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Abstract. We provide with criteria for a family of sequences of operators to share a frequently universal vector. These criteria are variants of the classical Frequent Hypercyclicity Criterion and of a recent criterion due to Grivaux, Matheron and Menet where periodic points play the central role. As an application, we obtain for any operator T in a specific class of operators acting on a separable Banach space, a necessary and sufficient condition on a subset Λ of the complex plane for the family $\{\lambda T : \lambda \in \Lambda\}$ to have a common frequently hypercyclic vector. In passing, this allows us to exhibit frequently hypercyclic weighted shifts which do not possess common frequently hypercyclic vectors. We also provide with criteria for families of the recently introduced operators of C-type to share a common frequently hypercyclic vector. Further, we prove that the same problem of common α-frequent hypercyclicity may be vacuous, where the notion of α-frequent hypercyclicity extends that of frequent hypercyclicity replacing the natural density by more general weighted densities. Finally, it is already known that any operator satisfying the classical Frequent Universality Criterion is α-frequently universal for any sequence α satisfying a suitable condition. We complement this result by showing that for any such operator, there exists a vector x which is α-frequently universal for T, with respect to all such densities α.

1. Introduction

For two separable Fréchet spaces X and Y, let us denote by $\mathcal{L}(X,Y)$ the set of all continuous operators from X to Y. If $X = Y$, we simply write $\mathcal{L}(X) = \mathcal{L}(X,Y)$. A sequence $\mathcal{T} = (T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X,Y)$, where $\mathbb{N}$ stands for the set $\{0,1,2,\ldots\}$, is said to be *universal* provided there exists a vector $x \in X$ such that for any non-empty open subset U of Y, the set

$$N(x,U,\mathcal{T}) := \{n \in \mathbb{N} : T_n(x) \in U\}$$

is infinite. The vector x is also called *universal* and the set of all universal vectors for $\mathcal{T}$ is denoted by $\mathcal{U}(\mathcal{T})$. A single operator $T \in \mathcal{L}(X)$ is called *hypercyclic* if the sequence $(T^n)_{n\in\mathbb{N}}$ of its iterates is universal. In this case, we write $N(x,U,\mathcal{T}) = N(x,U,T)$ and $\mathcal{U}(\mathcal{T}) = HC(T)$. In 2006, Bayart and Grivaux [4] introduced the important notion of *frequently hypercyclic operator*. An operator $T \in \mathcal{L}(X)$ is said to be *frequently hypercyclic* if there exists $x \in X$ such that for any non-empty open subset U of X, the *lower density* $\underline{d}(N(x,U,T))$ is positive, where for any $E \subset \mathbb{N}$,

$$\underline{d}(E) := \liminf_{n\to\infty} \frac{\mathrm{card}([0,n] \cap E)}{n+1} > 0.$$

Such a vector x is said to be *frequently hypercyclic vector* for T and the set of such vectors is denoted by $FHC(T)$. The notion of *frequent universality* for a sequence $\mathcal{T}$ of operators in $\mathcal{L}(X,Y)$ can be similarly defined (see, for *e.g.*, [10]) and we denote by $\mathcal{FU}(\mathcal{T})$ the set of universal vectors. For a rich source of information about linear dynamics, we refer to the monographs [7, 24].

A problem which has been extensively studied during the last decades is that of *common hypercyclicity*. For a given family $(T_\lambda)_{\lambda\in\Lambda}$ of hypercyclic operators in $\mathcal{L}(X)$, it asks when the

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set of *common hypercyclic vectors*, $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$, is empty and when it is not. Chapter 7 of [7] and Chapter 11 of [24] are entirely devoted to this topic. On the one hand, since $HC(T)$ is a dense G_δ subset of X whenever it is non-empty, the Baire Category Theorem ensures that $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$ is non-empty whenever Λ is a countable non-empty set and each T_λ is hypercyclic. On the other hand, it is not difficult to exhibit families of hypercyclic operators with no common hypercyclic vectors (for example the family of all hypercyclic weighted shifts on $\ell^2(\mathbb{N})$, see [7, Example 7.1]). The first positive important result in this direction was given by Abakumov and Gordon [1] who showed that $\bigcap_{\lambda>1} HC(\lambda B) \neq \emptyset$, where B is the backward shift on $\ell^2(\mathbb{N})$ defined by $B(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$. Later on, Costakis and Sambarino [17] provided with the first *criterion of common hypercyclicity* that they applied to show the residuality of the set of common hypercyclic vectors for multiples of the backward shift or differential operators, and for uncountable families of translation operators or specific weighted shifts. Constructions or the approach used by Costakis and Sambarino, based on the Baire Category Theorem, were developed by many authors to produce new criteria or prove common hypercyclicity for other uncountable families of classical operators, such as adjoint of multiplication operators, or composition and convolution operators (see, for *e.g.*, [2, 5, 6, 12, 15, 22]). A second approach to the problem, more algebraic, produced some of the most striking results. This approach is based on an interpretation in terms of common hypercyclicity of a result by León and Müller which asserts that for any $T \in \mathcal{L}(X)$ and any $\lambda \in \mathbb{C}$, $|\lambda| = 1$, $HC(T) = HC(\lambda T)$ [25]. Their idea, which exploits the group structure of the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, was extended by several authors to families of operators forming groups or semigroups, and then combined with the first approach to produce some new and strong results (for *e.g.*, [3, 9, 14, 31, 33]). We should say that the non-existence of common universal vectors has also been studied (see, for *e.g.*, [3, 7, 18, 24]).

In comparison, the existence of *common frequently hypercyclic vectors* for families of operators has been considered in only a very few amount of papers. Probably, it is partly because the Baire Category approach drastically fails for this notion: by [8, Corollary 19], the set $FHC(T)$ is always meager (*i.e.*, contained in the complement of a residual set). Moreover, it turns out that the frequently hypercyclic multiples λB, $\lambda \in \Lambda \subset (0, +\infty)$, of the backward shift on $\ell^2(\mathbb{N})$ have no common frequently hypercyclic vectors as soon as Λ is uncountable [4, Theorem 4.5]. This result has a straightforward extension to any $T \in \mathcal{L}(X)$ instead of B [3, Proposition 6.4]. However, the algebraic approach to common hypercyclicity perfectly fits to frequent hypercyclicity. For example, Bayart and Matheron proved that $FHC(\lambda T) = FHC(T)$ for any $\lambda \in \mathbb{T}$, obtaining a *frequent hypercyclicity version* of León-Müller's result [7, Theorem 6.28]. This approach has been pursued further in [3] (see also [14]) and led to several nice results of common frequent hypercyclicity for families of operators forming strongly continuous groups or semigroups (translation operators on $H(\mathbb{C}^d)$, composition operators induced by non-constant Heisenberg translations on the Hardy space of the Siegel half-space, etc...). Furthermore, in specific classes of operators, hypercyclic basically means frequently hypercyclic in a strong sense. For example, if Λ_{hyp} denotes the set of all hyperbolic automorphisms of the unit disc $\mathbb{D}$ having the same boundary attractive point, then the same argument as in [7, Example 7.3] gives that there exists $\phi_0 \in \Lambda_{hyp}$ such that for any $\phi \in \Lambda_{hyp}$, $FHC(C_{\phi_0}) \subset FHC(C_\phi)$, where C_ϕ denotes the composition operator with symbol ϕ on the Hardy space $H^2(\mathbb{D})$. Combined with the algebraic approach, this yields $\bigcap_{\phi\in\Lambda} FHC(C_\phi) \neq \emptyset$ where Λ stands for the set of all automorphisms having a common boundary attractive point. All in all, except when actions by strongly continuous groups or semigroups are involved, so far no criteria for common frequent hypercyclicity are known. In particular, we do not know under which non-trivial conditions on $\Lambda \subset (0, +\infty)$ and $T \in \mathcal{L}(X)$ the set $\bigcap_{\lambda\in\Lambda} FHC(\lambda T)$ may be non-empty.

In this paper we aim to contribute in filling these gaps. Our first result is a criterion of common frequent universality (Theorem 2.2) which is a natural strengthening of the *Frequent Universality Criterion* given in [11] (and of the classical Frequent Hypercyclicity Criterion [4]).

Theorem. *Let X be a Fréchet space, Y a separable Fréchet space and let $\mathcal{T}_i = (T_{i,n})_{n\in\mathbb{N}}$, $i \in \mathbb{N}$, be countably many sequences of continuous linear operators from X to Y. We assume that there exist a dense subset Y_0 of Y, mappings $S_{i,n} : Y_0 \to X$, $i, n \in \mathbb{N}$, and a real number $c > 1$ such that for every $y \in Y_0$,*

(1) the series $\sum_{0\leq n\leq m} T_{i,m}(S_{i,m-n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \in \mathbb{N}$;
(2) the series $\sum_{n\geq 0} T_{i,m}(S_{i,m+n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \in \mathbb{N}$;
(3) the series $\sum_{n\geq (c-1)m} T_{i,m}(S_{j,m+n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \neq j \in \mathbb{N}$;
(4) the series $\sum_{\frac{c-1}{c}m\leq n\leq m} T_{i,m}(S_{j,m-n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \neq j \in \mathbb{N}$;
(5) the series $\sum_{n\geq 0} S_{i,n}(y)$ converges unconditionally, uniformly for $i \in \mathbb{N}$;
(6) the sequence $(T_{i,n}(S_{i,n}(y)))_{n\in\mathbb{N}}$ converges to y, uniformly for $i \in \mathbb{N}$.

Then there exists a vector $x \in X$ which is frequently universal for every $\mathcal{T}_i$, $i \in \mathbb{N}$.

As an application, we get necessary or sufficient conditions on a subset Λ of $\mathbb{C}$ for the set $\bigcap_{\lambda\in\Lambda} FHC(\lambda B)$ to be non-empty, when X is a Banach space and $T \in \mathcal{L}(X)$. For example, we will get the following:

Theorem. *Let B be the backward shift on $\ell^2(\mathbb{N})$ and let $\Lambda \subset \mathbb{C}$. The set $\bigcap_{\lambda\in\Lambda} FHC(\lambda B)$ is non-empty if and only if the set $\{|\lambda| : \lambda \in \Lambda\}$ is a countable relatively compact non-empty subset of $(1, +\infty)$.*

This theorem is obtained for more general classes of (unilateral) weighted shifts on $\ell^2(\mathbb{N})$. For any operator $T \in \mathcal{L}(X)$, sufficient or necessary conditions on Λ are given, involving, for *e.g.*, the spectral radius of T. In full generality, our sufficient condition coincides with the assumption of a criterion of *common hypercyclicity* given by Bayart and Matheron [6, Proposition 4.2]. Our general criterion of common frequent universality is also applied to countable families of weighted shifts, differential operators or adjoint of multiplication operators (which may not be multiples of a single operator). In passing, we easily produce two frequently hypercyclic weighted shifts *without* common frequently hypercyclic vectors.

Recently, Grivaux, Matheron and Menet provided with a new frequent hypercyclicity criterion, based on the periodic points of the operator [23]. They proved that this criterion is theoretically better than the classical Frequent Hypercyclicity Criterion since any operator satisfying the assumptions of the latter automatically satisfies that of the new one. In practice, the classical criterion turns out to be much simpler to apply to most of the explicit and usual operators. However, Menet introduced a new class of operators, the so-called operators of C-type [26], conceived as a very rich source of counter-examples to difficult problems, such as the exhibition of a chaotic operator on $\ell^p(\mathbb{N})$ which is not frequently hypercyclic [26] (see also [23]), to which their new criterion for frequent hypercyclicity is very well adapted. In the present paper, based on this criterion, we establish another general criterion for common frequent hypercyclicity, involving the periodic points of the operators of the family. We show how this can be applied to classes of operators of C-type.

Furthermore, Ernst and Mouze recently proved [19, 20] that any operator satisfying the usual Frequent Universality Criterion in fact enjoys a stronger form of frequent universality

related to *generalized lower density* $\underline{d}_\alpha$ where α is a sequence of non-negative real numbers satisfying suitable conditions. The usual lower density corresponds to any constant sequence $(a, a, a, \ldots)$, $a > 0$. Moreover, if $\alpha \lesssim \beta$ (meaning α_k/β_k is eventually decreasing to 0), then $\underline{d}_\beta(E) \leq \underline{d}_\alpha(E)$ for every $E \subset \mathbb{N}$. The relation $\lesssim$ thus allows oneself to define scales of generalized lower densities. It then appears natural to define α-frequent universality as the usual frequent universality, replacing the sequence $(1, 1, 1, \ldots)$ by any α as above. One of the main results of [19, 20] tells us that any operator $T \in \mathcal{L}(X)$ which satisfies the Frequent Universality Criterion is α-frequently universal whenever there exists $s \geq 2$ such that $\alpha \lesssim \mathcal{D}^s$, where $\mathcal{D}^s := (\exp(k/(\log_{(s)}(k))))_{k \geq k_0}$ for some $k_0 \geq 1$ depending on s, and $\log_{(s)} = \log \circ \log \circ \ldots \circ \log$, log appearing s times. Moreover, they proved that no operator can be α-frequently hypercyclic for $\alpha = (e^k)_{k \geq 1}$. In view of the topic of the paper, two natural questions arise. For $T \in \mathcal{L}(X)$ we denote by $FHC_\alpha(T)$ the set of all α-frequently hypercyclic vectors for T.

Questions. *Let A denote the set of sequences α such that $\alpha \lesssim \mathcal{D}^s$ for some $s \geq 2$ and let $T \in \mathcal{L}(X)$.*

(1) Let $\Lambda \subset (0, +\infty)$ and $B \subset A$ be non-trivial. Do we have $\bigcap_{(\lambda,\beta) \in \Lambda \times B} FHC_\beta(\lambda T) \neq \emptyset$?
(2) If T satisfies the Frequent Hypercyclicity Criterion, do we have $\bigcap_{\alpha \in A} FHC_\alpha(T) \neq \emptyset$?

We will give a positive answer to the second question (Proposition 4.8) and show that the first one has a strongly negative answer if Λ is any non-trivial subset of $(0, +\infty)$ and B is reduced to a single generalized density which grows faster than $(e^{\log(k)\log_{(s)}(k)})_{k \geq k_0}$ for some positive integer s (Proposition 4.2). We should mention that, by [19, Lemma 2.10], $FHC_\beta(T) = FHC(T)$ whenever β has a growth at most polynomial (*i.e.*, $\beta \lesssim (k^r)_{k \geq 1}$ for some $r > -1$). Combined with our first common frequent hypercyclicity criterion, this gives a positive answer to (1) for some non-trivial Λ and the set B of sequences with at most polynomial growth.

We should conclude by mentioning that the problem of common hypercyclicity has also been considered for $\mathcal{U}$-frequent hypercyclicity. This intermediate notion between hypercyclicity and frequent hypercyclicity was introduced by Shkarin [32]. A sequence $\mathcal{T} \subset \mathcal{L}(X, Y)$ is said to be $\mathcal{U}$-frequently universal if for some $x \in X$ and any non-empty open set U in Y, $\overline{d}(N(x, U, \mathcal{T}))$ is positive. By definition, $\overline{d}(E) = 1 - \underline{d}(\mathbb{N} \setminus E)$ is the *upper density* of $E \subset \mathbb{N}$. In some sense, $\mathcal{U}$-frequent hypercyclicity is *closer* to hypercyclicity than to frequent hypercyclicity. For example, Bayart and Ruzsa proved that the set $\mathcal{U}FHC(T)$ of all $\mathcal{U}$-frequently hypercyclic vectors for T is residual whenever it is non-empty [8, Proposition 21]. However, they also proved that any $\mathcal{U}$-frequently hypercyclic weighted shift on $\ell^p(\mathbb{N})$, $1 \leq p < +\infty$, is frequently hypercyclic. Common $\mathcal{U}$-frequent hypercyclicity has been rather well-studied and criteria have been given. We refer to [27, 28] and the references therein for a complete and up-to-date overview on the subject.

The paper is organized as follows. Section 2 is devoted to our first general criteria of common frequent universality and their developments in various directions. In Section 3, we focus on our criterion for common frequent hypercyclicity involving periodic points. We finally give answers to the above two questions in Section 4.

2. Common frequent universality for countable families of operators

2.1. **A general criterion.** The main result of this section is a general common frequent universality criterion. We state it for F-spaces, *i.e.*, for complete and metrizable topological vector spaces. In this setting, the notation $\|\cdot\|$ will stand for any F-norm defining the topology of the F-space. By definition, a Fréchet space is a locally convex F-space (see for *e.g.*, [29]).

We recall the definition of uniform unconditional convergence.

Definition 2.1. Let Λ be a set. We say that the series $\sum x_{\lambda,n}$, $\lambda \in \Lambda$ in X converges unconditionally uniformly for $\lambda \in \Lambda$ if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for any finite set $F \subset \{N, N+1, \ldots\}$, one has

$$\left\| \sum_{n \in F} x_{\lambda,n} \right\| < \varepsilon$$

for every $\lambda \in \Lambda$.

Our general criterion of common frequent universality for countable families of operators states as follows.

Theorem 2.2. *Let X be a F-space, Y a separable F-space and $\mathcal{T}_i = (T_{i,n})_{n\in\mathbb{N}}$, $i \in \mathbb{N}$, be countably many sequences of continuous linear operators from X to Y. We assume that there exist a dense subset Y_0 of Y, mappings $S_{i,n} : Y_0 \to X$, $i, n \in \mathbb{N}$, and a real number $c > 1$ such that for every $y \in Y_0$,*

(1) the series $\sum_{0\leq n\leq m} T_{i,m}(S_{i,m-n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \in \mathbb{N}$;
(2) the series $\sum_{n\geq 0} T_{i,m}(S_{i,m+n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \in \mathbb{N}$;
(3) the series $\sum_{n\geq (c-1)m} T_{i,m}(S_{j,m+n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \neq j \in \mathbb{N}$;
(4) the series $\sum_{\frac{c-1}{c}m\leq n\leq m} T_{i,m}(S_{j,m-n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \neq j \in \mathbb{N}$;
(5) the series $\sum_{n\geq 0} S_{i,n}(y)$ converges unconditionally, uniformly for $i \in \mathbb{N}$;
(6) the sequence $(T_{i,n}(S_{i,n}(y)))_{n\in\mathbb{N}}$ converges to y, uniformly for $i \in \mathbb{N}$.

