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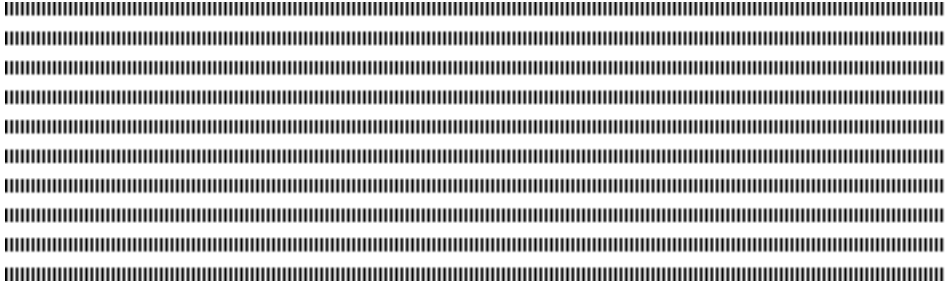
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Rubrique

# A Nash-game approach to joint data completion and location of small inclusions in Stokes flow

Abderrahmane HABBAL<sup>1</sup> — Moez KALLEL<sup>2</sup> — Marwa OUNI<sup>1,2</sup>

<sup>1</sup> Université Côte d'Azur, Inria, CNRS,  
LJAD, UMR 7351, Parc Valrose 06108 NICE CEDEX 2, France  
Abderrahmane.HABBAL@unice.fr

<sup>2</sup> Université de Tunis El Manar, Ecole Nationale d'Ingénieur de Tunis,  
Laboratoire de Modélisation Mathématique et Numérique dans les Sciences de l'Ingénieur B.P. 37,  
1002 Tunis, Tunisie  
moez.kallel@enit.utm.tn, marwa.ouni@enit.utm.tn



**RÉSUMÉ.** Nous étudions le problème de détection des petites inclusions immergées dans un fluide visqueux et incompressible, lorsque le mouvement de celui-ci est régi par les équations de Stokes. Des données du type Cauchy seront ainsi fournies seulement sur une partie frontière de l'écoulement. A cet égard, nous essayons de développer une méthode originale basée sur une approche de théorie des jeux, pour résoudre notre problème inverse. Un nouvel algorithme a été donc présenté traitant simultanément la question de la reconstruction des données manquantes avec celle de détection d'objets. La notion de gradient topologique a été utilisé afin de déterminer le nombre d'objets présents et leurs localisations approximatives. Dans cet objectif, une étude numérique présentée, a été effectuée pour prouver l'efficacité de la méthode.

**ABSTRACT.** We consider the inverse problem of determining the locations of some small objects immersed in a stationary viscous fluid, using incomplete boundary data. We carefully introduce a novel method to solve this problem based on a game theory approach. A new algorithm is provided to recover jointly the missing data and the locations of these objects. The detection problem is formulated as a topological one. The efficiency of the proposed method is illustrated by numerical experiments.

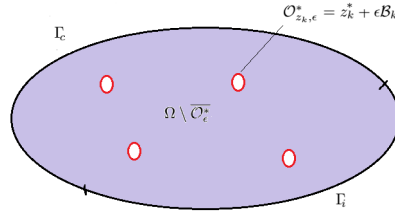
**MOTS-CLÉS :** Problème inverse géométrique, Complétion des données, Calculs des variations, Sensibilité topologique, Jeux de Nash

**KEYWORDS :** Geometric inverse problem, Data completion, Calculus of variations, Topological sensitivity, Nash games



## 1. Introduction and motivation

The geometric inverse problem is central in various fields industrial, biological and biomedical processes, such as the medical ultrasound imaging. In particular, in fluid mechanics, it is considered as a challenging problem and has in common been a subject of many investigations. As a case in point, in the filling of molds, small gas bubbles can be generated and trapped inside the material during its industrialization. Thus, the inverse problem has the special objective of determining and detecting the approximate location of these defects, as it is mentioned in [4].



**Figure 1.** An example of the geometric configuration of the problem.

Let us introduce a preliminary mathematical description of the problem. Consider a bounded open domain  $\Omega \subset \mathbb{R}^d$  ( $d=2,3$ ), which is filled with a viscous incompressible fluid. We assume that a finite number of the objects included in this domain, and we also suppose that these unknown objects are well separated and have the geometry form :

$$\mathcal{O}_{z_k, \epsilon} = z_k + \epsilon \mathcal{B}_k; \quad \forall k \in \{1, \dots, m\},$$

where  $\epsilon$  is the diameter and  $\mathcal{B}_k$  is bounded and smooth domain containing the origin. The points  $z_k \in \Omega$  determine the location of the unknown objects inside  $\Omega$ . Finally, we suppose that for  $k \in \{1, \dots, m\}$ ,  $\mathcal{O}_{z_k, \epsilon}$  is far from the boundary  $\partial\Omega$ , which is composed of two disjoint components  $\Gamma_i$  and  $\Gamma_c$ , see Figure-1.

The problem we study is to detect some small objects (location of the inclusions), from given velocity  $f$  and fluid stress forces  $\Phi$  prescribed only on the accessible part of the boundary. Then, we denote  $\mathcal{O}_\epsilon^* = \cup_{k=1}^m \mathcal{O}_{z_k, \epsilon}^*$  and we consider the following problem :

$$\begin{cases} \Delta u - \nabla p = 0 & \text{in } \Omega \setminus \overline{\mathcal{O}_\epsilon^*}, \\ \operatorname{div} u = 0 & \text{in } \Omega \setminus \overline{\mathcal{O}_\epsilon^*}, \\ \sigma(u, p)n = 0 & \text{on } \partial\mathcal{O}_\epsilon^*, \\ u = f & \text{on } \Gamma_c, \\ \sigma(u, p)n = \Phi & \text{on } \Gamma_c, \end{cases} \quad (1)$$

where  $u$  denote the velocity filed,  $p$  the pressure, and  $\sigma(u, p)$  represents the stress tensor defined by :

$$\sigma(u, p) = 2\nu \mathcal{D}(u) - pI_d,$$

with  $\nu$  is the kinematic viscosity of the fluid,  $\mathcal{D}(u) = 1/2(\nabla u + {}^t \nabla u)$  is the deformation tensor, and  $I_d$  is the identity matrix.

In this work, we study a geometric identification problem related to the Stokes equations. The problem consists in determining the number of some small objects located in a fluid flow domain and their approximate locations, using incomplete boundary data. Thanks to an identifiability result of Habbal et al [8], demonstrated for the inclusion Cauchy-Stokes problem, with a homogeneous Neumann condition imposed on the boundary of the inclusions, we suggest here an alternating minimization approach.

The Cauchy problem is a difficult issue and is ill-posed in the Hadamard sense [12]. In view of the fact, the existence of solutions for arbitrary Cauchy data cannot be guaranteed and even if the solution exists, it may not depend continuously on the given data. This problem is already discussed in literature by several methods and various algorithms which were carefully considered like optimal control methods and game theory approaches [2, 7, 8].

