



HAL
open science

A biharmonic transmission problem in L_p -spaces

Alexandre Thorel

► **To cite this version:**

| Alexandre Thorel. A biharmonic transmission problem in L_p -spaces. 2020. hal-02927622v1

HAL Id: hal-02927622

<https://hal.science/hal-02927622v1>

Preprint submitted on 1 Sep 2020 (v1), last revised 29 Nov 2021 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A biharmonic transmission problem in L^p -spaces

Alexandre Thorel

Normandie Univ, UNIHAVRE, LMAH, FR-CNRS-3335, ISCN, 76600 Le Havre, France.

alexandre.thorel@univ-lehavre.fr

Abstract

In this work we study, by a semigroup approach, a transmission problem based on biharmonic equations with boundary and transmission conditions, in two juxtaposed habitats. We give a result of existence and uniqueness of the classical solution in L^p -spaces, for $p \in (1, +\infty)$, using analytic semigroups and operators sum theory in Banach spaces. To this end, we invert explicitly the determinant operator of the transmission system in L^p -spaces using the \mathcal{E}_∞ -calculus and the Dore-Venni sums theory.

Key Words and Phrases: Analytic semigroups, biharmonic equations, functional calculus, interpolation spaces, maximal regularity.

2010 Mathematics Subject Classification: 35B65, 35J48, 35R20, 47A60, 47D06.

1 Introduction

In this work, we consider a system of linear biharmonic equations posed on two juxtaposed domains and coupled through boundary conditions at the interface. Throughout the paper we shall impose the continuity of the flux, of the dispersal and of its flux across the interface. Using an operator approach, we investigate the existence, uniqueness as well as maximal L^p -regularity for such a problem.

Transmission problems arise in various applicative fields including engineering, physics and biology. Here, we refer the reader for instance to [2], [11] or [17] for applications in plate theory, to [20], [29] or [37] for applications in electromagnetism and to [12], [23] or [36] for other applications in population dynamics. Let us also mention that mathematical models involving biharmonic operators also arise in various fields such as elasticity for instance see [8], [18] or [33], electrostatic see [5], [15] or [24], plate theory [2], [11] or [17] or population dynamics [7], [21], [22] or [28].

In this work, we consider a n -dimensional (with $n \geq 2$) straight cylinder of the form $\Omega = (a, b) \times \omega$ where $a < b$ are two given real numbers while the section $\omega \subset \mathbb{R}^{n-1}$ denotes a smooth bounded domain. This cylinder Ω is split into two (open) sub-cylinders Ω_\pm and an interface Γ given for some $\gamma \in (a, b)$ by

$$\Omega_- = (a, \gamma) \times \omega, \quad \Omega_+ = (\gamma, b) \times \omega \quad \text{and} \quad \Gamma = \{\gamma\} \times \omega,$$

so that $\Omega = \Omega_- \cup \Gamma \cup \Omega_+$.

We consider the following biharmonic equations,

$$(EQ_{pde}) \begin{cases} k_- \Delta^2 u_- = g_- & \text{in } \Omega_- \\ k_+ \Delta^2 u_+ = g_+ & \text{in } \Omega_+, \end{cases}$$

where $g_- \in L^p(\Omega_-)$, $g_+ \in L^p(\Omega_+)$ are given and $k_+, k_- > 0$.

We denote by (x, y) the spatial variables with $x \in (a, b)$ and $y \in \omega$. Then, we consider the following conditions on $\partial\Omega \setminus \Gamma$, the lateral boundary of Ω ,

$$(BC_{pde}) \begin{cases} (1) \begin{cases} u_-(x, \zeta) = 0, & x \in (a, \gamma), \zeta \in \partial\omega, & u_+(x, \zeta) = 0, & x \in (\gamma, b), \zeta \in \partial\omega \\ \Delta u_-(x, \zeta) = 0, & x \in (a, \gamma), \zeta \in \partial\omega, & \Delta u_+(x, \zeta) = 0, & x \in (\gamma, b), \zeta \in \partial\omega \end{cases} \\ (2) \begin{cases} u_-(a, y) = \varphi_1^-(y), & u_+(b, y) = \varphi_1^+(y), & y \in \omega \\ \frac{\partial u_-}{\partial x}(a, y) = \varphi_2^-(y), & \frac{\partial u_+}{\partial x}(b, y) = \varphi_2^+(y), & y \in \omega, \end{cases} \end{cases}$$

where φ_1^\pm and φ_2^\pm will be given in appropriated spaces. The system is coupled on the interface Γ where we impose the following continuity conditions,

$$(TC_{pde}) \left\{ \begin{array}{ll} u_- = u_+ & \text{on } \Gamma \\ \frac{\partial u_-}{\partial x} = \frac{\partial u_+}{\partial x} & \text{on } \Gamma \\ k_- \Delta u_- = k_+ \Delta u_+ & \text{on } \Gamma \\ k_- \frac{\partial \Delta u_-}{\partial x} = k_+ \frac{\partial \Delta u_+}{\partial x} & \text{on } \Gamma. \end{array} \right.$$

Let us now explain, for instance, in population dynamics framework, the boundary and transmission conditions.

The first line of $(BC_{pde})-(1)$, means that the individuals could not lie on the boundaries $(a, b) \times \partial\omega$, because, for instance, they die or the edge is impassable. The second line means that there is no dispersal in the normal direction. We deduce that the dispersal vanishes on $(a, b) \times \partial\omega$.

In $(BC_{pde})-(2)$, the population density and the flux are given on $\{a, b\} \times \omega$. This means that the habitats are not isolated.

In (TC_{pde}) , the two first transmission conditions mean the continuity of the density and its flux at the interface, while the two second express, in some sense, the continuity of the dispersal and its flux at Γ .

This work is a natural continuation of that done in [34]. Moreover, it completes the study realized in [22] where the authors have considered equations

$$k_\pm \Delta^2 u_\pm - l_\pm \Delta u_\pm = f_\pm \quad \text{in } \Omega_\pm$$

and on the interface Γ , the last condition of (TC_{pde}) is replaced by

$$\frac{\partial}{\partial x} (k_+ \Delta u_+ - l_+ u_+) \quad \text{on } \Gamma.$$

In the present work, $l_\pm = 0$, which must be treated differently, since the proof in [22] uses the representation formula obtained in [21] which is not defined when $l_\pm = 0$. Thus, we use the representation formula obtained in [34] which is different.

Note that, due to the transmission conditions, we can not obtain a solution $u \in W^{4,p}(\Omega)$ in all Ω . We only can obtain a solution u such that $u|_{\Omega_-} \in W^{4,p}(\Omega_-)$ and $u|_{\Omega_+} \in W^{4,p}(\Omega_+)$.

The paper is organized as follows.

First, in section 2, we recall the PDE transmission problem (P_{pde}) and we define the Laplace operator A_0 , then we rewrite problem (P_{pde}) under operational form which will be generalized replacing A_0 by a more general operator A in problem (P).

Then, in section 3, we recall some definitions about BIP operator and interpolation spaces. We give our hypotheses and their consequences. We explain how to solve our problem (P) by introducing two auxiliary problems (P_-) and (P_+) . Finally, we present our main result in Theorem 3.9 and as a consequence of this theorem, we obtain the Corollary 3.11 which states existence and uniqueness of the solution of problem (P_{pde}) that is $(EQ_{pde}) - (BC_{pde}) - (TC_{pde})$ quoted above.

In section 4, we state technical results which allow us to prove our main result. In Proposition 4.2 and in Proposition 4.4 we give necessary and sufficient condition to solve problems (P_-) and (P_+) . Moreover, we give an explicit representation formula of the solutions of (P_-) and (P_+) . We establish (see Theorem 4.6) a useful technical result which help us to prove Theorem 3.9. Then, we show some technical lemmas which lead us to apply functional calculus.

In Section 5, we give the proof of Theorem 3.9. This section is composed of three parts: in the first part, we use Theorem 4.6 to make explicit the determinant of the transmission system. In the second part, we explicitly inverse the determinant of the transmission system using functional calculus. Finally, in the last part, we show that the unique solution of the transmission problem (P) is a classical solution (see (2) in Section 2 below) by showing that this solution have the expected regularity.

2 Operational formulation

In this section, we first recall the PDE problem (P_{pde}) composed by $(EQ_{pde}) - (BC_{pde}) - (TC_{pde})$. Then, we define the Laplace operator A_0 and we use it to rewrite (P_{pde}) using A_0 . Note that this operational form is a vector values problem. Finally, we generalize this problem replacing A_0 by a more general operator A .

We consider the following problem

$$(P_{pde}) \left\{ \begin{array}{ll} k_- \Delta^2 u_- = g_- & \text{in } \Omega_- \\ k_+ \Delta^2 u_+ = g_+ & \text{in } \Omega_+ \\ u_-(x, \zeta) = 0, \quad \Delta u_-(x, \zeta) = 0, & x \in (a, \gamma), \zeta \in \partial\omega \\ u_+(x, \zeta) = 0, \quad \Delta u_+(x, \zeta) = 0, & x \in (\gamma, b), \zeta \in \partial\omega \\ u_-(a, y) = \varphi_1^-(y), \quad u_+(b, y) = \varphi_1^+(y), & y \in \omega \\ \frac{\partial u_-}{\partial x}(a, y) = \varphi_2^-(y), \quad \frac{\partial u_+}{\partial x}(b, y) = \varphi_2^+(y), & y \in \omega \\ u_- = u_+ & \text{on } \Gamma \\ \frac{\partial u_-}{\partial x} = \frac{\partial u_+}{\partial x} & \text{on } \Gamma \\ k_- \Delta u_- = k_+ \Delta u_+ & \text{on } \Gamma \\ k_- \frac{\partial \Delta u_-}{\partial x} = k_+ \frac{\partial \Delta u_+}{\partial x} & \text{on } \Gamma. \end{array} \right.$$

Let us define A_0 , the Laplace operator in \mathbb{R}^{n-1} , $n \in \mathbb{N} \setminus \{0, 1\}$, as follows

$$\left\{ \begin{array}{l} D(A_0) := \{\psi \in W^{2,p}(\omega) : \psi = 0 \text{ on } \partial\omega\} \\ \forall \psi \in D(A_0), \quad A_0 \psi = \Delta_y \psi. \end{array} \right. \quad (1)$$

