

# $L^p$ -estimates of the Stokes resolvent with nonhomogeneous boundary conditions in 3D exterior domains.

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## Abstract

The article deals with the homogeneous Stokes resolvent system in a 3D exterior domain, under inhomogeneous Dirichlet boundary conditions. Solutions to this boundary value problem are estimated in  $L^p$ -norms, with the bounds in these estimates depending on the absolute value of the resolvent parameter  $\lambda$  in an explicit way. Two types of boundary data are considered, that is,  $L^p$ - and  $W^{2-1/p,p}$ -data. It is shown in particular that the  $L^p$ -norm of the velocity outside a vicinity of the boundary, after subtraction of the gradient of a certain harmonic function, is bounded by a constant times  $|\lambda| \|b\|_p$ . This estimate carries over to the Oseen resolvent system, leading to a result which has applications in the theory of spatial asymptotics of solutions to the 3D time-dependent Navier-Stokes system with Oseen term.

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**Key words.** Stokes resolvent system. Oseen resolvent system.  $L^p$ -estimates. Inhomogeneous boundary conditions

## 1 Introduction

We consider the Stokes resolvent problem

$$-\Delta u + \lambda u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad (1.1)$$

in an exterior domain  $\overline{\Omega}^c := \mathbb{R}^3 \setminus \overline{\Omega}$ , where  $\Omega$  is an open, bounded set with  $C^2$ -boundary. The system in (1.1) is supplemented by Dirichlet boundary conditions

$$u|_{\partial\Omega} = b. \quad (1.2)$$

The velocity  $u : \overline{\Omega}^c \mapsto \mathbb{C}^3$  and the pressure  $\pi : \overline{\Omega}^c \mapsto \mathbb{C}$  are unknown, whereas the volume force  $f : \overline{\Omega}^c \mapsto \mathbb{C}^3$ , the resolvent parameter  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$  and the boundary data  $b : \partial\Omega \mapsto \mathbb{C}^3$  are given. In the case of homogeneous Dirichlet boundary data ( $b = 0$ ), problem (1.1), (1.2) has been studied extensively [32], [10] – [12], [47]. The results in these articles provide a functional analytic access to the study of the time-dependent Navier-Stokes system in  $\overline{\Omega}^c$ . Some of these results will be needed here, too. They are stated in Theorem 4.1 and Corollary 4.1 below.

The case  $f = 0$  and  $b$  nonvanishing has been investigated much less. In fact, to the best of our knowledge, as concerns  $L^p$ -theory, there are only some technical results in [11], where  $b$  is supposed to be given by  $b = w|_{\partial\Omega}$ , where  $w$  is some function from  $C^2(\mathbb{R}^3)^3 \cap L^p(\mathbb{R}^3)^3$  with  $\operatorname{div} w = 0$ . However, the case  $b \neq 0$  is of interest, too. In fact, the work at hand is motivated by a study [16] of spatial decay of time-dependent flows, where estimates of solutions to (1.1), (1.2) with  $b \neq 0$  play a key role. We will come back to this point further below.

In this work we consider two cases, that is,  $b \in L^p(\partial\Omega)^3$  and  $b \in W^{2-1/p,p}(\partial\Omega)^3$ . In the first case we estimate  $\|u\|_p$  by  $\|b\|_p$ , and in the second we give a bound of  $\|u\|_{2,p}$  in terms

of  $\|b\|_{2-1/p,p} + |\lambda| \|b\|_p$ . Each of these estimates is explicit with respect to  $|\lambda|$ , and uniform with respect to  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq \lambda_0$  and  $|\arg(\lambda)| \leq \vartheta$ , where  $\lambda_0 \in (0, \infty)$  and  $\vartheta \in [0, \pi)$  are given. More details are stated in the ensuing theorem.

**Theorem 1.1** *Let  $n^{(\Omega)}$  denote the outward unit normal to  $\Omega$ . Fix a function  $N^{(\Omega)} \in C^1(\overline{\Omega})^3$  with  $N^{(\Omega)}|_{\partial\Omega} = n^{(\Omega)}$  (Lemma 2.2). Put  $\mathfrak{N}(z) := (4\pi|z|)^{-1}$  for  $z \in \mathbb{R}^3 \setminus \{0\}$  (Newton kernel), and define  $\mathfrak{F}^{(\Omega)}(x) := -\int_{\Omega} (\nabla \mathfrak{N})(x-y) \operatorname{div} N^{(\Omega)}(y) dy$  for  $x \in \mathbb{R}^3$ . Then  $\mathfrak{F}^{(\Omega)} \in L^r(\mathbb{R}^3)^3$  for  $r \in (3/2, \infty)$ .*

*Let  $p \in (1, \infty)$ ,  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $b \in L^p(\partial\Omega)^3$ . Then there is a pair of functions  $(u, \pi) = (u(\lambda, b), \pi(\lambda, b)) \in C^\infty(\overline{\Omega}^c)^3 \times C^\infty(\overline{\Omega}^c)$  such that  $(u, \pi)$  solves (1.1) with  $f = 0$ , and  $u$  satisfies (1.2) in the  $L^p$ -sense, that is,  $\int_{\partial\Omega} |b(x) - u(x + \epsilon n^{(\Omega)}(x))|^p d\sigma_x \rightarrow 0$  ( $\epsilon \downarrow 0$ ), where (2.4) is used implicitly.*

*If  $b \in W^{2-1/p,p}(\partial\Omega)^3$ ,  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ , and  $\Omega_R := B_R \setminus \overline{\Omega}$ , then  $u|_{\Omega_R} \in W^{2,p}(\Omega_R)^3$ ,  $\pi|_{\Omega_R} \in W^{1,p}(\Omega_R)$ , and (1.2) is satisfied in the trace sense.*

*Let  $\vartheta \in [0, \pi)$  and  $\lambda_0 \in (0, \infty)$ . Then*

$$\|u - |\partial\Omega|^{-1} \int_{\partial\Omega} b \cdot n^{(\Omega)} d\sigma_x \mathfrak{F}^{(\Omega)}\|_p \leq \mathfrak{C}(p, \vartheta, \lambda_0) \|b\|_p \quad (1.3)$$

*for  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq \lambda_0$ ,  $|\arg(\lambda)| \leq \vartheta$  and  $b \in L^p(\partial\Omega)^3$ . In particular*

$$\|u\|_p \leq \mathfrak{C}(p, \vartheta, \lambda_0) \|b\|_p \quad (1.4)$$

*for  $\lambda$  as in (1.3) and for  $b \in L^p(\partial\Omega)^3$  with  $\int_{\partial\Omega} b \cdot n^{(\Omega)} d\sigma_x = 0$  (zero flux condition). If  $p > 3/2$ , inequality (1.4) holds for  $\lambda$  and  $b$  as in (1.3).*

*Let  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ . If  $r_1 \in (3/2, \infty)$ , then  $\|u|_{B_R^c}\|_{r_1} \leq \mathfrak{C}(r_1) \|b\|_p$ ,  $\|\nabla \pi|_{B_R^c}\|_{r_1} \leq \mathfrak{C}(r_1) (1 + |\lambda|) \|b\|_p$ ;*

*if  $r_2 \in (1, \infty)$  and  $\alpha \in \mathbb{N}_0^3$  with  $1 \leq |\alpha| \leq 2$ , then  $\|\partial^\alpha u|_{B_R^c}\|_{r_2} \leq \mathfrak{C}(r_2) \|b\|_p$ ;*

*if  $r_3 \in (3, \infty)$ , then  $\|\pi|_{B_R^c}\|_{r_3} \leq \mathfrak{C}(r_3) (1 + |\lambda|) \|b\|_p$ , each time for  $\lambda, b$  as in (1.3), and with  $\mathfrak{C}(r_j) = \mathfrak{C}(r_j, \vartheta, p, \lambda_0, R)$  for  $j \in \{1, 2, 3\}$ .*

*If  $a \in (0, 2/p)$ , then  $\|u|_{\Omega_R}\|_{(1/p-a/2)^{-1}} \leq \mathfrak{C}(p, \vartheta, \lambda_0, a, R) \|b\|_p$  for  $\lambda, b$  as in (1.3). Finally*

$$\|u|_{\Omega_R}\|_{2,p} + \|\pi|_{\Omega_R}\|_{1,p} \leq \mathfrak{C}(p, \vartheta, \lambda_0, R) (\|b\|_{2-1/p,p} + |\lambda| \|b\|_p) \quad (1.5)$$

*for  $\lambda$  as in (1.3) and for  $b \in W^{2-1/p,p}(\partial\Omega)^3$ .*

The usual approach for proving Theorem 1.1 would consist in extending  $b$  to a suitable function  $\tilde{b}$  on  $\overline{\Omega}^c$ , and then estimating  $u - \tilde{b}$  by means of the theory pertaining to the case  $f \neq 0$ ,  $b = 0$ . But in order to obtain the estimates stated in Theorem 1.1, it would be necessary that every function  $b \in L^p(\partial\Omega)^3$  admits a solenoidal extension  $B$  to at least  $\Omega_R$  with the property that  $\|B\|_p$  may be estimated by  $\|b\|_p$ . The only way we found to obtain such an extension consists in directly constructing a solution to (1.1), (1.2) with  $f = 0$ . To this end, we used the same approach as in [11], that is, the method of integral equations.

More precisely, we construct  $u(x)$  and  $\pi(x)$  as boundary potentials – integrals on  $\partial\Omega$  depending on  $x$  (Lemma 5.2) –, with layer functions solving an integral equation with right-hand side  $b$  (equation (5.3)). Theorem 1.1 may then be established by estimating these potential functions (Section 5 and 6).

This approach provides some further properties – not mentioned in Theorem 1.1 – of our solutions. For example, for any  $\phi \in L^p(\partial\Omega)^3$ , we define a function  $\mathfrak{F}(\phi) : \mathbb{R}^3 \setminus \partial\Omega \mapsto \mathbb{C}^3$  by setting

$$\mathfrak{F}(\phi)(x) := \int_{\partial\Omega} (\nabla \mathfrak{N})(x-y) (n^{(\Omega)} \cdot \phi)(y) \, do_y \quad \text{for } x \in \mathbb{R}^3 \setminus \Omega. \quad (1.6)$$

(The kernel  $\mathfrak{N}$  was introduced in Theorem 1.1.) Note that  $\mathfrak{F}(\phi)$  is the gradient of a harmonic function in  $\mathbb{R}^3 \setminus \partial\Omega$ . Then, for any  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$  and  $b \in L^p(\partial\Omega)^3$ , we take the solution  $\phi = \phi_{\lambda,b} \in L^p(\partial\Omega)^3$  of the integral equation (5.3) and consider  $\mathfrak{F}(\phi)$  for such  $\phi$ . Fixing numbers  $\vartheta \in [0, \pi)$ ,  $\lambda_0, R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ , we will prove that  $\|\lambda(u - \mathfrak{F}(\phi))\|_{B_R^c} \leq \mathfrak{C}(p, \vartheta, \lambda_0, R) \|\phi\|_p$  for  $\lambda$  et  $b$  as in (1.3) (Corollary 6.2). This inequality is remarkable because of the presence of the factor  $\lambda$  on the left-hand side, contrary to the situation in (1.3) and (1.4). It turned out (Theorem 7.2) this inequality, with an additional term  $\|\nabla u\|_p$  on the right-hand side, carries over to weak solutions of the Oseen resolvent system

$$-\Delta u + \tau u + \lambda u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \overline{\Omega}^c, \quad (1.7)$$

under Dirichlet boundary conditions (1.2), where  $\tau \in (0, \infty)$  is given (Reynolds number). A slightly generalized version (Corollary 7.1) of this Oseen resolvent estimate plays an important role in the article [16] mentioned above, which deals with spatial decay of  $L^2$ -strong solutions to the time-dependent Navier Stokes system in  $\overline{\Omega}^c$  with Oseen term and with some additional terms arising in stability estimates. In fact, for technical reasons we cannot elaborate here, the theory in [16] gives rise to the question as to whether certain weak solutions to the time-dependent Oseen system admit a time derivative that may be considered as an  $L^2$ -integrable function of the time variable with values in certain Banach spaces. By means of a Fourier transform, this question translates into the problem whether for a weak solution  $u$  to (1.7), (1.2) with  $\lambda = i\xi$  and  $f = 0$ , the quantity  $\|i\xi u\|_p$  is bounded by  $\mathfrak{C}|\xi|^{-1} \|b\|_p$ , uniformly with respect to  $\xi \in \mathbb{R}^3$  with  $|\xi| \geq 1$ . Unfortunately such an estimate cannot be expected to hold, as is already obvious by the situation in the Stokes case; see (1.3) and (1.4). However, it turned out that the estimate of  $\|\lambda(u - \mathfrak{F}(\phi))\|_{B_R^c}$  we are able to derive (Corollary 7.1) is sufficient for the purposes of [16]. This is the reason why we think our results on the Oseen resolvent (Section 7) are interesting.

We remark that reference [18], frequently used in this work, elaborates some sections of Ladyzhenskaya's monograph [37]. We further indicate that in literature, there is a great number of articles on the Stokes resolvent system, dealing with various aspect of this system. As examples, we cite [1] – [9], [14], [20] – [29], [33], [35], [36], [39] – [45], [49], [50]. The Oseen resolvent system has been studied much less [15], [17], [19], [34].

## 2 Notation. Local coordinates of $\Omega$ .

The bounded open set  $\Omega \subset \mathbb{R}^3$  with  $C^2$ -boundary will be kept fixed throughout. Recall that its outward unit normal is denoted by  $n^{(\Omega)}$  (Theorem 1.1).

The symbol  $|\cdot|$  denotes the Euclidean norm of  $\mathbb{R}^n$  for any  $n \in \mathbb{N}$ , the length  $\alpha_1 + \alpha_2 + \alpha_3$  of a multi-index  $\alpha \in \mathbb{N}_0^3$ , and the Borel measure of measurable subsets of  $\mathbb{R}^3$  or  $\partial\Omega$ . For  $R \in (0, \infty)$ ,  $x \in \mathbb{R}^3$ , put  $B_R(x) := \{y \in \mathbb{R}^3 : |x - y| < R\}$ . In the case  $x = 0$ , we write  $B_R$  instead of  $B_R(0)$ , and we set  $\Omega_R := B_R \setminus \overline{\Omega}$ .

If  $A \subset \mathbb{R}^3$ , we denote by  $A^c$  the complement  $\mathbb{R}^3 \setminus A$  of  $A$  in  $\mathbb{R}^3$ . If  $A$  is some nonempty set and  $\gamma : A \mapsto \mathbb{R}$  a function, we set  $|\gamma|_\infty := \sup\{|\gamma(x)| : x \in A\}$ .

Let  $p \in [1, \infty)$ ,  $m, n \in \mathbb{N}$ . For  $A \subset \mathbb{R}^n$  open, the notation  $\|\cdot\|_p$  stands for the usual norm of the Lebesgue space  $L^p(A)$ , and  $\|\cdot\|_{m,p}$  for the usual norm of the Sobolev space  $W^{m,p}(A)$  of order  $m$  and exponent  $p$ . If  $A \subset \mathbb{R}^3$  is open and bounded with  $C^2$ -boundary, and if  $s \in (0, 2]$ , analogous notation are used for the Lebesgue space  $L^p(\partial A)$  and the Sobolev space  $W^{s,p}(\partial A)$  (of fractional order if  $s \notin \{1, 2\}$ ; see [30, Section 6.8.6]). Again for an open set  $A \subset \mathbb{R}^n$ , we write  $L^p_{loc}(A)$  and  $W^{m,q}_{loc}(A)$  for the set of all functions  $v$  from  $A$  into  $\mathbb{C}$  such that  $v|_B \in L^p(B)$  and  $v|_B \in W^{m,q}(B)$ , respectively, for any open, bounded set  $B \subset \mathbb{R}^3$  with  $\bar{B} \subset A$ . We put  $\nabla v := (\partial_k v_j)_{1 \leq j, k \leq 3}$  for  $v \in W^{1,1}_{loc}(A)^3$ . Moreover the notation  $C^\infty_0(A)$  stands for the set of all functions  $v \in C^\infty_0(\mathbb{R}^3)$  with  $\text{supp}(v) \subset A$ . In the case of a closed subset  $A$  of  $\mathbb{R}^3$ , we write  $C^\infty_0(A)$  for the set of all functions  $v|_A$  with  $v \in C^\infty_0(\mathbb{R}^3)$ .

Let  $\mathcal{V}$  be a normed space, and let the norm of  $\mathcal{V}$  be denoted by  $\|\cdot\|$ . Take  $n \in \mathbb{N}$ . Then we will use the same notation  $\|\cdot\|$  for the norm on  $\mathcal{V}^n$  defined by  $\|(f_1, \dots, f_n)\| := (\sum_{j=1}^n \|f_j\|^2)^{1/2}$  for  $(f_1, \dots, f_n) \in \mathcal{V}^n$ . The space  $\mathcal{V}^{3 \times 3}$ , as concerns its norm, is identified with  $\mathcal{V}^9$ .

For open sets  $A \subset \mathbb{R}^3$ , we define  $C^\infty_{0,\sigma}(A) := \{V \in C^\infty_0(A)^3 : \text{div} V = 0\}$ , and we write  $L^p_\sigma(A)$  for the closure of  $C^\infty_{0,\sigma}(A)$  with respect to the norm of  $L^p(A)^3$ , where  $p \in (1, \infty)$ . This function space  $L^p_\sigma(A)$  ("space of solenoidal  $L^p$ -functions") is equipped with the norm  $\|\cdot\|_p$ .

We write  $C$  for numerical constants and  $C(\gamma_1, \dots, \gamma_n)$  for constants depending exclusively on parameters  $\gamma_1, \dots, \gamma_n \in [0, \infty)$  for some  $n \in \mathbb{N}$ . However, such a precise bookkeeping will be possible only at some places. Mostly we will use the symbol  $\mathfrak{C}$  for constants whose dependence on parameters must be traced from context. Sometimes we write  $\mathfrak{C}(\gamma_1, \dots, \gamma_n)$  in order to indicate that the constant in question is influenced by the quantities  $\gamma_1, \dots, \gamma_n$ . But in such cases, this constant depends on other parameters as well. In most cases, these implicit dependencies are associated with  $\Omega$ .

Following [30, Section 6.2], and recalling the assumption that  $\Omega$  is  $C^2$ -bounded, we choose numbers  $k(\Omega) \in \mathbb{N}$ ,  $\alpha(\Omega) \in (0, \infty)$ , orthonormal matrices  $A_1^{(\Omega)}, \dots, A_{k(\Omega)}^{(\Omega)} \in \mathbb{R}^{3 \times 3}$ , vectors  $C_1^{(\Omega)}, \dots, C_{k(\Omega)}^{(\Omega)} \in \mathbb{R}^3$ , and functions  $a_1^{(\Omega)}, \dots, a_{k(\Omega)}^{(\Omega)} \in C^2([-\alpha(\Omega), \alpha(\Omega)]^2)$  with the following properties:

Put  $\Delta^\Omega := (-\alpha(\Omega), \alpha(\Omega))^2$ ,  $H^{(j)}(\eta, r) := A_j^{(\Omega)} \cdot (\eta, a_j^{(\Omega)}(\eta) + r) + C_j^{(\Omega)}$  and  $h^{(j)}(\eta) := H^{(j)}(\eta, 0) = A_j^{(\Omega)} \cdot (\eta, a_j^{(\Omega)}(\eta)) + C_j^{(\Omega)}$  for  $\eta \in \Delta^\Omega$ ,  $r \in (-\alpha(\Omega), \alpha(\Omega))$ , and define the set  $U_j$  by  $U_j := \{H^{(j)}(\eta, r) : \eta \in \Delta^\Omega, r \in (-\alpha(\Omega), \alpha(\Omega))\}$ , for  $1 \leq j \leq k(\Omega)$ . Then we assume that

$$\begin{aligned} \bar{\Omega}^c \cap U_j &= \{H^{(j)}(\eta, r) : \eta \in \Delta^\Omega, r \in (-\alpha(\Omega), 0)\}, \quad \partial\Omega \cap U_j = \{h^{(j)}(\eta) : \eta \in \Delta^\Omega\} \\ &\text{for } 1 \leq j \leq k(\Omega), \quad \partial\Omega = \cup\{h^{(j)}(\eta) : \eta \in (-\alpha(\Omega)/4, \alpha(\Omega)/4)^2\}. \end{aligned}$$

It is obvious that for  $j$  as before, the function  $H^{(j)}$  is a  $C^2$ -diffeomorphism so that  $U_j$  is an open set in  $\mathbb{R}^3$ . We further set

$$J_j^{(\Omega)}(\eta) := \left[1 + \sum_{l=1}^2 (\partial_l a_j^{(\Omega)}(\eta))^2\right]^{1/2} \text{ for } \eta \in \Delta^\Omega, 1 \leq j \leq k(\Omega),$$

$$\Lambda_j^\gamma := \{h^{(j)}(\varrho) : \varrho \in (-\gamma\alpha(\Omega), \gamma\alpha(\Omega))^2\} \quad \text{for } \gamma \in (0, 1], j \text{ as before,}$$

and  $C^m(\partial\Omega) := \{v : \partial\Omega \mapsto \mathbb{C} : v \circ h^{(j)} \in C^m(\Delta^\Omega) \text{ for } 1 \leq j \leq k(\Omega)\}$  for  $m \in \{1, 2\}$ . Note that

$$\int_{\Lambda_j^1} g \, do_x = \int_{\Delta^\Omega} (g \circ h^{(j)})(\eta) J_j^{(\Omega)}(\eta) \, d\eta \quad \text{for } g \in L^1(\partial\Omega), 1 \leq j \leq k(\Omega). \quad (2.1)$$

Moreover

$$\int_{U_j} g \, dx = \int_{-\alpha(\Omega)}^{\alpha(\Omega)} \int_{\Delta^\Omega} (g \circ H^{(j)})(\varrho, r) \, d\varrho \, dr \quad \text{for } g \in L^1(U_j), 1 \leq j \leq k(\Omega). \quad (2.2)$$

The key properties of  $\partial\Omega$  used in the following are collected in the ensuing lemma. It also serves to introduce some further notation. For the convenience of the reader, we give some indications on the proof of (2.4) and (2.5). The other claims of the lemma are obvious or were proved in [18].