Then there exists a vector $x \in X$ which is frequently universal for every $\mathcal{T}_i$, $i \in \mathbb{N}$.

One may check that each $\mathcal{T}_i$, $i \in \mathbb{N}$, satisfies (1), (2), (5) and (6) if and only if it satisfies the Frequent Universality Criterion given in [11].

For the proof, we will make use of the following refinement of [7, Lemma 6.19] and of ideas developed in [8].

Lemma 2.3. *For every $K > 1$ and every countable family $(N_p(i))_{p\in\mathbb{N}}$, $i \in \mathbb{N}$, of increasing sequences of positive integers, there exists a countable family $(E_p(i))_{p\in\mathbb{N}}$, $i \in \mathbb{N}$, of sequences of subsets of $\mathbb{N}$ with positive lower density, such that for every $(p, i), (q, j) \in \mathbb{N}^2$ and every $(n, m) \in E_p(i) \times E_q(j)$,*

(1) $\min(E_p(i)) \geq N_p(i)$;
(2) if $n \neq m$, then $|n - m| \geq \max(N_p(i), N_q(j))$;
(3) if $(p, i) \neq (q, j)$ and $n > m$, then $n \geq Km$.

Proof. Let $K > 1$ and for every $i \in \mathbb{N}$, let $(N_p(i))_{p\in\mathbb{N}}$ be an increasing sequence of positive integers. For every $i \in \mathbb{N}$, let us denote by $(A_p(i))_{p\in\mathbb{N}}$ a sequence of pairwise disjoint syndetic subsets of $\mathbb{N}$. We recall that a subset $A \subset \mathbb{N}$ is *syndetic* if it has bounded gaps, that is

$$\mathbb{N} = \bigcup_{n\in F} A - n$$

for some finite set $F \subset \mathbb{N}$. We fix two real numbers $0 < \varepsilon < 1/2$ and $a > 1$ satisfying

$$\frac{1 - 2\varepsilon}{1 + 2\varepsilon} a > K. \tag{2.1}$$

For $0 < \eta < 1$, let us set $I_u^\eta = [(1-\eta)a^u, (1+\eta)a^u]$, $u \in \mathbb{N}$. Then, for every $(p,i) \in \mathbb{N}^2$, we define

$$E_p(i) = \bigcup_{u \in A_p(i)} \left(I_u^\varepsilon \cap (N_p(i)\mathbb{N})\right).$$

By definition, (1) clearly holds and (2) is satisfied whenever $(p,i) = (q,j)$. To prove that (2) also holds for any $(p,i) \neq (q,j)$, we first remark that for every $u \in A_p(i)$, the inclusion

$$I_u^\varepsilon + [-N_p(i), N_p(i)] \subseteq I_u^{2\varepsilon}$$

is equivalent to $N_p(i) \leq \varepsilon a^u$. From now on we assume, up to removing finitely many elements from each set $A_p(i)$, that the previous inclusion holds for any $(p,i) \in \mathbb{N}^2$ and any $u \in A_p(i)$.

Now, we observe that $I_u^{2\varepsilon} \cap I_v^{2\varepsilon} = \emptyset$ for every $u \neq v$. Indeed, one may check that if $u > v$, $I_u^{2\varepsilon}$ and $I_v^{2\varepsilon}$ are disjoint if and only if

$$\frac{1-2\varepsilon}{1+2\varepsilon} a^{u-v} > 1$$

which holds by (2.1). Altogether we deduce that the sets $E_p(i)$, $p, i \in \mathbb{N}$, are disjoint and that (2) is satisfied.

To check that (3) also holds, observe first that (2.1) implies that $K(1+\varepsilon) < (1-\varepsilon)a$. Let $(p,i),(q,j) \in \mathbb{N}^2$, $n \in E_p(i)$ and $m \in E_q(j)$ such that $n > m$. Then, there exist $u \in A_p(i)$ and $v \in A_q(j)$ with $u > v$ so that:

$$\begin{cases} (1-\varepsilon)a^u \leq n \leq (1+\varepsilon)a^u \\ (1-\varepsilon)a^v \leq m \leq (1+\varepsilon)a^v. \end{cases}$$

Thus, we have

$$Km \leq K(1+\varepsilon)a^v < (1-\varepsilon)a^{v+1} \leq (1-\varepsilon)a^u \leq n.$$

This proves (3).

Finally, it remains to prove that each set $E_p(i)$ has positive lower density. Let $p, i \in \mathbb{N}$ and $(u_k)_{k\in\mathbb{N}}$ be an enumeration of the set $A_p(i)$ and let M be the maximal size of a gap in $A_p(i)$. Then,

$$\begin{aligned} \underline{d}(E_p(i)) &\geq \liminf_{k\to\infty} \frac{\operatorname{card}(E_p(i) \cap [0, \lceil (1+\varepsilon)a^{u_k} \rceil])}{\lceil (1+\varepsilon)a^{u_{k+1}} \rceil} \\ &\geq \liminf_{k\to\infty} \left(\frac{2\varepsilon a^{u_k}}{2N_p(i)} - 2 \right) \frac{1}{a^{u_{k+1}} + 1} \\ &\geq \liminf_{k\to\infty} \left(\frac{\varepsilon a^{u_k}}{N_p(i)} - 2 \right) \frac{1}{a^{u_k+M} + 1} \\ &= \frac{\varepsilon}{N_p(i)a^M + 1} > 0. \end{aligned}$$

This ends the proof of the lemma. □

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Within the proof, the notation $\|\cdot\|$ will be indifferently used to denote an F-norm defining the topologies of X or Y. Since Y is separable, we can assume that $Y_0 = \{y_0, y_1, \ldots\}$. Let $(\varepsilon_p)_{p\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that $\sum_{p\geq 0} \varepsilon_p < 1$ and $p\varepsilon_p \to 0$ as $p \to \infty$. We also fix an increasing sequence $(J_p)_{p\in\mathbb{N}}$ such that $\sum_{i\geq J_p} \varepsilon_i < \varepsilon_p$. The assumptions of the theorem imply the existence of a sequence $(N_p(i))_{i,p\in\mathbb{N}}$, increasing with respect to $p \in \mathbb{N}$ such that for every $i, p \in \mathbb{N}$, every finite set $F \subset \{N_p(i), N_p(i)+1, \ldots\}$, every $m \in \mathbb{N}$, every $q \in \{0, \ldots, p\}$, every $k \in \mathbb{N}$, every $l \neq k \in \mathbb{N}$, and every $N \geq N_p(i)$,

(i) $\Big\| \sum\limits_{\substack{n\in F\\ n<m}} T_{k,m}(S_{k,m-n}(y_q))\Big\| < \varepsilon_p$;

(ii) $\Big\| \sum\limits_{n\in F} T_{k,m}(S_{k,m+n}(y_q))\Big\| < \varepsilon_p$;

(iii) $\Big\| \sum\limits_{\substack{n\in F\\ n\geq (c-1)m}} T_{k,m}(S_{l,m+n}(y_q))\Big\| < \varepsilon_p\varepsilon_i$;

(iv) $\Big\| \sum\limits_{\substack{n\in F\\ n\geq (c-1)m}} T_{k,m}(S_{l,m+n}(y_q))\Big\| < \varepsilon_{J_p}\varepsilon_p$;

(v) $\Big\| \sum\limits_{\substack{n\in F\\ \frac{c-1}{c}m\leq n\leq m}} T_{k,m}(S_{l,m-n}(y_q))\Big\| < \varepsilon_p\varepsilon_i$;

(vi) $\Big\| \sum\limits_{\substack{n\in F\\ \frac{c-1}{c}m\leq n\leq m}} T_{k,m}(S_{l,m-n}(y_q))\Big\| < \varepsilon_{J_p}\varepsilon_p$;

(vii) $\Big\| \sum\limits_{n\in F} S_{k,n}(y_q)\Big\| < \varepsilon_p\varepsilon_i$;

(viii) $\big\|T_{k,N}(S_{k,N}(y_p)) - y_p\big\| < \varepsilon_p$.

Let $(E_p(i))_{i,p\in\mathbb{N}}$ be a sequence of sets given by Lemma 2.3 applied to the sequence $(N_p(i))_{i,p\in\mathbb{N}}$ and to $K = c$. We put

$$x = \sum_{i\in\mathbb{N}}\sum_{p\in\mathbb{N}}\sum_{n\in E_p(i)} S_{i,n}(y_p).$$

One easily checks that $x \in X$. Indeed, since for every $p, i \in \mathbb{N}$, $\min(E_p(i)) \geq N_p(i)$, (vii) gives

$$\sum_{i\in\mathbb{N}}\sum_{p\in\mathbb{N}}\Big\| \sum_{n\in E_p(i)} S_{i,n}(y_p)\Big\| < \sum_{i\in\mathbb{N}}\sum_{p\in\mathbb{N}} \varepsilon_p\varepsilon_i < \infty.$$

Note that x is even unconditionally convergent. Our goal is now to prove that x is a frequently universal vector for each sequence $(T_{i,n})_{n\in\mathbb{N}}$, $i \in \mathbb{N}$. We fix $j \in \mathbb{N}$. Let $(r_q)_{q\in\mathbb{N}}$ be a sequence of positive real numbers with $r_q \to 0$ as $q \to \infty$, to be chosen later. Since the sets $E_p(i)$, $i, p \in \mathbb{N}$, have positive lower density, it is sufficient to prove that

$$\|T_{j,m}(x) - y_q\| < r_q \text{ for every } q \in \mathbb{N} \text{ and every } m \in E_q(j). \tag{2.2}$$

Let us then fix $q \in \mathbb{N}$ and $m \in E_q(j)$. Using that $E_p(i) \cap E_q(j) = \emptyset$ if $(i, p) \neq (j, q)$ and that x is unconditionally convergent in X, we can decompose $T_{j,m}(x)$ as follows:

$$T_{j,m}(x) = T_{j,m}(S_{j,m}(y_q)) + \overbrace{\sum_{p\in\mathbb{N}}\sum_{\substack{n\in E_p(j)\\ n\neq m}} T_{j,m}(S_{j,n}(y_p))}^{A_m} + \overbrace{\sum_{\substack{i\in\mathbb{N}\\ i\neq j}}\sum_{p\in\mathbb{N}}\sum_{\substack{n\in E_p(i)\\ n\neq m}} T_{j,m}(S_{i,n}(y_p))}^{B_m}.$$

First, since $m \geq N_q(j)$, (viii) gives

$$\|T_{j,m}(S_{j,m}(y_q)) - y_q\| < \varepsilon_q. \tag{2.3}$$

We next estimate $\|A_m\|$:

$$\|A_m\| \leq \sum_{p\in\mathbb{N}} \left(\Big\| \sum_{\substack{n\in E_p(j)\\ n<m}} T_{j,m}(S_{j,m-(m-n)}(y_p))\Big\| + \Big\| \sum_{\substack{n\in E_p(j)\\ n>m}} T_{j,m}(S_{j,m+(n-m)}(y_p))\Big\| \right).$$

Given that $|n - m| \geq \max(N_p(j), N_q(j))$ for any $n \in E_p(j)$, $n \neq m$, (i) and (ii) yield

$$\|A_m\| < 2\sum_{p<q} \varepsilon_q + 2\sum_{p\geq q} \varepsilon_p =: r_{1,q}. \tag{2.4}$$

We now turn to estimating $\|B_m\|$. Again, by unconditional convergence of the series, we have

$$\|B_m\| \leq \overbrace{\sum_{p\in\mathbb{N}}\sum_{\substack{i\in\mathbb{N}\\ i\neq j}}\Big\| \sum_{\substack{n\in E_p(i)\\ n<m}} T_{j,m}(S_{i,m-(m-n)}(y_p))\Big\|}^{B_m^1} + \overbrace{\sum_{p\in\mathbb{N}}\sum_{\substack{i\in\mathbb{N}\\ i\neq j}}\Big\| \sum_{\substack{n\in E_p(i)\\ n>m}} T_{j,m}(S_{i,m+(n-m)}(y_p))\Big\|}^{B_m^2}.$$

We deal first with B_m^2. We have

$$B_m^2 \leq \sum_{p\geq q}\sum_{\substack{i\in\mathbb{N}\\ i\neq j}}\Big\|\sum_{\substack{n\in E_p(i)\\ n>m}} T_{j,m}(S_{i,m+(n-m)}(y_p))\Big\|$$

$$+\sum_{p<q}\left(\sum_{\substack{i<J_q\\ i\neq j}}\Big\|\sum_{\substack{n\in E_p(i)\\ n>m}} T_{j,m}(S_{i,m+(n-m)}(y_p))\Big\| + \sum_{\substack{i\geq J_q\\ i\neq j}}\Big\|\sum_{\substack{n\in E_p(i)\\ n>m}} T_{j,m}(S_{i,m+(n-m)}(y_p))\Big\|\right). \tag{2.5}$$

We recall that Lemma 2.3 was applied to $K = c$. So, for $n \in E_p(i)$ with $(i,p) \neq (j,q)$, we have $|n-m| \geq \max(N_p(i), N_q(j))$. Moreover, $n > m$ implies $n \geq cm$, hence $n-m \geq (c-1)m$. In particular, $n - m \geq \max(N_p(i), (c-1)m)$. It follows from (iii) that

$$\sum_{p\geq q}\sum_{\substack{i\in\mathbb{N}\\ i\neq j}}\Big\|\sum_{\substack{n\in E_p(i)\\ n>m}} T_{j,m}(S_{i,m+(n-m)}(y_p))\Big\| \leq \sum_{p\geq q}\sum_{i\in\mathbb{N}}\varepsilon_i\varepsilon_p \leq \sum_{p\geq q}\varepsilon_p \tag{2.6}$$

and that

$$\sum_{p<q}\sum_{\substack{i\geq J_q\\ i\neq j}}\Big\|\sum_{\substack{n\in E_p(i)\\ n>m}} T_{j,m}(S_{i,m+(n-m)}(y_p))\Big\| \leq \sum_{p<q}\varepsilon_p\sum_{i\geq J_q}\varepsilon_i \leq q\varepsilon_q. \tag{2.7}$$

In the last inequality, we use that $0 < \varepsilon_q < 1$ and the fact that $\sum_{i\geq J_q}\varepsilon_i \leq \varepsilon_q$. Now, using that $n - m \geq N_q(j)$ for any $n \in E_p(i)$ with $(i,p) \neq (j,q)$, we get from (iv) that

$$\sum_{p<q}\sum_{\substack{i<J_q\\ i\neq j}}\Big\|\sum_{\substack{n\in E_p(i)\\ n>m}} T_{j,m}(S_{i,m+(n-m)}(y_p))\Big\| \leq \sum_{p<q}\sum_{i<J_q}\varepsilon_{J_q}\varepsilon_q \leq q\varepsilon_q J_q\varepsilon_{J_q}. \tag{2.8}$$

Thus, (2.5), (2.6), (2.7) and (2.8) altogether give

$$B_m^2 \leq \sum_{p\geq q}\varepsilon_p + q\varepsilon_q + q\varepsilon_q J_q\varepsilon_{J_q} =: r_{2,q}. \tag{2.9}$$

To finish, we consider B_m^1. We have

$$B_m^1 \leq \sum_{p\geq q}\sum_{\substack{i\in\mathbb{N}\\ i\neq j}}\Big\|\sum_{\substack{n\in E_p(i)\\ n<m}} T_{j,m}(S_{i,m-(m-n)}(y_p))\Big\|$$

$$+\sum_{p<q}\left(\sum_{\substack{i<J_q\\ i\neq j}}\Big\|\sum_{\substack{n\in E_p(i)\\ n<m}} T_{j,m}(S_{i,m-(m-n)}(y_p))\Big\| + \sum_{\substack{i\geq J_q\\ i\neq j}}\Big\|\sum_{\substack{n\in E_p(i)\\ n<m}} T_{j,m}(S_{i,m-(m-n)}(y_p))\Big\|\right). \tag{2.10}$$

For $n \in E_p(i)$ with $(i,p) \neq (j,q)$, we have $|n-m| \geq \max(N_p(i), N_q(j))$. Moreover, $n < m$ gives $n \leq m/c$, hence $\frac{c-1}{c}m \leq m - n \leq m$. So (v) implies

$$\sum_{p\geq q}\sum_{\substack{i\in\mathbb{N}\\ i\neq j}}\Big\|\sum_{\substack{n\in E_p(i)\\ n<m}} T_{j,m}(S_{i,m-(m-n)}(y_p))\Big\| \leq \sum_{p\geq q}\sum_{i\in\mathbb{N}}\varepsilon_i\varepsilon_p \leq \sum_{p\geq q}\varepsilon_p \tag{2.11}$$

and

$$\sum_{p<q}\sum_{\substack{i\geq J_q\\ i\neq j}}\Big\|\sum_{\substack{n\in E_p(i)\\ n<m}} T_{j,m}(S_{i,m-(m-n)}(y_p))\Big\| \leq \sum_{p<q}\varepsilon_p\sum_{i\geq J_q}\varepsilon_i \leq q\varepsilon_q. \tag{2.12}$$

Now, since $m-n \geq N_q(j)$, for $n \in E_p(i)$, $(p,i) \in \mathbb{N}^2$, (vi) yields

$$\sum_{p<q} \sum_{\substack{i<J_q \\ i\neq j}} \left\| \sum_{\substack{n\in E_p(i) \\ n<m}} T_{j,m}(S_{i,m-(m-n)}(y_p)) \right\| \leq \sum_{p<q} \sum_{i<J_q} \varepsilon_{J_q}\varepsilon_q \leq q\varepsilon_q J_q \varepsilon_{J_q}. \tag{2.13}$$

Thus, (2.10), (2.11), (2.12) and (2.13) imply $B_m^1 \leq r_{2,q}$ (see (2.9) for the definition of $r_{2,q}$).

The previous inequality, together with (2.3), (2.4) and (2.9) give (2.2), setting $r_q = \varepsilon_q + r_{1,q} + 2r_{2,q}$ which, by assumption, tends to 0 as $q \to +\infty$. □

It will often happen that in the assumptions of Theorem 2.2, S_i will be self-mappings of X_0 and right inverses of the operators T_i on X_0. It is in particular the case if T satisfies the so-called Frequent Hypercyclicity Criterion. Because we will refer to it several times in the paper, we recall its statement below.