Thus, using operator A_0 , problem (P_{pde}) becomes the following vector valued problem

$$\left\{ \begin{array}{l} u_-^{(4)}(x) + 2A_0 u_-''(x) + A_0^2 u_-(x) = f_-(x), \quad \text{for a.e. } x \in (a, \gamma) \\ u_+^{(4)}(x) + 2A_0 u_+''(x) + A_0^2 u_+(x) = f_+(x), \quad \text{for a.e. } x \in (\gamma, b) \\ u_-(a) = \varphi_1^-, \quad u_+(b) = \varphi_1^+ \\ u'_-(a) = \varphi_2^-, \quad u'_+(b) = \varphi_2^+ \\ u_-(\gamma) = u_+(\gamma) \\ u'_-(\gamma) = u'_+(\gamma) \\ k_- u_-''(\gamma) + k_- A_0 u_-(\gamma) = k_+ u_+''(\gamma) + k_+ A_0 u_+(\gamma) \\ k_- u_-^{(3)}(\gamma) + k_- A_0 u'_-(\gamma) = k_+ u_+^{(3)}(\gamma) + k_+ A_0 u'_+(\gamma), \end{array} \right.$$

where

$$f_- \in L^p(a, \gamma; L^p(\omega)), \quad f_+ \in L^p(\gamma, b; L^p(\omega)) \quad \text{and} \quad p \in (1, +\infty),$$

with

$$u_{\pm}(x) := u(x, \cdot) \quad \text{and} \quad f_{\pm}(x) := g_{\pm}(x, \cdot)/k_{\pm}.$$

Note that the boundary conditions on $\partial\omega$ in (P_{pde}) do not appear in the previous system since they are already included in the domain of A_0 . Thus, a classical solution of this problem satisfies the boundary conditions on $\partial\omega$ in (P_{pde}) .

Then, using operator $(A, D(A))$ instead of $(A_0, D(A_0))$ and X instead of $L^p(\omega)$, we can write that $f_- \in L^p(a, \gamma; X)$ and $f_+ \in L^p(\gamma, b; X)$.

Moreover, in all the sequel, we will study the following more general transmission problem (P):

$$(P) \left\{ \begin{array}{l} (EQ) \begin{cases} u_-^{(4)}(x) + 2Au_-''(x) + A^2u_-(x) = f_-(x), & \text{for a.e. } x \in (a, \gamma) \\ u_+^{(4)}(x) + 2Au_+''(x) + A^2u_+(x) = f_+(x), & \text{for a.e. } x \in (\gamma, b) \end{cases} \\ (BC) \begin{cases} u_-(a) = \varphi_1^-, & u_+(b) = \varphi_1^+ \\ u'_-(a) = \varphi_2^-, & u'_+(b) = \varphi_2^+ \end{cases} \\ (TC) \begin{cases} u_-(\gamma) = u_+(\gamma) \\ u'_-(\gamma) = u'_+(\gamma) \\ k_-u_-''(\gamma) + k_-Au_-(\gamma) = k_+u_+''(\gamma) + k_+Au_+(\gamma) \\ k_-u_-^{(3)}(\gamma) + k_-Au'_-(\gamma) = k_+u_+^{(3)}(\gamma) + k_+Au'_+(\gamma). \end{cases} \end{array} \right.$$

The transmission conditions (TC) will be split into

$$(TC1) \begin{cases} u_-(\gamma) = u_+(\gamma) \\ u'_-(\gamma) = u'_+(\gamma), \end{cases}$$

and

$$(TC2) \begin{cases} k_-u_-''(\gamma) + k_-Au_-(\gamma) = k_+u_+''(\gamma) + k_+Au_+(\gamma) \\ k_-u_-^{(3)}(\gamma) + k_-Au'_-(\gamma) = k_+u_+^{(3)}(\gamma) + k_+Au'_+(\gamma). \end{cases}$$

Note that (TC2) is well defined from Lemma 3.7, see Section 3.2 below.

We will search a classical solution of problem (P), that is a solution u such that

$$\begin{cases} u_- := u|_{(a,\gamma)} \in W^{4,p}(a, \gamma; X) \cap L^p(a, \gamma; D(A^2)), & u_-'' \in L^p(a, \gamma; D(A)), \\ u_+ := u|_{(\gamma,b)} \in W^{4,p}(\gamma, b; X) \cap L^p(\gamma, b; D(A^2)), & u_+'' \in L^p(\gamma, b; D(A)), \end{cases} \quad (2)$$

and which satisfies (EQ) – (BC) – (TC).

3 Assumptions, consequences and statement of results

3.1 The class BIP(X, θ)

Throughout the article, $(X, \|\cdot\|)$ is a complex Banach space. We give some definitions about UMD spaces, sectorial operators and BIP operators.

Definition 3.1. A Banach space X is a UMD space if and only if for all $p \in (1, +\infty)$, the Hilbert transform is bounded from $L^p(\mathbb{R}, X)$ into itself (see [3] and [4]).

Definition 3.2. A closed linear operator T_1 is called sectorial of angle $\theta \in [0, \pi)$ if

$$\begin{array}{l} i) \quad \sigma(T_1) \subset \overline{S_\theta}, \\ ii) \quad \forall \theta' \in (\theta, \pi), \quad \sup \left\{ \|\lambda(\lambda I - T_1)^{-1}\|_{\mathcal{L}(X)} : \lambda \in \mathbb{C} \setminus \overline{S_{\theta'}} \right\} < +\infty, \end{array}$$

where

$$S_\omega := \begin{cases} \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg(z)| < \omega\} & \text{if } \omega \in (0, \pi), \\ (0, +\infty) & \text{if } \omega = 0, \end{cases} \quad (3)$$

see [16], p. 19,

Remark 3.3. From [19], p. 342, we know that any injective sectorial operator T_1 admits imaginary powers T_1^{is} , $s \in \mathbb{R}$, but, in general, T_1^{is} is not bounded.

Definition 3.4. Let $\theta \in [0, \pi)$. We denote by $\text{BIP}(X, \theta)$, the class of sectorial injective operators T_1 such that

- i) $\overline{D(T_1)} = \overline{R(T_1)} = X$,
- ii) $\forall s \in \mathbb{R}, \quad T_1^{is} \in \mathcal{L}(X)$,
- iii) $\exists C \geq 1, \forall s \in \mathbb{R}, \quad \|T_1^{is}\|_{\mathcal{L}(X)} \leq Ce^{|s|\theta}$,

see [30], p. 430.

Remark 3.5. In the previous case, from [16], proof of Proposition 3.2.1, c), p. 71, $\overline{D(T_1)} \cap \overline{R(T_1)} = X$.

3.2 Interpolation spaces

Here we recall a definition given, for instance, in [9], [14], [25] or in [35] and a result from [14] concerning real interpolation spaces.

Definition 3.6. Let $T_2 : D(T_2) \subset X \rightarrow X$ be a linear operator such that

$$(0, +\infty) \subset \rho(T_2) \quad \text{and} \quad \exists C > 0 : \forall t > 0, \quad \|t(T_2 - tI)^{-1}\|_{\mathcal{L}(X)} \leq C. \quad (4)$$

Let $k \in \mathbb{N} \setminus \{0\}$, $\theta \in (0, 1)$ and $q \in [1, +\infty]$. We will use the real interpolation spaces

$$(D(T_2^k), X)_{\theta, q} = (X, D(T_2^k))_{1-\theta, q},$$

defined, for instance, in [25], or in [26].

In particular, for $k = 1$, we have the following characterization

$$(D(T_2), X)_{\theta, q} := \left\{ \psi \in X : t \mapsto t^{1-\theta} \|T_2(T_2 - tI)^{-1}\psi\|_X \in L_*^q(0, +\infty) \right\},$$

where $L_*^q(0, +\infty)$ is given by

$$L_*^q(0, +\infty) := \left\{ f \in L^q(0, +\infty) : \left(\int_0^{+\infty} \|f(t)\|^q \frac{dt}{t} \right)^{1/q} < +\infty \right\}, \quad \text{for } q \in [1, +\infty),$$

and for $q = +\infty$, by

$$L_*^\infty(0, +\infty) := \sup_{t \in (0, +\infty)} \|f(t)\|,$$

see [9] p. 325, or [14], p. 665, Teorema 3, or section 1.14 of [35], where this space is denoted by $(X, D(T_2))_{1-\theta, q}$. Note that we can also characterize the space $(D(T_2), X)_{\theta, q}$ taking into account the Osservazione, p. 666, in [14].

We set also, for any $k \in \mathbb{N} \setminus \{0\}$

$$\begin{aligned} (D(T_2), X)_{k+\theta, q} &:= \left\{ \psi \in D(T_2^k) : T_2^k \psi \in (D(T_2), X)_{\theta, q} \right\}, \\ (X, D(T_2))_{k+\theta, q} &:= \left\{ \psi \in D(T_2^k) : T_2^k \psi \in (X, D(T_2))_{\theta, q} \right\}. \end{aligned}$$

We recall the following lemma.

Lemma 3.7 ([14]). *Let T_2 be a linear operator satisfying (4). Let u such that*

$$u \in W^{n,p}(a_1, b_1; X) \cap L^p(a_1, b_1; D(T_2^k)),$$

where $a_1, b_1 \in \mathbb{R}$ with $a_1 < b_1$, $n, k \in \mathbb{N} \setminus \{0\}$ and $p \in (1, +\infty)$. Then for any $j \in \mathbb{N}$ satisfying the Poulsen condition $0 < \frac{1}{p} + j < n$ and $s \in \{a_1, b_1\}$, we have

$$u^{(j)}(s) \in (D(T_2^k), X)_{\frac{j}{n} + \frac{1}{np}, p}.$$

This result is proved in [14], p. 678, Teorema 2.

3.3 Hypotheses

Throughout this article, $k_+, k_- \in \mathbb{R}_+ \setminus \{0\}$ and A denotes a closed linear operator in X .

We assume the following hypotheses:

- (H₁) X is a UMD space,
- (H₂) $0 \in \rho(A)$,
- (H₃) $-A \in \text{BIP}(X, \theta_A)$ for some $\theta_A \in (0, \pi/2)$,
- (H₄) $\sigma(A) \subset (-\infty, 0)$ and $\forall \theta \in (0, \pi)$, $\sup_{\lambda \in S_\theta} \|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} < +\infty$.

Note that assumption (H₄) means that $-A$ is a sectorial operator of any angle $\theta \in (0, \pi)$. Now, we give some consequences on our hypotheses.

