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COMMON FREQUENT HYPERCYCLICITY

S. CHARPENTIER, R. ERNST, M. MESTIRI, A. MOUZE

ABSTRACT. We provide with criteria for a family of sequences of operators to share a frequently universal vector. These criteria are variants of the classical Frequent Hypercyclicity Criterion and of a recent criterion due to Grivaux, Matheron and Menet where periodic points play the central role. As an application, we obtain for any operator T in a specific class of operators acting on a separable Banach space, a necessary and sufficient condition on a subset Λ of the complex plane for the family $\{\lambda T : \lambda \in \Lambda\}$ to have a common frequently hypercyclic vector. In passing, this permits us to easily exhibit frequent hypercyclic weighted shifts which do not possess common frequent hypercyclic vectors. We also provide with criteria for families of the recently introduced operators of C -type to share a common frequently hypercyclic vector. Further, we prove that the same problem of common α -frequent hypercyclicity may be vacuous, where the notion of α -frequent hypercyclicity extends that of frequent hypercyclicity replacing the natural density by more general weighted densities. Finally, it is already known that any operator satisfying the classical Frequent Universality Criterion is α -frequently universal for any sequence α satisfying a suitable condition. We complement this result by showing that for any such operator, there exists a vector x which is α -frequently universal for T , with respect to *all* such α .

1. INTRODUCTION

For two separable Fréchet spaces X and Y , let us denote by $\mathcal{L}(X, Y)$ the set of all continuous operators from X to Y . If $X = Y$, we simply write $\mathcal{L}(X) = \mathcal{L}(X, Y)$. A sequence $\mathcal{T} = (T_n)_{n \in \mathbb{N}} \subset \mathcal{L}(X, Y)$ (where $\mathbb{N} := \{0, 1, 2, \dots\}$) is said to be *universal* provided there exists a vector $x \in X$ such that for any non-empty open subset U of Y , the set

$$N(x, U, \mathcal{T}) := \{n \in \mathbb{N} : T_n x \in U\}$$

is infinite. The vector x is also called *universal* and the set of all universal vectors for \mathcal{T} is denoted by $\mathcal{U}(\mathcal{T})$. A single operator $T \in \mathcal{L}(X)$ is called *hypercyclic* if the sequence $(T^n)_{n \in \mathbb{N}}$ of its iterates is universal. In this case, we write $N(x, U, \mathcal{T}) = N(x, U, T)$ and $\mathcal{U}(\mathcal{T}) = HC(T)$. In 2006, Bayart and Grivaux [4] introduced the important notion of *frequently hypercyclic operator*. An operator $T \in \mathcal{L}(X)$ is said to be *frequently hypercyclic* if there exists $x \in X$ such that for any non-empty open subset U of X , the *lower density* $\underline{d}(N(x, U, T))$ of $N(x, U, T)$ is positive, where for any $E \subset \mathbb{N}$,

$$\underline{d}(E) := \liminf_n \frac{\text{card}([0, n] \cap E)}{n+1} > 0.$$

Such a vector x is a *frequently hypercyclic vector* for T and the set of such vectors is denoted by $FHC(T)$. The notion of *frequent universality* for a sequence \mathcal{T} of operators in $\mathcal{L}(X, Y)$ can obviously be defined (see, for e.g., [11]). The set of *frequently universal vectors* for \mathcal{T} will be denoted by $F\mathcal{U}(\mathcal{T})$. For a rich source of information about Linear Dynamics, we refer to the monographs [7, 26].

A problem which has been extensively studied during the last decades is that of *common hypercyclicity*. For a given family $(T_\lambda)_{\lambda \in \Lambda}$ of hypercyclic operators in $\mathcal{L}(X)$, it asks when the

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set of *common hypercyclic vectors*, $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$, is empty and when it is not. [7, Chapter 7] and [26, Chapter 11] are entirely devoted to this topic. On the one hand, since $HC(T)$ is a dense G_δ subset of X whenever it is non-empty, the Baire Category Theorem trivially ensures that $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is non-empty whenever Λ is countable. On the other hand, it is not difficult to exhibit families of hypercyclic operators with no common hypercyclic vectors (for example the family of all hypercyclic operators on a given space X). The first positive important result in this direction was given by Abakumov and Gordon [1] who showed that $\bigcap_{\lambda > 1} HC(\lambda B) \neq \emptyset$, where B is the backward shift on $\ell^2(\mathbb{N})$ defined by $B(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$. Later on, Costakis and Sambarino [18] provided with the first *criterion of common hypercyclicity* that they applied to show the residuality of the set of common hypercyclic vectors for multiples of the backward shift or differential operators, and for uncountable families of translation operators or specific weighted shifts. Constructions or the approach used by Costakis and Sambarino, based on the Baire Category Theorem, were developed by many authors to produce new criteria or prove common hypercyclicity for other uncountable families of classical operators, such as adjoint of multipliers, or composition and convolution operators (see, for e.g., [2, 5, 6, 13, 16, 24]). A second approach to the problem, more algebraic, produced some of the most striking results. León and Müller proved that for any $T \in \mathcal{L}(X)$ and any $\lambda \in \mathbb{C}$, $|\lambda| = 1$, $HC(T) = HC(\lambda T)$. Their idea, which exploits the group structure of the torus $\mathcal{T} = \{z \in \mathbb{C} : |z| = 1\}$, was extended by several authors to families of operators forming groups or semigroups, and then combine with the first approach to produce some new and strong results (for e.g., [3, 9, 15, 32, 34]). We should say that the non-existence of common universal vectors has also been studied (see, for e.g., [3, 7, 19, 26]).

In comparison, *common frequent hypercyclicity* has been considered in only a very few amount of papers. Probably, it is partly because the Baire Category approach drastically fails for this notion: by [8, Corollary 19], the set $FHC(T)$ is always meager (i.e., contained in the complement of a residual set). Moreover, for any $T \in \mathcal{L}(X)$, it turns out that the set $\bigcap_{\lambda \in \Lambda} FHC(\lambda T)$ is empty, as soon as $\Lambda \subset (0, +\infty)$ is uncountable ([3, Proposition 6.4]). However, the algebraic approach to common hypercyclicity perfectly fits to frequent hypercyclicity. For example, Bayart and Matheron proved that $FHC(\lambda T) = FHC(T)$ for any $\lambda \in \mathcal{T}$, obtaining a *frequent version* of León-Müller's result. This approach has been pursued further in [3] (see also [15]) and led to several nice results of common frequent hypercyclicity for families of operators forming strongly continuous groups or semigroups (translation operators on $H(\mathbb{C}^d)$, composition operators induced by non-constant Heisenberg translations on the Hardy space of the Siegel half-space, etc...). Moreover, in specific classes of operators, hypercyclic basically means frequently hypercyclic in a strong sense. For example, if Λ_{hyp} denotes the set of all hyperbolic automorphisms of the unit disc \mathbb{D} having the same boundary attractive point, then the same argument as in [7, Example 7.3] gives that there exists $\phi_0 \in \Lambda_{hyp}$ such that for any $\phi \in \Lambda_{hyp}$, $FHC(C_{\phi_0}) \subset FHC(C_\phi)$, where C_ϕ denotes the composition operator with symbol ϕ on the Hardy space H^2 of \mathbb{D} . Combined with the algebraic approach, this yields $\bigcap_{\phi \in \Lambda} FHC(C_\phi) \neq \emptyset$ where Λ stands for the set of all automorphisms having a common boundary attractive point. All in all, except when action by strongly continuous groups or semigroups is involved, so far no criteria for common frequent hypercyclicity are known. In particular, we do not know under which non-trivial conditions on $\Lambda \subset (0, +\infty)$ and $T \in \mathcal{L}(X)$ the set $\bigcap_{\lambda \in \Lambda} FHC(\lambda T)$ may be non-empty.

In this paper we aim to contribute in filling these gaps. Our first result is a criterion of common frequent universality (Theorem 2.3) which is a natural strengthening of the *Frequent Universality Criterion* given in [12] (and of the classical Frequent Hypercyclicity Criterion [4]). As an application, we get necessary and/or sufficient conditions on a subset Λ of \mathbb{C} for the set $\bigcap_{\lambda \in \Lambda} FHC(\lambda T)$ to be non-empty, when X is a Banach space and $T \in \mathcal{L}(X)$. For example, we will get the following:

Theorem. *Let B be the backward shift on $\ell^2(\mathbb{N})$ and let $\Lambda \subset \mathbb{C}$. The set $\bigcap_{\lambda \in \mathbb{C}} FHC(\lambda T)$ is non-empty if and only if the set $\{|\lambda| : \lambda \in \Lambda\}$ is a countable relatively compact subset of $(1, +\infty)$.*

This theorem is obtained for more general classes of (unilateral) weighted shifts on $\ell^2(\mathbb{N})$. For any operator $T \in \mathcal{L}(X)$, sufficient or necessary conditions on Λ are given, involving the spectral or the local spectral radius of T . In full generality, our sufficient condition exactly coincides with the assumption of a criterion of *common hypercyclicity* given by Bayart and Matheron [6, Proposition 4.2]. Our general criterion of common frequent universality is also applied to countable families of weighted shifts, differential operators or adjoint of multipliers (which may not be multiples of a single operator). In passing, we deduce a simple way to produce two frequently hypercyclic weighted shifts *without* common frequently hypercyclic vectors.

Recently, Grivaux, Matheron and Menet provided with a new frequent hypercyclicity criterion, based on the periodic points of the operator [25]. They prove that this criterion is theoretically better than the classical Frequent Hypercyclicity Criterion since any operator satisfying the assumptions of the latter automatically satisfies that of the new one. In practice, the classical criterion turns out to be much simpler to apply to most of the explicit and usual operators. However, Menet introduced a new class of operators, the so-called operators of C -type [27], conceived as a very rich source of counter-examples to difficult problems (such as the exhibition of a chaotic operator on ℓ^p which is not frequently hypercyclic [27], see also [25]), to which their new criterion for frequent hypercyclicity is very well adapted. In the present paper, based on this criterion, we establish another general criterion for common frequent hypercyclicity, involving the periodic points of the family of operators. Once again, we show how this can be easily applied to classes of operators of C -type.

Furthermore, Ernst and Mouze recently proved [20, 21] that any operator satisfying the usual Frequent Universality Criterion in fact enjoys a stronger form of frequent universality. Let $\alpha = (\alpha_k)_{k \geq 1}$ be a sequence of non-negative real numbers with $\sum_{k \geq 1} \alpha_k = +\infty$. In [23], Freedman and Sember show that if a matrix $(\alpha_{n,k})_{n,k \geq 1}$ is given by

$$\alpha_{n,k} = \begin{cases} \alpha_k / (\sum_{j=1}^n \alpha_j) & \text{for } 1 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

then the function $\underline{d}_\alpha : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ ($\mathcal{P}(\mathbb{N})$ denotes the set of all subsets of \mathbb{N}) defined for $E \subset \mathbb{N}$ by

$$\underline{d}_\alpha(E) = \liminf_n \left(\sum_{k \geq 1} \alpha_{n,k} \mathbb{1}_E(k) \right)$$

is a *generalized lower density* (see [23] for the abstract definition of a (generalized lower) density). We call $\underline{d}_\alpha(E)$ the *lower α -density* of E . The usual lower density encountered above corresponds to the constant sequence $(1, 1, 1, \dots)$. Moreover, if $\alpha \lesssim \beta$ (meaning α_k / β_k is eventually decreasing to 0), then $\underline{d}_\beta(E) \leq \underline{d}_\alpha(E)$, $E \subset \mathbb{N}$ ([20, Lemma 2.8]). The order \lesssim thus allows to define (ordered) scales of generalized lower densities. We refer to [20, 21] for examples of sequences α defined by usual functions and well-ordered with respect to \lesssim . It then appears natural to define α -frequent universality as the usual frequent universality, replacing the sequence $(1, 1, 1, \dots)$ by any α as above. One of the main results of [20, 21] is that any operator $T \in \mathcal{L}(X)$ which satisfies the Frequent Universality Criterion is \underline{d}_α -frequently universal whenever there exists $s \geq 2$ such that $\alpha \lesssim (\exp(k / (\log_{(s)}(k))))_{k \geq 1}$ where $\log_{(s)} = \log \circ \log \circ \dots \circ \log$, \log appearing s times. Moreover, they prove that no operator can be α -frequent hypercyclic for $\alpha_k = e^k$. In view of the topic of the paper, two natural

questions arise. For $T \in \mathcal{L}(X)$ we denote by $FHC_\alpha(T)$ the set of all α -frequent hypercyclic vectors for T .

Questions. Let A denote the set of sequences α such that $\alpha \lesssim d^s$ for some $s \geq 2$ and let $T \in \mathcal{L}(X)$.

- 1) Let $\Lambda \in (0, +\infty)$ and $B \subset A$ be non-trivial. Do we have $\bigcap_{(\lambda, \beta) \in \Lambda \times B} FHC_\beta(\lambda T) \neq \emptyset$?
- 2) If T satisfies the Frequent Hypercyclicity Criterion, do we have $\bigcap_{\alpha \in A} FHC_\alpha(T) \neq \emptyset$?

We will give a positive answer to the second question (Proposition 4.6) and show that the first one has a strongly negative answer if Λ is any non-trivial subset of $(0, +\infty)$ and B is reduced to a single generalized density which grows faster than $(e^{k^\varepsilon})_{k \geq 1}$ for some $\varepsilon > 0$ (Proposition 4.2). We should mention that, by [20, Lemma 2.10], $FHC_\beta(T) = FHC(T)$ whenever β has a growth at most polynomial (i.e., $\beta \lesssim (k^r)_{k \geq 1}$ for some $r \geq -1$). Combined with our first common frequent hypercyclicity criterion, this thus gives a positive answer to (1) for some non-trivial Λ and the set B of sequences with at most polynomial growth.

We should conclude by mentioning that the problem of common hypercyclicity has been considered for the upper (or \mathcal{U} -)frequent hypercyclicity. This intermediate notion between hypercyclicity and frequent hypercyclicity was introduced by Shkarin [33]. A sequence $\mathcal{T} \subset \mathcal{L}(X, Y)$ is said to be \mathcal{U} -frequently universal if for some $x \in X$ and any non-empty open set U in Y , $\bar{d}(N(x, U, \mathcal{T}))$ is positive. By definition, $\bar{d}(E) = 1 - \underline{d}(\mathbb{N} \setminus E)$ is the *upper density* of $E \subset \mathbb{N}$. In some sense, \mathcal{U} -frequent hypercyclicity is *closer* to hypercyclicity than to frequent hypercyclicity. For example, Bayart and Ruzsa proved that the set $\mathcal{U}FHC(T)$ of all \mathcal{U} -frequently hypercyclic vectors for T is residual whenever it is non-empty [8, Proposition 21]. Common \mathcal{U} -frequent hypercyclicity has been rather well-studied and criteria have been given. We refer to [28, 29] and the references therein for an up-to-date and complete overview on the subject. In the sequel, we shall (almost) not come back to this notion.

The paper is organized as follows. Section 2 is devoted to our first general criteria of common frequent universality and their applications. In Section 3, we focus on the statement of our second criterion for common frequent hypercyclicity involving periodic points. We finally give the answers to the last two questions in Section 4.

2. COMMON FREQUENT UNIVERSALITY FOR COUNTABLE FAMILIES OF OPERATORS

2.1. A general criterion. For the proof of the main result of this section, we will make use of the following refinement of [7, Lemma 6.19] and of ideas developed in [8].

