

Dispatching to Parallel Servers

Solutions of Poisson's Equation for First-Policy Improvement

Supplementary material

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The author acknowledges support from the French National Research Agency (project ORACLESS, ANR-16-CE33-0004-01). Part of this work was completed at the Department of Communications and Networking, Aalto University, Espoo, Finland, with support from the Academy of Finland in the project FQ4BD (Grant No. 296206). The author is now with the Department of Data Science and Knowledge Engineering, Maastricht University, Netherlands (E-mail: o.bilenne@maastrichtuniversity.nl).

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F Uniform approximation schemes and error bounds (Table 2)

In Sections F.1, F.2 and F.3 we discuss the methods for deriving the finite sum $[\hat{f}^{(n)}]$ when the cost function f is continuous on any support $[0, \tau]$.

F.1 Bernstein polynomials

The function f can be approximated on $[0, \tau]$ by the *Bernstein polynomial* [1]

$$b^{(n)}(u) = \sum_{l=0}^n \binom{n}{l} \left(\frac{u}{\tau}\right)^l \left(1 - \frac{u}{\tau}\right)^{n-l} f(l\tau/n), \quad \forall u \in [0, \tau]. \quad (\text{F.1})$$

Notice that (F.1) rewrites as $b^{(n)}(u) = \mathbb{E}[f(K\tau/n)]$, where the random variable $K \sim \text{B}(n, u/\tau)$ is distributed according to the binomial distribution with n trials and success probability u/τ . The quantity K/n has mean u/τ and variance $(u/\tau)(1 - u/\tau)/n \leq 1/(4n)$, which vanishes uniformly on $[0, \tau]$. It follows from continuity arguments that $\mathbb{E}[f(K\tau/n)]$ converges uniformly towards $f(u)$ on that interval, [7, proof of Theorem 2.7]. So does (F.1), with rate

$$\|f - b^{(n)}\| \leq (3/2) \omega(f; [0, \tau]; \tau/\sqrt{n}), \quad (\text{F.2})$$

[10, Theorem 1.2], where

$$\omega(f; [0, \tau]; \delta) = \sup\{|f(u_1) - f(u_2)| : u_1, u_2 \in [0, \tau], |u_1 - u_2| \leq \delta\}$$

defines the *modulus of continuity* of f on the interval $[0, \tau]$, [6, §21]. To conform with (IV.4), we rewrite (F.1) as¹

$$b^{(n)}(u) = \sum_{k=0}^n \beta_{n,k} u^k, \quad \forall u \in [0, \tau], \quad (\text{F.3})$$

where

$$\beta_{n,k} = (-\tau)^{-k} \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^l f\left(\frac{l\tau}{n}\right), \quad (k = 0, \dots, n). \quad (\text{F.4})$$

From (F.2) and (F.3), we infer bounds for the relative value function.

¹ The coefficients $\beta_{n,0}, \dots, \beta_{n,n}$ can be computed recursively. Indeed, one show by induction that $\beta_{n,k} = (1/k!) \check{f}_n^{(k)}(0)$, where, for $k = 1, \dots, n$,

$$\check{f}_n^{(0)}(l) = f(l\tau/n), \quad \check{f}_n^{(k)}(l) = [(n-k+1)/\tau] [\check{f}_n^{(k-1)}(l+1) - \check{f}_n^{(k-1)}(l)], \quad (l = 0, \dots, n-k). \quad (\text{F.5})$$

Indeed, $\beta_{n,0} = \check{f}_n^{(0)}(0)/0!$ is immediate. Suppose now that

$$\check{f}_n^{(k)}(l) = \frac{k!}{(-\tau)^k} \binom{n}{k} \sum_{t=0}^k \binom{k}{t} (-1)^t f\left(\frac{t\tau}{n}\right), \quad (l = 0, \dots, n-k), \quad (\text{F.6})$$

holds for $k = 1, \dots, q-1$, where $1 \leq q \leq n-1$. Then, for $l = 0, \dots, n-q$,

$$\begin{aligned} \check{f}_n^{(q)}(l) &\stackrel{(\text{F.5})}{=} \frac{(n-q+1)}{\tau} [\check{f}_n^{(q-1)}(l+1) - \check{f}_n^{(q-1)}(l)] \\ &\stackrel{(\text{F.4})}{=} \frac{n-q+1}{\tau} \frac{(q-1)!}{(-\tau)^{q-1}} \binom{n}{q-1} \sum_{t=0}^{q-1} \binom{q-1}{t} (-1)^t \left[f\left(\frac{(t+l+1)\tau}{n}\right) - f\left(\frac{(t+l)\tau}{n}\right) \right] \\ &= \frac{(q!)}{(-\tau)^q} \frac{(n-q+1)}{q} \binom{n}{q-1} \left[\sum_{t=1}^q \binom{q-1}{t-1} (-1)^t f\left(\frac{(t+l)\tau}{n}\right) + \sum_{t=0}^{q-1} \binom{q-1}{t} (-1)^t f\left(\frac{(t+l)\tau}{n}\right) \right] \\ &= \frac{(q!)}{(-\tau)^q} \binom{n}{q} \left\{ f\left(\frac{l\tau}{n}\right) + (-1)^q f\left(\frac{(q+l)\tau}{n}\right) + \sum_{t=1}^{q-1} \left[\binom{q-1}{t-1} + \binom{q-1}{t} \right] (-1)^t f\left(\frac{(t+l)\tau}{n}\right) \right\} \\ &= \frac{(q!)}{(-\tau)^q} \binom{n}{q} \sum_{t=0}^q \binom{q}{t} (-1)^t f\left(\frac{(t+l)\tau}{n}\right) \end{aligned}$$