3.4 Consequences

1. Note that A_0 satisfies all the previous hypotheses with $X = L^q(\omega)$, with $q \in (1, +\infty)$. From [32], Proposition 3, p. 207, X satisfies (H₁) and from [13], Theorem 9.15, p. 241 and Lemma 9.17, p. 242, A_0 satisfies (H₂). Moreover, from [31], Theorem C, p. 166-167, (H₃) is satisfied for every $\theta_A \in (0, \pi)$, thus (H₄) is also satisfied.
2. In the scalar case, to solve each equation of (EQ), it is necessary to introduce the square roots $\pm\sqrt{-A}$ of the characteristic equations

$$x^4 + 2Ax^2 + A^2 = 0,$$

this is why, in our operational case, we consider,

$$M := -\sqrt{-A}. \quad (5)$$

From (H₃), $-A$ is a sectorial operator, thus the existence of M is ensured, see for instance [16], e), p. 25.

3. From (H₃), we have $-A \in \text{BIP}(X, \theta_A)$, then, from [16], Proposition 3.2.1, e), p. 71, we deduce

$$-M \in \text{BIP}(X, \theta_A/2).$$

Since $0 < \theta_A/2 < \pi/2$, we deduce that M generate a bounded analytic semigroup $(e^{xM})_{x \geq 0}$, see [30], Theorem 2, p. 437.

Furthermore, due to [30], Theorem 4, p. 441, for $n \in \mathbb{N} \setminus \{0\}$, we get $-nM \in \text{BIP}(X, \theta_A/4 + \varepsilon)$, for any $\varepsilon \in (0, \pi/2 - \theta_A/4)$. Then, due to [30], Theorem 2, p. 437, nM generate a bounded analytic semigroup $(e^{nxM})_{x \geq 0}$. The last results use also the works of [9] and [10].

3.5 The main results

To solve our transmission problem (P), we introduce two auxiliary problems:

$$(P_-) \begin{cases} u_-^{(4)}(x) + 2Au_-''(x) + A^2u_-(x) = f_-(x), & \text{for a.e. } x \in (a, \gamma) \\ u_-(a) = \varphi_1^-, \quad u_-(\gamma) = \psi_1 \\ u'_-(a) = \varphi_2^-, \quad u'_-(\gamma) = \psi_2, \end{cases}$$

and

$$(P_+) \begin{cases} u_+^{(4)}(x) + 2Au_+''(x) + A^2u_+(x) = f_+(x), & \text{for a.e. } x \in (\gamma, b) \\ u_+(\gamma) = \psi_1, \quad u_+(b) = \varphi_1^+ \\ u'_+(\gamma) = \psi_2, \quad u'_+(b) = \varphi_2^+. \end{cases}$$

Remark 3.8. We say that u is a classical solution of (P) if and only if there exist $\psi_1, \psi_2 \in X$ such that

- (i) u_- is a classical solution of (P_-) ,
- (ii) u_+ is a classical solution of (P_+) ,
- (iii) u_- and u_+ satisfy (TC2).

Note that by construction, if there exist a classical solution u_- of (P_-) and u_+ of (P_+) , then u_- and u_+ satisfy (TC1).

Our goal is to prove that there exists a unique couple (ψ_1, ψ_2) which satisfies (i), (ii) and (iii). This will lead us to obtain our main result.

Theorem 3.9. *Let $f_- \in L^p(a, \gamma; X)$ and $f_+ \in L^p(\gamma, b; X)$. Assume that (H_1) , (H_2) and (H_3) be satisfied. Then, there exists a unique classical solution u of the transmission problem (P) if and only if*

$$\varphi_1^+, \varphi_1^- \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_2^+, \varphi_2^- \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}. \quad (6)$$

Remark 3.10.

1. In the proof of Theorem 3.9, we use operator M and interpolation spaces $(D(M), X)_{3-j+\frac{1}{p}, p}$, $j = 0, 1, 2, 3$. But from the reiteration Theorem, we get

$$\begin{cases} (D(M), X)_{3+\frac{1}{p}, p} = (D(A), X)_{1+\frac{1}{2p}, p}, & (D(M), X)_{2+\frac{1}{p}, p} = (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} \\ (D(M), X)_{1+\frac{1}{p}, p} = (D(A), X)_{\frac{1}{2p}, p}, & (D(M), X)_{\frac{1}{p}, p} = (D(A), X)_{\frac{1}{2}+\frac{1}{2p}, p}. \end{cases} \quad (7)$$

2. We can generalize the previous Theorem by considering a transmission problem between n juxtaposed habitats, with $n \in \mathbb{N} \setminus \{0\}$. For instance, with $n = 3$, it suffices to use Theorem 3.9 on the two first habitats and then apply it on the transmission problem between the second and third habitat to solve the problem. By recurrence, we obtain the result for n habitats.

As consequence of Theorem 3.9, we deduce the following result for problem (P_{pde}) . Consider the case $A = A_0$ (other cases can be treated).

Corollary 3.11. *Assume that ω is a bounded open set of \mathbb{R}^{n-1} where $n \geq 2$ with C^2 -boundary. Let $g_+ \in L^p(\Omega_+)$ and $g_- \in L^p(\Omega_-)$ with $p \in (1, +\infty)$ and $p > n$; let $k_+, k_- \in \mathbb{R}_+ \setminus \{0\}$. Then, there exists a unique solution u of (P_{pde}) , such that*

$$u_- \in W^{4,p}(\Omega_-), \quad u_+ \in W^{4,p}(\Omega_+),$$

if and only if

$$\varphi_1^\pm, \varphi_2^\pm \in W^{2,p}(\omega) \cap W_0^{1,p}(\omega), \quad \Delta\varphi_1^\pm \in W^{2-\frac{1}{p},p}(\omega) \cap W_0^{1,p}(\omega) \quad \text{and} \quad \Delta\varphi_2^\pm \in W^{1-\frac{1}{p},p}(\omega) \cap W_0^{1,p}(\omega).$$

Proof. The proof is quite similar to the one of Corollary 3.6 in [22], see also Corollary 2.7 in [21]. \square

Taking into account the result of Theorem 3.9, we can also obtain anisotropic results by considering $f_- \in L^p(a, \gamma; L^q(\omega))$ and $f_+ \in L^p(\gamma, b; L^q(\omega))$ with $p, q \in (1, +\infty)$.

4 Preliminary results

Throughout this article, we set

$$c = \gamma - a > 0 \quad \text{and} \quad d = b - \gamma > 0.$$

From Remark 3.8, to solve problem (P), we first have to study problems (P_-) and (P_+) . To this end, we need the following invertibility result obtained in [34].

Lemma 4.1 ([34]). *The operators $U_+, U_-, V_+, V_- \in \mathcal{L}(X)$ defined by*

$$\begin{cases} U_+ & := I - e^{2dM} + 2dMe^{dM} \\ U_- & := I - e^{2cM} + 2cMe^{cM} \\ V_+ & := I - e^{2dM} - 2dMe^{dM} \\ V_- & := I - e^{2cM} - 2cMe^{cM}, \end{cases} \quad (8)$$

are invertible with bounded inverse.

All these exponentials are analytic semigroups and they are well defined due to statement 3 of Section 3.4. For a detailed proof, see [34], Proposition 4.5.

4.1 Problem (P_-)

Proposition 4.2. *Let $f_- \in L^p(a, \gamma; X)$. Assume that (H_1) , (H_2) and (H_3) hold. Then there exists a unique classical solution u_- of problem (P_-) if and only if*

$$\varphi_1^-, \psi_1 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_2^-, \psi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}. \quad (9)$$

Moreover

$$\begin{aligned} u_-(x) &= \left(e^{(x-a)M} - e^{(\gamma-x)M} \right) \alpha_1^- + \left((x-a)e^{(x-a)M} - (\gamma-x)e^{(\gamma-x)M} \right) \alpha_2^- \\ &+ \left(e^{(x-a)M} + e^{(\gamma-x)M} \right) \alpha_3^- + \left((x-a)e^{(x-a)M} + (\gamma-x)e^{(\gamma-x)M} \right) \alpha_4^- + F_-(x), \end{aligned} \quad (10)$$

where

$$\begin{cases} \alpha_1^- & := -\frac{1}{2}U_-^{-1} \left(\left(I + (I + cM) e^{cM} \right) \psi_1 - ce^{cM} \psi_2 \right) + \tilde{\varphi}_1^- \\ \alpha_2^- & := \frac{1}{2}U_-^{-1} \left(\left(I + e^{cM} \right) M\psi_1 + \left(I - e^{cM} \right) \psi_2 \right) + \tilde{\varphi}_2^- \\ \alpha_3^- & := \frac{1}{2}V_-^{-1} \left(\left(I - (I + cM) e^{cM} \right) \psi_1 + ce^{cM} \psi_2 \right) + \tilde{\varphi}_3^- \\ \alpha_4^- & := -\frac{1}{2}V_-^{-1} \left(\left(I - e^{cM} \right) M\psi_1 + \left(I + e^{cM} \right) \psi_2 \right) + \tilde{\varphi}_4^-, \end{cases} \quad (11)$$

with

$$\begin{cases} \tilde{\varphi}_1^- & := \frac{1}{2}U_-^{-1} \left(\varphi_1^- + e^{cM} \left(\varphi_1^- + c \left(M\varphi_1^- + \varphi_2^- - F'_-(a) - F'_-(\gamma) \right) \right) \right) \\ \tilde{\varphi}_2^- & := -\frac{1}{2}U_-^{-1} \left(M\varphi_1^- - \varphi_2^- + F'_-(a) + F'_-(\gamma) \right) \\ & \quad - \frac{1}{2}U_-^{-1} e^{cM} \left(M\varphi_1^- + \varphi_2^- - F'_-(a) - F'_-(\gamma) \right) \\ \tilde{\varphi}_3^- & := \frac{1}{2}V_-^{-1} \left(\varphi_1^- - e^{cM} \left(\varphi_1^- + c \left(M\varphi_1^- + \varphi_2^- - F'_-(a) + F'_-(\gamma) \right) \right) \right) \\ \tilde{\varphi}_4^- & := -\frac{1}{2}V_-^{-1} \left(M\varphi_1^- - \varphi_2^- + F'_-(a) - F'_-(\gamma) \right) \\ & \quad + \frac{1}{2}V_-^{-1} e^{cM} \left(M\varphi_1^- + \varphi_2^- - F'_-(a) + F'_-(\gamma) \right), \end{cases} \quad (12)$$

and F_- is the unique classical solution of problem

$$\begin{cases} u_-^{(4)}(x) + 2Au_-''(x) + A^2u_-(x) = f_-(x), & \text{a.e. } x \in (a, \gamma) \\ u_-(a) = u_-(\gamma) = u_-''(a) = u_-''(\gamma) = 0. \end{cases} \quad (13)$$