Lemma 2.1. *For every $K > 1$ and every countable family $(N_p(i))_p$, $i \in \mathbb{N}$, of increasing sequences of positive integers, there exists a countable family $(E_p(i))_p$, $i \in \mathbb{N}$, of sequences of sets $E_p(i) \subset \mathbb{N}$ with positive lower density, such that for every $(p, i), (q, j) \in \mathbb{N}^2$,*

- (1) $\min(E_p(i)) \geq N_p(i)$;
- (2) For every $n \in E_p(i)$, $m \in E_q(j)$, $[n \neq m \implies |n - m| \geq N_p(i) + N_q(j) \geq \max(N_p(i), N_q(j))]$;
- (3) If $(p, i) \neq (q, j)$, then for every $n \in E_p(i)$, $m \in E_q(j)$, $[n > m \implies n \geq Km]$;

Proof. Let $K > 1$ and for every $i \in \mathbb{N}$, let $(N_p(i))_p$ be an increasing sequences of positive integers. Moreover, for every $i \in \mathbb{N}$, let us denote by $(A_p(i))_p$ a sequence of syndetic sets forming a partition of \mathbb{N} . Let also $0 < \varepsilon < \frac{1}{2}$ and $a > 1$ be such that

$$\frac{1 - 2\varepsilon}{1 + 2\varepsilon} a > K.$$

For every $u \in \mathbb{N}$, we pose:

$$I_u^{a, \varepsilon} = [(1 - \varepsilon)a^u; (1 + \varepsilon)a^u]$$

and we set:

$$E_p(i) = \cup_{u \in A_p(i)} (I_u^{a,\varepsilon} \cap (2N_p(i)\mathbb{N})).$$

Remark that by definition, (2) is satisfied when $(p, i) = (q, j)$. Remark also that for every $u \in A_p(i)$, we have the following equivalence:

$$I_u^{a,\varepsilon} + [-N_p(i); N_p(i)] \subseteq I_u^{a,2\varepsilon} \Leftrightarrow N_p(i) \leq \varepsilon a^u.$$

Thus, it suffices to remove a finite number of elements in each $A_p(i)$ to ensure that both conditions above are satisfied. Moreover, such a modification of the sets $A_p(i)$ has no influence on rest the proof.

In the same spirit, one may check that for every $u > v$,

$$I_u^{a,2\varepsilon} \cap I_v^{a,2\varepsilon} = \emptyset \Leftrightarrow 1 < \frac{1-2\varepsilon}{1+2\varepsilon} a^{u-v}.$$

Moreover, the choice we made on a and ε gives

$$1 < K < \frac{1-2\varepsilon}{1+2\varepsilon} a < \frac{1-2\varepsilon}{1+2\varepsilon} a^{u-v}.$$

Therefore, for every $u > v$, $I_u^{a,2\varepsilon} \cap I_v^{a,2\varepsilon} = \emptyset$. This last relation and the previous one prove that (2) is satisfied.

To check that (3) is satisfied, remark first that our assumptions on a and ε implies that $\frac{1-\varepsilon}{1+\varepsilon} a > K$. Then, for every $n > m$ with $n \in E_p(i)$ and $m \in E_q(j)$, there exists $u \in A_p(i)$ and $v \in A_q(j)$ with $u > v$ so that:

$$\begin{cases} (1-\varepsilon)a^u \leq n \leq (1+\varepsilon)a^u \\ (1-\varepsilon)a^v \leq m \leq (1+\varepsilon)a^v \end{cases}$$

Thus, we have:

$$Km \leq K(1+\varepsilon)a^v < (1-\varepsilon)a^{v+1} \leq (1-\varepsilon)a^u \leq n.$$

This proves (3)

Condition (1) is easy to obtain, up to removing a finite number of elements from each set $E_p(i)$ which does not modify the other conditions.

Finally, it remains to prove that each set $E_p(i)$ has positive lower density. Let $p, i \in \mathbb{N}$ and $(n_k)_{k \in \mathbb{N}}$ be an enumeration of the set $A_p(i)$ and M be the maximal size of a gap in $A_p(i)$. Then,

$$\begin{aligned} d(E_p(i)) &= \liminf_{k \rightarrow \infty} \frac{\#(E_p(i) \cap [0; (1+\varepsilon)a^{n_k}])}{(1-\varepsilon)a^{n_{k+1}}} \\ &\geq \liminf_{k \rightarrow \infty} \left(\frac{2\varepsilon a^{n_k}}{2N_p(i)} - 2 \right) \frac{1}{a^{n_{k+1}}} \\ &\geq \liminf_{k \rightarrow \infty} \left(\frac{\varepsilon a^{n_k}}{N_p(i)} - 2 \right) \frac{1}{a^{n_k+M}} \\ &= \frac{\varepsilon}{N_p(i)a^M} > 0 \end{aligned}$$

This ends the proof of the lemma. \square

We recall the definition of uniform unconditional convergence.

Definition 2.2. Let Λ be a set. We say that the series $\sum x_{\lambda,n}$, $\lambda \in \Lambda$ in X converges unconditionally uniformly for $\lambda \in \Lambda$ if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for any finite set $F \subset \{N, N+1, \dots\}$, one has

$$\left\| \sum_{n \in F} x_{\lambda,n} \right\| < \varepsilon$$

for every $\lambda \in \Lambda$.

Our general common frequent universality criterion for countable families of operators states as follows.

Theorem 2.3. *Let X be an F -space, Y a separable F -space and $(T_{i,n})_{n \in \mathbb{N}}$, $i \in \mathbb{N}$, be countably many sequences of continuous linear operators from X to Y . We assume that there exists a dense subset Y_0 of Y , mappings $S_{i,n} : Y_0 \rightarrow X$, $i, n \in \mathbb{N}$, and a real number $c > 1$ such that for every $y \in Y_0$,*

- (1) *The series $\sum_{n=0}^m T_{i,m-n}(S_{i,m-n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \in \mathbb{N}$;*
- (2) *The series $\sum_{n \geq 0} T_{i,m+n}(S_{i,m+n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \in \mathbb{N}$;*
- (3) *The series $\sum_{n \geq (c-1)m} T_{i,m}(S_{j,m+n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \neq j \in \mathbb{N}$;*
- (4) *The series $\sum_{\frac{c-1}{c}m \leq n \leq m} T_{i,m}(S_{j,m-n}(y))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \neq j \in \mathbb{N}$;*
- (5) *The series $\sum_{n \geq 0} S_{i,n}(y)$ converges unconditionally, uniformly for $i \in \mathbb{N}$;*
- (6) *The sequence $(T_{i,n}(S_{i,n}(y)))$ converges to x , uniformly for every $i \in \mathbb{N}$.*

Then there exists a vector $x \in X$ frequently universal for every $(T_{i,n})_n$, $i \in \mathbb{N}$.

One can easily check that each $(T_{i,n})_n$, $i \in \mathbb{N}$ satisfies (1), (2), (5) and (6) if and only if it satisfies the Frequent Universality Criterion given in [12].

Proof. Since Y is separable, we can assume that $Y_0 = \{y_0, y_1, \dots\}$. Let $(\varepsilon_p)_{p \in \mathbb{N}}$ be a decreasing sequence of positive real numbers such that $\sum_p \varepsilon_p < 1$ and $p\varepsilon_p \rightarrow 0$ as $p \rightarrow \infty$. We also fix an increasing sequence $(J_p)_p$ such that $\sum_{i \geq J_p} \varepsilon_i < \varepsilon_p$. The assumptions of the theorem imply the existence of a sequence $(N_p(i))_{i,p \in \mathbb{N}}$ such that for every $i, p \in \mathbb{N}$, every finite set $F \subset \{N_p(i), N_p(i) + 1, \dots\}$, every $m \in \mathbb{N}$, every $q \in \{0, \dots, p\}$, every $k \in \mathbb{N}$ and every $n \geq N_p(i)$,

$$\begin{aligned}
\text{(i)} \quad & \left\| \sum_{\substack{n \in F \\ n < m}} T_{k,m}(S_{k,m-n}(y_q)) \right\| < \varepsilon_p; & \text{(v)} \quad & \left\| \sum_{\substack{n \in F \\ \frac{c-1}{c}m \leq n \leq m}} T_{k,m}(S_{l,m-n}(y_q)) \right\| < \varepsilon_p \varepsilon_i \text{ for} \\
& & & \text{every } l \neq k \in \mathbb{N}; \\
\text{(ii)} \quad & \left\| \sum_{n \in F} T_{k,m}(S_{k,m+n}(y_q)) \right\| < \varepsilon_p; & \text{(vi)} \quad & \left\| \sum_{\substack{n \in F \\ \frac{c-1}{c}m \leq n \leq m}} T_{k,m}(S_{l,m-n}(y_q)) \right\| < \varepsilon_{J_p} \varepsilon_p \\
\text{(iii)} \quad & \left\| \sum_{\substack{n \in F \\ n \geq (c-1)m}} T_{k,m}(S_{l,m+n}(y_q)) \right\| < \varepsilon_p \varepsilon_i \text{ for} & & \text{for every } l \in \{0, \dots, J_p\} \text{ and } l \neq k \in \mathbb{N}; \\
& \text{every } l \neq k \in \mathbb{N}; & \text{(vii)} \quad & \left\| \sum_{n \in F} S_{k,n}(y_q) \right\| < \varepsilon_p \varepsilon_i; \\
\text{(iv)} \quad & \left\| \sum_{\substack{n \in F \\ n \geq (c-1)m}} T_{k,m}(S_{l,m+n}(y_q)) \right\| < \varepsilon_{J_p} \varepsilon_p \text{ for} & \text{(viii)} \quad & \|T_{k,n}(S_{k,n}(y_q)) - y_q\| < \varepsilon_q. \\
& \text{every } l \in \{0, \dots, J_p\} \text{ and } l \neq k \in \mathbb{N}; & &
\end{aligned}$$

Let $(E_p(i))_{i,p \in \mathbb{N}}$ be a sequence of sets given by Lemma 2.1 applied to the sequence $(N_p(i))_{i,p}$ and to $K = c$. We put

$$x = \sum_{i \in \mathbb{N}} \sum_{p \in \mathbb{N}} \sum_{n \in E_p(i)} S_{i,n}(y_p).$$

One easily checks that $x \in X$. Indeed, since $\min(E_p(i)) \geq N_p(i)$, (vii) gives

$$\sum_{i \in \mathbb{N}} \sum_{p \in \mathbb{N}} \left\| \sum_{n \in E_p(i)} S_{i,n}(y_p) \right\| < \sum_{i \in \mathbb{N}} \sum_{p \in \mathbb{N}} \varepsilon_p \varepsilon_i < \infty.$$

Note that x is even unconditionally convergent. Our goal is now to prove that x is a frequently universal vector for each sequence $(T_{i,n})_n$, $i \in \mathbb{N}$. We fix $j \in \mathbb{N}$. Let $(r_q)_q$ be a sequence of positive real numbers with $r_q \rightarrow 0$ as $q \rightarrow \infty$, to be chosen later. Since the sets $E_p(i)$, $i, p \in \mathbb{N}$, have positive lower density, it is sufficient to prove that

$$(2.1) \quad \|T_{j,m}(x) - y_q\| < r_q \text{ for every } j \in \mathbb{N}, q \in \mathbb{N} \text{ and every } m \in E_q(j).$$

Using that $E_p(i) \cap E_q(j) = \emptyset$ if $(i, p) \neq (j, q)$ and that x is unconditionally convergent in X , if $m \in E_q(j)$ then we can decompose $T_{j,m}(x)$ as follows:

$$T_{j,m}(x) = T_{j,m}(S_{j,m}(y_q)) + \overbrace{\sum_{p \in \mathbb{N}} \sum_{\substack{n \in E_p(j) \\ n \neq m}} T_{j,m}(S_{j,n}(y_p))}^{A_m} + \overbrace{\sum_{\substack{i \in \mathbb{N} \\ i \neq j}} \sum_{p \in \mathbb{N}} \sum_{\substack{n \in E_p(i) \\ n \neq m}} T_{j,m}(S_{i,n}(y_p))}^{B_m}.$$

First, since $m \geq N_q(j)$ for any $m \in E_q(j)$, (viii) gives

$$(2.2) \quad \|T_{j,m}(S_{j,m}(y_q)) - y_q\| < \varepsilon_q.$$

We next estimate A_m :

$$\|A_m\| \leq \sum_{p \in \mathbb{N}} \left(\left\| \sum_{\substack{n \in E_p(j) \\ n < m}} T_{j,m}(S_{j,m-(m-n)}(y_p)) \right\| + \left\| \sum_{\substack{n \in E_p(j) \\ n > m}} T_{j,m}(S_{j,m+(n-m)}(y_p)) \right\| \right).$$

Given that $|n - m| \geq \max(N_p(j), N_q(j))$ for any $n \in E_p(j)$ and $m \in E_q(j)$, $n \neq m$, (i) and (ii) yield

$$(2.3) \quad \|A_m\| < 2 \sum_{p \leq q} \varepsilon_q + 2 \sum_{p > q} \varepsilon_p =: r_{1,q}.$$

We now turn to estimating B_m . Again, by unconditional convergence of the series, we have

$$\|B_m\| \leq \overbrace{\sum_{p \in \mathbb{N}} \sum_{\substack{i \in \mathbb{N} \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n < m}} T_{j,m}(S_{i,m-(m-n)}(y_p)) \right\|}^{B_m^1} + \overbrace{\sum_{p \in \mathbb{N}} \sum_{\substack{i \in \mathbb{N} \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n > m}} T_{j,m}(S_{i,m+(n-m)}(y_p)) \right\|}^{B_m^2}.$$

We deal first with B_m^2 . We have

$$(2.4) \quad \begin{aligned} \|B_m^2\| &\leq \sum_{p > q} \sum_{\substack{i \in \mathbb{N} \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n > m}} T_{j,m}(S_{i,m+(n-m)}(y_p)) \right\| + \\ &+ \sum_{p \leq q} \left(\sum_{\substack{i \leq J_q \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n > m}} T_{j,m}(S_{i,m+(n-m)}(y_p)) \right\| + \sum_{\substack{i > J_q \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n > m}} T_{j,m}(S_{i,m+(n-m)}(y_p)) \right\| \right) \end{aligned}$$

We recall that Lemma 2.1 was applied to $K = c$. So, for $n \in E_p(i)$ and $m \in E_q(j)$ with $(i, p) \neq (j, q)$, we have $|n - m| \geq \max(N_p(i), N_q(j))$. Moreover, $n > m$ implies $n \geq cm$, hence $n - m \geq (c - 1)m$. In particular, $n - m \geq \max(N_p(i), (c - 1)m)$. It follows from (iii) that

$$(2.5) \quad \sum_{p > q} \sum_{\substack{i \in \mathbb{N} \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n > m}} T_{j,m}(S_{i,m+(n-m)}(y_p)) \right\| \leq \sum_{p > q} \sum_{i \in \mathbb{N}} \varepsilon_i \varepsilon_p \leq \sum_{p > q} \varepsilon_p$$

and that

$$(2.6) \quad \sum_{p \leq q} \sum_{\substack{i > J_q \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n > m}} T_{j,m}(S_{i,m+(n-m)}(y_p)) \right\| \leq \sum_{p \leq q} \varepsilon_p \sum_{i > J_q} \varepsilon_i \leq q\varepsilon_q.$$

In the last inequality, we use that $0 < \varepsilon_p < 1$ and the definition of $(J_p)_p$ (i.e., $\sum_{i > J_q} \varepsilon_i \leq \varepsilon_q$). Now, using that $n - m \geq N_q(j)$, we get from (iv) that

$$(2.7) \quad \sum_{p \leq q} \sum_{\substack{i \leq J_q \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n > m}} T_{j,m}(S_{i,m+(n-m)}(y_p)) \right\| \leq \sum_{p \leq q} \sum_{i \leq J_q} \varepsilon_{J_q} \varepsilon_q \leq q\varepsilon_q J_q \varepsilon_{J_q}.$$