Theorem 2.4 (Frequent Hypercyclicity Criterion (see Theorem 6.18 in [7]))**.** *Let X be a separable F-space and T a continuous linear operator on X. We assume that there exist a dense subset X_0 of X and a mapping $S : X_0 \to X_0$ such that for every $x \in X_0$,*

(1) the series $\sum_{n\geq 0} T^n(x)$ and $\sum_{n\geq 0} S^n(x)$ converge unconditionally;
(2) the equality $T^n(S^n(x)) = x$ holds.

Then T is frequently hypercyclic.

In this context, Theorem 2.2 reads as follows.

Corollary 2.5. *Let X be a separable F-space and let $(T_i)_{i\in\mathbb{N}}$ be countably many continuous linear operators on X. We assume that there exist a dense subset X_0 of X, mappings $S_i : X_0 \to X_0$, $i \in \mathbb{N}$, and a real number $c > 1$ such that for every $x \in X_0$,*

(1) the series $\sum_{n\geq 0} T_i^n(x)$ and $\sum_{n\geq 0} S_i^n(x)$ converge unconditionally, uniformly for $i \in \mathbb{N}$;
(2) the series $\sum_{n\geq (c-1)m} T_i^m(S_j^{m+n}(x))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \neq j \in \mathbb{N}$;
(3) the series $\sum_{\frac{c-1}{c}m\leq n\leq m} T_i^m(S_j^{m-n}(x))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \neq j \in \mathbb{N}$;
(4) the sequence $T_i(S_i(x)) = x$ for every $i \in \mathbb{N}$.

Then there exists a common frequently hypercyclic vector for the family $(T_i)_{i\in\mathbb{N}}$.

Note that Corollary 2.5 coincides with the Frequent Hypercyclicity Criterion when the family $(T_i)_{i\in\mathbb{N}}$ is reduced to a single operator. Moreover, observe that the second part of (1) is a consequence of (2) by taking $m = 0$.

These two results apply to many situations and are sometimes sharp. This is described in the next paragraphs.

2.2. **Common frequent hypercyclicity for multiples of a single operator.** Let us first give necessary conditions for the existence of common frequently hypercyclic vectors for multiples of a given operator.

2.2.1. **Necessary conditions.** In this paragraph, we assume that X is a Banach space. We recall that if T is a bounded linear operator on X and $\Lambda \subset (0,+\infty)$, then for the family $(\lambda T)_{\lambda\in\Lambda}$ to have a common frequently hypercyclic vector, Λ has to be countable [3, Proposition 6.4]. The following proposition shows that Λ must also satisfy two other non-trivial conditions. We will denote by $r(T)$ the spectral radius of T and we recall the spectral radius formula [29]:

$$r(T) = \inf_{n\geq 1} \|T^n\|^{1/n} = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Proposition 2.6. *Let X be a Banach space, T a bounded linear operator on X and $\Lambda \subset (0,+\infty)$ be a set with at least two elements. If Λ is unbounded or $1/r(T) \geq \inf(\Lambda)$, then*

$$\bigcap_{\lambda\in\Lambda} FHC(\lambda T) = \emptyset.$$

Proof. First of all, let us recall that if $\lambda < 1/r(T)$, then λT is not hypercyclic. We only prove the case where $1/r(T) = \inf(\Lambda)$, the case Λ unbounded being treated very similarly. To start with, let us first assume that $1/r(T)$ is an accumulation point of Λ. We fix $\lambda_0 \in \Lambda$. Upon taking a subsequence, we can suppose that $\Lambda = (\lambda_k)_{k\in\mathbb{N}}$ is decreasing to $1/r(T)$. We may and shall also assume that there exists $x \in X$ which is hypercyclic for all $\lambda_k T$, $k \in \mathbb{N}$. We fix $x_0 \in X \setminus \{0\}$ with $\|x_0\| = 1$ and denote by $\mathcal{N}_k$, $k \geq 0$, the sets respectively given by

$$\mathcal{N}_0 := \left\{n \in \mathbb{N} : \|\lambda_0^n T^n(x)\| < 1\right\} \quad \text{and} \quad \mathcal{N}_k := \left\{m \in \mathbb{N} : \|\lambda_k^m T^m(x) - x_0\| < \frac{1}{2}\right\},\ k \geq 1.$$

By assumption, each $\mathcal{N}_k$, $k \geq 0$, is infinite. So there exist two increasing sequences $(m_k)_{k\geq1}$ and $(n_k)_{k\geq1}$ such that for every $k \geq 1$, $m_k \in \mathcal{N}_k$ and

$$n_k = \max\{n < m_k : n \in \mathcal{N}_0\}.$$

Then, from the definition of n_k, $k \geq 1$, we get

$$\underline{d}(\mathcal{N}_0) \leq \limsup_{k\to\infty} \frac{\mathrm{card}(\mathcal{N}_0 \cap \{0,\dots,m_k-1\})}{m_k} \leq \limsup_{k\to\infty} \frac{n_k}{m_k}. \tag{2.14}$$

Now, by construction, we have for any $k \geq 1$,

$$\|T^{n_k}(x)\| < \lambda_0^{-n_k} \quad \text{and} \quad \frac{\lambda_k^{-m_k}}{2} < \|T^{m_k}(x)\| \leq \|T^{m_k-n_k}\|\|T^{n_k}(x)\|.$$

It follows for any $k \geq 1$,

$$\frac{\lambda_0^{n_k}}{\lambda_k^{m_k}} < 2\|T^{m_k-n_k}\|,$$

whence

$$\left(\frac{\lambda_0}{\lambda_k}\right)^{n_k} \leq 2\lambda_k^{m_k-n_k}\|T^{m_k-n_k}\| \leq 2(\lambda_0\|T\|)^{m_k-n_k}. \tag{2.15}$$

Since $(\lambda_k)_{k\in\mathbb{N}}$ is decreasing and $n_k \to +\infty$, we first deduce from the last inequality that $m_k - n_k \to +\infty$. This gives $r(T) = \lim_{k\to\infty} \|T^{m_k-n_k}\|^{1/(m_k-n_k)}$. We also derive from (2.15) the following:

$$\left(\frac{\lambda_0}{\lambda_k}\right)^{n_k/m_k} \leq 2^{1/m_k}\lambda_k^{1-n_k/m_k}\|T^{m_k-n_k}\|^{1/m_k} \text{ for any } k \geq 1,$$

which implies, using that $m_k \to +\infty$ and $m_k - n_k \to +\infty$,

$$\limsup_{k\to\infty} \frac{n_k}{m_k} \leq \frac{1}{\ln(r(T)\lambda_0)} \limsup_{k\to\infty} \left(\ln(\lambda_k\|T^{m_k-n_k}\|^{\frac{1}{m_k-n_k}})\right) = 0,$$

since, by assumption, $(\lambda_k)_{k\in\mathbb{N}}$ is decreasing to $1/r(T)$. This with (2.14) shows that x is not frequently hypercyclic for $\lambda_0 T$ when $1/r(T)$ is an accumulation point of Λ.

Let us deal with the remaining case, *i.e.*, $1/r(T) \in \Lambda$ but $1/r(T)$ is not an accumulation point of Λ. We will in fact prove the stronger fact that, if $1/r(T)$ and λ are distinct and both in Λ, then $r(T)^{-1}T$ and λT share no frequently hypercyclic vectors. The proof goes along the same lines as above. Let us denote $\mu = 1/r(T)$. Let $\lambda \in \Lambda$ such that $\lambda \neq \mu$. By

assumption $\lambda/\mu > 1$. We may and shall assume that some $x \in X$ is hypercyclic for λT and μT and we set, for some vector $x_0 \in X$ with $\|x_0\| = 1$,

$$\mathcal{N}_\lambda := \left\{ n \in \mathbb{N} : \ \|\lambda^n T^n(x)\| < 1 \right\} \quad \text{and} \quad \mathcal{N}_\mu := \left\{ m \in \mathbb{N} : \ \|\mu^m T^m(x) - x_0\| < \frac{1}{2} \right\}, \ k \geq 1.$$

As above, since these sets are infinite, one can define an increasing sequence of integers $(m_k)_{k\in\mathbb{N}} \subset \mathcal{N}_\mu$, tending to $+\infty$, such that the sequence $(n_k)_{k\in\mathbb{N}}$ defined by

$$n_k := \max\{n < m_k : \ n \in \mathcal{N}_\lambda\}$$

is increasing. We have $\underline{d}(\mathcal{N}_\lambda) \leq \limsup_{k\to\infty} \frac{n_k}{m_k}$ and, proceeding exactly as in the first part of the proof, $m_k - n_k \to +\infty$ and

$$\left(\frac{\lambda}{\mu}\right)^{n_k} \leq 2\mu^{m_k - n_k} \|T^{m_k - n_k}\|, \quad k \in \mathbb{N}.$$

Therefore

$$\underline{d}(\mathcal{N}_\lambda) \leq \limsup_{k\to\infty} \frac{n_k}{m_k} \leq \frac{1}{\ln(r(T)\lambda)} \limsup_{k\to\infty} \left(\ln(\mu \|T^{m_k - n_k}\|^{\frac{1}{m_k - n_k}}) \right) = 0,$$

so x is not frequently hypercyclic for λT. □

Let us make two remarks.

Remark 2.7. The proof of the previous proposition tells us a bit more than its statement. More precisely, we have shown that, if Λ is unbounded or if $1/r(T) \in \overline{\Lambda}$, and if $x \in X$ is a common hypercyclic vector for all λT, then it is not *frequently* hypercyclic for any λT, $\lambda \neq 1/r(T)$. If, for *e.g.*, T is the backward shift B on $\ell^2(\mathbb{N})$, this is not difficult to see that the set $\bigcap_{\lambda>1} HC(\lambda B)$ is different from the set $HC(\mu B)$ for any $\mu > 1$. In fact, by the previous, we have, for $\mu > 1$,

$$FHC(\mu B) \subset HC(\mu B) \setminus \bigcap_{\lambda>1} HC(\lambda B).$$

Another interesting feature of Proposition 2.6 (more precisely of the second part of its proof) is that it gives an idea of how to build two frequently hypercyclic operators having no common frequently hypercyclic vectors. This will be detailed later, see Corollary 2.28).

Remark 2.8. One can wonder whether non-trivial conditions for common frequent hypercyclicity of families of non-zero real multiples of a single operator remain true for more general families of operators.

Recall that Bayart proved in [3] that a family of multiples of a single operator can admit a common frequently hypercyclic vector only this family is countable. However, we already know that some uncountable families of operators may have common frequently hypercyclic vectors (*e.g.*, translation operators or composition operators, see [3, 4]). Moreover, the León-Müller theorem for frequent hypercyclicity ([7, Theorem 6.28]), which asserts that $FHC(\lambda T) = FHC(T)$ for any $\lambda \in \mathbb{C}$, $|\lambda| = 1$, shows that uncountable families of complex multiples of an operator on a complex separable Banach space may have common frequently hypercyclic vector.

Now, let us focus on the necessary conditions given by Proposition 2.6. Note that requiring the index set to be bounded is equivalent to impose the family of real numbers $(\|\lambda T\|)_{\lambda\in\Lambda}$ to be bounded. Besides, the second condition is equivalent for the family $(r(\lambda T))_{\lambda\in\Lambda}$ to be bounded away from 1. Therefore, our question can be rephrased as follow: for a given family $(T_i)_{i\in\mathbb{N}}$ of bounded linear operators on X to share common frequently hypercyclic vectors, is it necessary for the families $(\|T_i\|)_{i\in\mathbb{N}}$ and $(r(T_i))_{i\in\mathbb{N}}$ to be respectively bounded and bounded away from 1? *The answer is no: consider the family* $(T^p)_{p\geq 1}$ *of the positive iterates of a*

frequently hypercyclic operator T on X. By Ansari's theorem for frequent hypercyclicity, $FHC(T) = FHC(T^p)$ for any $p \geq 1$. Since a hypercyclic operator cannot be power-bounded, the family $(\|T^p\|)_{p\geq 1}$ is not bounded. Moreover, if $r(T) = 1$, then by the spectral radius formula, $r(T^p) = r(T)^p = 1$ for any $p \geq 1$. Note that, if X is complex, by León-Müller's theorem, the multiples λT with $|\lambda| = 1$ also have spectral radii equal to 1 and yet have common frequently hypercyclic vectors. An example of a frequently hypercyclic operator whose spectral radius is one is given in Corollary 2.28.

Given any bounded linear operator T on X and any $\lambda > 1/r(T)$, there is no reason in general for λT to be frequently hypercyclic and, even if λT is frequently hypercyclic, it may not satisfy the Frequent Hypercyclicity Criterion. In the next paragraph we search for condition on a countable set $\Lambda \subset (0, +\infty)$ for multiples λT of some operator T to have common frequently hypercyclic vectors.

2.2.2. **Sufficient conditions.** Let us fix a separable Fréchet space X and a continuous linear operator T on X. We introduce some quantities which will play an important role in the sequel. Given X_0 a dense subset of X and a mapping $S : X_0 \to X_0$ such that $T(S(x)) = x$ for $x \in X_0$, we denote by

$$\begin{aligned} a_T(X_0, S) &= \inf\Big\{\lambda > 0 : \sum_{n\geq 0} \frac{S^n}{\lambda^n}(x) \text{ converges unconditionally for all } x \in X_0\Big\} \\ &= \inf\{\lambda > 0 : (\lambda^{-n} S^n(x))_{n\in\mathbb{N}} \text{ is bounded for all } x \in X_0\} \end{aligned}$$

and

$$\begin{aligned} b_T(X_0, S) &= \sup\Big\{\lambda > 0 : \sum_{n\geq 0} (\lambda T)^n(x) \text{ converges unconditionally for all } x \in X_0\Big\} \\ &= \sup\{\lambda > 0 : ((\lambda T)^n(x))_{n\in\mathbb{N}} \text{ is bounded for all } x \in X_0\}. \end{aligned}$$

When X is a Banach space, one easily checks that

$$a_T(X_0, S) = \sup_{x\in X_0} \limsup_{n\to\infty} \|S^n(x)\|^{1/n} \quad \text{and} \quad b_T(X_0, S) = \inf_{x\in X_0} \frac{1}{\limsup_{n\to\infty} \|T^n(x)\|^{1/n}}.$$

So, by the spectral radius formula, we have

$$a_T(X_0, S) \geq r(T)^{-1} \quad \text{and} \quad b_T(X_0, S) \geq r(T)^{-1}. \tag{2.16}$$

Note that $b_T(X_0, S)$ may be infinite, for *e.g.*, if $X_0 = \bigcup_{n\geq 0} \ker(T^n)$ is dense in X. This is for example the case if T is any weighted backward shift acting on a Fréchet space X with an unconditional basis. More specifically, if T is the backward shift B on $\ell^2(\mathbb{N})$, then S can be taken as the forward shift F and we have equalities in (2.16) with $a_B(X_0, F) = 1/\|B\| = 1$ (see Paragraph 2.3 for a focus on weighted shifts).

With these notations, a criterion of common hypercyclicity, due to Bayart and Matheron, can be rephrased as follows.

Theorem 2.9 (Proposition 4.2 in [6])**.** *Let X be a separable Fréchet space and let T be a continuous linear operator on X. We assume that there exist $X_0 \subset \bigcup_{n\in\mathbb{N}} \ker(T^n)$ and a mapping $S : X_0 \to X_0$ such that X_0 is dense in X and $T(S(x)) = x$ for all $x \in X_0$. Then $\bigcap_{\lambda > a_T(X_0,S)} HC(\lambda T)$ is a dense G_δ subset of X.*

Now let us observe that, by definition, for any $a_T(X_0, S) < \lambda < b_T(X_0, S)$ the family $(\lambda^n T^n)_{n\in\mathbb{N}}$ satisfies the *Frequent Universality Criterion* [11]. So it is natural to wonder under which extra condition the family λT, $a_T(X_0, S) < \lambda < b_T(X_0, S)$, has a common frequently hypercyclic vector. In virtue of the necessary conditions given in Paragraph 2.2.1, the following criterion is a quite natural extension of Bayart and Matheron's result.

Theorem 2.10. *Let X be a separable Fréchet space and T a continuous linear operator on X. We assume that there exist a dense subset X_0 of X and a mapping $S : X_0 \to X_0$ such that $T(S(x)) = x$ for all $x \in X_0$. If Λ is a countable relatively compact non-empty subset of $(a_T(X_0, S), b_T(X_0, S))$, then $\bigcap_{\lambda\in\Lambda} FHC(\lambda T) \neq \emptyset$.*

The proof is based on the following lemma, where it is assumed that $E \subset (a, b)$ with $a < b$ means $E = \emptyset$.

Lemma 2.11. *With the assumptions of Theorem 2.10, let Λ be a relatively compact subset of $(a_T(X_0, S), b_T(X_0, S))$. Then there exists $c > 1$ such that for any $x \in X_0$,*

(i) the series $\sum_{n\geq 0}(\lambda T)^n(x)$ converges unconditionally, uniformly for $\lambda \in \Lambda$;
(ii) the series $\sum_{n\geq 0}\left(\frac{S}{\lambda}\right)^n(x)$ converges unconditionally, uniformly for $\lambda \in \Lambda$;
(iii) the series $\sum_{n\geq(c-1)m}(\frac{\lambda}{\mu})^m(\frac{S}{\mu})^n(x)$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $\lambda, \mu \in \Lambda$;
(iv) the series $\sum_{m\geq n\geq \frac{c-1}{c}m}(\frac{\lambda}{\mu})^{m-n}(\lambda T)^n(x)$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $\lambda, \mu \in \Lambda$.

Proof. Let us denote by $\|\cdot\|$ any continuous semi-norm on X. For notational simplicity, we shall denote $a = a_T(X_0, S)$ and $b = b_T(X_0, S)$. We only prove (ii) and (iii). The conditions (i) and (iv) are respectively proved in a similar way. Let $a < d < \inf(\Lambda)$. To get (ii), it is enough to write, for $\lambda \in \Lambda$ and $m \in \mathbb{N}$, $(\frac{S}{\lambda})^n(x) = (\frac{d}{\lambda})^n(\frac{S}{d})^n(x)$, and use that $\frac{d}{\lambda} \leq \frac{d}{\inf(\Lambda)} < 1$ and that $(\frac{S}{d})^n(x)$ is bounded for any $x \in X_0$ by some constant independent of $\lambda \in \Lambda$ and $n \in \mathbb{N}$.