**Lemma 2.1** *For  $v \in C^1(\partial\Omega)$ , there is  $M(v) > 0$  with  $|v(x) - v(y)| \leq M(v)|x - y|$  for  $x, y \in \partial\Omega$ . (The definition of  $C^1(\partial\Omega)$  involves local coordinates, whereas the Lipschitz continuity stated here does not.) In particular  $C^\alpha(\partial\Omega) \subset C^1(\partial\Omega)$  for  $\alpha \in (0, 1)$ . There are constants  $\mathfrak{D}_1, \mathfrak{D}_2, \epsilon(\Omega) \in (0, \infty)$  such that*

$$|\varrho - \eta| + r \leq \mathfrak{D}_1 |H^{(j)}(\varrho, r) - h^{(j)}(\eta)| \quad \text{for } \varrho, \eta \in \Delta^\Omega, r \in (-\alpha(\Omega), \alpha(\Omega)), \quad (2.3)$$

$1 \leq j \leq k(\Omega)$ , as well as  $|(x - y) \cdot n^{(\Omega)}(y)| \leq \mathfrak{D}_1 |x - y|^2$  and  $|n^{(\Omega)}(x) - n^{(\Omega)}(y)| \leq \mathfrak{D}_1 |x - y|$  for  $x, y \in \partial\Omega$ ,

$$x + \epsilon n^{(\Omega)}(x) \in \overline{\Omega}^c, \quad x - \epsilon n^{(\Omega)}(x) \in \Omega \quad \text{for } x \in \partial\Omega, \epsilon \in (0, \epsilon(\Omega)], \quad (2.4)$$

$$|x + \kappa n^{(\Omega)}(x) - (x' + \kappa' n^{(\Omega)}(x'))| \geq \mathfrak{D}_2 (|x - x'| + |\kappa - \kappa'|) \quad \text{for } x, x' \in \partial\Omega, \quad (2.5)$$

$\kappa, \kappa' \in [-\epsilon(\Omega), \epsilon(\Omega)]$ .

Let  $\gamma \in (-2, \infty)$ . Then  $\int_{\partial\Omega} |x - y|^\gamma \, do_x \leq \mathfrak{C}$  and  $\int_{\partial\Omega} \epsilon (|x - y| + \epsilon)^{-3} \, do_x \leq \mathfrak{C}$  uniformly in  $x \in \partial\Omega$  and  $\epsilon \in (0, \infty)$ .

**Proof:** By [18, (2.24)], there is  $\tilde{\epsilon}(\Omega) > 0$  such that the two relations in (2.4) hold for  $\epsilon \in (0, \tilde{\epsilon}(\Omega)]$ ,  $x \in \partial\Omega$ .

Put  $\mathfrak{D}_0 := \max\{\sum_{l=1}^2 |\partial_l a_j^{(\Omega)}|_\infty : 1 \leq j \leq k(\Omega)\}$ . Let  $j \in \{1, \dots, k(\Omega)\}$ ,  $\varrho, \eta \in \Delta^\Omega$  and  $r \in (-\alpha(\Omega), \alpha(\Omega))$ . It is obvious that  $|\varrho - \eta| \leq |H^{(j)}(\varrho, r) - h^{(j)}(\eta)|$ , so if  $r \leq 2\mathfrak{D}_0|\varrho - \eta|$ , we get  $r \leq 2\mathfrak{D}_0 |H^{(j)}(\varrho, r) - h^{(j)}(\eta)|$ . In the case  $r > 2\mathfrak{D}_0|\varrho - \eta|$ , we use that  $|a_j^{(\Omega)}(\varrho) - a_j^{(\Omega)}(\eta)| \leq \mathfrak{D}_0|\varrho - \eta|$ , hence  $|a_j^{(\Omega)}(\varrho) + r - a_j^{(\Omega)}(\eta)| \geq |r| - \mathfrak{D}_0|\varrho - \eta| \geq |r|/2$ , so  $|r| \leq 2 |H^{(j)}(\varrho, r) - h^{(j)}(\eta)|$ . Since by [18, (2.22), (2.23)], there is  $\tilde{\mathfrak{D}} > 0$  with  $|(x - y) \cdot n^{(\Omega)}(y)| \leq \tilde{\mathfrak{D}} |x - y|^2$  and  $|n^{(\Omega)}(x) - n^{(\Omega)}(y)| \leq \tilde{\mathfrak{D}} |x - y|$  for  $x, y \in \partial\Omega$ , we see there is a constant  $\mathfrak{D}_1 > 0$  with the properties listed in the lemma.

Recall the sets  $\Lambda_j^\gamma$  defined above. Set  $\tilde{\delta} := \min\{\text{dist}(\Lambda_j^{1/4}, \partial\Omega \setminus \Lambda_j^{1/2}) : 1 \leq j \leq k(\Omega)\}$ . Then  $\tilde{\delta} > 0$  by [18, (2.12)]. Moreover, inequality [18, (2.48)] yields constants  $\mathfrak{D}, \epsilon'(\Omega) \in (0, \infty)$  such that for  $\kappa, \kappa' \in [-\epsilon'(\Omega), \epsilon'(\Omega)]$ ,  $1 \leq j \leq k(\Omega)$  and  $x, y \in \Lambda_j^1$ , we have  $|x + \kappa n^{(\Omega)}(x) - (x' + \kappa' n^{(\Omega)}(x'))| \geq \mathfrak{D} (|x - x'| + |\kappa - \kappa'|)$ . The assumption  $\kappa, \kappa' \geq 0$  in [18, (2.48)] is not needed. Now let  $x, y \in \partial\Omega$  be such that there is no  $j \in \{1, \dots, k(\Omega)\}$  with

$x, y \in \Lambda_j^1$ . By our assumptions on the functions  $h^{(j)}$ , there is some  $j \in \{1, \dots, k(\Omega)\}$  such that  $x \in \Lambda_j^{1/4}$ . Then  $y \notin \Lambda_j^{1/2}$  in view of the case we are considering, so  $|x - y| \geq \tilde{\delta}$  by the definition of  $\tilde{\delta}$ . Let  $\kappa, \tilde{\kappa} \in \mathbb{R}$  with  $|\kappa|, |\kappa'| \leq \tilde{\delta}/8$ . Then  $|x + \kappa n^{(\Omega)}(x) - (y + \kappa' n^{(\Omega)}(y))| \geq |x - y| - (|\kappa| + |\kappa'|) \geq |x - y| - \tilde{\delta}/4 \geq 3|x - y|/4 \geq |x - y|/2 + \tilde{\delta}/4 \geq |x - y|/2 + |\kappa| + |\kappa'|$ . Thus (2.5) and (2.4) hold with  $\mathfrak{D}_2 := \min\{1/2, \mathfrak{D}\}$  and  $\epsilon(\Omega) := \min\{\tilde{\epsilon}(\Omega), \epsilon'(\Omega), \delta/8\}$ .

Let us prove the last inequality in the lemma. The second from last follows with a similar argument. So take  $x \in \partial\Omega$  and  $\epsilon \in (0, \infty)$ . There is  $j \in \{1, \dots, k(\Omega)\}$  and  $\varrho \in (-\alpha(\Omega)/4, \alpha(\Omega)/4)^2$  with  $x = h^{(j)}(\varrho)$ . Thus by (2.1) and because  $|h^{(j)}(\varrho) - h^{(j)}(\eta)| \geq |\varrho - \eta|$  for  $\eta \in \Delta^\Omega$ , the integral  $\int_{\Lambda_j^1} \epsilon (|x - y| + \epsilon)^{-3} d\varrho_y$  is bounded by  $\mathfrak{C} \int_{\Delta^\Omega} \epsilon (|\varrho - \eta| + \epsilon)^{-3} d\eta$ , and therefore by a constant  $\mathfrak{C}$  independent of  $x$  and  $\epsilon$ . Obviously  $\text{dist}(\partial\Omega \setminus \Lambda_j^1, \Lambda_j^{1/4}) > 0$ , so  $\int_{\partial\Omega \setminus \Lambda_j^1} \epsilon (|x - y| + \epsilon)^{-3} d\varrho_y \leq \mathfrak{C}$  with the same type of constant  $\mathfrak{C}$ . Altogether we obtain the estimate at the end of the lemma.  $\square$

**Lemma 2.2** *The relation  $n^{(\Omega)} \in C^1(\partial\Omega)^3$  holds. There is a function  $N^{(\Omega)} \in C^1(\partial\Omega)^3$  with  $N^{(\Omega)}|_{\partial\Omega} = n^{(\Omega)}$ .*

**Proof:** We have  $(n^{(\Omega)} \circ h^{(j)})(\varrho) = J_j^{(\Omega)}(\varrho)^{-1} A_j^{(\Omega)} \cdot (-\nabla a_j^{(\Omega)}(\varrho), 1)^T$  for  $\varrho \in \Delta^\Omega$ ,  $1 \leq j \leq k(\Omega)$  ([18, (2.15)]), in particular  $n^{(\Omega)} \in C^1(\partial\Omega)^3$ . For  $j$  as before, put  $N^{(j)}(\varrho, r) := (n^{(\Omega)} \circ h^{(j)})(\varrho)$  for  $\varrho \in \Delta^\Omega$ ,  $r \in (-\alpha(\Omega), \alpha(\Omega))$ . Then we define  $N^{(\Omega)}$  by setting  $N^{(\Omega)}(y) := \sum_{j=1}^{k(\Omega)} \varphi_j^{(\Omega)}(y) (N^{(j)} \circ (H^{(j)})^{-1})(y)$  for  $y \in \bar{\Omega} \cap (\cup\{U_j : 1 \leq j \leq k(\Omega)\})$ ,  $N^{(\Omega)}(y) := 0$  for any other  $y \in \Omega$ .  $\square$

**Lemma 2.3** *Let  $p \in (1, \infty)$ ,  $b \in L^p(\partial\Omega)^3$ . Then there is a sequence  $(b_n)$  in  $C^2(\partial\Omega)$  such that  $\|b - b_n\|_p \rightarrow 0$ .*

**Proof:** For any  $j \in \{1, \dots, k(\Omega)\}$ , choose a sequence  $(b_n^{(j)})_{n \geq 1}$  in  $C^\infty(\Delta^\Omega)$  such that  $\|b \circ h^{(j)} - b_n^{(j)}\|_p \rightarrow 0$  ( $n \rightarrow \infty$ ). Then define  $B_n^{(j)}(\varrho, r) := b_n^{(j)}(\varrho)$  for  $\varrho \in \Delta^\Omega$ ,  $r \in (-\alpha(\Omega), \alpha(\Omega))$ ,  $n \in \mathbb{N}$  and  $j$  as before. For  $n \in \mathbb{N}$ , we define the function  $b_n$  by setting  $b_n(y) := \sum_{j=1}^{k(\Omega)} \varphi_j^{(\Omega)}(y) (B_n^{(j)} \circ (H^{(j)})^{-1})(y)$  for  $y \in \partial\Omega$ . Then the sequence  $(b_n)$  has the properties claimed in the lemma.  $\square$

### 3 Some auxiliary results.

In this section, we state some known results for which do not know a direct reference, or which we state in a form adapted to our purposes. We begin with a simple application of Hölder's inequality.

**Lemma 3.1** *Let  $K : \partial\Omega \times \partial\Omega \mapsto [0, \infty)$  be measurable. Assume that the two terms*

$$\mathfrak{A}_1 := \sup \left\{ \int_{\partial\Omega} |K(x, y)| d\varrho_y : x \in \partial\Omega \right\} \quad \text{and} \quad \mathfrak{A}_2 := \sup \left\{ \int_{\partial\Omega} |K(x, y)| d\varrho_x : y \in \partial\Omega \right\}$$

*are both finite. Then  $\int_{\partial\Omega} \left( \int_{\partial\Omega} K(x, y) |\phi(y)| d\varrho_y \right)^p d\varrho_x \leq \mathfrak{A}_1^{p-1} \mathfrak{A}_2 \|\phi\|_p^p$  for  $p \in (1, \infty)$ ,  $\phi \in L^p(\partial\Omega)$ .*

**Proof:** For  $p$  and  $\phi$  as in the lemma, Hölder's inequality yields that the left-hand side of the estimate in the lemma is bounded by  $\int_{\partial\Omega} \left( \int_{\partial\Omega} K(x, y) d\varrho_y \right)^{p-1} \int_{\partial\Omega} K(x, y) |\phi(y)|^p d\varrho_y d\varrho_x$ . The lemma follows from this observation.  $\square$

Next we study the  $L^p$ -integrability of certain surface potentials, considered as functions of  $x \in B_R$ .

**Lemma 3.2** *Let  $p \in (1, \infty)$ ,  $a \in (0, 2/p)$ ,  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ . Then, for  $\phi \in L^p(\partial\Omega)$ ,*

$$\mathfrak{A} := \left( \int_{B_R} \left( \int_{\partial\Omega} |x-y|^{-2} |\phi(y)| d\sigma_y \right)^{(1/p-a/2)^{-1}} dx \right)^{1/p-a/2} \leq C(a, p, R) \|\phi\|_p.$$

**Proof:** We use the functions  $H^{(j)}$  and  $h^{(j)}$  and the sets  $U_j$  and  $\Lambda_j^\gamma$  introduced at the beginning of Section 2. Take  $\phi \in L^p(\partial\Omega)$ . Proceeding as in [13, proof of Lemma 13.1], we start with the estimate  $\mathfrak{A} \leq \mathfrak{C} \sum_{j=1}^{k(\Omega)} (\mathfrak{A}_j + \mathfrak{B}_j)$ , where

$$\mathfrak{A}_j := \left( \int_{U_j \cap B_R} \left( \int_{\Lambda_j^{1/2}} |x-y|^2 |\phi(y)| d\sigma_y \right)^{(1/p-a/2)^{-1}} dx \right)^{1/p-a/2},$$

and with  $\mathfrak{B}_j$  defined in the same way as  $\mathfrak{A}_j$ , except that the domain of integration  $U_j \cap B_R$  is replaced by  $B_R \setminus U_j$  ( $1 \leq j \leq k(\Omega)$ ). Let  $j \in \{1, \dots, k(\Omega)\}$ . Since  $H^{(j)}$  is a diffeomorphism and  $H^{(j)}(\varrho, 0) = h^{(j)}(\varrho)$  for  $\varrho \in \Delta^\Omega$ , we have  $\text{dist}(B_R \setminus U_j, \Lambda_j^{1/2}) > 0$ , so  $\mathfrak{B}_j \leq \mathfrak{C} \|\phi\|_{\Lambda_j^{1/2}} \leq \mathfrak{C} \|\phi\|_p$ . Moreover, by (2.1), (2.2) and the first inequality in (2.3), we get

$$\mathfrak{A}_j \leq \mathfrak{C} \left( \int_{-\alpha(\Omega)}^{\alpha(\Omega)} \int_{\Delta^\Omega} \left( \int_{\Delta_{1/2}^\Omega} |\varrho - \eta|^{-2+a} |r|^{-a} |\phi \circ h^{(j)}|(\eta) d\eta \right)^{(1/p-a/2)^{-1}} d\varrho dr \right)^{1/p-a/2},$$

with the abbreviation  $\Delta_{1/2}^\Omega := (-\alpha(\Omega)/2, \alpha(\Omega)/2)^2$ . But  $a < 2/p$ , so  $-a(1/p - a/2)^{-1} > -1$ . Thus we may integrate with respect to  $r$ , to obtain

$$\mathfrak{A}_j \leq \mathfrak{C} \left( \int_{\Delta^\Omega} \left( \int_{\Delta_{1/2}^\Omega} |\varrho - \eta|^{-2+a} |\phi \circ h^{(j)}|(\eta) d\eta \right)^{(1/p-a/2)^{-1}} d\varrho \right)^{1/p-a/2}.$$

Therefore  $\mathfrak{A}_j \leq \mathfrak{C} \|\phi \circ h^{(j)}\|_p$  by the Hardy-Littlewood-Sobolev inequality ([48, p. 119]). Since  $\partial\Omega = \cup \{ \Lambda_j^{1/4} : 1 \leq j \leq k(\Omega) \}$  by our assumptions on the functions  $h^{(j)}$ , Lemma 3.2 is implied by the preceding inequalities.  $\square$

Next we state a theorem on solenoidal lifting of functions in  $W^{1-1/p, p}(\partial\Omega)^3$ .

**Theorem 3.1** ([31, Exercise III.3.5]) *Let  $p \in (1, \infty)$ ,  $b \in W^{1-1/p, p}(\partial\Omega)^3$  with  $\int_{\partial\Omega} b \cdot n^{(\Omega)} d\sigma_x = 0$ . Then there is  $B \in W^{1, p}(\Omega)^3$  with  $\text{div} B = 0$  and  $B|_{\partial\Omega} = b$ .*

Functions in exterior domains with  $L^q$ -integrable gradient are  $L^q$ -integrable in a neighbourhood of the boundary:

**Lemma 3.3** ([31, Lemma II.6.1]) *Let  $A \subset \mathbb{R}^3$  be open and bounded, with Lipschitz boundary,  $q \in (1, \infty)$ ,  $R \in (0, \infty)$  with  $\overline{A} \subset B_R$ ,  $V \in W_{loc}^{1,1}(\overline{A}^c)$  with  $\nabla V \in L^q(\overline{A}^c)^3$ . Then  $V|_{B_R \setminus \overline{A}} \in W^{1, q}(B_R \setminus \overline{A})$ .*

We will use a class of functions in  $\overline{\Omega}^c$  which, in a certain sense, vanish at infinity.

**Theorem 3.2** ([31, Theorem II.6.1]) *Let  $p \in (1, 3)$ . Then  $\|g\|_{3p/(3-p)} \leq \mathfrak{C} \|\nabla g\|_p$  for  $g \in L^{3p/(3-p)}(\overline{\Omega}^c) \cap W_{loc}^{1,1}(\overline{\Omega}^c)$  with  $\nabla g \in L^p(\overline{\Omega}^c)^3$ .*

In the ensuing theorem, we specify in which way functions  $u \in W_{loc}^{1,1}(\overline{\Omega}^c)$  with  $L^p$ -integrable gradient may be approximated by functions from  $C_0^\infty(\Omega^c)$ :

**Theorem 3.3** *Let  $p \in (1, \infty)$ ,  $v \in W_{loc}^{1,1}(\overline{\Omega}^c)$  with  $\nabla v \in L^p(\overline{\Omega}^c)^3$ . In the case  $p < 3$ , further suppose that  $v \in L^{3p/(3-p)}(\overline{\Omega}^c)$ . Then there is a sequence  $(v_n)$  in  $C_0^\infty(\Omega^c)$  such that  $\|\nabla(v - v_n)\|_p \rightarrow 0$  and  $\|v - v_n|_{\Omega_R}\|_p \rightarrow 0$  ( $n \rightarrow \infty$ ) for any  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ .*

**Proof:** According to Lemma 3.3, we have  $v|_{\Omega_R} \in L^p(\Omega_R)$  for  $R$  as in the theorem. Fix such a number  $R$  and denote it by  $R_0$ . Let  $\varphi_0 \in C_0^\infty(\Omega_{2R_0})$  with  $\varphi_0|_{B_{R_0}} = 1$ . Then  $(1 - \varphi_0)v \in W_{loc}^{1,p}(\overline{\Omega}^c)$ ,  $(1 - \varphi_0)v|_{\Omega_R} \in L^p(\Omega_R)$  for  $R$  as in the theorem, and  $\nabla((1 - \varphi_0)v) \in L^p(\overline{\Omega}^c)^3$ . Moreover, in the case  $p < 3$  we have  $(1 - \varphi_0)v \in L^{3p/(3-p)}(\overline{\Omega}^c)$ . Obviously  $(1 - \varphi_0)v|_{\partial\Omega} = 0$ . At this point, [46, Theorem 2.7 and 2.8] yield there is a sequence  $(\varphi_n)$  in  $C_0^\infty(\overline{\Omega}^c)$  with  $\|\nabla(\varphi_n - (1 - \varphi_0)v)\|_p \rightarrow 0$  and  $\|\varphi_n - (1 - \varphi_0)v|_{\Omega_R}\|_p \rightarrow 0$  for  $R$  as in the theorem. On the other hand,  $\text{supp}(\varphi_0 v) \subset B_{2R_0}$ , so  $\varphi_0 v \in W^{1,p}(\overline{\Omega}^c)$ . Let  $\tilde{v} \in W^{1,p}(\mathbb{R}^3)$  be an extension of  $\varphi_0 v$  to  $\mathbb{R}^3$ , and let  $(\psi_n)$  be a sequence in  $C_0^\infty(\mathbb{R}^3)$  with  $\|\tilde{v} - \psi_n\|_{1,p} \rightarrow 0$ . Put  $v_n := \varphi_n + \psi_n$  for  $n \in \mathbb{N}$ . Then the sequence  $(v_n)$  has all the properties stated in the theorem.  $\square$

We introduce the Helmholtz decomposition of  $L^q(\overline{\Omega}^c)^3$ .

