{M}} = \Sigma_{\mathrm{I}} \cup \Sigma_{\mathrm{I}'} \cup S(r_+).$$

Since $S(r_+)$ is of measure 0 in Σ_{M} for the induced Riemannian metric, this implies for the corresponding L^2 spaces of Cauchy data that:

$$L^2(\Sigma_{\mathrm{M}}) \ni f \mapsto f_{|\Sigma_{\mathrm{I}}} \oplus f_{|\Sigma_{\mathrm{I}'}} \in L^2(\Sigma_{\mathrm{I}}) \oplus L^2(\Sigma_{\mathrm{I}'})$$

is unitary, where we denote by $L^2(\Sigma)$ the completion of $C^\infty_{\mathrm{c}}(\Sigma;\mathbb{S}^*_\Sigma)$ for the scalar product ν_Σ, see (3.17). By Proposition 3.3 this yields a unitary map:

$$\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}) \ni \phi \mapsto \phi_{\mathrm{I}} \oplus \phi_{\mathrm{I}'} \in \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}) \oplus \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_{I'}}) \tag{7.35}$$

where ϕ_{I}, resp. $\phi_{\mathrm{I}'}$ are the restrictions of ϕ to $\mathrm{M_I}$, resp. $\mathrm{M_{I'}}$.

Next, we observe that while M'_{I} and $\mathrm{M}_{\mathrm{I}'}$ are strictly speaking different, they can be identified as spacetimes through the isometry

$$R : \mathrm{M}_{\mathrm{I}'} \ni (U, V, \theta, \varphi^{\#}) \mapsto (-U, -V, \theta, \varphi^{\#}) \in \mathrm{M}'_{\mathrm{I}}$$

which preserves the orientation and time-orientation. We also denote by $R : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}_{\mathrm{I}'}) \to \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}'_{\mathrm{I}})$ the unitary map

$$R\phi = \phi \circ R.$$

7.5. **Traces on the long horizon and at infinity.**

7.5.1. *Traces on the long horizon.* Recall that $\mathscr{H}_L = \{V = 0\}$ is the left long horizon in M, which decomposes as

$$\mathscr{H}_L = \mathscr{H}_- \cup \mathscr{H}'_- \cup S(r_+), \quad \mathscr{H}_- = \{V = 0,\ U > 0\}, \quad \mathscr{H}'_- = \{V = 0,\ U < 0\} = R(\mathscr{H}_-).$$

To ease notation $\mathscr{H}_L$ will be simply denoted by $\mathscr{H}$ in the sequel.

Let us denote by $\mathbf{L}^2(\mathscr{H})$ the completion of $C^\infty_{\mathrm{c}}(\mathscr{H} \setminus S(r_+); \mathbb{C}^2)$ for the scalar product

$$(\phi|\phi)_{\mathscr{H}} = -\mathrm{i} \int_{\mathscr{H}} \overline{\phi} \cdot \Gamma(\nabla V) \phi |g|^{\frac{1}{2}} dU d\theta d\varphi^{\#}.$$

Clearly we have

$$\mathbf{L}^2(\mathscr{H}) \sim \mathbf{L}^2(\mathscr{H}_-) \oplus \mathbf{L}^2(\mathscr{H}'_-)$$

with obvious notations. We denote

$$\mathrm{T}_{\mathscr{H}_-} \phi := \mathrm{T}_{\mathscr{H}_-} \phi_{\mathrm{I}}, \quad \mathrm{T}_{\mathscr{H}'_-} \phi := R \mathrm{T}_{\mathscr{H}_-} R \phi_{\mathrm{I}'},$$

and

$$\mathrm{T}_{\mathscr{H}} := \mathrm{T}_{\mathscr{H}_-} \oplus \mathrm{T}_{\mathscr{H}'_-} : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}) \to \mathbf{L}^2(\mathscr{H})$$

Similarly we denote by $L^2(\mathscr{H}; \mathbb{C})$ the closure of $C^\infty_{\mathrm{c}}(\mathscr{H}; \mathbb{C})$ for the scalar product

$$(\mathfrak{f}_1|\mathfrak{f}_1) = \kappa_+^{-1} \frac{1}{\sqrt{2}} C_1^2 \int_{\mathbb{R}_U \times \mathbb{S}^2_{\theta, \varphi^{\#}}} |p_+|(\theta) |\mathfrak{f}_1|^2 \sin\theta \, dU d\theta d\varphi^{\#},$$

so that

$$L^2(\mathscr{H}; \mathbb{C}) \sim L^2(\mathscr{H}_-; \mathbb{C}) \oplus L^2(\mathscr{H}'_-; \mathbb{C}).$$

We set

$$\mathcal{S}_{\mathscr{H}_-} \phi := \mathcal{S}_{\mathscr{H}_-} \phi_{\mathrm{I}}, \quad \mathcal{S}_{\mathscr{H}'_-} \phi := R \mathcal{S}_{\mathscr{H}_-} R \phi_{\mathrm{I}'},$$

and

$$\mathcal{S}_{\mathscr{H}} := \mathcal{S}_{\mathscr{H}_-} \oplus \mathcal{S}_{\mathscr{H}'_-} : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}) \to L^2(\mathscr{H}; \mathbb{C}).$$

7.5.2. *Traces at infinity.* We set

$$\mathrm{T}_{\mathscr{I}_-} \phi := \mathrm{T}_{\mathscr{I}_-} \phi_{\mathrm{I}}, \quad \mathrm{T}_{\mathscr{I}'_-} \phi := \mathrm{T}_{\mathscr{I}_-} R \phi_{\mathrm{I}'},$$

$$\mathcal{S}_{\mathscr{I}_-} \phi := \mathcal{S}_{\mathscr{I}_-} \phi_{\mathrm{I}}, \quad \mathcal{S}_{\mathscr{I}'_-} \phi := \mathcal{S}_{\mathscr{I}_-} R \phi_{\mathrm{I}'},$$

7.5.3. *Summary.* The results of this section are summarized in the following theorem.

Theorem 7.8. *The maps*

$$\begin{aligned}\mathrm{T}_{\mathrm{M}} &:= \mathrm{T}_{\mathscr{H}} \oplus \mathrm{T}_{\mathscr{I}_-} \oplus \mathrm{T}_{\mathscr{I}'_-} : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}) \to \mathrm{L}^2(\mathscr{H}) \oplus \mathrm{L}^2(\mathscr{I}_-) \oplus \mathrm{L}^2(\mathscr{I}_-) \\ \mathcal{S}_{\mathrm{M}} &:= \mathcal{S}_{\mathscr{H}} \oplus \mathcal{S}_{\mathscr{I}_-} \oplus \mathcal{S}_{\mathscr{I}'_-} : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}) \to L^2(\mathscr{H};\mathbb{C}) \oplus L^2(\mathscr{I}_-;\mathbb{C}) \oplus L^2(\mathscr{I}_-;\mathbb{C})\end{aligned} \tag{7.36}$$

are unitary.

The geometric situation is illustrated in Figure 4.

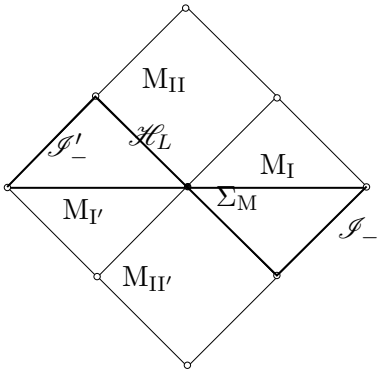

FIGURE 4. The Kerr-Kruskal spacetime with Cauchy surface Σ_{M}.

8. The Unruh state in the Kerr-Kruskal spacetime

We recall that the solution space covariances of a quasi-free state were defined in Subsect. 4.1. They will frequently be simply called covariances in the sequel.

8.1. The Unruh state in M.

Definition 8.1. The *Unruh state* ω_{M} is the quasi-free state on CAR(M) with solution space covariances:

$$C^{\pm}_{\mathrm{M}} = \mathcal{S}^{-1}_{\mathrm{M}} \left(\mathbf{1}_{\mathbb{R}^{\pm}}(-\mathrm{i}^{-1}\partial_U) \oplus \mathbf{1}_{\mathbb{R}^{\pm}}(\mathrm{i}^{-1}\partial_{t^*}) \oplus \mathbf{1}_{\mathbb{R}^{\mp}}(\mathrm{i}^{-1}\partial_{t^*}) \right) \mathcal{S}_{\mathrm{M}}. \tag{8.1}$$

Theorem 8.2. (1) *The Unruh state ω_{M} is a pure state.*
(2) *The restriction $\omega_{\mathrm{M_{I\cup II}}}$ of ω_{M} to $\mathrm{M_{I\cup II}}$ is a pure state.*
(3) *The restriction $\omega_{\mathrm{M_I}}$ of ω_{M} to $\mathrm{M_I}$ has covariances*

$$C^{\pm}_{\mathrm{M_I}} = \mathcal{S}^{-1}_{\mathrm{M_I}} \left(\chi^{\pm}_{\mathscr{H}_-}(-\mathrm{i}^{-1}\kappa_+(U\partial_U + \frac{1}{2})) \oplus \chi^{\pm}_{\mathscr{I}_-}(\mathrm{i}^{-1}\partial_{t^*}) \right) \mathcal{S}_{\mathrm{M_I}}$$

for

$$\chi^{\pm}_{\mathscr{I}_-}(\lambda) = \mathbf{1}_{\mathbb{R}^{\pm}}(\lambda), \quad \chi^{\pm}_{\mathscr{H}_-}(\lambda) = \left(1 + \mathrm{e}^{\mp T_{\mathrm{H}}^{-1}\lambda}\right)^{-1}, \tag{8.2}$$

where $T_{\mathrm{H}} = (2\pi)^{-1}\kappa_+$ is the Hawking temperature.
(4) *The restriction $\omega_{\mathrm{M_{I'}}}$ of ω_{M} to $\mathrm{M_{I'}}$ is the image of $\omega_{\mathrm{M_I}}$ under R, and has covariances*

$$C^{\pm}_{\mathrm{M_{I'}}} = R \circ C^{\mp}_{\mathrm{M_I}} \circ R.$$

Theorem 8.3. *There exists $0 < a_0 \leq 1$ such that if $|a|M^{-1} < a_0$ then the restriction $\omega_{\mathrm{M_{I\cup II}}}$ of the Unruh state ω_{M} to $\mathrm{M_{I\cup II}}$ is a Hadamard state.*

Remark 8.4. There are good reasons to expect that ω_{M} is not a Hadamard state in the whole of M, and that the Hadamard condition fails precisely on the long horizon $\mathscr{H}_L$, and in fact computations of the renormalized stress-energy tensor show that it diverges at $\mathscr{H}_L$ [6]. In our formalism, the problem comes from the wavefront set estimate in Proposition E.5, more precisely from (E.5). In general one has $N^*\operatorname{supp} u \subset \mathrm{WF}(\delta_S \otimes u)$, where N^*S is the conormal bundle to S and $N^*\operatorname{supp} u = N^*S \cap T^*_{\operatorname{supp} u}M$. Since S is null, N^*S is invariant under the bicharacteristic flow, so $N^*S \subset \mathrm{WF}(G(\delta_S \otimes u))$, and therefore $\mathrm{WF}(G(\delta_S) \otimes u)$ intersects both $\mathcal{N}^+$ and $\mathcal{N}^-$ over S. One applies this observation to the long horizon $\mathscr{H}_L$, which is included in M (but not in $\mathrm{M_I}$ or $\mathrm{M_{I\cup II}}$), see for example the key identity (8.25) below, and one sees that the form of the 'boundary data' of ω_{M} on $\mathscr{H}_L$ given in (8.1) cannot imply the Hadamard condition over $\mathscr{H}_L$.

8.2. **Proof of Theorem 8.2.** (1) follows from Lemma B.1 (which says that the state is pure if the covariances are projections). To prove (3), we note that the image of $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$ under $\mathcal{S}_{\mathrm{M}}$ equals $L^2(\mathscr{H}_-;\mathbb{C}) \oplus L^2(\mathscr{I}_-;\mathbb{C}) \oplus \{0\}$ and hence the restriction of $C^{\pm}_{\mathrm{M}}$ to $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$ (viewed as a sesquilinear form using the scalar product $(\cdot|\cdot)_{\mathbb{D}}$) equals

$$\mathcal{S}^{-1}_{\mathscr{H}_-}\left(\imath^*\mathbf{1}_{\mathbb{R}^\pm}(-\mathrm{i}^{-1}\partial_U)\imath\right)\mathcal{S}_{\mathscr{H}_-} \oplus \mathcal{S}^{-1}_{\mathscr{I}_-}\mathbf{1}_{\mathbb{R}^\pm}(\mathrm{i}^{-1}\partial_{t^*})\mathcal{S}_{\mathscr{I}_-},$$

where $\imath : L^2(\mathscr{H}_-;\mathbb{C}) \to L^2(\mathscr{H};\mathbb{C})$ is the canonical injection. By Lemma D.1 we have

$$\imath^*\mathbf{1}_{\mathbb{R}^\pm}(-\mathrm{i}^{-1}\partial_U)\imath = (1+\mathrm{e}^{\pm 2\pi A}),$$

where $A = \mathrm{i}^{-1}(U\partial_U + \frac{1}{2})$ is the generator of dilations in U acting on $L^2(\mathscr{H}_-;\mathbb{C})$, see Appendix D. This completes the proof of (3).

Let us now prove (2). We claim that the image of $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_{I\cup II}})$ under $\mathcal{S}_M$ is included in $L^2(\mathscr{H};\mathbb{C}) \oplus L^2(\mathscr{I}_-;\mathbb{C}) \oplus \{0\}$, which will imply (2) by Lemma B.1 and the fact that $\mathbf{1}_{\mathbb{R}^\pm}(-\mathrm{i}^{-1}\partial_U) \oplus \mathbf{1}_{\mathbb{R}^\pm}(\mathrm{i}^{-1}\partial_{t^*}) \oplus \mathbf{1}_{\mathbb{R}^\mp}(\mathrm{i}^{-1}\partial_{t^*})$ restricted to $L^2(\mathscr{H};\mathbb{C}) \oplus L^2(\mathscr{I}_-;\mathbb{C}) \oplus \{0\}$ is a projection. To prove our claim we take $\phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_{I\cup II}})$, which has hence compactly supported Cauchy data on one of the space-like Cauchy surfaces Z_T defined in C.6.2. The conformal trace of ϕ on $\mathscr{I}'_-$ vanishes, which by density implies our claim.

It remains to prove (4). We note that

$$RC^{\pm}_{\mathrm{M_I}}R = \mathcal{S}^{-1}_{\mathrm{M_I}}\left(\mathbf{1}_{\mathbb{R}^\mp}(-\mathrm{i}^{-1}\partial_U) \oplus \mathbf{1}_{\mathbb{R}^\pm}(\mathrm{i}^{-1}\partial_{t^*}) \oplus \mathbf{1}_{\mathbb{R}^\mp}(\mathrm{i}^{-1}\partial_{t^*})\right)\mathcal{S}_{\mathrm{M_I}}.$$

Then (4) follows from (3) and the fact that the image of $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_{I'}})$ under $\mathcal{S}_{\mathrm{M}}$ equals $L^2(\mathscr{H}'_-;\mathbb{C}) \oplus \{0\} \oplus L^2(\mathscr{I}_-;\mathbb{C})$. □

8.3. **Preparations.** We now collect some preparations for the proof of Theorem 8.3.

8.3.1. *Cauchy surfaces in* $\mathrm{M_I}$. In the proof of the Hadamard condition we will need two convenient families of Cauchy surfaces $\overline{\Sigma}_T$ and $\widetilde{\Sigma}_T$ in $\mathrm{M_I}$.