To prove (iii), let us fix $x \in X_0$. By assumption, the series $\sum_{n\geq 0}\left(\frac{S}{d}\right)^n(x)$ is unconditionally convergent. We also let $c > 1$ be such that

$$\frac{\sup(\Lambda)}{\inf(\Lambda)}\left(\frac{d}{\inf(\Lambda)}\right)^{c-1} \leq 1,$$

and we write

$$\sum_{n\geq(c-1)m}\left(\frac{\lambda}{\mu}\right)^m\left(\frac{S}{\mu}\right)^n(x) = \sum_{n\geq(c-1)m}\left(\frac{\lambda}{\mu}\left(\frac{d}{\mu}\right)^{c-1}\right)^m\left(\frac{d}{\mu}\right)^{n-(c-1)m}\left(\frac{S}{d}\right)^n(x).$$

Since the quantity

$$\left(\frac{\lambda}{\mu}\left(\frac{d}{\mu}\right)^{c-1}\right)^m\left(\frac{d}{\mu}\right)^{n-(c-1)m}$$

is bounded by 1 uniformly for $\lambda, \mu \in \Lambda$, $m \in \mathbb{N}$ and $n \geq (c-1)m$, we get (iii) by unconditional convergence of the series $\sum_{n\geq 0}\left(\frac{S}{d}\right)^n(x)$. □

Let us now prove Theorem 2.10.

Proof of Theorem 2.10. It is enough to check that the sequences $((\lambda T)^n)_{n\in\mathbb{N}}$ and $((S/\lambda)^n)_{n\in\mathbb{N}}$, $\lambda \in \Lambda$, satisfy the assumptions (1)–(6) of Theorem 2.2. (6) is trivial, while (1), (2) and (5) are direct consequences of (i) and (ii) of Lemma 2.11. Now, (3) and (4) follow from (iii) and (iv) of Lemma 2.11, after observing that for any $\lambda \neq \mu \in \Lambda$, $x \in X_0$,

$$\sum_{n\geq(c-1)m}(\lambda T)^m\left(\frac{S}{\mu}\right)^{m+n}(x) = \sum_{n\geq(c-1)m}\left(\frac{\lambda}{\mu}\right)^m\left(\frac{S}{\mu}\right)^n(x)$$

and

$$\sum_{\frac{c-1}{c}m\leq n\leq m}(\lambda T)^m\left(\frac{S}{\mu}\right)^{m-n}(x) = \sum_{\frac{c-1}{c}m\leq n\leq m}\left(\frac{\lambda}{\mu}\right)^{m-n}(\lambda T)^n(x).$$

□

A slight modification of the proof of Theorem 2.10 yields to the following *universal* version.

Proposition 2.12. *Let X be a separable Fréchet space, T a bounded linear operator on X, X_0 a dense subset of X and a mapping $S : X_0 \to X_0$ such that $T(S(x)) = x$ for all $x \in X_0$. Let also $(\lambda_{i,n})_{n\geq 1}$, $i \in \mathbb{N}$, be a countable family of sequences in $(0, +\infty)$. We assume that*

(1) there exist $c, d \in (a_T(X_0, S), b_T(X_0, S))$ such that $\lambda_{i,n} \in (c^n, d^n)$ for any $i \in \mathbb{N}$, $n \geq 1$;
(2) there exists $C > 0$ such that $C^{-1}\lambda_{i,n+m} \leq \lambda_{i,n}\lambda_{i,m} \leq C\lambda_{i,n+m}$ for any $n, m, i \in \mathbb{N}$.

Then

$$\bigcap_{i\in\mathbb{N}} \mathcal{FU}((\lambda_{i,n}T^n)_n) \neq \emptyset.$$

Together with the result of Paragraph 2.2.1, Theorem 2.10 gives a necessary and sufficient condition on a set $\Lambda \subset (0, +\infty)$ for common frequent hypercyclicity of the family λT, $\lambda \in \Lambda$, for any T in a certain subclass of operator acting on a Banach space. Recall that if $\lambda < 1/r(T)$, then λT is not hypercyclic.

Corollary 2.13. *Let X be a separable Banach space, T be a bounded linear operator on X and $\Lambda \subset (0, +\infty)$ with at least two elements. We assume that there exist a dense subset X_0 of X and a mapping $S : X_0 \to X_0$ such that $T(S(x)) = x$ for all $x \in X_0$. We also suppose that $a_T(S, X_0) = 1/r(T)$ and $b_T(S, X_0) = +\infty$. Then,*

$$\bigcap_{\lambda\in\Lambda} FHC(\lambda T) \neq \emptyset$$

if and only if Λ is countable and relatively compact in $(1/r(T), +\infty)$.

The following question arises. It will be investigated later for the class of weighted shifts, see Paragraph 2.3.2.

Question 2.14. *For those operators T such that $a_T(S, X_0) > 1/r(T)$ for some S and X_0 as in Corollary 2.13, can one improve the necessary condition on $\Lambda \subset (0, +\infty)$, given in Proposition 2.6, for the multiples λT to have common frequently hypercyclic vectors?*

To conclude the paragraph, let us combine the previous results with León-Müller's theorem and Ansari's theorem for frequent hypercyclicity, see respectively [7, Theorem 6.28] and [4, Theorem 4.7]. We recall that they tell us that $FHC(\lambda T) = FHC(T) = FHC(T^p)$ for any $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and any positive integer p. These with Theorem 2.10 thus imply:

Corollary 2.15. *Let X be a separable complex Fréchet space, T a bounded linear operator on X and Λ a non-empty subset of $\mathbb{C}$. We assume that there exist a dense subset X_0 of X and a mapping $S : X_0 \to X_0$ such that $T(S(x)) = x$ for all $x \in X_0$. If the set $\{|\lambda| : \lambda \in \Lambda\}$ is a countable relatively compact subset of $(a_T(X_0, S), b_T(X_0, S))$, then*

$$\bigcap_{\lambda\in\Lambda} FHC(\lambda T) \neq \emptyset.$$

Moreover, if the set $\{|\lambda|^{1/p} : p \in \mathbb{N}^*, \lambda \in \Lambda\}$ is a countable relatively compact subset of $(a_T(X_0, S), b_T(X_0, S))$, then

$$\bigcap_{\lambda\in\Lambda,\, p\in\mathbb{N}^*} FHC(\lambda T^p) \neq \emptyset.$$

Note that the second condition is a consequence of the first one if $a_T(X_0, S) < 1$ and $b_T(X_0, S) = +\infty$ (for *e.g.*, for a large class of weighted shifts, see the next paragraph).

Remark 2.16. When X is a separable Banach space, if $a_T(X_0, S) = 1/r(T)$ and $b_T(X_0, S) = +\infty$ and Λ has at least two elements, then the sufficient conditions on $\Lambda \subset \mathbb{C}$ given in Corollary 2.15 are also necessary.

We shall make another remark.

Remark 2.17. It should be noticed that the definitions of $a_T(X_0, S)$ and $b_T(X_0, S)$ depend *a priori* on X_0 and S. In particular, it could happen that for some $T \in \mathcal{L}(X)$, there exist couples (X_0, S_0) and (X_1, S_1) such that $a_T(X_1, S_1) < a_T(X_0, S_0)$. Thus this is tempting to introduce the quantities

$$a_T := \inf_{(X_0,S)} a_T(X_0, S) \quad \text{and} \quad b_T := \sup_{(X_0,S)} b_T(X_0, S),$$

where the infimum and the supremum are taken over all couples (X_0, S) such that X_0 is dense in X and $S : X_0 \to X_0$ is such that $T(S(x)) = x$ for every $x \in X_0$. But the conclusions of the previous results may not hold true replacing $a_T(X_0, S)$ by a_T and $b_T(X_0, S)$ by b_T. Indeed, this might happen that for some (X_0, S), $a_T(X_0, S)$ is very close to a_T but $b_T(X_0, S)$ is very small compared to b_T.

However, if $T := B_w$ is a frequently hypercyclic weighted shift acting on $\ell^p(\mathbb{N})$, $1 \leq p < +\infty$, it turns out that $a_T = a_T(c_{00}(\mathbb{N}), F_w)$ and $b_T = b_T(c_{00}(\mathbb{N}), F_w) = +\infty$, see the next paragraph for the formal definitions of $c_{00}(\mathbb{N})$ and F_w. This is a consequence of Bayart and Ruzsa's theorem [8].

In the next section we concentrate our attention on common frequent hypercyclicity for the important class of weighted shifts.

2.3. **Common frequent hypercyclicity for weighted shifts.** In this whole section, we assume that X is a Fréchet space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$. We call *weight* a sequence of nonzero real numbers. Given a weight $w = (w_n)_{n\in\mathbb{N}}$, the *weighted shift* B_w is defined, for $x = \sum_{n\geq 0} x_n e_n \in X$, by

$$B_w(x) = \sum_{n\geq 0} w_{n+1} x_{n+1} e_n.$$

The series $\sum_{n\geq 0} w_{n+1} x_{n+1} e_n$ may not be convergent in X for all $x \in X$ yet, by the Closed Graph Theorem, B_w maps X into itself if and only if it is continuous on X. In this case, it is equivalently defined by $B_w(e_n) = w_n e_{n-1}$, $n \geq 0$, with the convention $e_{-1} = 0$.

For any weight w, B_w admits a (formal) right inverse, that we denote F_w, given by

$$F_w(x) = \sum_{n\geq 1} \frac{x_{n-1}}{w_n} e_n$$

for $x = \sum_{n\geq 0} x_n e_n \in X$. The series $\sum_{n\geq 1} \frac{x_{n-1}}{w_n} e_n$ may not belong to X, but F_w is well-defined from $c_{00}(\mathbb{N}) := \mathrm{span}(e_n : n \geq 0)$ into itself and $F_w(e_n) = e_{n+1}/w_{n+1}$, $n \geq 0$. Note that the map F_w is referred to as the *forward shift* associated to the weight $w^{-1} := (w_n^{-1})_{n\geq 0}$.

We recall that a continuous weighted shift B_w on X is frequently hypercyclic whenever the series

$$\sum_{n\geq 1} (w_1 \dots w_n)^{-1} e_n$$

is convergent in X, see [24, Corollary 9.14].

2.3.1. **General criteria.** We first state a criterion of common frequent hypercyclicity for general families of weighted shifts, derived from Corollary 2.5.

Theorem 2.18. *Let X be a separable Fréchet space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$ and $w(i) = (w_n(i))_{n\in\mathbb{N}}$, $i \in \mathbb{N}$, be countably many weights for which every $B_{w(i)}$, $i \in \mathbb{N}$, is a continuous operator on X. We assume that there exist a weight $\omega = (\omega_n)_{n\in\mathbb{N}}$, constants $M \geq 1$ and $0 < \eta \leq 1$ with either $M = \eta = 1$ or $M \neq 1$ and $\eta \neq 1$, and a constant $C > 0$, such that for any $i \in \mathbb{N}$ and any $n \geq 0$, $m \geq 1$,*

(i) the series $\sum_{k\geq 1}(\omega_1 \dots \omega_k)^{-1}e_k$ is unconditionally convergent in X;
(ii) $|\omega_n \dots \omega_{n+m}| \leq C\eta^m |w_n(i) \dots w_{n+m}(i)|$;
(iii) $C^{-1}M^{-m} \leq |w_n(i) \dots w_{n+m}(i)| \leq CM^m$.

Then there exists a common frequently hypercyclic vector for the family $(B_{w(i)})_{i\in\mathbb{N}}$.

Proof. We consider $X_0 = \mathrm{span}(e_k : k \geq 0)$. Since X_0 is dense in X, up to taking $S_i := F_{w(i)}$, $i \in \mathbb{N}$, we need only check that the assumptions (1)–(4) of Corollary 2.5 are satified for any $x = e_k$, $k \in \mathbb{N}$. Let us then fix $k \in \mathbb{N}$. Observe that (4) is trivially satisfied. From now on, for $l < 0$, we use the notations $e_l = 0$ and $w_l(i) = 0$, $i \in \mathbb{N}$. For any $i, j, l, m \in \mathbb{N}$, let us write

$$B^m_{w(i)}(F^l_{w(j)}(e_k)) = \frac{w_{k+l-m+1}(i) \dots w_{k+l}(i)}{w_{k+1}(j) \dots w_{k+l}(j)} e_{k+l-m}.$$

Note that $B^n_{w(i)}(e_k) = 0$ whenever $n > k$. This gives the first part of (1) in Corollary 2.5. Moreover, for every $i \in \mathbb{N}$,

$$\sum_{n\geq 0} F^n_{w(i)}(e_k) = \sum_{n\geq 0} \frac{1}{w_{k+1}(i) \dots w_{k+n}(i)} e_{k+n}. \tag{2.17}$$

By assumption (ii), we have $|w_{k+1}(i) \dots w_{k+n}(i)| > |\omega_{k+1} \dots \omega_{k+n}|$ for every $n \geq 1$ and $i \in \mathbb{N}$. So, by condition (i) and using that $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis, we get that the left-hand side term in (2.17) is unconditionally convergent in X, uniformly for i, hence the second part of (1) in Corollary 2.5.

Let us now turn to proving that (2) in Corollary 2.5 holds. We write

$$\begin{aligned} B^m_{w(i)}(F^{m+n}_{w(j)}(e_k)) &= \frac{w_{k+n+1}(i) \dots w_{k+n+m}(i)}{w_{k+1}(j) \dots w_{k+n+m}(j)} e_{k+n} \\ &= \frac{w_{k+n+1}(i) \dots w_{k+n+m}(i)}{w_{k+n+1}(j) \dots w_{k+n+m}(j)} \frac{\omega_{k+1} \dots \omega_{k+n}}{w_{k+1}(j) \dots w_{k+n}(j)} (\omega_{k+1} \dots \omega_{k+n})^{-1} e_{k+n}. \end{aligned}$$

Let us first assume that $M = \eta = 1$. By (i), (ii) and (iii), the series $\sum_{n\geq 0} B^m_{w(i)}(F^{m+n}_{w(j)}(e_k))$ is unconditionally convergent uniformly with respect to $m \geq 1$ and $i, j \in \mathbb{N}$. Therefore, the assumption (2) is satisfied for every $c > 1$.

Let us now suppose that $M > 1$ so that $\eta < 1$. Let $\varrho \in (\eta, 1)$ and $c > 1$ be such that $M^2\varrho^{c-1} \leq 1$. By the condition (i) and unconditionality of $(e_n)_{n\in\mathbb{N}}$, the sequence $((\omega_{k+1} \dots \omega_{k+n})^{-1}e_{k+n})_{n\in\mathbb{N}}$ is bounded. We denote by $\|\cdot\|$ any continuous semi-norm on X. Then, for some constant K (depending only on η, C, k and the constant of unconditionality of $(e_n)_{n\in\mathbb{N}}$) and thanks to the assumptions (ii) and (iii), we have for any $n, i, j \in \mathbb{N}$ and $m \geq 1$,

$$\left\| B^m_{w(i)}(F^{m+n}_{w(j)}(e_k)) \right\| \leq KM^{2m}\eta^n.$$

Let us now write

$$M^{2m}\eta^n = (M^2\varrho^{c-1})^m \varrho^{n-(c-1)m} \left(\frac{\eta}{\varrho}\right)^n \leq \left(\frac{\eta}{\varrho}\right)^n$$

for any $n \geq (c-1)m$ and $m \geq 1$. As $\varrho \in (\eta, 1)$, the series $\sum_{n\geq(c-1)m} M^{2m}\eta^n$ is absolutely convergent, uniformly for $m \geq 0$, and so the series $\sum_{n\geq(c-1)m} B^m_{w(i)}(F^{m+n}_{w(j)}(e_k))$ converges unconditionally, uniformly for $m \geq 0$ and $i, j \in \mathbb{N}$. This implies (2) from Corollary 2.5.

That (3) in Corollary 2.5 holds in this setting is left to the reader. □

Remark 2.19. *As a corollary of the proof, one may check that Theorem 2.18 remains true if we suppose that there exist a weight $\omega = (\omega_n)_{n\in\mathbb{N}}$ and a constant $C > 0$, such that for any $i, j \in \mathbb{N}$ and any $n \geq 0$, $m \geq 1$,*

(i) the series $\sum_{k\geq1}(\omega_1 \dots \omega_k)^{-1}e_k$ is unconditionally convergent in X;
(ii) $|\omega_n \dots \omega_{n+m}| \leq C|w_n(i) \dots w_{n+m}(i)|$;
(iii) $C^{-1} \leq \left|\frac{w_0(i)\dots w_n(i)}{w_0(j)\dots w_n(j)}\right| \leq C$.

In particular, if the family is composed of a finite (non-zero) number of frequently hypercyclic operators, then it suffices to check (iii). Moreover, if two such operators satisfy that the product of their weights are equivalent then they share frequent hypercyclic operators.

Let us give an example.

Example 2.20. Let $1 \leq p < +\infty$. For $\lambda \in (0, +\infty)$, let $B_{w(\lambda)}$ be the weighted shift on $\ell^p(\mathbb{N})$, defined by $w_n(\lambda) = 1 + \lambda/n$, $n \geq 1$. In [17], it is proven that $\bigcap_{\lambda>1} HC(B_{w(\lambda)})$ is residual. Now, one may check that the series

$$\sum_{n\geq1} \frac{1}{w_1(\lambda)\dots w_n(\lambda)} e_n$$

is unconditionally convergent in $\ell^p(\mathbb{N})$ if and only if $\lambda > 1/p$ (where $(e_n)_{n\in\mathbb{N}}$ is the unit sequence in $\ell^p(\mathbb{N})$). We can thus deduce from Theorem 2.18, applied with $M = \eta = 1$, that for any countable relatively compact subset Λ of $(\frac{1}{p}, +\infty)$, one has

$$\bigcap_{\lambda\in\Lambda} FHC(B_{w(\lambda)}) \neq \emptyset.$$

Observe that, by Bayart and Ruzsa's theorem (see Remark 2.22 below), $B_{w(\lambda)}$ is not frequently hypercyclic on $\ell^p(\mathbb{N})$ if $\lambda \leq 1/p$.