The aim of this work is to reconstruct the missing data, fluid velocity and stress forces on the inaccessible part of the boundary, from available measurements on the accessible part, in addition to determine the number of the small objects included in the domain, and their approximate locations. In Section 2, we present our original approach to solve the coupled problem of data completion and several objects detection. A topological sensitivity analysis method is used in order to determine the optimal locations of these inclusions. In that case a new algorithm is provided. In section 3, we illustrate the efficiency of the proposed method by treating two different situations.

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## 2. Data completion and localization of small objects

In this section, we present an approach to solve our inverse coupled problem by solving jointly the data completion and the determination of the number of the unknown objects and their relative locations, using an original Nash game strategy. Analyzing this strategy, three players are defined, playing non-cooperatively a Nash game : the two first players are associated to the data completion while the third one is in charge of the identification problem.

Therefore, we define for any set of  $\mathcal{O}_\epsilon = \cup_{k=1}^m z_k + \epsilon \mathcal{B}_k \in \Omega$  three boundary value problems :

$$\begin{cases}
 (\mathcal{P}_1^\epsilon) \left\{ \begin{array}{l}
 \text{Find } (u_1^\epsilon, p_1^\epsilon) \in (H^1(\Omega \setminus \overline{\mathcal{O}_\epsilon}))^d \times L^2(\Omega \setminus \overline{\mathcal{O}_\epsilon}) \text{ such that :} \\
 -\text{div}(\sigma(u_1^\epsilon, p_1^\epsilon)) = 0 \quad \text{in } \Omega \setminus \overline{\mathcal{O}_\epsilon}, \\
 \text{div} u_1^\epsilon = 0 \quad \text{in } \Omega \setminus \overline{\mathcal{O}_\epsilon}, \\
 \sigma(u_1^\epsilon, p_1^\epsilon) n = 0 \quad \text{on } \partial \mathcal{O}_\epsilon, \\
 u_1^\epsilon = f \quad \text{on } \Gamma_c, \\
 \sigma(u_1^\epsilon, p_1^\epsilon) n = \eta \quad \text{on } \Gamma_i.
 \end{array} \right. \\
 \\
 (\mathcal{P}_2^\epsilon) \left\{ \begin{array}{l}
 \text{Find } (u_2^\epsilon, p_2^\epsilon) \in (H^1(\Omega \setminus \overline{\mathcal{O}_\epsilon}))^d \times L^2(\Omega \setminus \overline{\mathcal{O}_\epsilon}) \text{ such that :} \\
 -\text{div}(\sigma(u_2^\epsilon, p_2^\epsilon)) = 0 \quad \text{in } \Omega \setminus \overline{\mathcal{O}_\epsilon}, \\
 \text{div} u_2^\epsilon = 0 \quad \text{in } \Omega \setminus \overline{\mathcal{O}_\epsilon}, \\
 \sigma(u_2^\epsilon, p_2^\epsilon) n = 0 \quad \text{on } \partial \mathcal{O}_\epsilon, \\
 u_2^\epsilon = \tau \quad \text{on } \Gamma_i, \\
 \sigma(u_2^\epsilon, p_2^\epsilon) n = \Phi \quad \text{on } \Gamma_c.
 \end{array} \right.
 \end{cases}$$

$$(\mathcal{P}_3^\epsilon) \left\{ \begin{array}{l} \text{Find } (u_3^\epsilon, p_3^\epsilon) \in (H^1(\Omega \setminus \overline{\mathcal{O}_\epsilon}))^d \times L_0^2(\Omega \setminus \overline{\mathcal{O}_\epsilon}) \text{ such that :} \\ -\text{div}(\sigma(u_3^\epsilon, p_3^\epsilon)) = 0 \quad \text{in } \Omega \setminus \overline{\mathcal{O}_\epsilon}, \\ \text{div} u_3^\epsilon = 0 \quad \text{in } \Omega \setminus \overline{\mathcal{O}_\epsilon}, \\ \sigma(u_3^\epsilon, p_3^\epsilon) n = 0 \quad \text{on } \partial \mathcal{O}_\epsilon, \\ u_3^\epsilon = \tau \quad \text{on } \Gamma_i, \\ u_3^\epsilon = f \quad \text{on } \Gamma_c, \end{array} \right.$$

where  $\eta \in (H_{00}^{\frac{1}{2}}(\Gamma_i)^d)'$  and  $\tau \in H^{\frac{1}{2}}(\Gamma_i)^d$  are given functions. Let us present the following three costs : for  $\mathcal{Z} = \{z_1, \dots, z_m\} \subset \Omega$

$$\mathcal{J}_1(\eta, \tau; \mathcal{Z}) = \frac{1}{2} \|\sigma(u_1^\epsilon, p_1^\epsilon) n - \Phi\|_{(H_{00}^{\frac{1}{2}}(\Gamma_c)^d)'}^2 + \frac{1}{2} \|u_1^\epsilon - u_2^\epsilon\|_{H^{\frac{1}{2}}(\Gamma_i)^d}^2, \quad (2)$$

$$\mathcal{J}_2(\eta, \tau; \mathcal{Z}) = \frac{1}{2} \|u_2^\epsilon - f\|_{H^{\frac{1}{2}}(\Gamma_c)^d}^2 + \frac{1}{2} \|u_1^\epsilon - u_2^\epsilon\|_{H^{\frac{1}{2}}(\Gamma_i)^d}^2, \quad (3)$$

$$\mathcal{J}_3(\eta, \tau; \mathcal{Z}) = 2\nu \|\mathcal{D}(u_3^\epsilon) - \mathcal{D}(u_2^\epsilon)\|_{L^2(\Omega \setminus \overline{\mathcal{O}_\epsilon})}^2. \quad (4)$$

In a few words, there are three players : the two first players, player 1 and player 2, solve the associated mixed boundary value problem  $(\mathcal{P}_1^\epsilon)$  and  $(\mathcal{P}_2^\epsilon)$ , using as strategies variables the respective Neumann condition  $\eta \in (H_{00}^{\frac{1}{2}}(\Gamma_i)^d)'$  and Dirichlet condition  $\tau \in H^{\frac{1}{2}}(\Gamma_i)^d$ . Each of these two players tries to minimize its own cost, namely,  $\mathcal{J}_1$  for player 1 and  $\mathcal{J}_2$  for player 2, while the third player, player 3, controls the strategy variable  $\mathcal{Z} = \{z_1, \dots, z_m\} \subset \Omega$ , where no information on the number  $m$  is given, and tries to minimize the Kohn-Vogeluis functional.