and (F.6) holds for $k = q$. By induction, (F.6) is true for $0 \leq k \leq n$. By setting $l = 0$ in (F.6), we infer from (F.4) that $\beta_{n,k} = \check{f}_n^{(k)}(0)/k!$ for $0 \leq k \leq n$.

Corollary F.1 (Bernstein polynomials) *Proposition 3 holds for $\hat{f}^{(n)} \equiv b^{(n)}$ defined by (F.3) with the uniform error bound $\eta^{(n)} = 3/2\omega(f; [0, \tau]; \tau/\sqrt{n})$.*

F.2 Approximation by trigonometric sums

A better convergence rate for $\hat{f}^{(n)}$ can be obtained using trigonometric sums; we refer to [10, §1.1] for details on this topic. Consider the continuous, 2τ -periodic function $\check{f} : \mathbb{R} \mapsto \mathbb{R}$ defined on $[-\tau, \tau]$ by $\check{f}(u) = f(|u|)$. The Weierstrass approximation theorem (see, e.g., [8, Weierstrass first theorem], [10, Theorem 1.1], [7, Theorem 2.7]) claims that \check{f} can be approximated by a trigonometric sum with arbitrary precision with respect to the uniform norm $\|\check{f}\| = \sup_{u \in [-\tau, \tau]} |\check{f}(u)|$. This implies that for any $\epsilon > 0$ one can find $n < \infty$ and a trigonometric sum $t^{(n)}$ such that $\eta^{(n)} = \|\check{f}(u) - t^{(n)}(u)\| < \epsilon$. It then follows that $f \in [f] = t^{(n)} + [-\eta^{(n)}, \eta^{(n)}]$. Such a trigonometric sum is given by the partial Fourier series, which for the real, even function \check{f} reduces to

$$t^{(n)}(u) = \check{\alpha}_0 + 2 \sum_{k=1}^n \check{\alpha}_k \cos(k\pi u/\tau), \quad (\text{F.7})$$

where

$$\check{\alpha}_k = \frac{1}{\tau} \int_0^\tau f(u) \cos(k\pi u/\tau) du, \quad (k \in \mathbb{N}). \quad (\text{F.8})$$

are the Fourier coefficients. With the modulus of continuity of \check{f} defined by

$$\omega(\check{f}; \delta) = \sup\{|\check{f}(u_1) - \check{f}(u_2)| : u_1, u_2 \in \mathbb{R}, |u_1 - u_2| \leq \delta\}, \quad (\text{F.9})$$

the Fourier series (F.7) converges towards the periodic function \check{f} with rate $O(\eta^{(n)}) = \log(n)\omega(\check{f}; \tau/(n\pi))$, [6, §21]. Faster convergence can be obtained by slightly modifying the Fourier coefficients in (F.7). For this, consider

$$\check{t}^{(n)}(u) = \varrho_{n,0} \check{\alpha}_0 + 2 \sum_{k=0}^n \varrho_{n,k} \check{\alpha}_k \cos(k\pi u/\tau), \quad (\text{F.10})$$

where $\varrho_{n,0}, \dots, \varrho_{n,n} \in \mathbb{R}$. The choice of parameters proposed in [8, §3],

$$\varrho_{n,0} = 1, \quad \varrho_{n,1} = \cos\left(\frac{\pi}{n+2}\right), \quad \varrho_{n,k} = \frac{\sum_{q=0}^{n-k} \sin\left(\frac{q+1}{n+2}\pi\right) \sin\left(\frac{q+k+1}{n+2}\pi\right)}{\sum_{q=0}^n \sin^2\left(\frac{q+1}{n+2}\pi\right)} \quad \text{for } k = 2, \dots, n, \quad (\text{IV.6})$$

leads (F.10) the convergence rate

$$\eta^{(n)} \leq 6\omega(\check{f}; \frac{\tau}{\pi n}), \quad (\text{F.11})$$

(see [8, *first Jackson Theorem*], or [10, Theorem 1.3]). Since $\check{t}^{(n)} \in \Xi$ and by construction $\omega(\check{f}; \cdot) \equiv \omega(f; [0, \tau]; \cdot)$, $\check{t}^{(n)}$ is a candidate finite sum for Proposition 3 and (F.11) gives us bounds for the relative value function.