The main result of Paragraph 2.2 can be also applied to weighted shifts. Let X be a separable Fréchet space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$ and set $c_{00}(\mathbb{N}) = \text{span}(e_n : n \geq 0)$. With the notations of Paragraph 2.2, a slight generalization of Abakumov and Gordon's theorem states that the set of common hypercyclic vectors for the multiples λB_w of a continuous weighted shift B_w on X, $\lambda > a_{B_w}(c_{00}(\mathbb{N}), F_w)$, is G_δ and dense in X see [7, p. 178] or [6]. Note that this result can also be deduced from Theorem 2.9.

In this context, Theorem 2.10 reads as follows.

Corollary 2.21. *Let X be a separable Fréchet space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$ and let B_w be a continuous weighted shift on X. Then the set $\bigcap_{\lambda\in\Lambda} FHC(\lambda B_w)$ is non-empty whenever Λ is any countable relatively compact non-empty subset of $(a_{B_w}(c_{00}(\mathbb{N}), F_w), +\infty)$.*

Similarly, Corollary 2.13 tells us that if in Corollary 2.21 we assume in addition that $(a_{B_w}(c_{00}(\mathbb{N}), F_w) = r(B_w)$ and Λ has at least two elements, then the condition becomes also necessary . Moreover Question 2.14 makes sense, and seems to be a bit more accessible in this setting, especially when $X = \ell^p(\mathbb{N})$, $1 \leq p < +\infty$. This particular case will be investigated in the next paragraph.

To finish, we shall make a remark.

Remark 2.22. Bayart and Ruzsa [8] proved in 2015 that, when acting on $\ell^p(\mathbb{N})$ spaces, $1 \leq p < \infty$, weighted shifts are frequently hypercyclic if and only if they satisfy the Frequent Hypercyclicity Criterion (*i.e.*, they are chaotic). This result was extended to more general classes of spaces in [13]. For instance, it is proved there that Bayart-Ruzsa's theorem extends to any Banach space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$ whenever $(e_n)_{n\in\mathbb{N}}$ is *boundedly complete*. We recall that a basis $(e_n)_{n\in\mathbb{N}}$ in X is called boundedly complete if, for any sequence of scalars $(x_n)_{n\in\mathbb{N}}$, whenever the sequence

$$\left(\sum_{n=0}^{N} x_n e_n\right)_{N\geq 0}$$

is bounded in X, then it is convergent in X. Examples of such Banach spaces are given among Köthe sequence spaces (including of course $\ell^p(\mathbb{N})$ spaces). Note that the usual basis of c_0 is not boundedly complete.

Therefore, if in Theorem 2.18 we assume in addition that X is a Banach space and that $(e_n)_{n\in\mathbb{N}}$ is boundedly complete, then the condition (i) can simply be replaced by "B_ω frequently hypercyclic on X".

In Fréchet spaces, bounded completeness of the unconditional basis $(e_n)_{n\in\mathbb{N}}$ is not sufficient any more, and some other conditions are given in [13]. As an application, it is shown that on the space $H(\mathbb{D})$ of analytic functions in the unit disc $\mathbb{D}$, endowed with the locally uniform Fréchet topology, a weighted shift is frequently hypercyclic if and only if it satisfies the Frequent Hypercyclicity Criterion. Thus the previous remark also holds if the Banach space X is replaced with $H(\mathbb{D})$.

Moreover, if we assume that X is a Banach space with $(e_n)_{n\in\mathbb{N}}$ as a boundedly complete unconditional basis, then in the statement of Corollary 2.21, the quantity $a_{B_w}(c_{00}(\mathbb{N}), F_w)$ can be replaced by 0. This occurs in particular when $X = \ell^p(\mathbb{N})$, $1 \leq p < +\infty$; see the next paragraph.

2.3.2. **Common frequent hypercyclicity for multiples of weighted shifts on $\ell^p(\mathbb{N})$.** In this paragraph we specify the study led in Section 2.2 and in the previous paragraph to multiples of a single weighted shift acting on $\ell^p(\mathbb{N})$, $1 \leq p < +\infty$.

Let us fix $1 \leq p < +\infty$. We recall that $\ell^p(\mathbb{N})$ stands for the space of all sequences $x = (x_n)_{n\in\mathbb{N}}$ of scalars for which $\|x\| := (\sum_{n\in\mathbb{N}} |a_n|^p)^{1/p} < +\infty$. Endowed with the norm $\|\cdot\|$, it is a Banach space. The unit sequence $(e_n)_{n\in\mathbb{N}}$ is a boundedly complete unconditional basis of $\ell^p(\mathbb{N})$ and the subspace $X_0 := c_{00}(\mathbb{N}) = \mathrm{span}(e_n : n \geq 0)$ is dense in $\ell^p(\mathbb{N})$. A weighted shift B_w is bounded on $\ell^p(\mathbb{N})$ if and only if the sequence w is bounded, i.e. $\sup_{n\in\mathbb{N}} w_n < +\infty$, in which case $\|B_w\| = \sup_{n\in\mathbb{N}} w_n$.

Most of the important quantities introduced in Section 2.2 can be explicitly computed when working with weighted shifts on $\ell^p(\mathbb{N})$. We keep the notations of Paragraph 2.2.2 except for the spectral radius $r(B_w)$ of a weighted shift B_w that we will simply denote by r_w. We also set

$$r_{p,w} := \sup\{|\lambda| : \lambda \in \sigma_p(B_w)\},$$

where $\sigma_p(B_w)$ denotes de point spectrum of B_w (*i.e.*, the eigenvalues of B_w). Then some calculations give:

- $X_0 := \cup_{n\geq 1} \ker(B_w^n)$, hence $b_{B_w}(X_0, F_w) = +\infty$;
- $a_{B_w}(X_0, F_w) = r_{p,w}^{-1} = \limsup_{n\to\infty}(w_1 \dots w_n)^{-1/n}$, see *e.g.*, [30, Theorem 8, p. 70];
- $r_w = \lim_{n\to\infty} (\sup_k w_k \dots w_{k+n})^{1/n}$.

Let us also introduce the quantity:

- $\lambda_w := \limsup_{n\to\infty}(w_1 \dots w_n)^{1/n}$.

We thus have

$$\|B_w\|^{-1} \leq r_w^{-1} \leq \lambda_w^{-1} \leq r_{p,w}^{-1}.$$

On the one hand, if w is a monotonic sequence (hence a convergent sequence to some positive real number w_∞), then $r_w^{-1} = \lambda_w^{-1} = r_{p,w}^{-1} = w_\infty^{-1}$. Note that if w is increasing, then these quantities are also equal to $\|B_w\|^{-1}$. On the other hand, as shown by the next example, it is not difficult to provide with weights w which allows to distinguish all or some of the quantities $\|B_w\|^{-1}$, r_w^{-1}, λ_w^{-1} and $r_{p,w}^{-1}$.

Example 2.23. Let $a \leq b \leq c \leq d$ be four positive real numbers, and let us define, for any $n \in \mathbb{N}$,

$$w_n := \begin{cases} a & \text{if } n \in \{1, \dots, 4\} \cup \{k2^{(k-1)^2}, \dots, 2^{k^2} - 1\} \\ d & \text{if } n = 2^{k^2} \\ c & \text{if } n \in \{2^{k^2} + 1, 2^{k^2} + k + 1\} \\ b & \text{if } n \in \{2^{k^2} + k + 2, (k+1)2^{k^2}\} \end{cases}, \ k \geq 2.$$

Then, one may check that

$$\|B_w\|^{-1} = 1/d \leq r_w^{-1} = 1/c \leq \lambda_w^{-1} = 1/b \leq r_{p,w}^{-1} = 1/a.$$

We recall that by Bayart and Ruzsa's theorem [8], a weighted shift is frequently hypercyclic on $\ell^p(\mathbb{N})$, $1 \leq p < +\infty$, if and only if it satisfies the Frequent Hypercyclicity Criterion. Then, for any $0 \leq \lambda < r_{p,w}^{-1}$, λB_w is not frequently hypercyclic. Together with Proposition 2.6 and Corollary 2.13, we thus have the following so far:

Corollary 2.24. *Let B_w be a bounded weighted shift on $\ell^p(\mathbb{N})$, $1 \leq p < +\infty$, and let $\Lambda \subset (0, +\infty)$ be a non-empty set. Then*

(1) the set $\bigcap_{\lambda \in \Lambda} FHC(\lambda B_w)$ is non-empty whenever Λ is a countable relatively compact subset of $(r_{p,w}^{-1}, +\infty)$;

(2) the set $\bigcap_{\lambda \in \Lambda} FHC(\lambda B_w)$ is empty whenever Λ is unbounded, or Λ has at least two elements and $r_w^{-1} \geq \inf(\Lambda)$.

In particular, if $r_{p,w} = r_w$ and Λ has at least two elements, then the sufficient condition in (1) *is also necessary.*

The next proposition is a slight improvement of (2) in the previous corollary, and a partial answer to Question 2.14 in the present context.

Proposition 2.25. *Let B_w be a weighted shift acting on $\ell^p(\mathbb{N})$ and let Λ be a subset of $(0, +\infty)$ with at least two elements. If $\lambda_w^{-1} \geq \inf(\Lambda)$, then*

$$\bigcap_{\lambda \in \Lambda} FHC(\lambda T) = \emptyset.$$

Proof. It is very similar to that of Proposition 2.6, so we only give the outline in the case where Λ is a sequence $(\lambda_k)_{k \in \mathbb{N}}$ decreasing to some $\lambda_\infty \leq \lambda_w^{-1}$, and that there exists $x = (x_n)_{n \in \mathbb{N}} \in \ell^p(\mathbb{N})$ which is a hypercyclic vector for each $\lambda_k B_w$, $k \in \mathbb{N}$. As in the proof of Proposition 2.6, we introduce the sets

$$\mathcal{N}_0 := \left\{ n \in \mathbb{N} : \ \|\lambda_0^n B_w^n(x)\| < 1 \right\} \quad \text{and} \quad \mathcal{N}_k := \left\{ m \in \mathbb{N} : \ \|\lambda_k^m B_w^m(x) - e_0\| < \frac{1}{2} \right\}, \ k \geq 1.$$

Then we similarly define increasing sequences $(n_k)_{k \geq 1} \subset \mathcal{N}_0$ and $(m_k)_{k \geq 1}$, tending to $+\infty$, with $m_k \in \mathcal{N}_k$, for $k \geq 1$, and such that $\underline{d}(\mathcal{N}_0) \leq \limsup_{k \to \infty} n_k/m_k$ and, for any $k \geq 1$,

$$\lambda_0^{n_k} |w_{m_k - n_k + 1} \dots w_{m_k}| |x_{m_k}| < 1 \quad \text{and} \quad \lambda_k^{m_k} |w_1 \dots w_{m_k}| |x_{m_k}| > \frac{1}{2}.$$

It follows, for any $k \geq 1$,

$$\frac{\lambda_0^{n_k}}{\lambda_k^{m_k}} < 2|w_1 \dots w_{m_k - n_k}|.$$

In particular $m_k - n_k \to +\infty$ and for any $k \geq 1$,

$$(\lambda_0/\lambda_k)^{n_k/m_k} < 2^{1/m_k} \lambda_k^{1-n_k/m_k} |w_1 \dots w_{m_k-n_k}|^{1/m_k},$$

whence

$$\underline{d}(\mathcal{N}_0) \leq \limsup_{k\to\infty} \frac{n_k}{m_k} \leq C(\limsup_{k\to\infty} \ln(\lambda_k) - \ln(\lambda_w^{-1})) \leq 0,$$

for some constant $C \geq 0$. Thus x is not frequently hypercyclic for $\lambda_0 B_w$. □

We then deduce the following.

Corollary 2.26. *Let B_w be a weighted shift on $\ell^p(\mathbb{N})$ and let Λ be a non-empty subset of $(0, +\infty)$ with at least two elements. We assume that $\lambda_w = r_{p,w}$. Then*

$$\bigcap_{\lambda\in\Lambda} FHC(\lambda B_w) \neq \emptyset$$

if and only if Λ is relatively compact in $(r_{p,w}^{-1}, +\infty)$.

The question whether the last corollary holds true for any weighted shift remains open. More precisely,

Question 2.27. *Does the conclusion of Proposition 2.25 hold true if λ_w^{-1} is replaced by $r_{p,w}^{-1}$?*

We conclude by applying the results of this paragraph in order to exhibit explicit frequently hypercyclic weighted shifts which share no frequently hypercyclic vector.

Corollary 2.28. *There exist two frequently hypercyclic weighted shifts on $\ell^p(\mathbb{N})$, $1 \leq p < +\infty$, with no common frequent hypercyclic vector.*

Proof. Let $(w_n)_{n\geq 1} = \left((\frac{n+1}{n})^2\right)_{n\geq 1}$. Since $(w_n)_{n\geq 1}$ is decreasing to 1, one has $r_w^{-1} = r_{p,w}^{-1} = \lambda_w^{-1} = 1$. Moreover, B_w is frequently hypercyclic, since $\sum_{n\geq 1}(w_1 \dots w_n)^{-1} < \infty$. Thus, applying Proposition 2.25 with $\Lambda = \{1, \lambda\}$, $\lambda > 1$, we get $FHC(B_w) \cap FHC(\lambda B_w) = \emptyset$. □

Remark 2.29. The proof of Proposition 2.25 shows a bit more striking fact: for any monotonic weight w converging to $w_\infty > 0$, $w_\infty^{-1} B_w$ shares a common frequently hypercyclic vector with none of its multiple (different from itself).

2.4. **Other examples.** In this paragraph, we apply our general common frequent hypercyclicity criterion (Theorem 2.2) to classical frequently universal sequences of operators which are not weighted shifts.

Since almost all the classical examples of frequently hypercyclic operators satisfy the Frequent Hypercyclicity Criterion, the range of applications of Theorem 2.10 is quite large.

Example 2.30 (Differential operators on $H(\mathbb{C})$)**.** Let D be the differentiation operator on $H(\mathbb{C})$, $Df(z) = f'(z)$. Costakis and Mavroudis showed [16] that for any non-constant polynomial P, $P(D)$ satisfies Bayart and Matheron's criterion (Theorem 2.9) with $a_{P(D)}(X_0, S) = 0$ and $b_{P(D)}(X_0, S) = +\infty$ for some dense subset X_0 of X and some right inverse S of $P(D)$ on X_0. Thus, with the frequent hypercyclicity version of the León-Müller Theorem and Theorem 2.10, we can deduce that

$$\bigcap_{\lambda\in\Lambda} FHC(\lambda P(D)) \neq \emptyset,$$

for any countable relatively compact subset Λ of $\mathbb{C}^*$.

We shall now focus on applications of Theorem 2.2 to families of operators which are not multiples of a single one.

Example 2.31 (Adjoint of a multiplication operator on the Hardy space)**.** We denote by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the unit disc, by H^∞ the space of bounded analytic functions in $\mathbb{D}$, and by H^2 the classical Hardy space,

$$H^2 := \left\{ f = \sum_{k\geq 0} a_k z^k \in H(\mathbb{D}) : \|f\|_2 := \left(\sum_{k\geq 0} |a_k|^2 \right)^{1/2} < \infty \right\}.$$

We recall that H^∞ and H^2 are Banach spaces, endowed respectively with the sup-norm $\|\cdot\|_\infty$ and $\|\cdot\|_2$. Let $\Phi \in H^\infty$ be such that Φ is not outer and $1/\Phi \in H^\infty$. We denote by $M_\Phi : H^2 \to H^2$ the multiplication operator with symbol Φ, $M_\Phi(f) = \Phi f$, and by M_Φ^* its adjoint. It is known [7] that λM_Φ^* is frequently hypercyclic on H^2 for any $\lambda > \|1/\Phi\|_\infty$ and that

$$\bigcap_{\lambda > \|1/\Phi\|_\infty} HC(\lambda M_\Phi^*)$$

is a dense G_δ-subset of H^2 [22].

Now, let us write the inner-outer decomposition $\Phi = u\theta$, with u outer and θ the non-constant inner part of Φ. Let us define $X_0 := \cup_{n\geq 1} K_n$ with $K_n := H^2 \ominus \theta^n H^2$. Then X_0 is the generalized kernel of M_Φ^* and is dense in X_0. Moreover, if we define $S := M_{1/u}^* M_\theta$, then $M_\Phi^* S = \mathrm{Id}$ and $\|S\| = \|1/\Phi\|_\infty$. We refer, for *e.g.*, to the proof of [22, Theorem 3.1] for the details concerning the previous claims. It is also known that $r(M_\Phi^*) = \|\Phi\|_\infty$. Thus we have

$$\|\Phi\|_\infty^{-1} = r(M_\Phi^*)^{-1} \leq a_{M_\Phi^*}(X_0, S) \leq \|S\| = \|\Phi^{-1}\|_\infty$$

and $b_{M_\Phi^*}(X_0, S) = +\infty$. Therefore, Theorem 2.10 directly implies that

$$\bigcap_{\lambda\in\Lambda} FHC(\lambda M_\Phi^*) \neq \emptyset,$$

whenever Λ is a countable relatively compact non-empty subset of $(\|1/\Phi\|_\infty, +\infty)$.

In fact, we can deduce from Corollary 2.5 the following more general result.