**Theorem 3.4** *Let  $A \subset \mathbb{R}^3$  be open and bounded, with Lipschitz boundary. Then for  $q \in (1, \infty)$ , there is a constant  $C_q > 0$ , and for any  $f \in L^q(\overline{A}^c)^3$  there are uniquely determined functions  $\mathcal{P}_q(f) = \mathcal{P}_q^{(A)}(f) \in L_\sigma^q(\overline{A}^c)$  and  $\mathcal{G}_q(f) = \mathcal{G}_q^{(A)}(f) \in W_{loc}^{1,q}(\overline{A}^c)$  such that  $f = \mathcal{P}_q(f) + \nabla \mathcal{G}_q(f)$  and  $\|\mathcal{P}_q(f)\|_q + \|\nabla \mathcal{G}_q(f)\|_q \leq C_q \|f\|_q$ . Moreover  $\mathcal{P}_{q'} = \mathcal{P}'_q$ ,  $\mathcal{P}_q(f) = f$  for  $f \in L_\sigma^q(\overline{A}^c)$ , and  $\mathcal{P}_q(\nabla \pi) = 0$  for  $\pi \in W_{loc}^{1,q}(\overline{A}^c)$  with  $\nabla \pi \in L^q(\overline{A}^c)^3$ , for any  $q \in (1, \infty)$ .*

**Proof:** [31, Theorem III.1.2], [15, Corollary 2.3].  $\square$

**Corollary 3.1** *Let  $q \in (1, \infty)$ . Fix some  $R_0 \in (0, \infty)$  with  $\overline{\Omega} \subset B_{R_0}$ . Then for any  $f \in L^q(\overline{\Omega}^c)^3$ , the function  $\mathcal{G}_q(f) \in W_{loc}^{1,q}(\overline{\Omega}^c)$  in Theorem 3.4 may be chosen in such a way that  $\mathcal{G}_q(f) \in L^{3q/(3-q)}(\overline{\Omega}^c)$  in the case  $q < 3$  and  $\int_{\Omega_{R_0}} \mathcal{G}_q(f) dx = 0$  if  $q \geq 3$ .*

**Proof:** Let  $f \in L^q(\overline{\Omega}^c)^3$ , and take  $\mathcal{G}_q(f)$  as in Theorem 3.4. According to [31, Theorem II.6.1], in the case  $q < 3$ , there is  $c_f \in \mathbb{C}$  with  $\mathcal{G}_q(f) - c_f \in L^{3q/(3-q)}(\overline{\Omega}^3)$ . Thus, for any  $f$  as before, we replace  $\mathcal{G}_q(f)$  by  $\mathcal{G}_q(f) - c_f$  if  $q < 3$ , and by  $\mathcal{G}_q(f) - |\Omega_{R_0}|^{-1} \int_{\Omega_{R_0}} f dx$  in the case  $q \geq 3$ .  $\square$

The following well known theorem on the Poisson equation in the whole space  $\mathbb{R}^3$  follows from the Hard-Littlewood-Sobolev and the Calderon-Zygmund inequality, Lebesgue's theorem and some approximation arguments. The kernel  $\mathfrak{N}$  was introduced in Theorem 1.1.

**Theorem 3.5** *Let  $p \in (1, 3/2)$  and  $f \in L^p(\mathbb{R}^3)$ . Then  $\int_{\mathbb{R}^3} |(\partial^\alpha \mathfrak{N})(x - y) f(y)| dy < \infty$  for  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$  and for a. e.  $x \in \mathbb{R}^3$ . As a consequence, we may define  $(\mathfrak{N} * f)(x) := \int_{\mathbb{R}^3} \mathfrak{N}(x - y) f(y) dy$  for  $x \in \mathbb{R}^3$ . Then  $\mathfrak{N} * f \in W_{loc}^{2,p}(\mathbb{R}^3)$ ,  $-\Delta(\mathfrak{N} * f) = f$  and  $\partial_l(\mathfrak{N} * f)(x) = \int_{\mathbb{R}^3} (\partial_l \mathfrak{N})(x - y) f(y) dy$  ( $1 \leq l \leq 3$ ,  $x \in \mathbb{R}^3$ ). Moreover, for any  $r \in (1, 3)$  with  $f \in L^r(\mathbb{R}^3)$ , the inequality  $\|\nabla(\mathfrak{N} * f)\|_{(1/r-1/3)^{-1}} \leq C(r) \|f\|_r$  holds, and for any  $s \in (1, \infty)$  with  $f \in L^s(\mathbb{R}^3)$ , we have  $\|\partial_l \partial_m(\mathfrak{N} * f)\|_s \leq C(s) \|f\|_s$  ( $1 \leq l, m \leq 3$ ).*

*If  $\text{supp}(f)$  is compact, then  $\partial_l \partial_m(\mathfrak{N} * f)(x) = \int_{\mathbb{R}^3} (\partial_l \partial_m \mathfrak{N})(x - y) f(y) dy$  for  $l, m$  as before and for  $x \in \mathbb{R}^3 \setminus \text{supp}(f)$ .*

We end this section with a remark on the link between solutions of the Laplace equation in  $C^0(\overline{U}) \cap C^\infty(U)$  on the one hand and in  $W^{2,r}(U)$  on the other, if  $U \subset \mathbb{R}^3$  is open, bounded and with smooth boundary.

**Lemma 3.4** *Let  $U$  be a bounded open set with  $C^2$ -boundary, let  $q \in (3/2, \infty)$ ,  $b \in W^{2-1/q, q}(\partial U)$ ,  $f \in L^q(U)$ ,  $u \in C^0(\bar{U})$  with  $u|_U \in C^2(U)$ ,  $\Delta(u|U) = f$  and  $u|_{\partial U} = b$ . Then for  $r \in (1, q]$ , the relations  $u \in W^{2, r}(U)$ ,  $u|_{\partial U} = b$  in the trace sense and  $\|u\|_{2, r} \leq \mathfrak{C}(r) (\|f\|_q + \|b\|_{2-1/r, r})$  hold, with the constant  $\mathfrak{C}(r)$  only depending on  $U$  and  $r$ .*

**Proof:** By standard results (see [46, Theorem II.3.1, II.9.1, II.10.1] for example), for any  $r \in (1, q]$  there is a uniquely determined function  $u^{(r)} \in W^{2, r}(U)$  with  $\Delta u^{(r)} = f$  and  $u^{(r)}|_{\partial U} = b$  in the trace sense. This function  $u^{(r)}$  satisfies the estimate  $\|u^{(r)}\|_{2, r} \leq \mathfrak{C}(r) (\|f\|_r + \|b\|_{2-1/r, r})$ , with a constant  $\mathfrak{C}(r)$  only depending on  $r$  and  $U$ . Since  $q > 3/2$ , a Sobolev inequality yields  $u^{(q)} \in C^0(\bar{U})$ . At this point we may conclude that  $u^{(q)}|_{\partial U} = b$  in the sense of  $C^0(\bar{U})$ -functions. Obviously  $\int_U (u^{(q)} - u) \Delta \varphi \, dx = 0$  for  $\varphi \in C_0^\infty(U)$ , so  $u^{(q)} - u \in C^\infty(U)$  by Weil's lemma ([46, Appendix]). As a consequence  $u^{(q)} \in C^2(U)$ . It follows by the maximum principle that  $u^{(q)} = u$ . Let  $r \in (1, q)$ . Since  $U$  is bounded, we have  $u^{(q)} \in W^{2, r}(U)$ , so  $u^{(q)} = u^{(r)}$  by the uniqueness property of  $u^{(r)}$ , and thus  $u^{(r)} = u$ .  $\square$

## 4 Some known results about the Stokes resolvent system in $\bar{\Omega}^c$ with homogeneous Dirichlet boundary conditions.

We start with an existence result and the basic estimate of the Stokes resolvent.

**Theorem 4.1** *Let  $A \subset \mathbb{R}^3$  be open and bounded, with  $C^2$ -boundary, and let  $B \in \{\mathbb{R}^3, \bar{A}^c\}$  and  $q \in (1, \infty)$ . Then, for any  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$  and for any  $f \in L^q(B)^3$ , there is a unique function  $u = u(\lambda, f) \in W^{2, q}(B)^3 \cap W_0^{1, q}(B)^3$  and a function  $\pi = \pi(\lambda, f) \in W_{loc}^{1, q}(B)$ , unique up to a constant, such that  $\nabla \pi \in L^q(B)^3$  and  $-\Delta u + \lambda u + \nabla \pi = f$ ,  $\operatorname{div} u = 0$ .*

*Let  $\vartheta \in [0, \pi)$ . Then  $\|\lambda u(\lambda, f)\| \leq \mathfrak{C} \|f\|_q$  for  $f \in L^q(B)^3$ ,  $\lambda \in \mathbb{C} \setminus \{0\}$  with  $|\arg \lambda| \leq \vartheta$ .*

**Proof:** See [38] in the case  $B = \mathbb{R}^3$ , and [32] or [10] – [12] or [47] else.  $\square$

We turn to uniqueness, first considering solutions to (1.1) in  $\mathbb{R}^3$ , then solutions to (1.1) in  $\bar{B}^c$  with  $u|_{\partial B} = 0$ , for  $B \subset \mathbb{R}^3$  open and bounded.

**Theorem 4.2** *Let  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $R \in (0, \infty)$ ,  $n \in \mathbb{N}$ ,  $q_j, r_j \in (1, \infty)$ ,  $u^{(j)} \in W_{loc}^{1, 1}(\mathbb{R}^3)^3$  with  $u^{(j)}|_{B_R^c} \in L^{r_j}(B_R^c)^3$ ,  $\nabla u^{(j)} \in L^{q_j}(\mathbb{R}^3)^9$  for  $j \in \{1, \dots, n\}$ . Put  $u := \sum_{j=1}^n u^{(j)}$ , and suppose that  $\operatorname{div} u = 0$  and  $\int_{\mathbb{R}^3} (\nabla u \cdot \nabla \vartheta + \lambda u \cdot \vartheta) \, dx = 0$  for  $\vartheta \in C_{0, \sigma}^\infty(\mathbb{R}^3)$ . Then  $u = 0$ .*

**Proof:** The function  $u$  given in the theorem belongs to  $C^\infty(\mathbb{R}^3)^3$ , and there is  $\pi \in C^\infty(\mathbb{R}^3)$  such that  $-\Delta u + \lambda u + \nabla \pi = 0$ ,  $\operatorname{div} u = 0$ . This follows by the same arguments and references as in the proof of [15, Theorem 3.1, Theorem 3.2, Corollary 3.2] (associate pressure, interior  $W^{2, q}$ -regularity,  $C^\infty$ -regularity), where the Oseen system and the Oseen resolvent system (1.7) are considered. But with this result on  $u$  available, Theorem 4.2 may be proved in exactly the same way as [15, Theorem 5.1], dealing with uniqueness of weak solutions to either the Oseen system or to the Oseen resolvent system in  $\mathbb{R}^3$ .  $\square$

**Theorem 4.3** *Let  $A \subset \mathbb{R}^3$  be open, bounded, with Lipschitz boundary,  $R_0 \in (0, \infty)$  with  $\bar{A} \subset B_{R_0}$ ,  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $n \in \mathbb{N}$ ,  $q_j, s_j \in (1, \infty)$ ,  $u^{(j)} \in W_{loc}^{1, 1}(\bar{A}^c)^3$ ,  $\nabla u^{(j)} \in L^{q_j}(\bar{A}^c)^9$  (hence  $u^{(j)}|_{B_R \setminus \bar{A}} \in W^{1, q_j}(B_R \setminus \bar{A})^3$  for  $R \in (0, \infty)$  with  $\bar{\Omega} \subset B_R$  by Lemma 3.3), and  $u^{(j)}|_{B_{R_0}^c} \in L^{s_j}(B_{R_0}^c)^3$  for  $j \in \{1, \dots, n\}$ . Put  $u := \sum_{j=1}^n u^{(j)}$ , and suppose that  $u|_{\partial A} = 0$ ,  $\operatorname{div} u = 0$  and  $\int_{\mathbb{R}^3} (\nabla u \cdot \nabla \vartheta + \lambda u \cdot \vartheta) \, dx = 0$  for  $\vartheta \in C_{0, \infty}^\infty(\bar{A}^c)$ . Then  $u = 0$ .*

**Proof:** As in the proof of Theorem 4.2, we refer to the proof of [15, Theorem 3.1, Theorem 3.2, Corollary 3.2] dealing with the Oseen system and the Oseen resolvent system, in order to obtain that  $u \in C^\infty(\overline{A^c})^3$  and there is  $\pi \in C^\infty(\overline{A^c})$  such that  $-\Delta u + \lambda u + \nabla \pi = 0$ ,  $\operatorname{div} u = 0$ . Similarly, the arguments applied in the proof of [15, Theorem 3.3, Corollary 3.1] (boundary regularity) to the Oseen system and to the Oseen resolvent system yield in our situation that  $u|_{B_R \setminus \overline{A}} \in W^{2,p}(B_R \setminus \overline{A})^3$  and  $\pi|_{B_R \setminus \overline{A}} \in W^{1,p}(B_R \setminus \overline{A})$  for  $R \in (0, \infty)$  with  $\overline{A} \subset B_R$ , where  $p := \min\{q_1, \dots, q_n\}$ . Moreover, for any  $f \in C_0^\infty(\mathbb{R}^3)^3$ , Theorem 4.1 and the proof of [15, Corollary 3.2] furnish existence of a solution  $(w, \gamma)$  to (1.1) in  $\mathbb{R}^3$  with  $u, \pi$  replaced by  $w$  and  $\gamma$ , respectively, where  $w \in W^{2,2}(\mathbb{R}^3)^3 \cap C^\infty(\mathbb{R}^3)^3$  and  $\gamma \in C^\infty(\mathbb{R}^3)$ . With these ingredients available, Theorem 4.3 may be shown in the same way as [15, Theorem 5.2] (uniqueness of weak solutions to the Oseen system or to the Oseen resolvent system), but with Theorem 4.2 in the role of [15, Theorem 5.1].  $\square$

We introduce the Stokes operator.

**Corollary 4.1** *Let  $A \subset \mathbb{R}^3$  be open, bounded, with  $C^2$ -boundary. Let  $q \in (1, \infty)$ , and define  $\mathcal{D}(\mathcal{A}_q) := \mathcal{D}(\mathcal{A}_q^{(A)}) := W^{2,q}(\overline{A^c})^3 \cap W_0^{1,q}(\overline{A^c})^3 \cap L_\sigma^q(\overline{A^c})$ ,  $\mathcal{A}_q(u) := \mathcal{A}_q^{(A)}(u) := -\mathcal{P}_q(\Delta u)$  for  $u \in \mathcal{D}(\mathcal{A}_q)$ , with the operator  $\mathcal{P}_q = \mathcal{P}_q^{(A)}$  introduced in Theorem 3.4.*

*Then  $\mathcal{A}_q$  is a linear and densely defined operator from  $\mathcal{D}(\mathcal{A}_q)$  into  $L_\sigma^q(\overline{A^c})$ . The set  $\mathbb{C} \setminus (-\infty, 0]$  is contained in the resolvent set  $\varrho(\mathcal{A}_q)$  of  $\mathcal{A}_q$ . Let  $\mathcal{I}_q = \mathcal{I}_q^{(A)}$  denote the identical mapping of  $L_\sigma^q(\overline{A^c})$  onto itself. Then the operator  $(\lambda \mathcal{I}_q + \mathcal{A}_q)^{-1}$  is holomorphic as a function of  $\lambda \in \varrho(\mathcal{A}_q)$  with values in the space of linear bounded operators from  $L_\sigma^q(\overline{A^c})$  into  $L_\sigma^q(\overline{A^c})$ .*

*Let  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $f \in L^q(\overline{A^c})^3$ ,  $u \in W^{2,q}(\overline{A^c})^3 \cap W_0^{1,q}(\overline{A^c})^3$ ,  $\pi \in W_{loc}^{1,q}(\overline{A^c})$  with  $\nabla \pi \in L^q(\overline{A^c})^3$ ,  $-\Delta u + \lambda u + \nabla \pi = f$ ,  $\operatorname{div} u = 0$ . Then  $u = (\lambda \mathcal{I}_q + \mathcal{A}_q)^{-1}(\mathcal{P}_q(f))$ . For  $\vartheta \in [0, \pi)$ , the inequality  $\|(\lambda \mathcal{I}_q + \mathcal{A}_q)^{-1}(f)\|_q \leq \mathfrak{C} |\lambda|^{-1} \|f\|_q$  holds for  $f \in L_\sigma^q(\overline{A^c})$ ,  $\lambda \in \mathbb{C} \setminus \{0\}$  with  $|\arg \lambda| \leq \vartheta$ .*

**Proof:** Let  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $f \in L^q(\overline{A^c})^3$ . Then, by Theorem 4.1, there is a pair of functions  $(u, \pi) = (u(\lambda, f), \pi(\lambda, f))$  with properties as stated in that theorem. In particular  $u \in W_0^{1,q}(\overline{A^c})^3$  and  $\operatorname{div} u = 0$ , so  $u \in L_\sigma^q(\overline{A^c})$  by [31, Theorem III.4.2]. Since in addition  $u \in W^{2,q}(\overline{A^c})^3$ , we have  $u \in \mathcal{D}(\mathcal{A}_q)$ . Applying the operator  $\mathcal{P}_q$  to the equation  $-\Delta u + \lambda u + \nabla \pi = f$ , recalling that  $\nabla \pi \in L^q(\overline{A^c})^3$  and referring to Theorem 3.4, we get  $(\lambda \mathcal{I}_q + \mathcal{A}_q)(u) = \mathcal{P}_q(f)$ . Since  $\mathcal{P}_q(f) = f$  if  $f \in L_\sigma^q(\overline{A^c})$  (Theorem 3.4), we may conclude that  $\lambda \mathcal{I}_q + \mathcal{A}_q : \mathcal{D}(\mathcal{A}_q) \mapsto L_\sigma^q(\overline{A^c})$  is onto. Let  $\tilde{u} \in \mathcal{D}(\mathcal{A}_q)$  satisfy the equation  $(\lambda \mathcal{I}_q + \mathcal{A}_q)(\tilde{u}) = 0$ . The relation  $\tilde{u} \in L_\sigma^q(\overline{A^c}) \cap W_{loc}^{1,1}(\overline{A^c})^3$  implies  $\operatorname{div} \tilde{u} = 0$ . Since  $\mathcal{P}'_q = \mathcal{P}_q$  and  $\mathcal{P}'_q(v) = v$  for  $v \in C_{0,\sigma}^\infty(\overline{A^c})$  (Theorem 3.4), we see that  $\int_{\mathbb{R}^3} (\nabla \tilde{u} \cdot \nabla v + \lambda \tilde{u} \cdot v) dx = 0$  for  $v$  as before. Thus Theorem 4.3 implies  $\tilde{u} = 0$ , so the operator  $\lambda \mathcal{I}_q + \mathcal{A}_q$  is one-to-one. Now we may conclude that the operator  $(\lambda \mathcal{I}_q + \mathcal{A}_q)^{-1}$  exists, has domain  $L_\sigma^q(\overline{A^c})$  and  $(\lambda \mathcal{I}_q + \mathcal{A}_q)^{-1}(f) = u(\lambda, f)$  for  $f \in L_\sigma^q(\overline{A^c})$ . By Theorem 4.1 with  $\vartheta := |\arg \lambda|$ , we have  $\|u(\lambda, f)\|_q \leq \mathfrak{C}(\lambda) \|f\|_q$  for  $f$  as before, so  $(\lambda \mathcal{I}_q + \mathcal{A}_q)^{-1}$  is bounded. Therefore we get  $\lambda \in \varrho(\mathcal{A}_q)$ . The estimate at the end of the corollary now follows from Theorem 4.1. Abstract theory yields that the mapping  $\lambda \mapsto (\lambda \mathcal{I}_q + \mathcal{A}_q)^{-1}$  ( $\lambda \in \varrho(\mathcal{A}_q)$ ) is holomorphic as described in the corollary.  $\square$

## 5 Some surface potentials related to the Stokes resolvent problem.

In this section we recreate the framework used in [11] in order to deal with the boundary value problem (1.1), (1.2). In addition we state some results proved in [10] or [11]. We begin by introducing the fundamental solution to the Stokes resolvent system (1.1) constructed in [38].