Thus, (2.4), (2.5), (2.6) and (2.7) altogether give, for any $m \in E_q(j)$,

$$(2.8) \quad \|B_m^2\| \leq \sum_{p > q} \varepsilon_p + q\varepsilon_q + q\varepsilon_q J_q \varepsilon_{J_q} =: r_{2,q}.$$

To finish, we consider B_m^1 . We have

$$(2.9) \quad \begin{aligned} \|B_m^1\| &\leq \sum_{p > q} \sum_{\substack{i \in \mathbb{N} \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n < m}} T_{j,m}(S_{i,m-(m-n)}(y_p)) \right\| + \\ &+ \sum_{p \leq q} \left(\sum_{\substack{i \leq J_q \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n < m}} T_{j,m}(S_{i,m-(m-n)}(y_p)) \right\| + \sum_{\substack{i > J_q \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n < m}} T_{j,m}(S_{i,m-(m-n)}(y_p)) \right\| \right) \end{aligned}$$

For $n \in E_p(i)$ and $m \in E_q(j)$ with $(i, p) \neq (j, q)$, we have $|n - m| \geq \max(N_p(i), N_q(j))$. Moreover, $n < m$ gives $n \leq m/c$, hence $\frac{c-1}{c}m \leq m - n \leq m$. So (v) implies

$$(2.10) \quad \sum_{p > q} \sum_{\substack{i \in \mathbb{N} \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n < m}} T_{j,m}(S_{i,m-(m-n)}(y_p)) \right\| \leq \sum_{p > q} \sum_{i \in \mathbb{N}} \varepsilon_i \varepsilon_p \leq \sum_{p > q} \varepsilon_p$$

and

$$(2.11) \quad \sum_{p \leq q} \sum_{\substack{i > J_q \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n < m}} T_{j,m}(S_{i,m-(m-n)}(y_p)) \right\| \leq \sum_{p \leq q} \varepsilon_p \sum_{i > J_q} \varepsilon_i \leq q\varepsilon_q.$$

Now, since $n - m \geq N_q(j)$, (vi) yields

$$(2.12) \quad \sum_{p \leq q} \sum_{\substack{i \leq J_q \\ i \neq j}} \left\| \sum_{\substack{n \in E_p(i) \\ n < m}} T_{j,m}(S_{i,m-(m-n)}(y_p)) \right\| \leq \sum_{p \leq q} \sum_{i \leq J_q} \varepsilon_{J_q} \varepsilon_q \leq q\varepsilon_q J_q \varepsilon_{J_q}.$$

Thus, (2.9), (2.10), (2.11) and (2.12) imply, for any $m \in E_q(j)$, $\|B_m^1\| \leq r_{2,q}$ (see (2.8) for the definition of $r_{2,q}$).

The previous inequality, together with (2.2), (2.3) and (2.8) give (2.1), setting $r_q = \varepsilon_q + r_{1,q} + 2r_{2,q}$ which, by assumption, tends to 0 as $q \rightarrow +\infty$. \square

In Linear Dynamics, it often happens that in the assumptions of Theorem 2.3, S_i are self-mappings of X_0 and right inverses of the operators T_i on X_0 . It is in particular the case if T satisfies the so-called Frequent Hypercyclicity Criterion ([7, Theorem 6.18]). In this context, Theorem 2.3 reads as follows.

Corollary 2.4. *Let $(T_i)_{i \in \mathbb{N}}$ be countably many bounded linear operators on X . We assume that there exists a dense subset X_0 of X , mappings $S_i : X_0 \rightarrow X_0$, $i \in \mathbb{N}$, and a real number $c > 1$ such that for every $x \in X_0$,*

- (1) The series $\sum_{n \geq 0} T_i^n(x)$ and $\sum_{n \geq 0} S_i^n(x)$ converge unconditionally, uniformly for $i \in \mathbb{N}$;
- (2) The sequence $T_i S_i(x) = x$ for every $i \in \mathbb{N}$;
- (3) The series $\sum_{n \geq (c-1)m} T_i^m(S_j^{m+n}(x))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \neq j \in \mathbb{N}$;
- (4) The series $\sum_{\frac{c-1}{c}m \leq n \leq m} T_i^m(S_j^{m-n}(x))$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $i \neq j \in \mathbb{N}$.

Then there exists a vector $x \in X$ frequently hypercyclic for every T_i , $i \in \mathbb{N}$.

In the previous statement, (1) and (2) exactly say that each T_i satisfies the Frequent Hypercyclicity Criterion. Note that the second part of (1) is a consequence of (3) by taking $m = 0$.

These two results apply to many situations, that we describe below.

2.2. Application to multiples of a single operator. Let fix a continuous linear operator T on X . Given X_0 a dense subset of X and $S : X_0 \rightarrow X_0$ such that $TS(x) = x$ for $x \in X_0$, we denote by

$$\begin{aligned} a_T(X_0, S) &= \inf \left\{ \lambda : \sum \frac{S^n}{\lambda^n}(x) \text{ converges unconditionally for all } x \in X_0 \right\} \\ &= \inf \left\{ \lambda : (\lambda^{-n} S^n(x))_{n \in \mathbb{N}} \text{ is bounded for all } x \in X_0 \right\}. \end{aligned}$$

and

$$\begin{aligned} b_T(X_0, S) &= \sup \left\{ \lambda : \sum (\lambda T)^n(x) \text{ converges unconditionally for all } x \in X_0 \right\} \\ &= \sup \left\{ \lambda : ((\lambda T)^n(x))_{n \in \mathbb{N}} \text{ is bounded for all } x \in X_0 \right\}. \end{aligned}$$

One easily checks that

$$a_T(X_0, S) = \sup_{x \in X_0} \limsup_n \|S^n(x)\|^{1/n} \text{ and } b_T(X_0, S) = \inf_{x \in X_0} \frac{1}{\limsup_n \|T^n(x)\|^{1/n}}.$$

In particular, if X is a Banach space and $r(T)$ denotes the spectral radius of T ,

$$(2.13) \quad a_T(X_0, S) \geq \frac{1}{\inf_n \|T^n\|^{1/n}} = \frac{1}{\lim_n \|T^n\|^{1/n}} = \frac{1}{r(T)} \geq \frac{1}{\|T\|},$$

where the equality follows from the spectral radius formula. Also note that $b_T(X_0, S)$ may be infinite, for e.g., if $X_0 = \bigcup_{n \geq 0} \ker T^n$ is dense in X . This is for example the case if T is any weighted backward shift acting on the Fréchet space X with an unconditional basis. Even more specifically, if T is the unweighted backward shift B on $\ell^2(\mathbb{N})$, then S can be taken as the unweighted forward shift F and we have equalities in (2.13) with $a_B(X_0, F) = 1/\|B\| = 1$ (see Paragraph 2.3 for a focus on weighted shifts).

It is not difficult to check (see Lemma 2.7 below) that if $a_T(X_0, S) < \lambda < b_T(X_0, S)$ then λT satisfies the *Frequent Hypercyclicity Criterion* ([7, Theorem 6.18]). The following criterion of common hypercyclicity, due to Bayart and Matheron ([6, Proposition 4.2]), can be rephrased as follows.

Theorem 2.5 (Bayart-Matheron). *Let X be a separable Fréchet space and let $T : X \rightarrow X$ be a continuous linear operator. Assume that there exist $X_0 \subset \bigcup_{n \in \mathbb{N}} \ker(T^n)$ and $S : X_0 \rightarrow X_0$ such that X_0 is dense in X and $TS(x) = x$ for all $x \in X_0$. Then $\bigcap_{\lambda > a_T(X_0, S)} HC(\lambda T)$ is a dense G_δ subset of X .*

It is known that for any continuous operator T on X , $\bigcap_{\lambda \in \Lambda} FHC(\lambda T) = \emptyset$ whenever Λ is an uncountable subset of $(0, +\infty)$ (of course, even if λT is frequently hypercyclic for any $\lambda \in \Lambda$), see [26, Exercise 9.2.7]. Under the same (in fact, a bit weaker) assumptions as in Bayart-Matheron's criterion, we have the following *countably* common frequent hypercyclicity.

Theorem 2.6. *Let X be a separable Banach space and let $T : X \rightarrow X$ be a continuous linear operator. Assume that there exists a dense subset X_0 of X and $S : X_0 \rightarrow X_0$ such that $TS(x) = x$ for all $x \in X_0$. If Λ is a countable relatively compact subset of $(a_T(X_0, S), b_T(X_0, S))$, then $\bigcap_{\lambda \in \Lambda} FHC(\lambda T) \neq \emptyset$.*

The proof of this theorem is based on the following easy lemma, where it is assumed that $E \subset (a, b)$ with $b < a$ means $E = \emptyset$.

Lemma 2.7. *With the notations of Theorem 2.6, let E be a relatively compact subset of $(a_T(X_0, S), b_T(X_0, S))$. Then there exists $c > 1$ such that for any $x \in X_0$,*

- (i) *The series $\sum_{n \geq 0} (\lambda T)^n(x)$ converges unconditionally, uniformly for $\lambda \in E$;*
- (ii) *The series $\sum_{n \geq 0} \left(\frac{S}{\lambda}\right)^n(x)$ converges unconditionally, uniformly for $\lambda \in E$;*
- (iii) *The series $\sum_{n \geq (c-1)m} \left(\frac{\lambda}{\mu}\right)^m \left(\frac{S}{\mu}\right)^n(x)$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $\lambda, \mu \in E$.*
- (iv) *The series $\sum_{n \geq \frac{c-1}{c}m} \left(\frac{\lambda}{\mu}\right)^{m-n} (\lambda T)^n(x)$ converges unconditionally, uniformly for $m \in \mathbb{N}$ and $\lambda, \mu \in E$.*

Proof. For notational simplicity, we shall denote $a = a_T(X_0, S)$ and $b = b_T(X_0, S)$. We only prove (ii) and (iv). (i) and (iii) are respectively proved in the same way. To get (ii), let $a < d < \inf(E)$. Then, it is enough to write, for $\lambda \in E$, $\left(\frac{S}{\lambda}\right)^n(x) = \left(\frac{d}{\lambda}\right)^n \left(\frac{S}{d}\right)^n(x)$, and use that $\frac{d}{\lambda} \leq \frac{d}{\inf(E)} < 1$ and that $\left(\frac{S}{d}\right)^n(x)$ is bounded for any $x \in X_0$ by some constant independent of $\lambda \in E$ and $n \in \mathbb{N}$.

To prove (iv), let us now fix $d \in (a, b)$ such that $\sup(E) < d < b$. Then $\sum (dT)^n(x)$ is convergent in X_0 and the sequence $((dT)^n(x))_n$ is bounded for any $x \in X_0$ by some constant M independent of m and $\lambda, \mu \in E$. Then, for any $c > 1$ and any $\lambda, \mu \in E$, we have, writing $n = \frac{c-1}{c}m + s + k$ with $k \in \mathbb{N}$ and $0 \leq s < 1$ which does depend on m and c but not on n ,

$$\begin{aligned} \left(\frac{\lambda}{\mu}\right)^{m-n} \|(\lambda T)^n(x)\| &= \frac{\lambda^m}{\mu^{m - (\frac{c-1}{c}m + s + k)} d^{\frac{c-1}{c}m + s + k}} \|(dT)^n(x)\| \\ &\leq M \left(\frac{\lambda}{(\mu d^{c-1})^{1/c}}\right)^m \left(\frac{\sup(E)}{d}\right)^{k+s}. \end{aligned}$$

Now, one can observe that $\frac{\lambda}{(\mu d^{c-1})^{1/c}}$ is less than 1 uniformly for $\lambda, \mu \in E$ whenever $\left(\frac{\sup(E)}{d}\right)^c < \frac{a}{b}$, which in turn holds true whenever c is large enough. \square

Let us now finish the proof of Theorem 2.6.

Proof of Theorem 2.6. It is enough to check that the sequences $((\lambda T)^n)_n$ and $((S/\lambda)^n)_n$, $\lambda \in \Lambda$, satisfy the assumptions (1)–(6) of Theorem 2.3. (6) is trivial, while (1), (2) and (5) are direct consequences of (i) and (ii) of Lemma 2.7. Now, (3) and (4) follow from (iii) and (iv) of Lemma 2.7, after observing that for any $\lambda \neq \mu \in \Lambda$, $x \in X_0$,

$$\sum_{n \geq (c-1)m} (\lambda T)^m \left(\frac{S}{\mu}\right)^{m+n}(x) = \sum_{n \geq (c-1)m} \left(\frac{\lambda}{\mu}\right)^m \left(\frac{S}{\mu}\right)^n(x)$$

and

$$\sum_{\frac{c-1}{c}m \leq n \leq m} (\lambda T)^m \left(\frac{S}{\mu}\right)^{m-n}(x) = \sum_{\frac{c-1}{c}m \leq n \leq m} \left(\frac{\lambda}{\mu}\right)^{m-n} (\lambda T)^n(x).$$

\square

Remark 2.8. 1) The first two points of Lemma 2.7 tell us that, whenever $a_T(X_0, S) < b_T(X_0, S)$, then for any $\lambda \in (a_T(X_0, S), b_T(X_0, S))$, the sequences $((\lambda T)^n)_n$ and $((S/\lambda)^n)_n$

satisfy the assumptions of the Frequent Universality Criterion given in [12] for X_0 , where $(S/\lambda)^n : X_0 \rightarrow X$ is well-defined by $(S/\lambda)^n(x) = S^n(x)/\lambda^n$, $x \in X_0$. Yet observe that S/λ may not map X_0 into itself, so λT may not satisfy the usual Frequent Hypercyclicity Criterion with S/λ as a right inverse. However it does if X_0 can be chosen as a dense vector subspace of X .

2) An easy modification of the proof of Theorem 2.6 yields to the following *universal* version:

Proposition 2.9. *Let X_0 be a dense subset of X and $S : X_0 \rightarrow X_0$ such that $TS(x) = x$ for all $x \in X_0$. Let also $(\lambda_n^i)_n$, $i \in \mathbb{N}$, be a countable family of sequences in (a, b) . We assume that*

- (1) *There exist $c, d \in (a, b)$ such that $\lambda_n^i \in (c^n, d^n)$ for any $i \in \mathbb{N}$, $n \geq 1$;*
- (2) *There exists $C > 0$ such that $C^{-1}\lambda_{n+m}^i \leq \lambda_n^i \lambda_m^i \leq C\lambda_{n+m}^i$ for any $n, m, i \in \mathbb{N}$.*

Then

$$\bigcap_{i \in \mathbb{N}} FHC((\lambda_n^i T^n)_n) \neq \emptyset.$$

The next proposition tells us that, when X is a Banach space, Theorem 2.6 is not so far from being optimal. We will see in the next paragraph that it is optimal for a rather standard class of weighted shifts.