Corollary F.2 (Trigonometric sums) *Proposition 3 holds for $\hat{f}^{(n)} \equiv b^{(n)}$ defined by (F.3) with the uniform error bound $\eta^{(n)} = 6\omega(f; [0, \tau]; \tau/(\pi n))$.*

In particular, if for some $\alpha \in (0, 1]$ the cost function satisfies the α -Hölder condition $|f(u_1) - f(u_2)| \leq h|u_1 - u_2|^\alpha$ for all $u_1, u_2 \in [0, \tau]$, then $\omega(f; [0, \tau]; \delta) \leq h\delta^\alpha$, and (F.7) converges uniformly towards f on $[0, \tau]$ with $\eta^{(n)} = O((\tau/n)^\alpha)$. If f is Lipschitz continuous on $[0, \tau]$ with modulus L , then $\eta^{(n)} < 2L\tau/n$.

F.3 Near-optimal polynomial approximation

Alternatively, the convergence rate of Corollary F.3 can be obtained using polynomials. Set $\tilde{f} : \mathbb{R} \mapsto \mathbb{R}$ to be the continuous, 2π -periodic function defined by $\tilde{f}(\theta) = f(\tilde{u}(\theta))$, where $\tilde{u}(\theta) = (\tau/2)(1 + \cos(\theta))$. It follows from the $(\tau/2)$ -Lipschitz continuity of \tilde{u} and the definition (F.9) of the modulus of continuity, that $\omega(\tilde{f}; \delta) \leq \omega(f; [0, \tau]; \tau\delta/2)$ for $\delta > 0$. Proceeding as in (F.10), we consider the trigonometric sum for \tilde{f} given by the modified Fourier series

$$\tilde{t}^{(n)}(\theta) = \sum_{k=0}^n \varrho_{n,k} \beta_k \cos(k\theta), \quad \forall \theta \in \mathbb{R}, \quad (\text{F.12})$$

where $\beta_0 = \tilde{\alpha}_0$, $\beta_k = 2\tilde{\alpha}_k$ if $k \geq 1$, and $\tilde{\alpha}_k = (1/\pi) \int_0^\pi \tilde{f}(\theta) \cos(k\theta) d\theta$ for $k \in \mathbb{N}$. Defining $\varrho_{n,k}$ as in (IV.6), yields the uniform convergence rate

$$\|f - \tilde{t}^{(n)}\| = \stackrel{(\text{F.11})}{\leq} 6\omega(\tilde{f}; 1/n) \leq 6\omega(f; [0, \tau]; \tau/(2n)). \quad (\text{F.13})$$

It remains to rewrite (F.12) as a polynomial in u by returning to the backlog domain. For this, we develop $\cos(k\theta) = \Re((\cos(\theta) + i \sin(\theta))^k)$ and find $\cos(k\theta) = p_k(\cos(\theta))$, where the polynomial $p_k(x)$, characterized by its k real roots located in $(-1, 1)$, is defined by

$$p_k(x) = \left\{ \begin{array}{ll} \sum_{q=0}^{k/2} \nu(k, q) x^{2q}, & \text{if } k \text{ is even} \\ \sum_{q=0}^{(k-1)/2} \nu(k, q) x^{2q+1}, & \text{if } k \text{ is odd} \end{array} \right\}, \quad (\text{F.14})$$

where $\nu(0, 0) = 1$, and

$$\nu(k, q) = (-1)^{\lfloor \frac{k}{2} \rfloor - q} \sum_{t=0}^q \binom{k}{2(\lfloor \frac{k}{2} \rfloor - t)} \binom{\lfloor \frac{k}{2} \rfloor - t}{\lfloor \frac{k}{2} \rfloor - q}, \quad (q = 0 \dots, \lfloor \frac{k}{2} \rfloor, k \in \mathbb{N}_{>0}).$$

Since $\tilde{f}(\theta) = f(\tau(1 + \cos(\theta))/2)$, a polynomial approximation of f on $[0, \tau]$ is obtained by setting $\cos(k\theta) = p_k(2u/\tau - 1)$ in (F.12), and we find, after straightforward computations,

$$\hat{t}^{(n)}(u) = \sum_{k=0}^n \gamma(n, k) u^k, \quad \forall u \in [0, \tau], \quad (\text{F.15})$$

where we define

$$\gamma(n, k) = (2/\tau)^k \sum_{t=0}^{n-k} \binom{t+k}{k} (-1)^t \bar{\gamma}(n, t+k), \quad (k = 0 \dots, n).$$

and $\bar{\gamma}(n, t) = \sum_{k \in \bar{\sigma}(n, t)} \varrho_{n,k} \beta_k \nu(k, \lfloor t/2 \rfloor)$ for $t = 0, \dots, n$, in which $\bar{\sigma}(n, t) = \{t, t+2, t+4, \dots, n\}$ if $n-t$ is even, and $\bar{\sigma}(n, t) = \{t, t+2, t+4, \dots, n-1\}$ otherwise ($0 \leq t \leq n$). As for the Fourier coefficients of \tilde{f} , they reduce to

$$\tilde{\alpha}_k = \frac{1}{\pi} \int_0^\pi f(\tilde{u}(\theta)) \cos(k\theta) d\theta \stackrel{(\text{F.14})}{=} \frac{1}{\pi} \int_0^\tau f(u) \frac{p_k(\frac{2u}{\tau} - 1)}{\sqrt{u(\tau-u)}} du, \quad (\text{F.16})$$

where we have used the change of variable $u = \tilde{u}(\theta)$. For many cost functions, the coefficients $\{\tilde{\alpha}_k\}$ can be derived exactly. See Lemma for expressions of these coefficients in the case when f is given as a quotient of polynomials.