Define  $g_1(r) := e^{-r} + r^{-2}(e^{-r} + r e^{-r} - 1)$ ,  $g_2(r) := e^{-r} + 3r^{-2}(e^{-r} + r e^{-r} - 1)$  ( $r \in \mathbb{C} \setminus \{0\}$ ),  $\tilde{E}_{jk}^{(\lambda)}(z) := (4\pi|z|)^{-1}(\delta_{jk}g_1(\lambda^{1/2}|z|) - z_j z_k |z|^{-2}g_2(\lambda^{1/2}|z|))$  ( $z \in \mathbb{R}^3 \setminus \{0\}$ ,  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $1 \leq j, k \leq 3$ ) The matrix-valued function  $(\tilde{E}_{jk}^{(\lambda)})_{1 \leq j, k \leq 3}$  is the velocity part of a fundamental solution to (1.1). Its associated pressure part is given by  $-\nabla \mathfrak{N}$ , where  $\mathfrak{N}$  was defined in Theorem 1.1.

Put  $E_{jk}(z) := (8\pi|z|)^{-1}(\delta_{jk} + z_j z_k |z|^{-2})$  for  $z \in \mathbb{R}^3 \setminus \{0\}$ ,  $1 \leq j, k \leq 3$  (velocity part of a fundamental solution to the Stokes system  $-\Delta u + \nabla \pi = f$ ,  $\operatorname{div} u = 0$ ). Define  $\mathring{E}_{jk}^{(\lambda)} := \tilde{E}_{jk}^{(\lambda)} - E_{jk}$ ,  $\tilde{S}_{jkl}^{(\lambda)} := -\delta_{jk} \partial_l \mathfrak{N} - \partial_k \tilde{E}_{jl}^{(\lambda)} - \partial_j \tilde{E}_{kl}^{(\lambda)}$ ,  $S_{jkl} := -\delta_{jk} \partial_l \mathfrak{N} - \partial_k E_{jl} - \partial_j E_{kl}$ ,  $\mathring{S}_{jkl} := \tilde{S}_{jkl}^{(\lambda)} - S_{jkl} = -\partial_k \mathring{E}_{jl}^{(\lambda)} - \partial_j \mathring{E}_{kl}^{(\lambda)}$  for  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $1 \leq j, k, l \leq 3$ . The ensuing lemma collects some properties of the preceding functions.

**Lemma 5.1** *The functions  $\tilde{E}_{jk}^{(\lambda)}$ ,  $E_{jk}$ ,  $\mathring{E}_{jk}^{(\lambda)}$ ,  $\mathfrak{N}$ ,  $\tilde{S}_{jkl}^{(\lambda)}$ ,  $S_{jkl}$ ,  $\mathring{S}_{jkl}^{(\lambda)}$  belong to  $C^\infty(\mathbb{R}^3 \setminus \{0\})$  and the equations  $-\Delta \tilde{E}_{jk}^{(\lambda)} + \lambda \tilde{E}_{jk}^{(\lambda)} - \partial_j \partial_k \mathfrak{N} = 0$ ,  $\sum_{\mu=1}^3 \partial_\mu \tilde{E}_{j\mu}^{(\lambda)} = 0$  hold for  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $1 \leq j, k, l \leq 3$ . Let  $\vartheta \in [0, \pi)$ . Then*

$$|\partial^\alpha \tilde{E}_{jk}^{(\lambda)}(z)| \leq C(\vartheta, \gamma) |\lambda|^{-\gamma} |z|^{-1-2\gamma-|\alpha|}, \quad |\partial^\beta \mathring{E}_{jk}^{(\lambda)}(z)| \leq C(\vartheta, \gamma) |\lambda|^\gamma |z|^{-1+2\gamma-|\beta|},$$

and  $|\partial^{\alpha'} \tilde{S}_{jkl}^{(\lambda)}(z)| \leq C(\vartheta) |z|^{-2-|\alpha'|}$ , for  $1 \leq j, k, l \leq 3$ ,  $\lambda \in \mathbb{C} \setminus \{0\}$  with  $|\arg \lambda| \leq \vartheta$ ,  $z \in \mathbb{R}^3 \setminus \{0\}$ ,  $\gamma \in [0, 1]$ ,  $\alpha, \beta, \alpha' \in \mathbb{N}_0^3$  with  $|\alpha| \leq 3$ ,  $1 \leq |\beta| \leq 3$ ,  $|\alpha'| \leq 1$ . Moreover

$$S_{jkl}(z) = 3z_j z_k z_l |z|^{-5}/(4\pi) \quad \text{for } z \in \mathbb{R}^3 \setminus \{0\}, 1 \leq j, k, l \leq 3. \quad (5.1)$$

**Proof:** The lemma follows from the properties of the exponential function, in particular its series expansion; compare [11, (1.7), (3.2), (3.3)].  $\square$

Next we define the potential functions we will use in the following.

**Lemma 5.2** *Let  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $\phi \in L^1(\partial\Omega)^3$  and  $l \in \{1, 2, 3\}$ . Then, for  $x \in \mathbb{R}^3 \setminus \partial\Omega$ , set*

$$\tilde{W}^{(\lambda)}(\phi)_l(x) := \int_{\partial\Omega} \sum_{j,k=1}^3 -\tilde{S}_{jkl}^{(\lambda)}(x-y) \phi_j(y) n_k^{(\Omega)}(y) do_y,$$

and define  $W_l(\phi)(x)$  and  $\mathring{W}^{(\lambda)}(\phi)_l(x)$  in the same way as  $\tilde{W}^{(\lambda)}(\phi)_l(x)$ , but with  $\tilde{S}_{jkl}$  replaced by  $S_{jkl}$  and  $\mathring{S}_{jkl}^{(\lambda)}$ , respectively. Further define

$$\Pi(\phi)(x) := \int_{\partial\Omega} \sum_{j,k=1}^3 -2(\partial_j \partial_k \mathfrak{N})(x-y) \phi_j(y) n_k^{(A)}(y) do_y,$$

$$\mathring{\Pi}(\phi)(x) := \int_{\partial\Omega} -\mathfrak{N}(x-y) \cdot \phi(y) do_y, \quad \tilde{\Pi}^{(\lambda)}(\phi)(x) := \Pi(\phi)(x) + \lambda \mathring{\Pi}(\phi)(x),$$

$$\tilde{J}^{(\lambda)}(\phi)_l(x) := \int_{\partial\Omega} \sum_{j,k=1}^3 (\partial_j \tilde{E}_{kl}^{(\lambda)} + \partial_k \tilde{E}_{jl}^{(\lambda)}) \phi_j(y) n_k^{(\Omega)}(y) do_y$$

and  $\mathfrak{F}(\phi)_l(x) := \int_{\partial\Omega} (\partial_l \mathfrak{N})(x-y) (n^{(\Omega)} \cdot \phi)(y) do_y$  for  $x \in \mathbb{R}^3 \setminus \partial\Omega$ ,  $1 \leq l \leq 3$ .

Then the functions  $\widetilde{W}^{(\lambda)}(\phi)_l$ ,  $W(\phi)_l$ ,  $\dot{W}^{(\lambda)}(\phi)_l$ ,  $\Pi(\phi)$ ,  $\mathring{\Pi}(\phi)$ ,  $\widetilde{\Pi}^{(\lambda)}(\phi)$ ,  $\widetilde{J}^{(\lambda)}(\phi)_l$  and  $\mathfrak{F}(\phi)_l$  belong to  $C^\infty(\mathbb{R}^3 \setminus \partial\Omega)$  for  $1 \leq l \leq 3$ ,

$$\begin{aligned} -\Delta \widetilde{W}^{(\lambda)}(\phi) + \lambda \widetilde{W}^{(\lambda)}(\phi) + \nabla \widetilde{\Pi}^{(\lambda)}(\phi) &= 0, \quad \operatorname{div} \widetilde{W}^{(\lambda)}(\phi) = \operatorname{div} \widetilde{J}^{(\lambda)}(\phi) = 0, \quad (5.2) \\ \widetilde{W}^{(\lambda)}(\phi) &= W(\phi) + \dot{W}^{(\lambda)}(\phi) = \widetilde{J}^{(\lambda)}(\phi) + \mathfrak{F}(\phi). \end{aligned}$$

**Proof:** Lebesgue's theorem and the relations  $\widetilde{E}_{jk}^{(\lambda)}$ ,  $E_{jk}$ ,  $\mathring{E}_{jk}^{(\lambda)}$ ,  $\mathfrak{N} \in C^\infty(\mathbb{R}^3 \setminus \{0\})$  ( $1 \leq j, k \leq 3$ ) yield the claims about  $C^\infty$ -regularity. The differential equations satisfied by  $\widetilde{E}^{(\lambda)}$  according to Lemma 5.1 and the equation  $\Delta \mathfrak{N} = 0$  imply (5.2).  $\square$

**Theorem 5.1** Let  $p \in (1, \infty)$ ,  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $\phi \in L^p(\partial\Omega)^3$ . For  $Z \in \{\widetilde{S}^{(\lambda)}, S, \mathring{S}^{(\lambda)}\}$ ,  $l \in \{1, 2, 3\}$  and for a. e.  $x \in \partial\Omega$ , we have  $\int_{\partial\Omega} \sum_{j,k=1}^3 |Z_{jkl}(x-y) n_k^{(\Omega)}(y) \phi_j(y)| do_y < \infty$ . Thus we may set

$$\widetilde{\mathfrak{F}}^{(\lambda)}(\phi)_l(x) := \int_{\partial\Omega} 2 \sum_{j,k=1}^3 \widetilde{S}_{jkl}^{(\lambda)}(x-y) n_k^{(\Omega)}(y) \phi_j(y) do_y$$

for a. e.  $x \in \partial\Omega$  and for  $1 \leq l \leq 3$ , and we may define  $\mathfrak{T}(\phi)_l(x)$  and  $\mathring{\mathfrak{T}}^{(\lambda)}(\phi)_l(x)$  in the same way as  $\widetilde{\mathfrak{T}}^{(\lambda)}(\phi)_l(x)$ , but with the function  $\widetilde{S}^{(\lambda)}$  replaced by  $S$  and  $\mathring{S}^{(\lambda)}$ , respectively. Note that  $\widetilde{\mathfrak{T}}^{(\lambda)}(\phi) = \mathfrak{T}(\phi) + \mathring{\mathfrak{T}}^{(\lambda)}(\phi)$ .

The inequalities  $\|\mathfrak{T}(\phi)\|_p \leq \mathfrak{C} \|\phi\|_p$  and  $\|\widetilde{\mathfrak{T}}^{(\lambda)}(\phi)\|_p + \|\mathring{\mathfrak{T}}^{(\lambda)}(\phi)\|_p \leq \mathfrak{C}(\lambda) \|\phi\|_p$  hold for  $\phi \in L^p(\partial\Omega)^3$ . If  $b \in L^p(\partial\Omega)^3$ , there is a unique function  $\phi = \phi(\lambda, b) \in L^p(\partial\Omega)^3$  such that

$$(-1/2) (\phi + \widetilde{\mathfrak{T}}^{(\lambda)}(\phi)) = b. \quad (5.3)$$

If  $b \in C^a(\partial\Omega)^3$  for some  $a \in [0, 1)$ , then  $\phi$  also belongs to  $C^a(\partial\Omega)^3$ . Moreover the estimate  $\|\phi\|_p \leq \mathfrak{C}(\lambda) \|\phi + \widetilde{\mathfrak{T}}^{(\lambda)}(\phi)\|_p$  holds for  $\phi \in L^p(\partial\Omega)^3$ .

**Proof:** The statements related to  $S_{jkl}$  and  $\mathfrak{T}(\phi)$  follow from (5.1) and [18, Lemma 5.1], those associated with  $\mathring{S}^{(\lambda)}$  and  $\mathring{\mathfrak{T}}^{(\lambda)}(\phi)$  are a consequence of the estimate  $|\partial_j \mathring{E}_{kl}^{(\lambda)}(z)| \leq C(\lambda)$  for  $z \in \mathbb{R}^3 \setminus \{0\}$ ,  $1 \leq j, k, l \leq 3$  (Lemma 5.1). This implies all the claims about  $\widetilde{S}_{jkl}^{(\lambda)}$  and  $\widetilde{\mathfrak{T}}^{(\lambda)}(\phi)$ , except those related to equation (5.3). In this respect we refer to [11, Lemma 1.1].  $\square$

The estimate at the end of Theorem 5.1 actually is valid uniformly with respect to  $\lambda$  with  $|\lambda|$  larger than some positive constant and  $|\arg(\lambda)|$  bounded by some  $\vartheta \in [0, \pi)$ :

**Theorem 5.2** Let  $p \in (1, \infty)$ ,  $\vartheta \in [0, \pi)$ . Then there are constants  $\Lambda_0, C_0 \in (0, \infty)$  such that  $\|\phi\|_p \leq C_0 \|\phi + \widetilde{\mathfrak{T}}^{(\lambda)}(\phi)\|_p$  for  $\phi \in L^p(\partial\Omega)^3$ ,  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq \Lambda_0$  and  $|\arg(\lambda)| \leq \vartheta$ .

**Proof:** A proof of this theorem is the content of [10].  $\square$

The lower bound  $\Lambda_0$  in Theorem 5.2 may be replaced by any other  $\lambda_0 \in (0, \infty)$ :

**Corollary 5.1** Let  $p \in (1, \infty)$ ,  $\vartheta \in [0, \pi)$  and  $\lambda_0 \in (0, \infty)$ . Then the conclusion of Theorem 5.2 remains valid with  $\lambda_0$  in the place of  $\Lambda_0$ .

**Proof:** Choose  $R \in (0, \infty)$  so large that  $x - y \in B_R$  for  $x, y \in \partial\Omega$ . Let  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$  and  $\epsilon \in (0, \infty)$ . Put  $\delta_\lambda := \text{dist}(\lambda, (-\infty, 0])/2$  and

$$K_\lambda := \{(\mu, z) \in \mathbb{C} \times \overline{B_R} : |\mu - \lambda| \leq \delta_\lambda, z \neq 0\}.$$

By Lemma 5.1, there is  $M_\lambda > 0$  with  $|\partial_j \mathring{E}_{kl}^{(\mu)}(z)| \leq M_\lambda$  for  $(\mu, z) \in K_\lambda$ ,  $1 \leq j, k, l \leq 3$ . Thus we may choose  $\delta_1 \in (0, \infty)$  with  $\int_{\partial\Omega} \chi_{(0, \delta_1]}(|x - y|) |\partial_j \mathring{E}_{kl}^{(\mu)}(x - y)| do_y \leq \epsilon/3$  for  $x \in \partial\Omega$ ,  $\mu \in \mathbb{C}$  with  $|\mu - \lambda| \leq \delta_\lambda$  and  $j, k, l$  as before. The function  $(\mu, z) \mapsto \partial_j \mathring{E}_{kl}^{(\mu)}(z)$  with  $(\mu, z) \in K_\lambda$  and  $|z| \geq \delta_1$  is uniformly continuous. Therefore we may choose  $\delta_2 \in (0, \delta_\lambda]$  such that  $\int_{\partial\Omega} \chi_{(\delta_1, \infty)}(|x - y|) |\partial_j \mathring{E}_{kl}^{(\mu)}(x - y) - \partial_j \mathring{E}_{kl}^{(\lambda)}(x - y)| do_y \leq \epsilon/3$  for  $x \in \partial\Omega$ ,  $\mu \in \mathbb{C}$  with  $|\mu - \lambda| \leq \delta_2$ ,  $1 \leq j, k, l \leq 3$ . Altogether  $\int_{\partial\Omega} |\partial_j \mathring{E}_{kl}^{(\mu)}(x - y) - \partial_j \mathring{E}_{kl}^{(\lambda)}(x - y)| do_y \leq \epsilon$  for  $x, \mu, j, k, l$  as in the preceding inequality. When the roles of  $x$  and  $y$  are exchanged, we obtain the same inequality. Thus Lemma 3.1 yields that  $\|\mathring{\mathfrak{F}}^{(\mu)}(\phi) - \mathring{\mathfrak{F}}^{(\lambda)}(\phi)\|_p \leq 48\epsilon \|\phi\|_p$  for  $\mu \in \mathbb{C}$  with  $|\mu - \lambda| \leq \delta_2$ ,  $\phi \in L^p(\partial\Omega)^3$ . As a consequence,  $\|\mathring{\mathfrak{F}}^{(\mu)}(\phi) - \mathring{\mathfrak{F}}^{(\lambda)}(\phi)\|_p \leq 48\epsilon \|\phi\|_p$  for such  $\mu$  and  $\phi$  by Theorem 5.1. On the other hand, by the last statement in that theorem, we have  $\|\phi\|_p \leq \mathfrak{C}(\lambda) \|\phi + \mathring{\mathfrak{F}}^{(\lambda)}(\phi)\|_p$  for  $\phi \in L^p(\partial\Omega)^3$ . Now Corollary 5.1 follows from Theorem 5.2 by a standard compactness argument.  $\square$

The role of the operator  $\phi \mapsto \phi + \mathring{\mathfrak{F}}^{(\lambda)}(\phi)$  becomes apparent from the following theorem.

**Theorem 5.3** *Let  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$  and  $\phi \in C^0(\partial\Omega)^3$ . Define the functions  $W_{ex}(\phi) : \Omega^c \mapsto \mathbb{C}^3$  and  $W_{in}(\phi) : \overline{\Omega} \mapsto \mathbb{C}^3$  by setting*

$$W_{ex}(\phi)|_{\overline{\Omega}^c} := W(\phi)|_{\overline{\Omega}^c}, \quad W_{ex}(\phi)|_{\partial\Omega} := (-1/2) (\phi + \mathfrak{F}(\phi)), \quad W_{in}(\phi)|_{\Omega} := W(\phi)|_{\Omega},$$

$W_{in}(\phi)|_{\partial\Omega} := (-1/2) (-\phi + \mathfrak{F}(\phi))$ . The functions  $\widetilde{W}_{ex}^{(\lambda)}(\phi)$  and  $\widetilde{W}_{in}^{(\lambda)}(\phi)$  are to be defined in an analogous way. (Replace  $W(\phi)$  by  $\widetilde{W}^{(\lambda)}(\phi)$  and  $\mathfrak{F}(\phi)$  by  $\mathring{\mathfrak{F}}^{(\lambda)}(\phi)$ .)

Then  $W_{ex}(\phi), \widetilde{W}_{ex}^{(\lambda)}(\phi) \in C^0(\Omega^c)^3$  and  $W_{in}(\phi), \widetilde{W}_{in}^{(\lambda)}(\phi) \in C^0(\overline{\Omega})^3$  ("jump relation").

The function  $\mathring{W}^{(\lambda)}(\phi)$  may be extended to a  $C^\infty$ -function in  $\mathbb{R}^3$ . We denote this extension also by  $\mathring{W}^{(\lambda)}(\phi)$ . Then  $\mathring{W}^{(\lambda)}(\phi)|_{\partial\Omega} = (-1/2) \mathring{\mathfrak{F}}^{(\lambda)}(\phi)$ .