Proposition 2.10. *We keep the notations of Theorem 2.6. Let us assume that X is a Banach space. If $\Lambda \subset [1/r(T), +\infty)$ is unbounded or $1/r(T) \in \bar{\Lambda}$, then*

$$\bigcap_{\lambda \in \Lambda} FHC(\lambda T) = \emptyset.$$

Proof. We only prove the case where $1/r(T) \in \bar{\Lambda}$, the case Λ unbounded being treated very similarly. To start with, let us first assume that $1/r(T)$ is an accumulation point of Λ . We fix $\lambda_0 \in \Lambda$. Upon taking a subsequence, we can assume that $\Lambda = (\lambda_k)_{k \in \mathbb{N}}$ is decreasing to $1/r(T)$. By contradiction, we assume that there exists $x \in X$ which is frequently hypercyclic for all $\lambda_k T$, $k \in \mathbb{N}$. We fix $e_0 \in X \setminus \{0\}$ and denote by N_k , $k \geq 0$, the sets respectively given by

$$N_0 := \{n \in \mathbb{N} : \|\lambda_0^n T^n x\| < 1\} \quad \text{and} \quad N_k := \{m \in \mathbb{N} : \|\lambda_k^m T^m x - e_0\| < \frac{\|e_0\|}{2}\}, \quad k \geq 1.$$

By assumption, $\underline{d}(N_0) \geq \varepsilon > 0$ and each N_k , $k \in \mathbb{N}$, is infinite. So there exists an increasing sequence $(m_k)_{k \geq 1}$ with $m_k \in N_k$ such that $m_k \rightarrow +\infty$, and one can define $(n_k)_{k \geq 1}$ by

$$n_k := \max\{n < m_k : n \in N_0\}.$$

Note that $(m_k)_k$ can be chosen so that $(n_k)_k$ is also increasing and tends to $+\infty$. Moreover, from the definition of n_k , $k \geq 1$, we get

$$(2.14) \quad \varepsilon \leq \underline{d}(N_0) \leq \limsup_k \frac{\text{card}(N_0 \cap \{0, \dots, m_k - 1\})}{m_k} \leq \limsup_k \frac{n_k}{m_k}.$$

Now, by construction, we have for any $k \geq 1$,

$$\|T^{n_k} x\| < \lambda_0^{-n_k} \quad \text{and} \quad \lambda_k^{-m_k} \frac{\|e_0\|}{2} < \|T^{m_k} x\| \leq \|T^{m_k - n_k}\| \|T^{n_k}(x)\|.$$

It follows,

$$\frac{2}{\|e_0\|} \|T^{m_k - n_k}\| > \frac{\lambda_0^{n_k}}{\lambda_k^{m_k}},$$

whence

$$(2.15) \quad \left(\frac{\lambda_0}{\lambda_k}\right)^{n_k} \leq \frac{2}{\|e_0\|} \lambda_k^{m_k - n_k} \|T^{m_k - n_k}\| \leq \frac{2}{\|e_0\|} (\lambda_0 \|T\|)^{m_k - n_k}.$$

Since $(\lambda_k)_k$ is decreasing and $n_k \rightarrow +\infty$, we first deduce from the last inequality that $m_k - n_k \rightarrow +\infty$. This gives $r(T) = \lim_k \|T^{m_k - n_k}\|^{1/(m_k - n_k)}$. We also derive from (2.15) the following:

$$\left(\frac{\lambda_0}{\lambda_k}\right)^{n_k/m_k} \leq \left(\frac{2}{\|e_0\|}\right)^{1/m_k} \lambda_k^{1 - n_k/m_k} \|T^{m_k - n_k}\|^{1/m_k},$$

which implies, using that $m_k \rightarrow +\infty$ and $m_k - n_k \rightarrow +\infty$,

$$\limsup_k \frac{n_k}{m_k} \leq \frac{1}{\ln(r(T)\lambda_0)} \lim_k \left(\ln(\lambda_k \|T^{m_k - n_k}\|^{1/m_k}) \right) = 0,$$

since by assumption $(\lambda_k)_k$ is decreasing to $1/r(T)$. This contradicts (2.14) and concludes the proof when $1/r(T)$ is an accumulation point of Λ .

Let us deal with the remaining case, i.e., $1/r(T) \in \Lambda$ but $1/r(T)$ is not an accumulation point of Λ . We will in fact prove the stronger fact that, if $1/r(T) \neq \lambda$ are both in Λ , then $r(T)^{-1}T$ and λT share no frequent hypercyclic vectors. The proof goes along the same lines as above. Let us denote $\mu = 1/r(T)$. By assumption $\lambda/\mu > 1$. We assume by contradiction that x is hypercyclic for λT and μT and we set

$$N_\lambda := \{n \in \mathbb{N} : \|\lambda^n T^n x\| < 1\} \quad \text{and} \quad N_\mu := \{m \in \mathbb{N} : \|\mu^m T^m x - e_0\| < \frac{\|e_0\|}{2}\}, \quad k \geq 1.$$

As above, since these sets are infinite, one can define an increasing sequence of integers $(m_k)_{k \in \mathbb{N}} \subset N_\mu$, tending to ∞ , such that the sequence $(n_k)_{k \in \mathbb{N}}$ defined by

$$n_k := \max\{n < m_k : n \in N_\lambda\}$$

is increasing. We have $\underline{d}(N_\lambda) \leq \limsup_k \frac{n_k}{m_k}$ and, proceeding exactly as in the first part of the proof, $m_k - n_k \rightarrow \infty$ and

$$\left(\frac{\lambda}{\mu}\right)^{n_k} \leq \frac{2}{\|e_0\|} (\mu \|T\|)^{m_k - n_k}, \quad k \in \mathbb{N}.$$

Therefore

$$\underline{d}(N_\lambda) \leq \limsup_k \frac{n_k}{m_k} \leq \frac{1}{\ln(r(T)\lambda)} \lim_k \left(\ln(\mu \|T^{m_k - n_k}\|^{1/m_k}) \right) = 0,$$

and x is not frequently hypercyclic for λT . □

Remark 2.11. The proof of the previous proposition tells us a bit more than its statement. More precisely, we have shown that, if Λ is unbounded or if $1/r(T) \in \bar{\Lambda}$, and if $x \in X$ is a common hypercyclic vector for all λT , then it can be a *frequently* hypercyclic vector for none of the λT . This complements for instance the result saying that the set $\bigcap_{\lambda > 1} HC(\lambda B)$ is different from the set $HC(\mu B)$ for any $\mu > 1$. Indeed, for $\mu > 1$, we have

$$FHC(\mu B) \subset HC(\mu B) \setminus \bigcap_{\lambda > 1} HC(\lambda B).$$

Another interesting feature of the previous result (more precisely of the one proved in the second part of the proof) is the idea that it gives to build two frequently hypercyclic operators (in fact satisfying the Frequent Hypercyclicity Criterion) sharing no frequently hypercyclic vectors. This will be detailed in the end of the next Paragraph, see Corollary 2.23).

In the whole paragraph, we have considered *positive real* scalar multiples of a single operator T . In virtue of the León-Müller theorem for frequent hypercyclicity ([7, Theorem 6.28]), $FHC(\lambda T) = FHC(T)$ for any $\lambda \in \mathbb{C}$, $|\lambda| = 1$. It is also known that the Ansari theorem for frequent hypercyclicity holds true: For any positive integer p , $FHC(T) = FHC(T^p)$ (see [4]). This with Theorem 2.6 thus implies:

Corollary 2.12. *Let X_0 be a dense set of X and $S : X_0 \rightarrow X_0$ such that $TS(x) = x$ for all $x \in X_0$. Let also T be a bounded linear operator on X and $\Lambda \subset \{z \in \mathbb{C} : a_T(X_0, S) < |z| < b_T(X_0, S)\}$. If $\{|\lambda| : \lambda \in \Lambda\}$ is a countable relatively compact subset of $(a_T(X_0, S), b_T(X_0, S))$, then*

$$\bigcap_{\lambda \in \Lambda} FHC(\lambda T) \neq \emptyset.$$

If in addition $\{|\lambda|^{1/p} : p \in \mathbb{N}^, \lambda \in \Lambda\}$ is a relatively compact subset of $(a_T(X_0, S), b_T(X_0, S))$, then*

$$\bigcap_{\lambda \in \Lambda, p \in \mathbb{N}^*} FHC(\lambda T^p) \neq \emptyset.$$

Note that the additional assumption above occurs for example if $a_T(X_0, S) < 1$ and $b_T(X_0, S) = +\infty$ (for e.g., for a large class of weighted shifts, see the next paragraph).

2.3. Application to weighted shifts.

Theorem 2.13. *Let X be a Fréchet space with unconditional basis $(e_n)_n$, and let $\{w(\lambda) = (w_n(\lambda))_n, \lambda \in \Lambda\}$ be a countable family of weights. We assume that there exist a weight $\omega = (\omega_n)_n$ and constants $0 < \eta < 1$ and $M > 1$ such that for any $\lambda \in \Lambda$ and any $n \geq 0$, $m \geq 1$,*

- (i) *The series $\sum_{n \geq 0} (\omega_1 \dots \omega_n)^{-1} e_n$ is convergent in X ;*
- (ii) $\omega_n \dots \omega_{n+m} \leq \eta^m \omega_n \dots \omega_{n+m}(\lambda)$;
- (iii) $M^{-m} \leq \omega_n \dots \omega_{n+m}(\lambda) \leq M^m$.

Then $\bigcap_{\lambda \in \Lambda} FHC(B_{w(\lambda)})$ is non-empty.

Proof. For notational simplicity, let us denote $\{w(\lambda) = (w_n(\lambda))_n, \lambda \in \Lambda\} = (w(i))_{i \in \mathbb{N}}$. We consider

$$X_0 = \text{span}(e_k : k \geq 0) = \bigcup_{n \geq 0} \ker(T^n)$$

and, for a weight w , the operator F_w on X given by

$$F_w(e_k) = \frac{1}{w_{k+1}}(e_{k+1}).$$

By definition of X_0 , we need only check that $\{B_{w(i)} : i \in \mathbb{N}\}$ satisfies the assumptions (1)–(4) of Corollary 2.4 for any $x = e_k$, $k \in \mathbb{N}$. Observe that (2) is trivially satisfied. From now on, for $l < 0$, we use the notations $e_l = 0$ and $w_l(i) = 0$, $i \in \mathbb{N}$. For any $i, j, m, l \in \mathbb{N}$, let us write

$$B_{w(i)}^m F_{w(j)}^l(e_k) = \frac{w_{k+l}(i) \dots w_{k+l-m+1}(i)}{w_{k+l}(j) \dots w_{k+1}(j)} e_{k+l-m}.$$

Note that $B_{w(i)}^n(e_k) = 0$ whenever n is large enough, uniformly for $i \in \mathbb{N}$. This gives the first part of (1) in Corollary 2.4. Moreover,

$$(2.16) \quad \sum_{n \geq 0} F_{w(i)}^n(e_k) = \sum_{n \geq 0} \frac{1}{w_{k+n}(i) \dots w_{k+1}(i)} e_{k+n}$$

By assumption (ii), we have $w_{k+n}(i) \dots w_{k+1}(i) > \omega_{k+n} \dots \omega_{k+1}$. So, by assumption (i) and using that $(e_k)_k$ is an unconditional basis, we get that the left-hand side term in (2.16) is

unconditionally convergent in X , uniformly for i , hence the second part of (1) in Corollary 2.4.

Let us now turn to proving that (3) in Corollary 2.4 holds. By the assumption (i) and unconditionality of $(e_k)_k$, the sequence $(\omega_{k+1} \dots \omega_{k+n})^{-1} e_{k+n}$ is bounded uniformly for $n \geq 0$. We denote by $\|\cdot\|$ the F -norm associated to the Fréchet distance of X . Then, for some constant K (depending only on k and the constant of unconditionality of $(e_k)_k$) and by the assumptions (ii) and (iii), we have

(2.17)

$$\|B_{w(i)}^m F_{w(j)}^{m+n}(e_k)\| = \left\| \frac{w_{k+m+n}(i) \dots w_{k+n+1}(i)}{w_{k+m+n}(j) \dots w_{k+1}(j)} e_{k+n} \right\|$$

$$(2.18) \quad = \left\| \frac{w_{k+m+n}(i) \dots w_{k+n+1}(i)}{w_{k+m+n}(j) \dots w_{k+n+1}(j)} \frac{\omega_{k+1} \dots \omega_{k+n}}{w_{k+1}(j) \dots w_{k+n}(j)} (\omega_{k+1} \dots \omega_{k+n})^{-1} e_{k+n} \right\|$$

$$(2.19) \quad \leq KM^{2m} \eta^n.$$

So, after writing $n = (c-1)m + l$, $l \geq 0$, we easily check that there exists some $c > 1$ such that $M^{2m} \eta^{(c-1)m} \leq 1$ for any $m \geq 0$. Since $\eta < 1$, the series $\sum_{n \geq (c-1)m} B_{w(i)}^m F_{w(j)}^{m+n}(e_k)$ is absolutely convergent, uniformly for $m \in \mathbb{N}$, which implies (3).

The proof of Corollary 2.4, (4) is left to the reader. \square

At this point, we shall make a remark.

Remark 2.14. Bayart-Ruzsa [8] proved in 2015 that, when acting on ℓ^p spaces, $1 \leq p < \infty$, weighted shifts are frequently hypercyclic if and only if they satisfy the Frequent Hypercyclicity Criterion (i.e., they are chaotic). This result was extended to more general classes of spaces in [14]. For instance, it is proved there that Bayart-Ruzsa theorem extends to any Banach space with unconditional basis $(e_k)_k$ whenever $(e_k)_k$ is *boundedly complete*. We recall that a basis $(e_k)_k$ in X is called boundedly complete if, for any sequence of scalars $(x_k)_k$, whenever the sequence

$$\left(\sum_{k=0}^K x_k e_k \right)_{K \geq 0}$$

is bounded in X , then it is convergent in X . Examples of such Banach spaces are given among Köthe sequences spaces (including of course ℓ^p spaces). Note that the usual basis of c_0 is not boundedly complete.

Anyway, in the situation given by Remark 2.14, Theorem 2.13 can be rephrased as follows.

Corollary 2.15. *Let X be a Banach space with boundedly complete unconditional basis $(e_n)_n$, and let $\{w(\lambda) = (w_n(\lambda))_n, \lambda \in \Lambda\}$ be a countable family of weights. We assume that there exist a frequently hypercyclic weighted shift B_w , $w = (w_n)_n$, and constants $0 < \eta < 1$ and $M > 0$ such that for any $\lambda \in \Lambda$ and any $n \geq 0$, $m \geq 1$,*

$$(i) \quad w_n \dots \omega_{m+n} \leq \eta^m w_n \dots w_{m+n}(\lambda);$$

$$(ii) \quad M^{-m} \leq w_n \dots w_{m+n}(\lambda) \leq M^m.$$

Then $\bigcap_{\lambda \in \Lambda} FHC(B_{w(\lambda)})$ is non-empty.

In Fréchet spaces, bounded completeness of the unconditional basis $(e_n)_n$ is not sufficient any more, and some other conditions are given in [14]. As an application, it is shown that on the space $H(\mathbb{D})$ of analytic functions in the unit disc \mathbb{D} , endowed with the locally uniform Fréchet topology, a weighted shift is frequently hypercyclic if and only if it satisfies the Frequent Hypercyclicity Criterion. Thus the previous corollary holds if the Banach space X is replaced with $H(\mathbb{D})$.

Let us give an example.

Example 2.16. For $\lambda \in (0, +\infty)$, let $B_{w(\lambda)}$ be the weighted shift on $\ell^2(\mathbb{N})$ defined by $w_n(\lambda) = 1 + \lambda/n$. In [18], it is proven that $\bigcap_{\lambda > 1} HC(B_{w(\lambda)})$ is residual. We can easily deduce from Corollary 2.15 that for any countable relatively compact subset Λ of $(\frac{1}{2}, +\infty)$, one has

$$\bigcap_{\lambda \in \Lambda} FHC(B_{w(\lambda)}) \neq \emptyset.$$

Furthermore Theorems 2.7 or 2.13 can be applied to a family of multiples of a single weighted shift. We keep the notations of Paragraph 2.2. We fix a weight $w = (w_n)_n$ and consider $X_0 = \text{span}(e_k : k \geq 0)$. First observe that $b(B_w, X_0, F_w) = +\infty$. Let us simply denote $\lambda_w = a(B_w, X_0, F_w)$, that is

$$\lambda_w = \inf\{\lambda > 0 : \text{the series } \sum \frac{\lambda^{-n}}{w_1 \dots w_n} e_n \text{ is convergent in } X\}.$$

Note that $\lambda_w = \limsup_n (\frac{\|e_n\|}{w_1 \dots w_n})^{1/n}$. We recall that a slight generalization of Abakumov-Gordon theorem states that $\bigcap_{\lambda > \lambda_w} HC(\lambda B_w)$ is a dense G_δ subset of X , see [7, p. 178] or [6]. In this context, Theorem 2.6 reads as follows.