To approximate the solution of the original coupled problem, we seek to find a Nash equilibrium, defined as follows :

**Definition 1** A triplet  $(\eta_N, \tau_N, \mathcal{Z}_N) \in (H_{00}^{\frac{1}{2}}(\Gamma_i)^d)' \times H^{\frac{1}{2}}(\Gamma_i)^d \times \Omega$  is a Nash equilibrium for the three players game involving the costs  $\mathcal{J}_1$ ,  $\mathcal{J}_2$  and  $\mathcal{J}_3$  if :

$$(NE) \left\{ \begin{array}{l} \mathcal{J}_1(\eta_N, \tau_N, \mathcal{Z}_N) \leq \mathcal{J}_1(\eta, \tau_N, \mathcal{Z}_N), \quad \forall \eta \in (H_{00}^{\frac{1}{2}}(\Gamma_i)^d)', \\ \mathcal{J}_2(\eta_N, \tau_N, \mathcal{Z}_N) \leq \mathcal{J}_2(\eta_N, \tau, \mathcal{Z}_N), \quad \forall \tau \in H^{\frac{1}{2}}(\Gamma_i)^d, \\ \mathcal{J}_3(\eta_N, \tau_N, \mathcal{Z}_N) \leq \mathcal{J}_3(\eta_N, \tau_N, \mathcal{Z}), \quad \forall \mathcal{Z} \in \Omega, \end{array} \right.$$

The minimization problem  $\min_{\mathcal{Z} \subset \Omega} \mathcal{J}_3$  can be formulated as a topological optimization problem as follows : for fixed  $\eta \in (H_{00}^{\frac{1}{2}}(\Gamma_i)^d)'$  and  $\tau \in H^{\frac{1}{2}}(\Gamma_i)^d$ ,

$$(\mathcal{P}_\epsilon) \left\{ \begin{array}{l} \text{Find } \mathcal{Z}^* = \{z_1^*, \dots, z_m^*\} \subset \Omega, \text{ such that :} \\ \mathcal{J}(\Omega \setminus \overline{\mathcal{O}_\epsilon^*}) = \min_{z_k \in \Omega} \mathcal{J}(\Omega \setminus \overline{\mathcal{O}_\epsilon}), \end{array} \right.$$

where  $\mathcal{O}_\epsilon^* = \cup_{k=1}^m z_k^* + \epsilon \mathcal{B}_k \subset \Omega$ , and  $\mathcal{J}$  is defined by

$$\mathcal{J}(\Omega \setminus \overline{\mathcal{O}_\epsilon}) = \mathcal{J}_{KV}(u_2^\epsilon, u_3^\epsilon) := \mathcal{J}_3(\eta, \tau; \mathcal{Z}). \quad (5)$$

In order to solve the optimization problem  $(\mathcal{P}_\epsilon)$  above, we will use the notion of topological gradient. The topological gradient method has been known as an efficient

approach to solve shape optimization problems. It consists in studying the variation of a cost function with respect to the modification of the topology of the domain  $\Omega$ . For simplicity in what follows, we will consider the case of a single object  $\mathcal{O}_{z,\epsilon}$ . Notice that in the case of several inclusions, the results presented below are still valid.

## 2.1. The topological gradient method

The topological sensitivity analysis consists in the study of the variations of a design functional  $\mathcal{J}$  with respect to the insertion of a small inclusion  $\mathcal{O}_{z,\epsilon}$  at the point  $z$ . Then, an asymptotic expansion of the function  $\mathcal{J}$  can be obtained in the following form : for  $\epsilon > 0$

$$\begin{aligned} \mathcal{J}(\Omega_{z,\epsilon}) &= \mathcal{J}(\Omega) + \rho(\epsilon)\delta\mathcal{J}(z) + o(\rho(\epsilon)), \quad \forall z \in \Omega, \\ \lim_{\epsilon \rightarrow 0} \rho(\epsilon) &= 0, \quad \rho(\epsilon) > 0, \end{aligned}$$

where  $\Omega_{z,\epsilon} = \Omega \setminus \overline{\mathcal{O}_{z,\epsilon}}$  and the function  $\delta\mathcal{J}(z)$  is the so-called topological gradient. This function  $\delta\mathcal{J}(z)$  provides an information for creating a small hole located at  $z$ . Hence, it can be used like a descent direction in an optimization process. Therefore, to minimize the cost function, one has to create small hole at the location  $z$  when  $\delta\mathcal{J}$  is the most negative. The concept of the topological derivative was introduced by Schumacher [5] in the case of compliance minimization. Next, Sokolowski et al [11] extended it to a more general shape functionals. Since that, it has been widely applied in literature for arbitrary shaped perturbations and a general class of cost functionals related to PDEs. One can cite the work related to : the Laplace equations [1], the elasticity equations [3], the Maxwell equations [10], and the Stokes system [4], with a homogeneous boundary conditions prescribed on the boundary of the objects.

An adaptation of the adjoint method to the topological context [3] is developed. Next, we present the following proposition that describes a generalized adjoint method for the computation of the first variation of a given cost functional.

**Proposition 1** *Let  $\mathcal{V}$  be a Hilbert space. For  $\epsilon \in [0, \xi]$ ,  $\xi \geq 0$ , consider a function  $u_\epsilon \in \mathcal{V}$  that's solution of a variational problem of the form*

$$\mathcal{A}_\epsilon(u_\epsilon, v) = l_\epsilon(v), \quad \forall v \in \mathcal{V}$$

where  $\mathcal{A}_\epsilon$  and  $l_\epsilon$  are a bilinear form and a linear form on  $\mathcal{V}$ , respectively. For all  $\epsilon \in [0, \xi]$ , consider a functional  $j(\epsilon) = J_\epsilon(u_\epsilon)$ , where  $J_0$  is Fréchet differentiable at the point  $u_0$  and its derivative being denoted  $DJ(u_0)$ . Suppose that the following hypotheses hold :

(i)- *There exist two numbers  $\delta a$ ,  $\delta l$  and a function  $\rho(\epsilon) \geq 0$  such that*

$$(\mathcal{A}_\epsilon - \mathcal{A}_0)(u_0, v_\epsilon) = \rho(\epsilon)\delta a + o(\rho(\epsilon)),$$

$$(l_\epsilon - l_0)(v_\epsilon) = \rho(\epsilon)\delta l + o(\rho(\epsilon)),$$

where  $v_\epsilon$  is an adjoint state satisfying,

$$\mathcal{A}_\epsilon(w, v_\epsilon) = -DJ(u_0)w, \quad \forall w \in \mathcal{V}.$$

(ii)- *There exist a real number  $\delta J$  such that*

$$J_\epsilon(u_\epsilon) = J_0(u_0) + DJ(u_0)(u_\epsilon - u_0) + \rho(\epsilon)\delta J + o(\rho(\epsilon)).$$

Then, the first variation of the cost function with respect to  $\epsilon$  is given by

$$j(\epsilon) = j(0) + \rho(\epsilon)(\delta a + \delta l + \delta J) + o(\rho(\epsilon)).$$

## 2.2. Application to the model problem

The aim here is to derive an asymptotic expansion for our functional  $\mathcal{J}$  defined in (5) following the same steps described in the proposition 1 above. Then, we shall give explicitly the variations  $\delta a$ ,  $\delta l$ ,  $\delta \mathcal{J}_{KV}$ .