From (F.13), we infer the following bounds for the relative value function.

Corollary F.3 (Near-optimal polynomials) *Proposition 3 holds for $\hat{f}^{(n)} \equiv \hat{t}^{(n)}$ defined by (F.15) with uniform error bound $\eta^{(n)} = 6\omega(f; [0, \tau]; \tau/(2n))$.*

Without further assumptions on f , the convergence rate $O(\omega(f; [0, \tau]; \tau/(2n)))$ guaranteed by (F.15) is non-improvable. The performance of $\hat{t}^{(n)}$ and $\hat{t}^{(n)}$ in Corollaries F.2 and F.3 are then really close, and the choice of either approach (Section F.2 or F.3), mostly dependent on the computability of the Fourier coefficients (F.8) or (F.16), respectively, is left to the

appreciation of the reader. The second approach nevertheless prevails in the event the cost function has a k th derivative $f^{(k)}$ on $[0, \tau]$. Then, the convergence rate in Corollary F.3 can be lowered to $O(n^{-k}\omega(f^{(k)}; [0, \tau]; \tau/[2(n-k)]))$ by using the derivatives as the targets of approximation, [10, Theorem 1.5]. This distinguishing property of approach F.3 stems from the fact that $\tilde{f}(\theta)$ retains the smoothness of the cost function, whereas $\tilde{f}(u)$ shows irregularities at $u = (2k+1)\tau$. We refer to [10, §1.1] and references therein for further considerations on the optimality of (F.13) as a convergence rate for polynomial approximations.

G Proof of Proposition B.1

Proposition B.1 (Analyticity of W^* and pole location) *Under Assumption 2:*

- (i) The dominant singularity p_W of W^* (i.e., that with largest real value) is a pole with degree 1 lying on the negative real axis $\mathbb{R}_{<0}$. The dominant singularity p_X of X^* is real, negative (possibly infinite) and satisfies $p_X < p_W$. X^* is analytic on $\{s \in \mathbb{C} \mid \Re(s) > p_W\}$.
- (ii) W^* is analytic on $\{s \in \mathbb{C}_0 \mid \Re(s) > p_W\}$, where $\lim_{s \rightarrow \infty} |W^*(s)| \leq 1$.
- (iii) One can find $\epsilon > 0$ such that W^* is analytic on $\{s \in \mathbb{C}_0 \setminus \{p_W\} \mid \Re(s) > p_W - \epsilon\}$.
- (iv) W^* is analytic in a neighborhood of 0, where it rewrites as the series

$$W^*(s) = \sum_{k=0}^{\infty} w_k (-s)^k, \quad \forall s \in \{\sigma \in \mathbb{C}_0 : |\sigma - a| < |p_W|\}, \quad (\text{B.1})$$

in which the coefficients $\{w_k\}$ are given by (III.3) in Table 1, and satisfy $w_k = \mathbb{E}[W^k]/k!$, for $k \in \mathbb{N}$. The series $\{w_k\}$ is asymptotically geometric with asymptotic rate $|p_W|^{-1}$.

- (v) At any point $a \in \mathbb{C}_0$ where W^* is analytic, W^* rewrites as the series

$$W^*(s) = \sum_{k=0}^{\infty} w_{a:k} (a-s)^k, \quad \forall s \in \{\sigma \in \mathbb{C} : |\sigma - a| < r_a\}, \quad (\text{B.2})$$

where r_a denotes the distance from a to the closest singularity of W^* . The coefficients $\{w_{a:k}\}$ are given by (III.4) in Table 1.

Proof (i) First observe in

$$W^*(s) = \frac{(1-\rho)s}{s - \lambda(1 - X^*(s))} \quad (\text{PK})$$

that 0 is a removable singularity of $W^*(s)$. Since X^* and W^* are the Laplace transforms of probability density functions on $\mathbb{R}_{\geq 0}$, their dominant singularities p_X, p_W are real, nonpositive, and possibly infinite $(-\infty)^2$. Besides, $X^*(s)$ is, by definition, continuous and strictly decreasing on $(p_W, +\infty)$ with $\lim_{s \rightarrow \infty} X^*(s) = 0$. It intersects with the straight line $1 - s/\lambda$ at $s = 0$ (removable singularity) with slope strictly larger than $-1/\lambda$ ($\rho < 1$). Since the limit value of the derivative of $X^*(s)$ at $-\infty$, given by $\lim_{s \rightarrow -\infty} \mathbb{E}[-Xe^{-sX}]$ is infinite, $X^*(s)$ must necessarily cross $1 - s/\lambda$ at another negative value $p_W > p_X$, which is the dominant (non-removable) singularity of W^* . Moreover, since the slopes of the curves at p_W are different, the derivative of $s - \lambda(1 - X^*(s))$ at p_W is nonzero. It follows that p_W is a pole of degree 1. Because X^* is a Laplace transform, it is analytic on its domain, which includes the half-plane to the right of p_W .