Corollary 2.17. *Let B_w be a weighted shift on a Fréchet space X with unconditional basis $(e_n)_n$. Then $\bigcap_{\lambda \in \Lambda} FHC(\lambda B_w)$ is non-empty for any countable relatively compact subset Λ of $(\lambda_w, +\infty)$.*

As already said, *countability* in the previous corollary is necessary. A natural question is whether it is possible to get common frequent hypercyclicity in the case Λ is bounded but $\lambda_w \in \bar{\Lambda}$. We can give partial answers in two different directions. First of all, as mentioned in Remark 2.14, if we additionally assume that X is a Banach space and that the unconditional basis $(e_n)_n$ is boundedly complete, then the convergence of the series $\frac{\lambda^{-n}}{w_1 \dots w_n} e_n$ is necessary for B_w to be frequently hypercyclic [14]. So, in such a case, for any weighted shift B_w ,

$$\bigcap_{\lambda \in \Lambda} FHC(\lambda B_w) = \emptyset$$

whenever $\lambda_w > \inf \Lambda$. Second, for any X with unconditional basis $(e_n)_n$ and any B_w such that $\lambda_w = r_w^{-1}$ (where r_w denotes the spectral radius $r(B_w)$ of B_w)

$$\bigcap_{\lambda \in \Lambda} FHC(\lambda B_w) = \emptyset$$

whenever $\lambda_w \in \bar{\Lambda}$ (Proposition 2.10).

Altogether, these two observations thus give the following:

Proposition 2.18. *Let X be a separable Banach space with unconditional basis $(e_n)_n$, w a bounded weight and $\Lambda \subset [\lambda_w, +\infty)$. We assume that $(e_n)_n$ is boundedly complete and that $\lambda_w = r_w^{-1}$. Then*

$$\bigcap_{\lambda \in \Lambda} FHC(\lambda B_w) \neq \emptyset$$

if and only if Λ is countable, bounded and $\lambda_w \notin \bar{\Lambda}$.

Special case of ℓ^p spaces. There of course exist many general situations where the assumptions of Proposition 2.18 do not hold. However, bounded completeness of course holds when $X = \ell^p$, $1 \leq p < \infty$. In this context, it makes sense to examine in which extend the condition $\lambda_w = r_w^{-1}$ can be relaxed. From now on, p is fixed in $[1, \infty)$.

Observe that for a weighted shift B_w acting (boundedly) on ℓ^p , one has

$$\lambda_w = \frac{1}{\liminf (w_1 \dots w_n)^{1/n}}.$$

It is not difficult to check that $\lambda_w = r_{p,w}^{-1}$ with $r_{p,w} := r_p(B_w)$ where

$$r_p(T) := \sup\{\lambda : \lambda \in \sigma_p(T)\}.$$

Here $\sigma_p(T)$ denotes the point spectrum of an operator T (see for e.g., [31, Theorem 8, P. 70]). We consider that there cannot be confusion due to the notation p in ℓ^p and the other p appearing in $r_{p,w}(T)$ (where it is for *pointwise*). Thus λ_w is in general larger than r_w^{-1} . Moreover, it is easily seen that, whenever the weight sequence $(w_n)_n$ is bounded,

$$r_w = \lim_n \left(\sup_k w_k \dots w_{k+n} \right)^{1/n} = \limsup_n \left(\sup_k w_k \dots w_{k+n} \right)^{1/n}.$$

Proposition 2.10 then tells us that if

$$\Lambda \subset (1/r_w, \infty)$$

is unbounded or admits $1/r_w$ as an accumulation point, then $\bigcap_{\lambda \in \Lambda} FHC(B_w) = \emptyset$. Of course, by Corollary 2.17 (or as in Proposition 2.18), this is only interesting when $\lambda_w = r_w^{-1}$. This equality happens quite often, but this can also fail to occur (see Example 2.20 below). In fact, for weighted shifts, one can get a little improvement of Proposition 2.10. To state it, let us introduce the quantity

$$\lambda_w^0 = \frac{1}{\limsup (w_1 \dots w_n)^{1/n}}.$$

We observe that $r_w^{-1} \leq \lambda_w^0 \leq \lambda_w$. Examples of weights w for which $r_w^{-1} \neq \lambda_w^0 = \lambda_w$ or $r_w^{-1} \neq \lambda_w^0 \neq \lambda_w$ are easily built (Example 2.20).

Corollary 2.19. *Let B_w be a weighted shift acting on ℓ^p and let Λ be a countable subset of $[\lambda_w^0, +\infty)$. If Λ is unbounded or $\lambda_w^0 \in \bar{\Lambda}$, then*

$$\bigcap_{\lambda \in \Lambda} FHC(\lambda T) = \emptyset.$$

Proof. It is very similar to that of Proposition 2.10. Let us only give the outline of the proof in the case where Λ is any sequence $(\lambda_k)_{k \in \mathbb{N}}$ decreasing to λ_w^0 . By contradiction, let us assume that $x = (x_n)_{n \in \mathbb{N}} \in \ell^p$ is some frequent hypercyclic vector for each $\lambda_k B_w$, $k \in \mathbb{N}$. As in the proof of Proposition 2.10, we introduce the sets

$$N_0 := \{n \in \mathbb{N} : \|\lambda_0^n B_w^n x\| < 1\} \quad \text{and} \quad N_k := \{m \in \mathbb{N} : \|\lambda_k^m B_w^m x - e_0\| < \frac{1}{2}\}, \quad k \geq 1.$$

Then, replacing T by B_w , we similarly define increasing sequences $(n_k)_{k \geq 1} \subset N_0$ and $(m_k)_{k \geq 1}$, tending to $+\infty$, with $m_k \in N_k$ and such that $\underline{d}(N_0) \leq \limsup_k n_k/m_k$ and, for any $k \geq 1$,

$$\lambda_0^{n_k} w_{m_k - n_k + 1} \dots w_{m_k} x_{m_k} < 1 \quad \text{and} \quad \lambda_k^{m_k} w_1 \dots w_{m_k} x_{m_k} > \frac{1}{2}.$$

It follows

$$\frac{\lambda_0^{n_k}}{\lambda_k^{m_k}} < 2w_1 \dots w_{m_k - n_k};$$

In particular $m_k - n_k \rightarrow +\infty$ and

$$(\lambda_0/\lambda_k)^{n_k/m_k} < 2^{1/m_k} \lambda_k^{1 - n_k/m_k} (w_1 \dots w_{m_k - n_k})^{1/m_k}$$

hence

$$\limsup_k \frac{n_k}{m_k} \leq C (\limsup_k \ln(\lambda_k) - \ln(\lambda_w^0)) = 0,$$

for some constant $C \geq 0$. This contradicts $\underline{d}(N_0) > 0$. □

Let us provide with some examples.

Example 2.20. 1) For the class \mathcal{W}_0 of bounded weights w satisfying $\lambda_w = \lambda_w^0$, then Corollaries 2.17 and 2.19 give the following necessary and sufficient condition:

Proposition 2.21. *Let $w \in \mathcal{W}_0$ and $\Lambda \subset [\lambda_w, +\infty)$. The set $\bigcap_{\lambda \in \Lambda} FHC(\lambda B_w)$ is non-empty if and only if Λ is countable, bounded and $\lambda_w \notin \bar{\Lambda}$.*

The class \mathcal{W}_0 contains, for e.g., all weights $w = (w_n)_n$ such that $(w_n)_n$ is convergent. Yet, for such weights, we even have $\lambda_w = r_w^{-1}$ and Proposition 2.18 applies.

It is in fact not difficult to provide with a set of examples of weights w which allows to distinguish all the quantities $\|B_w\|^{-1}$, r_w^{-1} , λ_w^0 and λ_w . Indeed, let $a \leq b \leq c \leq d$ be four positive real numbers, and let us define

$$w_n := \begin{cases} a & \text{if } n \in \{1, \dots, 4\} \cup \{k2^{(k-1)^2}, \dots, 2^{k^2} - 1\} \\ d & \text{if } n = 2^{k^2} \\ c & \text{if } n \in \{2^{k^2} + 1, 2^{k^2} + k + 1\} \\ b & \text{if } n \in \{2^{k^2} + k + 2, (k+1)2^{k^2}\} \end{cases}, \quad k \geq 2.$$

Then one easily checks that

$$\|B_w\|^{-1} = 1/d \leq r_w^{-1} = 1/c \leq \lambda_w^0 = 1/b \leq \lambda_w = 1/a.$$

Therefore, if we choose $a = b \neq c$, then Proposition 2.21 tells us that $\bigcap_{\lambda \in \Lambda} FHC(\lambda B_w)$ is non-empty if and only if Λ is countable, bounded and $\lambda_w \notin \bar{\Lambda}$. Yet Proposition 2.18 cannot be used here.

In view of the previous discussion, it seems to be reasonable to wonder whether $1/r(T)$ could be replaced by $1/r_p(T)$ in Proposition 2.10. More precisely, for the weighted shifts on ℓ^p , one can pose the following:

Question 2.22. *Do there exist weighted shifts B_w on ℓ^p , $1 \leq p < \infty$, with $\lambda_w^0 < \lambda_w$ and some countable $\Lambda \subset (\lambda_w, +\infty)$ such that $\lambda_w \in \bar{\Lambda}$ and $\bigcap_{\lambda \in \Lambda} FHC(\lambda B_w)$ is non-empty?*

We conclude the paragraph by showing how the results of this paragraph permit to easily exhibit two (or more) explicit frequently hypercyclic weighted shifts which share no frequently hypercyclic vector. We can then state the following.

Corollary 2.23. *There exist two frequently hypercyclic weighted shifts on ℓ^p , $1 \leq p < +\infty$ (hence satisfying the Frequent Hypercyclicity Criterion), with no common frequent hypercyclic vector.*

Proof. Let $w_n = (\frac{n+1}{n})^2$. Since $(w_n)_n$ is decreasing to 1, one has $r_w^{-1} = \lambda_w^0 = \lambda_w = 1$. Moreover, B_w is frequently hypercyclic, since $\sum_{n \geq 1} (w_1 \dots w_n)^{-1} < \infty$. Thus, applying Proposition 2.21 with $\Lambda = \{1, \lambda\}$, $\lambda > 1$, we get $FHC(B_w) \cap FHC(\lambda B_w) = \emptyset$. \square

2.4. Other examples. In this paragraph, we apply our general common frequent hypercyclicity criteria to classical frequently universal sequences of operators which are not weighted shifts.

Since almost all the classical examples of frequent hypercyclic operators satisfy the Frequent Hypercyclicity Criterion, the range of applications of Theorem 2.6 is almost the largest. Let us give one example.

Example 2.24 (Differential operators on $H(\mathbb{C})$). Let D be the differentiation operator on $H(\mathbb{C})$, $Df(z) = f'(z)$. Costakis and Mavroudis showed [17] that for any non-constant polynomial P , $P(D)$ satisfies the Bayart-Matheron criterion (Theorem 2.5) with $a_{P(D)}(X_0, S) = 0$ and $b_{P(D)}(X_0, S) = +\infty$ for some dense subset X_0 of X and some right inverse S of $P(D)$ on

X_0 . Thus, with the frequent hypercyclicity version of the León-Müller theorem and Theorem 2.6, we can deduce that

$$\bigcap_{\lambda \in \Lambda} FHC(\lambda P(D)) \neq \emptyset,$$

for any countable relatively compact subset Λ of \mathbb{C}^* .

We shall now focus on applications of Theorem 2.3 to families of operators which are not multiples of a single one.

Example 2.25 (Adjoint of multipliers on the Hardy space). We denote by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the unit disc, by H^∞ the space of bounded analytic functions in \mathbb{D} , and by H^2 the classical Hardy space,

$$H^2 := \left\{ f = \sum_{k \geq 0} a_k z^k \in H(\mathbb{D}) : \|f\|_2 := \left(\sum_{k \geq 0} |a_k|^2 \right)^{1/2} < \infty \right\}.$$

We recall that H^∞ and H^2 are Banach spaces, endowed respectively by the sup-norm $\|\cdot\|_\infty$ and the $\|\cdot\|_2$. Let $\Phi \in H^\infty$ and $\Phi^* \in L^\infty(\mathbb{T})$ its boundary value. Let us assume that Φ is not outer and that $1/\Phi \in H^\infty$. We denote by $M_\Phi : H^2 \rightarrow H^2$ the multiplication operator with symbol Φ , $M_\Phi(f) = \Phi f$, and by M_Φ^* its adjoint. It is known [7] that λM_Φ^* is frequently hypercyclic on H^2 for any $\lambda > \|1/\Phi\|_\infty$ and that

$$\bigcap_{\lambda > \|1/\Phi\|_\infty} HC(\lambda M_\Phi^*)$$

is a G_δ -subset of H^2 [24].

Now, let us write the inner-outer decomposition $\Phi = u\theta$, with u outer and θ the non-constant inner part of Φ . Let us define $X_0 := \cup_{n \geq 1} K_n$ with $K_n := H^2 \ominus \theta^n H^2$. Then X_0 is the generalized kernel of M_Φ^* and is dense in X_0 . Moreover, if we define $S := M_{1/u}^* M_\theta$, then $M_\Phi^* S = \text{Id}$ and $\|S\| = \|1/\Phi\|_\infty$. We refer, for e.g., to the proof of [24, Theorem 3.1] for the details concerning the previous claims. So, with the notations introduced before Theorem 2.6, we have $a(M_\Phi^*, X_0, S) \leq \|S\| = \|1/\Phi\|_\infty$ and $b(T, X_0, S) = +\infty$. Therefore, Theorem 2.6 directly implies that

$$\bigcap_{\lambda \in \Lambda} FHC(\lambda M_\Phi^*) \neq \emptyset,$$

whenever Λ is a countable relatively compact subset of $(\|1/\Phi\|_\infty, +\infty)$.

In fact, we can deduce from Corollary 2.4 the following more general result.