Since $\frac{r^2+a^2}{\Delta} = 1 - 2r^{-1} + O(r^{-2})$ when $r \to +\infty$ we can choose $x(r)$ in (5.3) such that

$$x(r) = r - 2\ln r + O(r^{-1}), \text{ when } r \to +\infty.$$

It follows that the hypersurface $\{t + r - 2\ln r = T\}$ intersects $\mathscr{I}_-$ (inside $\hat{\mathrm{M}}_{\mathrm{I}}$, the conformal extension of $\mathrm{M_I}$, see 5.4), transversally along the submanifold $\{w = 0, t^* = T\}$. We set then

$$\overline{\Sigma}_T := \{t + r - 2\ln r = T\} \cup \mathscr{I}_- \cap \{t^* \geq T\},$$

which we smooth out near the intersection, to make it smooth and space-like inside $\mathrm{M_I}$, null on $\mathscr{I}_-$ (see Figure 5).

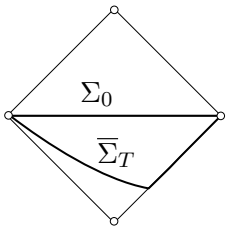

FIGURE 5. The Cauchy surfaces $\overline{\Sigma}_T$.

To define the hypersurfaces $\tilde{\Sigma}_T$ we set $f(r) = \dfrac{(r^2+a^2)^2}{\Delta^2} - \dfrac{a^2+M^2}{\Delta}$. We have

$$f > 0 \text{ in }]r_+, +\infty[\text{ and } f^{\frac{1}{2}}(r) - \frac{r^2+a^2}{\Delta} \in O(1) \text{ when } r \to r_+. \tag{8.3}$$

We define the function $y(r)$ by

$$y'(r) = f^{\frac{1}{2}}(r), \ \lim_{r\to r_+} y(r) - x(r) = 0, \tag{8.4}$$

noting that the second condition can be imposed due to (8.3). Finally we define

$$\tilde{x}(r) = \mathbf{1}_{]1,3]}(r)y(r) + \mathbf{1}_{]3,+\infty[}(r)y(3), \tag{8.5}$$

making $y(r)$ constant in $\{r \geq 3\}$.

The hypersurface $\{t - \tilde{x}(r) = T\}$ intersects $\mathscr{H}_-$ (inside ${}^*\mathrm{K}$, see 5.3) transversally along the submanifold $\{r = r_+, {}^*t = T\}$. We set

$$\widetilde{\Sigma}_T := \{t - \tilde{x}(r) = T\} \cup \mathscr{H}_- \cap \{{}^*t \geq T\}$$

again smoothed out near the intersection, to make it smooth and space-like inside $\mathrm{M_I}$, null on $\mathscr{H}_-$ (see Figure 6).

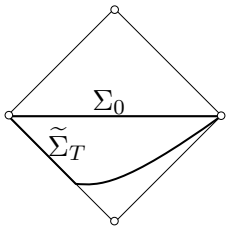

FIGURE 6. The Cauchy surfaces $\widetilde{\Sigma}_T$.

The fact that $\overline{\Sigma}_T$ and $\widetilde{\Sigma}_T$ are Cauchy surfaces follows from the constructions in Subsect. C.6, which provide a way of approximating $\overline{\Sigma}_T$ and $\widetilde{\Sigma}_T$ by families of space-like Cauchy surface $\overline{\Sigma}_{T,n}$ and $\widetilde{\Sigma}_{T,n}$. The space-like property of $\overline{\Sigma}_T$ and $\widetilde{\Sigma}_T$ inside of $\mathrm{M_I}$ can be also deduced from the computations in Subsect. C.6.

8.3.2. *Cauchy surfaces in* M. We will also need Cauchy surfaces $\widehat{\Sigma}_T$ in M, obtained by gluing together along $S(r_+)$ the surface $\widetilde{\Sigma}_T$ with its image under the wedge reflection R, see Figure 7.

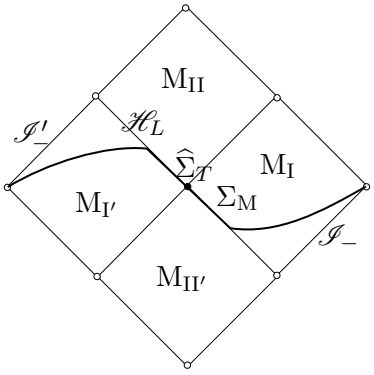

FIGURE 7. The Cauchy surfaces $\widehat{\Sigma}_T$.

8.3.3. *Wavefront set estimates.* The next lemma provides a way of producing solutions with wave front set in $\mathcal{N}^+$ or $\mathcal{N}^-$ using a Killing vector field X, in a region where it is assumed to be time-like. For the sake of simplicity we formulate it here in the specific case of the Killing vector fields $v_{\mathscr{H}}$ and $v_{\mathscr{I}}$ on $\mathrm{M_I}$ which will be needed later on. Below, $\langle\lambda\rangle = (1+\lambda^2)^{\frac{1}{2}}$.

Lemma 8.5. *Let $X = v_{\mathscr{H}}$ or $v_{\mathscr{I}}$ and recall that $\mathrm{i}^{-1}\mathcal{L}_X$ is the selfadjoint generator of a unitary group on $\mathrm{Sol_{L^2}(M_I)}$. Let $\chi^\pm \in L^\infty(\mathbb{R})$ be such that $\chi^\pm - \mathbf{1}_{\mathbb{R}^\pm} \in O(\langle\lambda\rangle^{-\infty})$ and $\operatorname{sing\,supp}\chi^\pm$ is compact. Then, if X is future directed time-like near $x_0 \in \mathrm{M_I}$, one has:*

$$\mathrm{WF}(\chi^\pm(\mathrm{i}^{-1}\mathcal{L}_X)\phi) \subset \mathcal{N}^\pm \text{ over a neighborhood of } x_0,$$

for all $\phi \in \mathrm{Sol_{L^2}(M_I)}$.

Proof. We consider the case of $v_{\mathscr{I}} = \partial_t$, the other one being analogous. Let us denote $\phi^+ = \chi^+(\mathrm{i}^{-1}\mathcal{L}_X)\phi$. Let $q_0 = (x_0, \xi_0) \in \mathcal{N}^-$.

Let $\vartheta \in C^\infty(\mathbb{R})$ be such that $\vartheta(\lambda) = 1$ for $\lambda \ll 0$ and $\vartheta(\lambda) = 0$ for $\lambda > 0$ and $\lambda \in \operatorname{sing\,supp}\chi^+$. Setting $\chi_\infty = \vartheta\chi^+ \in O(\langle\lambda\rangle^{-\infty})$, by functional calculus we have

$$\vartheta(\mathrm{i}^{-1}\mathcal{L}_X)\phi^+ = \chi_\infty(\mathrm{i}^{-1}\mathcal{L}_X)\phi. \tag{8.6}$$

In coordinates, $\mathcal{L}_X\phi = M(t)\partial_t M(t)^{-1}\phi$ for some smooth family of invertible fiber endomorphisms $M(t)$. It follows that

$$\vartheta(\mathrm{i}^{-1}\mathcal{L}_X)\phi^+ = M(t)\big(\vartheta(\mathrm{i}^{-1}\partial_t)\otimes\mathbf{1}_y\big)M(t)^{-1}\phi^+,$$

where $\vartheta(\mathrm{i}^{-1}\partial_t)\otimes\mathbf{1}_y$ acts as a product-type pseudo-differential operator. Let $A \in \Psi^0$ be a properly supported pseudo-differential operator (in the sense of the usual calculus on manifolds), elliptic at q_0 and microsupported in a small conic neighborhood Γ_0 of q_0. Since $q_0 \in \mathcal{N}^-$ and ∂_t is future directed time-like in the relevant region, the symbol of $\vartheta(\mathrm{i}^{-1}\partial_t)\otimes\mathbf{1}_y$ equals 1 on Γ_0 away from o. Therefore, the operator

$$B := A \circ M(t)\big(\vartheta(\mathrm{i}^{-1}\partial_t)\otimes\mathbf{1}_y\big)M(t)^{-1}$$

belongs to Ψ^0 and is elliptic at q_0, and similarly one can show that $A\circ\big(\chi_\infty(\mathrm{i}^{-1}\partial_t)\otimes\mathbf{1}_y\big)$ belongs to $\Psi^{-\infty}$. Thus, by acting on both sides of (8.6) with A we find $B\phi^+ \in C^\infty$. Consequently, $q_0 \notin \mathrm{WF}(\phi^+)$.

The proof of the minus sign version of the statement is analogous. □

The other essential ingredient are the Green's formulas from Subsect. 3.3 (which roughly speaking, allow to represent a solutions locally as $\phi = -\mathbb{G}(\delta_S\otimes v)$, with S

a Cauchy surface and v a trace on S) combined with Proposition E.5 in the appendix (which allows to estimate $\mathrm{WF}\big(\mathbb{G}(\delta_S \otimes v)\big)$ in terms of $\mathrm{WF}(v)$ if v is compactly supported).

8.4. **Hadamard property in $\mathrm{M_I}$.** Our objective is to prove the Hadamard property of $\omega_{\mathrm{M_{I\cup II}}}$. We start by considering the restriction $\omega_{\mathrm{M_I}}$ to $\mathrm{M_I}$. From Proposition 7.7 we obtain the key formula

$$C^{\pm}_{\mathrm{M_I}} = \mathrm{P}_{\mathscr{H}_-} \circ \chi^{\pm}_{\mathscr{H}_-}(\mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}}) + \mathrm{P}_{\mathscr{I}_-} \circ \chi^{\pm}_{\mathscr{I}_-}(\mathrm{i}^{-1}\mathcal{L}_{\mathscr{I}}), \tag{8.7}$$

where the projections $\mathrm{P}_{\mathscr{H}_-/\mathscr{I}_-}$ are defined in (7.2).

Proposition 8.6. *There exists $0 < a_0 \leq 1$ such that if $|a|M^{-1} < a_0$ then $\omega_{\mathrm{M_I}}$ is a Hadamard state.*

Remark 8.7. The Hadamard property extends to any state in $\mathrm{M_I}$ of the form (8.7) where $\chi^{\pm}_{\mathscr{H}_-}, \chi^{\pm}_{\mathscr{I}_-} \in L^\infty(\mathbb{R})$ are such that $\operatorname{sing\,supp}\chi^{\pm}$ is compact, and:

$$\begin{aligned} &\chi^{+}_{\mathscr{H}_-/\mathscr{I}_-} + \chi^{-}_{\mathscr{H}_-/\mathscr{I}_-} = 1 \\ &\chi^{\pm}_{\mathscr{H}_-/\mathscr{I}_-}(\lambda) \in O(\langle\lambda\rangle^{-\infty}) \text{ in } \mathbb{R}^{\mp}. \end{aligned} \tag{8.8}$$

Such states can be called *asymptotically passive*. The class of *passive states* was introduced by Pusz–Woronowicz [51], and was investigated by Sahlmann–Verch [53] in settings with a global time-like Killing vector field v (cf. the work of Hack–Verch [31] for the related notion of non-equilibrium steady states in a relativistic setting). Roughly speaking, a passive state is a mixture of ground and thermal states with respect to v. The name *asymptotically passive* indicates that at $\mathscr{H}_-$, the state is passive with respect to $v_{\mathscr{H}}$, and asymptotically at $\mathscr{I}_-$ it is passive with respect to $v_{\mathscr{I}}$.

Proof of Proposition 8.6. By Theorem 4.3 it suffices to prove that

$$\mathrm{WF}((C^{\pm}_{\mathrm{M_I}})^{\frac{1}{2}}\phi) \subset \mathcal{N}^{\pm},\ \forall \phi \in \mathrm{Sol}_{\mathrm{L^2}}(\mathrm{M_I}). \tag{8.9}$$

We will prove only the $+$ case, the $-$ case being analogous. Let $q_0 = (x_0, \xi_0) \in T^*\mathrm{M_I} \cap \mathcal{N}^-$ and let γ be the null bicharacteristic in $T^*\mathrm{M}$ from q_0.

Case 1: suppose that $\pi(\gamma)$ (where $\pi : T^*\mathrm{M} \to \mathrm{M}$ is the base projection) intersects $\mathscr{I}_-$ at some point y_0. We will need a family $\overline{\Sigma}_{T,n}$ of smooth, space-like Cauchy surfaces in $\mathrm{M_I}$ that approximates $\overline{\Sigma}_T$. It is constructed in Proposition C.9 in the appendix.

We will use the conformal transformation $\hat{g} = c^2 g$, $c = r^{-1}$, see Subsects. 2.7 and 5.4. We recall from Subsect. 2.7 that $\hat{\Gamma}(X) = \hat{\beta}\hat{\gamma}(X) = \Gamma(X)$, $\hat{\mathbb{D}} = c^{-3}\mathbb{D}c$ and $\hat{\mathbb{G}}$ is the Pauli-Jordan operator for $\hat{\mathbb{D}}$, equal to $c^{-1}\mathbb{G}c^3$, see [27, (17.51)].

Applying (3.11) to the conformally rescaled Weyl operator $\hat{\mathbb{D}}$ and recalling that $\hat{\phi} = c^{-1}\phi$, this yields:

$$\hat{\phi}(x) = -\int_{\overline{\Sigma}_{T,n}} \hat{\mathbb{G}}(x,y)\hat{\Gamma}(\hat{g}^{-1}\nu)(y)\hat{\phi}^{\pm}(y) i_l^*(d\mathrm{vol}_{\hat{g}})(y),\ \phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I}). \tag{8.10}$$

The solution $\hat{\phi}$ extends across $\mathscr{I}_-$ to the conformal extension $\hat{\mathrm{M}}_{\mathrm{I}}$ of $\mathrm{M_I}$ defined in Subsect. 5.4, as a smooth solution of $\hat{\mathbb{D}}\hat{\phi} = 0$. We can then take the limit $n \to +\infty$ in (8.10) and obtain that:

$$\hat{\phi}(x) = -\int_{\overline{\Sigma}_{T}} \hat{\mathbb{G}}(x,y)\hat{\Gamma}(\hat{g}^{-1}\nu)(y)\hat{\phi}(y) i_l^*(d\mathrm{vol}_{\hat{g}})(y),\ \phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I}). \tag{8.11}$$

From Subsect. 7.3 we recall that the conformal trace on $\mathscr{I}_-$, $\mathrm{T}_{\mathscr{I}_-}\phi = \hat{\phi}_{|\mathscr{I}_-}$ is well defined on $\mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I})$ and extends as a bounded operator $\mathrm{T}_{\mathscr{I}_-} : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}) \to \mathrm{L}^2(\mathscr{I}_-)$. On the other hand the trace of $\hat{\phi}$ on $\overline{\Sigma}_T \setminus \mathscr{I}_-$ is also well defined and obviously bounded from $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$ to the appropriate L^2 space, because $\overline{\Sigma}_T \setminus \mathscr{I}_-$ is a part of a smooth, space-like Cauchy surface and the conformal factor $c = r^{-1}$ is bounded with bounded inverse on $\overline{\Sigma}_T \setminus \mathscr{I}_-$.

Summarizing we obtain a bounded operator $\mathrm{T}_{\overline{\Sigma}_T} : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}) \to \mathrm{L}^2(\overline{\Sigma}_T)$, extending the map $\phi \mapsto \hat{\phi}_{|\overline{\Sigma}_T}$, where we denote by $\mathrm{L}^2(\overline{\Sigma}_T)$ the appropriate L^2 space on $\overline{\Sigma}_T$. Let us also recall from (3.16) that $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}) \ni \phi \mapsto \hat{\phi} = c^{-1}\phi \in \mathrm{Sol}_{\mathrm{L}^2}(\hat{\mathrm{M}}_\mathrm{I})$ is unitary.