(ii) From the conclusions of (i), we find $\sigma - \lambda(1 - \mathbb{E}[e^{-\sigma X}]) \neq 0$ for every $\sigma \in \{s \in \mathbb{C}_0 \mid \Re(s) > p_W\}$, where $W^*(\sigma)$ is nonzero. Since we also have $\lim_{s \rightarrow \infty} X^*(s) = 0$, it follows from (PK) that $\lim_{s \rightarrow \infty} |W^*(s)| \leq 1$.

² If F_Y is the probability distribution of a random variable Y on $\mathbb{R}_{\geq 0}$, then

$$\left| \mathcal{L}_{F'_Y}(s) \right| \leq \int_0^{\infty} |e^{-su}| F_Y(du) = \int_0^{\infty} e^{-\Re(s)u} F_Y(du) = \mathcal{L}_{F'_Y}(\Re(s)), \quad \forall s \in \mathbb{C}.$$

It follows that $\mathcal{L}_{F'_Y}$ is absolutely dominated by its expression on the real axis, which is a real, nonnegative function. The dominant singularity p_Y of $\mathcal{L}_{F'_Y}$ lies therefore on the real axis. Since F'_Y is absolutely integrable, p_Y is negative.

(iii) On the vertical axis $s = p_W + it$, the denominator of $W^*(s)$ is given by

$$\{p_W - \lambda(1 - \mathbb{E}[e^{-p_W X} \cos(tX)]\} + i\{t - \lambda\mathbb{E}[e^{-p_W X} \sin(tX)]\}. \quad (\text{G.1})$$

The real part of (G.1) can only be 0 if $\cos(tX) = 1$ almost everywhere with respect to F_X , where $\sin(tX) = 0$ and the imaginary part of (G.1) reduces to t . Hence, the only singularity on the axis $s = p_W + it$ is p_W . Next, we show that it is not possible to find a sequence $\{\hat{s}_k\}$ of singularities of W^* such that $\Re(\hat{s}_k) \rightarrow 0$. Suppose it is: the sequence is either bounded or not. If it is bounded, then there exists a subsequence of $\{\hat{s}_k\}$ of poles converging towards a point of the imaginary axis, which can only be p_W , and consequently $\hat{s}_k - \lambda(1 - X^*(\hat{s}_k)) = 0$ for the subsequence converging towards p_W . By analytic continuation, $s - \lambda(1 - X^*(s)) = 0$ in a neighborhood of p_W , which is impossible. Suppose now that $\{\hat{s}_k\}$ is unbounded and converges to $i\infty$. If $\hat{s}_k = p_W - \delta_k + it_k$, the imaginary part of $s - \lambda(1 - X^*(s))$ is given by

$$t_k - \lambda\mathbb{E}[e^{-p_W X} \sin(t_k X)] - \lambda\mathbb{E}[X e^{-p_W X} \sin(t_k X)]\delta_k + o(\delta_k), \quad (\text{G.2})$$

where $|\mathbb{E}[e^{-p_W X} \sin(t_k X)]| \leq \mathbb{E}[e^{-p_W X}] = 1 - p_W/\lambda$ is a finite quantity, while the deviation $|\mathbb{E}[X e^{-p_W X} \sin(t_k X)]| \leq \mathbb{E}[X e^{-p_W X}]$ is dominated by the (finite) slope of $X^*(s)$ at p_W . Since $t_k \rightarrow \infty$, (G.2) diverges, and the sequence does not exist. It follows that there is no singularity with imaginary value arbitrarily close to that of p_W .

(iv) The series expansion of W^* at 0 is $W^*(s) = \sum_{k=0}^{\infty} (\mathbb{E}[W^k]/k!)(-s)^k$, where $\mathbb{E}[W^k]$ is the k^{th} moment of the waiting time distribution. These moments have known expressions which satisfy $\mathbb{E}[W^k] = k! w_k$ for all k —see e.g. [3, §5.1.5]. Hence (B.1) is the (unique) Taylor expansion of W^* at 0. Now, the ratio test for this Taylor series tells us that $\{\mathbb{E}[W^k]/k!\}$ grows asymptotically exponentially with asymptotic rate \underline{a} if and only if the Taylor series converges on the interior of a disc with radius \underline{a} centered at the origin, and diverges outside the disc, thus betraying the presence of a singularity of W^* on the circle. Besides, since $\{\mathbb{E}[W^k]/k!\}$ has only real, nonnegative values, the series takes its largest absolute value on the circle at the intersection with the negative branch of the real axis. It follows in that case that $-\underline{a} = p_W$ is the dominant singularity of W^* .