Proposition 2.26. *Let $\{\Phi_\lambda : \lambda \in \Lambda\}$ be a countable family of bounded analytic functions on \mathbb{D} with the same non-constant inner factor θ . We assume that*

$$a := \sup\{\|\Phi_\lambda^{-1}\|_\infty : \lambda \in \Lambda\} < 1 \quad \text{and} \quad M := \sup\{\|\Phi_\lambda/\Phi_\mu\|_\infty : \lambda \in \Lambda\} < \infty.$$

Then

$$\bigcap_{\lambda \in \Lambda} FHC(M_{\phi_\lambda}^*) \neq \emptyset.$$

Proof. We aim to apply Corollary 2.4. By the comment after its statement, we need only check items (2)–(4). Since the functions Φ_λ share the same non-constant inner factor, the set $X_0 := \cup_{n \geq 1} K_n$ with $K_n := H^2 \ominus \theta^n H^2$ is the generalized kernel of each $M_{\phi_\lambda}^*$. Let u_λ denote the outer factor of Φ_λ . As recalled above, setting $S_\lambda := M_{1/u_\lambda}^* M_\theta$, we have $T_\lambda^n S_\lambda^n = \text{Id}$ for

any $n \in \mathbb{N}$. So (2) and (4) of Corollary 2.4 are satisfied. Let $\lambda \neq \mu \in \Lambda$ and $f \in X_0$. By assumption, there exists $b \in (a, 1)$ such that for any $m \in \mathbb{N}$, writing $n = (c-1)m + k$, $k \in \mathbb{N}$,

$$\begin{aligned} \|T_\lambda^m S_\mu^{m+n}(f)\|_2 &= \sup_{\|g\|_2=1} \langle T_\lambda^m S_\mu^{m+n}(f), g \rangle \\ &= \sup_{\|g\|_2=1} \left\langle f, \left(\frac{u_\lambda}{u_\mu}\right)^m \left(\frac{\bar{\theta}}{u_\mu}\right)^n g \right\rangle \\ &\leq \|f\|_2 \left\| \frac{u_\lambda}{u_\mu} \right\|_\infty^m \left\| \frac{1}{u_\mu} \right\|_\infty^n \\ &\leq \|f\|_2 (M(a/b)^{(c-1)})^m \left(\frac{a}{b}\right)^k. \end{aligned}$$

Since $a/b < 1$, (3) of Corollary 2.4 then follows by taking $c > 1$ such that $M \leq (b/a)^{(c-1)}$. \square

3. PERIODIC POINTS AT THE SERVICE OF COMMON FREQUENT HYPERCYCLICITY

Despite its apparent unpleasant formulation, the classical Frequent Hypercyclicity Criterion turns out to be very useful for checking that natural operators are frequently hypercyclic (and chaotic). We saw in the previous section that it fits well to formulating easy-to-use sufficient conditions for common frequent hypercyclicity. In [25], the authors provided a quite appealing new criterion for frequent hypercyclicity and chaos involving the periodic points of the operator [25, Theorem 5.31]. It is shown there that all the operators which satisfy the Frequent Hypercyclicity Criterion satisfy the assumptions of this new one. However, it quickly appears from its statement that it is not so simple to use when dealing with natural operators (for e.g., weighted shifts). Yet it is very well adapted to certain type of operators which were introduced by Menet in [27] to build chaotic operators on ℓ^p which are not frequently hypercyclic. These operators have been extensively developed - and called operators of C-type - in [25, Section 6] to build several counter-examples.

In this section, we provide with a sufficient condition for common frequent hypercyclicity derived from [25, Theorem 5.31]. We recall that a vector $x \in X$ is a periodic point for $T \in \mathcal{L}(X)$ if there exists $p \in \mathbb{N}$ such $T^p x = x$. Let us denote by $\text{Per}(T)$ the set of all periodic points for T . For $x \in \text{Per}(T)$ we denote by $p_T(x)$ the period of x for T (i.e., the smallest positive integer p such that $T^p x = x$).

Theorem 3.1. *Let $\{T_s : s = 1, 2, 3, \dots\}$ be a countable family in $\mathcal{L}(X)$. We assume that there exists a dense linear subspace X_0 of X with $T_s(X_0) \subset X_0$ and $X_0 \subset \text{Per}(T_s)$ for any $s \geq 1$, and a constant $\alpha \in (0, 1)$ such that the following property holds true: For every $s \geq 1$, every $\varepsilon > 0$, every $x, y \in X_0$, every $q \geq 1$ and every $t_1, \dots, t_q \geq 1$, there exist $z \in X_0$ and integers $n, d \geq 1$ such that*

- (1) d is a multiple of $p_{T_{t_i}}(y)$ and of $p_{T_{t_i}}(z)$ for each $i = 1, \dots, q$;
- (2) $\|T_{t_i}^k z\| < \varepsilon$ for every $0 \leq k \leq \alpha d$ and every $t \geq 1$;
- (3) $\|T_s^{n+k} z - T_s^k x\| < \varepsilon$ for every $0 \leq k \leq \alpha d$.

Then there exists a vector in X which is frequently hypercyclic for each T_s , $s \geq 1$.

If the family $\{T_s : s \in \mathbb{N} \setminus \{0\}\}$ is reduced to a single operator, Theorem 3.1 is exactly [25, Theorem 5.31]. Yet one should mention that the previous statement does not only mean "each T_s satisfies the assumptions of [25, Theorem 5.31]". It would be interesting to know whether two operators satisfying the assumptions of [25, Theorem 5.31] automatically have a frequently hypercyclic vector in common. Note that we already saw that two operators T_1 and T_2 may have no common frequently hypercyclic vector, even if they both satisfy the classical Frequent Hypercyclicity Criterion (see Corollary 2.23). Finally note that Theorem

3.1 does not apply to families of multiples of a single operator, since $\text{Per}(T) \cap \text{Per}(\lambda T) = \emptyset$ in general.

Proof of Theorem 3.1. As one could expect, the proof is greatly inspired by that of [25, Theorem 5.31]. Let $(x_l)_{l \geq 1}$ be a sequence of vectors in X_0 , dense in X , and let $(I_p(s))_{p,s \geq 1}$ be a partition of \mathbb{N} such that each set $I_p(s)$ is infinite and has bounded gaps. Let us denote by $r_p(s)$ the maximal size of a gap for $I_p(s)$. We also let $(y_j)_{j \in \mathbb{N}}$ be given by $y_j = x_p$ if $j \in I_p(s)$. Now we use the assumptions of the theorem to build, by induction on $j \in \mathbb{N}$ a sequence $(z_j)_{j \in \mathbb{N}}$ of vectors in X_0 and increasing sequences of positive integers $(d_j)_{j \in \mathbb{N}}$ and $(n_j)_{j \in \mathbb{N}}$ such that the following properties hold, if $j \in I_p(s)$:

- (i) d_j is a multiple of $p_{T_t}(\sum_{k=1}^{j-1} z_k)$ and $p_{T_t}(z_j)$ for every t so that there exist $q \geq 1$ and $1 \leq i \leq j$ with $i \in I_q(t)$;
- (ii) $\|T_t^k(z_j)\| < 2^{-j}$ for every $0 \leq k \leq \alpha d_j$ and every $t \geq 1$;
- (iii) $\|T_s^{n_j+k} z_j - T_s^k(y_j - \sum_{i=1}^{j-1} z_i)\| < 2^{-j}$ for every $0 \leq k \leq \alpha d_j$;
- (iv) n_j is a multiple of $p_{T_s}(\sum_{i=1}^{j-1} z_i)$ and $\alpha d_j < n_j \leq d_j$;
- (v) $\alpha d_j > 4d_{j-1}$.

By (ii), the sum $z := \sum_{i \geq 1} z_i$ defines a vector in X . Let us check that z is frequently hypercyclic for every T_s , $s \geq 1$.

Let $p, s \geq 1$ be fixed. We set $I_p(s) := \{j_m : m \geq 1\}$, where $(j_m)_{m \geq 1}$ is increasing and satisfies $j_{m+1} - j_m \leq r_p(s)$ for every $m \geq 1$. Then, for every $m \geq 1$ we define by induction on $j \in \mathbb{N}$ a family of sets $(A_{m,j,s})_{0 \leq j < j_{m+1} - j_m}$ as follows:

$$A_{m,0,s} := \{n_{j_m} + kd_{j_m} + k'p_{T_s}(x_p) : 0 \leq k' \leq \frac{\alpha d_{j_m}}{p_{T_s}(x_p)}, 0 \leq k \leq \frac{\alpha d_{j_m+1}}{d_{j_m}} - 2\},$$

and, for $1 \leq j < j_{m+1} - j_m$,

$$A_{m,j,s} := \bigcup_{1 \leq k \leq \frac{\alpha d_{j_m+j+1}}{d_{j_m+j}} - 1} (A_{m,j-1} + kd_{j_m+j}).$$

As in the proof of [25, Theorem 5.31, Equation (16)], one easily checks by induction that $\max(A_{m,j,s}) \leq \alpha d_{j_m+j+1}$. Moreover, by [25, Fact 5.35] (in fact exactly reproducing its proof), we have $\underline{d}(A_s) > 0$ where

$$A_s := \bigcup_{m \geq 1} \bigcup_{0 \leq j < j_{m+1} - j_m} A_{m,j,s}.$$

Thus to finish the proof of the theorem, we need only prove that for every $m \geq 1$ and every $0 \leq j < j_{m+1} - j_m$, we have

$$\|T_s^n z - x_p\| \leq 2^{-(j_m-1)}, \quad n \in A_{m,j,s}.$$

This shall be proven as in [25, Fact 5.34] up to some modifications. For $m \geq 1$ and $0 \leq j < j_{m+1} - j_m$, we first observe that for any $n \in A_{m,j,s}$ we have

$$\|T_s^n z - x_p\| \leq \|T_s^n(\sum_{i=1}^{j_m+j} z_i) - x_p\| + \sum_{i > j_m+j} \|T_s^n z_i\|.$$

Since $\max(A_{m,j,s}) \leq \alpha d_{j_m+j+1}$, we have $n \leq \alpha d_{j_m+j+1} \leq \alpha d_i$ for every $i > j_m + j$, and it follows from (ii) that

$$\sum_{i > j_m+j} \|T_s^n z_i\| < \sum_{i > j_m+j} 2^{-i} \leq \frac{1}{2^{j_m+j}}.$$

To conclude we now turn to proving that for every $n \in A_{m,j,s}$

$$(3.1) \quad \left\| T_s^n \left(\sum_{i=1}^{j_m+j} z_i \right) - x_p \right\| \leq \sum_{i=0}^j 2^{-(j_m+i)}.$$

To do so, we proceed by induction on $0 \leq j < j_{m+1} - j_m$. If $n \in A_{m,0,s}$, then $j = 0$ and $n = n_{j_m} + kd_{j_m} + k'p_{T_s}(x_p)$ with $0 \leq k \leq \frac{\alpha d_{j_m+1}}{d_{j_m}} - 2$ and $0 \leq k' \leq \frac{\alpha d_{j_m}}{p_{T_s}(x_p)}$ and by (i) and (iv)

$$\begin{aligned} T_s^n \left(\sum_{i=1}^{j_m} z_i \right) - x_p &= T_s^{n_{j_m} + kd_{j_m} + k'p_{T_s}(x_p)} \left(\sum_{i=1}^{j_m} z_i \right) - x_p \\ &= T_s^{n_{j_m} + k'p_{T_s}(x_p)}(z_{j_m}) - T_s^{k'p_{T_s}(x_p)} \left(x_p - \sum_{i=1}^{j_m-1} z_i \right). \end{aligned}$$

By (iii) we get

$$\left\| T_s^n \left(\sum_{i=1}^{j_m} z_i \right) - x_p \right\| \leq 2^{-j_m}.$$

Assume now that (3.1) has been proven up to $j-1$ for some $1 \leq j < j_{m+1} - j_m$. For $n \in A_{m,j,s}$, we write $n = kd_{j_m+j} + l$ with $l \in A_{m,j-1,s}$ and

$$0 \leq k \leq \frac{\alpha d_{j_m+j+1}}{d_{j_m+j}} - 1.$$

Then, by (i) we have

$$\begin{aligned} T_s^n \left(\sum_{i=1}^{j_m+j} z_i \right) - x_p &= T_s^{kd_{j_m+j} + l} \left(\sum_{i=1}^{j_m+j} z_i \right) - x_p \\ &= T_s^l \left(\sum_{i=1}^{j_m+j-1} z_i \right) - x_p + T_s^l(z_{j_m+j}) \end{aligned}$$

Since $l \in A_{m,j-1,s}$, we deduce from the induction hypothesis and (ii) that

$$\left\| T_s^n \left(\sum_{i=1}^{j_m+j} z_i \right) - x_p \right\| \leq \sum_{i=0}^{j-1} 2^{-(j_m+i)} + 2^{-(j_m+j)},$$

and (3.1) as desired. \square

Application to operators of C-type. We will apply Theorem 3.1 to operators of C-type on $\ell^p(\mathbb{N})$. First we shall recall their definition, following the formalism of [25, Section 6]. As usual, we denote by $(e_k)_{k \in \mathbb{N}}$ the canonical basis of $\ell^p(\mathbb{N})$. An operator of C-type is associated a data of four parameters v, w, φ and b :

- $v = (v_n)_{n \geq 1}$ is a sequence of non-zero complex numbers with $\sum_{n \geq 1} |v_n| < \infty$;
- $w = (w_n)_{n \geq 1}$ is a sequence of complex numbers such that

$$0 < \inf_{n \geq 1} |w_n| \leq \sup_{n \geq 1} |w_n| < \infty;$$

- $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ is such that $\varphi(0) = 0$, $\varphi(n) < n$ for every $n \geq 1$, and the set $\{n \in \mathbb{N} : \varphi(n) = l\}$ is infinite for every $l \geq 0$;
- $b = (b_n)_{n \geq 0}$ is a strictly increasing sequence of positive integers with $b_0 = 0$ and $b_{n+1} - b_n$ is a multiple of $2(b_{\varphi(n)+1} - b_{\varphi(n)})$ for every $n \geq 1$.

Now, for a data v, w, φ and b as above, the operator of C -type $T_{v,w,\varphi,b}$ is defined by

$$T_{v,w,\varphi,b}e_k = \begin{cases} w_{k+1}e_{k+1} & \text{if } k \in [b_n, b_{n+1} - 1), n \geq 0 \\ v_n e_{b_{\varphi(n)}} - \left(\prod_{j=b_{n+1}}^{b_{n+1}-1} w_j \right)^{-1} e_{b_n} & \text{if } k = b_{n+1} - 1, n \geq 1 \\ - \left(\prod_{j=b_0+1}^{b_1-1} w_j \right)^{-1} e_0 & \text{if } k = b_1 - 1. \end{cases}$$

Here, by convention, an empty product is equal to 1. From now on, we assume that the condition

$$\inf_{n \geq 0} \prod_{b_n < j < b_{n+1}} |w_j| > 0$$

is satisfied. As shown by [25, Fact 6.2], this assumption ensures that $T_{v,w,\varphi,b}$ is a bounded operator from $\ell^p(\mathbb{N})$ into itself. It can also be checked that each element of c_{00} is a periodic point for $T_{v,w,\varphi,b}$, see [25, Fact 6.4].

In order to deal with frequent hypercyclicity, the authors of [25] introduce a subclass of operators of C -type. As we are interested in common frequent hypercyclicity, we will work within this subclass. It consists in operators of C -type for which the data has a special structure. More precisely, an operator of C -type $T_{v,w,\varphi,b}$ is of C_+ -type if the following conditions hold for every $k \geq 1$:

- $\varphi(n) = n - 2^{k-1}$ for every $n \in [2^{k-1}, 2^k)$;
- There exists $\Delta^{(k)} \in \mathbb{N}$ such that the size of the block $[b_n, b_{n+1})$, i.e. the quantity $b_{n+1} - b_n$, is equal to $\Delta^{(k)}$ for every $n \in [2^{k-1}, 2^k)$;
- There exists $v^{(k)} \in \mathbb{C} \setminus \{0\}$ such that $v_n = v^{(k)}$ for every $n \in [2^{k-1}, 2^k)$;
- There exists a sequence $(w_i^{(k)})_{1 \leq i < \Delta^{(k)}}$ such that $w_{b_n+i} = w_i^{(k)}$ for every $1 \leq i < \Delta^{(k)}$ and every $n \in [2^{k-1}, 2^k)$.

An operator of C -type which satisfies the previous conditions is called an operator of C_+ -type. The following result is a criterion for a countable family of operators of C_+ -type to share a common frequently hypercyclic vector.

Theorem 3.2. *Let $\{T_{v(s),w(s),\varphi,b} : s \in \mathbb{N}\}$ be a countable family of operators of C_+ -type on $\ell^p(\mathbb{N})$ where b does not depend on s . We assume that there exists a constant $\alpha > 0$ such that for every $s \geq 1$, every $C \geq 1$ and every $k_0 \geq 1$, there exists an integer $k \geq k_0$ such that, for every $0 \leq n \leq \alpha \Delta^{(k)}$,*

$$(3.2) \quad |v^{(k)}(s)| \prod_{i=n+1}^{\Delta^{(k)}-1} |w_i^{(k)}(s)| > C.$$

If there exists a constant $K > 0$ such that for any $s, t \geq 1$ and any $r \geq p \geq 1$,

$$(3.3) \quad \left| \frac{w_p(s)w_{p+1}(s) \cdots w_r(s)}{w_p(t)w_{p+1}(t) \cdots w_r(t)} \right| \leq K,$$

then $\bigcap_{s \geq 1} FHC(T_{v(s),w(s),\varphi,b})$ is non-empty.