**Proof:** The claims about  $W_{ex}(\phi)$  and  $W_{in}(\phi)$  hold by [18, Satz 4.1]. By Lemma 5.1, the function  $\partial_j \mathring{E}_{kl}^{(\lambda)} : \mathbb{R}^3 \setminus \{0\} \mapsto \mathbb{C}$  is continuous and bounded. It follows by Lebesgue's theorem that  $\mathring{W}^{(\lambda)}(\phi)$  may be continuously extended from  $\mathbb{R}^3 \setminus \partial\Omega$  to  $\mathbb{R}^3$  and the equation at the end of Theorem 5.3 holds. Since  $\widetilde{W}^{(\lambda)}(\phi) = W(\phi) + \mathring{W}^{(\lambda)}(\phi)$  (Lemma 5.2) and  $\mathring{\mathfrak{F}}^{(\lambda)}(\phi) = \mathfrak{F}(\phi) + \mathring{\mathfrak{F}}^{(\lambda)}(\phi)$  (Theorem 5.1), the remaining claims of Theorem 5.3 follow.  $\square$

If  $\phi$  belongs to  $L^p(\partial\Omega)^3$  but may not be continuous, then  $\widetilde{W}^{(\lambda)}(\phi)|_{\overline{\Omega}^c}$  takes the boundary value  $(-1/2) (\phi + \mathring{\mathfrak{F}}^{(\lambda)}(\phi))$  only in the  $L^p$ -sense:

**Corollary 5.2** *Let  $p \in (1, \infty)$ ,  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $\phi \in L^p(\partial\Omega)^3$ . Then*

$$\int_{\partial\Omega} \left| \widetilde{W}^{(\lambda)}(\phi)(x \pm \epsilon n^{(\Omega)}(x)) + (1/2) (\pm\phi + \mathring{\mathfrak{F}}^{(\lambda)}(\phi))(x) \right|^p do_x \rightarrow 0 \quad (\epsilon \downarrow 0). \quad (5.4)$$

**Proof:** For any  $\psi \in C^0(\partial\Omega)^3$ , we know by Theorem 5.3 and the relations in (2.4) that  $\widetilde{W}^{(\lambda)}(\psi)(x \pm \epsilon n^{(\Omega)}(x)) \rightarrow (-1/2) (\pm\psi + \mathring{\mathfrak{F}}^{(\lambda)}(\psi))(x)$  ( $\epsilon \downarrow 0$ ) uniformly in  $x \in \partial\Omega$ . Also,

recall that  $\|\tilde{\mathfrak{I}}^{(\lambda)}(\psi)\|_p \leq \mathfrak{C}(\lambda) \|\psi\|_p$  for  $\psi \in L^p(\partial\Omega)^3$  (Theorem 5.1). This leaves us to show that for  $\psi \in L^p(\partial\Omega)^3$ ,  $\epsilon \in (0, \epsilon(\Omega)]$ ,

$$\left( \int_{\partial\Omega} |\widetilde{W}^{(\lambda)}(\psi)(x \pm \epsilon n^{(\Omega)}(x))|^p dx \right)^{1/p} \leq \mathfrak{C}(\lambda) \|\psi\|_p, \quad (5.5)$$

with a constant independent of  $\epsilon$ . (The parameter  $\epsilon(\Omega)$  was introduced in Lemma 2.1.) The corollary then follows from a density argument based on Lemma 2.3 and the remark on  $C^1(\partial\Omega)$  in Lemma 2.1. In view of a proof of (5.5), we recall that  $\tilde{S}_{jkl}^{(\lambda)} = S_{jkl} + \mathring{S}_{jkl}^{(\lambda)}$ , the function  $\partial_j \mathring{E}_{kl}^{(\lambda)}$  is bounded for  $1 \leq j, k, l \leq 3$  (Lemma 5.1), and  $|(x - y) \cdot n^{(\Omega)}(y)| \leq \mathfrak{D}_1 |x - y|^2$  for  $x, y \in \partial\Omega$  (Lemma 2.1). Using (5.1) and (2.5), we thus see that

$$\left| \sum_{k=1}^3 S_{jkl}(x \pm \epsilon n^{(\Omega)}(x) - y) n_k^{(\Omega)}(y) \right| \leq \mathfrak{C} (|x - y|^{-1} + \epsilon (|x - y| + \epsilon)^{-3} + 1)$$

for  $x, y, j, l$  as before and  $\epsilon \in (0, \epsilon(\Omega)]$ . Therefore by the last two inequalities in Lemma 2.1, we get  $\int_{\partial\Omega} |\sum_{k=1}^3 S_{jkl}(x \pm \epsilon n^{(\Omega)}(x) - y) n_k^{(\Omega)}(y)| dy \leq \mathfrak{C}$  for  $x \in \partial\Omega$ ,  $\epsilon \in (0, \epsilon(\Omega)]$ . If we integrate with respect to  $x \in \partial\Omega$  instead of  $y$ , the same argument yields a bound uniform in  $y \in \partial\Omega$  and  $\epsilon \in (0, \epsilon(\Omega)]$ . Thus inequality (5.5) follows from Lemma 3.1.  $\square$

The function  $\widetilde{W}^{(\lambda)}(\psi)|_{\overline{\Omega}^c}$  may rather easily be approximated by functions that are  $C^\infty$  in a domain slightly larger than  $\overline{\Omega}^c$ . An analogous remark is true with respect to  $\widetilde{W}^{(\lambda)}(\psi)|_{\Omega}$ . Here are the details:

**Lemma 5.3** *Let  $\phi \in L^1(\partial\Omega)^3$ ,  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$  and  $\epsilon \in (0, \epsilon(\Omega)]$ , with  $\epsilon(\Omega)$  introduced in Lemma 2.1. Define  $U_\epsilon := \{x \in \mathbb{R}^3 : \text{dist}(x, \overline{\Omega}) < \mathfrak{D}_2 \epsilon/2\}$ ,  $U_{-\epsilon} := \{x \in \mathbb{R}^3 : \text{dist}(x, \Omega^c) < \mathfrak{D}_2 \epsilon/2\}$ , with  $\mathfrak{D}_2$  also introduced in Lemma 2.1. Note that  $U_\epsilon$  and  $U_{-\epsilon}$  are open,  $\overline{\Omega} \subset U_\epsilon$  and  $\Omega^c \subset U_{-\epsilon}$ . Further define*

$$\widetilde{W}^{(\lambda, \epsilon)}(\phi)_l(x) := \int_{\partial\Omega} - \sum_{j,k=1}^3 \tilde{S}_{jkl}^{(\lambda)}(x - [y + \epsilon n^{(\Omega)}(y)]) \phi_j(y) n_k^{(\Omega)}(y) dy \quad \text{for } x \in U_\epsilon,$$

$1 \leq l \leq 3$ , and let  $\widetilde{W}^{(\lambda, -\epsilon)}(\phi)_l(x)$  be defined in the same way as  $\widetilde{W}^{(\lambda, \epsilon)}(\phi)_l(x)$ , but for  $x \in U_{-\epsilon}$  and with the term  $y + \epsilon n^{(\Omega)}(y)$  replaced by  $y - \epsilon n^{(\Omega)}(y)$ .

Then  $\widetilde{W}^{(\lambda, \pm\epsilon)}(\phi) \in C^\infty(U_{\pm\epsilon})^3$  and  $\text{div } \widetilde{W}^{(\lambda, \pm\epsilon)}(\phi) = 0$ .

**Proof:** Let  $x \in U_\epsilon \setminus \overline{\Omega}$ . Then there is  $x' \in \partial\Omega$  with  $|x - x'| = \text{dist}(x, \overline{\Omega})$ . Since  $x \in U_\epsilon$ , we have  $|x' - x| < \mathfrak{D}_2 \epsilon/2$ . Thus with (2.5), for  $y \in \partial\Omega$ ,

$$|x - [y + \epsilon n^{(\Omega)}(y)]| \geq |x' - [y + \epsilon n^{(\Omega)}(y)]| - |x - x'| \geq \mathfrak{D}_2 \epsilon - |x - x'| \geq \mathfrak{D}_2 \epsilon/2.$$

Suppose that  $x \in \overline{\Omega}$ , and let  $y \in \partial\Omega$ . By (2.4), we have  $y + \epsilon n^{(\Omega)}(y) \in \overline{\Omega}^c$ , so there is  $x' \in \partial\Omega$  with  $|x - [y + \epsilon n^{(\Omega)}(y)]| \geq |x' - [y + \epsilon n^{(\Omega)}(y)]|$ . Hence due to (2.5) again,  $|x - [y + \epsilon n^{(\Omega)}(y)]| \geq \mathfrak{D}_2 \epsilon$ . Altogether  $|x - [y + \epsilon n^{(\Omega)}(y)]| \geq \mathfrak{D}_2 \epsilon/2$  for any  $x \in U_\epsilon$  and for any  $y \in \partial\Omega$ . Since  $\tilde{S}_{jkl}^{(\lambda)} \in C^\infty(\mathbb{R}^3 \setminus \{0\})$  for  $1 \leq j, k, l \leq 3$ , the claims of the lemma related to  $\widetilde{W}^{(\lambda, \epsilon)}$  follow by the estimate of  $\tilde{S}_{jkl}^{(\lambda)}$  in Lemma 5.2 and Lebesgue's theorem. Analogous arguments are valid for the function  $\widetilde{W}^{(\lambda, -\epsilon)}$ .  $\square$

**Corollary 5.3** *Let  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$  and  $\phi \in C^0(\partial\Omega)^3$ . Then  $\widetilde{W}^{(\lambda, \pm\epsilon)}(\phi)(x)$  converges to  $(-1/2) (\mp\phi + \widetilde{\mathfrak{I}}^{(\lambda)}(\phi))(x)$  for  $\epsilon \downarrow 0$  uniformly in  $x \in \partial\Omega$ .*

*Let  $p \in (1, \infty)$  and  $\phi \in L^p(\partial\Omega)^3$ . Then the preceding convergence result holds in the  $L^p$ -sense.*

**Proof:** Put  $\mathfrak{G}(x, y, \kappa)_{jkl} := |\widetilde{S}_{jkl}^{(\lambda)}(x + \kappa n^{(\Omega)}(x) - y) - \widetilde{S}_{jkl}^{(\lambda)}(x - [y - \kappa n^{(\Omega)}(y)])|$  for  $x, y \in \partial\Omega$ ,  $\kappa \in [-\epsilon(\Omega), \epsilon(\Omega)] \setminus \{0\}$ ,  $1 \leq j, k, l \leq 3$ . Then

$$\mathfrak{G}(x, y, \kappa)_{jkl} = \left| \int_0^1 \Lambda_{jkl}(x, y, \kappa, \vartheta) d\vartheta \kappa (n^{(\Omega)}(x) - n^{(\Omega)}(y)) \right|,$$

with  $\Lambda_{jkl}(x, y, \kappa, \vartheta) := \sum_{m=1}^3 \partial_m \widetilde{S}_{jkl}^{(\lambda)}(x - y + \vartheta \kappa n^{(\Omega)}(x) + (1 - \vartheta) \kappa n^{(\Omega)}(y))$  for  $x, y, j, k, l$  and  $\kappa$  as above and for  $\vartheta \in [0, 1]$ . But  $|\Lambda_{jkl}(x, y, \kappa, \vartheta)| \leq \mathfrak{C}(\lambda) \mathfrak{D}_2^{-3} (|x - y| + |\kappa|)^{-3}$  by Lemma 5.2 and (2.5). Using the estimate  $|n^{(\Omega)}(x) - n^{(\Omega)}(y)| \leq \mathfrak{D}_1 |x - y|$  for  $x, y \in \partial\Omega$  (Lemma 2.1), we may conclude with the second from last inequality in Lemma 2.1 that  $\int_{\partial\Omega} G(x, y, \kappa) do_y \leq \mathfrak{C}(\lambda) |\kappa|^{-1/2}$ . Therefore  $|\widetilde{W}^{(\lambda)}(\phi)(x \pm \epsilon n^{(\Omega)}(x)) - \widetilde{W}^{(\lambda, \mp\epsilon)}(\phi)(x)| \leq \mathfrak{C}(\lambda) \|\phi\|_1 \epsilon^{1/2}$  for  $\epsilon \in (0, \epsilon(\Omega)]$ . Now the first part of the lemma follows with Theorem 5.3. The second part is a consequence of (5.4), Lemma 3.1 and the preceding estimate of  $\int_{\partial\Omega} \mathfrak{G}(x, y, \kappa) do_y$  and an analogous inequality for  $\int_{\partial\Omega} \mathfrak{G}(x, y, \kappa) do_x$ .  $\square$

**Lemma 5.4** *Let  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $p \in (1, \infty)$ ,  $\phi \in L^p(\partial\Omega)^3$ ,  $a \in (0, 2/p)$ ,  $r \in (1, \infty)$  and  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ . Then the relations  $\|\widetilde{W}^{(\lambda)}(\phi) - \widetilde{W}^{(\lambda, \epsilon)}(\phi)|_{\Omega}\|_{(1/p-a/2)^{-1}} \rightarrow 0$ ,  $\|\widetilde{W}^{(\lambda)}(\phi) - \widetilde{W}^{(\lambda, -\epsilon)}(\phi)|_{\Omega_R}\|_{(1/p-a/2)^{-1}} \rightarrow 0$  and  $\|\widetilde{W}^{(\lambda)}(\phi) - \widetilde{W}^{(\lambda, -\epsilon)}(\phi)|_{\partial B_R}\|_r \rightarrow 0$  hold for  $\epsilon \downarrow 0$ .*

**Proof:** Let  $x \in \Omega$ ,  $y \in \partial\Omega$  and  $\epsilon \in (0, \epsilon(\Omega)]$ . If  $\epsilon < |x - y|/2$ , we have  $|x - y - \epsilon n^{(\Omega)}(y)| \geq |x - y| - \epsilon \geq |x - y|/2$ . Suppose that  $\epsilon \geq |x - y|/2$ . Since  $y + \epsilon n^{(\Omega)}(y) \in \overline{\Omega}^c$  (see (2.4)), there is  $x' \in \partial\Omega$  such that  $|x - y - \epsilon n^{(\Omega)}(y)| \geq |x' - y - \epsilon n^{(\Omega)}(y)|$ . Using (2.5) and our assumption  $\epsilon \geq |x - y|/2$ , it follows that  $|x - y - \epsilon n^{(\Omega)}(y)| \geq \mathfrak{D}_2 \epsilon \geq \mathfrak{D}_2 |x - y|/2$ . So we have in any case that  $|x - y - \epsilon n^{(\Omega)}(y)| \geq \mathfrak{C} |x - y|$ , hence by Lemma 5.1,  $|\widetilde{S}_{jkl}^{(\lambda)}(x - [y + \epsilon n^{(\Omega)}(y)])| \leq \mathfrak{C} |x - y|^{-2}$  for  $x, y, \epsilon$  as above and for  $1 \leq j, k, l \leq 3$ . If  $\epsilon = 0$ , the preceding estimate is also valid by Lemma 5.1.

Let  $x \in \Omega$ . Since  $\text{dist}(x, \partial\Omega) > 0$ , the function  $y \mapsto |x - y|^{-2}$  ( $y \in \partial\Omega$ ) is integrable, so we get  $\mathfrak{A}(\epsilon, x) := |\widetilde{W}^{(\lambda)}(\phi) - \widetilde{W}^{(\lambda, \epsilon)}(\phi)|(x) \rightarrow 0$  ( $\epsilon \downarrow 0$ ) by the preceding estimates and Lebesgue's theorem. But the function  $x \mapsto (\int_{\partial\Omega} |x - y|^{-2} |\phi(y)| do_y)^{(1/p-a/2)^{-1}}$  ( $x \in \Omega$ ) is integrable by Lemma 3.2. So, since  $\mathfrak{A}(x, \epsilon) \rightarrow 0$  ( $\epsilon \downarrow 0$ ) and because of the estimates obtained in the first part of this proof, we may apply Lebesgue's theorem a second time, obtaining that  $(\int_{\Omega} |\mathfrak{A}(x, \epsilon)|^{(1/p-a/2)^{-1}} dx)^{1/p-a/2} \rightarrow 0$  for  $\epsilon \downarrow 0$ . This proves the first relation stated in the lemma. The other two follow by an analogous argument.  $\square$

We end this section by three estimates taken from [18] or [11].

**Theorem 5.4 ([11, (5.2)])** *Let  $\vartheta \in [0, \pi)$  and  $p \in (1, \infty)$ . Then for  $\phi \in L^p(\partial\Omega)^3$  and  $\lambda \in \mathbb{C} \setminus \{0\}$  with  $|\arg(\lambda)| \leq \vartheta$ , the inequality  $\|\widetilde{\mathcal{J}}^{(\lambda)}(\phi)|_{\overline{\Omega}^c}\|_p \leq \mathfrak{C}(\vartheta, p) |\lambda|^{-1/(2p)} \|\phi\|_p$  holds.*

**Theorem 5.5 ([18, Lemma 7.8])** *Let  $p \in (1, \infty)$ . Then, for  $\varrho \in (0, 1)$ ,  $b \in C^{\varrho}(\partial\Omega)^3 \cap W^{2-1/p, p}(\partial\Omega)^3$ ,  $\phi \in C^{\varrho}(\partial\Omega)^3$  with  $b = (-1/2) (\phi + \mathfrak{I}(\phi))$ , we have  $\phi \in W^{2-1/p, p}(\partial\Omega)^3$  and  $\|\phi\|_{2-1/p, p} \leq \mathfrak{C} (\|b\|_{2-1/p, p} + \|\phi\|_p)$ .*

**Theorem 5.6** *Let  $r \in (1, \infty)$  and  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ . Then  $\|\Pi(\phi)|_{\Omega_R}\|_{1, r} \leq$*

$\mathfrak{C} \|\phi\|_{2-1/r, r}$  for  $\phi \in W^{2-1/r, r}(\partial\Omega)^3$  with  $\phi \in C^{\varrho}(\partial\Omega)^3 \cap W^{2-1/p, p}(\partial\Omega)^3$  for some  $\varrho \in (0, 1)$  and some  $p \in (3/2, \infty)$ .

**Proof:** Let  $\phi, \varrho, p$  be given as in the theorem. Define  $\psi^{(m, n)} := (\delta_{nj} \phi_m)_{1 \leq j \leq 3}$  for  $m, n \in \{1, 2, 3\}$ , and set  $u(x) := (-2/3) \left( \sum_{j=1}^3 \widetilde{W}^{(\lambda)}(\psi^{(m, j)})_j(x) \right)_{1 \leq m \leq 3}$  for  $x \in \Omega_R$ . Obviously (Lemma 5.2)  $u \in C^\infty(\Omega_R)^3$ . According to the proof of [18, Lemma 7.15], we have  $\Delta u = 0$ ,  $u$  may be continuously extended to a function  $\tilde{u} \in C^0(\overline{\Omega_R})^3$ , and  $\|\tilde{u}|_{\partial\Omega_R}\|_{2-1/q, q} \leq \mathfrak{C}(q) \|\phi\|_{2-1/q, q}$  for  $q \in \{p, r\}$ . In this situation Lemma 3.4 yields  $\|u\|_{2, r} \leq \mathfrak{C}(r) \|\tilde{u}|_{\partial\Omega}\|_{2-1/r, r}$ . Since  $\Pi(\phi)|_{\Omega_R} = \operatorname{div} u$  by the proof of [18, Lemma 7.15], the theorem follows from the two preceding inequalities.  $\square$

## 6 Proof of Theorem 1.1.

We recall that for any  $\phi \in L^p(\partial\Omega)^3$ , the function  $\widetilde{W}^{(\lambda)}(\phi)$  splits into the sum  $\widetilde{J}^{(\lambda)}(\phi) + \mathfrak{F}(\phi)$  (Lemma 5.2). We begin by taking a closer look at the function  $\mathfrak{F}(\phi)$ , considering functions  $\phi \in L^p(\partial\Omega)^3$  that solve equation (5.3) with a right-hand side  $b$  falling into one of the following three categories: either  $b \in L^p(\partial\Omega)^3$  with  $\int_{\partial\Omega} b \cdot n^{(\Omega)} do_x = 0$ , or secondly  $b = B|_{\partial\Omega}$  with  $B \in W_{loc}^{1, p}(\mathbb{R}^3)^3 \cap L^p(\mathbb{R}^3)^3$  and  $\operatorname{div} B = 0$ , or thirdly  $b = n^{(\Omega)}$ . The second case is of interest if  $f$  does not vanish. Then  $B$  is chosen as a volume potential involving  $f$ . This is the situation considered in [11], for more regular  $B$ .