Proposition 2.32. *Let $\{\Phi_\lambda : \lambda \in \Lambda\}$ be a countable family of bounded analytic functions on $\mathbb{D}$ with the same non-constant inner factor θ. We assume that*

$$a := \sup\{\|\Phi_\lambda^{-1}\|_\infty : \lambda \in \Lambda\} < 1 \quad \text{and} \quad M := \sup\{\|\Phi_\lambda/\Phi_\mu\|_\infty : \lambda \in \Lambda\} < \infty.$$

Then

$$\bigcap_{\lambda\in\Lambda} FHC(M_{\phi_\lambda}^*) \neq \emptyset.$$

Proof. We aim to apply Corollary 2.5. By the comment after its statement, we need only check items (2)–(4). Since the functions Φ_λ share the same non-constant inner factor, the set $X_0 := \cup_{n\geq 1} K_n$ with $K_n := H^2 \ominus \theta^n H^2$ is the generalized kernel of each $M_{\phi_\lambda}^*$. Let u_λ denote the outer factor of Φ_λ. As recalled above, setting $S_\lambda := M_{1/u_\lambda}^* M_\theta$, we have $T_\lambda^n S_\lambda^n = \mathrm{Id}$ for any $n \in \mathbb{N}$. So (2) and (4) of Corollary 2.5 are satisfied. Let $\lambda \neq \mu \in \Lambda$ and $f \in X_0$. By

assumption, there exists $b \in (a,1)$ such that for any $m \in \mathbb{N}$, writing $n = (c-1)m + s$, $s \geq 0$,

$$\begin{aligned}\|T_\lambda^m(S_\mu^{m+n}(f))\|_2 &= \sup_{\|g\|_2=1} \left\langle T_\lambda^m(S_\mu^{m+n}(f)), g\right\rangle \\ &= \sup_{\|g\|_2=1} \left\langle f, \left(\frac{u_\lambda}{u_\mu}\right)^m \left(\frac{\bar\theta}{u_\mu}\right)^n g\right\rangle \\ &\leq \|f\|_2 \left\|\frac{u_\lambda}{u_\mu}\right\|_\infty^m \left\|\frac{1}{u_\mu}\right\|_\infty^n \\ &\leq \|f\|_2 \left(Mb^{(c-1)}\right)^m b^{n-(c-1)m} \left(\frac{a}{b}\right)^n.\end{aligned}$$

Since $b \in (a,1)$, (3) of Corollary 2.5 then follows by taking $c > 1$ such that $Mb^{(c-1)} \leq 1$. □

3. Periodic points at the service of common frequent hypercyclicity

Despite its apparent unpleasant formulation, the classical Frequent Hypercyclicity Criterion turns out to be very useful for checking that natural operators are frequently hypercyclic (and chaotic). We saw in the previous section that it fits well to formulating easy-to-use sufficient conditions for common frequent hypercyclicity. In [23], the authors provided a quite appealing new criterion for frequent hypercyclicity and chaos involving the periodic points of the operator [23, Theorem 5.31]. It is shown there that all the operators which satisfy the Frequent Hypercyclicity Criterion do also satisfy the assumptions of this new one. However, it quickly appears from its statement that it is not so simple to use when dealing with natural operators (for *e.g.*, weighted shifts). Yet it is very well adapted to certain type of operators which were introduced by Menet in [26] to build chaotic operators on $\ell^p(\mathbb{N})$ which are not frequently hyperyclic. These operators have been extensively developed - and called operators of C-type in [23, Section 6] to build several counter-examples.

In this section, we provide with a sufficient condition for common frequent hypercyclicity derived from [23, Theorem 5.31]. In the whole section, X is a separable Banach space. We recall that a vector $x \in X$ is a periodic point for $T \in \mathcal{L}(X)$ if there exists $p \in \mathbb{N}$ such $T^p(x) = x$. Let us denote by $\mathrm{Per}(T)$ the set of all periodic points for T. For $x \in \mathrm{Per}(T)$ we denote by $p_T(x)$ the period of x for T (*i.e.*, the smallest positive integer p such that $T^p(x) = x$).

Theorem 3.1. *Let X be a separable Banach and let $(T_s)_{s\geq 1}$ be a countable family of bounded linear operators on X. We assume that there exist a dense linear subspace X_0 of X with $T_s(X_0) \subset X_0$ and $X_0 \subset Per(T_s)$ for any $s \geq 1$, and a constant $\alpha \in (0,1)$ such that the following property holds true: for every $s, q \geq 1$, every $\varepsilon > 0$ and every $x, y \in X_0$, there exist $z \in X_0$ and integers $n, d \geq 1$ such that, for every $1 \leq t \leq q$,*

(1) d is a multiple of $p_{T_t}(y)$ and of $p_{T_t}(z)$;
(2) $\|T_t^k(z)\| < \varepsilon$ for every $0 \leq k \leq \alpha d$;
(3) $\|T_s^{n+k}(z) - T_s^k(x)\| < \varepsilon$ for every $0 \leq k \leq \alpha d$.

Then there exists a common frequently hypercyclic vector for the family $(T_s)_{s\geq 1}$.

If the family $(T_s)_{s\geq 1}$ is reduced to a single operator, Theorem 3.1 is exactly [23, Theorem 5.31]. Yet one should mention that the previous statement does not only mean "each T_s satisfies the assumptions of [23, Theorem 5.31] with the same X_0 and α". It would be interesting to know whether two operators satisfying the assumptions of [23, Theorem 5.31] with the same X_0 and α automatically have a frequently hypercyclic vector in common. Note that we already saw that an operator may share common frequently hypercyclic vectors with none of its multiples (different from itself), even if they all satisfy the classical Frequent Hypercyclicity Criterion (see Corollary 2.28). These operators satisfy [23, Theorem 5.31],

but not with the same X_0. Indeed, $\mathrm{Per}(T) \cap \mathrm{Per}(\lambda T) = \emptyset$ in general. In passing, observe then that Theorem 3.1 does not apply to families of multiples of a single operator.

Proof of Theorem 3.1. The proof is greatly inspired by that of [23, Theorem 5.31]. Let $(x_l)_{l\geq 1}$ be a sequence of vectors in X_0, dense in X, and let $(I_p(s))_{p,s\geq 1}$ be a partition of $\mathbb{N}$ such that each set $I_p(s)$ is infinite and has bounded gaps. Let us denote by $r_p(s)$ the maximal size of a gap for $I_p(s)$. We set $I_p(s) := \{j_m(p,s) : m \geq 1\}$, where $(j_m(p,s))_{m\geq 1}$ is increasing and satisfies $j_{m+1}(p,s) - j_m(p,s) \leq r_p(s)$ for every $m \geq 1$. We also let $(y_j)_{j\in\mathbb{N}}$ be given by $y_j = x_p$ if $j \in I_p(s)$. Now we use the assumptions of the theorem to build, by induction on $j \in \mathbb{N}$, a sequence $(z_j)_{j\in\mathbb{N}}$ of vectors in X_0 and increasing sequences of positive integers $(d_j)_{j\in\mathbb{N}}$ and $(n_j)_{j\in\mathbb{N}}$ such that the following properties hold, if $j \in I_p(s)$:

(i) d_j is a multiple of $p_{T_t}(\sum_{k=1}^{j-1} z_k)$ and $p_{T_t}(z_j)$ for every $t \geq 1$ so that there exist $q \geq 1$ and $1 \leq i \leq j$ with $i \in I_q(t)$;
(ii) $\|T_t^k(z_j)\| < 2^{-j}$ for every $0 \leq k \leq \alpha d_j$ and every $t \geq 1$ so that there exist $q \geq 1$ and $1 \leq i \leq j$ with $i \in I_q(t)$;
(iii) $\|T_s^{n_j+k}(z_j) - T_s^k(y_j - \sum_{i=1}^{j-1} z_i)\| < 2^{-j}$ for every $0 \leq k \leq \alpha d_j$;
(iv) n_j is a multiple of $p_{T_s}(\sum_{i=1}^{j-1} z_i)$ and $\alpha d_j < n_j \leq d_j$;
(v) $\alpha d_j > 4 d_{j-1}$.

Note that the choice of n_j is made possible thanks to several elementary facts, see the first lines of the proof of Theorem 5.31 in [23]; (i) and (ii) are possible using the assumptions of the theorem and the fact that the set

$$\bigcup_{q\geq 1} \bigcup_{1\leq i\leq j} \{t \geq 1 : i \in I_q(t)\}$$

is finite for any $j \geq 1$. By (ii), the sum $z := \sum_{i\geq 1} z_i$ defines a vector in X. Let us check that z is frequently hypercyclic for every T_s, $s \geq 1$.

Let $p, s \geq 1$ be fixed. For notational simplicity, we will denote $j_m(p,s)$ by j_m. Then, for every $m \geq 1$ we define by induction on $j \in \mathbb{N}$ a family of sets $(A_{m,j,s})_{0\leq j<j_{m+1}-j_m}$ as follows:

$$A_{m,0,s} := \left\{ n_{j_m} + k d_{j_m} + k' p_{T_s}(x_p) : 0 \leq k' \leq \frac{\alpha d_{j_m}}{p_{T_s}(x_p)},\ 0 \leq k \leq \frac{\alpha d_{j_m+1}}{d_{j_m}} - 2 \right\},$$

and, for $1 \leq j < j_{m+1} - j_m$,

$$A_{m,j,s} := \bigcup_{1\leq k\leq \frac{\alpha d_{j_m+j+1}}{d_{j_m+j}}-1} (A_{m,j-1} + k d_{j_m+j}).$$

As in the proof of [23, Theorem 5.31, Equation (16)], one easily checks by induction that $\max(A_{m,j,s}) \leq \alpha d_{j_m+j+1}$. Moreover, by [23, Fact 5.35] (in fact exactly reproducing its proof), we have $\underline{d}(A_s) > 0$ where

$$A_s := \bigcup_{m\geq 1} \bigcup_{0\leq j<j_{m+1}-j_m} A_{m,j,s}.$$

Thus to finish the proof of the theorem, we need only prove that for every $m \geq 1$ and every $0 \leq j < j_{m+1} - j_m$, we have

$$\|T_s^n(z) - x_p\| \leq 2^{-(j_m-1)}, \quad n \in A_{m,j,s}.$$

This shall be proven as in [23, Fact 5.34] up to some modifications. For $m \geq 1$ and $0 \leq j < j_{m+1} - j_m$, we first observe that for any $n \in A_{m,j,s}$ we have

$$\|T_s^n(z) - x_p\| \leq \left\| T_s^n\left(\sum_{i=1}^{j_m+j} z_i \right) - x_p \right\| + \sum_{i>j_m+j} \left\| T_s^n(z_i) \right\|.$$

Since $\max(A_{m,j,s}) \leq \alpha d_{j_m+j+1}$, we have $n \leq \alpha d_{j_m+j+1} \leq \alpha d_i$ for every $i > j_m + j$, and it follows from (ii) that

$$\sum_{i>j_m+j} \|T_s^n(z_i)\| < \sum_{i>j_m+j} 2^{-i} \leq \frac{1}{2^{j_m+j}}.$$

To conclude we now turn to proving that for every $n \in A_{m,j,s}$,

$$\left\| T_s^n\left(\sum_{i=1}^{j_m+j} z_i \right) - x_p \right\| \leq \sum_{i=0}^{j} 2^{-(j_m+i)}. \tag{3.1}$$

To do so, we proceed by induction on $0 \leq j < j_{m+1} - j_m$. If $n \in A_{m,0,s}$ (*i.e.*, $j = 0$) then $n = n_{j_m} + kd_{j_m} + k' p_{T_s}(x_p)$ with $0 \leq k \leq \frac{\alpha d_{j_m+1}}{d_{j_m}} - 2$ and $0 \leq k' \leq \frac{\alpha d_{j_m}}{p_{T_s}(x_p)}$, and by (i) and (iv)

$$\begin{aligned} T_s^n\left(\sum_{i=1}^{j_m} z_i \right) - x_p &= T_s^{n_{j_m}+kd_{j_m}+k' p_{T_s}(x_p)}\left(\sum_{i=1}^{j_m} z_i \right) - x_p \\ &= T_s^{n_{j_m}+k' p_{T_s}(x_p)}(z_{j_m}) - T_s^{k' p_{T_s}(x_p)}\left(x_p - \sum_{i=1}^{j_m-1} z_i \right). \end{aligned}$$

By (iii) we get

$$\left\| T_s^n\left(\sum_{i=1}^{j_m} z_i \right) - x_p \right\| \leq 2^{-j_m}.$$

Assume now that (3.1) has been proven up to $j-1$ for some $1 \leq j < j_{m+1} - j_m$. For $n \in A_{m,j,s}$, we write $n = kd_{j_m+j} + l$ with $l \in A_{m,j-1,s}$ and

$$0 \leq k \leq \frac{\alpha d_{j_m+j+1}}{d_{j_m+j}} - 1.$$

Then, by (i) we have

$$\begin{aligned} T_s^n\left(\sum_{i=1}^{j_m+j} z_i \right) - x_p &= T_s^{kd_{j_m+j}+l}\left(\sum_{i=1}^{j_m+j} z_i \right) - x_p \\ &= T_s^l\left(\sum_{i=1}^{j_m+j-1} z_i \right) - x_p + T_s^l(z_{j_m+j}) \end{aligned}$$

Since $l \in A_{m,j-1,s}$, we deduce from the induction hypothesis and (ii) that

$$\left\| T_s^n\left(\sum_{i=1}^{j_m+j} z_i \right) - x_p \right\| \leq \sum_{i=0}^{j-1} 2^{-(j_m+i)} + 2^{-(j_m+j)},$$

and (3.1) as desired. □

Application to operators of C-type. We will apply Theorem 3.1 to operators of C-type on $\ell^p(\mathbb{N})$. First we shall recall their definition, following the formalism of [23, Section 6]. As usual, we denote by $(e_k)_{k\in\mathbb{N}}$ the unit sequence of $\ell^p(\mathbb{N})$. An operator of C-type is associated with a data of four parameters v, w, φ and b:

- $v = (v_n)_{n\geq 1}$ is a sequence of non-zero complex numbers with $\sum_{n\geq 1} |v_n| < \infty$;
- $w = (w_n)_{n\geq 1}$ is a sequence of complex numbers such that
$$0 < \inf_{n\geq 1} |w_n| \leq \sup_{n\geq 1} |w_n| < \infty;$$
- $\varphi : \mathbb{N} \to \mathbb{N}$ is such that $\varphi(0) = 0$, $\varphi(n) < n$ for every $n \geq 1$, and the set $\{n \in \mathbb{N} : \varphi(n) = l\}$ is infinite for every $l \geq 0$;

- $b = (b_n)_{n\geq 0}$ is a strictly increasing sequence of positive integers with $b_0 = 0$ and $b_{n+1} - b_n$ is a multiple of $2(b_{\varphi(n)+1} - b_{\varphi(n)})$ for every $n \geq 1$.

Now, for a data v, w, φ and b as above, the operator of C-type $T_{v,w,\varphi,b}$ is defined by

$$T_{v,w,\varphi,b}(e_k) = \begin{cases} w_{k+1}e_{k+1} & \text{if } k \in [b_n, b_{n+1} - 1),\ n \geq 0 \\ v_n e_{b_{\varphi(n)}} - \left(\prod_{j=b_n+1}^{b_{n+1}-1} w_j\right)^{-1} e_{b_n} & \text{if } k = b_{n+1} - 1,\ n \geq 1 \\ -\left(\prod_{j=b_0+1}^{b_1-1} w_j\right)^{-1} e_0 & \text{if } k = b_1 - 1. \end{cases}$$

Here, by convention, en empty product is equal to 0. From now on, we assume that the condition

$$\inf_{n\geq 0} \prod_{b_n<j<b_{n+1}} |w_j| > 0$$

is satisfied. As shown by [23, Fact 6.2], this assumption ensures that $T_{v,w,\varphi,b}$ is a bounded operator from $\ell^p(\mathbb{N})$ into itself. It can also be checked that each element of c_{00} is a periodic point for $T_{v,w,\varphi,b}$, see [23, Fact 6.4].

In order to deal with frequent hypercyclicity, the authors of [23] introduce a subclass of operators of C-type. As we are interested in common frequent hypercyclicity, we will work within this subclass. It consists in those operators of C-type for which the data v, w, φ, b has the following special structure: for every $k \geq 1$,

- $\varphi(n) = n - 2^{k-1}$ for every $n \in [2^{k-1}, 2^k)$;
- there exists $\Delta^{(k)} \in \mathbb{N}$ such that the size of the block $[b_n, b_{n+1})$, i.e. the quantity $b_{n+1} - b_n$, is equal to $\Delta^{(k)}$ for every $n \in [2^{k-1}, 2^k)$;
- there exists $v^{(k)} \in \mathbb{C} \setminus \{0\}$ such that $v_n = v^{(k)}$ for every $n \in [2^{k-1}, 2^k)$;
- there exists a sequence $(w_i^{(k)})_{1\leq i<\Delta^{(k)}}$ such that $w_{b_n+i} = w_i^{(k)}$ for every $1 \leq i < \Delta^{(k)}$ and every $n \in [2^{k-1}, 2^k)$.

An operator of C-type which satisfies the previous conditions is called an operator of C_+-type. The next result is a criterion for a countable family of operators of C_+-type to share a common frequently hypercyclic vector.

Theorem 3.2. *Let $(T_{v(s),w(s),\varphi,b})_{s\in\mathbb{N}}$ be a countable family of operators of C_+-type on $\ell^p(\mathbb{N})$ where b does not depend on s. We assume that there exists a constant $\alpha > 0$ such that for every $s \geq 1$, every $C \geq 1$ and every $k_0 \geq 1$, there exists an integer $k \geq k_0$ such that, for every $0 \leq n \leq \alpha\Delta^{(k)}$,*

$$|v^{(k)}(s)| \prod_{i=n+1}^{\Delta^{(k)}-1} |w_i^{(k)}(s)| > C. \tag{3.2}$$

If, for any $s, t \geq 1$, there exists a constant $K_{s,t} > 0$ such that for any $r \geq p \geq 1$,

$$\left|\frac{w_p(t)w_{p+1}(t)\dots w_r(t)}{w_p(s)w_{p+1}(s)\dots w_r(s)}\right| \leq K_{s,t}, \tag{3.3}$$

then $\bigcap_{s\geq 1} FHC(T_{v(s),w(s),\varphi,b})$ is non-empty.

Note that since b does not depend on s, by definition the $\Delta^{(k)}$, $k \geq 1$, do not depend on s either. It is plainly checked that condition (3.2) is equivalent to saying that each T_s satisfies the assumption of [23, Theorem 6.9]. In particular, if $\{T_{v(s),w(s),\varphi,b} : s \in \mathbb{N}\}$ is reduced to a single operator (*i.e.*, $v(s)$ and $w(s)$ do not depend on s), then the previous criterion is exactly [23, Theorem 6.9].

For the proof of Theorem 3.2, we recall [23, Fact 6.8] below.