Proof. Due to [34], Theorem 2.8, statement 2., there exists a unique classical solution u_- of problem (P_-) if and only if (9) holds. Moreover, adapting the representation formula of u given by (31) in [34], where $u, f, b, F_{0,f}, \varphi_1, \varphi_2, \varphi_3$ and φ_4 are respectively replaced by $u_-, f_-, \gamma, F_-, \varphi_1^-, \psi_1, \varphi_2^-$ and ψ_2 , we obtain that the representation formula of u_- is given by (10), (11) and (12). \square

Remark 4.3. In the previous proposition, since (9), (11) and (12) hold, we have

$$\alpha_1^-, \alpha_3^- \in D(M^3) \quad \text{and} \quad \alpha_2^-, \alpha_4^- \in D(M^2).$$

Moreover, since F_- is a classical solution of (13), from Lemma 3.7, for $j = 0, 1, 2, 3$ and $s = a$ or γ

$$F_-^{(j)}(s) \in (D(M), X)_{3-j+\frac{1}{p}, p}.$$

4.2 Problem (P_+)

Proposition 4.4. Let $f_+ \in L^p(\gamma, b; X)$. Assume that (H_1) , (H_2) and (H_3) hold. Then, there exists a unique classical solution u_+ of problem (P_+) if and only if

$$\varphi_1^+, \psi_1 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_2^+, \psi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}. \quad (14)$$

Moreover

$$\begin{aligned} u_+(x) = & \left(e^{(x-\gamma)M} - e^{(b-x)M} \right) \alpha_1^+ + \left((x-\gamma)e^{(x-\gamma)M} - (b-x)e^{(b-x)M} \right) \alpha_2^+ \\ & + \left(e^{(x-\gamma)M} + e^{(b-x)M} \right) \alpha_3^+ + \left((x-\gamma)e^{(x-\gamma)M} + (b-x)e^{(b-x)M} \right) \alpha_4^+ + F_+(x), \end{aligned} \quad (15)$$

where

$$\begin{cases} \alpha_1^+ := \frac{1}{2}U_+^{-1} \left((I + (I + dM)e^{dM}) \psi_1 + de^{dM} \psi_2 \right) + \tilde{\varphi}_1^+ \\ \alpha_2^+ := -\frac{1}{2}U_+^{-1} \left((I + e^{dM}) M \psi_1 - (I - e^{dM}) \psi_2 \right) + \tilde{\varphi}_2^+ \\ \alpha_3^+ := \frac{1}{2}V_+^{-1} \left((I - (I + dM)e^{dM}) \psi_1 - de^{dM} \psi_2 \right) + \tilde{\varphi}_3^+ \\ \alpha_4^+ := -\frac{1}{2}V_+^{-1} \left((I - e^{dM}) M \psi_1 - (I + e^{dM}) \psi_2 \right) + \tilde{\varphi}_4^+, \end{cases} \quad (16)$$

with

$$\begin{cases} \tilde{\varphi}_1^+ := -\frac{1}{2}U_+^{-1} \left(\varphi_1^+ + e^{dM} \left(\varphi_1^+ + d \left(M\varphi_1^+ - \varphi_2^+ + F_+'(\gamma) + F_+'(b) \right) \right) \right) \\ \tilde{\varphi}_2^+ := \frac{1}{2}U_+^{-1} \left(M\varphi_1^+ + \varphi_2^+ - F_+'(\gamma) - F_+'(b) \right) \\ \quad + \frac{1}{2}U_+^{-1} e^{dM} \left(M\varphi_1^+ - \varphi_2^+ + F_+'(\gamma) + F_+'(b) \right) \\ \tilde{\varphi}_3^+ := \frac{1}{2}V_+^{-1} \left(\varphi_1^+ - e^{dM} \left(\varphi_1^+ + d \left(M\varphi_1^+ - \varphi_2^+ - F_+'(\gamma) + F_+'(b) \right) \right) \right) \\ \tilde{\varphi}_4^+ := -\frac{1}{2}V_+^{-1} \left(M\varphi_1^+ + \varphi_2^+ + F_+'(\gamma) - F_+'(b) \right) \\ \quad + \frac{1}{2}V_+^{-1} e^{dM} \left(M\varphi_1^+ - \varphi_2^+ - F_+'(\gamma) + F_+'(b) \right), \end{cases} \quad (17)$$

and F_+ is the unique classical solution of problem

$$\begin{cases} u_+^{(4)}(x) + 2Au_+''(x) + A^2u_+(x) = f_+(x), & a.e. x \in (\gamma, b) \\ u_+(\gamma) = u_+(b) = u_+'(\gamma) = u_+'(b) = 0. \end{cases} \quad (18)$$

Proof. Due to [34], Theorem 2.8, statement 2., there exists a unique classical solution u_+ of problem (P_+) if and only if (14) holds. Moreover, adapting the representation formula of u given by (31) in [34], where $u, f, a, F_{0,f}, \varphi_1, \varphi_2, \varphi_3$ and φ_4 are respectively replaced by $u_+, f_+, \gamma, F_+, \psi_1, \varphi_1^+, \psi_2$ and φ_2^+ , we obtain that the representation formula of u_+ is given by (15), (16) and (17). \square

Remark 4.5. In the previous proposition, since (14), (16) and (17) hold, we have

$$\alpha_1^+, \alpha_3^+ \in D(M^3) \quad \text{and} \quad \alpha_2^+, \alpha_4^+ \in D(M^2).$$

Moreover, since F_+ is a classical solution of (18), from Lemma 3.7, for $j = 0, 1, 2, 3$ and $s = \gamma$ or b

$$F_+^{(j)}(s) \in (D(M), X)_{3-j+\frac{1}{p}, p}.$$

4.3 The transmission system

In this section we give the proof of Theorem 4.6 stated below. This theorem make the link between problem (P) and the following system

$$\begin{cases} (P_1^+ + P_1^-) M\psi_1 - (P_2^+ - P_2^-) \psi_2 = S_1 \\ (P_2^+ - P_2^-) M\psi_1 - (P_3^+ + P_3^-) \psi_2 = S_2. \end{cases} \quad (19)$$

The coefficients of the previous system are given by

$$\begin{cases} P_1^+ = k_+ \left(U_+^{-1} (I + e^{dM})^2 + V_+^{-1} (I - e^{dM})^2 \right) \\ P_2^+ = k_+ (U_+^{-1} + V_+^{-1}) (I - e^{2dM}) \\ P_3^+ = k_+ \left(U_+^{-1} (I - e^{dM})^2 + V_+^{-1} (I + e^{dM})^2 \right) \end{cases} \quad (20)$$

and

$$\begin{cases} P_1^- = k_- \left(U_-^{-1} (I + e^{cM})^2 + V_-^{-1} (I - e^{cM})^2 \right) \\ P_2^- = k_- (U_-^{-1} + V_-^{-1}) (I - e^{2cM}) \\ P_3^- = k_- \left(U_-^{-1} (I - e^{cM})^2 + V_-^{-1} (I + e^{cM})^2 \right). \end{cases} \quad (21)$$

The seconds members are given by

$$S_1 = 2k_+ \left((\tilde{\varphi}_2^+ + \tilde{\varphi}_4^+) + e^{dM} (\tilde{\varphi}_2^+ - \tilde{\varphi}_4^+) \right) - 2k_- \left((\tilde{\varphi}_2^- - \tilde{\varphi}_4^-) + e^{cM} (\tilde{\varphi}_2^- + \tilde{\varphi}_4^-) \right) - M^{-2} \check{S} \quad (22)$$

where

$$\check{S} = -k_+ F_+^{(3)}(\gamma) + k_+ M^2 F_+'(\gamma) + k_- F_-^{(3)}(\gamma) - k_- M^2 F_-'(\gamma) \quad (23)$$

and

$$S_2 = 2k_+ \left((\tilde{\varphi}_2^+ + \tilde{\varphi}_4^+) - e^{dM} (\tilde{\varphi}_2^+ - \tilde{\varphi}_4^+) \right) + 2k_- \left((\tilde{\varphi}_2^- - \tilde{\varphi}_4^-) - e^{cM} (\tilde{\varphi}_2^- + \tilde{\varphi}_4^-) \right). \quad (24)$$

Theorem 4.6. Let $f_- \in L^p(a, \gamma; X)$ and $f_+ \in L^p(\gamma, b; X)$. Assume that (H_1) , (H_2) and (H_3) hold. Then, problem (P) has a unique classical solution if and only if the data $\varphi_1^+, \varphi_1^-, \varphi_2^+, \varphi_2^-$ satisfy (6) and system (19) has a unique solution (ψ_1, ψ_2) such that

$$(\psi_1, \psi_2) \in (D(A), X)_{1+\frac{1}{2p}, p} \times (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}. \quad (25)$$

Proof. Assume that problem (P) has a unique classical solution u . We set

$$\psi_1 = u_-(\gamma) = u_+(\gamma) \quad \text{and} \quad \psi_2 = u'_-(\gamma) = u'_+(\gamma).$$

We get that u_- (respectively u_+) is the classical solution of (P_-) (respectively (P_+)). Then, applying Proposition 4.2 (respectively Proposition 4.4), we obtain (6). Moreover, from (7), we have

$$\psi_1 \in (D(M), X)_{3+\frac{1}{p}, p} \quad \text{and} \quad \psi_2 \in (D(M), X)_{2+\frac{1}{p}, p}.$$

It remains to prove that (ψ_1, ψ_2) is solution of system (19). To this end, since u satisfies the transmission conditions $(TC2)$. Then, we obtain the following system

$$\begin{cases} k_- u_-^{(3)}(\gamma) + k_- A u'_-(\gamma) = k_+ u_+^{(3)}(\gamma) + k_+ A u'_+(\gamma) \\ k_- u_-''(\gamma) + k_- A u_-(\gamma) = k_+ u_+''(\gamma) + k_+ A u_+(\gamma). \end{cases}$$

Hence

$$\begin{cases} k_+ \left(u_+^{(3)}(\gamma) - M^2 u'_+(\gamma) \right) - k_- \left(u_-^{(3)}(\gamma) - M^2 u'_-(\gamma) \right) = 0 \\ k_+ \left(u_+''(\gamma) - M^2 u_+(\gamma) \right) - k_- \left(u_-''(\gamma) - M^2 u_-(\gamma) \right) = 0. \end{cases}$$