We start by defining the -control free- Sobolev state spaces : Given  $g \in H^{\frac{1}{2}}(\Gamma_i)^d$  and  $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ ,

$$\mathcal{V}_{1,g}^\epsilon = \{v \in H^1(\Omega_{z,\epsilon})^d, \text{ such that } \operatorname{div} v = 0 \text{ in } \Omega_{z,\epsilon} \text{ and } v|_{\Gamma_i} = g\},$$

and

$$\mathcal{V}_{2,\phi}^\epsilon = \{v \in H^1(\Omega_{z,\epsilon})^d, \text{ such that } \operatorname{div} v = 0 \text{ in } \Omega_{z,\epsilon} \text{ and } v|_{\partial\Omega} = \phi\}.$$

The variational formulations associated to problems  $(\mathcal{P}_2^\epsilon)$  and  $(\mathcal{P}_3^\epsilon)$  can be stated respectively as follows :

$$\left\{ \begin{array}{l} \text{Find } u_2^\epsilon \in \mathcal{V}_{1,f}^\epsilon \text{ such that :} \\ \mathcal{A}_{1,\epsilon}(u_2^\epsilon, v) = l_{1,\epsilon}(v), \quad \forall v \in \mathcal{V}_{1,0}^\epsilon, \end{array} \right. \quad (6)$$

$$\left\{ \begin{array}{l} \text{Find } u_3^\epsilon \in \mathcal{V}_{2,f}^\epsilon \text{ such that :} \\ \mathcal{A}_{2,\epsilon}(u_3^\epsilon, v) = l_{2,\epsilon}(v), \quad \forall v \in \mathcal{V}_{2,0}^\epsilon, \end{array} \right. \quad (7)$$

where

$$\mathcal{A}_{1,\epsilon}(u_2^\epsilon, v) = 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon) : \mathcal{D}(v) \, dx,$$

$$\mathcal{A}_{2,\epsilon}(u_3^\epsilon, v) = 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_3^\epsilon) : \mathcal{D}(v) \, dx,$$

$$l_{1,\epsilon}(v) = \int_{\Gamma_c} \Phi \phi \, ds,$$

and

$$l_{2,\epsilon}(v) = 0.$$

Note that for  $\epsilon = 0$ , we have  $\Omega_0 = \Omega$ , and  $(u_2^0, p_2^0) \in H^1(\Omega)^d \times L^2(\Omega)$  and  $(u_3^0, p_3^0) \in H^1(\Omega)^d \times L^2(\Omega)$  solve the respective boundary value problems :

$$(\mathcal{P}_2^0) \left\{ \begin{array}{l} \text{Find } (u_2^0, p_2^0) \in H^1(\Omega)^d \times L^2(\Omega) \text{ such that :} \\ -\operatorname{div}(\sigma(u_2^0, p_2^0)) = 0 \quad \text{in } \Omega, \\ \operatorname{div} u_2^0 = 0 \quad \text{in } \Omega, \\ u_2^0 = \tau \quad \text{on } \Gamma_i, \\ \sigma(u_2^0, p_2^0)n = \Phi \quad \text{on } \Gamma_c, \end{array} \right.$$

$$(\mathcal{P}_3^0) \left\{ \begin{array}{l} \text{Find } (u_3^0, p_3^0) \in H^1(\Omega)^d \times L^2(\Omega) \text{ such that :} \\ -\text{div}(\sigma(u_3^0, p_3^0)) = 0 \quad \text{in } \Omega, \\ \text{div} u_3^0 = 0 \quad \text{in } \Omega, \\ u_3^0 = \tau \quad \text{on } \Gamma_i, \\ u_3^0 = f \quad \text{on } \Gamma_c. \end{array} \right.$$

VARIATION OF THE BILINEAR FORM  $\mathcal{A}_{1,\epsilon}$  AND  $\mathcal{A}_{2,\epsilon}$  :

In order to obtain an asymptotic expansion of the variation of the bilinear form, we will use a simplified technique proposed in [4] for the Stokes system. We can also use a truncation technique, which is developed in [3] for elasticity equations, with a Neumann boundary condition on  $\partial\mathcal{O}_{z,\epsilon}$ .

*Variation of  $\mathcal{A}_{1,\epsilon}$  :* We are interested in the asymptotic analysis of the variation

$$\begin{aligned} (\mathcal{A}_{1,\epsilon} - \mathcal{A}_{1,0})(u_2^0, v_1^\epsilon) &= -\mathcal{A}_{1,\epsilon}(u_2^\epsilon - u_2^0, v_1^\epsilon) + (l_{1,\epsilon}(v_1^\epsilon) - l_{1,0}(v_1^\epsilon)) \\ &= -\mathcal{A}_{1,\epsilon}(u_2^\epsilon - u_2^0, v_1^\epsilon) \\ &= -\int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon - u_2^0) : \mathcal{D}(v_1^\epsilon) dx \\ &= \int_{\partial\mathcal{O}_{z,\epsilon}} \sigma(u_2^0, p_2^0) n v_1^\epsilon ds. \end{aligned}$$

According to fundamental assumption (i) carefully formulated in proposition 1, we search to find a real number  $\delta a_1 \in \mathbb{R}$  and a scalar function positive  $\rho$  such that

$$\begin{aligned} \int_{\partial\mathcal{O}_{z,\epsilon}} \sigma(u_2^0, p_2^0) n v_1^\epsilon ds &= \rho(\epsilon) \delta a_1 + o(\rho(\epsilon)), \\ \lim_{\epsilon \rightarrow 0} \rho(\epsilon) &= 0. \end{aligned}$$

To this end, we start by splitting the integral above,

$$\begin{aligned} \int_{\partial\mathcal{O}_{z,\epsilon}} \sigma(u_2^0, p_2^0) n v_1^\epsilon ds &= \int_{\partial\mathcal{O}_{z,\epsilon}} \sigma(u_2^0, p_2^0) n v_1^0 ds + \int_{\partial\mathcal{O}_{z,\epsilon}} \sigma(u_2^0, p_2^0) n (v_1^\epsilon - v_1^0) ds \\ &= \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