**Theorem 6.1** *Let  $p \in (1, \infty)$  and  $\vartheta \in [0, \pi)$ . Then*

$$\|\mathfrak{F}(\phi)|_{\overline{\Omega}^c}\|_p \leq \mathfrak{C}(p, \vartheta) (\|b\|_p + |\lambda|^{-1/(2p)} \|\phi\|_p) \quad \text{for } \lambda \in \mathbb{C} \setminus \{0\} \text{ with } |\arg(\lambda)| \leq \vartheta, \quad (6.1)$$

*$b \in L^p(\partial\Omega)^3$  with  $\int_{\partial\Omega} b \cdot n^{(\Omega)} do_x = 0$ , and  $\phi \in L^p(\partial\Omega)^3$  the unique solution of (5.3). In addition*

$$\|\mathfrak{F}(\phi)|_{\overline{\Omega}^c}\|_p \leq \mathfrak{C}(p, \vartheta) (\|B\|_p + |\lambda|^{-1/(2p)} \|\phi\|_p) \quad (6.2)$$

*for  $\lambda$  as before,  $b = B|_{\partial\Omega}$  for some function  $B \in W_{loc}^{1, p}(\mathbb{R}^3)^3 \cap L^p(\mathbb{R}^3)^3$  with  $\operatorname{div} B = 0$ , and  $\phi$  as before.*

*For any  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ , there exists a unique function  $\phi^{(\lambda)} \in C^0(\partial\Omega)^3$  such that  $n^{(\Omega)} = (-1/2) (\phi^{(\lambda)} + \widetilde{\mathfrak{F}}^{(\lambda)}(\phi^{(\lambda)}))$  (Theorem 5.1). The function  $\mathfrak{F}(\phi^{(\lambda)}) - \mathfrak{F}^{(\Omega)}|_{\overline{\Omega}^c}$  belongs to  $L^p(\overline{\Omega}^c)^3$ , where  $\mathfrak{F}^{(\Omega)}$  was introduced in Theorem 1.1. This latter function is in  $L^r(\mathbb{R}^3)^3$  for any  $r \in (3/2, \infty)$ .*

**Proof:** Take  $\lambda$  as in (6.1),  $g \in C^1(\partial\Omega)^3$  and  $\psi \in C^0(\partial\Omega)^3$  with  $(-1/2) (\psi + \widetilde{\mathfrak{F}}^{(\lambda)}(\psi)) = g$  (Theorem 5.1). Let  $x \in \overline{\Omega}^c$ . The function  $y \mapsto (\partial^\alpha \mathfrak{N})(x - y)$  ( $y \in \overline{\Omega}$ ) belongs in particular to  $C^1(\overline{\Omega}^c)$  ( $\alpha \in \mathbb{N}_0^3$ ,  $|\alpha| \leq 2$ ). Since  $\psi = -g - (1/2) (-\psi + \widetilde{\mathfrak{F}}^{(\lambda)}(\psi))$ , we thus get with Corollary 5.3 that  $\mathfrak{F}(\psi)_l(x) = \lim_{\epsilon \downarrow 0} \int_{\partial\Omega} (\partial_l \mathfrak{N})(x - y) n^{(\Omega)}(y) \cdot (-g + \widetilde{W}^{(\lambda, \epsilon)}(\psi))(y) do_y$ . Take some  $G \in W^{1, 1}(\Omega)^3$  with  $G|_{\partial\Omega} = g$ . Then, due to the smoothness of  $\widetilde{W}^{(\lambda, \epsilon)}(\psi)$  on an open set larger than  $\overline{\Omega}$ , and because in addition this function is solenoidal (Lemma 5.3), we get  $\mathfrak{F}(\psi)(x) = \mathfrak{F}^{(1)}(\psi)(x) + \mathfrak{F}^{(2)}(G)(x)$ , where

$$\begin{aligned} \mathfrak{F}^{(1)}(\psi)_l(x) &= \lim_{\epsilon \downarrow 0} \int_{\Omega} \sum_{k=1}^3 \partial y_k ((\partial_l \mathfrak{N})(x - y)) (-G_k + \widetilde{W}^{(\lambda, \epsilon)}(\psi)_k)(y) dy, \\ \mathfrak{F}^{(2)}(G)_l(x) &= - \int_{\partial\Omega} (\partial_l \mathfrak{N})(x - y) \operatorname{div} G(y) dy \quad (1 \leq l \leq 3). \end{aligned}$$

Lemma 5.4 yields that  $\mathfrak{F}^{(1)}(\psi)_l(x) = \int_{\Omega} \sum_{k=1}^3 (\partial_k \partial_l \mathfrak{N})(x-y) (G_k - \widetilde{W}^{(\lambda)}(\psi)_k)(y) dy$ . Here  $x$  was arbitrarily chosen in  $\overline{\Omega}^c$ . If  $\operatorname{div} G \in L^r(\Omega)$  for some  $r \in (1, 3)$ , Theorem 3.5 implies that  $\mathfrak{F}^{(2)}(G) \in L^{(1/r-1/3)^{-1}}(\overline{\Omega}^c)^3$ . Suppose that  $G \in L^p(\Omega)^3$  for some  $p \in (1, \infty)$ . Then again Theorem 3.5 allows us to conclude that  $\|\mathfrak{F}^{(1)}(\psi)\|_p \leq C(p) (\|G\|_p + \|\widetilde{W}^{(\lambda)}(\psi)|_{\Omega}\|_p)$ . But  $|\widetilde{S}_{jkl}^{(\lambda)}(z)| \leq C(\vartheta) |z|^{-2}$  for  $z \in \mathbb{R}^3 \setminus \{0\}$ ,  $1 \leq j, k, l \leq 3$  according to Lemma 5.1, so Lemma 3.2 implies  $\|\widetilde{W}^{(\lambda)}(\psi)|_{\Omega}\|_p \leq \mathfrak{C} \|\psi\|_p$ . Thus we arrive at the estimate  $\|\mathfrak{F}^{(1)}(\psi)\|_p \leq C(p) (\|G\|_p + \|\psi\|_p)$ . Here  $\psi$  was arbitrarily taken in  $C^0(\partial\Omega)^3$ .

Recall that in Theorem 1.1, we chose  $N^{(\Omega)} \in C^1(\overline{\Omega})^3$  with  $N^{(\Omega)}|_{\partial\Omega} = n^{(\Omega)}$ . Thus we may apply the preceding results to the case that  $g, \psi$  and  $G$  are replaced by  $n^{(\Omega)}, \phi^{(\lambda)}$  and  $N^{(\Omega)}$ . Note that  $\mathfrak{F}^{(2)}(N^{(\Omega)}) = \mathfrak{F}^{(\Omega)}|\overline{\Omega}^c$ . Thus the last part of Theorem 6.1 follows from what has been proved above for  $\mathfrak{F}^{(1)}(\psi)$  and  $\mathfrak{F}^{(2)}(G)$ , and because  $\mathfrak{F}^{(\Omega)} \in L^r(\mathbb{R}^3)^3$  for  $r > 3/2$  by Theorem 3.5.

Let  $b \in C^1(\partial\Omega)^3$  with  $\int_{\partial\Omega} b \cdot n^{(\Omega)} do_x = 0$ . Theorem 5.1 yields a function  $\phi \in C^0(\partial\Omega)^3$  with  $b = (-1/2) (\phi + \widetilde{\mathfrak{I}}^{(\lambda)}(\phi))$ . Moreover, by Theorem 3.1, we may choose  $B \in W^{1,p}(\Omega)^3$  with  $\operatorname{div} B = 0$  and  $B|_{\partial\Omega} = b$ . In this situation, we may apply the first part of this proof with  $g, \psi$  and  $G$  replaced by  $b, \phi$  and  $B$ , respectively. Then  $\mathfrak{F}^{(2)}(B) = 0$ , so  $\mathfrak{F}(\phi)|\overline{\Omega}^c = \mathfrak{F}^{(1)}(\phi)$ . The first part of this proof thus yields  $\mathfrak{F}(\phi)|\overline{\Omega}^c \in L^p(\overline{\Omega}^c)^3$ .

In order to show (6.1) for this  $b$ , we proceed as in [11, p. 346-347]. So let  $\varphi \in C_0^\infty(\Omega^c)$ . With Lemma 5.4, 5.3 and Corollary 5.3, we obtain

$$\int_{\overline{\Omega}^c} \widetilde{W}^{(\lambda)}(\phi) \cdot \nabla \varphi dx = - \lim_{\epsilon \downarrow 0} \int_{\Omega^\epsilon} \widetilde{W}^{(\lambda, -\epsilon)}(\phi) \cdot \nabla \varphi dx = \int_{\partial\Omega} (1/2) (\phi + \widetilde{\mathfrak{I}}^{(\lambda)}(\phi)) \cdot \varphi n^{(\Omega)} do_x.$$

By the choice of  $\phi$ , the right-hand side of this equation equals  $-\int_{\partial\Omega} b \cdot \varphi n^{(\Omega)} do_x$ . Since  $\mathfrak{F}(\phi) = \widetilde{W}^{(\lambda)}(\phi) - \widetilde{J}^{(\lambda)}(\phi)$  (Lemma 5.2), we may conclude with Theorem 5.4 that

$$\left| \int_{\overline{\Omega}^c} \mathfrak{F}(\phi) \cdot \nabla \varphi dx \right| \leq \mathfrak{C} (\|b\|_p \|\varphi|_{\partial\Omega}\|_{p'} + |\lambda|^{-1/(2p)} \|\phi\|_p \|\nabla \varphi\|_{p'}). \quad (6.3)$$

Fix some  $R \in (0, \infty)$  with  $\Omega \subset B_R$ . Let  $g \in W_{loc}^{1,p'}(\overline{\Omega}^c)^3$  with  $\nabla g \in L^{p'}(\overline{\Omega}^c)^3$ . If  $p' < 3$ , further assume that  $g \in L^{3p'/(3-p')}(\overline{\Omega}^c)^3$ . Else suppose that  $\int_{\Omega_R} g dx = 0$ . Then, by Theorem 3.3, there is a sequence  $(\varphi_n)$  in  $C_0^\infty(\Omega^c)$  with  $\|\nabla(\varphi_n - g)\|_{p'} \rightarrow 0$  and  $\|\varphi_n - g|_{\Omega_R}\|_{p'} \rightarrow 0$ . Since  $\mathfrak{F}(\phi)|\overline{\Omega}^c \in L^p(\overline{\Omega}^c)^3$ , as shown above, we get that  $\int_{\overline{\Omega}^c} \mathfrak{F}(\phi) \cdot \nabla \varphi_n dx \rightarrow \int_{\overline{\Omega}^c} \mathfrak{F}(\phi) \cdot \nabla g dx$ . On the other hand, by a standard trace estimate,  $\|\varphi_n|_{\partial\Omega}\|_{p'} \leq \mathfrak{C} \|\varphi_n|_{\Omega_R}\|_{1,p'}$  for  $n \in \mathbb{N}$ , so  $\|\varphi_n|_{\partial\Omega}\|_{p'} \leq \mathfrak{C} (\|\varphi_n - g|_{\Omega_R}\|_{1,p'} + \|g|_{\Omega_R}\|_{1,p'})$ . If  $\nabla g \neq 0$ , we may choose  $n$  so large that  $\|\varphi_n - g|_{\Omega_R}\|_{1,p'} \leq \|\nabla g\|_{p'}$ . Then  $\|\varphi_n|_{\partial\Omega}\|_{p'} \leq \mathfrak{C} (\|g|_{\Omega_R}\|_{p'} + \|\nabla g\|_{p'})$ . In the case  $p' < 3$ , we have  $\|g|_{\Omega_R}\|_{p'} \leq C(R, p) \|g|_{\Omega_R}\|_{3p'/(3-p')} \leq \mathfrak{C} \|\nabla g\|_{p'}$ , with the last inequality being valid by Theorem 3.2. If  $p' \geq 3$ , Poincaré's inequality yields  $\|g|_{\Omega_R}\|_{p'} \leq \mathfrak{C} \|\nabla g|_{\Omega_R}\|_{p'}$  because  $\int_{\Omega_R} g dx = 0$  by assumption. So in any case we obtain for  $n$  large enough that  $\|\varphi_n|_{\partial\Omega}\|_{p'} \leq \mathfrak{C} \|\nabla g\|_{p'}$ . Thus, after replacing  $\varphi$  by  $\varphi_n$  in (6.3) and then letting  $n$  tend to infinity, we arrive at the estimate

$$\left| \int_{\overline{\Omega}^c} \mathfrak{F}(\phi) \cdot \nabla g dx \right| \leq \mathfrak{C} (\|b\|_p + |\lambda|^{-1/(2p)} \|\phi\|_p) \|\nabla g\|_{p'}. \quad (6.4)$$

Define  $Z(\phi)(x) := \int_{\partial\Omega} \mathfrak{N}(x-y) n^{(\Omega)}(y) \cdot \phi(y) do_y$  for  $x \in \overline{\Omega}^c$ . Obviously  $Z(\phi) \in C^\infty(\overline{\Omega}^c)$  and  $\nabla Z(\phi) = \mathfrak{F}(\phi)|\overline{\Omega}^c$ . Thus we get for  $h \in C_{0,\sigma}^\infty(\overline{\Omega}^c)$  that  $\int_{\overline{\Omega}^c} \mathfrak{F}(\phi) \cdot h dx = 0$ . Since  $\mathfrak{F}(\phi)|\overline{\Omega}^c \in$

$L^p(\overline{\Omega}^c)^3$ , as shown above, it follows that  $\int_{\overline{\Omega}^c} \mathfrak{F}(\phi) \cdot h \, dx = 0$  even for  $L^p_\sigma(\overline{\Omega}^c)$ . Therefore from (6.4) and Corollary 3.1,  $|\int_{\overline{\Omega}^c} \mathfrak{F}(\phi) \cdot f \, dx| \leq \mathfrak{C}(\|b\|_p + |\lambda|^{-1/(2p)} \|\phi\|_p) \|\nabla \mathcal{G}_{p'}(f)\|_{p'}$  for  $f \in L^{p'}(\overline{\Omega}^c)^3$ . Since  $\|\nabla \mathcal{G}_{p'}(f)\|_{p'} \leq \mathfrak{C}\|f\|_{p'}$  for  $f$  as before (Corollary 3.1), we obtain inequality (6.1) for the function  $b$  chosen above.

If  $b = B|\partial\Omega$  for some  $B \in C^1(\mathbb{R}^3)^3 \cap L^p(\mathbb{R}^3)^3$  with  $\operatorname{div} B = 0$ , then  $\int_{\partial\Omega} b \cdot \varphi n^{(\Omega)} \, do_x = \int_{\overline{\Omega}^c} B \cdot \nabla \varphi \, dx$  for  $\varphi \in C^\infty(\Omega^c)$ . Thus we obtain (6.2) by a simpler reasoning than the one leading to (6.1) for the function  $b$  given above; compare [11, p. 346-347].

A density argument now yields (6.1) and (6.2) for  $b$  and  $B$  as in the theorem. In fact, take  $\lambda$ ,  $b$  and  $\phi$  as in the first part of the theorem. Lemma 2.2 yields a sequence  $(b_n)$  in  $C^1(\partial\Omega)^3$  with  $\|b - b_n\|_p \rightarrow 0$ . For  $n \in \mathbb{N}$ , put  $\tilde{b}_n := b_n - |\partial\Omega|^{-1} \int_{\partial\Omega} b_n \cdot n^{(\Omega)} \, do_x n^{(\Omega)}$ . Then  $\tilde{b}_n \in C^1(\partial\Omega)^3$ ,  $\int_{\partial\Omega} \tilde{b}_n \cdot n^{(\Omega)} \, do_x = 0$  for  $n \in \mathbb{N}$ , and  $\|\tilde{b}_n - b\|_p \rightarrow 0$ . Let  $\phi_n \in C^0(\partial\Omega)^3$  with  $\tilde{b}_n = (-1/2)(\phi_n + \tilde{\mathfrak{F}}^{(\lambda)}(\phi_n))$  for  $n \in \mathbb{N}$  (Theorem 5.1). Then  $\|\phi_n - \phi\|_p \rightarrow 0$  also by Theorem 5.1. Inequality (6.1) is already proved for  $b \in C^1(\partial\Omega)^3$  and is therefore valid with  $b$ ,  $\phi$  replaced by  $b_n - b_m$  and  $\phi_n - \phi_m$ , respectively, for  $m, n \in \mathbb{N}$ . Thus  $\|\mathfrak{F}(\phi_n - \phi_m)|\overline{\Omega}^c\|_p \rightarrow 0$  for  $m, n \rightarrow \infty$ . On the other hand, the relation  $\|\phi_n - \phi\|_p \rightarrow 0$  and Lebesgue's theorem yield for  $A \subset \mathbb{R}^3$  open with  $\overline{A} \subset \overline{\Omega}^c$  that  $\|\mathfrak{F}(\phi_n - \phi)|A\|_p \rightarrow 0$ . Altogether we may conclude that  $\|\mathfrak{F}(\phi_n - \phi)|\overline{\Omega}^c\|_p \rightarrow 0$ . Inequality (6.1) now follows by applying it to  $b_n$  and  $\phi_n$  instead of  $b$  and  $\phi$  and then letting  $n$  tend to infinity.

Let  $\lambda$ ,  $B$ ,  $\phi$  be given as in (6.2). Using Friedrich's mollifier, we may choose a family  $(B_\epsilon)_{\epsilon>0}$  of functions in  $C^\infty(\mathbb{R}^3)^3 \cap L^p(\mathbb{R}^3)^3$  such that  $\|B_\epsilon - B\|_p \rightarrow 0$  and  $\|\nabla(B_\epsilon - B)|A\|_p \rightarrow 0$  for  $\epsilon \downarrow 0$  for any open bounded set  $A \subset \mathbb{R}^3$ , and  $\operatorname{div} B_\epsilon = 0$  for  $\epsilon > 0$ . In particular  $B_\epsilon|\partial\Omega \in C^1(\partial\Omega)^3$  ( $\epsilon > 0$ ) and  $\|B_\epsilon - B|\partial\Omega\|_p \rightarrow 0$  ( $\epsilon \downarrow 0$ ). With this type of approximation, inequality (6.2) for  $B$  as above may be deduced from (6.2) for  $B \in C^1(\mathbb{R}^3)^3 \cap L^p(\mathbb{R}^3)^3$ , that is, for the case already considered.  $\square$

In the ensuing corollary, we collect our various previous estimates in order to obtain  $L^p$ -estimates in particular of  $\widetilde{W}^{(\lambda)}(\phi)$ .

**Corollary 6.1** *For  $b \in C^1(\partial\Omega)^3$ , abbreviate  $\gamma(b) := |\partial\Omega|^{-1} \int_{\partial\Omega} b \cdot n^{(\Omega)} \, do_x$ . Let  $p \in (1, \infty)$ ,  $\vartheta \in [0, \pi)$ . Then*

$$\begin{aligned} & \|\mathfrak{F}(\phi) - \gamma(b) \mathfrak{F}^{(\Omega)}|\overline{\Omega}^c\|_p + \|\widetilde{W}^{(\lambda)}(\phi) - \gamma(b) \mathfrak{F}^{(\Omega)}|\overline{\Omega}^c\|_p \\ & \leq \mathfrak{C}(p, \vartheta) (\|b\|_p + |\lambda|^{-1/(2p)} \|\phi\|_p + |\gamma(b)| (|\lambda|^{-1/(2p)} + 1)) \end{aligned} \quad (6.5)$$

for  $\lambda \in \mathbb{C} \setminus \{0\}$  with  $|\arg(\lambda)| \leq \vartheta$  and  $b \in L^p(\partial\Omega)^3$ , where  $\phi \in L^p(\partial\Omega)^3$  with  $b = (-1/2)(\phi + \tilde{\mathfrak{F}}^{(\lambda)}(\phi))$  (Theorem 5.1). The function  $\mathfrak{F}^{(\Omega)}$  was introduced in Theorem 1.1. Moreover  $\|\gamma(b) \mathfrak{F}^{(\Omega)}|\overline{\Omega}^c\|_r \leq \mathfrak{C}(r) \|b\|_p$  for  $r \in (3/2, \infty)$ ,  $b \in L^p(\partial\Omega)^3$  (Theorem 6.1). Thus, if  $p > 3/2$ , the estimate

$$\|\mathfrak{F}(\phi)|\overline{\Omega}^c\|_p + \|\widetilde{W}^{(\lambda)}(\phi)|\overline{\Omega}^c\|_p \leq \mathfrak{C}(p, \vartheta) ((|\lambda|^{-1/(2p)} + 1) \|b\|_p + |\lambda|^{-1/(2p)} \|\phi\|_p) \quad (6.6)$$

holds for  $\lambda$ ,  $\phi$ ,  $b$  as in (6.5).

If  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ , the term  $\|\mathfrak{F}(\phi)|\Omega_R\|_p + \|\widetilde{W}^{(\lambda)}(\phi)|\Omega_R\|_p$ , too, is bounded by the right-hand side of (6.6) for  $\lambda$ ,  $\phi$ ,  $b$  as in (6.5), except that the constant additionally depends on  $R$ .

Moreover  $\|\mathfrak{F}(\phi)|\overline{\Omega}^c\|_p + \|\widetilde{W}^{(\lambda)}(\phi)|\overline{\Omega}^c\|_p \leq \mathfrak{C}(p, \vartheta) (\|b\|_p + |\lambda|^{-1/(2p)} \|\phi\|_p)$  for  $\lambda$  as in (6.5),  $b \in L^p(\partial\Omega)^3$  with  $\int_{\partial\Omega} b \cdot n^{(\Omega)} \, do_x = 0$ , and  $\phi$  as in (6.5).

In addition,  $\|\widetilde{W}^{(\lambda)}(\phi)|\overline{\Omega}^c\|_p \leq \mathfrak{C}(\|B\|_p + |\lambda|^{-1/(2p)} \|\phi\|_p)$  for  $\lambda$  as in (6.5) and for  $B \in W_{loc}^{1,p}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)^3$  with  $\operatorname{div} B = 0$ , where  $\phi \in L^p(\partial\Omega)^3$  solves (5.3) with  $b = B|_{\partial\Omega}$ .

For  $R$  as above and  $a \in (0, 2/p)$ , the inequality  $\|\widetilde{W}^{(\lambda)}(\phi)|_{\Omega_R}\|_{(1/p-a/2)^{-1}} \leq \mathfrak{C}\|\phi\|_p$  holds for  $\lambda$  as in (6.5) and  $\phi \in L^p(\partial\Omega)^3$ .