Moreover, for all $x \in (a, \gamma)$, we have

$$\begin{aligned} u_-(x) &= \left(e^{(x-a)M} - e^{(\gamma-x)M} \right) \alpha_1^- + \left((x-a)e^{(x-a)M} - (\gamma-x)e^{(\gamma-x)M} \right) \alpha_2^- \\ &\quad + \left(e^{(x-a)M} + e^{(\gamma-x)M} \right) \alpha_3^- + \left((x-a)e^{(x-a)M} + (\gamma-x)e^{(\gamma-x)M} \right) \alpha_4^- + F_-(x), \end{aligned}$$

$$\begin{aligned} u'_-(x) &= M \left(e^{(x-a)M} + e^{(\gamma-x)M} \right) \alpha_1^- + M \left(e^{(x-a)M} - e^{(\gamma-x)M} \right) \alpha_3^- + F'_-(x) \\ &\quad + \left((I + (x-a)M)e^{(x-a)M} + (I + (\gamma-x)M)e^{(\gamma-x)M} \right) \alpha_2^- \\ &\quad + \left((I + (x-a)M)e^{(x-a)M} - (I + (\gamma-x)M)e^{(\gamma-x)M} \right) \alpha_4^-, \end{aligned}$$

$$\begin{aligned} u_-''(x) &= M^2 \left(e^{(x-a)M} - e^{(\gamma-x)M} \right) \alpha_1^- + M^2 \left(e^{(x-a)M} + e^{(\gamma-x)M} \right) \alpha_3^- + F_-''(x) \\ &\quad + \left((2M + (x-a)M^2)e^{(x-a)M} - (2M + (\gamma-x)M^2)e^{(\gamma-x)M} \right) \alpha_2^- \\ &\quad + \left((2M + (x-a)M^2)e^{(x-a)M} + (2M + (\gamma-x)M^2)e^{(\gamma-x)M} \right) \alpha_4^-, \end{aligned}$$

and

$$\begin{aligned} u_-^{(3)}(x) &= M^3 \left(e^{(x-a)M} + e^{(\gamma-x)M} \right) \alpha_1^- + M^3 \left(e^{(x-a)M} - e^{(\gamma-x)M} \right) \alpha_3^- + F_-^{(3)}(x) \\ &\quad + \left((3M^2 + (x-a)M^3)e^{(x-a)M} + (3M^2 + (\gamma-x)M^3)e^{(\gamma-x)M} \right) \alpha_2^- \\ &\quad + \left((3M^2 + (x-a)M^3)e^{(x-a)M} - (3M^2 + (\gamma-x)M^3)e^{(\gamma-x)M} \right) \alpha_4^-. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} k_- \left(u_-^{(3)}(\gamma) - M^2 u'_-(\gamma) \right) &= k_- \left(2M^2 \left(I + e^{cM} \right) \alpha_2^- - 2M^2 \left(I - e^{cM} \right) \alpha_4^- \right) \\ &\quad + k_- F_-^{(3)}(\gamma) - k_- M^2 F'_-(\gamma). \end{aligned} \tag{26}$$

Moreover, from (13), we have

$$k_- \left(u_-''(\gamma) - M^2 u_-(\gamma) \right) = -k_- \left(2M \left(I - e^{cM} \right) \alpha_2^- - 2M \left(I + e^{cM} \right) \alpha_4^- \right). \tag{27}$$

Note that, from Remark 4.3, all terms of the previous equality are well defined.

In the same way, for all $x \in (\gamma, b)$, we have

$$\begin{aligned} u_+(x) &= \left(e^{(x-\gamma)M} - e^{(b-x)M} \right) \alpha_1^+ + \left((x-\gamma)e^{(x-\gamma)M} - (b-x)e^{(b-x)M} \right) \alpha_2^+ \\ &\quad + \left(e^{(x-\gamma)M} + e^{(b-x)M} \right) \alpha_3^+ + \left((x-\gamma)e^{(x-\gamma)M} + (b-x)e^{(b-x)M} \right) \alpha_4^+ + F_+(x), \end{aligned}$$

$$\begin{aligned} u'_+(x) &= M \left(e^{(x-\gamma)M} + e^{(b-x)M} \right) \alpha_1^+ + M \left(e^{(x-\gamma)M} - e^{(b-x)M} \right) \alpha_3^+ + F'_+(x) \\ &\quad + \left((I + (x-\gamma)M)e^{(x-\gamma)M} + (I + (b-x)M)e^{(b-x)M} \right) \alpha_2^+ \\ &\quad + \left((I + (x-\gamma)M)e^{(x-\gamma)M} - (I + (b-x)M)e^{(b-x)M} \right) \alpha_4^+, \end{aligned}$$

$$\begin{aligned} u''_+(x) &= M^2 \left(e^{(x-\gamma)M} - e^{(b-x)M} \right) \alpha_1^+ + M^2 \left(e^{(x-\gamma)M} + e^{(b-x)M} \right) \alpha_3^+ + F''_+(x) \\ &\quad + \left((2M + (x-\gamma)M^2)e^{(x-\gamma)M} - (2M + (b-x)M^2)e^{(b-x)M} \right) \alpha_2^+ \\ &\quad + \left((2M + (x-\gamma)M^2)e^{(x-\gamma)M} + (2M + (b-x)M^2)e^{(b-x)M} \right) \alpha_4^+, \end{aligned}$$

and

$$\begin{aligned} u_+^{(3)}(x) &= M^3 \left(e^{(x-\gamma)M} + e^{(b-x)M} \right) \alpha_1^+ + M^3 \left(e^{(x-\gamma)M} - e^{(b-x)M} \right) \alpha_3^+ + F_+^{(3)}(x) \\ &\quad + \left((3M^2 + (x-\gamma)M^3)e^{(x-\gamma)M} + (3M^2 + (b-x)M^3)e^{(b-x)M} \right) \alpha_2^+ \\ &\quad + \left((3M^2 + (x-\gamma)M^3)e^{(x-\gamma)M} - (3M^2 + (b-x)M^3)e^{(b-x)M} \right) \alpha_4^+. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} k_+ \left(u_+^{(3)}(\gamma) - M^2 u'_+(\gamma) \right) &= k_+ \left(2M^2 \left(I + e^{dM} \right) \alpha_2^+ + 2M^2 \left(I - e^{dM} \right) \alpha_4^+ \right) \\ &\quad + k_+ F_+^{(3)}(\gamma) - k_+ M^2 F'_+(\gamma). \end{aligned} \quad (28)$$

Moreover, from (18), we have

$$k_+ \left(u_+''(\gamma) - M^2 u_+(\gamma) \right) = k_+ \left(2M \left(I - e^{dM} \right) \alpha_2^+ + 2M \left(I + e^{dM} \right) \alpha_4^+ \right). \quad (29)$$

As previously, from Remark 4.5, all terms of the previous equality are justified. Then, from (23), (26), (27), (28) and (29), we deduce that system (19) is equivalent to

$$\begin{cases} k_+ \left(2M^2 \left(I + e^{dM} \right) \alpha_2^+ + 2M^2 \left(I - e^{dM} \right) \alpha_4^+ \right) \\ -k_- \left(2M^2 \left(I + e^{cM} \right) \alpha_2^- - 2M^2 \left(I - e^{cM} \right) \alpha_4^- \right) = \check{S} \\ k_+ \left(2M \left(I - e^{dM} \right) \alpha_2^+ + 2M \left(I + e^{dM} \right) \alpha_4^+ \right) \\ +k_- \left(2M \left(I - e^{cM} \right) \alpha_2^- - 2M \left(I + e^{cM} \right) \alpha_4^- \right) = 0. \end{cases}$$

Thus, we obtain the following system

$$\begin{cases} k_+ \left(\left(I + e^{dM} \right) \alpha_2^+ + \left(I - e^{dM} \right) \alpha_4^+ \right) \\ -k_- \left(\left(I + e^{cM} \right) \alpha_2^- - \left(I - e^{cM} \right) \alpha_4^- \right) = \frac{1}{2} M^{-2} \check{S} \\ k_+ \left(\left(I - e^{dM} \right) \alpha_2^+ + \left(I + e^{dM} \right) \alpha_4^+ \right) \\ +k_- \left(\left(I - e^{cM} \right) \alpha_2^- - \left(I + e^{cM} \right) \alpha_4^- \right) = 0. \end{cases} \quad (30)$$

Now, we regroup all source terms provided from (11), (12), (16) and (17) in S_1 and S_2 , defined by (22), (23) and (24). Then, we deduce that system (30) writes

$$\left\{ \begin{array}{l} -\frac{k_+}{2}U_+^{-1}(I+e^{dM})\left(\left(I+e^{dM}\right)M\psi_1-\left(I-e^{dM}\right)\psi_2\right) \\ -\frac{k_+}{2}V_+^{-1}\left(I-e^{dM}\right)\left(\left(I-e^{dM}\right)M\psi_1-\left(I+e^{dM}\right)\psi_2\right) \\ -\frac{k_-}{2}U_-^{-1}\left(I+e^{cM}\right)\left(\left(I+e^{cM}\right)M\psi_1+\left(I-e^{cM}\right)\psi_2\right) \\ -\frac{k_-}{2}V_-^{-1}\left(I-e^{cM}\right)\left(\left(I-e^{cM}\right)M\psi_1+\left(I+e^{cM}\right)\psi_2\right) \end{array} \right. = -\frac{S_1}{2}$$

$$\left\{ \begin{array}{l} -\frac{k_+}{2}U_+^{-1}\left(I-e^{dM}\right)\left(\left(I+e^{dM}\right)M\psi_1-\left(I-e^{dM}\right)\psi_2\right) \\ -\frac{k_+}{2}V_+^{-1}\left(I+e^{dM}\right)\left(\left(I-e^{dM}\right)M\psi_1-\left(I+e^{dM}\right)\psi_2\right) \\ +\frac{k_-}{2}U_-^{-1}\left(I-e^{cM}\right)\left(\left(I+e^{cM}\right)M\psi_1+\left(I-e^{cM}\right)\psi_2\right) \\ +\frac{k_-}{2}V_-^{-1}\left(I+e^{cM}\right)\left(\left(I-e^{cM}\right)M\psi_1+\left(I+e^{cM}\right)\psi_2\right) \end{array} \right. = -\frac{S_2}{2}.$$

Finally, we obtain

$$\left\{ \begin{array}{l} k_+\left(U_+^{-1}\left(I+e^{dM}\right)^2+V_+^{-1}\left(I-e^{dM}\right)^2\right)M\psi_1 \\ +k_-\left(U_-^{-1}\left(I+e^{cM}\right)^2+V_-^{-1}\left(I-e^{cM}\right)^2\right)M\psi_1 \\ \quad -k_+\left(U_+^{-1}+V_+^{-1}\right)\left(I-e^{2dM}\right)\psi_2 \\ \quad +k_-\left(U_-^{-1}+V_-^{-1}\right)\left(I-e^{2cM}\right)\psi_2 \end{array} \right. = S_1 \tag{31}$$

$$\left\{ \begin{array}{l} k_+\left(U_+^{-1}+V_+^{-1}\right)\left(I-e^{2dM}\right)M\psi_1 \\ -k_-\left(U_-^{-1}+V_-^{-1}\right)\left(I-e^{2cM}\right)M\psi_1 \\ -k_+\left(U_+^{-1}\left(I-e^{dM}\right)^2+V_+^{-1}\left(I+e^{dM}\right)^2\right)\psi_2 \\ -k_-\left(U_-^{-1}\left(I-e^{cM}\right)^2+V_-^{-1}\left(I+e^{cM}\right)^2\right)\psi_2 \end{array} \right. = S_2.$$

Then, using (20) and (21) system (31) become system (19). So, (ψ_1, ψ_2) is solution of system (19).