It follows then from Proposition 3.4 that (8.11) extends to $\phi \in \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$. More precisely if $\chi \in C^\infty_{\mathrm{c}}(\mathrm{M_I})$ is a cutoff function supported near x_0, there exists $\chi_1 \in C^\infty_{\mathrm{c}}(\overline{\Sigma}_T)$ such that

$$\chi\hat{\phi}(x) = -\int_{\overline{\Sigma}_T} \hat{\mathbb{G}}(x,y)\hat{\Gamma}(\hat{g}^{-1}\nu)(y)\chi_1(y)\mathrm{T}_{\overline{\Sigma}_T}\phi(y) i_l^*(d\mathrm{vol}_{\hat{g}})(y),\ \phi \in \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}).$$

We then choose $T \ll -1$ such that $t^*(y_0) > T$. Since one has

$$\mathrm{WF}(\hat{\mathbb{G}})' \subset \{(q,q') \,:\, q \sim q'\},$$

we can find $\chi_2 \in C^\infty_{\mathrm{c}}(\mathscr{I}_-)$, equal to 1 near y_0 such that:

$$\chi\hat{\phi}(x) = -\int_{\overline{\Sigma}_T} \hat{\mathbb{G}}(x,y)\hat{\Gamma}(\hat{g}^{-1}\nu)(y)\chi_2(y)\mathrm{T}_{\mathscr{I}_-}\phi(y) i_l^*(d\mathrm{vol}_{\hat{g}})(y) + u,\ \phi \in \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}), \tag{8.12}$$

with $q_0 \notin \mathrm{WF}(u)$. We will drop the error term u in the sequel. Using (3.13) we can rewrite this identity as:

$$\chi\hat{\phi}(x) = -\int_{\mathscr{I}_-} \hat{\mathbb{G}}(x,y)\hat{\Gamma}(\hat{\nabla}w)\chi_2\mathrm{T}_{\mathscr{I}_-}\phi|\hat{g}|^{\frac{1}{2}}dt^*d\theta d\varphi^*.$$

We can assume that χ_2 depends only on t^*. Since $\hat{\nabla}w = -\partial_{t^*}$ and $|\hat{g}|^{\frac{1}{2}} = \sin\theta$ on $\mathscr{I}_-$, we get:

$$\chi\hat{\phi}(x) = \int_{\mathscr{I}_-} \hat{\mathbb{G}}(x,t^*,\theta,\varphi^*)\hat{\Gamma}(\partial_{t^*})\chi_2(t^*)\mathrm{T}_{\mathscr{I}_-}\phi \sin\theta\, dt^*d\theta d\varphi^*. \tag{8.13}$$

We use the spinor basis $(\hat{o},\hat{\imath})$ associated to the null tetrad $(\hat{l},\hat{n},\hat{m},\overline{\hat{m}})$ introduced in Subsect. 7.3, see (7.23), (7.24) to express the r.h.s. of (8.13) in terms of the components

$$\hat{f}_0 = \mathrm{T}_{\mathscr{I}_-}\phi \cdot \hat{o} = \mathcal{S}_{\mathscr{I}_-}\phi, \quad \hat{f}_1 = \mathrm{T}_{\mathscr{I}_-}\phi \cdot \hat{\imath}.$$

We have $\partial_{t^*} = \sqrt{2}^{-1}l$ hence the matrix of $\hat{\Gamma}(\partial_{t^*})$ equals $\mathrm{i}^{-1}\sqrt{2}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, see the proof of Prop. 7.4 . We set

$$\hat{\mathbb{G}}_0(x,y) = \hat{\mathbb{G}}(x,y)\hat{o}^*(y), \quad \hat{\mathbb{G}}_1(x,y) = \hat{\mathbb{G}}(x,y)\hat{\imath}^*(y),$$

and obtain finally:

$$\chi\phi(x) = Cr^{-1}\int_{\mathscr{I}_-} \hat{\mathbb{G}}_0(x,t^*,\theta,\varphi^*)\chi_2(t^*)\mathcal{S}_{\mathscr{I}_-}\phi \sin\theta\, dt^*d\theta d\varphi^*. \tag{8.14}$$

We apply (8.14) to $(C^+_{\mathrm{M_I}})^{\frac{1}{2}}\phi$, using that $\mathcal{S}_{\mathscr{I}_-}\mathrm{P}_{\mathscr{H}_-} = 0$ and Proposition 7.6 to obtain:

$$\chi(C^+_{\mathrm{M_I}})^{\frac{1}{2}}\phi = Cr^{-1}\int_{\mathscr{I}_-} \hat{\mathbb{G}}_0(x,t^*,\theta,\varphi^*)\chi_2(t^*)(\chi^+_{\mathscr{I}_-})^{\frac{1}{2}}(\mathrm{i}^{-1}\partial_{t^*})\mathcal{S}_{\mathscr{I}_-}\phi \sin\theta\, dt^*d\theta d\varphi^*.$$

We apply Prop. E.5 with $S = \mathscr{I}_-$ and $X = v_{\mathscr{I}} = \partial_{t^*}$. We claim that for

$$g^+ = \chi_2(t^*)(\chi^+_{\mathscr{I}_-})^{\frac{1}{2}}(\mathrm{i}^{-1}\partial_{t^*})\mathcal{S}_{\mathscr{I}_-}\phi \in \mathcal{E}'(\mathscr{I}_-),$$

we have

$$\mathrm{WF}(g^+) \subset \{(y,\eta) \in T^*\mathscr{I}_- \setminus o \,:\, +\eta\cdot v_{\mathscr{I}}(y) \geq 0\},$$

which is immediate, since by (8.8) $\chi^+_{\mathscr{I}_-}$ decays rapidly in $\mathbb{R}^-$.

By Prop. E.5 this implies that $q_0 \notin \mathrm{WF}(\phi^+)$.

Case 2: suppose that $\pi(\gamma)$ intersects $\mathscr{H}_-$ at some y_0. We use now the family $\widetilde{\Sigma}_{T,n}$ of Cauchy surfaces in $\mathrm{M_I}$ and write

$$\phi(x) = -\int_{\widetilde{\Sigma}_{T,n}} \mathbb{G}(x,y)\Gamma(g^{-1}\nu)(y)\phi(y) i_l^*(d\mathrm{vol}_g)(y),\ \phi \in \mathrm{Sol_{sc}}(\mathrm{M_I}). \tag{8.15}$$

The smooth solutions ϕ extends across $\mathscr{H}_-$ to ${}^*\mathrm{K}$ as smooth solutions of $\mathbb{D}\phi = 0$. We can again pass to the limit $n \to +\infty$ in (8.15) and obtain:

$$\phi(x) = -\int_{\widetilde{\Sigma}_T} \mathbb{G}(x,y)\Gamma(g^{-1}\nu)(y)\phi(y) i_l^*(d\mathrm{vol}_g)(y),\ \phi \in \mathrm{Sol_{sc}}(\mathrm{M_I}). \tag{8.16}$$

As before the traces on $\mathscr{H}_-$ and on $\widetilde{\Sigma}_T \setminus \mathscr{H}_-$ extend as a bounded operator $\mathrm{T}_{\widetilde{\Sigma}_T} : \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}) \to \mathrm{L}^2(\widetilde{\Sigma}_T)$, by Proposition 7.1 for the trace on $\mathscr{H}_-$ and using that $\widetilde{\Sigma}_T \setminus \mathscr{H}_-$ is a part of a smooth, space-like Cauchy surface for the trace on $\widetilde{\Sigma}_T \setminus \mathscr{H}_-$.

Again we denote by $\mathrm{L}^2(\widetilde{\Sigma}_T)$ the appropriate L^2 space on $\widetilde{\Sigma}_T$.

By Proposition 3.4 we can extend (8.16) to $\phi \in \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$ and obtain:

$$\chi\phi(x) = -\int_{\widetilde{\Sigma}_T} \mathbb{G}(x,y)\Gamma(g^{-1}\nu)(y)\chi_1 \mathrm{T}_{\widetilde{\Sigma}_T}\phi(y) i_l^*(d\mathrm{vol}_g)(y),\ \phi \in \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}).$$

We choose $T \ll -1$ such that ${}^*t(y_0) > T$. Again we can find $\chi_2 \in C^\infty_{\mathrm{c}}(\mathscr{H}_-)$ equal to 1 near y_0 such that

$$\chi\phi(x) = -\int_{\mathscr{H}_-} \mathbb{G}(x,y)\Gamma(g^{-1}\nu)(y)\chi_2(y)\mathrm{T}_{\mathscr{H}_-}\phi(y) i_l^*(d\mathrm{vol}_g)(y) + u,\ \phi \in \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}), \tag{8.17}$$

with $q_0 \notin \mathrm{WF}(u)$, and we will forget the error term u in the sequel. Using (3.13) we can rewrite this identity as

$$\chi\phi(x) = -\int_{\mathscr{H}_-} \mathbb{G}(x,y)\Gamma(\nabla V)\chi_2 \mathrm{T}_{\mathscr{H}_-}\phi\, |g|^{\frac{1}{2}}\, dU d\theta d\varphi^{\#}, \tag{8.18}$$

where $\mathscr{H}_- \sim \mathbb{R}^+_U \times \mathbb{S}^2_{\theta,\varphi^{\#}}$.

We can assume that χ_2 depends only on the variable U. Since $\nabla V = (g_{UV})^{-1}\partial_U$ and $|g|^{\frac{1}{2}} = g_{UV}(r_+^2 + a^2)\sin\theta$, see (7.16), we obtain:

$$\chi\phi(x) = -(r_+^2 + a^2)\int_{\mathscr{H}_-} \mathbb{G}(x,U,\theta,\varphi^{\#})\Gamma(\partial_U)\chi_2(U)\mathrm{T}_{\mathscr{H}_-}\phi\, \sin\theta\, dU d\theta d\varphi^{\#}. \tag{8.19}$$

We now use the spinor basis $(\mathfrak{o},\mathfrak{i})$ associated to the null tetrad $(\mathfrak{l},\mathfrak{n},m,\overline{m})$ introduced in Subsect. 7.2, see (7.9), (7.10), and express (8.19) in terms of the components

$$\mathfrak{f}_0 = \mathrm{T}_{\mathscr{H}_-}\phi\cdot\mathfrak{o},\quad \mathfrak{f}_1 = \mathrm{T}_{\mathscr{H}_-}\phi\cdot\mathfrak{i} = \mathcal{S}_{\mathscr{H}_-}\phi.$$

We have $\partial_U = C|p_+(\theta)|\mathfrak{n}$, hence the matrix of $\Gamma(\partial_U)$ in the basis $(\mathfrak{o},\mathfrak{i})$ is equal to $\mathrm{i}^{-1}C|p_+(\theta)|\begin{pmatrix}0&0\\0&1\end{pmatrix}$, see the proof of Prop. 7.1. We also set

$$\mathbb{G}_0(x,y)=\mathbb{G}(x,y)\mathfrak{o}^*(y),\quad \mathbb{G}_1(x,y)=\mathbb{G}(x,y)\mathfrak{i}^*(y),$$

where we recall that $(\mathfrak{o}^*,\mathfrak{i}^*)$ is the dual basis of $(\mathfrak{o},\mathfrak{i})$. This yields

$$\chi\phi(x)=C\int_{\mathscr{H}_-}\mathbb{G}_1(x,U,\theta,\varphi^\#)\chi_2(U)\mathcal{S}_{\mathscr{H}_-}\phi\sin\theta\,dUd\theta d\varphi^\#. \tag{8.20}$$

Finally we apply (8.20) to $(C^+_{\mathrm{M_I}})^{\frac12}\phi$, using that $\mathcal{S}_{\mathscr{H}_-}\mathrm{P}_{\mathscr{I}_-}=0$ and Proposition 7.3 to obtain:

$$\chi(C^+_{\mathrm{M_I}})^{\frac12}\phi(x)=C\int_{\mathscr{H}_-}\mathbb{G}_1(x,U,\theta,\varphi^\#)g^+(U,\theta,\varphi^\#)\sin\theta\,dUd\theta d\varphi^\#,$$

where

$$g^+=\chi_2(U)(\chi^+_{\mathscr{H}_-})^{\frac12}\big(-\mathrm{i}^{-1}\kappa_+(U\partial_U+\tfrac12)\big)\mathcal{S}_{\mathscr{H}_-}\phi\in\mathcal{E}'(\mathscr{H}_-).$$

We now apply Proposition E.5, with $S=\mathscr{H}_-$ and $X=v_{\mathscr{H}}=-\kappa_+U\partial_U$. This yields $\mathrm{WF}(\chi(C^+_{\mathrm{M_I}})^{\frac12}\phi)\subset\mathcal{N}^+$ and hence $q_0\notin\mathrm{WF}((C^+_{\mathrm{M_I}})^{\frac12}\phi)$ provided that

$$\mathrm{WF}(g^+)\subset\{(y,\eta)\in T^*\mathscr{H}_-\setminus o\,:\,+\eta\cdot v_{\mathscr{H}}(y)\geq0\}. \tag{8.21}$$

To prove (8.21) we use the unitary map:

$$B:L^2(]0,+\infty[_U\times\mathbb{S}^2)\ni f\mapsto Bf(s,\omega)=\kappa_+^{\frac12}\mathrm{e}^{-\kappa_+s/2}f(\mathrm{e}^{-\kappa_+s},\omega),$$

implementing the change of coordinates $U=\mathrm{e}^{-\kappa_+s}$, which satisfies:

$$B\circ(-\mathrm{i}^{-1}\kappa_+(U\partial_U+\tfrac12))=(\mathrm{i}^{-1}\partial_s)\circ B.$$

Since by (8.8) $\chi^+_{\mathscr{H}_-}(\lambda)$ decays rapidly in $\mathbb{R}^-$, we have

$$\mathrm{WF}\big(B(\chi^+_{\mathscr{H}_-})^{\frac12}(-\mathrm{i}^{-1}\kappa_+(U\partial_U+\tfrac12))\mathcal{S}_{\mathscr{H}_-}\phi\big)\subset\{(y,\eta)\in T^*\mathscr{H}_-\setminus o\,:\,+\eta\cdot\partial_s\geq0\},$$

which implies (8.21) by the covariance of the wavefront set under change of coordinates.

Case 3: suppose that $\pi(\gamma)$ does not end up at $\mathscr{H}_-$ nor at $\mathscr{I}_-$.

By Lemma C.4, there exists $x_1\in\pi(\gamma)$ such that $v_{\mathscr{H}}$ and $v_{\mathscr{I}}$ are future directed time-like near x_1. We have

$$C^+_{\mathrm{M_I}}\phi=(\chi^+_{\mathscr{H}_-})^{\frac12}(\mathrm{i}^{-1}\mathcal{L}_{\mathscr{H}})\mathrm{P}_{\mathscr{H}_-}\phi+(\chi^+_{\mathscr{I}_-})^{\frac12}(\mathrm{i}^{-1}\mathcal{L}_{\mathscr{I}})\mathrm{P}_{\mathscr{I}_-}\phi=:\phi_{\mathscr{H}}+\phi_{\mathscr{I}}$$

By Lemma 8.5 we get that

$$\mathrm{WF}(\phi_{\mathscr{H}/\mathscr{I}})\subset\mathcal{N}^+\text{ above a neighborhood of }x_1.$$

By propagation of singularities this implies that $q_0\notin\mathrm{WF}(C^+_{\mathrm{M_I}}\phi)$. □

8.5. **Hadamard property in $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$.** Finally, we prove the Hadamard property in $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$.