Next, properly using the obtained estimates by Ben Abda et al [4] for each term  $\mathcal{I}_1$  and  $\mathcal{I}_2$ , which are written as follows

$$\mathcal{I}_1 = -2\nu\epsilon^2 |\mathcal{B}| \mathcal{D}(u_2^0)(z) : \mathcal{D}(v_1^0)(z) + o(\epsilon^2),$$

and

$$\mathcal{I}_2 = 2\nu\epsilon^2 |\mathcal{B}| \mathcal{D}(u_2^0)(z) : \mathcal{D}(v_1^0)(z) - \epsilon^2 \mathcal{D}(u_2^0)(z) : \int_{\partial\mathcal{B}} \mu(y) y^T ds + o(\epsilon^2),$$

where  $\mu \in H^{-\frac{1}{2}}(\partial\mathcal{B})^d$  is the solution to the boundary integral equation :  $\forall y \in \partial\mathcal{B}$ ,

$$-\frac{\mu(y)}{2} + \int_{\partial\mathcal{B}} [2\nu \mathcal{D}_y(E)(x-y)\mu(x)n(y) - P(x-y)\mu(x)n(y)] ds(x) = -2\nu \mathcal{D}(v_1^0)(z)n(y),$$

with  $(E, P)$  is the fundamental solution to the stokes system in  $\mathbb{R}^2$ . Therefore, we deduce

$$(\mathcal{A}_{1,\epsilon} - \mathcal{A}_{1,0})(u_2^0, v_1^\epsilon) = -\epsilon^2 \mathcal{D}(u_2^0)(z) : \int_{\partial\mathcal{B}} \mu(y) y^T ds + o(\epsilon^2).$$



If  $\mathcal{B} = \mathcal{B}(0, 1)$ , using the same technique as that used in [4], we obtain

$$\delta a_1 = -4\nu \mathcal{D}(u_2^0)(z) : \mathcal{D}(v_1^0)(z),$$

where  $v_1^0$  is the solution to the associated adjoint problem :

$$\mathcal{A}_{1,0}(w, v_1^0) = -\partial_{u_2^0} \mathcal{J}_{KV}(u_2^0, u_3^0)w, \quad \forall v_1^0 \in \mathcal{V}_{1,0}^0.$$

*Variation of  $\mathcal{A}_{2,\epsilon}$*  : Let us mention that the same bilinear form is available also for the  $(\mathcal{P}_3^\epsilon)$ , namely,  $\mathcal{A}_{1,\epsilon} \equiv \mathcal{A}_{2,\epsilon}$ . Thus, the variation of  $\mathcal{A}_{1,\epsilon}$  associated to  $(\mathcal{P}_3^\epsilon)$  is written as follows :

$$(\mathcal{A}_{2,\epsilon} - \mathcal{A}_{2,0})(u_3^0, v_2^\epsilon) = -4\nu\pi\epsilon^2 \mathcal{D}(u_3^0)(z) : \mathcal{D}(v_2^0)(z),$$

where  $v_2^0$  is the solution to the associated adjoint problem :

$$\mathcal{A}_{2,0}(w, v_2^0) = -\partial_{u_3^0} \mathcal{J}_{KV}(u_2^0, u_3^0)w, \quad \forall v_2^0 \in \mathcal{V}_{2,0}^0.$$

VARIATION OF THE LINEAR FORM  $l_{1,\epsilon}$  AND  $l_{2,\epsilon}$  :

Since  $l_{1,\epsilon}$  and  $l_{2,\epsilon}$  are independent of  $\epsilon$ , it follows trivially that  $\delta l_1 = \delta l_2 = 0$ .

VARIATION OF THE COST FUNCTIONAL  $\mathcal{J}_{KV}$  :

Let us now turn to the asymptotic analysis of the variation of the Kohn-Vogelius functional given by

$$\mathcal{J}_{kv}(u_2^\epsilon, u_3^\epsilon) = 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_3^\epsilon - u_2^\epsilon) : \mathcal{D}(u_3^\epsilon - u_2^\epsilon) dx.$$

One can decompose this above functional as follows :

$$\mathcal{J}_{KV}(u_2^\epsilon, u_3^\epsilon) = \mathfrak{J}_1(u_2^\epsilon) + \mathfrak{J}_2(u_3^\epsilon) - 2\mathfrak{J}_{12}(u_2^\epsilon, u_3^\epsilon),$$

where

$$\mathfrak{J}_1(u_2^\epsilon) = 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon) : \mathcal{D}(u_2^\epsilon) dx,$$

$$\mathfrak{J}_2(u_3^\epsilon) = 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_3^\epsilon) : \mathcal{D}(u_3^\epsilon) dx,$$

$$\mathfrak{J}_{12}(u_2^\epsilon, u_3^\epsilon) = 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon) : \mathcal{D}(u_3^\epsilon) dx.$$

*Variation of  $\mathfrak{J}_1$*  : The variation of  $\mathfrak{J}_1$  reads

$$\begin{aligned} \mathfrak{J}_1(u_2^\epsilon) - \mathfrak{J}_1(u_2^0) &= 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon) : \mathcal{D}(u_2^\epsilon) dx - 2\nu \int_{\Omega} \mathcal{D}(u_2^0) : \mathcal{D}(u_2^0) dx \\ &= 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon - u_2^0) : \mathcal{D}(u_2^\epsilon) dx + 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon - u_2^0) : \mathcal{D}(u_2^0) dx \\ &\quad - 2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_2^0) : \mathcal{D}(u_2^0) dx \end{aligned}$$

Using the Green formula applied to the problem  $(\mathcal{P}_2^\epsilon)$ , we get

$$2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon - u_2^0) : \mathcal{D}(u_2^\epsilon) dx = \int_{\Gamma} \Phi(u_2^\epsilon - u_2^0) ds.$$

Then, it follows that

$$\begin{aligned} \mathfrak{J}_1(u_2^\epsilon) - \mathfrak{J}_1(u_2^0) &= 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon - u_2^0) : \mathcal{D}(u_2^0) dx - 2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_2^0) : \mathcal{D}(u_2^0) dx \\ &\quad + \int_{\Gamma} \Phi(u_2^\epsilon - u_2^0) ds. \end{aligned} \quad (8)$$

*Variation of  $\mathfrak{J}_2$*  : The variation of  $\mathfrak{J}_2$  reads

$$\begin{aligned} \mathfrak{J}_2(u_3^\epsilon) - \mathfrak{J}_2(u_3^0) &= 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_3^\epsilon) : \mathcal{D}(u_3^\epsilon) dx - 2\nu \int_{\Omega} \mathcal{D}(u_3^0) : \mathcal{D}(u_3^0) dx \\ &= 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_3^\epsilon - u_3^0) : \mathcal{D}(u_3^\epsilon) dx + 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_3^\epsilon - u_3^0) : \mathcal{D}(u_3^0) dx \\ &\quad - 2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_3^0) : \mathcal{D}(u_3^0) dx \end{aligned}$$

Using the Green formula, one can get from  $(\mathcal{P}_3^\epsilon)$  that

$$2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_3^\epsilon - u_3^0) : \mathcal{D}(u_3^\epsilon) dx = 0.$$