Conversely, if (6) holds and system (19) has a unique solution (ψ_1, ψ_2) such that

$$\psi_1 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \psi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p},$$

Then, considering u_- (respectively u_+) the unique classical solution of problem (P_-) (respectively problem (P_+)), we obtain that

$$u = \begin{cases} u_- & \text{in } \Omega_- \\ u_+ & \text{in } \Omega_+, \end{cases}$$

is the unique classical solution of problem (P) . □

4.4 Functional calculus

To prove Theorem 3.9, it remains, from Theorem 4.6, to solve system (19). To this end, we have to inverse the determinant operator of system (19) by using functional calculus.

We first recall some classical notations. Let $\theta \in (0, \pi)$, we denote by $H(S_\theta)$ the space of holomorphic functions on S_θ (defined by (3)). Moreover, we consider the following subspace of $H(S_\theta)$:

$$\mathcal{E}_\infty(S_\theta) := \{f \in H(S_\theta) : f = O(|z|^{-s}) \text{ } (|z| \rightarrow +\infty) \text{ for some } s > 0\}.$$

Thus, $\mathcal{E}_\infty(S_\theta)$ is the space of polynomial decreasing holomorphic functions at $+\infty$.

Let T be an invertible sectorial operator of angle $\theta_T \in (0, \pi)$ and let $f \in \mathcal{E}_\infty(S_\theta)$, with $\theta \in (\theta_T, \pi)$, then, by functional calculus, we can define $f(T) \in \mathcal{L}(X)$, see [16], p. 45. In this work, we use functional calculus, as classically done, see for instance [6] or for analytic semigroup generation [1] and [27].

We recall a useful invertibility result from [21], Lemma 5.3.

Lemma 4.7 ([21]). *Let P be an invertible sectorial operator in X with angle θ , for all $\theta \in (0, \pi)$. Let $G \in H(S_\theta)$, for some $\theta \in (0, \pi)$, such that*

$$(i) \quad 1 - G \in \mathcal{E}_\infty(S_\theta),$$

$$(ii) \quad G(x) \neq 0 \text{ for any } x \in \mathbb{R}_+ \setminus \{0\}.$$

Then, $G(P) \in \mathcal{L}(X)$, is invertible with bounded inverse.

Now, to inverse the determinant of system (19), we introduce some holomorphic functions which will play the role of G in the previous lemma. Then, we study them on the positive real axis.

Let $\delta > 0$ and $z \in \mathbb{C} \setminus \mathbb{R}_-$. We set

$$\begin{cases} u_\delta(z) &= 1 - e^{-2\delta\sqrt{z}} - 2\delta\sqrt{z}e^{-\delta\sqrt{z}} \\ v_\delta(z) &= 1 - e^{-2\delta\sqrt{z}} + 2\delta\sqrt{z}e^{-\delta\sqrt{z}}. \end{cases}$$

Then, we have

$$U_+ = u_d(-A), \quad U_- = u_c(-A), \quad V_+ = v_d(-A) \quad \text{and} \quad V_- = v_c(-A).$$

We set

$$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\}.$$

Moreover, from [34], Lemma 4.4, we have $u_\delta(z) \neq 0$ and $v_\delta(z) \neq 0$ for $z \in \mathbb{C}_+ \setminus \{0\}$, thus we note

$$\begin{cases} f_{\delta,1}(z) &= u_\delta^{-1}(z) \left(1 + e^{-\delta\sqrt{z}}\right)^2 + v_\delta^{-1}(z) \left(1 - e^{-\delta\sqrt{z}}\right)^2 \\ f_{\delta,2}(z) &= \left(u_\delta^{-1}(z) + v_\delta^{-1}(z)\right) \left(1 - e^{-2\delta\sqrt{z}}\right) \\ f_{\delta,3}(z) &= u_\delta^{-1}(z) \left(1 - e^{-\delta\sqrt{z}}\right)^2 + v_\delta^{-1}(z) \left(1 + e^{-\delta\sqrt{z}}\right)^2 \\ g_\delta(z) &= 16u_\delta^{-1}(z)v_\delta^{-1}(z)e^{-2\delta\sqrt{z}}. \end{cases}$$

Thus, we obtain that

$$\begin{cases} P_1^+ = k_+ f_{d,1}(-A), & P_2^+ = k_+ f_{d,2}(-A), & P_3^+ = k_+ f_{d,3}(-A) \\ P_1^- = k_- f_{c,1}(-A), & P_2^- = k_- f_{c,2}(-A), & P_3^- = k_- f_{c,3}(-A). \end{cases}$$

Moreover, for $z \in \mathbb{C}_+ \setminus \{0\}$, we define

$$f(z) := k_+^2 g_d(z) + k_-^2 g_c(z) + k_+ k_- (f_{d,1}(z) f_{c,3}(z) + f_{c,1}(z) f_{d,3}(z) + 2f_{d,2}(z) f_{c,2}(z)).$$

Note that $f \in H(S_\theta)$, for all $\theta \in (0, \pi)$. It follows that

$$f(-A) = 16k_+^2 U_+^{-1} V_+^{-1} e^{2dM} + 16k_-^2 U_-^{-1} V_-^{-1} e^{2cM} + P_1^+ P_3^- + P_1^- P_3^+ + 2P_2^+ P_2^-. \quad (32)$$

Lemma 4.8. *For all $x \in \mathbb{R}_+ \setminus \{0\}$, then $f(x)$ do not vanish.*

Proof. Let $\delta > 0$, from Lemma 4.4, in [34], for all $z \in \mathbb{C}_+ \setminus \{0\}$, we have

$$0 < |u_\delta(z)| \quad \text{and} \quad 0 < |v_\delta(z)|.$$

Thus, for all $x > 0$, one has $u_\delta(x), v_\delta(x) > 0$. Moreover, for all $x > 0$, we deduce that

$$f_{\delta,1}(x), f_{\delta,2}(x), f_{\delta,3}(x), g_\delta(x) > 0$$

and since $k_+k_- > 0$, it follows that

$$f(x) = k_+^2 g_d(x) + k_-^2 g_c(x) + k_+k_- (f_{d,1}(x)f_{c,3}(x) + f_{c,1}(x)f_{d,3}(x) + 2f_{d,2}(x)f_{c,2}(x)) > 0.$$

□

5 Proof of the main result

Assume that (P) has a unique classical solution, then, (6) holds from Theorem 4.6.

Conversely, Assume that (6) holds. From Theorem 4.6, it suffices to show that system (19) has a unique solution such that (25) is satisfied.

The proof is divided in three parts. In the first part, we make explicit the determinant of system (19). Then, in the second part, we show the uniqueness of the solution, to this end, we inverse the determinant with the help of functional calculus. Finally, in the last part, we prove that ψ_1 and ψ_2 have the expected regularity.

5.1 Calculus of the determinant

In this section, we make explicit the determinant. Recall system (19):

$$\begin{cases} (P_1^+ + P_1^-) M\psi_1 - (P_2^+ - P_2^-) \psi_2 = S_1 \\ (P_2^+ - P_2^-) M\psi_1 - (P_3^+ + P_3^-) \psi_2 = S_2. \end{cases}$$

Moreover, we write the previous system as the matrix equation $\Lambda\Psi = S$, where

$$\Lambda = \begin{pmatrix} M(P_1^+ + P_1^-) & -(P_2^+ - P_2^-) \\ M(P_2^+ - P_2^-) & -(P_3^+ + P_3^-) \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}.$$

To solve system (19), we calculate the determinant of the associated matrix Λ :

$$\begin{aligned} \det(\Lambda) &:= -M(P_1^+ + P_1^-)(P_3^+ + P_3^-) + M(P_2^+ - P_2^-)^2 \\ &= -M(P_1^+P_3^+ + P_1^+P_3^- + P_1^-P_3^+ + P_1^-P_3^- - (P_2^+)^2 - (P_2^-)^2 + 2P_2^+P_2^-) \\ &= -M(P_1^+P_3^+ - (P_2^+)^2 + P_1^-P_3^- - (P_2^-)^2 + P_1^+P_3^- + P_1^-P_3^+ + 2P_2^+P_2^-). \end{aligned}$$

We begin to calculate each term separately. Then, we have

$$\begin{aligned} P_1^+P_3^+ &= k_+^2 \left(U_+^{-1} (I + e^{dM})^2 + V_+^{-1} (I - e^{dM})^2 \right) \left(U_+^{-1} (I - e^{dM})^2 + V_+^{-1} (I + e^{dM})^2 \right) \\ &= k_+^2 \left((U_+^{-2} + V_+^{-2}) (I - e^{2dM})^2 + U_+^{-1}V_+^{-1} \left((I + e^{dM})^4 + (I - e^{dM})^4 \right) \right) \end{aligned}$$

and

$$\begin{aligned}
(I + e^{dM})^4 + (I - e^{dM})^4 &= (I + 2e^{dM} + e^{2dM})^2 + (I - 2e^{dM} + e^{2dM})^2 \\
&= (I + e^{2dM})^2 + 4e^{dM} (I + e^{2dM}) + 4e^{2dM} \\
&\quad + (I + e^{2dM})^2 - 4e^{dM} (I + e^{2dM}) + 4e^{2dM} \\
&= 2(I + e^{2dM})^2 + 8e^{2dM}.
\end{aligned}$$