(v) For $n \in \mathbb{N}$ and the specified point a , consider the cost function $f(u) = (-u)^n e^{-au} = (\partial^n / \partial a^n)[e^{-au}]$. It follows from Table 1 and the Leibniz integral rule that

$$c'(u) = \frac{\lambda}{1-\rho} \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k} W^*(a)}{da^{n-k}} (-u)^k e^{-au}, \quad (\text{G.3})$$

and from Table 1 that

$$\begin{aligned} c'(u) &= (-1)^n \frac{\lambda n!}{1-\rho} \sum_{k=0}^n w_{a:n-k} \frac{u^k e^{-au}}{k!} \\ &= \frac{\lambda}{1-\rho} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (n-k)! w_{a:n-k} (-u^k) e^{-au}. \end{aligned} \quad (\text{G.4})$$

Inspection of (G.3) and (G.4) then yields $(d^k/da^k)[W^*(a)]/k! = (-1)^k w_{a:k}$ for all k , and (B.2) follows from the Taylor series of W^* at a . \square

H Proof of Proposition 2 (variant)

This appendix contains an alternate proof for Proposition 2. First, we recall basic properties of the M/G/1 queue.

Lemma H.1 *Let $0 \leq u_2 \leq u_1$ and suppose that the M/G/1 queue, initially at state u_1 , reaches state u_2 for the first time after a random period of time T , during which N jobs have arrived. Then,*

$$(a) \quad \mathbb{E}[T] = \frac{1}{1-\rho} (u_1 - u_2), \quad (b) \quad \mathbb{E}[N] = \frac{\lambda}{1-\rho} (u_1 - u_2). \quad (\text{H.1})$$

Proof Lemma H.1 is a consequence of the law of large numbers. Consider n realizations of the setting, and denote by T_1, \dots, T_n the random values observed for the variable T , by N_1, \dots, N_n those observed for the variable N , and by $\{X_{k,l}\}_{l=1}^{N_k}, \dots, \{X_{k,l}\}_{l=1}^{N_n}$. Since the rate of the Poisson process is equal to the density of arrivals per unit of time,

$$\lambda = \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n N_k}{\sum_{k=1}^n T_k} = \left(\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n N_k}{n} \right) \left(\lim_{n \rightarrow \infty} \frac{n}{\sum_{k=1}^n T_k} \right) = \frac{\mathbb{E}[N]}{\mathbb{E}[T]}. \quad (\text{H.2})$$

Similarly,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{l=1}^{N_k} X_{k,l} = \left(\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \sum_{l=1}^{N_k} X_{k,l}}{\sum_{k=1}^n N_k} \right) \left(\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n N_k}{n} \right) = \mathbb{E}[X] \mathbb{E}[N]. \quad (\text{H.3})$$

By definition of the variables, we also have $u_2 = u_1 + \sum_{l=1}^{N_k} X_{k,l} - T_k$ for $k = 1, \dots, n$, and it follows that

$$\mathbb{E}[T] = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n T_k = u_1 - u_2 + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{l=1}^{N_k} X_{k,l} \stackrel{(\text{H.3})}{=} u_1 - u_2 + \mathbb{E}[X] \mathbb{E}[N] \stackrel{(\text{H.2})}{=} u_1 - u_2 + \rho \mathbb{E}[T]$$

which yields (H.1a), and (H.1b) follows from (H.2). \square

Proposition 2 (Relative value function) *Let f satisfy Assumption 2 and be piecewise continuous.*

(i) *The relative value function (VF) is continuous, almost everywhere continuously differentiable, and semi-differentiable with right derivative*

$$\partial_+ v(u) = \lambda (f^+(u) - \bar{f} + \mathbb{E}[v(u+X) - v(u)]), \quad \forall u \in \mathbb{R}_{>0}, \quad (\text{DE})$$

where $f^+(u) := \lim_{t \rightarrow u^+} f(t)$. At $u = 0$, one has

$$v(0) = f(0) - \bar{f} + \mathbb{E}[v(X_0)], \quad (\text{BCa})$$

$$v'(0) = \lambda (f^+(0) - f(0) + \mathbb{E}[v(X) - v(X_0)]). \quad (\text{BCb})$$

(ii) *The relative value function is given by*

$$v(u) = v(0) + c(u) - \frac{\lambda \bar{f}}{1 - \rho} u, \quad \forall u \in \mathbb{R}_{\geq 0}, \quad (\text{S})$$

where $c : \mathbb{R} \mapsto \mathbb{R}$ is continuous, almost everywhere continuously differentiable, and semi-differentiable with right-derivative

$$\partial_+ c(u) = \frac{\lambda}{1 - \rho} \mathbb{E}[f(u+W)], \quad \forall u \in \mathbb{R}. \quad (\text{CVF})$$

Proof (Proposition 2) Start the queue at state u . The quantity $V(u, t)$ appearing in (VF) rewrites, for any $T \geq 0$ and for t large enough, as $V(u, t) = V(u, T) + V(U(T), t - T)$, where $U(T)$ denotes the backlog observed after time T . It follows from the Markov property of the system and from the definition VF of the relative value function that

$$v(u) = \mathbb{E}[V(u, T) - \lambda \bar{f} T] + \mathbb{E}[v(U(T))]. \quad (\text{H.4})$$

We now turn to show the claims, starting with (ii), from which useful properties of v (existence, continuity) can be inferred.