Then, we obtain

$$\mathfrak{J}_2(u_3^\epsilon) - \mathfrak{J}_2(u_3^0) = 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_3^\epsilon - u_3^0) : \mathcal{D}(u_3^0) dx - 2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_3^0) : \mathcal{D}(u_3^0) dx. \quad (9)$$

*Variation of  $\mathfrak{J}_{12}$*  : The variation of  $\mathfrak{J}_{12}$  reads

$$\mathfrak{J}_{12}(u_2^\epsilon, u_3^\epsilon) - \mathfrak{J}_{12}(u_2^0, u_3^0) = 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon) : \mathcal{D}(u_3^\epsilon) dx - 2\nu \int_{\Omega} \mathcal{D}(u_2^0) : \mathcal{D}(u_3^0) dx.$$

Using the Green formula applied to  $(\mathcal{P}_2^\epsilon)$  and  $(\mathcal{P}_2^0)$ , we obtain

$$\begin{aligned} 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon) : \mathcal{D}(u_3^\epsilon) dx &= \int_{\Gamma} \Phi f ds + \int_{\Gamma_i} \sigma(u_2^\epsilon, p_2^\epsilon) n \tau ds. \\ 2\nu \int_{\Omega} \mathcal{D}(u_2^0) : \mathcal{D}(u_3^0) dx &= \int_{\Gamma} \Phi f ds + \int_{\Gamma} \sigma(u_2^0, p_2^0) n \tau ds. \end{aligned}$$

Then, we deduce

$$\mathfrak{J}_{12}(u_2^\epsilon, u_3^\epsilon) - \mathfrak{J}_{12}(u_2^0, u_3^0) = \int_{\Gamma_i} \sigma(u_2^\epsilon - u_2^0, p_2^\epsilon - p_2^0) n \tau ds. \quad (10)$$

Combining the variation (8), (9) and (10), the variation of the functional  $\mathcal{J}_{KV}$  becomes

$$\begin{aligned} \mathcal{J}_{KV}(u_2^\epsilon, u_3^\epsilon) - \mathcal{J}_{KV}(u_2^0, u_3^0) &= -2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_2^0) : \mathcal{D}(u_2^0) dx - 2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_3^0) : \mathcal{D}(u_3^0) dx \\ &\quad + 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon - u_2^0) : \mathcal{D}(u_2^0) dx + 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_3^\epsilon - u_3^0) : \mathcal{D}(u_3^0) dx \\ &\quad + \int_{\Gamma} \Phi(u_2^\epsilon - u_2^0) ds - 2 \int_{\Gamma} \sigma(u_2^\epsilon - u_2^0, p_2^\epsilon - p_2^0) n \tau ds. \end{aligned}$$

Then for  $\epsilon \in [0, \xi)$ , we get

$$\begin{aligned} D\mathcal{J}_{KV}(u_2^0, u_3^0)(u_2^\epsilon - u_2^0, u_3^\epsilon - u_3^0) &= 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_2^\epsilon - u_2^0) : \mathcal{D}(u_2^0) dx \\ &+ 2\nu \int_{\Omega_{z,\epsilon}} \mathcal{D}(u_3^\epsilon - u_3^0) : \mathcal{D}(u_3^0) dx + \int_{\Gamma_c} \Phi(u_2^\epsilon - u_2^0) ds \\ &- 2 \int_{\Gamma} \sigma(u_2^\epsilon - u_2^0, p_2^\epsilon - p_2^0) n \tau ds. \end{aligned}$$

Thus, we have

$$\begin{aligned} \mathcal{J}_{KV}(u_2^\epsilon, u_3^\epsilon) - \mathcal{J}_{KV}(u_2^0, u_3^0) &= D\mathcal{J}_{KV}(u_2^0, u_3^0)(u_2^\epsilon - u_2^0, u_3^\epsilon - u_3^0) \\ &- 2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_2^0) : \mathcal{D}(u_2^0) dx - 2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_3^0) : \mathcal{D}(u_3^0) dx. \end{aligned} \quad (11)$$

Next, the second term on the right hand side of (11) may be written as

$$\begin{aligned} 2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_2^0) : \mathcal{D}(u_2^0) dx &= 2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_2^0)(z) : \mathcal{D}(u_2^0)(z) dx \\ &+ 2\nu \int_{\mathcal{O}_{z,\epsilon}} [\mathcal{D}(u_2^0) - \mathcal{D}(u_2^0)(z)] : \mathcal{D}(u_2^0) dx \\ &+ 2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_2^0)(z) : [\mathcal{D}(u_2^0) - \mathcal{D}(u_2^0)(z)] dx. \end{aligned}$$

Using the Taylor theorem and the change of variables  $x = z + \epsilon y$ , we obtain

$$2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_2^0) : \mathcal{D}(u_2^0) dx = 2\nu\epsilon^2 |\mathcal{B}| \mathcal{D}(u_2^0)(z) : \mathcal{D}(u_2^0)(z) + o(\epsilon^2).$$

The same way for the third term, we have

$$2\nu \int_{\mathcal{O}_{z,\epsilon}} \mathcal{D}(u_3^0) : \mathcal{D}(u_3^0) dx = 2\nu\epsilon^2 |\mathcal{B}| \mathcal{D}(u_3^0)(z) : \mathcal{D}(u_3^0)(z) + o(\epsilon^2).$$

Thus,

$$\delta\mathcal{J}_{KV} = -2\nu\pi(|\mathcal{D}(u_2^0)(z)|^2 + |\mathcal{D}(u_3^0)(z)|^2).$$

Now, we are ready to give the main result of this paper.

**Theorem 1** *If  $d = 2$ , the function  $\mathcal{J}$  has the following asymptotic expansion*

$$\mathcal{J}(\Omega_{z,\epsilon}) - \mathcal{J}(\Omega) = \pi\epsilon^d (\delta a_1(u_2^0, v_1^0) + \delta a_2(u_3^0, v_2^0) + \delta\mathcal{J}_{KV}(u_2^0, u_3^0)) + o(\epsilon^d),$$

where  $\forall z \in \Omega$ , we have

$$\begin{cases} \delta a_1(u_2^0, v_1^0) &= -4\nu \mathcal{D}(u_2^0)(z) : \mathcal{D}(v_1^0)(z), \\ \delta a_2(u_3^0, v_2^0) &= -4\nu \mathcal{D}(u_3^0)(z) : \mathcal{D}(v_2^0)(z), \\ \delta\mathcal{J}_{KV}(u_2^0, u_3^0) &= -2(|\mathcal{D}(u_2^0)(z)|^2 + |\mathcal{D}(u_3^0)(z)|^2), \end{cases}$$

with  $v_1^0 \in \mathcal{V}_{1,0}^0$  and  $v_2^0 \in \mathcal{V}_{2,0}^0$  are solutions to the adjoint equations associated respectively to the  $(\mathcal{P}_2^0)$  and  $(\mathcal{P}_3^0)$ :

$$\mathcal{A}_{1,0}(w, v_1^0) = -\partial_{u_2} \mathcal{J}(u_2^0, u_3^0)w, \quad \forall w \in \mathcal{V}_{1,0}^\epsilon,$$

$$\mathcal{A}_{2,0}(w, v_2^0) = -\partial_{u_3} \mathcal{J}(u_2^0, u_3^0)w, \quad \forall w \in \mathcal{V}_{2,0}^\epsilon.$$

The main steps in computing the Nash equilibrium are described in Algorithm below. The gradient method used to solve the partial optimization problems, a fixed step to minimize  $\mathcal{J}_1$  and  $\mathcal{J}_2$ , and the topological derivative finding the approximate locations to optimize  $\mathcal{J}_3$ .