Thus

$$P_1^+ P_3^+ = k_+^2 \left((U_+^{-2} + V_+^{-2}) (I - e^{2dM})^2 + 2U_+^{-1} V_+^{-1} (I + e^{2dM})^2 + 8U_+^{-1} V_+^{-1} e^{2dM} \right).$$

Moreover, we have

$$\begin{aligned}
(P_2^+)^2 &= k_+^2 (U_+^{-1} + V_+^{-1})^2 (I - e^{2dM})^2 \\
&= k_+^2 (U_+^{-2} + V_+^{-2}) (I - e^{2dM})^2 + 2k_+^2 U_+^{-1} V_+^{-1} (I - e^{2dM})^2.
\end{aligned}$$

It follows

$$\begin{aligned}
P_1^+ P_3^+ - (P_2^+)^2 &= k_+^2 \left((U_+^{-2} + V_+^{-2}) (I - e^{2dM})^2 + 2U_+^{-1} V_+^{-1} (I + e^{2dM})^2 + 8U_+^{-1} V_+^{-1} e^{2dM} \right) \\
&\quad - k_+^2 (U_+^{-2} + V_+^{-2}) (I - e^{2dM})^2 - 2k_+^2 U_+^{-1} V_+^{-1} (I - e^{2dM})^2 \\
&= 2k_+^2 U_+^{-1} V_+^{-1} \left((I + e^{2dM})^2 - (I - e^{2dM})^2 + 4e^{2dM} \right) \\
&= 2k_+^2 U_+^{-1} V_+^{-1} \left((I + 2e^{2dM} + e^{4dM}) - (I - 2e^{2dM} + e^{4dM}) + 4e^{2dM} \right) \\
&= 2k_+^2 U_+^{-1} V_+^{-1} (4e^{2dM} + 4e^{2dM}) \\
&= 16k_+^2 U_+^{-1} V_+^{-1} e^{2dM}.
\end{aligned}$$

In the same way, replacing respectively k_+ , U_+^{-1} , V_+^{-1} and d by k_- , U_-^{-1} , V_-^{-1} and c , we obtain

$$P_1^- P_3^- - (P_2^-)^2 = 16k_-^2 U_-^{-1} V_-^{-1} e^{2cM}.$$

Thus, the determinant of Λ writes

$$\det(\Lambda) = -M \left(16k_+^2 U_+^{-1} V_+^{-1} e^{2dM} + 16k_-^2 U_-^{-1} V_-^{-1} e^{2cM} + P_1^+ P_3^- + P_1^- P_3^+ + 2P_2^+ P_2^- \right). \quad (33)$$

Finally, from (32) and (33), we obtain

$$\det(\Lambda) = -Mf(-A). \quad (34)$$

5.2 Inversion of the determinant

Let C_1, C_2 two linear operators in X . We note $C_1 \sim C_2$ to mean that $C_1 = C_2 + \Sigma$, where Σ is a finite sum of term of type $kM^n e^{\alpha M}$, with $k \in \mathbb{R}$, $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+ \setminus \{0\}$. Note that Σ is a regular term in the sense:

$$\Sigma \in \mathcal{L}(X) \quad \text{with} \quad \Sigma(X) \subset D(M^\infty) := \bigcap_{k \geq 0} D(M^k).$$

Since $U_{\pm} \sim I$ and $V_{\pm} \sim I$, then setting $W = U_+U_-V_+V_- \sim I$, we obtain

$$\begin{cases} WP_1^+ \sim 2k_+I, & WP_1^- \sim 2k_-I, \\ WP_2^+ \sim 2k_+I, & WP_2^- \sim 2k_-I, \\ WP_3^+ \sim 2k_+I, & WP_3^- \sim 2k_-I. \end{cases}$$

Moreover, we have

$$16k_+^2W^2U_+^{-1}V_+^{-1}e^{2dM} \sim 0 \quad \text{and} \quad 16k_-^2W^2U_-^{-1}V_-^{-1}e^{2cM} \sim 0.$$

We then deduce the following relation

$$\begin{aligned} -M^{-1}W^2 \det(\Lambda) &= 16k_+^2W^2U_+^{-1}V_+^{-1}e^{2dM} + 16k_-^2W^2U_-^{-1}V_-^{-1}e^{2cM} \\ &\quad + WP_1^+WP_3^- + WP_1^-WP_3^+ + 2WP_2^+WP_2^- \\ &\sim 4k_+k_-I + 4k_+k_-I + 8k_+k_-I \\ &\sim 16k_+k_-I. \end{aligned}$$

Thus, we get

$$\det(\Lambda) = -W^{-2}M \left(16k_+k_-I + \sum_{j \in J} k_j M^{n_j} e^{\alpha_j M} \right), \quad (35)$$

where J is a finite set and for all $j \in J$:

$$k_j \in \mathbb{R}, n_j \in \mathbb{N} \quad \text{and} \quad \alpha_j \in \mathbb{R}_+ \setminus \{0\}.$$

From (35), we have

$$\det(\Lambda) = -16k_+k_-W^{-2}MF, \quad (36)$$

where

$$F = I + \sum_{j \in J} \frac{k_j}{16k_+k_-} M^{n_j} e^{\alpha_j M}. \quad (37)$$

For $z \in \mathbb{C} \setminus \mathbb{R}_-$, we set

$$\tilde{f}(z) = 1 + \sum_{j \in J} \frac{k_j}{16k_+k_-} (-\sqrt{z})^{n_j} e^{-\alpha_j \sqrt{z}}.$$

Then, $F = \tilde{f}(-A)$ and from (34) and (36), we have

$$f(-A) = -M^{-1} \det(\Lambda) = 16k_+k_-W^{-2} \tilde{f}(-A).$$

Thus, by construction, for $z \in \mathbb{C} \setminus \mathbb{R}_-$, the link between f and \tilde{f} is

$$f(z) = 16k_+k_-u_d^{-2}(z)u_c^{-2}(z)v_d^{-2}(z)v_c^{-2}(z)\tilde{f}(z). \quad (38)$$

Proposition 5.1. *The operator $F \in \mathcal{L}(X)$ defined by (37), is invertible with bounded inverse.*

Proof. Note that $f, \tilde{f} \in H(S_\theta)$, for a given $\theta \in (0, \pi)$. Moreover, for $z \in \mathbb{C} \setminus \mathbb{R}_-$ and $j \in J$, functions $\frac{k_j}{16k_+k_-} (-\sqrt{z})^{n_j}$ are polynomial. Thus, $1 - \tilde{f} \in \mathcal{E}_\infty(S_\theta)$.

From Lemma 4.4 in [34], u_d, u_c, v_d and v_c do not vanish on $\mathbb{C}_+ \setminus \{0\}$. Moreover, due to Lemma 4.8, f does not vanish on $\mathbb{R}_+ \setminus \{0\}$. Thus, we deduce from (38) that \tilde{f} do not vanish on $\mathbb{R}_+ \setminus \{0\}$.

Finally, we apply Lemma 4.7 with $P = -A$ and $G = \tilde{f}$ to obtain that $F = \tilde{f}(-A) \in \mathcal{L}(X)$ is invertible with bounded inverse. \square

We are now in position to prove the main result of this section.

Proposition 5.2. *The operator $\det(\Lambda)$, defined by (36) is invertible with bounded inverse.*

Proof. From Lemma 4.1, U_+, U_-, V_+ and V_- are bounded invertible operators with bounded inverse. So we deduce that W^{-2} is invertible with bounded inverse. Moreover, from (H_2) , (36) and Proposition 5.1, we obtain that $\det(\Lambda)$ is invertible with bounded inverse. \square

5.3 Regularity

To study the regularity, we need to recall the following technical result from [21], Lemma 5.1.

Lemma 5.3 ([21]). *Let $V \in \mathcal{L}(X)$ such that $0 \in \rho(I + V)$. Then, there exists $W \in \mathcal{L}(X)$ such that*

$$(I + V)^{-1} = I - W,$$

and $W(X) \subset V(X)$. Moreover, if T is a linear operator in X such that $V(X) \subset D(T)$ and for $\psi \in D(T)$, $TV\psi = VT\psi$, then

$$\forall \psi \in D(T), \quad WT\psi = TW\psi.$$

From Proposition 5.2, system (19) has a unique solution (ψ_1, ψ_2) . From Theorem 4.6, it remains to prove that

$$\psi_1 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \psi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}.$$

To this end, we have to study the regularity of the inverse of the determinant $\det(\Lambda)$.

Lemma 5.4. *There exists $R_d \in D(M^\infty)$ such that*

$$[\det(\Lambda)]^{-1} = -\frac{1}{16k_+k_-}M^{-1} + R_d.$$

Proof. From (36) and Proposition 5.2, we have

$$[\det(\Lambda)]^{-1} = -\frac{1}{16k_+k_-}M^{-1}W^2F^{-1}.$$

From Lemma 5.3, there exists $R_F \in D(M^\infty)$, such that

$$F^{-1} = I + R_F.$$

Moreover, for $\delta > 0$, we know that

$$e^{\delta M}\psi \in D(M^\infty). \tag{39}$$

Since $W = U_+U_-V_+V_-$, from (39), there exists $R_W \in D(M^\infty)$ such that

$$W^2 = I + R_W.$$

We deduce that there exists $R_d \in D(M^\infty)$, such that

$$[\det(\Lambda)]^{-1} = -\frac{1}{16k_+k_-}M^{-1} + R_d.$$

□

Now, we study the regularity of ψ_1 and ψ_2 . We recall that

$$\Lambda\Psi = S,$$

where Λ is invertible from Proposition 5.2. From Lemma 5.3, there exist $R_{U_\pm}, R_{V_\pm} \in D(M^\infty)$, such that

$$U_\pm^{-1} = I + R_{U_\pm} \quad \text{and} \quad V_\pm^{-1} = I + R_{V_\pm}. \tag{40}$$

From (39) and (40), there exist $R_i \in D(M^\infty)$, $i = 1, 2, 3, 4$, such that

$$\Lambda = \begin{pmatrix} 2(k_+ + k_-)M + R_1 & -2(k_+ - k_-)I + R_2 \\ 2(k_+ - k_-)M + R_3 & -2(k_+ + k_-)I + R_4 \end{pmatrix}.$$

It follows that there exist $\mathcal{R}_1, \mathcal{R}_2 \in D(M^\infty)$, such that

$$\begin{cases} \psi_1 &= \frac{(k_+ + k_-)}{8k_+k_-} M^{-1} S_1 - \frac{(k_+ - k_-)}{8k_+k_-} M^{-1} S_2 + \mathcal{R}_1 \\ \psi_2 &= \frac{(k_+ - k_-)}{8k_+k_-} S_1 - \frac{(k_+ + k_-)}{8k_+k_-} S_2 + \mathcal{R}_2, \end{cases} \quad (41)$$

where S_1 is given by (22) and S_2 is given by (24).