Fact 1. *Let T be an operator of C_+-type on $\ell^p(\mathbb{N})$ and $k \geq 1$. For any $l < 2^{k-1}$ and $1 \leq m \leq \Delta^{(k)}$, we have*

$$T^m(e_{b_{2^{k-1}+l+1}-m}) = v^{(k)} \left(\prod_{i=\Delta^{(k)}-m+1}^{\Delta^{(k)}-1} w_i^{(k)} \right) e_{b_l} - \left(\prod_{i=1}^{\Delta^{(k)}-m} w_i^{(k)} \right)^{-1} e_{b_{2^{k-1}+l}}.$$

Proof of Theorem 3.2. Without loss of generality, we can assume that $0 < \alpha < 1$. It suffices to check that the assumptions of Theorem 3.1 are satisfied. Let us define $X_0 := \mathrm{span}(e_k : k \in \mathbb{N})$ and fix $x, y \in X_0$, $\varepsilon > 0$ and $s, q \geq 1$. There exists $k_0 \geq 1$ such that

$$x = \sum_{l<2^{k_0}} \sum_{j=b_l}^{b_{l+1}-1} x_j e_j.$$

By (3.2), for any $C > 0$, there exists $k \geq k_0$ such that

$$|v^{(k)}(s)| \prod_{i=n+1}^{\Delta^{(k)}-1} |w_i^{(k)}(s)| > C, \quad 0 \leq n \leq \alpha\Delta^{(k)}.$$

Since $v(s)$ and $w(s)$ are bounded, upon choosing C large enough, we may assume that k is so large that the following holds true:

(a) $\Delta^{(k)}$ is a multiple of $p_{T_t}(y)$ for any $t \geq 1$;
(b) $\Delta^{(k_0)} < \min((1 - \frac{\alpha}{2})\Delta^{(k)}, \frac{\alpha}{2}\Delta^{(k)} - 1)$.

Note that, by the definition of b and φ for operators of C_+-type, and since the period of any vector in X_0 depends only on the sequence b, (a) is satisfied whenever y is supported in $[0, b_{2^{k-1}}[$. Let us now set $n := \Delta^{(k)} - 1$, $d := 2\Delta^{(k)}$ and

$$z := \sum_{l<2^{k_0}} \sum_{j=b_l}^{b_{l+1}-1} x_j \left(v^{(k)}(s) \prod_{i=j-b_l+2}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right)^{-1} \left(\prod_{i=1}^{j-b_l} w_{b_l+i}(s) \right)^{-1} e_{b_{2^{k-1}+l+1}-n+j-b_l}.$$

Like for (a) above, d is a multiple of $p_{T_s}(z)$ for any $s \geq 1$. Thus condition (1) of Theorem 3.1 is satisfied.

Let us now fix $0 \leq m \leq \frac{\alpha d}{4}$ and $1 \leq t \leq q$. We observe that for every $l < 2^{k_0}$ and $b_l \leq j \leq b_{l+1} - 1$, we have

$$b_{2^{k-1}+l+1} - n + j - b_l + m \in [b_{2^{k-1}+l}, b_{2^{k-1}+l+1}).$$

Indeed, by definition $b_{2^{k-1}+l+1} - b_{2^{k-1}+l} = \Delta^{(k)}$ and by (b), $-\Delta^{(k)} \leq -n + j - b_l + m < 0$. So for every $t \geq 1$, we have

$$T_t^m(e_{b_{2^{k-1}+l+1}-n+j-b_l}) = \left(\prod_{i=\Delta^{(k)}-n+j-b_l+1}^{\Delta^{(k)}-n+j-b_l+m} w_i^{(k)}(t) \right) e_{b_{2^{k-1}+l+1}-n+j-b_l+m},$$

hence the expression

$$T_t^m(z) = \sum_{l<2^{k_0}} \sum_{j=b_l}^{b_{l+1}-1} x_j \left(v^{(k)}(s) \prod_{i=j-b_l+m+2}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right)^{-1} \left(\prod_{i=1}^{j-b_l} w_{b_l+i}(s) \right)^{-1} \left(\prod_{i=j-b_l+2}^{j-b_l+m+1} \frac{w_i^{(k)}(t)}{w_i^{(k)}(s)} \right) e_{b_{2^{k-1}+l+1}-n+j-b_l+m}. \tag{3.4}$$

Using (b), we know that $0 \leq j - b_l + m + 1 \leq \alpha\Delta^{(k)}$ which, by (3.2), (3.3) and the definition of C_+-type operators, implies that for some constant $A > 0$ (independent of k),

$$\|T_t^m(z)\| \leq \|x\| C^{-1} \max_{1 \leq t' \leq q} (K_{s,t'}) A^{\Delta^{(k_0)}}.$$

Up to choose C large enough, we get (2) in Theorem 3.1.

Let us now estimate the norm of $T_s^{n+m} z - T_s^m(x)$ for $0 \leq m \leq \frac{\alpha d}{4}$. By Fact 1, we obtain

$$T_s^{n-(j-b_l)}(e_{b_{2^{k-1}+l+1}-n+j-b_l}) = v^{(k)}(s) \left(\prod_{i=\Delta^{(k)}-n+j-b_l+1}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right) e_{b_l} - \left(\prod_{i=1}^{\Delta^{(k)}-n+j-b_l} w_i^{(k)}(s) \right)^{-1} e_{b_{2^{k-1}+l}}.$$

Applying $T_s^{j-b_l}$ yields

$$T_s^n(e_{b_{2^{k-1}+l+1}-n+j-b_l}) = \left(v^{(k)}(s) \prod_{i=j-b_l+2}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right) \left(\prod_{i=1}^{j-b_l} w_{b_l+i}(s) \right) e_j - \left(w_{j-b_l+1}^{(k)}(s) \right)^{-1} e_{b_{2^{k-1}+l}+j-b_l}.$$

Moreover, since $m + j - b_l < \Delta^{(k)}$, we have

$$T_s^m(e_{b_{2^{k-1}+l}+j-b_l}) = \left(\prod_{i=j-b_l+1}^{j-b_l+m} w_i^{(k)}(s) \right) e_{b_{2^{k-1}+l}+j-b_l+m},$$

hence

$$T_s^{n+m}(e_{b_{2^{k-1}+l+1}-n+j-b_l}) = \left(v^{(k)}(s) \prod_{i=j-b_l+2}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right) \left(\prod_{i=1}^{j-b_l} w_{b_l+i}(s) \right) T_s^m(e_j) - \left(w_{j-b_l+1}^{(k)}(s) \right)^{-1} \left(\prod_{i=j-b_l+1}^{j-b_l+m} w_i^{(k)}(s) \right) e_{b_{2^{k-1}+l}+j-b_l+m}.$$

By definition of z, it follows that

$$T_s^{n+m}(z) = T_s^m(x) - \sum_{l<2^{k_0}} \sum_{j=b_l}^{b_{l+1}-1} x_j \left(v^{(k)}(s) \prod_{i=j-b_l+m+1}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right)^{-1} \left(\prod_{i=1}^{j-b_l} w_{b_l+i}(s) \right)^{-1} e_{b_{2^{k-1}+l}+j-b_l+m}.$$

By assumption, we thus get

$$\|T_s^{n+m}(z) - T_s^m(x)\| \leq \|x\| C^{-1} A^{\Delta^{(k_0)}}$$

and condition (3) of Theorem 3.1 with $\alpha' = \frac{\alpha}{4}$, as desired. □

Remark 3.3. *It is clear from the proof that the conclusion of Theorem 3.2 remains true under the following weaker (but less nice) assumption: we assume that there exists a constant*

$0 < \alpha < 1$ such that for every integers $s, q \geq 1$, every $C \geq 1$ and every $k_0 \geq 1$, there exists $k \geq k_0$ such that for every $0 \leq n \leq \alpha\Delta^{(k)}$,

$$|v^{(k)}(s)| \prod_{i=n+1}^{\Delta^{(k)}-1} |w_i^{(k)}(s)| > C; \tag{3.5}$$

$$\sup_{\substack{1\leq t\leq q\\ 0\leq j<\Delta^{(k_0)}\\ 0\leq m\leq\alpha\Delta^{(k)}}} \left(|v^{(k)}(s)| \prod_{i=j+2}^{\Delta^{(k)}-1} |w_i^{(k)}(s)|\right)^{-1} \left(\prod_{i=j+2}^{j+m+1} |w_i^{(k)}(t)|\right) < \frac{1}{C}. \tag{3.6}$$

Moreover, it is clear that (3.6) *is satisfied whenever there exists a constant $0 < \alpha < 1$ such that for every integers $s, q \geq 1$, every $C \geq 1$ and every $k_0 \geq 1$, there exists $k \geq k_0$ and $A > 1$ such that*

$$\sup_{i,t\geq 1} \max\left(|w_i(t)|; \frac{1}{|w_i(t)|}\right) \leq A; \tag{3.7}$$

and

$$\sup_{\substack{1\leq t\leq q\\ 0\leq m\leq\alpha\Delta^{(k)}}} \left(|v^{(k)}(s)| \prod_{i=1}^{\Delta^{(k)}-1} |w_i^{(k)}(s)|\right)^{-1} \left(\prod_{i=1}^{m+1} |w_i^{(k)}(t)|\right) < \frac{1}{C}. \tag{3.8}$$

Thus the conclusion of Theorem 3.2 holds true under the assumptions of Remark 3.3 with (3.6) *replaced by* (3.7) *and* (3.8).

It turns out that for a certain subclass of operators of C_+-type, for which (3.7) automatically holds true, some rather simple condition for frequent hypercyclicity is given in [23]. We shall now see that a similar condition for a family of operators in this subclass implies (3.8) and thus common frequent hypercyclicity.

Application to operators of $C_{+,1}$-type. Operators of $C_{+,1}$-type are introduced in [23, Section 6.5] as those operators of C_+-type for which the parameters v and w satisfy the following extra condition: for every $k \geq 1$,

$$v^{(k)} = 2^{-\tau^{(k)}} \quad \text{and} \quad w_i^{(k)} = \begin{cases} 2 & \text{if } 1 \leq i \leq \delta^{(k)} \\ 1 & \text{if } \delta^{(k)} < i < \Delta^{(k)} \end{cases},$$

where $\tau := (\tau^{(k)})_{k\geq 1}$ and $\delta := (\delta^{(k)})_{k\geq 1}$ are two strictly increasing sequences of integers such that $\delta^{(k)} < \Delta^{(k)}$, $k \geq 1$. Within this class of operators of $C_{+,1}$-type, that we simply denote by $T_{\tau,\delta,\varphi,b}$, examples of frequently hypercyclic operators which are not ergodic were provided in [23].

Theorem 3.4. *Let $(T_{\tau(s),\delta(s),\varphi,b})_{s\geq 1}$ be a countable family of operators of $C_{+,1}$-type on $\ell^p(\mathbb{N})$ where b does not depend on s. If*

$$\inf_{t\geq 1} \limsup_{k\to\infty} \frac{\delta^{(k)}(t) - \tau^{(k)}(t)}{\Delta^{(k)}} > 0,$$

then $\bigcap_{s\geq 1} FHC(T_{\tau(s),\delta(s),\varphi,b})$ is non-empty.

Proof. Remark that (3.7) in Remark 3.3 trivially holds, thus it is enough to check (3.8) and (3.5). To do so, we define

$$\alpha < \min\left(1, \frac{1}{2} \inf_{t\geq 1} \limsup_{k\to\infty} \frac{\delta^{(k)}(t) - \tau^{(k)}(t)}{\Delta^{(k)}}\right).$$

Let $s, k_0 \geq 1$ and $C \geq 1$, and let us set $n = \Delta^{(k)} - 1$. Since $\Delta^{(k)} \to \infty$ as $k \to \infty$, there exists $k \geq k_0$ such that

$$\frac{\delta^{(k)}(s) - \tau^{(k)}(s)}{\Delta^{(k)}} > 2\alpha \quad \text{and} \quad \alpha\Delta^{(k)} > \log_2(C).$$

Then it follows from the definition of operators of $C_{+,1}$-type that (3.8) in Remark 3.3 is equivalent to

$$2^{\tau^{(k)}(s)-\delta^{(k)}(s)} \sup_{\substack{t\geq 1\\ 0\leq m\leq \alpha\Delta^{(k)}}} \prod_{i=1}^{m+1} |w_i^{(k)}(t)| < \frac{1}{C}.$$

Now, we have

$$\sup_{\substack{t\geq 1\\ 0\leq m\leq \alpha\Delta^{(k)}}} \prod_{i=1}^{m+1} |w_i^{(k)}(t)| \leq 2^{\alpha\Delta^{(k)}} \leq 2^{\frac{1}{2}(\delta^{(k)}(s)-\tau^{(k)}(s))}.$$

Hence,

$$2^{\tau^{(k)}(s)-\delta^{(k)}(s)} \sup_{\substack{t\geq 1\\ 0\leq m\leq \alpha\Delta^{(k)}}} \prod_{i=1}^{m+1} |w_i^{(k)}(t)| \leq 2^{\frac{1}{2}(\tau^{(k)}(s)-\delta^{(k)}(s))} < 2^{-\alpha\Delta^{(k)}} < \frac{1}{C}.$$

It remains to check that for every $0 \leq n \leq \alpha\Delta^{(k)}$,

$$|v^{(k)}(s)| \prod_{i=n+1}^{\Delta^{(k)}-1} |w_i^{(k)}(s)| > C$$

which works the same as in the proof of [23, Theorem 6.17]. □

Remark 3.5. When the family is reduced to a single operator, Theorem 3.4 is exactly [23, Theorem 6.17].

4. Common frequent hypercyclicity with respect to densities

We refer to [21] for the abstract definitions and the study of *generalized lower and upper densities*. In particular it is proven there that to any sequence of non-negative real numbers $\alpha = (\alpha_k)_{k\geq 1}$ such that $\sum_{k\geq 1} \alpha_k = +\infty$, one can associate generalized lower and upper densities $\underline{d}_\alpha$ and $\overline{d}_\alpha$ by the formulae

$$\underline{d}_\alpha(E) = \liminf_{n\to\infty} \sum_{k\geq 1} \alpha_{n,k} \mathbb{1}_E(k) \quad \text{and} \quad \overline{d}_\alpha(E) = 1 - \underline{d}_\alpha(\mathbb{N} \setminus E), \quad E \subset \mathbb{N},$$

where $(\alpha_{n,k})_{n,k\geq 1}$ is the matrix given by

$$\alpha_{n,k} = \begin{cases} \alpha_k / (\sum_{j=1}^n \alpha_j) \text{ for } 1 \leq k \leq n, \\ 0 \text{ otherwise.} \end{cases}$$

Then we also have $\overline{d}_\alpha(E) = \limsup_{n\to\infty} \sum_{k=1}^{+\infty} \alpha_{n,k} \mathbb{1}_E(k)$. By [19, Lemma 2.7], if we assume in addition that the sequence $(\alpha_n / (\sum_{j=1}^n \alpha_j))_{n\geq 1}$ converges to 0, then for any set $E \subset \mathbb{N}$ enumerated by an increasing sequence $(n_k)_{k\geq 1}$, we have

$$\underline{d}_\alpha(E) = \liminf_{k\to\infty} \frac{\sum_{j=1}^k \alpha_{n_j}}{\sum_{j=1}^{n_k} \alpha_j}.$$

For α and β two sequences as above, let us write $\alpha \lesssim \beta$ if there exists $k_0 \in \mathbb{N}$ such that $(\alpha_k/\beta_k)_{k\geq k_0}$ is non-increasing. Then we have

$$\underline{d}_\beta(E) \leq \underline{d}_\alpha(E) \leq \overline{d}_\alpha(E) \leq \overline{d}_\beta(E), \quad E \subset \mathbb{N},$$

whenever $\alpha \lesssim \beta$ (see [19, Lemma 2.8]). Thus one can define scales of well-ordered densities with respect to the type of growth of the defining sequences. As we aim to study densities $\underline{d}_\alpha$ which are less than or equal to the natural one, it will be natural to assume that α is non-decreasing.

From now on, a sequence $\alpha = (\alpha_k)_{k\geq 1}$ of non-negative numbers will be called *completely admissible* if it satisfies the following two properties:

- $\sum_{k\geq 1} \alpha_k = +\infty$;
- α is non-decreasing;
- $(\alpha_n/(\sum_{j=1}^n \alpha_j))_{n\geq 1} \to 0$ as $n \to \infty$.

A generalized density $\underline{d}_\alpha$ or $\overline{d}_\alpha$ will be also called *completely admissible* if it is associated to a completely admissible sequence α. Finally, the function $\varphi_\alpha : (0, +\infty) \to (0, +\infty)$ defined by $\varphi_\alpha(x) = \sum_{k\leq x} \alpha_k$, $n \geq 1$, will play an important role in the sequel.

Several examples of generalized densities can be found in [19, 20]. In this work, we will mainly be interested in four types of such sequences.

(1) For $0 \leq \varepsilon \leq 1$, $\mathcal{E}_\varepsilon := (\exp(k^\varepsilon))_{k\geq 1}$. By a summation by parts, one can see that for $0 < \varepsilon < 1$, $\varphi_{\mathcal{E}_\varepsilon}(n) \sim \frac{n^{1-\varepsilon}}{\varepsilon} \exp(n^\varepsilon)$ (where $u_k \sim v_k$ means $u_k/v_k \to 1$);

(2) For $s \in \mathbb{N} \cup \{\infty\}$, $\mathcal{D}_s := (\exp(k/\log_{(s)}(k)))_{k\geq k_0}$ with k_0 large enough, where $\log_{(s)} = \log \circ \cdots \circ \log$, log appearing s times, with the conventions $\log_{(0)}(x) = x$ and $\log_{(\infty)}(x) = 1$ for any $x > 0$. One can check that $\varphi_{\mathcal{D}_s}(n) \sim \log_{(s)}(n) \exp(x/\log_{(s)}(n))$ for $s \in \mathbb{N}$ (see [20, Remark 3.10]) and $\varphi_{\mathcal{D}_\infty}(n) \sim \frac{e}{e-1} \exp(n)$;

(3) For all $l \geq 1$, let us consider the sequence $\mathcal{L}_l = (e^{\log(k)\log_{(l)}(k)})_{k\geq k_0}$, with k_0 large enough. A simple calculation leads to $\varphi_{\mathcal{L}_l}(n) \sim \frac{n e^{\log(n)\log_{(l)}(n)}}{\log_{(l)}(n)}$;

(4) For $r \geq -1$ we shall also write $\mathcal{P}_r := (k^r)_{k\geq 1}$. Then $\varphi_{\mathcal{P}_r}(k) \sim \frac{k^{r+1}}{r+1}$.