**Algorithm :**

Set  $k = 0$  and choose an initial guess  $S^{(0)} = (\eta^{(0)}, \tau^{(0)}) \in (H_{00}^{\frac{1}{2}}(\Gamma_i)^d)' \times H^{\frac{1}{2}}(\Gamma_i)^d$  :

- Step I : Fix an initial shape  $\mathcal{O}_k = \emptyset$  and use the one-shot algorithm to determine the set

$$\mathcal{Z}^{(k+1)} = \operatorname{argmin}_{\mathcal{Z} \in \Omega} \mathcal{J}_3(\eta^{(k)}, \tau^{(k)}; \mathcal{Z}).$$

- Step II : Solve the Nash game between  $\eta$  and  $\tau$  : Set  $p = 0$ .

Set  $\Omega^{(k+1)} = \Omega \setminus \overline{\mathcal{O}_{k+1}}$ , where  $\mathcal{O}_{k+1} = \bigcup_{i=1}^{m_k} \mathcal{B}(z_i, r)$ .

1. Compute  $\bar{\eta}^{(p)}$ , which solves  $\min_{\eta} \mathcal{J}_1(\eta, \tau^{(p)}; \mathcal{Z}^{(k+1)})$ .

Evaluate  $\eta^{(p+1)} = \alpha \eta^{(p)} + (1 - \alpha) \bar{\eta}^{(p)}$ , with  $0 \leq \alpha < 1$ .

2. Compute  $\bar{\tau}^{(p)}$ , which solves  $\min_{\tau} \mathcal{J}_2(\eta^{(p)}, \tau; \mathcal{Z}^{(k+1)})$ .

Evaluate  $\tau^{(p+1)} = \alpha \tau^{(p)} + (1 - \alpha) \bar{\tau}^{(p)}$ , with  $0 \leq \alpha < 1$ .

3. Set  $S^{(p+1)} = (\eta^{(p+1)}, \tau^{(p+1)})$ .

While  $\|S^{(p+1)} - S^{(p)}\| > \varepsilon_S$ , set  $p = p + 1$ , return back to step 1.

- Step III : Compute  $r_k = \|u_2^{(k)} - f\|_{0, \Gamma_c}$ , where  $(u_2^{(k)}, p_2^{(k)})$  is the solution of the problem  $(\mathcal{P}_2^\varepsilon)$ . If  $r_k < \varepsilon_N$  stop. Otherwise  $k = k + 1$ , go to Step I.

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### 3. Numerical simulations.

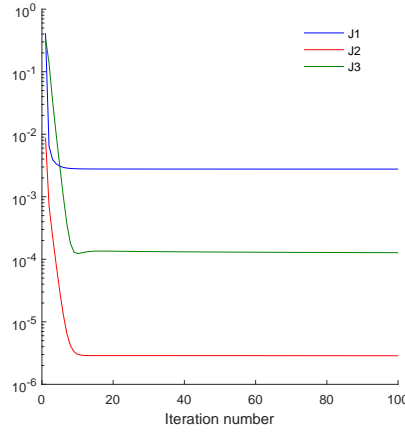
In this section, we will present some numerical reconstructions in two dimensions to show the efficiency of our novel approach, using synthetic data generated via a finite element resolution of boundary value problem, corresponding to a homogeneous Neumann condition on the boundary  $\partial\mathcal{O}_\varepsilon^*$  with the code *FreeFem++* [9].

The exterior boundary is assumed to be the rectangle  $\Omega = [-0.5, 0.5] \times [-0.25, 0.25]$ , which will be split in two components : the inaccessible part of the boundary  $\Gamma_i = \{0.5\} \times [-0.25, 0.25]$  and the accessible part  $\Gamma_c = \partial\Omega \setminus \Gamma_i$ , where the Cauchy data are available. In the following, the subscripts *ex* and *opt* denote, respectively, exact and optimal values.

**Case A- Single object :** First, we start testing the detection of a single circle  $\mathcal{O}^* = \mathcal{C}(z_{ex}, r)$  centered at  $z_{ex}$  and with radius  $r$  where  $z_{ex} = (-0.3, -0.15)$  and  $r = 0.025$ .

Figure 2 presents the evolution of the three costs functionals as functions of overall Nash iterations for unnoisy data. We remark that the three players show a fast decrease of their costs before stagnating. The detection is quite efficient, see Figure 3(a)-(b), where 3(a) presents the iso-values of the topological gradient and 3(b) presents the obtained domain at convergence. Figure 4 show the reconstructed Dirichlet and Neumann boundary data. It can be seen that the reconstructed Dirichlet data give a good approximation for the exact one, while the reconstructed Neumann data deviate from the exact one, especially near the endpoints of the unspecified boundary, which is the region of singularities. At convergence, the approximate location  $z_{opt}$  is equal to  $(-0.294, -0.174)$  and the relative error

on Dirichlet data  $\frac{\|\tau_{opt} - u_{ex}|_{\Gamma}\|_{L^2(\Gamma_i)}}{\|u_{ex}|_{\Gamma_i}\|_{L^2(\Gamma_i)}}$  and Neumann data  $\frac{\|\eta_{opt} - \sigma(u_{ex}, p_{ex})n|_{\Gamma_i}\|_{L^2(\Gamma_i)}}{\|\sigma(u_{ex}, p_{ex})n|_{\Gamma_i}\|_{L^2(\Gamma_i)}}$  are equal to 0.003 and 0.073 respectively.