(ii) If in (H.4) u is taken to be the initial backlog of the queue and T is defined as the time when the queueing system reaches the empty state ($u = 0$) for the first time, we get (S), with $c(u) = \mathbb{E}[V(u, T)]$ defined as the expected total cost incurred until the queue is first empty. In order to compute $\mathbb{E}[V(u_1, T)]$, we introduce the following notion: given any random set S containing a random number of stochastic real variables, we define the measure $\Psi_S : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ as $\Psi_S((-\infty, t]) = \mathbb{E}[\sum_{t' \in S} \theta(t - t')]$ for all $t \in \mathbb{R}$, where θ denotes the

step function with the convention $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ otherwise. If a function f is defined on the same domain as S and measurable with respect to Ψ_S , then

$$\Psi_S f = \int_{-\infty}^{\infty} f(\xi) \Psi_S(d\xi) = \mathbb{E}[\sum_{t' \in S} f(t')] \quad (\text{H.5})$$

provided that f is integrable on \mathbb{R} with respect to Ψ_S . In particular, if S denotes the set of the times of the successive arrivals of a Poisson process with density λ initiated at a time t_0 , we have $\Psi_S(dt) = \lambda dt$ for $t \geq t_0$, and find $\Psi_S((-\infty, t]) = \lambda(t - t_0)\theta(t - t_0)$.

To compute $c(u)$, we report the arrival times of all the jobs landed during the random period of time T , and classify the coinciding backlog values as follows:

$$Y_1, Y_1 + U_{11}, \dots, Y_1 + U_{1M_1}, Y_2, Y_2 + U_{21}, \dots, Y_2 + U_{2M_2}, Y_3, \dots, Y_N, Y_N + U_{N1}, \dots, Y_N + U_{NM_N},$$

in which $B = \{Y_1, \dots, Y_N\}$ is the longest nonincreasing sequence of backlogs and, for $p = 1, \dots, N$, $A_p = \{U_{p0}, U_{p1}, \dots, U_{pM_p}\}$ is the sequence of relative backlog values interposed in-between, where we have used the convention $U_{p0} = 0$ for $p = 1, \dots, N$. First observe that Y_1, \dots, Y_N rewrite as $Y_p = u - T_p$ for $p = 1, \dots, N$, where the variables T_1, T_2, \dots form a Poisson process with rate λ . It follows that Ψ_B is given by

$$\Psi_B(dt) = \lambda \mathbf{1}_{[0, u]}(t) dt. \quad (\text{H.6})$$

We compute the quantities $\Psi_{A_1}, \dots, \Psi_{A_N}$, by applying the law of large numbers. First observe, for $p \in \{1, \dots, N\}$, that $U_{p0}, \dots, U_{pM_p}, U_{p0}, \dots, U_{pM_p}, U_{p0}, \dots$ form the successive waiting times of an analogous M/G/1 queue with service time convention (X, X) . Taking n samples of the process, we find, for $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} \Psi_{A_p}([0, t]) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\sum_{q=0}^{M_p^{(k)}} \theta(t - U_{pq}^{(k)}) \right) \\ &= \left(\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n (M_p^{(k)} + 1)}{n} \right) \left(\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \sum_{q=0}^{M_p^{(k)}} \theta(t - U_{pq}^{(k)})}{\sum_{k=1}^n (M_p^{(k)} + 1)} \right) \\ &= (1 + \mathbb{E}[M_p]) \mathbb{E}[\theta(t - W)] \stackrel{(\text{H.1b})}{=} \left(1 + \frac{\lambda \mathbb{E}[X_p]}{1 - \rho} \right) \mu_W([0, t]) = \frac{\mu_W([0, t])}{1 - \rho}. \end{aligned} \quad (\text{H.7})$$

We are now able to compute $c(u)$, and we find

$$\begin{aligned} c(u) &= \mathbb{E} \left[\sum_{p=1}^N \sum_{q=0}^{M_p} f(Y_p + U_{pq}) \right] \\ &= \mathbb{E} \left[\sum_{p=1}^N \mathbb{E} \left[\sum_{q=0}^{M_p} f(Y_p + U_{pq}) | Y_p \right] \right] \\ &\stackrel{(\text{H.5})}{=} \mathbb{E} \left[\sum_{p=1}^N \int_{-\infty}^{\infty} f(Y_p + t) \Psi_{A_p}(dt) \right] \\ &\stackrel{(\text{H.7})}{=} \frac{1}{1 - \rho} \mathbb{E} \left[\sum_{p=1}^N \int_0^{\infty} f(Y_p + t) \mu_W(dt) \right] \\ &\stackrel{(\text{H.5})}{=} \frac{1}{1 - \rho} \int_0^u \left(\int_0^{\infty} f(\xi + t) \mu_W(dt) \right) \Psi_B(d\xi) \\ &\stackrel{(\text{H.6})}{=} \frac{\lambda}{1 - \rho} \int_0^u \left(\int_0^{\infty} f(\xi + t) \mu_W(dt) \right) d\xi \\ &= \frac{\lambda}{1 - \rho} \int_0^u \mathbb{E} [f(\xi + W)] d\xi. \end{aligned} \quad (\text{H.8})$$