Since F_+ is a classical solution of problem (P_+) and F_- is a classical solution of problem (P_-) , then from Remark 4.3 and Remark 4.5, we obtain that \check{S} defined by (23) has the following regularity

$$\check{S} \in (D(M), X)_{\frac{1}{p}, p}. \quad (42)$$

Moreover, from (7), we have

$$\varphi_1^+, \varphi_1^- \in (D(A), X)_{1+\frac{1}{2p}, p} = (D(M), X)_{3+\frac{1}{p}, p}$$

and

$$\varphi_2^+, \varphi_2^- \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} = (D(M), X)_{2+\frac{1}{p}, p}.$$

Thus, from (12), (17), Lemma 5.3, Remark 4.3 and Remark 4.5, we deduce that

$$\tilde{\varphi}_1^+, \tilde{\varphi}_1^-, \tilde{\varphi}_3^+, \tilde{\varphi}_3^- \in (D(M), X)_{3+\frac{1}{p}, p} \quad \text{and} \quad \tilde{\varphi}_2^+, \tilde{\varphi}_2^-, \tilde{\varphi}_4^+, \tilde{\varphi}_4^- \in (D(M), X)_{2+\frac{1}{p}, p}. \quad (43)$$

So, from (22), (24), (39), (42) and (43), we have

$$S_1 \in (D(M), X)_{2+\frac{1}{p}, p} \quad \text{et} \quad S_2 \in (D(M), X)_{2+\frac{1}{p}, p}. \quad (44)$$

Finally, from (7), (41) and (44), we obtain

$$\psi_1 \in (D(M), X)_{3+\frac{1}{p}, p} = (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \psi_2 \in (D(M), X)_{2+\frac{1}{p}, p} = (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}.$$

Acknowledgments

This research is supported by CIFRE contract 2014/1307 with Qualiom Eco company and partially by the LMAH and the european funding ERDF through grant project Xterm.

References

- [1] W. ARENDT & R. SCHÄTZLE, “Semigroup generated by elliptic operators in non-divergence form on $C_0(\Omega)$ ”, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, Vol. XIII, issue 2, 2014, pp. 417-434.
- [2] B. BARRAZA MARTÍNEZ, R. DENK, J. HERNÁNDEZ MONZÓN, F. KAMMERLANDER & M. NENDEL, “Regularity and asymptotic behavior for a damped plate–membrane transmission problem”, *J. Math. Anal. Appl.*, Volume 474, Issue 2, 2019, pp. 1082-1103.
- [3] J. BOURGAIN, “Some remarks on Banach spaces in which martingale difference sequences are unconditional”, *Ark. Mat.*, vol. 21, 1983, pp. 163-168.
- [4] D.L. BURKHOLDER, “A geometrical characterisation of Banach spaces in which martingale difference sequences are unconditional”, *Ann. Probab.*, vol. 9, 1981, pp. 997-1011.
- [5] F. CAKONI, G. C. HSIAO & W. L. WENDLAND, “On the boundary integral equation method for a mixed boundary value problem of the biharmonic equation”, *Complex variables*, Vol. 50, No. 7-11, 2005, pp. 681-696.

- [6] A. CARBONARO & O. DRAGIČEVIĆ, “Bounded holomorphic functional calculus for nonsymmetric ornstein-uhlenbeck operators”, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, Vol. XIX, issue 4, 2019, pp. 1497-1533.
- [7] D. S. COHEN & J. D. MURRAY, “A generalized diffusion model for growth and dispersal in population”, *Journal of Mathematical Biology*, 12, Springer-Verlag, 1981, pp. 237-249.
- [8] M. COSTABEL, E. STEPHAN & W. L. WENGLAND, “On boundary integral equations of the first kind for the bi-Laplacian in a polygonal plane domain”, *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, 4^e serie, tome 10, no 2, 1983, pp. 197-241.
- [9] G. DA PRATO & P. GRISVARD, “Sommes d’opérateurs linéaires et équations différentielles opérationnelles”, *J. Math. pures et appl.*, 54, 1975, pp. 305-387.
- [10] G. DORE & A. VENNI, “On the closedness of the sum of two closed operators”, *Math. Z.*, 196, 1987, pp. 189-201.
- [11] A. FAVINI, R. LABBAS, K. LEMRABET, S. MAINGOT & H. SIDIBÉ, “Transmission Problem for an Abstract Fourth-order Differential Equation of Elliptic Type in UMD Spaces”, *Advances in Differential Equations*, Volume 15, Numbers 1-2, pp. 43-72, 2010.
- [12] A. FAVINI, R. LABBAS, A. MEDEGHRI & A. MENAD, “Analytic semigroups generated by the dispersal process in two habitats incorporating individual behavior at the interface”, *J. Math. Anal. Appl.*, 471, 2019, pp. 448–480.
- [13] D. GILBARG & N. TRUDINGER, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [14] P. GRISVARD, “Spazi di tracce e applicazioni”, *Rendiconti di Matematica*, (4) Vol.5, Serie VI, 1972, pp. 657-729.
- [15] Z. GUO, B. LAI & D. YE, “Revisiting the biharmonic equation modelling electrostatic actuation in lower dimensions”, *Proc. Amer. Math. Soc.*, 142, 2014, pp. 2027-2034.
- [16] M. HAASE, *The functional calculus for sectorial Operators*, Birkhauser, 2006.
- [17] F. HASSINE, “Logarithmic stabilization of the Euler–Bernoulli transmission plate equation with locally distributed Kelvin–Voigt damping”, *Journal of Mathematical Analysis and Applications*, Volume 455, Issue 2, 15 November 2017, pp. 1765-1782.
- [18] M. A. JASWON & G. T. SYMM, “Integral equation methods in potential theory and elastostatics”, *Academic Press*, New York, San Francisco, London, 1977, pp. 1-10.
- [19] H. KOMATSU, “Fractional powers of operators”, *Pacific Journal of Mathematics*, Vol. 19, No. 2, 1966, pp. 285-346.
- [20] M. KOTSCHOTE, “Maximal L^p -regularity for a linear three-phase problem of parabolic–elliptic type”, *Journal of Evolution Equations*, Volume 10, Issue 2, 2010, pp. 293-318.
- [21] R. LABBAS, S. MAINGOT, D. MANCEAU & A. THOREL, “On the regularity of a generalized diffusion problem arising in population dynamics set in a cylindrical domain”, *Journal of Mathematical Analysis and Applications*, 450, 2017, pp. 351-376.
- [22] R. LABBAS, K. LEMRABET, S. MAINGOT & A. THOREL, “Generalized linear models for population dynamics in two juxtaposed habitats”, *Discrete and Continuous Dynamical Systems - A*, Volume 39, Number 5, May 2019, pp. 2933-2960.

- [23] K. LIMAM, R. LABBAS, K. LEMRABET, A. MEDEGHRI & M. MEISNER, “On Some Transmission Problems Set in a Biological Cell, Analysis and Resolution”, *Journal of Differential Equations*, Volume 259, issue 7, 2015, p. 2695-2731.
- [24] F. LIN & Y. YANG, “Nonlinear non-local elliptic equation modelling electrostatic actuation”, *Proceedings of the Royal Society of London A*, 463, 2007, pp. 1323-1337.
- [25] J.-L. LIONS & J. PEETRE, “Sur une classe d’espaces d’interpolation”, *Publications mathématiques de l’I.H.É.S.*, tome 19, 1964, pp. 5-68.
- [26] A. LUNARDI, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhauser, Basel, Boston, Berlin, 1995.
- [27] G. METAFUNE & C. SPINA, “Elliptic operators with unbounded diffusion coefficients in L^p spaces”, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, (5) Vol. XI, 2012, pp. 303-340.
- [28] F.L. OCHOA, “A generalized reaction-diffusion model for spatial structures formed by motile cells”, *BioSystems*, 17, 1984, pp. 35-50.
- [29] K.-M. PERFEKT, “The transmission problem on a three-dimensional wedge”, *Arch. Ration. Mech. Anal.*, 231, No. 3, 2019, pp. 1745-1780.
- [30] J. PRÜSS & H. SOHR, “On operators with bounded imaginary powers in Banach spaces”, *Mathematische Zeitschrift, Springer-Verlag, Math. Z.*, 203, 1990, pp. 429-452.
- [31] J. PRÜSS & H. SOHR, “Imaginary powers of elliptic second order differential operators in L^p -spaces”, *Hiroshima Math. J.*, 23, no. 1, 1993, pp. 161-192.
- [32] J.L. RUBIO DE FRANCIA, *Martingale and integral transforms of Banach space valued functions*, in: J. Bastero, M. San Miguel (Eds.), *Probability and Banach Spaces*, Zaragoza, 1985, in: *Lecture Notes in Math.*, vol. 1221, Springer-Verlag, Berlin, 1986, pp. 195-222.
- [33] H. SAKER & N. BOUSELSAL, “On the bilaplacian problem with nonlinear boundary conditions”, *Indian J. Pure Appl. Math.*, Volume 47, Issue 3, 2016, pp. 425–435.
- [34] A. THOREL, “Operational approach for biharmonic equations in L^p -spaces”, *Journal of Evolution Equations*, 2019.
- [35] H. TRIEBEL, *Interpolation theory, function Spaces, differential Operators*, North-Holland publishing company Amsterdam New York Oxford, 1978.
- [36] J. WANG, J. LANG & Y. CHEN, “Global dynamics of an age-structured HIV infection model incorporating latency and cell-to-cell transmission”, *Discrete & Continuous Dynamical Systems - B*, Vol 22, 10, 2017, pp. 3721-3747.
- [37] C.-F. YANGA & S. BUTERIN, “Uniqueness of the interior transmission problem with partial information on the potential and eigenvalues”, *Journal of Differential Equations*, Vol. 260, Issue 6, 2016, pp. 4871-4887.