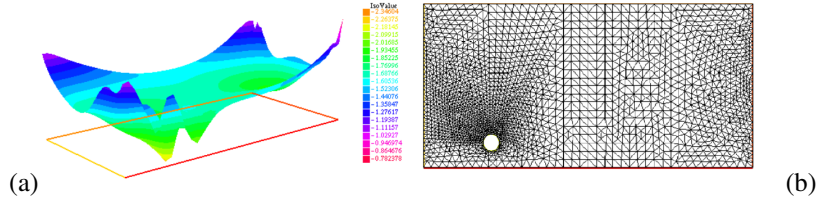


**Figure 2.** Case A- . Plots of the three costs  $\mathcal{J}_1$ ,  $\mathcal{J}_2$  and  $\mathcal{J}_3$  as functions of overall Nash iterations

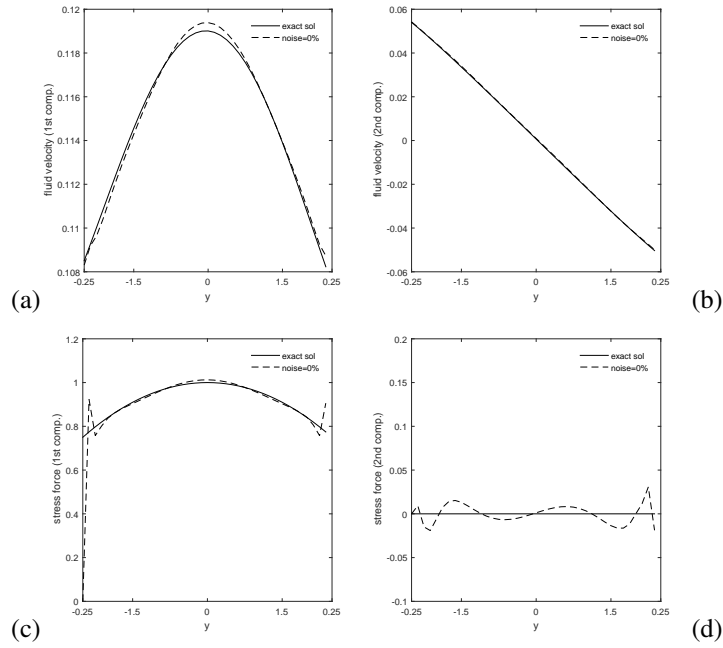
**Case B- Two objects :** For this test, we want to detect two circles  $\mathcal{O}_1^*$  and  $\mathcal{O}_2^*$  centered respectively at  $(-0.4, -0.15)$  and  $(-0.4, 0.15)$ , with shared radius  $r = 0.025$ . The numerical results are illustrated in Figure 5 and 6. Figure 5 shows the iso-values of the topological gradient and the obtained domain at convergence. The optimal locations are equal to  $(-0.422006, -0.160662)$  and  $(-0.416224, 0.159772)$ . In Figure 6, we present the reconstruction of the missing Dirichlet and Neumann boundary data.

## 4. Conclusion

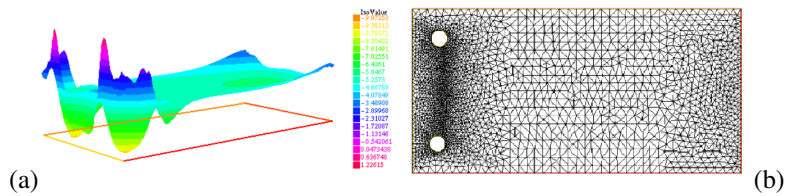
We addressed in this work the inverse problem of detecting unknown objects immersed in a viscous fluid in a larger bounded domain from partial boundary measurements. To treat this problem, we introduced an algorithm based on a game theory. In this case, a three-player Nash game is considered here as a properly formulation of the above problem : The two first players target the data completion while the third one determine the unknown locations of inclusions. The latter problem is formulated as a topological one. In view of this, the topological gradient of the considered Kohn-Vogelius functional was calculated. Then, we explored two different numerical tests which have shown precisely that small inclusions close to the accessible part of the boundary can be detected, simultaneously to recovering the missing data on the inaccessible part. This novel approach allows us to deal effectively with these problems, coupled with "ill-posedness."



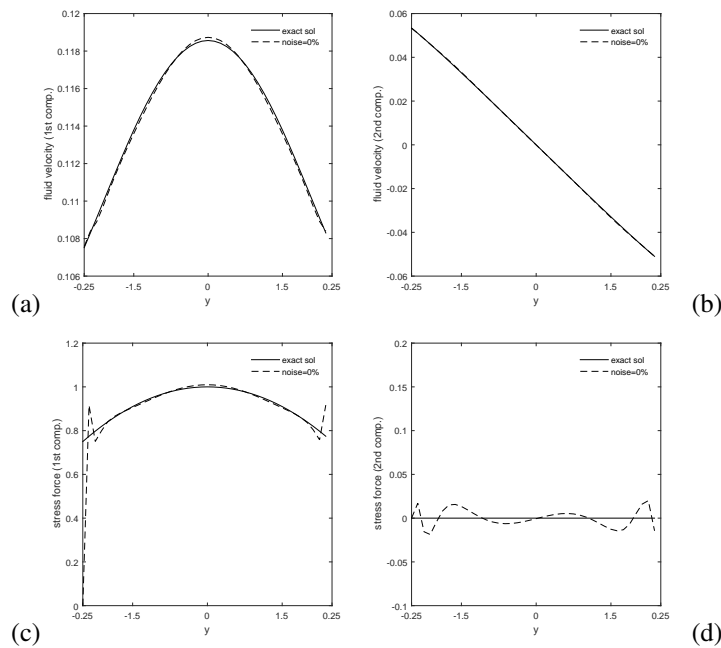
**Figure 3.** Case A- Detection of a single object (for unnoisy data). (a) the iso-values of the topological gradient at convergence (b) the approximate location  $z_{opt}$  is equal to  $(-0.294, -0.174)$ , while the exact one  $z_{ex}$  is equal to  $(-0.3, -0.15)$ .



**Figure 4.** Case A- Reconstruction of the missing boundary data (for unnoisy data). (a) exact -line- and computed -dashed line- first component of the velocity over  $\Gamma_i$  (b) exact -line- and computed -dashed line- second component of the velocity over  $\Gamma_i$  (c) exact -line- and computed -dashed line- first component of the normal stress over  $\Gamma_i$  (d) exact -line- and computed -dashed line- second component of the normal stress over  $\Gamma_i$ .



**Figure 5.** Case B- Detection of two objects (for unnoisy data). (a) the iso-values of the topological gradient at convergence (b) the approximate locations are equal to  $z_{opt}^1 = (-0.422006, -0.160662)$  and  $z_{opt}^2 = (-0.416224, 0.159772)$ , while the exact ones are equal to  $z_{ex}^1 = (-0.4, -0.15)$  and  $z_{ex}^2 = (-0.4, 0.15)$ .



**Figure 6.** Case B- Reconstruction of the missing boundary data (for unnoisy data). (a) exact -line- and computed -dashed line- first component of the velocity over  $\Gamma_i$  (b) exact -line- and computed -dashed line- second component of the velocity over  $\Gamma_i$  (c) exact -line- and computed -dashed line- first component of the normal stress over  $\Gamma_i$  (d) exact -line- and computed -dashed line- second component of the normal stress over  $\Gamma_i$ .

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