Proof of Theorem 8.3. We only consider the + case. By Theorem 4.3 it suffices to prove that

$$\mathrm{WF}(C^+_{\mathrm{M}}\phi)\cap T^*\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}\subset\mathcal{N}^+,\ \phi\in\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}),\tag{8.22}$$

where we recall that

$$C^+_{\mathrm{M}}\phi=C^+_{\mathrm{M}_{\mathrm{I}}}\phi_{\mathrm{I}}\oplus RC^-_{\mathrm{M}_{\mathrm{I}}}R\phi_{\mathrm{I}'}.$$

Let $q_0=(x_0,\xi_0)\in T^*\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}\cap\mathcal{N}^-$ and let γ the null bicharacteristic through q_0 *in* $T^*\mathrm{M}$, and let π be the base projection.

Case 1: suppose that $\pi(\gamma)$ intersects M_{I}. We have $C^+_{\mathrm{M}}\phi=C^+_{\mathrm{M}_{\mathrm{I}}}\phi_{\mathrm{I}}$ over M_{I}, with $\phi_{\mathrm{I}}\in\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}_{\mathrm{I}})$ and $C^\pm_{\mathrm{M}_{\mathrm{I}}}$ the covariances of $\omega_{\mathrm{M}_{\mathrm{I}}}$ in Theorem 8.2. We already know that $\omega_{\mathrm{M}_{\mathrm{I}}}$ is Hadamard in M_{I} by Proposition 8.6, which implies that $\mathrm{WF}(C^+_{\mathrm{M}}\phi)\subset\mathcal{N}^+$ over M_{I}, hence $q_0\notin\mathrm{WF}(C^+_{\mathrm{M}}\phi)$ by propagation of singularities.

Case 2: suppose that $\pi(\gamma)$ intersects $\mathrm{M}_{\mathrm{I}'}$. We have $C^+_{\mathrm{M}}\phi=RC^-_{\mathrm{M}_{\mathrm{I}}}R\phi_{\mathrm{I}'}$ over $\mathrm{M}_{\mathrm{I}'}$, with $R\phi_{\mathrm{I}'}\in\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}_{\mathrm{I}})$. Since $\omega_{\mathrm{M}_{\mathrm{I}}}$ is Hadamard, we have $\mathrm{WF}(C^-_{\mathrm{M}_{\mathrm{I}}}R\phi_{\mathrm{I}'})\subset\mathcal{N}^-$, hence $\mathrm{WF}(C^+_{\mathrm{M}}\phi)\subset\mathcal{N}^+$ over $\mathrm{M}_{\mathrm{I}'}$, and $q_0\notin\mathrm{WF}(C^+_{\mathrm{M}}\phi)$ by propagation of singularities.

Case 3: suppose that $\pi(\gamma)$ does not intersect M_{I} nor $\mathrm{M}_{\mathrm{I}'}$, which means that $\pi(\gamma)$ leaves $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$ through $S(r_+)$, see the proof of Proposition C.12. By propagation of singularities, we can assume that x_0 belongs to an arbitrarily small neighborhood of $S(r_+)$ in M. If $\chi\in C^\infty_{\mathrm{c}}(\mathrm{M})$ then as in the proof of Proposition 8.6, we can find $\chi_1\in C^\infty_{\mathrm{c}}(\widehat{\Sigma}_T)$ such that

$$\chi\phi(x)=-\int_{\widehat{\Sigma}_T}\mathbb{G}(x,y)\Gamma(g^{-1}\nu)(y)\chi_1 r_{\widetilde{\Sigma}_T}\phi(y)i_l^*(d\mathrm{vol}_g)(y),\ \phi\in\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}),$$

where $r_{\widehat{\Sigma}_T}:\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M})\to L^2(\widehat{\Sigma}_T)$ is the bounded extension of the trace on $\widehat{\Sigma}_T$.

We can then find $\chi_2\in C^\infty_{\mathrm{c}}(\mathscr{H}_L)$, equal to 1 near $S(r_+)$ such that

$$\chi\phi(x)=-\int_{\widehat{\Sigma}_T}\mathbb{G}(x,y)\Gamma(g^{-1}\nu)(y)\chi_2\mathrm{T}_{\mathscr{H}}\phi(y)i_l^*(d\mathrm{vol}_g)(y)+u,\ \phi\in\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M}),$$

where $q_0\notin\mathrm{WF}(u)$ and we will drop the error term u. We rewrite this identity as

$$\chi\phi(x)=-\int_{\mathscr{H}}\mathbb{G}(x,y)\Gamma(\nabla V)\chi_2\mathrm{T}_{\mathscr{H}}\phi\,|g|^{\frac{1}{2}}\,dUd\theta d\varphi^{\#},\tag{8.23}$$

where $\mathscr{H}\sim\mathbb{R}_U\times\mathbb{S}^2_{\theta,\varphi^{\#}}$, and assume that χ_2 depends only on the variable U. The same arguments as in the proof of Proposition 8.6 yield:

$$\chi\phi(x)=C\int_{\mathscr{H}}\mathbb{G}_1(x,U,\theta,\varphi^{\#})\chi_2(U)\mathcal{S}_{\mathscr{H}}\phi\,\sin\theta\,dUd\theta d\varphi^{\#}.\tag{8.24}$$

We apply (8.24) to $C^+\phi$ using that

$$\mathcal{S}_{\mathscr{H}}C^+\phi=\mathbf{1}_{\mathbb{R}^+}(-\mathrm{i}^{-1}\partial_U)\mathcal{S}_{\mathscr{H}}\phi,$$

and obtain:

$$\chi\phi(x)=C\int_{\mathscr{H}}\mathbb{G}_1(x,U,\theta,\varphi^{\#})\chi_2(U)\mathbf{1}_{\mathbb{R}^+}(-\mathrm{i}^{-1}\partial_U)\mathcal{S}_{\mathscr{H}}\phi\,\sin\theta\,dUd\theta d\varphi^{\#}.\tag{8.25}$$

We apply Proposition E.5 with $S=\mathscr{H}_L$ and $X=-\partial_U$. Clearly for

$$g^+=\chi_2(U)\mathbf{1}_{\mathbb{R}^+}(-\mathrm{i}^{-1}\partial_U)\mathcal{S}_{\mathscr{H}}\phi$$

we have

$$\mathrm{WF}(g^+) \subset \{(y,\eta) \in T^*\mathscr{H} \setminus o \,:\, \eta\cdot X(y) \geq 0\}.$$

Thus, Proposition E.5 combined with (8.25) imply that $q_0 \notin \mathrm{WF}(\phi)$. □

Appendix A. Properties of the spinorial Lie derivative

A.1. **Lie derivative of spinors.** We will use the notation introduced in Section 2 and assume that we are given a spinor bundle $\mathcal{S} \xrightarrow{\pi} M$ on an oriented and time oriented Lorentzian manifold of dimension 4. Recall the definition:

$$\mathcal{L}_X\psi = \nabla^{\mathcal{S}}_X\psi + \frac{1}{8}((\nabla_a X)_b - (\nabla_b X)_a)\gamma^a\gamma^b\psi, \quad \psi \in C^\infty(M,S), \tag{A.1}$$

for $X \in C^\infty(M;TM)$, and:

$$\mathcal{L}_X(\gamma(v)\psi) =: (\mathcal{L}_X\gamma)(v)\psi + \gamma(\mathcal{L}_X v)\psi + \gamma(v)\mathcal{L}_X\psi, \quad \psi \in C^\infty(M;\mathcal{S}),$$

for all $v \in C^\infty(M;TM)$. One has the identity ([43, (III.10)]):

$$(\mathcal{L}_X\gamma)^b = -\frac{1}{2}((\nabla_a X)^b + (\nabla^b X)_a)\gamma^a. \tag{A.2}$$

If X is Killing this yields:

$$\mathcal{L}_X(\gamma(v)\psi) = \gamma(\mathcal{L}_X v)\psi + \gamma(v)\mathcal{L}_X\psi. \tag{A.3}$$

We denote by $X(f)$ the action of the vector field X on the scalar function f. Recall that the divergence δX of X is defined by $\mathcal{L}_X\Omega_g =: \delta X\Omega_g$, where Ω_g is the volume form associated to g.

Lemma A.1. *One has*

$$X(\overline{\psi}_1\cdot\beta\psi_2) = \overline{\mathcal{L}_X\psi_1}\cdot\beta\psi_2 + \overline{\psi}_1\cdot\beta\mathcal{L}_X\psi_2,$$

for all $X \in C^\infty(M;TM)$, $\psi_i \in C^\infty(M;\mathcal{S})$. It follows that if $\delta X = 0$, then $\mathcal{L}_X$ is formally self-adjoint for the (non-positive) Hermitian scalar product

$$(\psi_1|\psi_2)_{M,\beta} = \int_M \overline{\psi}_1\cdot\beta\psi_2\, d\mathrm{vol}_g, \quad \psi_1,\psi_2 \in C^\infty_{\mathrm{c}}(M;\mathcal{S}).$$

Furthermore, $\nabla^{\mathcal{S}}_X$ is formally self-adjoint when $\delta X = 0$.

Proof. For $\psi_1,\psi_2 \in C^\infty(M;S)$ we have

$$\begin{aligned} X(\overline{\psi}_1\cdot\beta\psi_2) &= \overline{\nabla_X\psi_1}\cdot\beta\psi_2 + \overline{\psi}_1\cdot\beta\nabla_X\psi_2 \\ &= \overline{\mathcal{L}_X\psi_1}\cdot\beta\psi_2 + \overline{\psi}_1\cdot\beta\mathcal{L}_X\psi_2 + \overline{R(X)\psi}_1\cdot\beta\psi_2 + \overline{\psi}_1\cdot\beta R(X)\psi_2, \end{aligned}$$

for $R(X) = \frac{1}{8}((\nabla_a X)_b - (\nabla_b X)_a)\gamma^b\gamma^a$. Since $(\gamma^a)^*\beta = -\beta\gamma^a$, we obtain

$$\begin{aligned} R(X)^*\beta + \beta R(X) &= \frac{1}{8}(\nabla_a X)_b - (\nabla_b X)_a\beta(\gamma^a\gamma^b + \gamma^b\gamma^a) \\ &= \frac{1}{8}((\nabla_a X)_b - (\nabla_b X)_a)g^{ab}\beta = 0, \end{aligned}$$

since $g^{ab} = g^{ba}$. □

Lemma A.2. *One has $[\mathcal{L}_X,\kappa] = 0$.*

Proof. This follows from $[\nabla^{\mathcal{S}}_X,\kappa] = 0$, $[\gamma(v),\kappa] = 0$ and (2.16). □

Lemma A.3. *One has $[\mathcal{L}_X,H] = 0$ and thus $\mathcal{L}_X$ preserves $C^\infty(M;\mathcal{W}_{\mathrm{e/o}})$.*

Proof. We know that $[\nabla^{\mathcal{S}}_X, H] = 0$ and $\gamma(v)H = -\gamma(v)H$, so the result follows from (2.16). □

Lemma A.4. *If X is a Killing vector field then*

$$[\mathcal{L}_X, \not{D}] = 0.$$

Proof. By [43, eqs. (V.1), (III.50)] one has

$$[\mathcal{L}_X, \not{D}] = (\mathcal{L}_X\gamma)^a\nabla^{\mathcal{S}}_a - \frac{1}{2}(\partial_a(\delta X) + \frac{1}{2}\nabla_b(\mathcal{L}_X g)^b_a)\gamma^a.$$

If X is Killing then $\mathcal{L}_X\gamma = 0$ by (A.2) and $\delta X = 0$. □

A.2. **Lie derivative of Weyl spinors.** By Lemma A.3, $\mathcal{L}_X$ preserves $C^\infty(M;\mathcal{W}_{\mathrm{e}})$. Since $\mathbb{S} = \mathcal{W}^*_{\mathrm{e}}$ we can define $\mathcal{L}^*_X$ acting on $C^\infty(M;\mathbb{S})$ by:

$$X((w|s)) =: (\mathcal{L}_X W|s) + (w|\mathcal{L}^*_X s), \quad w \in C^\infty(M;\mathcal{W}_{\mathrm{e}}),\ s \in C^\infty(M;\mathbb{S}).$$

We recall that $L(\mathcal{W}_{\mathrm{e}}, \mathcal{W}^*_{\mathrm{e}})$ is identified with $\mathbb{S}\otimes\overline{\mathbb{S}}$ and we define $\tilde{\mathcal{L}}_X$ acting on $C^\infty(M;\mathbb{S}\otimes\mathbb{S}^*)$ by:

$$\tilde{\mathcal{L}}_X|s_1)(s_2| := |\mathcal{L}^*_X s_1)(s_2| + |s_1)(\mathcal{L}^*_X s_2|, \quad s_i \in C^\infty(M;\mathbb{S}).$$

We also recall that

$$\tau : C^\infty(M;\mathbb{C}TM) \ni v \mapsto \beta\gamma(v) \in C^\infty(M;\mathbb{S}\otimes\overline{\mathbb{S}})$$

is the canonical isomorphism.

The bundle $\mathbb{S}$ is equipped with the symplectic form $\epsilon = \frac{1}{\sqrt{2}}(\beta\kappa)^{-1}$. From Lemmas A.1, A.2 we obtain that:

$$X(s_1\cdot\epsilon s_2) = \mathcal{L}^*_X s_1\cdot\epsilon s_2 + s_1\cdot\epsilon\mathcal{L}^*_X s_2, \quad s_i \in C^\infty(M;\mathbb{S}). \tag{A.4}$$

Lemma A.5. *If X is Killing then we have:*

$$\tau\circ\mathcal{L}_X = \tilde{\mathcal{L}}_X\circ\tau.$$

Proof. For $w_i \in C^\infty(M;\mathcal{W}_{\mathrm{e}})$, $s_i \in C^\infty(M;\mathbb{S})$ and $A = |s_1)(s_2|$ we have:

$$\begin{aligned} X((w_1|Aw_2)) &= X((w_1|s_1)\times(s_2|w_2)) \\ &= ((\mathcal{L}_X w_1|s_1) + (w_1|\mathcal{L}^*_X s_1))(w_2|s_2) \\ &\quad + (w_1|s_1)((\mathcal{L}^*_X s_2|w_2) + (s_2|\mathcal{L}_X w_2)) \\ &= (\mathcal{L}_X w_1|s_1)(w_2|s_2) + (w_1|s_1)(s_2|\mathcal{L}_X w_2) + (w_1|\tilde{\mathcal{L}}_X A w_2). \end{aligned} \tag{A.5}$$

If $A = \tau(v)$ for $v \in C^\infty(M;\mathbb{C}TM)$, we have $(w_1|Aw_2) = \overline{w}_1\cdot\beta\gamma(v)w_2$ so by Lemma A.1:

$$X((w_1|Aw_2)) = \overline{\mathcal{L}_X w}_1\cdot\beta\gamma(v)w_2 + \overline{w}_1\cdot\beta\mathcal{L}_X(\gamma(v)w_2).$$

If X is Killing this yields using (A.3):

$$\begin{aligned} X((w_1|Aw_2)) &= \overline{\mathcal{L}_X w}_1\cdot\beta\gamma(v)w_2 + \overline{w}_1\cdot\beta\gamma(\mathcal{L}_X v)w_2 + \overline{w}_1\cdot\beta\gamma(v)\mathcal{L}_X w_2 \\ &= (\mathcal{L}_X w_1|s_1)(w_2|s_2) + (w_1|s_1)(s_2|\mathcal{L}_X w_2) + \overline{w}_1\cdot\beta\gamma(\mathcal{L}_X v)w_2. \end{aligned} \tag{A.6}$$

By comparing (A.5) with (A.6) we obtain that $\tau(\mathcal{L}_X v) = \tilde{\mathcal{L}}_X\tau v$. □