Note that since b does not depend on s , by definition the $\Delta^{(k)}$, $k \geq 1$, do not depend on s either. It is plainly checked that condition (3.2) is equivalent to saying that each T_s satisfies the assumption of [25, Theorem 6.9]. In particular, if $\{T_{v(s),w(s),\varphi,b} : s \in \mathbb{N}\}$ is reduced to a single operator (i.e., $v(s)$ and $w(s)$ do not depend on s), then the previous criterion is exactly [25, Theorem 6.9].

For the proof of Theorem 3.2, we recall [25, Fact 6.8] below.

Fact 1. Let T be an operator of C_+ -type on $\ell^p(\mathbb{N})$ and $k \geq 1$. For any $l < 2^{k-1}$ and $1 \leq m \leq \Delta^{(k)}$, we have

$$T^m e_{b_{2^{k-1+l+1}}-m} = v^{(k)} \left(\prod_{i=\Delta^{(k)}-m+1}^{\Delta^{(k)}-1} w_i^{(k)} \right) e_{b_l} - \left(\prod_{i=1}^{\Delta^{(k)}-m} w_i^{(k)} \right)^{-1} e_{b_{2^{k-1+l}}}.$$

Proof of Theorem 3.2. Without loss of generality, we can assume that $0 < \alpha < 1$. It suffices to check that the assumptions of Theorem 3.1 are satisfied. Let us define $X_0 := \text{span}(e_k; k \in \mathbb{N})$ and fix $x, y \in X_0$, $\varepsilon > 0$ and $s \geq 1$. There exists $k_0 \geq 1$ such that

$$x = \sum_{l < 2^{k_0}} \sum_{j=b_l}^{b_{l+1}-1} x_j e_j.$$

By (3.2), for any $C > 0$, there exists $k \geq k_0$ such that

$$|v^{(k)}(s)| \prod_{i=n+1}^{\Delta^{(k)}-1} |w_i^{(k)}(s)| > C, \quad 0 \leq n \leq \alpha \Delta^{(k)}.$$

Since $v(s)$ and $w(s)$ are bounded, upon choosing C large enough, we may assume that k is so large that the following holds true:

- (a) $\Delta^{(k)}$ is a multiple of $p_{T_t}(y)$ for any $t \geq 1$;
- (b) $\Delta^{(k_0)} \leq \min((1 - \frac{\alpha}{2})\Delta^{(k)}, \frac{\alpha}{2}\Delta^{(k)} - 1)$.

Note that, by the definition of b and φ for operators of C_+ -type, and since the period of any vector in X_0 depends only on the sequence b , (a) is satisfied whenever y is supported in $[0, b_{2^{k-1}}[$. Let us now set $n := \Delta^{(k)} - 1$, $d := 2\Delta^{(k)}$ and

$$z := \sum_{l < 2^{k_0}} \sum_{j=b_l}^{b_{l+1}-1} x_j \left(v^{(k)}(s) \prod_{i=j-b_l+2}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right)^{-1} \left(\prod_{i=1}^{j-b_l} w_{b_l+i}(s) \right)^{-1} e_{b_{2^{k-1+l+1}}-n+j-b_l}.$$

Like for (a) above, d is a multiple of $p_{T_s}(z)$ for any $s \geq 1$. Thus condition (1) of Theorem 3.1 is satisfied.

Let us now fix $0 \leq m \leq \frac{\alpha d}{4}$ and $t \geq 1$. We observe that for every $l < 2^{k_0}$ and $b_l \leq j \leq b_{l+1} - 1$, we have

$$b_{2^{k-1+l+1}} - n + j - b_l + m \in [b_{2^{k-1+l}}, b_{2^{k-1+l+1}}].$$

Indeed, by definition $b_{2^{k-1+l+1}} - b_{2^{k-1+l}} = \Delta^{(k)}$ and by (b), $-\Delta^{(k)} \leq -n + j - b_l + m \leq 0$. So for every $t \geq 1$, we have

$$T_t^m e_{b_{2^{k-1+l+1}}-n+j-b_l} = \left(\prod_{i=\Delta^{(k)}-n+j-b_l+1}^{\Delta^{(k)}-n+j-b_l+m} w_i^{(k)}(t) \right) e_{b_{2^{k-1+l+1}}-n+j-b_l+m},$$

hence the expression

$$(3.4) \quad T_t^m(z) = \sum_{l < 2^{k_0}} \sum_{j=b_l}^{b_{l+1}-1} x_j \left(v^{(k)}(s) \prod_{i=j-b_l+m+2}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right)^{-1} \left(\prod_{i=1}^{j-b_l} w_{b_l+i}(s) \right)^{-1} \left(\prod_{i=j-b_l+2}^{j-b_l+m+1} \frac{w_i^{(k)}(t)}{w_i^{(k)}(s)} \right) e_{b_{2^{k-1+l+1}}-n+j-b_l+m}.$$

Using (b), we know that $0 \leq j - b_l + m + 1 \leq \alpha \Delta^{(k)}$ which, by (3.2), (3.3) and the definition of C_+ -type operators, implies that for some constant $A > 0$ (independent of k),

$$\|T_t^m(z)\| \leq \|x\| C^{-1} K A^{\Delta^{(k_0)}}.$$

Up to choose C large enough, we get (2) in Theorem 3.1.

Let us now estimate the norm of $T_s^{n+m}z - T_s^m(x)$ for $0 \leq m \leq \frac{\alpha d}{4}$. By Fact 1, we obtain

$$T_s^{n-(j-b_l)} e_{b_{2^{k-1+l+1}}-n+j-b_l} = v^{(k)}(s) \left(\prod_{i=\Delta^{(k)}-n+j-b_l+1}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right) e_{b_l} \\ - \left(\prod_{i=1}^{\Delta^{(k)}-n+j-b_l} w_i^{(k)}(s) \right)^{-1} e_{b_{2^{k-1+l}}}.$$

Applying $T_s^{j-b_l}$ yields

$$T_s^m e_{b_{2^{k-1+l+1}}-n+j-b_l} = \left(v^{(k)}(s) \prod_{i=j-b_l+2}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right) \left(\prod_{i=1}^{j-b_l} w_{b_l+i}(s) \right) e_j \\ - \left(w_{j-b_l+1}^{(k)}(s) \right)^{-1} e_{b_{2^{k-1+l}}+j-b_l}.$$

Moreover, since $m + j - b_l < \Delta^{(k)}$, we have

$$T_s^m e_{b_{2^{k-1+l}}+j-b_l} = \left(\prod_{i=j-b_l+1}^{j-b_l+m} w_i^{(k)}(s) \right) e_{b_{2^{k-1+l}}+j-b_l+m},$$

hence

$$T_s^{n+m} e_{b_{2^{k-1+l+1}}-n+j-b_l} = \left(v^{(k)}(s) \prod_{i=j-b_l+2}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right) \left(\prod_{i=1}^{j-b_l} w_{b_l+i}(s) \right) T_s^m e_j \\ - \left(w_{j-b_l+1}^{(k)}(s) \right)^{-1} \left(\prod_{i=j-b_l+1}^{j-b_l+m} w_i^{(k)}(s) \right) e_{b_{2^{k-1+l}}+j-b_l+m}.$$

By definition of z , it follows that

$$T_s^{n+m}(z) \\ = T_s^m(x) - \sum_{l < 2^{k_0}} \sum_{j=b_l}^{b_{l+1}-1} x_j \left(v^{(k)}(s) \prod_{i=j-b_l+m+1}^{\Delta^{(k)}-1} w_i^{(k)}(s) \right)^{-1} \left(\prod_{i=1}^{j-b_l} w_{b_l+i}(s) \right)^{-1} e_{b_{2^{k-1+l}}+j-b_l+m}$$

By assumption, we thus get

$$\|T_s^{n+m}(z) - T_s^m(x)\| \leq \|x\| C^{-1} A^{\Delta^{(k_0)}}$$

and condition (3) of Theorem 3.1 with $\alpha' = \frac{\alpha}{4}$, as desired. \square

Remark 3.3. 1) It is clear from the proof that the conclusion of Theorem 3.2 remains true if condition (3.3) is replaced by the following weaker (but less nice) one:

$$(3.5) \quad \sup_{\substack{t \geq 1 \\ 0 \leq j < \Delta^{(k_0)} \\ 0 \leq m \leq \alpha \Delta^{(k)}}} \left(|v^{(k)}(s)| \prod_{i=j+2}^{\Delta^{(k)}-1} |w_i^{(k)}(s)| \right)^{-1} \left(\prod_{i=j+2}^{j+m+1} |w_i^{(k)}(t)| \right) < \frac{1}{C}$$

2) Moreover, if (3.3) is replaced by the following two conditions:

- (a) For some $A > 1$, $\sup_{i,t \geq 1} \max \left(|w_i(t)|; \frac{1}{|w_i(t)|} \right) \leq A$,
 (b)

$$(3.6) \quad \sup_{\substack{t \geq 1 \\ 0 \leq m \leq \alpha \Delta^{(k)}}} \left(|v^{(k)}(s)| \prod_{i=1}^{\Delta^{(k)}-1} |w_i^{(k)}(s)| \right)^{-1} \left(\prod_{i=1}^{m+1} |w_i^{(k)}(t)| \right) < \frac{1}{C}$$

then it is also clear that (3.5) holds, and the conclusion of Theorem 3.2 is still true.

It turns out that for a certain subclass of operators of C_+ -type, for which (2a) holds true, some rather simple condition for frequent hypercyclicity is given in [25]. We shall now see that a similar condition for a family of operators in this subclass implies (3.6) and thus common frequent hypercyclicity.

Application to operators of $C_{+,1}$ -type. Operators of $C_{+,1}$ -type are introduced in [25, Section 6.5] as those for which the parameters v and w satisfy the following extra condition: For every $k \geq 1$,

$$v^{(k)} = 2^{-\tau^{(k)}} \quad \text{and} \quad w_i^{(k)} = \begin{cases} 2 & \text{if } 1 \leq i \leq \delta^{(k)} \\ 1 & \text{if } \delta^{(k)} < i < \Delta^{(k)} \end{cases},$$

where $\tau := (\tau^{(k)})_{k \geq 1}$ and $\delta := (\delta^{(k)})_{k \geq 1}$ are two strictly increasing sequences of integers such that $\delta^{(k)} < \Delta^{(k)}$, $k \geq 1$. Within this class of operators of $C_{+,1}$ -type, that we simply denote by $T_{\tau, \delta, \varphi, b}$, examples of frequently hypercyclic operators which are not ergodic were provided in [25].

Theorem 3.4. *Let $\{T_{\tau(s), \delta(s), \varphi, b} : s \geq 1\}$ be a countable family of operators of $C_{+,1}$ -type on $\ell^p(\mathbb{N})$ where b does not depend on s . If*

$$\inf_{t \geq 1} \limsup_{k \rightarrow \infty} \frac{\delta^{(k)}(t) - \tau^{(k)}(t)}{\Delta^{(k)}} > 0,$$

then $\bigcap_{s \geq 1} FHC(T_{\tau(s), \delta(s), \varphi, b})$ is non-empty.

Proof. Remark that (2a) in Remark 3.3 trivially holds, thus it is enough to check (2b). To do so, we define

$$\alpha < \min \left(1, \frac{1}{2} \inf_{t \geq 1} \limsup_{k \rightarrow \infty} \frac{\delta^{(k)}(t) - \tau^{(k)}(t)}{\Delta^{(k)}} \right).$$

Let $s, k_0 \geq 1$ and $C \geq 1$, and let us set $n = \Delta^{(k)} - 1$. Since $\Delta^{(k)} \rightarrow \infty$ as $k \rightarrow \infty$, there exists $k \geq k_0$ such that:

$$\frac{\delta^{(k)}(s) - \tau^{(k)}(s)}{\Delta^{(k)}} > 2\alpha \quad \text{and} \quad \alpha \Delta^{(k)} > \ln_2(C).$$

Then it follows from the definition of operators of $C_{+,1}$ -type that (2b) in Remark 3.3 is equivalent to

$$2^{\tau^{(k)}(s) - \delta^{(k)}(s)} \sup_{\substack{t \geq 1 \\ 0 \leq m \leq \alpha \Delta^{(k)}}} \prod_{i=1}^{m+1} |w_i^{(k)}(t)| < \frac{1}{C}.$$

Now, we have

$$\sup_{\substack{t \geq 1 \\ 0 \leq m \leq \alpha \Delta^{(k)}}} \prod_{i=1}^{m+1} |w_i^{(k)}(t)| \leq 2^{\alpha \Delta^{(k)}} \leq 2^{\frac{1}{2}(\delta^{(k)}(s) - \tau^{(k)}(s))}.$$

Hence,

$$2^{\tau^{(k)}(s) - \delta^{(k)}(s)} \sup_{\substack{t \geq 1 \\ 0 \leq m \leq \alpha \Delta^{(k)}}} \prod_{i=1}^{m+1} |w_i^{(k)}(t)| \leq 2^{\frac{1}{2}(\tau^{(k)}(s) - \delta^{(k)}(s))} < 2^{-\alpha \Delta^{(k)}} < \frac{1}{C}$$

It remains to check that for every $0 \leq n \leq \alpha \Delta^{(k)}$,

$$|v^{(k)}(s)| \prod_{i=n+1}^{\Delta^{(k)}-1} |w_i^{(k)}(s)| > C$$

which works the same as in the proof of [25, Theorem 6.17]. \square

Remark 3.5. When one considers only a single operator, Theorem 3.4 is exactly [25, Theorem 6.17].

4. COMMON FREQUENT HYPERCYCLICITY WITH RESPECT TO DENSITIES

We refer to [23] for the abstract definitions and the study of *generalized lower/upper densities*. In particular it is proven there that to any sequence of non-negative real numbers α such that $\sum_{k \geq 1} \alpha_k = +\infty$, one can associate generalized lower and upper densities \underline{d}_α and \bar{d}_α by the formulae

$$\underline{d}_\alpha(E) = \liminf_n \sum_{k \geq 1} \alpha_{n,k} \mathbb{1}_F(k) \quad \text{and} \quad \bar{d}_\alpha(E) = 1 - \underline{d}_\alpha(\mathbb{N} \setminus E), \quad E \subset \mathbb{N},$$

where $(\alpha_{n,k})_{n,k \geq 1}$ is the matrix given by

$$\alpha_{n,k} = \begin{cases} \alpha_k / (\sum_{j=1}^n \alpha_j) & \text{for } 1 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then we also have $\bar{d}_\alpha(E) = \limsup_n \sum_{k=1}^{+\infty} \alpha_{n,k} \mathbb{1}_F(k)$. Let us also introduce the notation φ_α for the function defined by $\varphi_\alpha(x) = \sum_{k \leq x} \alpha_k$, $x \in [0, +\infty)$.

For α and β two sequences as above, let us write $\alpha \lesssim \beta$ if there exists $k_0 \in \mathbb{N}$ such that $(\alpha_k / \beta_k)_{k \geq k_0}$ is decreasing to 0. As recalled in the introduction, we have

$$\underline{d}_\beta(E) \leq \underline{d}_\alpha(E) \leq \bar{d}_\alpha(E) \leq \bar{d}_\beta(E), \quad E \subset \mathbb{N},$$

whenever $\alpha \lesssim \beta$ (see [20, Lemma 2.8]). Thus one can define scales of well-ordered densities with respect to the type of growth of the defining sequences. In this section, two types of sequences will play an important role.