**Proof:** Let  $\lambda$ ,  $b$  and  $\phi$  be given as in (6.5), and set  $\widetilde{b} := b - \gamma(b)n^{(\Omega)}$ . Then  $\widetilde{b} \in L^p(\partial\Omega)^3$  and  $\int_{\partial\Omega} \widetilde{b} n^{(\Omega)} d\sigma_x = 0$ . Let  $\widetilde{\phi} \in L^p(\partial\Omega)^3$  with  $\widetilde{b} = (-1/2)(\widetilde{\phi} + \mathfrak{F}(\widetilde{\phi}))$  (Theorem 5.1). Then inequality (6.1) yields that  $\|\mathfrak{F}(\widetilde{\phi})|\overline{\Omega}^c\|_p \leq \mathfrak{C}(\|\widetilde{b}\|_p + |\lambda|^{-1/(2p)} \|\widetilde{\phi}\|_p)$ . On the other hand, due to the uniqueness of the solution to (5.3) and the choice of  $\phi^{(\lambda)}$  in Theorem 6.1, we have  $\phi = \widetilde{\phi} + \gamma(b)\phi^{(\lambda)}$ , and thus  $\mathfrak{F}(\phi) - \gamma(b)\mathfrak{F}^{(\Omega)} = \mathfrak{F}(\widetilde{\phi}) + \gamma(b)(\mathfrak{F}(\phi^{(\lambda)}) - \mathfrak{F}^{(\Omega)})$ . Thus the estimate involving  $\mathfrak{F}(\phi)$  in (6.5) follows from the estimate of  $\mathfrak{F}(\widetilde{\phi})$  given above, from the relation between  $\phi$  and  $\widetilde{\phi}$  also given above, and because  $\mathfrak{F}(\phi^{(\lambda)}) - \mathfrak{F}^{(\Omega)} \in L^p(\overline{\Omega}^c)^3$  (Theorem 6.1). All the other statements of the corollary are now a direct consequence of the equation  $\widetilde{W}^{(\lambda)}(\phi) = \widetilde{J}^{(\lambda)}(\phi) + \mathfrak{F}(\phi)$  (Lemma 5.2), Theorem 6.1, 5.4, Lemma 3.2, (6.5) and the estimate  $|\widetilde{S}_{jkl}^{(\lambda)}(z)| \leq C(\vartheta)|z|^{-2}$  for  $\lambda$  as in (6.5) and  $z \in \mathbb{R}^3 \setminus \{0\}$  (Lemma 5.1).  $\square$

The structure of our solutions immediately provides pointwise decay estimates.

**Lemma 6.1** *Let  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ , and take  $\vartheta \in [0, \pi)$ . Then*

$$\begin{aligned} |\partial^\alpha \widetilde{W}^{(\lambda)}(\phi)(x)| + |\partial^\alpha \mathfrak{F}(\phi)(x)| &\leq \mathfrak{C}|x|^{-2-|\alpha|} \|\phi\|_1, \\ |(\widetilde{W}^{(\lambda)}(\phi) - \mathfrak{F}(\phi))(x)| &\leq \mathfrak{C}|\lambda|^{-1}|x|^{-4} \|\phi\|_1, \quad |\partial^\alpha \widetilde{\Pi}^{(\lambda)}(\phi)(x)| \leq \mathfrak{C}(1+|\lambda|)|x|^{-1-|\alpha|} \|\phi\|_1 \end{aligned}$$

for  $\lambda \in \mathbb{C} \setminus \{0\}$  with  $|\arg(\lambda)| \leq \vartheta$ ,  $\phi \in L^1(\partial\Omega)^3$ ,  $x \in B_R^c$ ,  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 2$ , where all constants  $\mathfrak{C}$  only depend on  $R$ ,  $\delta := \operatorname{dist}(B_R^c, \overline{\Omega})$  or  $\vartheta$ .

**Proof:** Since  $\overline{\Omega} \subset B_R$ , we have  $\delta > 0$ . Let  $x \in B_R^c$ ,  $y \in \partial\Omega$ . Then  $|x - y| \geq |x|\delta/R + |x|(1 - \delta/R) - |y| \geq |x|\delta/R + R - \delta - |y|$ . But  $R - |y| = |R|y|^{-1}y - y|$  and  $R|y|^{-1}y \in B_R^c$  if  $y \neq 0$ , so  $R - |y| \geq \delta$ . Thus we may conclude that  $|x - y| \geq |x|\delta/R$ . The lemma now follows from the estimates of  $\partial^\alpha \widetilde{E}_{jk}^{(\lambda)}$  given in Lemma 5.1.  $\square$

The preceding pointwise decay estimates immediately imply  $L^p$ -estimates.

**Corollary 6.2** *Take  $R$ ,  $\delta$  and  $\vartheta$  as in Lemma 6.1. Then*

$$\begin{aligned} \|\widetilde{W}^{(\lambda)}(\phi)|_{B_R^c}\|_{p_1} + \|\mathfrak{F}(\phi)|_{B_R^c}\|_{p_1} &\leq \mathfrak{C}\|\phi\|_1, \quad \|\partial^\alpha \widetilde{W}^{(\lambda)}(\phi)|_{B_R^c}\|_{p_2} \leq \mathfrak{C}\|\phi\|_1, \\ \|\nabla \widetilde{\Pi}^{(\lambda)}(\phi)|_{B_R^c}\|_{p_1} &\leq \mathfrak{C}(1+|\lambda|)\|\phi\|_1, \quad \|\widetilde{\Pi}^{(\lambda)}(\phi)|_{B_R^c}\|_{p_3} \leq \mathfrak{C}(1+|\lambda|)\|\phi\|_1, \end{aligned}$$

as well as  $\|(\widetilde{W}^{(\lambda)}(\phi) - \mathfrak{F}(\phi))|_{B_R^c}\|_{p_2} \leq \mathfrak{C}|\lambda|^{-1}\|\phi\|_1$ , for  $\lambda$ ,  $\phi$  as in Lemma 6.1,  $p_1 \in (3/2, \infty)$ ,  $p_2 \in (1, \infty)$ ,  $p_3 \in (3, \infty)$ , and  $\alpha \in \mathbb{N}_0^3$  with  $1 \leq |\alpha| \leq 2$ , with constants depending exclusively on  $R$ ,  $\delta$ ,  $\vartheta$ ,  $p_1$ ,  $p_2$  or  $p_3$ .

We show that if  $b$  satisfies a zero flux condition on  $\partial\Omega$ , and if  $\phi$  solves (5.3), then  $\mathfrak{F}(\phi)$  fulfills a zero flux condition on  $\partial B_R$ :

**Lemma 6.2** *Let  $p \in (1, \infty)$ ,  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $b, \phi \in L^p(\partial\Omega)^3$  with  $\int_{\partial\Omega} b \cdot n^{(\Omega)} d\sigma_x = 0$  and  $b = (-1/2)(\phi + \mathfrak{F}^{(\lambda)}(\phi))$ ,  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ . Then  $\int_{\partial B_R} \mathfrak{F}(\phi)(x) \cdot R^{-1}x d\sigma_x = 0$ .*

**Proof:** Since  $\widetilde{W}^{(\lambda, -\epsilon)}(\phi)$  is a smooth, solenoidal function on an open set somewhat larger than  $\Omega^c$  (Lemma 5.3), we get  $\int_{\partial B_R} \widetilde{W}^{(\lambda, -\epsilon)}(\phi)(x) \cdot R^{-1}x d\sigma_x = \int_{\partial\Omega} \widetilde{W}^{(\lambda, -\epsilon)}(\phi) \cdot n^{(\Omega)} d\sigma_x$ , for  $\epsilon \in (0, \epsilon(\Omega)]$ . Letting  $\epsilon$  tend to zero, we may conclude with Corollary 5.3 and Lemma 5.4

that  $\int_{\partial B_R} \widetilde{W}^{(\lambda)}(\phi)(x) \cdot R^{-1} x \, do_x = \int_{\partial \Omega} b \cdot n^{(\Omega)} \, do_x = 0$ . On the other hand, from Lemma 5.2 we know that  $\operatorname{div} \widetilde{J}^{(\lambda)}(\phi) = 0$ , so  $\int_{\partial B_R} \widetilde{J}^{(\lambda)}(\phi)(x) \cdot R^{-1} x \, do_x = \int_{\partial B_r} \widetilde{J}^{(\lambda)}(\phi)(x) \cdot r^{-1} x \, do_x$  for  $r \in (R, \infty)$ . But from the estimate of  $\partial_j \widetilde{E}_{kl}^{(\lambda)}$  provided by Lemma 5.1, we obtain that  $\int_{\partial B_r} \widetilde{J}^{(\lambda)}(\phi)(x) \cdot r^{-1} x \, do_x \rightarrow 0$  for  $r \rightarrow \infty$ . Thus Lemma 6.2 follows with the equation  $\widetilde{W}^{(\lambda)}(\phi) = \widetilde{J}^{(\lambda)}(\phi) + \mathfrak{F}(\phi)$  (Lemma 5.2).  $\square$

The ensuing theorem was already used in [11] but not proved there. For the convenience of the reader, we give a proof in the appendix, with an approach different from the indication on [11, p. 347].

**Theorem 6.2** *Let  $p \in (1, \infty)$ ,  $\vartheta \in [0, \pi)$ . Then  $\mathring{\mathfrak{I}}^{(\lambda)}(\phi) \in W^{2-1/p, p}(\partial \Omega)^3$  and the inequality  $\|\mathring{\mathfrak{I}}^{(\lambda)}(\phi)\|_{2-1/p, p} \leq \mathfrak{C} |\lambda|^{1-1/(2p)} \|\phi\|_p$  holds for  $\phi \in L^p(\partial \Omega)^3$  and  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$  with  $|\arg(\lambda)| \leq \vartheta$ .*

We turn to the case  $b \in W^{2-1/p, p}(\partial \Omega)^3$ .

**Theorem 6.3** *Let  $p \in (1, \infty)$ ,  $\vartheta \in [0, \pi)$  and  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ . Then*

$$\begin{aligned} & \|\widetilde{W}^{(\lambda)}(\phi)|_{\Omega_R}\|_{2,p} + \|\widetilde{\Pi}^{(\lambda)}(\phi)|_{\Omega_R}\|_{1,p} \\ & \leq \mathfrak{C}(p, \vartheta, R) \left( (|\lambda|^{1-1/(2p)} + 1) \|\phi\|_p + |\lambda| \|b\|_p + \|b\|_{2-1/p, p} \right) \end{aligned} \quad (6.7)$$

for  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$  with  $|\arg(\lambda)| \leq \vartheta$ ,  $b \in W^{2-1/p, p}(\partial \Omega)^3$  and  $\phi \in L^p(\partial \Omega)^3$  with  $b = (-1/2) (\phi + \widetilde{\mathfrak{I}}^{(\lambda)}(\phi))$ .

For  $B \in W^{2,p}(\mathbb{R}^3)^3$  with  $\operatorname{div} B = 0$  and  $\lambda$  as in (6.7), the left hand side in (6.7) is bounded by  $\mathfrak{C}(p, \vartheta, R) \left( (|\lambda|^{1-1/(2p)} + 1) \|\phi\|_p + |\lambda| \|B\|_p + \|B\|_{2,p} \right)$ , where  $\phi \in L^p(\partial \Omega)^3$  with  $B|_{\partial \Omega} = (-1/2) (\phi + \widetilde{\mathfrak{I}}^{(\lambda)}(\phi))$ .

**Proof:** Take  $\lambda$  as in the theorem and  $b \in C^2(\partial \Omega)^3$ . In particular  $b$  belongs to  $C^\varrho(\partial \Omega)^3$  for  $\varrho \in (0, 1)$  (Lemma 2.1) and to  $W^{2-1/r, r}(\partial \Omega)^3$  for  $r \in (1, \infty)$ . There is a function  $\phi$  as specified in the theorem, and this function additionally belongs to  $C^\varrho(\partial \Omega)^3$  for  $\varrho \in (0, 1)$  (Theorem 5.1). Recall that  $W := \widetilde{W}^{(\lambda)}(\phi)$  and  $\Pi := \widetilde{\Pi}^{(\lambda)}(\phi)$  are  $C^\infty$ -functions in  $\mathbb{R}^3 \setminus \partial \Omega$  (Lemma 5.2), and  $W|_{\overline{\Omega}^c}$  admits a continuous extension  $W_{ex} := \widetilde{W}_{ex}^{(\lambda)}(\phi)$  to  $\Omega^c$  with  $W_{ex}|_{\partial \Omega} = (-1/2) (\phi + \widetilde{\mathfrak{I}}^{(\lambda)}(\phi))$  (Theorem 5.3), that is,  $W_{ex}|_{\partial \Omega} = b$ .

For this smooth function  $b$ , let us briefly check whether (6.7) follows by an approach as in [11, p. 348], but without the assumption that  $b$  admits a solenoidal extension to  $\mathbb{R}^3$ , as required in [11]. Afterwards we will extend (6.7) to functions  $b$  as given in the theorem, via a density argument. Referring to the choice of  $\phi$ , and because  $\widetilde{\mathfrak{I}}^{(\lambda)}(\phi) = \mathfrak{I}(\phi) + \mathring{\mathfrak{I}}^{(\lambda)}(\phi)$  (Theorem 5.1), we get  $\phi + \mathfrak{I}(\phi) = -\mathring{\mathfrak{I}}^{(\lambda)}(\phi) - 2b$ . Moreover we have  $\mathring{\mathfrak{I}}^{(\lambda)}(\phi) \in W^{2-1/r, r}(\partial \Omega)^3$  for  $r \in (1, \infty)$  by Theorem 6.2, hence in view of the properties of  $b$ , and with Theorem 6.2 and 5.5, we conclude that  $\phi \in W^{2-1/r, r}(\partial \Omega)^3$  and

$$\|\phi\|_{2-1/r, r} \leq \mathfrak{C} \left( \|b\|_{2-1/r, r} + (|\lambda|^{1-1/(2r)} + 1) \|\phi\|_r \right) \quad \text{for } r \in (1, \infty). \quad (6.8)$$

At this point, Theorem 5.6 implies that  $\|\Pi(\phi)|_{\Omega_R}\|_{1,r} \leq \mathfrak{C} \|\phi\|_{2-1/r, r}$  and thus with (6.8),  $\|\Pi(\phi)|_{\Omega_R}\|_{1,r} \leq \mathfrak{C} \left( \|b\|_{2-1/r, r} + (|\lambda|^{1-1/(2r)} + 1) \|\phi\|_r \right)$  for  $r \in (1, \infty)$ . Moreover we observe that  $\nabla \dot{\Pi}(\phi) = -\mathfrak{F}(\phi)$ , so Corollary 6.1 yields that

$$\|\lambda W|_{\Omega_R}\|_r + \|\lambda \nabla \dot{\Pi}(\phi)|_{\Omega_R}\|_r \leq \mathfrak{C} \left( (|\lambda|^{1-1/(2r)} + |\lambda|) \|b\|_r + |\lambda|^{1-1/(2r)} \|\phi\|_r \right)$$

for  $r$  as before. Since  $W \in C^\infty(\mathbb{R}^3 \setminus \partial\Omega)^3$  and  $\bar{\Omega} \subset B_R$ , we have  $\|W|_{\partial B_R}\|_{2-1/r, r} \leq \mathfrak{C} \|\phi\|_r$  for  $r$  as before (Lemma 6.1); compare [18, Lemma 7.14]. We may thus conclude that  $\|W_{ex}|_{\partial\Omega_R}\|_{2-1/r, r} \leq \mathfrak{C} (\|b\|_{2-1/r, r} + \|\phi\|_r)$ . Recalling that  $\Pi = \Pi(\phi) + \lambda \tilde{\Pi}(\phi)$  and the pair  $(W, \Pi)$  solves (1.1) with  $f = 0$  (Lemma 5.2), we may thus apply Lemma 3.4, which together with the preceding estimates of  $\nabla \Pi(\phi)|_{\Omega_R}$ ,  $\lambda W|_{\Omega_R}$ ,  $\lambda \nabla \tilde{\Pi}(\phi)|_{\Omega_R}$  and  $W_{ex}|_{\partial\Omega_R}$  yields that  $W \in W^{2,p}(\Omega_R)^3$ ,  $W|_{\partial\Omega} = b$  in the trace sense and inequality (6.7) holds. Recall that we assumed  $b \in C^2(\partial\Omega)^3$ .

Now take  $\lambda, b$  and  $\phi$  as in the theorem. Choose  $B \in W^{2,p}(\Omega)^3$  with  $B|_{\partial\Omega} = b$ , and take a sequence  $(B_n)$  in  $C^\infty(\bar{\Omega})^3$  with  $\|B_n - B\|_{2,p} \rightarrow 0$ . Put  $b_n := B_n|_{\partial\Omega}$ . Then  $(b_n)$  is a sequence in  $C^2(\partial\Omega)^3$  with  $\|b - b_n\|_{2-1/p, p} \rightarrow 0$ . For  $n \in \mathbb{N}$ , take  $\phi_n \in L^p(\partial\Omega)^3$  with  $b_n = (-1/2) (\phi_n + \tilde{\mathfrak{T}}^{(\lambda)}(\phi_n))$ ; see Theorem 5.1. Due to the same theorem, we have  $\|\phi_n - \phi\|_p \rightarrow 0$ . By what has been proved above, the function  $\tilde{W}^{(\lambda)}(\phi_n)|_{\Omega_R}$  belongs to  $W^{2,p}(\Omega_R)^3$  and  $[\tilde{W}^{(\lambda)}(\phi_n)|_{\Omega_R}]|_{\partial\Omega} = b_n$  in the trace sense, and inequality (6.7) is valid with  $b, \phi$  replaced by  $b_n$  and  $\phi_n$ , respectively, for  $n \in \mathbb{N}$ . It follows from (6.7) that  $\|\tilde{W}^{(\lambda)}(\phi_n - \phi_m)|_{\Omega_R}\|_{2,p} + \|\tilde{\Pi}^{(\lambda)}(\phi_n - \phi_m)|_{\Omega_R}\|_{1,p} \leq \mathfrak{C}(\lambda) (\|b_n - b_m\|_{2-1/p, p} + \|\phi_n - \phi_m\|_p)$  for  $m, n \in \mathbb{N}$ . Therefore  $(\tilde{W}^{(\lambda)}(\phi_n)|_{\Omega_R})_{n \geq 1}$  and  $(\tilde{\Pi}^{(\lambda)}(\phi_n)|_{\Omega_R})_{n \geq 1}$  are Cauchy sequences in  $W^{2,p}(\Omega_R)^3$  and  $W^{1,p}(\Omega_R)$ , respectively. In addition, if  $A \subset \mathbb{R}^3$  with  $\bar{A} \subset \Omega_R$ , then Lebesgue's theorem and the relation  $\|\phi_n - \phi\|_p \rightarrow 0$  imply  $\|\tilde{W}^{(\lambda)}(\phi_n - \phi)|_A\|_p \rightarrow 0$  and  $\|\tilde{\Pi}^{(\lambda)}(\phi_n - \phi)|_A\|_p \rightarrow 0$ . Thus  $\tilde{W}^{(\lambda)}(\phi)|_{\Omega_R} \in W^{2,p}(\Omega_R)^3$ ,  $\tilde{\Pi}^{(\lambda)}(\phi)|_{\Omega_R} \in W^{1,p}(\Omega_R)$  and

$$\|\tilde{W}^{(\lambda)}(\phi_n) - \tilde{W}^{(\lambda)}(\phi)|_{\Omega_R}\|_{2,p} + \|\tilde{\Pi}^{(\lambda)}(\phi_n) - \tilde{\Pi}^{(\lambda)}(\phi)|_{\Omega_R}\|_{1,p} \rightarrow 0. \quad (6.9)$$

Since  $\|b_n - b\|_{2-1/p, p} \rightarrow 0$  and  $[\tilde{W}^{(\lambda)}(\phi_n)|_{\Omega_R}]|_{\partial\Omega} = b_n$  in the trace sense, as remarked above, we thus get that  $[\tilde{W}^{(\lambda)}(\phi)|_{\Omega_R}]|_{\partial\Omega} = b$  in the trace sense. Moreover the relations  $\|b_n - b_m\|_{2-1/p, p} \rightarrow 0$ ,  $\|\phi_n - \phi_m\|_p \rightarrow 0$  and those in (6.9), as well as the fact that (6.7) holds for  $b_n, \phi_n$  in the place of  $b$  and  $\phi$ , respectively, for  $n \in \mathbb{N}$ , allows us to conclude that (6.7) holds as stated in the theorem.

Under the additional condition  $B \in C^2(\mathbb{R}^3)^3$ , the last statement of Theorem 6.3 holds by [11, (1.15)]. The assumption  $B \in C^2(\mathbb{R}^3)^3$  may be removed by a density argument involving Friedrich's mollifier, in a similar way as in the proof of Theorem 6.1.  $\square$

Now we are in a position to carry out the

**Proof of Theorem 1.1:** Let  $p \in (1, \infty)$ ,  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ ,  $b \in L^p(\partial\Omega)^3$ . Theorem 5.1 yields a function  $\phi \in L^p(\partial\Omega)^3$  with  $b = (-1/2) (\phi + \tilde{\mathfrak{T}}^{(\lambda)}(\phi))$ . Put  $u := \tilde{W}^{(\lambda)}(\phi)|_{\bar{\Omega}^c}$ ,  $\pi := \tilde{\Pi}^{(\lambda)}(\phi)|_{\bar{\Omega}^c}$ . Then Lemma 5.2 yields that  $u_j, \pi \in C^\infty(\bar{\Omega}^c)$  for  $1 \leq j \leq 3$ , and the pair  $(u, \pi)$  solves (1.1) with  $f = 0$ . According to Corollary 5.2, the boundary condition (1.2) is fulfilled in the  $L^p$ -sense. If  $b \in W^{2-1/p, p}(\partial\Omega)^3$ , Theorem 6.3 states that  $u|_{\Omega_R} \in W^{2,p}(\Omega_R)^3$  and  $\pi|_{\Omega_R} \in W^{1,p}(\Omega_R)$  for  $R \in (0, \infty)$  with  $\bar{\Omega} \subset B_R$ , and equation (1.2) is satisfied in the trace sense.