Lemma A.6. *Let $l, n, m, \overline{m}$ be a normalized null tetrad, and let $(o, \imath)$ be the associated frame of $\mathbb{S}$, such that*

$$\mathrm{i}\tau l = o \otimes \overline{o}, \quad \mathrm{i}\tau n = \imath \otimes \overline{\imath}, \quad \mathrm{i}\tau m = o \otimes \overline{\imath}, \quad \mathrm{i}\tau \overline{m} = \imath \otimes \overline{o}.$$

Then if X is a Killing vector field such that $\mathcal{L}_X l = \mathcal{L}_X n = \mathcal{L}_X m = 0$, one has:

$$\mathcal{L}_X^* o = \mathcal{L}_X^* \imath = 0.$$

Proof. Let us denote $\mathcal{L}_X^* o, \mathcal{L}_X^* i$ simply by o', i'. Since $\mathcal{L}_X l = \mathcal{L}_X n = \mathcal{L}_X m = 0$, we obtain from Lemma A.5:

$$1)\ o' \otimes \overline{o} + o \otimes \overline{o}' = 0,$$

$$2)\ \imath' \otimes \overline{\imath} + \imath \otimes \overline{\imath}' = 0,$$

$$3)\ o' \otimes \overline{\imath} + o \otimes \overline{\imath}' = 0,$$

$$4)\ \imath' \otimes \overline{o} + \imath \otimes \overline{o}' = 0.$$

Using (A.4) and the fact that $o \cdot \epsilon \imath = 1$ we obtain $o' \cdot \epsilon \imath + o \cdot \epsilon \imath' = 0$ hence $o' \cdot \epsilon \imath = \imath' \cdot \epsilon o$. Composing then 1) and 3) to the right with $\mathbf{1} \otimes \overline{\epsilon \imath}$ and $\mathbf{1} \otimes \overline{\epsilon o}$ respectively gives:

$$o' + (\overline{o' \cdot \epsilon \imath}) o = 0,$$

$$-o' + (\overline{\imath' \cdot \epsilon o}) o = 0,$$

hence $o' = 0$ since $o' \cdot \epsilon \imath = \imath' \cdot \epsilon o$. Composing then 4) to the right with $\mathbf{1} \otimes \overline{\epsilon \imath}$ gives finally $\imath' = 0$. □

Appendix B. Algebraic quantization of the Weyl equation

B.1. **Quasi-free states on CAR $*$-algebras.** In this appendix of algebraic quantization of fermionic fields. Below, we follow [27, Sect. 17.14]. Let $(\mathcal{Y}, \nu)$ be a pre-Hilbert space. We denote by $\mathrm{CAR}(\mathcal{Y}, \nu)$ the unital complex $*$-algebra generated by elements $\psi(y)$, $\psi^*(y)$, $y \in \mathcal{Y}$, with the relations

$$\begin{aligned}
&\psi(y_1 + \lambda y_2) = \psi(y_1) + \overline{\lambda} \psi(y_2), \\
&\psi^*(y_1 + \lambda y_2) = \psi(y_1) + \lambda \psi^*(y_2),\ y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C}, \\
&[\psi(y_1), \psi(y_2)]_+ = [\psi^*(y_1), \psi^*(y_2)]_+ = 0, \\
&[\psi(y_1), \psi^*(y_2)]_+ = \overline{y}_1 \cdot \nu y_2 \mathbf{1},\ y_1, y_2 \in \mathcal{Y}, \\
&\psi(y)^* = \psi^*(y),\ y \in \mathcal{Y},
\end{aligned} \tag{B.1}$$

where $[A, B]_+ = AB + BA$ is the anti-commutator.

A quasi-free state ω on $\mathrm{CAR}(\mathcal{Y}, \nu)$ is determined by its covariances $\lambda^{\pm} \in L_{\mathrm{h}}(\mathcal{Y}, \mathcal{Y}^*)$, defined by

$$\omega(\psi(y_1)\psi^*(y_2)) =: \overline{y}_1 \cdot \lambda^+ y_2, \quad \omega(\psi^*(y_2)\psi(y_1)) =: \overline{y}_1 \cdot \lambda^- y_2, \quad y_1, y_2 \in \mathcal{Y}.$$

A pair of Hermitian sesquilinear forms $\lambda^{\pm}$ on $\mathcal{Y}$ are the covariances of a quasi-free state on $\mathrm{CAR}(\mathcal{Y}, \nu)$ iff

$$\lambda^{\pm} \geq 0, \quad \lambda^+ + \lambda^- = \nu. \tag{B.2}$$

It follows that $\lambda^{\pm}$ uniquely extend to the completion $\mathcal{Y}^{\mathrm{cpl}}$ of $\mathcal{Y}$ for ν. The following characterization of pure quasi-free states is well-known, see e.g. [16, Thm. 17.31, Subsect. 17.2.3].

Lemma B.1. *The quasi-free state ω on $\mathrm{CAR}(\mathcal{Y},\nu)$ is a pure state iff there exist projections $\pi^{\pm}$ on $\mathcal{Y}^{\mathrm{cpl}}$ such that*

$$\lambda^{\pm} = \nu \circ \pi^{\pm}.$$

Note that $\pi^{\pm}$ are selfadjoint for ν and $\pi^{+} + \pi^{-} = \mathbf{1}$.

B.2. **Time reversal.** In the remaining part of this section we discuss the notion of time reversal in the context of quantization of Weyl fields.

B.2.1. *Time reversal on* $\mathrm{CAR}(\mathcal{Y},\nu)$. Let $(\mathcal{Y},\nu)$ a pre-Hilbert space as in Subsect. B.1. A *time reversal* is a unitary involution $\tau \in U(\mathcal{Y},\nu)$ i.e.

$$\tau^2 = \mathbf{1}, \quad \overline{\tau y_1}\cdot\nu\tau y_2 = \overline{y}_1\cdot\nu y_2, \quad y_1, y_2 \in \mathcal{Y}.$$

We associate to τ the anti-linear $*$-involution $\hat{\tau}$ on $\mathrm{CAR}(\mathcal{Y},\nu)$ defined by

$$\hat{\tau}(\psi(y)) := \psi^*(\tau y), \ \hat{\tau}(\psi^*(y)) := \psi(\tau y), \ y \in \mathcal{Y}.$$

Note that this map is well defined in $\mathrm{CAR}(\mathcal{Y},\nu)$ since

$$\begin{aligned}\hat{\tau}([\psi(y_1),\psi^*(y_2)]_+) &= \hat{\tau}(\overline{y}_1\cdot\nu y_2\mathbf{1}) = \overline{y}_2\cdot\nu y_1\mathbf{1},\\ &= [\hat{\tau}(\psi(y_1)),\hat{\tau}(\psi^*(y_2))]_+\\ &= [\psi^*(\tau y_1),\psi(\tau y_2)]_+ = \overline{y}_2\cdot\nu y_1\mathbf{1}.\end{aligned}$$

If ω is a quasi-free state on $\mathrm{CAR}(\mathcal{Y},\nu)$ with covariances $\lambda^{\pm}$ we define $\hat{\tau}^*\omega = \hat{\omega}$ by

$$\hat{\omega}(A) := \omega(\hat{\tau}(A^*)), \ A \in \mathrm{CAR}(\mathcal{Y},\nu),$$

which is also a quasi-free state on $\mathrm{CAR}(\mathcal{Y},\nu)$. We have then

$$\begin{aligned}\hat{\omega}(\psi(y_1)\psi^*(y_2)) &= \omega(\psi^*(\tau y_2)\psi(\tau y_1)),\\ \hat{\omega}(\psi^*(y_2)\psi(y_1)) &= \omega(\psi(\tau y_1)\psi^*(\tau y_2)),\end{aligned}$$

so the covariances $\hat{\lambda}^{\pm}$ of $\hat{\omega}$ are:

$$\hat{\lambda}^{\pm} = \tau^*\lambda^{\mp}\tau.$$

B.2.2. *Time reversal on a spacetime.* Recall that if (M,g) is a spacetime, (M',g) is the same Lorentzian manifold with the reversed time orientation. Denoting with primes the objects associated to (M',g) we have:

$$\begin{aligned}&\gamma' = \gamma, \quad \kappa' = \kappa, \quad \beta' = -\beta, \quad \mathcal{N}^{\pm'} = \mathcal{N}^{\mp},\\ &\nabla^{S'} = \nabla^{S}, \quad \not{D}' = \not{D}, \quad \mathbb{D}' = -\mathbb{D}, \quad \mathbb{G}' = \mathbb{G},\\ &(\cdot|\cdot)_{M'} = (\cdot|\cdot)_{M}, \quad (\cdot|\cdot)_{\mathbb{D}'} = (\cdot|\cdot)_{\mathbb{D}},\end{aligned}$$

so $\mathrm{id} : (\mathrm{Sol}_{\mathrm{sc}}(M),(\cdot|\cdot)_{\mathbb{D}}) \to (\mathrm{Sol}_{\mathrm{sc}}(M'),(\cdot|\cdot)_{\mathbb{D}'})$ is a time reversal in the sense of B.2.1.

B.2.3. *Time reversal on quasi-free states.* If $\mathbb{\Lambda}^{\pm}$ resp. $C^{\pm}$ are the spacetime resp. solution space covariances of a quasi-free state ω on $\mathrm{CAR}(M)$, then

$$\mathbb{\Lambda}^{\pm'} = \mathbb{\Lambda}^{\mp}, \quad C^{\pm'} = C^{\mp}$$

are the spacetime resp. solution space covariances of the corresponding state ω' on $\mathrm{CAR}(M')$ (denoted by $\hat{\omega}$ in B.2.1). Since $\mathcal{N}^{\pm'} = \mathcal{N}^{\mp}$, we see that if ω is a Hadamard state on $\mathrm{CAR}(M)$, then ω' is a Hadamard state on $\mathrm{CAR}(M')$ (see Subsect. 4.2 for the definition of Hadamard states).

Appendix C. Null geodesics and Cauchy surfaces in Kerr spacetime

C.1. **Summary.** The first purpose of this appendix is to show that various hypersurfaces used throughout the text are Cauchy, and that the corresponding spacetime regions are therefore globally hyperbolic. We remark that the causal structure of the Kerr spacetime could be deduced from detailed knowledge of lower-dimensional diagrams, see [8] for the relevant framework. Here however we need to analyse the geodesics directly to treat the families of hypersurfaces that are of interest to us.

We also prove an important result used in the main text, namely (2) of Lemma C.4, which states that for sufficiently small a, null geodesics in $\mathrm{M_I}$ that do not meet $\mathscr{H}_-$ nor $\mathscr{I}_-$ always cross a region where the Killing vector fields $v_{\mathscr{H}}$ and $v_{\mathscr{I}}$ are time-like

C.2. **Null geodesics and first integrals.** We first recall some well-known facts about null geodesics in the Kerr spacetime, see [48, Ch. 4], [5, Ch. 7].

Denoting by $g_{a,M}$ the Kerr metric in the Boyer-Lindquist coordinates (r, t, θ, φ), the change of coordinates $r = M\tilde{r}$, $t = M\tilde{t}$ gives $g_{a,M} = M^2 g_{aM^{-1},1}$. After a trivial conformal transformation, we can assume that $M = 1$ to make the computations easier. However, we will state the main results about null geodesics for arbitrary values of M.

With the convention that $M = 1$, recall that we have the sub-extremality assumption $0 < a < 1$, and:

$$\Delta = r^2 - 2r + a^2, \quad \rho^2 = r^2 + a^2 \cos^2\theta,$$

and $r_\pm = 1 \pm \sqrt{1 - a^2}$ are the horizon radii.

Let $\gamma : I \ni s \mapsto x(s)$ be a null geodesic, where s is the affine parameter. We have the following three independent first integrals:

$$E = -\partial_t \cdot g\dot{x}, \quad L = \partial_\varphi \cdot g\dot{x}, \quad K = \text{ Carter constant},$$

which are constant along γ. One sets $Q = K - (L - aE)^2$, and one uses (E, L, Q) as independent first integrals. Any null geodesic starting away from the axis $\{\sin\theta = 0\}$ is entirely determined by its initial coordinates $(r, t, \theta, \varphi)(0)$, the signs of $r'(0)$ and $\theta'(0)$ and the constants (E, L, Q), see [48, Lem. 4.2.5].

C.3. **Equations of motion.** Let:

$$P(r) = (r^2 + a^2)E - aL, \quad D(\theta) = L - aE\sin^2\theta.$$

If $\gamma : I \ni s \mapsto x(s)$ is an affinely parametrized null geodesic, we have, see [48, Thm. 4.2.2]:

$$\begin{cases} \rho^4 \dot{r}^2 = R(r), \\ \rho^4 \dot{\theta}^2 = \Theta(\theta), \end{cases} \tag{C.1}$$

for

$$\begin{aligned} R(r) &= P^2(r) - \Delta(r)K, \\ &= E^2 r^4 + (a^2E^2 - L^2 - Q)r^2 + ((aE - L)^2 + Q)2r - a^2 Q, \\ \Theta(\theta) &= K - \sin^{-2}(\theta) D^2(\theta). \end{aligned}$$

Knowing the functions $r(s), \theta(s)$, the functions $t(s), \varphi(s)$ are then determined by the equations:

$$\begin{cases} \rho^2 \dot{\varphi} = \sin^{-2}\theta D(\theta) + a\dfrac{P(r)}{\Delta(r)}, \\ \rho^2 \dot{t} = aD(\theta) + \dfrac{(r^2 + a^2)P(r)}{\Delta(r)} =: T(r, \theta) \end{cases} \tag{C.2}$$

see [48, Prop. 4.1.5].

C.3.1. *Some more facts.* The *polar planes* $P = P(t_0, \varphi_0)$ are the submanifolds $\{t = t_0, \varphi = \varphi_0\}$. The *axis* is the submanifold $A = \{\sin\theta = 0\}$. We know, see [48, Cor. 4.2.8, Lem. 4.2.9], that if γ is a null geodesic then:

(1) $K \geq 0$,
(2) $K = 0$ iff γ is a principal null geodesic,
(3) $L = E = 0$ and $K > 0$ iff γ is in a time-like polar plane (hence in $\mathrm{M_{II}}$ (in order for the polar plane to be time-like),
(4) $K = L = 0$ and $E \neq 0$ iff γ is in $A \setminus (\mathscr{H}_+ \cup \mathscr{H}_-)$,
(5) $K = L = E = 0$ iff γ is in $\mathscr{H}_+ \cup \mathscr{H}_-$.

Moreover, see e.g. [48, Prop. 2.5.5], the horizons $\mathscr{H}_\pm$ are closed and totally geodesic, hence any geodesic tangent to $\mathscr{H}_\pm$ is entirely included in $\mathscr{H}_\pm$.

C.3.2. *Double zeros of R.* We now look at possible double zeros (roots) of R.

If $E = 0$, then $R(r)$ is a polynomial of degree 0 if $K = 0$, and 2 if $K \neq 0$. If $K = 0$, then $R(r) = -a^2 Q$ so R has double zeros only if $Q = 0$. If $K \neq 0$, then $K > 0$ and R has a double zero if $L^2 = (a^2 - 1)Q$, which implies that $Q < 0$ and $K = a^2 Q < 0$ which is a contradiction. Therefore if $E = 0$, R has a double zero only if $K = L = E = 0$, i.e. if γ is included in $\mathscr{H}_+ \cup \mathscr{H}_-$.

If $E \neq 0$ it is convenient to set

$$\xi = LE^{-1}, \quad \eta = QE^{-2}, \quad \kappa = KE^{-2} = \eta + (\xi - a)^2. \tag{C.3}$$

One has the following constraints on the integrals of motions for null geodesics:

$$\eta + (\xi - a)^2 \geq 0, \tag{C.4}$$

$$\eta < 0 \;\Rightarrow\; \eta^2 \leq a^2,\ 0 \leq |\xi| \leq |a| - \sqrt{|\eta|}, \tag{C.5}$$

see equations (194), (206) in [5, Ch. 7]. The following lemmas will be needed later on.