Notice that for any $0 \leq \varepsilon < 1$, any $s \in \mathbb{N}$, any $l > 1$ and any $r \geq 0$, the sequences $\mathcal{E}_\varepsilon$, $\mathcal{D}_s$, $\mathcal{L}_l$ and $\mathcal{P}_r$ are completely admissible. Observe that the usual lower density $\underline{d}$ (associated to any constant sequence $(a, a, a, \ldots)$, $a > 0$) corresponds to $\underline{d}_{\mathcal{E}_0}$, $\underline{d}_{\mathcal{D}_0}$ and $\underline{d}_{\mathcal{P}_0}$. Later on, the sequence $\mathcal{E}_1$ shall be simply denoted by $\mathcal{E}$; note that $\underline{d}_{\mathcal{E}} = \underline{d}_{\mathcal{D}_\infty}$. For any $0 < \delta \leq \varepsilon \leq 1$, any $s \leq t \in \mathbb{N}$, any $r \geq 0$ and any positive integer $l \leq l'$, we thus have

$$\underline{d}_{\mathcal{E}} \leq \underline{d}_{\mathcal{D}_t} \leq \underline{d}_{\mathcal{D}_s} \leq \underline{d}_{\mathcal{E}_\varepsilon} \leq \underline{d}_{\mathcal{E}_\delta} \leq \underline{d}_{\mathcal{L}_l} \leq \underline{d}_{\mathcal{L}_{l'}} \leq \underline{d}_{\mathcal{P}_r} \leq \underline{d}.$$

Let X be a separable Fréchet space. As for frequently hypercyclic operators, we now say that a continuous linear operator on T is α-frequently hypercyclic if there exists $x \in X$ such that for any non-empty open set U in X, $\underline{d}_\alpha(N(x, U, T))$ is positive. We denote by $FHC_\alpha(T)$ the set of all α-frequently hypercyclic vectors for T. As proven in [19], no operator can be $\mathcal{E}$-frequently hypercyclic (and hence α-frequently hypercyclic whenever $\mathcal{E} \lesssim \alpha$).

A first natural question arises:

Question 4.1. *Does common α-frequent hypercyclicity exist for some α?*

Let us recall that any operator satisfying the Frequent Universality Criterion is automatically α-universal whenever $\alpha \lesssim \mathcal{D}_s$ for some $s \geq 1$ [20]. Since each of the criteria given in Section 2 are natural strengthenings of the Frequent Hypercyclicity Criterion, we could expect a positive answer to this question for any such α. Moreover, it is easily seen that $FHC_{\mathcal{P}_r}(T) = FHC(T)$ for any $r > -1$ (see [19, Lemma 2.10]). So Question 4.1 has a *strong* positive answer for sequences with polynomial growth. In fact, the next proposition shows that for multiples of a single operator, the answer is either strongly positive, either strongly negative.

We will say that an increasing function $\varphi : (0, +\infty) \to (0, +\infty)$ satisfies the Δ_2-condition if there exists a constant $K > 0$ such that $\varphi(2x) \leq K\varphi(x)$ for any x large enough.

Proposition 4.2. *Let X be a separable Banach space, T a bounded linear operator on X and let $\alpha = (\alpha_k)_{k\geq 1}$ be a completely admissible sequence. Then*

(1) If φ_α satisfies the Δ_2-condition, then $FHC(T) = FHC_\alpha(T)$;
(2) If φ_α does not satisfy the Δ_2-condition, then $HC(\lambda T) \cap FHC_\alpha(\mu T) = \emptyset$ for any $0 < \lambda < \mu < +\infty$.

Proof. To prove (1), let us assume that φ_α satisfies the Δ_2-condition. We need only check that for any $E \subset \mathbb{N}$, if $\underline{d}(E)$ is positive then $\underline{d}_\alpha(E)$ is also positive. Let us enumerate some $E \subset \mathbb{N}$ with $\underline{d}(E) > 0$ by some increasing sequence $(n_k)_{k\geq 1}$. Thus, there exists $M > 0$ with $k \leq n_k \leq Mk$ for any $k \geq 1$. Since α is completely admissible and φ_α satisfies the Δ_2-condition, it is easy to check that there exists a constant K depending on M, such that

$$\sum_{j=1}^{n_k} \alpha_j \leq \sum_{j=1}^{Mk} \alpha_j \leq K \sum_{j=1}^{k} \alpha_j \leq K \sum_{j=1}^{k} \alpha_{n_j},$$

for k large enough, whence

$$\underline{d}_\alpha(E) = \liminf_{k\to\infty} \frac{\sum_{j=1}^{k} \alpha_{n_j}}{\sum_{j=1}^{n_k} \alpha_j} \geq \frac{1}{K}.$$

Let us now prove (2) and let us then fix α such that φ_α does not satisfy the Δ_2-condition. It is not difficult to check that φ_α equivalently satisfies that for all $C > 0$,

$$\liminf_{k\to\infty} \left(\frac{\varphi_\alpha(k)}{\varphi_\alpha((1+C)k)} \right) = 0, \tag{4.1}$$

Let us also fix $0 < \lambda < \mu < +\infty$. We shall assume that there exists $x \in HC(\lambda T) \cap HC(\mu T)$. Throughout the proof, $r > 0$ is fixed. It is enough to prove the following:
(a) $\overline{d}_\alpha(N(x, B(0, r), \lambda T)) = 1$;
(b) $\underline{d}_\alpha(N(x, B(0, r), \mu T)) = 0$.
We first prove (a). By assumption, there exists an increasing sequence $(p_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\|\mu^{p_k} T^{p_k}(x)\| < r$ for any $k \in \mathbb{N}$. Writing

$$\lambda^{p_k+i} T^{p_k+i}(x) = \lambda^i T^i \left(\left(\frac{\lambda}{\mu} \right)^{p_k} \mu^{p_k} T^{p_k}(x) \right), \quad i \in \mathbb{N},$$

we easily check that $\|\lambda^{p_k+i} T^{p_k+i}(x)\| < r$ whenever $(\lambda\|T\|)^i < (\mu/\lambda)^{p_k}$. Since by assumption λT is hypercyclic, we have $\lambda\|T\| > 1$. Thus there exists a constant $C > 0$ (depending on λ, μ and T, but not on k) such that for any $i < Cp_k$, $\|\lambda^{p_k+i} T^{p_k+i}(x)\| < r$. Therefore,

$$\bigcup_{k\in\mathbb{N}} \left\{ p_k, \ldots, \lfloor (1+C)p_k \rfloor \right\} \subset N(x, B(0, r), \lambda T).$$

It follows thanks to (4.1) that

$$\overline{d}_\alpha(N(x, B(0, r), \lambda T)) \geq 1 - \liminf_{k\to\infty} \left(\frac{\varphi_\alpha(p_k)}{\varphi_\alpha((1+C)p_k)} \right) = 1.$$

(b) is proved similarly. Since $x \in HC(\lambda T)$, there exists an increasing sequence $(p_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\|\lambda^{p_k} T^{p_k} x\| > r$. Writing $T^i \mu^{p_k-i} T^{p_k-i} = \mu^{-i}(\mu/\lambda)^{p_k} \lambda^{p_k} T^{p_k}$, $1 \leq i \leq p_k$, one can check that $\|\mu^{p_k-i} T^{p_k-i} x\| \geq r(\mu\|T\|)^{-i}(\mu/\lambda)^{p_k}$, $1 \leq i \leq p_k$. Thus $\|\mu^{p_k-i} T^{p_k-i} x\| > r$ whenever $(\mu\|T\|)^i > (\mu/\lambda)^{p_k}$. Since $\lambda\|T\| > 1$, the last inequality is equivalent to $i \in$

$\{\lfloor Cp_k \rfloor + 1, \dots, p_k\}$ for some constant $0 < C < 1$ not depending on k. Therefore we get, using (4.1) again,

$$\overline{d}_\alpha(N(x, X \setminus B(0,r), \mu T)) \geq 1 - \liminf_{k\to\infty} \left(\frac{\varphi_\alpha(\lfloor Cp_k \rfloor + 1)}{\varphi_\alpha(p_k)} \right) = 1.$$

□

Remark 4.3. *We shall mention that if the function φ_α is assumed to be convex and satisfies that there exists $\beta > 1$ such that $\varphi_\alpha(\beta x) \geq 2\beta\varphi_\alpha(x)$ for any x large enough, then φ_α satisfies the Δ_2-condition if and only if it has a growth controlled from below and from above by some polynomials;* i.e., *if and only if there exist $r, r' \geq 1$ such that*

$$cx^{r'} \leq \varphi_\alpha(x) \leq Cx^r$$

for some constants $c, C > 0$ and any $x \in (1, +\infty)$. Next observe that on the scale of weighted densities given above the only class of sequences α for which φ_α satisfies the Δ_2-condition is given by polynomials $\mathcal{P}_r$.

We can illustrate the preceding Proposition on our examples.

Corollary 4.4. *Let X be a separable Banach space, T a bounded linear operator on X and $l \geq 1$. Then for any $0 < \lambda < \mu < +\infty$,*

$$HC(\lambda T) \cap FHC_{\mathcal{L}_l}(\mu T) = \emptyset.$$

We shall mention that the non-existence of common frequently hypercyclic vectors in Proposition 4.2 concerns multiples λT and μT of the same operator T with $|\lambda| \neq |\mu|$. So we can still wonder whether common frequently hypercyclicity may exist for other kinds of families. Actually, this is the case if we consider families of unimodular multiples of a single operator, as the following extension of León-Müller's Theorem [7, Theorem 6.28] shows.

Theorem 4.5. *Let X be a complex F-space, T a α-frequently hypercyclic operator on X where α is a completely admissible sequence. Then λT is α-frequently hypercyclic for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, and $FHC_\alpha(\lambda T) = FHC_\alpha(T)$.*

The proof goes along the same lines as that of [7, Theorem 6.28], replacing Lemma 6.29 by the following.

Lemma 4.6. *Let $A \subset \mathbb{N}$ be a set of positive lower α-density where α is completely admissible. Let also $I_1, \dots, I_q \subset \mathbb{N}$ be such that $\cup_{j=1}^q I_j = \mathbb{N}$ and $n_1, \dots, n_q \in \mathbb{N}$. Then $B := \cup_{j=1}^q (n_j + A \cap I_j)$ has positive lower α-density.*

Proof. Let $N := \max_{1\leq i\leq q}(n_i)$. On the one hand, for any $M \geq N$,

$$\begin{aligned}
\frac{\sum_{k=1}^{M+N} \alpha_k \mathbb{1}_B(k)}{\sum_{k=1}^{M+N} \alpha_k} &\geq \frac{1}{q} \frac{\sum_{j=1}^q \sum_{k=1}^{M+N} \alpha_k \mathbb{1}_{n_j + A\cap I_j}(k)}{\sum_{k=1}^{M+N} \alpha_k} \\
&\geq \frac{1}{q} \frac{\sum_{j=1}^q \sum_{k=1}^{M} \alpha_k \mathbb{1}_{A\cap I_j}(k)}{\sum_{k=1}^{M+N} \alpha_k} \\
&\geq \frac{1}{q} \frac{\sum_{k=1}^{M} \alpha_k \mathbb{1}_{A}(k)}{\sum_{k=1}^{M+N} \alpha_k} \\
&= \frac{1}{q} \frac{\sum_{k=1}^{M} \alpha_k \mathbb{1}_{A}(k)}{\sum_{k=1}^{M} \alpha_k} \frac{\sum_{k=1}^{M} \alpha_k}{\sum_{k=1}^{M+N} \alpha_k}.
\end{aligned}$$

On the other hand,

$$\frac{\sum_{k=1}^{M}\alpha_k}{\sum_{k=1}^{M+N}\alpha_k} = 1 - \frac{\sum_{k=M+1}^{M+N}\alpha_k}{\sum_{k=1}^{M+N}\alpha_k} \geq 1 - \left(\sum_{j=M+1}^{M+N}\frac{\alpha_j}{\sum_{k=1}^{j}\alpha_k}\right) \longrightarrow 1, \quad \text{as } M \to +\infty.$$

Hence,

$$\underline{d}_\alpha(B) = \liminf_{M\to\infty}\frac{\sum_{k=1}^{M+N}\alpha_k \mathbb{1}_B(k)}{\sum_{k=1}^{M+N}\alpha_k} \geq \liminf_{M\to\infty}\frac{1}{q}\frac{\sum_{k=1}^{M}\alpha_k\mathbb{1}_A(k)}{\sum_{k=1}^{M}\alpha_k} = \frac{1}{q}\,\underline{d}_\alpha(A) > 0.$$

□

To conclude, we come back to the main result of [20]: any operator satisfying the Frequent Universality Criterion [11] is automatically α-universal if $\alpha \lesssim \mathcal{D}_s$ for some $s \geq 1$. The following question naturally arises:

Question 4.7. *For any operator $T \in \mathcal{L}(X)$ satisfying the Frequent Universality Criterion, does there exist a vector $x \in X$ which is α-frequently universal for T and for any $\alpha \lesssim \mathcal{E}$?*

The following proposition gives a positive answer.

Proposition 4.8. *We denote by $\mathcal{D}$ the set of all completely admissible sequences of non-negative numbers $\alpha = (\alpha_k)_{k\geq 1}$ such that $\alpha \lesssim \mathcal{D}_s$ for some $s \in \mathbb{N}$. If $T \in \mathcal{L}(X)$ satisfies the Frequent Universality Criterion, then*

$$\bigcap_{\alpha\in\mathcal{D}} FHC_\alpha(T) \neq \emptyset.$$

Proof. It is enough to prove that $\bigcap_{s\in\mathbb{N}} FHC_{\mathcal{D}_s}(T)$ is non-empty. The proof is based on the calculations led in [20, Section 3]. Let us consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(j) = m$ for all $j \in \{a_m, \dots, a_{m+1} - 1\}$ with

$$a_m = 2^{2^{\cdot^{\cdot^{\cdot^{2^m}}}}} \quad \text{where 2 appears } m \text{ times.}$$

Then we define the sequence $(n_k(f))_{k\geq 1}$ as follows:

$$n_1(f) = 2 \quad \text{and} \quad n_k(f) = 2\sum_{i=1}^{k-1} f(\delta_i) + f(\delta_k),\ k \geq 2,$$

where δ_j is the index of the first zero in the dyadic representation of j (for *e.g.*, if $k = 11 = 1.2^0 + 1.2^1 + 0.2^2 + 1.2^3$, then $\delta_k = 3$). Lemma 3.8 of [20] ensures that for all $s \geq 1$ there exist $C_1, C_2, C_3 > 0$ such that for all integers k large enough,

$$C_1 k - C_2 \log_{(s)}(k) \leq n_k(f) \leq C_1 k + C_3 \log_{(s)}(k).$$

A similar calculation as that of [19, Lemma 4.10] allows us to conclude that $d_{\mathcal{D}_s}((n_k(f))) > 0$ for all $s \geq 1$. Now one can use this sequence $(n_k(f))_{k\geq 1}$ to construct a universal vector for T which will be $\mathcal{D}_s$-frequently universal for all $s \geq 1$ (we refer the reader to the beginning of Section 2 of [20]). □

5. Remark: an ergodic approach?

We shall conclude this paper by a word on the possible approach to common frequent hypercyclicity by ergodic theory. The most natural way to conceive frequent hypercyclicity is probably through Birkhoff's ergodic theorem: if T is a bounded linear operator on some

separable Banach space X, which is an ergodic measure-preserving transformation with respect to some measure m with full support, then

$$\lim_{N\to\infty} \frac{\mathrm{card}(N(x,U,T)\cap[0,N])}{N+1} = m(U) > 0,$$

for any non-empty open set U of X. In particular T is frequently hypercyclic. We recall that T is measure-preserving for m if for any measurable set $A \subset X$, we have $m(A) = m(T^{-1}(A))$, and that T is ergodic with respect to m if for every measurable subsets A and B of X, with $m(A), m(B) > 0$, there exists an integer n such that $m(T^{-n}(A)\cap B) > 0$ (see [34] for instance).

Now, if T_1 and T_2 are two ergodic measure-preserving transformations of X for respectively two measures m_1 and m_2 which are absolutely continuous with respect to each other, then T_1 and T_2 automatically share a common frequently hypercyclic vector. Theorem 3.22 in [4] gives a sufficient condition for an operator T on X to be ergodic and measure-preserving for some Gaussian measure m_T. In general, for two operators T_1 and T_2 satisfying this condition, m_{T_1} and m_{T_2} are not absolutely continuous with respect to each other and one cannot conclude whether they share a common frequently hypercyclic vector or not.

But the opposite situation can also occur: for example, let B_{w_1} and B_{w_2} be two weighted shifts on $\ell^2(\mathbb{N})$ such that the supremum r_{p,w_i} of their point spectrum is greater than 1, $i = 1, 2$ (see Paragraph 2.3.2 where r_{p,w_i} appears). Then by [4, Theorem 3.22], B_{w_1} and B_{w_2} are ergodic and measure-preserving with respect to some Gaussian measures m_{w_1} and m_{w_2}. For $i = 1, 2$, let us define $w_{i,n} = \prod_{k=0}^{n} w_i(k)$. Now, as explained in [4, Pages 5111-5112], m_{w_1} and m_{w_2} are absolutely continuous with respect to each other if and only if the sequence $(1-\sqrt{w_{1,n}/w_{2,n}})_{n\in\mathbb{N}}$ is in $\ell^2(\mathbb{N})$. This condition is much stronger than the condition derived from the proof of Theorem 2.18 (see Remark 2.19) which ensures the existence of common frequently hypercyclic vectors for more general families of weighted shifts. Note also that an ergodic approach has not permitted so far to obtain common frequent hypercyclicity for general multiples of a single operator. Yet, the fact that $(1-\sqrt{w_{1,n}/w_{2,n}})_{n\in\mathbb{N}}$ is in $\ell^2(\mathbb{N})$ implies that $m_{w_1}(FHC(B_{w_1})\cap FHC(B_{w_2})) = 1$, while our results give no quantitative information on the size of the set of common frequently hypercyclic vectors.

It would be of interest to investigate further the problem of common frequent hypercyclicity from the point of view of ergodicity.

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