- (1) For $0 \leq \varepsilon \leq 1$, $\mathcal{E}_\varepsilon := (\exp(k^\varepsilon))_{k \geq 1}$. By a summation by parts, one can see that for $0 < \varepsilon < 1$, $\varphi_{\mathcal{E}_\varepsilon}(n) \sim \frac{n^{1-\varepsilon}}{\varepsilon} \exp(n^\varepsilon)$ (where $u_k \sim v_k$ means $u_k/v_k \rightarrow 1$);
- (2) For $s \in \mathbb{N} \cup \{\infty\}$, $\mathcal{D}_s := (\exp(k / \log_{(s)}(k)))_{k \geq 1}$ where $\log_{(s)} = \log \circ \dots \circ \log$, \log appearing s times, with the conventions $\log_{(0)}(x) = x$ and $\log_{(\infty)}(x) = 1$ for any $x > 0$. One can check that $\varphi_{\mathcal{D}_s}(n) \sim \log_{(s)}(n) \exp(x / \log_{(s)}(n))$ for $s \in \mathbb{N}$ (see [21, Remark 3.10]) and $\varphi_{\mathcal{D}_\infty}(n) \sim \frac{e}{e-1} \exp(n)$.

For $r \geq 1$ we shall also write $\mathcal{P}_r := (k^r)_{k \geq 1}$. More examples of generalized densities can be found in [20, 21]. Observe that the usual lower density \underline{d} (associated to the constant sequence $(1, 1, 1, \dots)$) corresponds to $\underline{d}_{\mathcal{E}_0}$, $\underline{d}_{\mathcal{D}_0}$ and $\underline{d}_{\mathcal{P}_1}$. Note that \mathcal{E}_1 shall simply be denoted by \mathcal{E} and $\underline{d}_{\mathcal{E}_1} = \underline{d}_{\mathcal{D}_\infty}$ by $\underline{d}_\mathcal{E}$. For any $0 < \delta \leq \varepsilon \leq 1$, any $s \leq t \in \mathbb{N}$ and any $r \geq 1$, we thus have

$$\underline{d}_\mathcal{E} \leq \underline{d}_{\mathcal{D}_t} \leq \underline{d}_{\mathcal{D}_s} \leq \underline{d}_{\mathcal{E}_\varepsilon} \leq \underline{d}_{\mathcal{E}_\delta} \leq \underline{d}_{\mathcal{P}_r} \leq \underline{d} \leq \bar{d} \leq \bar{d}_{\mathcal{P}_r} \leq \bar{d}_{\mathcal{E}_\delta} \leq \bar{d}_{\mathcal{E}_\varepsilon} \leq \bar{d}_{\mathcal{D}_s} \leq \bar{d}_{\mathcal{D}_t} \leq \bar{d}_\mathcal{E}.$$

As for (\mathcal{U} -)frequently hypercyclic operators, we now say that $T \in \mathcal{L}(X)$ is α -frequently hypercyclic (resp. \mathcal{U}_α -frequently hypercyclic) if there exists $x \in X$ such that for any non-empty open set U in X , $\underline{d}_\alpha(N(x, U, T))$ (resp. $\overline{d}_\alpha(N(x, U, T))$) is positive. We denote by $FHC_\alpha(T)$ (resp. $\mathcal{U}FHC_\alpha(T)$) the set of all α -frequently (resp. \mathcal{U}_α -frequently) hypercyclic vectors for T . As proven in [20], no operator can be \mathcal{E} -frequently hypercyclic (and hence α -frequently hypercyclic whenever $\mathcal{E} \lesssim \alpha$).

A first natural question arises:

Question 4.1. *Does the work done in Section 2 extend to α -frequent hypercyclicity for some α ?*

Let us recall that any operator satisfying the Frequent Universality Criterion is automatically α -universal whenever $\alpha \lesssim \mathcal{D}_s$ for some $s \geq 1$ [21]. Since each of the criteria given in Section 2 are natural strengthenings of the Frequent Hypercyclicity Criterion, we could expect a positive answer to this question for any such α . Moreover, it is easily seen that $FHC_{\mathcal{P}_r}(T) = FHC(T)$ for any $r \geq 1$ (see [20, Lemma 2.10]), so Question 4.1 has at least an obvious positive answer for sequences with polynomial growth.

Yet the next proposition shows that Theorem 2.6 for the multiples of a single operator completely fails to extend to α -frequent hypercyclicity as soon as $\mathcal{E}_\varepsilon \lesssim \alpha$ for some $\varepsilon > 0$. We will denote by $B(a, r)$ the open ball centered at a with radius r .

Proposition 4.2. *Let $0 < \varepsilon \leq 1$, $T \in L(X)$ and $0 < \lambda \neq \mu < +\infty$. Then for any $x \in HC(\mu T) \cap HC(\lambda T)$ and any $r > 0$,*

- (1) *If $\mu > \lambda$, then $\overline{d}_{\mathcal{E}_\varepsilon}(N(x, B(0, r), \lambda T)) = 1$;*
- (2) *If $\lambda > \mu$, then $\underline{d}_{\mathcal{E}_\varepsilon}(N(x, B(0, r), \lambda T)) = 0$.*

In particular, $HC(\mu T) \cap FHC_{\mathcal{E}_\varepsilon}(\lambda T) = \emptyset$.

Proof. Of course, we shall assume that $HC(\lambda T) \cap HC(\mu T) \neq \emptyset$. Throughout the proof, $r > 0$ is fixed. We first prove (1). By assumption, there exists an increasing sequence $(p_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that $\|\mu^{p_k} T^{p_k} x\| < r$ for any $k \in \mathbb{N}$. Writing

$$\lambda^{p_k+i} T^{p_k+i} x = \lambda^i T^i \left(\frac{\lambda}{\mu} \right)^{p_k} \mu^{p_k} T^{p_k} x, \quad i \in \mathbb{N},$$

we easily check that $\|\lambda^{p_k+i} T^{p_k+i} x\| < r$ whenever $(\lambda \|T\|)^i < (\mu/\lambda)^{p_k}$. Since by assumption λT is hypercyclic, we have $\lambda \|T\| > 1$. Thus there exists a constant $C > 0$ (depending on λ, μ and T , but not on k) such that for any $i < Cp_k$, $\|\lambda^{p_k+i} T^{p_k+i} x\| < r$. Therefore,

$$\bigcup_{k \in \mathbb{N}} \{p_k, \dots, \lfloor (1+C)p_k \rfloor\} \subset N(x, B(0, r), \lambda T).$$

It follows that

$$\overline{d}_{\mathcal{E}_\varepsilon}(N(x, B(0, r), \lambda T)) \geq 1 - \lim_k \left(\frac{\varphi_{\mathcal{E}_\varepsilon}(p_k)}{\varphi_{\mathcal{E}_\varepsilon}(\lfloor (1+C)p_k \rfloor)} \right) = 1.$$

(2) is proved similarly. Since $x \in HC(\mu T)$, there exists an increasing sequence $(p_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that $\|\mu^{p_k} T^{p_k} x\| > r$. Writing $T^i \lambda^{p_k-i} T^{p_k-i} = \lambda^{-i} (\lambda/\mu)^{p_k} \mu^{p_k} T^{p_k}$, $1 \leq i \leq p_k$, one can easily check that $\|\lambda^{p_k-i} T^{p_k-i} x\| \geq r (\lambda \|T\|)^{-i} (\lambda/\mu)^{p_k}$, $1 \leq i \leq p_k$. Thus $\|\lambda^{p_k-i} T^{p_k-i} x\| > r$ whenever $(\lambda \|T\|)^i > (\lambda/\mu)^{p_k}$. Since $\mu \|T\| > 1$, the last inequality is equivalent to $i \in \{\lfloor Cp_k \rfloor + 1, \dots, p_k\}$ for some constant $0 < C < 1$ not depending on k . Therefore,

$$\overline{d}_{\mathcal{E}_\varepsilon}(N(x, X \setminus B(0, r), \lambda T)) \geq 1 - \lim_k \left(\frac{\varphi_{\mathcal{E}_\varepsilon}(\lfloor Cp_k \rfloor + 1)}{\varphi_{\mathcal{E}_\varepsilon}(p_k)} \right) = 1.$$

□

Remark 4.3. 1) A trivial argument shows that any hypercyclic operator is automatically $\mathcal{U}_{\mathcal{E}}$ -frequently hypercyclic. This comes from the fact that any infinite subset of \mathbb{N} has positive upper \mathcal{E} -density. Indeed, if $E = (n_k)_{k \in \mathbb{N}}$ is an increasing sequence, then

$$\bar{d}_{\mathcal{E}}(E) \geq \lim_k \left(\frac{e^{n_k}}{\varphi_{\mathcal{E}}(n_k)} \right) = 1 - \frac{1}{e}.$$

At the opposite, it turns out that for any sequence of non-negative real numbers $\alpha = (\alpha_k)$ with $\sum \alpha_k = +\infty$ such that $\alpha_n / (\sum_{k=1}^n \alpha_k)$ tends to 0 as n tends to infinity, there exists a hypercyclic operator $T \in \mathcal{L}(X)$ which is not \mathcal{U}_{α} -frequently hypercyclic. This can be deduced from the fact that for such α , \mathcal{U}_{α} -frequent hypercyclicity implies *reiterative hypercyclicity* [22] and that reiteratively hypercyclic weighted shifts on ℓ^p , $1 \leq p < +\infty$ are automatically frequently hypercyclic [10] (and, of course, some hypercyclic weighted shifts are not frequently hypercyclic). For the definition of reiterative hypercyclicity, we refer to [10]. Thus, in particular, this observation applies to the weights $\mathcal{E}_{\varepsilon}$ for all $0 \leq \varepsilon < 1$ and \mathcal{D}_s for all $s \in \mathbb{N}$.

2) The algebraic approach to common (frequent) hypercyclicity mentioned in the introduction is still efficient when dealing with α -frequent hypercyclicity. In particular, the same proof as that of [7, Theorem 6.28] shows that for any weight sequence α of non-negative real numbers satisfying $\sum_{k \in \mathbb{N}} \alpha_k = +\infty$ and such that $\alpha_n / (\sum_{k=1}^n \alpha_k)$ decreases to 0 as n goes to infinity and any λ with modulus 1, $FHC_{\alpha}(\lambda T) = FHC_{\alpha}(T)$.

Indeed, it suffices to follow the same lines as in the proof for frequent hypercyclic operators that one may find in [7] replacing Lemma 6.29 by the following.

Lemma 4.4. *Let $A \subset \mathbb{N}$ have positive lower α -density with a non-decreasing weight sequence $(\alpha_k)_{k \in \mathbb{N}}$ so that $\sum_{k \in \mathbb{N}} \alpha_k = +\infty$ and $\alpha_n / (\sum_{k=1}^n \alpha_k)$ decreases to zero as n tends to infinity. Let also $I_1, \dots, I_q \subset \mathbb{N}$ be such that $\cup_{j=1}^q I_j = \mathbb{N}$ and $n_1, \dots, n_q \in \mathbb{N}$. Then, $B := \cup_{j=1}^q (n_j + A \cap I_j)$ has positive lower α -density.*

Proof. Let $N := \max_{1 \leq i \leq q} (n_i)$. Then, for any $M \geq N$,

$$\begin{aligned} \frac{\sum_{k=1}^{M+N} \alpha_k \mathbb{1}_B(k)}{\sum_{k=1}^{M+N} \alpha_k} &\geq \frac{1}{q} \frac{\sum_{j=1}^q \sum_{k=1}^{M+N} \alpha_k \mathbb{1}_{n_j + A \cap I_j}(k)}{\sum_{k=1}^{M+N} \alpha_k} \\ &\geq \frac{1}{q} \frac{\sum_{j=1}^q \sum_{k=1}^M \alpha_k \mathbb{1}_{A \cap I_j}(k)}{\sum_{k=1}^{M+N} \alpha_k} && \text{since } \alpha_{k+n_j} \geq \alpha_k \\ &\geq \frac{1}{q} \frac{\sum_{k=1}^M \alpha_k \mathbb{1}_A(k)}{\sum_{k=1}^{M+N} \alpha_k} \\ &= \frac{1}{q} \frac{\sum_{k=1}^M \alpha_k \mathbb{1}_A(k)}{\sum_{k=1}^M \alpha_k} \frac{\sum_{k=1}^M \alpha_k}{\sum_{k=1}^{M+N} \alpha_k} \end{aligned}$$

On the other hand,

$$\frac{\sum_{k=1}^M \alpha_k}{\sum_{k=1}^{M+N} \alpha_k} = 1 - \frac{\sum_{k=M+1}^{M+N} \alpha_k}{\sum_{k=1}^{M+N} \alpha_k} \geq 1 - \left(\sum_{j=M+1}^{M+N} \frac{\alpha_j}{\sum_{k=1}^j \alpha_k} \right) \xrightarrow{M \rightarrow \infty} 1.$$

Hence,

$$d_{\alpha}(B) = \liminf_{M \rightarrow \infty} \frac{\sum_{k=1}^{M+N} \alpha_k \mathbb{1}_B(k)}{\sum_{k=1}^{M+N} \alpha_k} \geq \liminf_{M \rightarrow \infty} \frac{1}{q} \frac{\sum_{k=1}^M \alpha_k \mathbb{1}_A(k)}{\sum_{k=1}^M \alpha_k} = \frac{1}{q} d_{\alpha}(A) > 0.$$

□

We conclude by an answer to the following natural question:

Question 4.5. *Does there exist an operator which is α -frequently hypercyclic for any $\alpha \lesssim \mathcal{E}$?*

This question is clearly a question of *common frequent universality* except that, this time, *common* refers to an uncountable family of densities. The following proposition strengthens one of the main results of [21] and gives an almost positive answer to Question 4.5.

Proposition 4.6. *We denote by \mathcal{D} the set of all sequences α (with $\sum_{k \geq 1} \alpha_k = +\infty$) such that $\alpha \lesssim \mathcal{D}_s$ for some $s \in \mathbb{N}$. If $T \in \mathcal{L}(X)$ satisfies the Frequent Hypercyclicity Criterion, then*

$$\bigcap_{\alpha \in \mathcal{D}} FHC_{\alpha}(T) \neq \emptyset.$$

Proof. It is clearly enough to prove that $\bigcap_{s \in \mathbb{N}} FHC_{\mathcal{D}_s}(T)$ is non-empty. The proof is based on the calculations led in [21, Section 3]. Let us consider the function $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(j) = m$ for all $j \in \{a_m, \dots, a_{m+1} - 1\}$ with

$$a_m = 2^{2^{\dots^{2^m}}} \quad \text{where 2 appears } m \text{ times.}$$

Then we define the sequence $(n_k(f))_{k \geq 1}$ as follows:

$$n_1(f) = 2 \text{ and } n_k(f) = 2 \sum_{i=1}^{k-1} f(\delta_i) + f(\delta_k) \text{ for } k \geq 2,$$

where δ_j is the index of the first zero in the dyadic representation of j (for e.g., if $k = 11 = 1.2^0 + 1.2^1 + 0.2^2 + 1.2^3$, then $\delta_k = 3$). Lemma 3.8 of [21] ensures that for all $s \geq 1$ there exist $C_1, C_2, C_3 > 0$ such that for all integer k large enough

$$C_1 k - C_2 \log_{(s)}(k) \leq n_k(f) \leq C_1 k + C_3 \log_{(s)}(k).$$

A similar calculation as that of [20, Lemma 4.10] allows to conclude that for all $s \geq 1$ $d_{\mathcal{D}_s}((n_k(f))) > 0$. Therefore this sequence $(n_k(f))$ allows to construct a hypercyclic vector for T which will be \mathcal{D}_s -frequently hypercyclic for all $s \geq 1$ (we refer the reader to the beginning of Section 2 of [21]). \square

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