Lemma C.1. (1) *There are no double zeros of R in $]r_-, r_+[$.*
(2) *There exists $0 < a_0 < 1$ and $C > 0$ such that for $|a| \leq a_0$ and $E \neq 0$ the double zeros of R in $[r_-, +\infty[$ belong to $[3 - Ca, 3 + Ca]$ and one has $|\xi| \leq C$, $|\eta| \leq C$.*

Proof. By the computations in [5, p. 351] the conditions $R(r) = 0$, $\partial_r R(r) = 0$ imply that:

$$\xi = \frac{1}{a(r-1)}((r^2 - a^2) - r\Delta), \tag{C.6}$$

$$\eta = \frac{r^3}{a^2(r-1)^2}(4a^2 - r(r-3)^2), \tag{C.7}$$

see equations (224), (225) in [5, Ch. 7]. The conditions $\eta < 0$ and (C.5) are incompatible with (C.6), (C.7) (see [5, p. 352]). It follows hence from (C.7) that if r is a double zero of R then $r(r-3)^2 \leq 4a^2$. If $r \in]r_-, r_+[$ we have $\Delta(r) < 0$, i.e. $a^2 < 2r - r^2$ hence

$$r(r-3)^2 < 4(2r - r^2)$$

and therefore $r(r-1)^2 < 0$, which is a contradiction. This proves (1).

Let us now prove (2). By studying the graph of $f(r) = r(r-3)^2$ we obtain that the double zeros of R belong to

$$\Big[0, \frac{4}{9}a^2 + O(a^3)\Big] \cup \Big[3 - \frac{2}{\sqrt{3}}a + O(a^2), 3 + \frac{2}{\sqrt{3}}a + O(a^2)\Big].$$

Since $r_- = \frac{1}{2}a^2 + O(a^3)$, the first interval above is disjoint from $[r_-, +\infty[$ for a small enough. If r is a double zero in $[r_-, +\infty[$ then from $r = 3 + O(a)$ and (C.6) we obtain that $\xi \in O(1)$, and since $0 \leq 4a^2 - r(r-3)^2 \leq 4a^2$ we obtain that $\eta \in O(1)$. □

Lemma C.2. *Let $r_0 > r_+$ be a double zero of R. Then*

$$T(r_0, \theta) \neq 0, \ \forall\, \theta \in [0, \pi].$$

Proof. We note that

$$T(r, \theta) = \frac{\sigma^2(r, \theta)E - 2arL}{\Delta(r)},$$

where $\sigma^2(r, \theta) = (r^2 + a^2)^2 - a^2(\sin^2\theta)\Delta(r)$. Assume that $T(r_0, \theta) = 0$. Setting $\sigma_0 = \sigma(r_0, \theta), \Delta_0 = \Delta(r_0)$, we obtain $\xi = \frac{\sigma_0^2}{2ar_0}$. By equation (224) in [5, Ch. 7] we have

$$\xi = \frac{1}{a(r_0 - 1)}(r_0^2 - a^2 - r_0\Delta_0), \tag{C.8}$$

which yields:

$$\sigma_0^2(r_0 - 1) - 2r_0(r_0^2 - a^2 - r_0\Delta_0) = 0.$$

We write

$$\begin{aligned}\sigma_0^2(r_0 - 1) - 2r_0(r_0^2 - a^2 - r_0\Delta_0) = (r_0 - 1)(r_0^2 + a^2)^2 - 2r_0(r_0^2 - a^2)\\ + (2r_0^2 - a^2\sin^2\theta(r_0 - 1))\Delta_0.\end{aligned}$$

We have

$$2r_0^2 - a^2\sin^2\theta(r_0 - 1) \geq 2r_0^2 - a^2 r_0 = r_0(2r_0 - a^2) > 0,$$

and

$$\begin{aligned}(r_0 - 1)(r_0^2 + a^2)^2 - 2r_0(r_0^2 - a^2) &\geq 2r_0((r_0 - 1)(r_0^2 + a^2) - (r_0^2 - a^2))\\ &= 2r_0(r_0^3 + a^2 r_0 - r_0^2 - a^2 - r_0^2 + a^2)\\ &= 2r_0^2(r_0^2 + a^2 - 2r_0) = 2r_0^2\Delta_0 > 0.\end{aligned}$$

Therefore $\xi \neq \frac{\sigma_0^2}{2ar_0}$ which proves the lemma. □

Corollary C.3. *There exists $0 < a_1 \leq 1$ such that for $|a|M^{-1} < a_1$ the Killing vector fields $v_{\mathscr{H}}$ and $v_{\mathscr{I}}$ are future directed time-like at all points $(r, \theta, \varphi) \in [r_+, +\infty[\times \mathbb{S}^2$ such that r is a double zero of R.*

Proof. We can assume without loss of generality $M = 1$ as explained at the beginning of the section. Let $f_{\mathscr{H}/\mathscr{I}}(a, r, \theta) = v_{\mathscr{H}/\mathscr{I}} \cdot g v_{\mathscr{H}/\mathscr{I}}$. We have $f_{\mathscr{H}/\mathscr{I}}(0, 3, \theta) = -1/3$, which implies that there exists $\epsilon, \delta > 0$ such that $v_{\mathscr{H}_-/\mathscr{I}_-}$ are time-like for $|a| \leq \epsilon$ and $|r - 3| \leq \delta$. By applying Lemma C.1 we obtain that $v_{\mathscr{H}}$ and $v_{\mathscr{I}}$ are time-like at all points $(r, \theta, \varphi) \in [r_+, +\infty[\times \mathbb{S}^2$ such that r is a double zero of R. They are clearly future directed since $-\nabla t$ is future directed. □

C.4. **Null geodesics in $\mathrm{M_I}$.** We recall the classification of null geodesics according to the behavior of $r(s)$ for $s \to \infty$. By (C.1) the region $\{R(r) < 0\}$ is not accessible to null geodesics. One can then discuss the possible connected components of $\{R(r) \geq 0\}$, depending of the values of the first integrals E, L, Q.

Another standard fact, see e.g. [48, Ch. 4], is that $r(s)$ can reach a first order zero of R only for some finite affine time s_0, and then $\dot{r}(s)$ changes sign at $s = s_0$. On the contrary $r(s)$ can reach a double zero r_0 of R only for $s = \pm\infty$, or else $r(s) \equiv r_0$. These

facts follow easily from (C.1). The horizon $\{r = r_+\}$ can be crossed only transversally, at finite affine time, see [48, Sect. 4.4].

We obtain the following classification for future directed null geodesics.

In the tables below, *type* $[a, b]$ for example means that γ starts at $r = a$ and ends at $r = b$ for finite values of the affine parameter, while *type* $[a, b[$ means that γ ends at $r = b$ for infinite value of the affine parameter. Furthermore, r_0 denotes some double zero of R, and $[r_0]$ corresponds to a geodesic for which $r(s) = r_0$ for all s.

Table 1. $E = 0$	
Type	Constants
$[r_+ \to r_+]$	$E = 0,\ K > 0,\ L \neq 0$
$[r_+ \to \infty[$ or $]\infty \to r_+]$	$E = 0,\ K = 0,\ L \neq 0$ (princ. null)

Table 2. $E \neq 0$	
Type	Constants
$[r_+ \to r_+]$	$E \neq 0,\ Q > 0,\ R(r_+) > 0$ $E \neq 0,\ Q = 0,\ L \neq aE,\ R(r_+) > 0$
$[r_+ \to \infty[$ or $]\infty \to r_+]$	$E \neq 0,\ Q \geq 0,\ R(r_+) > 0$ $E \neq 0,\ Q < 0$
$]\infty \to \infty[$	$E \neq 0,\ Q > 0$ $E \neq 0,\ Q = 0,\ L \neq aE$
$[r_+ \to r_0[$ or $]r_0 \to r_+]$	$E \neq 0,\ Q > 0,\ R(r_+) > 0$ $E \neq 0,\ Q = 0,\ L \neq aE,\ R(r_+) > 0$
$]r_0 \to \infty[$ or $]\infty \to r_0[$	$E \neq 0,\ Q > 0,\ R(r_+) > 0$
$[r_0]$	$L = \frac{1}{a(r_0 - M)}[M(r_0^2 - a^2) - r_0\Delta(r_0)]E,$ $Q = \frac{r_0^3}{a^2(r_0-M)^2}[4M\Delta(r_0) - r_0(r_0 - M)^2]E^2$

To summarize the tables above, let γ a future directed null geodesic in $\mathrm{M_I}$, parametrized by $s \in I$. We have $I =]s_-, s_+[$ for

$$
\begin{aligned}
&-\infty = s_-,\ s_+ = +\infty \text{ if } \gamma \text{ is of type }]\infty \to \infty[,\ ]r_0 \to \infty[,\ ]\infty \to r_0[,\\
&-\infty < s_-,\ s_+ = +\infty \text{ if } \gamma \text{ is of type } [r_+ \to \infty[,\ [r_+ \to r_0[,\\
&-\infty = s_-,\ s_+ < +\infty \text{ if } \gamma \text{ is of type }]\infty \to r_+],\ ]r_0 \to r_+]\\
&-\infty < s_-,\ s_+ < +\infty \text{ if } \gamma \text{ is of type } [r_+ \to r_+].
\end{aligned}
$$

Lemma C.4. (1) *Let γ be a future directed null geodesic. Then*

$$
\begin{aligned}
&i)\quad \lim_{s \to s_\pm} t(s) = \pm\infty,\\
&ii)\quad \lim_{s \to s_-} t^*(s) = -\infty \text{ if } \gamma \text{ starts at } r_+.
\end{aligned}
\tag{C.9}
$$

(2) *There exists $0 < a_0 \leq 1$ such that for $|a|M^{-1} < a_0$ any future directed null geodesic which does not start at r_+ nor at ∞ passes through a region where the Killing vector fields $v_{\mathscr{H}}$ and $v_{\mathscr{I}}$ are both time-like.*

Proof. Again we assume $M = 1$. Statement (2) follows from Corollary C.3 and the tables above. Let us now prove (1).

Let us first check *ii)*. If γ starts at r_+, then γ enters $\mathrm{M_I}$ from $\mathrm{M'_{II}}$, and since $\mathrm{M_I}$ and $\mathrm{M'_{II}}$ belong to a common ${}^*\mathrm{K}$ patch we have $\lim_{s\to s_-} {}^*t(s)$ finite, $\lim_{s\to s_-} r(s) = r_+$, so $\lim_{s\to s_-} t^*(s) = -\infty$.

Let us now prove *i)*. We first assume that $E \neq 0$.

From (C.1), (C.2) we have

$$t = \pm \int \frac{T(r,\theta)}{R^{\frac{1}{2}}(r)} dr. \tag{C.10}$$

Case 1: assume that $\lim_{s\to s_\pm} r(s) = +\infty$. We have:

$$\frac{T(r,\theta)}{R^{\frac{1}{2}}(r)} = \frac{E}{|E|} + O(r^{-1}),\ r \to +\infty. \tag{C.11}$$

Note that $E > 0$ since γ is future directed. Therefore by (C.10) $\lim_{s\to s_\pm} t(s) = +\infty$.

Case 2: assume next that $\lim_{s\to s_\pm} r(s) = r_+$. By Table 2 we have $R^{\frac{1}{2}}(r_+) = |P(r_+)| \neq 0$ hence

$$\frac{T(r,\theta)}{R^{\frac{1}{2}}(r)} = \frac{2r_+}{(r-r_+)(r_+-r_-)} + O(1),\ r \to r_+, \tag{C.12}$$

which by (C.10) shows that $\lim_{s\to s_\pm} t(s) = +\infty$.

Case 3: assume that $\lim_{s\to s_\pm} r(s) = r_0$, where $r_0 \neq r_+$ is a double root of R. By Lemma C.2, $T(r_0,\theta) \neq 0$ for $\theta \in [0,\pi]$. If $r(s)$ is not constant, then (C.10) implies that $\lim_{s\to s_\pm} t(s) = +\infty$. If $r(s) \equiv r_0$ we use (C.2) to obtain again that $\lim_{s\to s_\pm} t(s) = +\infty$.

Consider now the case $E = 0$. If $K > 0$ and $L \neq 0$ then γ is of type $[r_+ \to r_+]$ and

$$\begin{aligned} &\rho^{-2}T = -\tfrac{2a^2Lr_+}{(r-r_+)(r_+-r_-)} + O(1),\\ &\rho^{-2}R^{\frac{1}{2}}(r) = |aL| + O(r-r_+), \end{aligned} \quad \text{when } r \to r_+,\ \text{uniformly in } \theta, \tag{C.13}$$

Again, the assertion of the lemma is satisfied. If $K = 0, L \neq 0$, γ is a principal null geodesic and *i)* is satisfied. If $K = L = 0$ γ is not in $\mathrm{M_I}$. □

C.5. **Null geodesics in** $\mathrm{M_{II}}$**.** In $\mathrm{M_{II}}$ we have other possibilities: a future directed null geodesic can originate at the bifurcation sphere $S(r_+)$ and terminate at the bifurcation sphere $S(r_-)$. We obtain the following table of null geodesics in $\mathrm{M_{II}}$:

Table 3. $E = 0$

Type	Constants
$[r_\pm \to r_\mp]$	$E = 0,\ K > 0,\ L \neq 0$ $E = 0,\ K = 0,\ L \neq 0$ (princ. null)
$[S(r_\pm) \to S(r_\mp)]$	$E = 0,\ K > 0,\ L \neq 0$

Table 4. $E \neq 0$	
Type	Constants
$[r_\pm \to r_\mp]$	$E \neq 0$, $Q \geq 0$, $R(r_\pm) > 0$ $E \neq 0$, $Q < 0$, $R(r_\pm) > 0$
$[r_- \to r_-]$	$E \neq 0$, $Q > 0$, $R(r_-) > 0$ $E \neq 0$, $Q = 0$, $L \neq aE$, $R(r_-) > 0$
$[r_+ \to r_+]$	$E \neq 0$, $Q > 0$, $R(r_+) > 0$ $E \neq 0$, $Q = 0$, $L \neq aE$, $R(r_+) > 0$
$[r_+ \to S(r_-)]$	$E \neq 0$, $Q > 0$, $R(r_-) = 0$ $E \neq 0$, $Q = 0$, $L \neq aE$, $R(r_-) = 0$
$[S(r_+) \to r_-]$	$E \neq 0$, $Q > 0$, $R(r_+) = 0$ $E \neq 0$, $Q = 0$, $L \neq aE$, $R(r_+) = 0$

Let γ be a future directed null geodesic in $\mathrm{M_{II}}$, parametrized by $s \in I$. We have $I =]s_-, s_+[$ for $-\infty < s_- < s_+ < +\infty$. We have the following analogue of Lemma C.4.