Let  $\vartheta \in [0, \pi)$ ,  $\lambda_0 \in (0, \infty)$  and  $\lambda \in \mathbb{C} \setminus \{0\}$  with  $|\arg(\lambda)| \leq \vartheta$  and  $|\lambda| \geq \lambda_0$ . Then  $\|\phi\|_p \leq \mathfrak{C}(p, \vartheta, \lambda_0) \|\phi + \tilde{\mathfrak{T}}^{(\lambda)}(\phi)\|_p$  by Corollary 5.2. The estimates stated in Theorem 1.1 follow from this inequality and from Corollary 6.1, 6.2 and Theorem 6.3.  $\square$

## 7 An application to the Oseen resolvent problem (1.7).

First we recall a result on interior regularity of weak solutions to the stationary Oseen system and to the Oseen resolvent problem.

**Theorem 7.1** *Let  $A \subset \mathbb{R}^3$  be open,  $\lambda \in \mathbb{C}$ ,  $q, s \in (1, \infty)$ ,  $f \in L^q_{loc}(A)^3$ ,  $u \in W^{1,1}_{loc}(A)^3$  with  $\nabla u \in L^s_{loc}(A)^9$  such that*

$$\int_A (\nabla u \cdot \nabla \vartheta + (\tau \partial_1 u + \lambda u - f) \cdot \vartheta) dx = 0 \text{ for } \vartheta \in C^\infty_{0,\sigma}(A), \text{ div } u = 0. \quad (7.1)$$

Then  $u \in W^{2,q}_{loc}(A)^3$ .

**Proof:** The theorem is a consequence of interior regularity of solutions to the Stokes system; see [15, Theorem 3.2].  $\square$

Note that if  $u \in W^{2,1}_{loc}(\overline{\Omega}^c)^3$  and  $\pi \in W^{1,1}_{loc}(\overline{\Omega}^c)$  such that the pair  $(u, \pi)$  solves (1.7), then  $u$  is a solution to (7.1). This is, of course, because  $\int_{\overline{\Omega}^c} \nabla \pi \cdot \vartheta dx = 0$  for  $\vartheta \in C^\infty_{0,\sigma}(\overline{\Omega}^c)$ .

The main theorem of this section provides conditions allowing to represent an Oseen resolvent by a sum of solutions to problem (1.1), (1.2). In addition this theorem derives  $L^p$ -estimates from this representation.

**Theorem 7.2** *Let  $n_0, \mu_0 \in \mathbb{N}$ ,  $p_1, \dots, p_{n_0}, q^{(1)}, \dots, q^{(\mu_0)}, q_1 \in (1, \infty)$ , and define  $q := \min(\{p_1, \dots, p_{n_0}\} \cup \{q^{(1)}, \dots, q^{(\mu_0)}\} \cup \{q_1\})$ .*

*Let  $\vartheta \in [0, \pi)$ ,  $\lambda_0 \in (0, \infty)$ ,  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq \lambda_0$  and  $|\arg(\lambda)| \leq \vartheta$ ,*

*$f^{(j)} \in L^{p_j}(\overline{\Omega}^c)^3$  for  $1 \leq j \leq n_0$ ,  $v^{(\mu)} \in L^{q^{(\mu)}}(\overline{\Omega}^c)^3 \cap W^{1,1}_{loc}(\overline{\Omega}^c)^3$  and  $\nabla v^{(\mu)} \in L^{q_1}(\overline{\Omega}^c)^3$  for  $1 \leq \mu \leq \mu_0$ .*

*Put  $v := \sum_{\mu=1}^{\mu_0} v^{(\mu)}$  and suppose that  $v|_{\Omega_R} \in W^{2,q}(\Omega_R)^3$  for  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ , and (7.1) is satisfied with  $A = \overline{\Omega}^c$  and  $f = \sum_{j=1}^{n_0} f^{(j)}$ .*

*Put  $p_{n_0+1} := q_1$ ,  $f^{(n_0+1)} := -\tau \partial_1 v$  and  $u^{(j)} := (\lambda \mathcal{I}_{p_j} + \mathcal{A}_j)^{-1}(\mathcal{P}_{p_j}(f^{(j)}))$  for  $1 \leq j \leq n_0+1$  (notation as in Corollary 4.1). Let  $\phi \in L^q(\partial\Omega)^3$  with  $(-1/2)(\phi + \tilde{\mathfrak{F}}^{(\lambda)}(\phi)) = -v|_{\partial\Omega}$  (Theorem 5.1), and  $u^{(n_0+2)} := \widehat{W}^{(\lambda)}(\phi)|_{\overline{\Omega}^c}$ .*

*Then  $v = \sum_{j=1}^{n_0+2} u^{(j)}$ ,  $u^{(j)} \in W^{2,p_j}(\overline{\Omega}^c)^3$  for  $1 \leq j \leq n_0+1$ ,  $u^{(n_0+2)} \in C^\infty(\overline{\Omega}^c)^3$  and  $u^{(n_0+2)}|_{\Omega_R} \in W^{2,q}(\Omega_R)^3$  for  $R$  as above,  $u^{(n_0+2)}|_{\partial\Omega} = -v|_{\partial\Omega}$ ,*

$$\|\lambda u^{(j)}\|_{p_j} \leq \mathfrak{C}(p_j, \vartheta, \lambda_0) \|f^{(j)}\|_{p_j} \text{ for } 1 \leq j \leq n_0, \quad \|\lambda u^{(n_0+1)}\|_{q_1} \leq \mathfrak{C}(q_1, \vartheta, \lambda_0) \tau \|\nabla v\|_{q_1}.$$

*Moreover  $\|\lambda(u^{(n_0+2)} - \mathfrak{F}(\phi))\|_{B_R^c} \leq \mathfrak{C}(q, r, \vartheta, \lambda_0, R) \|v|_{\partial\Omega}\|_q$  if  $R$  is given as above and  $r \in (1, \infty)$ . If in addition  $r > 3/2$ , the estimate  $\|\mathfrak{F}(\phi)\|_{B_R^c} \leq \mathfrak{C}(q, r, \vartheta, \lambda_0, R) \|v|_{\partial\Omega}\|_q$  is valid, where  $\mathfrak{F}(\phi)$  was introduced in Lemma 5.2.*

**Proof:** All the claims of the lemma except the equation  $v = \sum_{j=1}^{n_0+2} u^{(j)}$  follow from Corollary 4.1, Theorem 1.1 and Corollary 6.2 (estimate of  $\lambda(u^{(n_0+2)} - \mathfrak{F}(\phi))|_{B_R^c}$  and of  $\mathfrak{F}(\phi)|_{B_R^c}$ ). Concerning these last two points, note that  $\|\phi\|_q \leq \mathfrak{C}(q, \vartheta, \lambda_0) \|v|_{\partial\Omega}\|_q$  by Corollary 5.1.

Turning to the proof of the equation  $v = \sum_{j=1}^{n_0+2} u^{(j)}$ , we fix  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ . Theorem 1.1 yields that  $u^{(n_0+2)}|_{B_R^c} \in L^r(B_R^c)^3$  for any  $r \in (3/2, \infty)$ . Thus each of the functions  $v^{(\mu)}|_{B_R^c}$  and  $u^{(j)}|_{B_R^c}$  for  $1 \leq \mu \leq \mu_0$  and  $1 \leq j \leq n_0+2$  belongs to

$L^r(B_R^c)^3$  for some  $r \in (1, \infty)$ . Theorem 1.1 further yields that  $u^{(n_0+2)}|_{\Omega_R} \in W^{2,q}(\Omega_R)^9$  and  $\nabla u^{(n_0+2)}|_{B_R^c} \in L^q(B_R^c)^9$ . So each of the functions  $\nabla v^{(\mu)}|_{B_R^c}$  and  $\nabla u^{(j)}|_{B_R^c}$  for  $\mu$  and  $j$  as before is in  $L^r(B_R^c)^9$  for some  $r \in (1, \infty)$ .

Put  $w := v - \sum_{j=1}^{n_0+2} u^{(j)}$ . Since  $u^{(j)}|_{\partial\Omega} = 0$  for  $1 \leq j \leq n_0+1$  (Corollary 4.1) and because  $u^{(n_0+2)}|_{\partial\Omega} = -v|_{\partial\Omega}$ , it follows that  $w|_{\partial\Omega} = 0$ . Due to Theorem 4.1 and Corollary 4.1 as concerns  $u^{(1)}$  to  $u^{(n_0+1)}$ , Theorem 1.1 with respect to  $u^{(n_0+2)}$ , and our assumptions on  $v$ , we may conclude that  $\operatorname{div} w = 0$  and  $\int_{\overline{\Omega_0^c}} (\nabla w \cdot \nabla \vartheta + \lambda w \cdot \vartheta) dx = 0$  for  $\vartheta \in C_{0,\sigma}^\infty(\overline{\Omega^c})$ . Thus we see that all conditions of the uniqueness result in Theorem 4.3 are satisfied with  $A = \Omega$ . It follows that  $w = 0$ , so  $v = \sum_{j=1}^{n_0+2} u^{(j)}$ .  $\square$

The version of Theorem 7.2 which is relevant in [16] (see [16, Theorem 3.2]) is stated as

**Corollary 7.1** *Let  $A \subset \mathbb{R}^3$  be open and bounded, with Lipschitz boundary. Fix  $S \in (0, \infty)$  with  $\overline{\Omega} \subset B_S$ . Let  $n_0, \mu_0, p_1, \dots, p_{n_0}, q^{(1)}, \dots, q^{(\mu_0)}, q, \vartheta, \lambda_0$  and  $\lambda$  be given as in Theorem 7.2. Take  $f^{(1)}, \dots, f^{(n_0)}, v^{(1)}, \dots, v^{(\mu)}$  also as in Theorem 7.2, but with  $\overline{\Omega^c}$  replaced by  $\overline{A^c}$ . Define  $v := \sum_{\mu=1}^{\mu_0} v^{(\mu)}$  and suppose that  $v$  satisfies (7.1) with  $A$  replaced by  $\overline{A^c}$  and with  $f = \sum_{j=1}^{n_0} f^{(j)}$ .*

*Then  $v|_{B_R \setminus \overline{B_S^c}} \in W^{2,q}(B_R \setminus \overline{B_S^c})^3$  for  $R \in (S, \infty)$ .*

*Define  $p_{n_0+1}$  and  $f^{(n_0+1)}$  as in Theorem 7.2. For  $j \in \{1, \dots, n_0+1\}$ , let  $\mathcal{P}_{p_j}$  be defined as in Theorem 3.4 and  $\mathcal{I}_{p_j}, \mathcal{A}_{p_j}$  as in Corollary 4.1, each time with  $A = B_S$ . Put  $u^{(j)} := (\lambda \mathcal{I}_{p_j} + \mathcal{A}_{p_j})^{-1}(\mathcal{P}_{p_j}(f|_{\overline{B_S^c}}))$  for  $1 \leq j \leq n_0+1$ . Let  $\phi \in L^p(\partial B_S)^3$  with  $-v|_{\partial B_S} = (-1/2)(\phi + \tilde{\mathfrak{F}}^{(\lambda)}(\phi))$ , and put  $u^{(n_0+2)} := \widetilde{W}^{(\lambda)}(\phi)|_{\overline{B_S^c}}$ , where the definitions of both  $\tilde{\mathfrak{F}}^{(\lambda)}(\phi)$  and  $\widetilde{W}^{(\lambda)}(\phi)$  refer to the case  $\Omega = B_S$  (Theorem 5.1 and Lemma 5.2, respectively).*

*Then  $v|_{\overline{B_S^c}} = \sum_{j=1}^{n_0+2} u^{(j)}$ ,  $u^{(j)} \in W^{2,p_j}(\overline{B_S^c})^3$  for  $1 \leq j \leq n_0+1$ ,  $u^{(n_0+2)} \in C^\infty(\overline{B_S^c})^3$  and  $u^{(n_0+2)}|_{B_R \setminus \overline{B_S^c}} \in W^{2,q}(B_R \setminus \overline{B_S^c})^3$  for  $R \in (S, \infty)$ ,  $u^{(n_0+2)}|_{\partial B_S} = -v|_{\partial B_S}$ ,*

$$\|\lambda u^{(j)}\|_{p_j} \leq C(p_j, \vartheta, \lambda_0, S) \|f^{(j)}|_{\overline{B_S^c}}\|_{p_j} \text{ for } 1 \leq j \leq n_0.$$

*Define  $\mathfrak{L} := \sum_{\mu=1}^{\mu_0} \|v^{(\mu)}\|_{q^{(\mu)}} + \|\nabla v\|_{q_1}$ . Then  $\|\lambda u^{(n_0+1)}\|_{q_1} \leq C(q_1, \vartheta, \lambda_0, S, \tau) \mathfrak{L}$  and  $\|\phi\|_q \leq C(q, \vartheta, \lambda_0, S) \mathfrak{L}$ . For  $R \in (S, \infty)$  and  $r \in (1, \infty)$ , the inequality  $\|\lambda(u^{(n_0+2)} - \mathfrak{F}(\phi))|_{B_R^c}\|_r \leq C(q, r, \vartheta, \lambda_0, R, S) \mathfrak{L}$  holds, and if  $r > 3/2$ , then  $\|\mathfrak{F}(\phi)|_{B_R^c}\|_r \leq C(q, r, \vartheta, \lambda_0, R, S) \mathfrak{L}$ .*

*If  $\int_{\partial A} v \cdot n^{(A)} do_x = 0$ , then  $\int_{\partial B_R} \mathfrak{F}(\phi)(y) \cdot R^{-1} y do_y = 0$  for  $R \in (S, \infty)$ , where  $n^{(A)}$  denotes the outward unit normal to  $A$ .*

**Proof:** By Theorem 7.1 and because  $q \leq p_j$  for  $1 \leq j \leq n_0$ , we know that  $u \in W_{loc}^{2,q}(\overline{A^c})^3$ . Thus  $v|_{B_R \setminus \overline{B_S^c}} \in W^{2,q}(B_R \setminus \overline{B_S^c})^3$  for  $R \in (S, \infty)$ . Obviously  $v|_{\overline{B_S^c}}$  satisfies (7.1) with  $A$  replaced by  $\overline{B_S^c}$ . By a trace theorem and because  $q \leq q^{(\mu)}$  ( $1 \leq \mu \leq \mu_0$ ) and  $q \leq q_1$ , we have  $\|v|_{\partial B_S}\|_q \leq C(S, q) \|v|_{B_{2S} \setminus \overline{B_S^c}}\|_{1,q} \leq \mathfrak{L}$ . Therefore the estimates stated in the corollary follow from those in Theorem 7.2 and, as concerns the inequality  $\|\phi\|_q \leq C(q, \vartheta, \lambda_0, S) \mathfrak{L}$ , from Corollary 5.1 with  $\Omega = B_S$ .

Suppose that  $\int_{\partial A} v \cdot n^{(A)} do_x = 0$ . Since  $q \leq \min(\{q^{(\mu)} : 1 \leq \mu \leq \mu_0\} \cup \{q_1\})$ , we have  $v|_{B_S \setminus \overline{A}} \in W^{1,q}(B_S \setminus \overline{A})^3$ . In addition  $\operatorname{div} v = 0$ , so we get that  $\int_{\partial B_S} v(y) \cdot S^{-1} y do_y = \int_{\partial A} v \cdot n^{(A)} do_x = 0$ . Therefore  $\int_{\partial B_R} \mathfrak{F}(\phi)(y) \cdot R^{-1} y do_x = 0$  by Lemma 6.2 with  $\Omega = B_S$ ,  $b = -v|_{\partial B_S}$ .  $\square$

## Appendix

**Proof of Theorem 6.2:** Take  $\phi$  and  $\lambda$  as in the theorem. The functions  $\mathring{W}^{(\lambda)}(\phi)$  and  $\tilde{J}^{(\lambda)}(\phi)$  are defined in an analogous way, with the role of the function  $\tilde{E}_{jk}^{(\lambda)}$  in the definition of  $\tilde{J}^{(\lambda)}(\phi)$  played by  $\mathring{E}_{jk}^{(\lambda)}$  in the definition of  $\mathring{W}^{(\lambda)}(\phi)$  (Lemma 5.2). Moreover the proof given in [11, Section 5] for the estimate  $\|\tilde{J}^{(\lambda)}(\phi)|\overline{\Omega}^c\|_p \leq \mathfrak{C}(\vartheta, p) |\lambda|^{-1/(2p)} \|\phi\|_p$  (Theorem 5.4) is based exclusively on the inequality  $|\partial_l \tilde{E}_{jk}^{(\lambda)}(z)| \leq C(\vartheta) \min\{|z|^{-2}, |\lambda|^{-1} |z|^{-4}\}$  ( $z \in \mathbb{R}^3$ ,  $1 \leq j, k, l \leq 3$ ). (Lemma 5.1). On the other hand, also by Lemma 5.1, we have  $|\partial_n \partial_m \partial_l \mathring{E}_{jk}^{(\lambda)}(z)| \leq C(\vartheta) |\lambda| \min\{|z|^{-2}, |\lambda|^{-1} |z|^{-4}\}$  for  $z, j, k, l$  as before, and  $m, n$  also in  $\{1, 2, 3\}$ . Thus the same proof as that of Theorem 5.4 yields  $\|\partial_n \partial_m \mathring{W}^{(\lambda)}(\phi)|\overline{\Omega}^c\|_p \leq \mathfrak{C}(\vartheta, p) |\lambda|^{1-1/(2p)} \|\phi\|_p$ . We fix a constant  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ . Due to the inequality  $|\partial_l \mathring{E}_{jk}^{(\lambda)}(z)| \leq C(\vartheta) |\lambda|^{1-1/(2p)} |z|^{-2+1/p}$  (Lemma 5.1), we may use Lemma 3.2 to obtain  $\|\mathring{W}^{(\lambda)}(\phi)|\Omega_R\|_p \leq \mathfrak{C} |\lambda|^{1-1/(2p)} \|\phi\|_p$ . Altogether we obtain that  $\|\mathring{W}^{(\lambda)}(\phi)|\Omega_R\|_{2,p} \leq \mathfrak{C} |\lambda|^{1-1/(2p)} \|\phi\|_p$ .

Let  $\mathring{W}_{tr}^{(\lambda)}(\phi)$  denote the restriction to  $\partial\Omega$  of the trace of  $\mathring{W}^{(\lambda)}(\phi)|\Omega_R$ . Then it follows that  $\mathring{W}_{tr}^{(\lambda)}(\phi) \in W^{2-1/p, p}(\partial\Omega)^3$  and  $\|\mathring{W}_{tr}^{(\lambda)}(\phi)\|_{2-1/p, p} \leq \mathfrak{C} |\lambda|^{1-1/(2p)} \|\phi\|_p$ . Recall that  $\phi$  was arbitrarily chosen from  $L^p(\partial\Omega)^3$ .

The proof is completed once we have shown that  $\mathring{W}_{tr}^{(\lambda)}(\phi) = (-1/2) (\phi + \tilde{\mathfrak{T}}(\phi))$  for such  $\phi$ . In fact, again take such a function  $\phi$ , and let  $(\phi_n)$  be a sequence in  $C^0(\partial\Omega)^3$  with  $\|\phi - \phi_n\|_p \rightarrow 0$  (Lemma 2.2). Then  $\|\phi - \phi_n + \tilde{\mathfrak{T}}(\phi - \phi_n)\|_p \rightarrow 0$  by Theorem 5.1 and  $\|\mathring{W}_{tr}^{(\lambda)}(\phi - \phi_n)\|_{2-1/p, p} \rightarrow 0$  by what was shown above. But  $\mathring{W}_{tr}^{(\lambda)}(\phi_n)|\overline{\Omega}^c$  may be continuously extended to  $\Omega^c$  with boundary value  $(-1/2) (\phi_n + \tilde{\mathfrak{T}}(\phi_n))$  on  $\partial\Omega$  (Theorem 5.3), so  $\mathring{W}_{tr}^{(\lambda)}(\phi_n) = (-1/2) (\phi_n + \tilde{\mathfrak{T}}(\phi_n))$ , for  $n \in \mathbb{N}$ . By the convergence results just mentioned, the preceding equation remains valid if  $\phi_n$  is replaced by  $\phi$ .  $\square$

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