Lemma C.5. *Let γ be a future directed null geodesic in $\mathrm{M_{II}}$. Then:*

1) $\lim_{s\to s_-} t^*(s) = -\infty$ *if γ enters $\mathrm{M_{II}}$ from $\mathrm{M_{I'}}$,*
2) $\lim_{s\to s_-} t^*(s) = -\infty$ *if γ enters $\mathrm{M_{II}}$ from $\mathrm{M_{II'}}$,*
3) $\lim_{s\to s_-} t^*(s)$ *finite if γ enters $\mathrm{M_{II}}$ from $\mathrm{M_{I}}$,*
4) $\lim_{s\to s_+} t^*(s) = +\infty$ *if γ leaves $\mathrm{M_{II}}$ into $\mathrm{M_{III'}}$,*
5) $\lim_{s\to s_+} t^*(s) = +\infty$ *if γ leaves $\mathrm{M_{II}}$ into $\mathrm{M_{II'}}$,*
6) $\lim_{s\to s_+} t^*(s)$ *finite if γ leaves $\mathrm{M_{II}}$ into $\mathrm{M_{III}}$.*

Proof. We recall that

$$t^* = {}^*t + 2x(r), \quad \lim_{r\to r_\pm} x(r) = \mp\infty. \tag{C.14}$$

The blocks $\mathrm{M_{I'}}$, $\mathrm{M_{II}}$ and $\mathrm{M_{III'}}$ belong to a common ${}^*\mathrm{K}$ patch. This implies that if γ enters $\mathrm{M_{II}}$ from $\mathrm{M_{I'}}$ then $\lim_{s\to s_-} {}^*t(s)$ finite, $\lim_{s\to s_-} r(s) = r_+$, and if γ leaves $\mathrm{M_{II}}$ into $\mathrm{M_{III'}}$ then $\lim_{s\to s^+} {}^*t(s)$ finite, $\lim_{s\to s^+} r(s) = r_-$. Using (C.14) this implies 1) and 4).

Since $\mathrm{M_I}$, $\mathrm{M_{II}}$ and $\mathrm{M_{III}}$ belong to a common K^* patch we obtain similarly 3) and 6).

If γ enters $\mathrm{M_{II}}$ from $\mathrm{M_{II'}}$, then γ passes through the crossing sphere $S(r_+)$, hence $\lim_{s\to s_-} V(s) = 0$, which proves 2), using (5.10). Finally if γ leaves $\mathrm{M_{II}}$ into $\mathrm{M_{II'}}$, then γ passes through the other crossing sphere $S(r_-)$. One uses then the KBL coordinates used to construct the Kerr-Kruskal extension $D(r_-)$, see [48, Def. 3.4.5]. The same argument as before shows that $\lim_{s\to s_+} t^*(s) = +\infty$, which proves 5). □

C.6. **Cauchy surfaces.** We now prove that several hypersurfaces are Cauchy surfaces in $\mathrm{M_I}, \mathrm{M_{I\cup II}}$ or M. To prove that a set S is a Cauchy surface in some spacetime (M, g) we use the following facts. We recall that a set $S \subset M$ is called *achronal* if each time-like curve intersects S at most once.

Theorem C.6. (1) *A closed achronal set S is a Cauchy surface iff each maximal null geodesic intersects S and enters $I^+(S)$ and $I^-(S)$.*

(2) *A closed connected non time-like hypersurface S is achronal if $M \setminus S$ is disconnected.*

Statement (1) can be found in [30], [61, Thm. 8.3.7], and statement (2) in [48, Corr. 3.10.2].

We will use the following corollary of Theorem C.6.

Corollary C.7. *Suppose $u \in C^\infty(M)$ is such that:*
(1) *∇u is time-like,*
(2) *$\sup_\gamma u = +\infty$ and $\inf_\gamma u = -\infty$ for any maximal future directed null geodesic γ.*
Then the level sets $S_T = u^{-1}(\{T\})$ are Cauchy surfaces in (M, g).

Proof. Since ∇u is time-like, S_T is space-like and achronal and clearly closed and connected. Possibly replacing u by $-u$, we can assume that ∇u is future directed. This implies that $\{u(x) > T\} \subset I_+(S_T)$ and $\{u(x) < T\} \subset I_-(S_T)$, since the integral curves of ∇u are time-like future directed. Using (2) this shows that any maximal null geodesic intersects S_T and enters $I^\pm(S_T)$. □

Proposition C.8. *The surfaces $\Sigma_T = \{t = T\}$ are Cauchy surfaces in $\mathrm{M_I}$.*

Proof. $g^{tt} < 0$ in $\mathrm{M_I}$, so ∇t is time-like in $\mathrm{M_I}$. We apply then Lemma C.4 *i)* and Corollary C.7. □

C.6.1. *Auxiliary Cauchy surfaces in* $\mathrm{M_I}$. Recall that in 8.3.1 we defined the family of hypersurfaces $\overline{\Sigma}_T, \widetilde{\Sigma}_T$. We now introduce families $\overline{\Sigma}_{T,n}, \widetilde{\Sigma}_{T,n}$ of Cauchy surfaces in $\mathrm{M_I}$ converging to $\overline{\Sigma}_T, \widetilde{\Sigma}_T$ when $n \to \infty$.

We start with defining $\overline{\Sigma}_{T,n}$. We set $x_n(r) = \min(r - 2\ln r, n)$ for $n \in \mathbb{N}$ and

$$\overline{\Sigma}_{T,n} = \{t + x_n(r) = T\}.$$

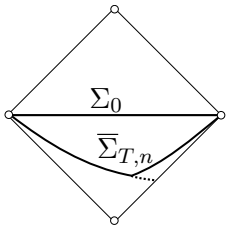

FIGURE 8. The Cauchy surfaces $\overline{\Sigma}_{T,n}$.

Proposition C.9. *The surfaces $\overline{\Sigma}_{T,n}$ for $T \ll -1, n \in \mathbb{N}$ are Cauchy surfaces in $\mathrm{M_I}$.*

Proof. Let $u_n(t, r) = t + x_n(r)$. We have using (5.1):

$$\begin{aligned} -du_n \cdot g^{-1} du_n &\geq \frac{1}{\Delta\rho^2}((r^2+a^2)^2 - a^2\Delta - \Delta^2(1-2r^{-1})^2) \\ &\geq \frac{1}{\Delta\rho^2}((r^2+a^2)^2 - a^2\Delta - \Delta^2) \\ &= \frac{1}{\Delta\rho^2}(2r(r^2+a^2) + (2r - a^2)\Delta) > 0 \text{ for } r > r_+, \end{aligned}$$

so ∇u_n is time-like in $\mathrm{M_I}$. For each n, $u_n(t, r) - t$ is bounded, so we conclude again by Lemma C.4 and Corollary C.7. □

Let us now define $\widetilde{\Sigma}_{T,n}$. We set $\tilde{x}_n(r) = \min(-n, \tilde{x}(r))$ for $n \in \mathbb{N}$, where $\tilde{x}(r)$ is defined in (8.5) and

$$\widetilde{\Sigma}_{T,n} = \{t + \tilde{x}_n(r) = T\}.$$

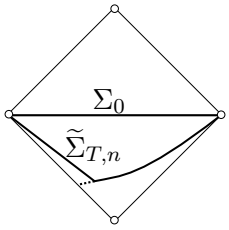

FIGURE 9. The Cauchy surfaces $\widetilde{\Sigma}_{T,n}$.

Proposition C.10. *The surfaces $\widetilde{\Sigma}_{T,n}$ for $T \ll -1, n \in \mathbb{N}$ are Cauchy surfaces in $\mathrm{M_I}$.*

Proof. Let $\tilde{u}_n(t,r) = t - \tilde{x}_n(r)$. We have

$$\begin{aligned} -d\tilde{u}_n \cdot g^{-1} d\tilde{u}_n &\geq \frac{1}{\Delta\rho^2}((r^2+a^2)^2 - a^2\Delta - \Delta^2(\tilde{x}_n')^2) \\ &\geq \frac{1}{\Delta\rho^2}((r^2+a^2)^2 - a^2\Delta - \Delta^2(y')^2) = \frac{1}{\rho^2}, \end{aligned}$$

where we recall that $y(r)$ is defined in (8.4). It follows that $\nabla\tilde{u}_n$ is time-like in $\mathrm{M_I}$.For each n $\tilde{u}_n(t,r) - t$ is bounded, and we complete the proof as for Proposition C.9. □

C.6.2. *Cauchy surfaces in* $\mathrm{M_{I\cup II}}$. We construct a family of space-like Cauchy surfaces in $\mathrm{M_{I\cup II}}$.

It is easy to see that for $T \gg 1$ the equation $x(r) - r = -T$ has a unique solution r_T in $]r_+, 2r_+[$ with $r_T - r_+ \in O(\mathrm{e}^{-cT})$ for some $c > 0$. We fix a smooth decreasing function χ with $\chi = 1$ in $]-\infty, r_- + \epsilon]$ and $\chi = 0$ in $]\frac{1}{2}(r_+ + r_-), +\infty[$. We define $v(r)$ by

$$v'(r) = 1 + \chi(r)\frac{1}{r - r_-}, \quad v(r_+) = r_+$$

so

$$v(r) = r \text{ in } [r_+, +\infty[, \quad v(r) \to -\infty \text{ when } r \to r_-. \tag{C.15}$$

We define the function u_T on $\mathrm{M_{I\cup II}}$ by

$$u_T = \begin{cases} t^* - v(r) + T, \text{ for } r \leq r_T, \\ t \quad \text{for } r_T < r. \end{cases}$$

We set

$$Z_T := \{u_T(x) = 0\} \subset \mathrm{M_{I\cup II}}.$$

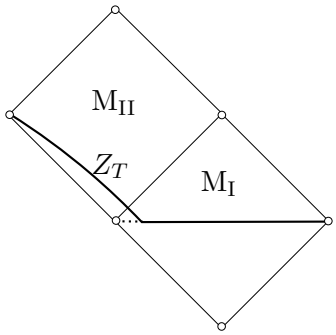

FIGURE 10. The Cauchy surfaces Z_T.

Proposition C.11. *The surfaces Z_T for $T \gg 1$ are Cauchy surfaces in* $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$.

Proof. In $\{r > r_T\}$ we have $du_T \cdot g^{-1} du_T = g^{tt} < 0$. In $\{r < r_T\}$ we have

$$\begin{aligned} du_T \cdot g^{-1} du_T &= \frac{1}{\rho^2}\big(\Delta(r)(v')^2(r) - 2(r^2+a^2)v'(r) + a^2\sin^2\theta\big) \\ &\leq \frac{1}{\rho^2}\big(\Delta(r)(v')^2(r) - 2(r^2+a^2)v'(r) + a^2\big). \end{aligned}$$

In $[r_+, +\infty[$ we have

$$\Delta(r)(v')^2(r) - 2(r^2+a^2)v'(r) + a^2 = \Delta(r) - 2(r^2+a^2) + a^2 = -r^2 - 2r < 0.$$

In $]r_-, r_+]$ we have $\Delta(r) \leq 0$ and $(v')^2(r) \geq v'(r)$ so

$$\begin{aligned} \Delta(r)(v')^2(r) - 2(r^2+a^2)v'(r) + a^2 &\leq (\Delta(r) - 2(r^2+a^2))v'(r) + a^2 \\ &= (-r^2-2r)v' + a^2 - a^2 v' \leq -r^2 - 2r < 0, \end{aligned}$$

since $v' \geq 1$. It follows that ∇u_T is time-like in $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$. Let now $\gamma : J \ni s \mapsto x(s)$ be a future directed null geodesic in $\mathrm{M}_{\mathrm{I}\cup\mathrm{II}}$. We claim that

$$\sup_\gamma u_T = +\infty, \tag{C.16}$$

$$\inf_\gamma u_T = -\infty, \tag{C.17}$$

which using Corollary C.7 will complete the proof.

Let us set $\gamma_{\mathrm{I}} = \gamma \cap \mathrm{M}_{\mathrm{I}}$, $\gamma_{\mathrm{II}} = \gamma \cap \mathrm{M}_{\mathrm{II}}$.

If $\gamma_{\mathrm{I}} = \gamma_{\mathrm{II}} = \emptyset$, then γ is included in the horizon $\mathscr{H}_+$ and γ is a rest photon, see [48, Lem. 4.2.9]. By [48, Lem. 3.4.10] we have $V(x(s)) = s$ which implies that $\sup_\gamma t^* = +\infty$, $\inf_\gamma t^* = -\infty$ and hence (C.16), (C.17), since $v(r_+)$ is finite.

It remains to check (C.16), (C.17) if γ_{I} or γ_{II} is not empty. Let us first check (C.16).

Case I: $\gamma_{\mathrm{II}} \neq \emptyset$. Using Lemma C.5 4), 5), 6) and the fact that $\lim_{r\to r_-} v(r) = -\infty$, we obtain (C.16).

Case II: $\gamma_{\mathrm{II}} = \emptyset$, $\gamma_{\mathrm{I}} \neq \emptyset$. Let $]s_-, s_+[$ be the interval of affine parameter s such that $x(s) \in \mathrm{M}_{\mathrm{I}}$. Note that γ_{I} cannot end at r_+ otherwise γ enters M_{II}. If γ_{I} ends at ∞ or some double root r_0, then by Lemma C.4 we have $\lim_{s\to s_+} t(s) = +\infty$ and since $r_T < r_0$ (at least for T large enough), we have $u_T(x(s)) = t(s)$ for s close to s_+, which shows (C.16).

Let us now check (C.17).

Case I: $\gamma_{\mathrm{I}} \neq \emptyset$.

If $\gamma_{\rm I}$ starts at r_+ then by Lemma C.4, we have $\lim_{s\to s_-} t^* = -\infty$, $\lim_{s\to s_-} r(s) = r_+$ which implies (C.17).

If $\gamma_{\rm I}$ starts at ∞ or some double root r_0, then by Lemma C.4 we have $\lim_{s\to s_-} t(s) = -\infty$ and since $r_T < r_0$ for T large enough, we have $u_T(x(s)) = t(s)$ for s close to s_- hence (C.17) holds also in this case.

Case II: $\gamma_{\rm I} = \emptyset$, $\gamma_{\rm II} \neq \emptyset$. We use Lemma C.5 1), 2) and the fact that $v(r_+)$ is finite to obtain (C.17). □

C.6.3. *Cauchy surfaces in* M.

Proposition C.12. *The surface* $\Sigma_{\rm M} := \{U = V\}$ *is a space-like Cauchy surface in* M.

Proof. Using (5.10), we obtain first that

$$\Sigma_{\rm M} = (\{t = 0\} \cap {\rm M_I}) \cup (\{t = 0\} \cap {\rm M_{I'}}) \cup S(r_+), \tag{C.18}$$

and that over $\Sigma_{\rm M} \cap {\rm M_I}$ we have $d(U-V) = -2\kappa_+ U dt$, while over $\Sigma_{\rm M} \cap {\rm M_{I'}}$ $d(U-V) = 2\kappa_+ U dt$. Since $g^{tt} < 0$ on ${\rm M_I}, {\rm M_{I'}}$ we obtain that $d(U-V)\cdot g^{-1} d(U-V) < 0$ on $\Sigma_{\rm M} \setminus S(r_+)$. On $S(r_+)$ we see directly that $d(U-V)\cdot g^{-1} d(U-V) < 0$, using (5.8). This implies that $\Sigma_{\rm M}$ is space-like, hence achronal by Theorem C.6 (2), since ${\rm M} \setminus \Sigma_{\rm M}$ is clearly disconnected.

Let us now γ a future directed null geodesic in M. If γ passes through ${\rm M_I}$, then by Proposition C.8 γ crosses $\{t = 0\} \cap {\rm M_I}$ and enters $I^{\pm}(\{t = 0\} \cap {\rm M_I})$, hence crosses $\Sigma_{\rm M}$ and enters $I^{\pm}(\Sigma_{\rm M})$. The similar conclusion holds if γ passes through ${\rm M_{I'}}$.

Assume now that γ passes through ${\rm M_{II}}$. If γ enters ${\rm M_{II}}$ from ${\rm M_I}$ or ${\rm M_{I'}}$ we argue as before. Otherwise γ enters ${\rm M_{II}}$ through $S(r_+)$. Since $S(r_+)$ is totally geodesic, γ is transverse to $S(r_+)$ and passes through ${\rm M_{II}}$ and ${\rm M_{II'}}$, hence enters $I^{\pm}(\Sigma_{\rm M})$. We argue similarly if γ passes through ${\rm M'_{II}}$.

Finally if γ passes through one horizon without entering any of the Boyer-Lindquist blocks, then γ is included in a horizon, for example in $\mathscr{H}_R = \{U = 0\}$ and γ is a rest photon, see [48, Lem. 4.2.9]. By [48, Lem. 3.4.10] we have $V(x(s)) = s$ so γ again crosses $\Sigma_{\rm M}$ and enters $I^{\pm}(\Sigma_{\rm M})$. Applying Theorem C.6 (1) we obtain that $\Sigma_{\rm M}$ is a Cauchy surface. □

Appendix D. Spectral projections in exponential coordinates

D.1. **A one-dimensional lemma.** For $\beta > 0$, let $\chi^{\pm}_{\beta}(s) = (1 + {\rm e}^{\mp\beta s})^{-1}$. Let $\mathbf{1}_{\mathbb{R}^{\pm}}(s)$ be the characteristic function of $\pm[0, +\infty[$.

We consider the momentum operator $D_x = {\rm i}^{-1}\partial_x$ acting in $L^2(\mathbb{R})$. Let us denote by

$$\chi^{\pm}_{\infty}(D_x) := \imath^* \circ \mathbf{1}_{\mathbb{R}^{\pm}}(D_x) \circ \imath$$

the restriction of $\mathbf{1}_{\mathbb{R}^{\pm}}(D_x)$ to $L^2(\mathbb{R}^+)$ using the canonical embedding $\imath : L^2(\mathbb{R}^+) \to L^2(\mathbb{R})$. We also consider the selfadjoint generator of dilations

$$A = \frac{1}{2}(xD_x + D_x x) = {\rm i}^{-1}(x\partial_x + \frac{1}{2})$$

defined on $L^2(\mathbb{R}^+)$ by ${\rm e}^{{\rm i}sA}u(x) = {\rm e}^{s/2}u({\rm e}^s x)$, $s \in \mathbb{R}$.

Lemma D.1. *On* $L^2(\mathbb{R}^+)$ *we have* $\chi^{\pm}_{\infty}(D_x) = \chi^{\pm}_{2\pi}(A)$.

Proof. Using the formula for the Fourier transform of $\mathbf{1}_{\mathbb{R}^{\pm}}$ (see e.g. [24, Ch. 2.3, (22)]) one finds that the Schwartz kernel of $\chi^{\pm}_{\infty}(D_x)$ equals

$$\chi^{\pm}_{\infty}(D_x)(x, y) = \pm{\rm i}(2\pi)^{-\frac{1}{2}}(x - y \pm {\rm i}0)^{-1}. \tag{D.1}$$

We will use the following convention for the Mellin transform on $C_c^\infty(\mathbb{R}^+)$ and the inverse Mellin transform:

$$\begin{aligned}(\mathcal{M}f)(\sigma) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^+} x^{-\frac{1}{2}-\mathrm{i}\sigma} f(x)dx,\\ (\mathcal{M}^{-1}g)(x) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^{-\frac{1}{2}+\mathrm{i}\sigma} g(\sigma)d\sigma.\end{aligned} \tag{D.2}$$

It is well known that $\mathcal{M}$ extends to a unitary map $\mathcal{M} : L^2(\mathbb{R}^+) \to L^2(\mathbb{R})$, and the extension of $\mathcal{M}^{-1}$ to $L^2(\mathbb{R})$ is the inverse of $\mathcal{M}$. Another essential fact is that $\mathcal{M}$ diagonalizes the generator of dilations, meaning that

$$\chi_\beta^\pm(A) = \mathcal{M}^{-1} \circ \chi_\beta^\pm(\sigma) \circ \mathcal{M}, \tag{D.3}$$

where $\chi_\beta^\pm(\sigma)$ denotes the operator of multiplication by $\chi_\beta^\pm$. A brief computation using (D.2) shows that the Schwartz kernel of (D.3) equals

$$\chi_\beta^\pm(A)(x,y) = \frac{1}{y}(\mathcal{M}^{-1}\chi_\beta^\pm)\big(\tfrac{x}{y}\big). \tag{D.4}$$

On the other hand, (as follows from e.g. Formula 2.4 in [47])

$$(\mathcal{M}^{-1}\chi_{2\pi}^\pm)(x) = \pm\mathrm{i}(2\pi)^{-\frac{1}{2}}(x-1\pm\mathrm{i}0)^{-1}$$

in the sense of distributions. Plugging this into (D.4) and comparing with (D.1) yields the result. □

Let us now rename the variable x on $\mathbb{R}^+$ to U, and let us consider the change of coordinates $U = \mathrm{e}^{-\kappa u}$ for some $\kappa > 0$. This change of coordinates is implemented by a unitary map $\Phi : L^2(\mathbb{R}^+, dU) \to L^2(\mathbb{R}, du)$, which satisfies

$$\Phi \circ (-\kappa(U\partial_U + \partial_U U)) = D_u \circ \Phi.$$

From Lemma D.1 we obtain immediately

$$\chi_\infty^\pm(D_U) = \Phi^{-1} \circ \chi_\beta^\mp\big(D_u\big) \circ \Phi, \text{ with } \beta = \frac{2\pi}{\kappa}. \tag{D.5}$$

This identity (together with its analogue for bosons) plays an important rôle in the description of the Unruh and Hawking effect.

Appendix E. Wavefront sets and oscillatory test functions

E.1. **Oscillatory test functions.** We first recall a well-known characterization of the wavefront set of a distribution using oscillatory test functions. The equivalence with other standard definitions, stated in Lemma E.1 below, follows from e.g. [20, Sect. 1.3].

Let $\Omega \subset \mathbb{R}^n$ be an open set. For $x \in \Omega$, $q = (y,\eta) \in T^*\Omega \setminus o$ and $\chi \in C_c^\infty(\Omega)$ we denote

$$v_q^\lambda(x) := \chi(x)\mathrm{e}^{\mathrm{i}\lambda(x-y)\cdot\eta}, \quad \lambda \geq 1. \tag{E.1}$$

We then extend the definition to manifolds by chart diffeomorphism pullback. We will say that a function v_q^λ of this form is an *oscillatory test function at* $q_0 = (x_0,\xi_0)$ if $v_q^\lambda(x_0) \neq 0$. Oscillatory test functions can be used to give the following elementary characterization of the wavefront set.

Let $(\cdot|\cdot)_M$ be the $L^2(M, d\mu)$ pairing associated to some arbitrary smooth density $d\mu$.

Lemma E.1. *Let $u \in \mathcal{D}'(M)$ and $(x_0, \xi_0) \in T^*M \setminus o$. Then $(x_0, \xi_0) \notin \mathrm{WF}(u)$ iff there exists an oscillatory test function v_q^λ at q_0 such that for all $N \in \mathbb{N}$,*

$$|(v_q^\lambda | u)_M| \leq C_N \lambda^{-N}, \ \lambda \geq 1, \tag{E.2}$$

uniformly for q in a neighborhood of (x_0, ξ_0) in $T^*M \setminus o$.

This fact extends in a straightforward way to distributional sections of a hermitian vector bundle.

A very simple, but useful observation is that microlocal regularity can be tested with more general oscillatory functions.

Definition E.2. We say that w_q^λ is a *generalized oscillatory test function at* $q_0 = (x_0, \xi_0) \in T^*M \setminus o$ if it is of the form $w_q^\lambda = A^* v_q^\lambda$, where $A \in \Psi^0(M)$ is properly supported and elliptic at q_0, and v_q^λ is an oscillatory test function at q_0.

Lemma E.3. *Let $u \in \mathcal{D}'(M)$ and $q_0 \in T^*M \setminus o$. Then $q_0 \notin \mathrm{WF}(u)$ iff there exists a generalized oscillatory test function w_q^λ at q_0 such that for all $N \in \mathbb{N}$,*

$$|(w_q^\lambda | u)_M| \leq C_N \lambda^{-N}, \ \lambda \geq 1,$$

uniformly for q in a neighborhood of q_0 in $T^*M \setminus o$.

Proof. Let A and v_q^λ be as in Definition E.2. Since A is elliptic at q_0, $q_0 \in \mathrm{WF}(u)$ iff $q_0 \in \mathrm{WF}(Au)$. We also have $(w_q^\lambda | u)_M = (v_q^\lambda | Au)_M$, so the assertion follows from Lemma E.1 applied to Au. □

The next lemma shows that the oscillatory function w_q^λ (and the neighborhood of q_0 where the estimate holds) can be chosen in a uniform way if u varies in some set.

Lemma E.4. *Let $\mathcal{X} \subset \mathcal{D}'(M)$ and let $\Gamma \subset T^*M \setminus o$ be closed. Then $\mathrm{WF}(u) \subset \Gamma$ for all $u \in \mathcal{X}$ iff for all non-zero $q_0 \in T^*M \setminus \Gamma$ there exists a generalized oscillatory test function w_q^λ at q_0 such that for all $u \in \mathcal{X}$ and $N \in \mathbb{N}$,*

$$|(w_q^\lambda | u)_M| \leq C_{u,N} \lambda^{-N}, \ \lambda \geq 1, \tag{E.3}$$

uniformly for q in a neighborhood of q_0 in $T^*M \setminus o$.

Proof. Suppose that $\mathrm{WF}(u) \subset \Gamma$ for all $u \in \mathcal{X}$. Let $q_0 \in T^*M \setminus \Gamma$. Let $A \in \Psi(M)$ be properly supported and such that $\mathrm{WF}'(A) \cap \Gamma = \emptyset$ and A is elliptic at q_0. Then $Au \in C^\infty(M)$ for all $u \in \mathcal{X}$. Thus, if v_q^λ is an arbitrary oscillatory test function at q_0, then for all $u \in \mathcal{X}$,

$$|(v_q^\lambda | Au)_M| \leq C_{u,N} \lambda^{-N}, \ \lambda \geq 1, \ N \in \mathbb{N},$$

uniformly in q. Thus, the assertion (E.3) follows by setting $w_q^\lambda = A^* v_q^\lambda$.

The opposite direction trivially follows from Lemma E.3. □

E.2. **Wavefront set estimates.** The proposition below provides a refinement of the general strategy used by Moretti in [45] and in subsequent works [11, 12, 2, 28] to estimate wave front sets of solutions in terms of their traces on a null hypersurface. Strictly speaking, we will use its generalization to distributional sections of vector bundles, which is immediate.

Proposition E.5. *Let (M, g) be an oriented and time oriented Lorentzian manifold of dimension n, and let $S \subset M$ be a null hypersurface equipped with a smooth density dm. For $u \in \mathcal{E}'(S)$ we define $\delta_S \otimes u \in \mathcal{E}'(M)$ by:*

$$\int_M (\delta_S \otimes u)\varphi \, d\mathrm{vol}_g := \int_S u\varphi \, dm, \quad \varphi \in C^\infty_{\mathrm{c}}(M).$$

Let also X be a vector field on M, tangent to S, null, future directed on S and suppose $G \in \mathcal{D}'(M \times M)$ satisfies $\mathrm{WF}(G)' \subset \{(q, q') : q \sim q'\}$. Then for any $u \in \mathcal{E}'(M)$ one has the implication:

$$\mathrm{WF}(u) \subset \{(y, \eta) \in T^*S \setminus o : \pm\eta \cdot X(y) \geq 0\} \Rightarrow \mathrm{WF}(G(\delta_S \otimes u)) \cap \pi^{-1}(M \setminus S) \subset \mathcal{N}^\pm.$$

Proof. Since $\mathrm{WF}(G)'_M = \emptyset$ (i.e. $\mathrm{WF}(G)'$ has no points of the form $(x_1, 0, x_2, \xi_2)$), we can apply G to $\delta_S \otimes u$ and we have (see e.g. [27, 7.2.7])

$$\mathrm{WF}(G(\delta_S \otimes u)) \subset \mathrm{WF}(G)'(\mathrm{WF}(\delta_S \otimes u)). \tag{E.4}$$

Denoting by $i : S \to M$ the canonical injection, we have:

$$\mathrm{WF}(\delta_S \otimes u) \subset (i^*)^{-1}(\mathrm{WF}(u)) \cup N^*S, \tag{E.5}$$

where $N^*S = \{(x, \xi) \in T^*M \setminus o : x \in S, \ \xi_{|T_xS} = 0\}$ is the conormal bundle to S.

Let now $(x_1, \xi_1) \in \mathrm{WF}(G(\delta_S \otimes u))$ with $x_1 \notin S$. By (E.4) there exists $(x_0, \xi_0) \in \mathrm{WF}(\delta_S \otimes u)$ such that $(x_1, \xi_1) \sim (x_0, \xi_0)$. Since $g_{|T_{x_0}S}$ is positive semi-definite with kernel $\mathbb{R}X(x_0)$, we can find $L \subset T_{x_0}S$ space-like with $T_{x_0}S = L \oplus \mathbb{R}X(x_0)$. The orthogonal $L^\perp$ is time-like and 2-dimensional, hence contains two null lines, $\mathbb{R}X(x_0)$ and $\mathbb{R}v$ for $v \in T_{x_0}M$ transverse to S. We can assume that $X(x_0) \cdot g(x_0)v = -1$ and v is future directed.

We fix a basis $(w_1, \ldots, w_{n-2})$ of L and denote by $x = (y_1, y_2, y')$, $y' \in \mathbb{R}^{n-2}$ the coordinates in the basis $(v, X(x_0), w_1, \ldots, w_{n-2})$ of $T_{x_0}M$. We have then

$$x \cdot g(x_0)x = -2y_1y_2 + y' \cdot hy', \text{ where } h > 0,$$

and consequently, for $\xi_0 = (\eta_1, \eta_2, \eta')$ expressed in dual coordinates, we have

$$\xi_0 \cdot g(x_0)^{-1}\xi_0 = -2\eta_1\eta_2 + \eta' \cdot h^{-1}\eta'. \tag{E.6}$$

Since $(x_0, \xi_0) \in \mathcal{N}$, we have $\xi_0 \cdot g^{-1}(x_0)\xi_0 = 0$. On the other hand, from (E.5) either $(x_0, \eta_2, \eta') \in \mathrm{WF}(u)$ or $\eta_2 = \eta' = 0$, i.e. $(x_0, \xi_0) \in N^*S$. Since h is positive definite and the l.h.s. of (E.6) vanishes, $\eta_2 = 0$ implies $\eta' = 0$. Therefore we have $\xi_0 = (\eta_1, \eta_2, \eta')$ with either

$$-2\eta_1\eta_2 + \eta' \cdot h\eta' = 0, \quad (x_0, \eta_2, \eta') \in \mathrm{WF}(u), \quad \eta_2 \neq 0, \tag{E.7}$$

or $(x_0, \xi_0) \in N^*S$. Let us first consider the second case. The fact that S is null is equivalent to $N^*S \subset \mathcal{N}$, which using the fact that N^*S is a Lagrangian submanifold of T^*M implies that N^*S is invariant under the bicharacteristic flow. Therefore the null bicharacteristic from (x_0, ξ_0) stays in N^*S, hence above S, and thus cannot reach the point (x_1, ξ_1) which is above $M \setminus S$.

Let us now consider the first case. Since by assumption $\mathrm{WF}(u) \subset \{(y, \eta) \in T^*S \setminus o : \pm\eta \cdot X(y) \geq 0\}$ we deduce from (E.7) that $\pm\eta_2 > 0$ and $\pm\eta_1 = \eta_2^{-1}\eta' \cdot h\eta' > 0$. Therefore $(x_0, \xi_0) \in \mathcal{N}^\pm$ hence $(x_1, \xi_1) \in \mathcal{N}^\pm$ since $(x_1, \xi_1) \sim (x_0, \xi_0)$. This completes the proof of the proposition. □

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