Then, (CVF) follows by right differentiation of (H.8).

(i) Let $T = \delta$, where $\delta > 0$ is a small time step. Since the inter-arrival times are exponentially distributed, $\text{Prob}(N = 0) = 1 - \lambda\delta + o(\delta)$, $\text{Prob}(N = 1) = \lambda\delta + o(\delta)$, and $\text{Prob}(N \geq 2) = o(\delta)$.

First consider $t > 0$ and initialize the system at time $u = t + \delta$. By considering separately the three events $N = 0$, $N = 1$ and $N > 1$, we find that $\mathbb{E}[V(u, \delta)] = \lambda\delta \mathbb{E}[f(\hat{U})] + o(\delta)$, with \hat{U} uniformly distributed over (t, u) , and $\mathbb{E}[v(U(\delta))] = (1 - \lambda\delta)v(t) + \lambda\delta \mathbb{E}[v(t + X)] + o(\delta)$. Introducing these two results into (H.4) gives

$$v(t + \delta) = v(t) + \lambda\delta \mathbb{E}[f(\hat{U}) - \bar{f}] + \lambda\delta \mathbb{E}[v(t + X) - v(t)] + o(\delta), \quad (\text{H.9})$$

where under Assumption 2 the expectations are necessarily finite (cf. (iii)). By letting $\delta \downarrow 0$ in (H.9), we find that v is continuous, and right differentiable at t with right derivative

$$\partial_+ v(u) = \lambda (f^+(u) - \bar{f} + \mathbb{E}[v(u + X)] - v(u)), \quad (\text{H.10})$$

Table I.1: Germ of $W^*(-s)$ at $s = 0$ for constant (M/D/1), exponentially-distributed (M/M/1) and Erlang-distributed (M/E $_q$ /1) service times.

M/D/1 (constant): $X = x$, with $x > 0$; $\mathbb{E}[X^k] = x^k$; $W^*(s) = \frac{(1-\lambda x)s}{s-\lambda(1-e^{-sx})}$; $\mathcal{P}(W^*) = \{\lambda[1 + \frac{1}{\lambda x} W_k(-\lambda x e^{-\lambda x})] \mid k \in \mathbb{Z}_0\}$; $-p_W = -\lambda[1 + \frac{1}{\lambda x} W_{-1}(-\lambda x e^{-\lambda x})]$;

$$w_k = [\sum_{t=1}^k (\frac{\lambda x}{1-\lambda x})^t \phi(t, k+t)] x^k, \quad (k \geq 1), \quad (I.1)$$

where W_n denotes the n th branch of the product logarithm function, [2], and

$$\phi(1, n) = \frac{1}{n!}, \quad (n \geq 2), \quad (I.2a)$$

$$\phi(m+1, n) = \sum_{p=2m}^{n-2} \frac{\phi(m, p)}{(n-p)!}, \quad (m = 1, \dots, \lfloor \frac{n-2}{2} \rfloor, n \geq 2), \quad (I.2b)$$

$$\phi(m, n+1) = \frac{m}{n+1} [\phi(m, n) + \phi(m-1, n-1)], \quad (m = 2, \dots, \lfloor \frac{n}{2} \rfloor, n \geq 4). \quad (I.2c)$$

M/M/1 (exponential): $F'_X(x) = \omega e^{-\omega x}$ with rate $\omega > \lambda$; $\mathbb{E}[X^k] = k! \omega^{-k}$; $W^*(s) = \frac{(\omega-\lambda)(s+\omega)}{\omega(s+\omega-\lambda)}$; $\mathcal{P}(W^*) = \{\lambda - \omega\}$; $-p_W = \omega - \lambda$;

$$w_k = \frac{\lambda}{\omega(\omega-\lambda)^k}, \quad (k \geq 1). \quad (I.3)$$

M/E $_q$ /1 (Erlang): $F'_X(x) = \frac{\omega}{(q-1)!} (\omega x)^{q-1} e^{-\omega x}$, with shape $q \geq 1$ and rate $\omega > q\lambda$; $\mathbb{E}[X^k] = \frac{(k+q-1)!}{(q-1)!} \omega^{-k}$; $W^*(s) = \frac{(1-\lambda x)(s+\omega)^q}{\omega^q [(\frac{s}{\omega}+1)^q - \frac{\lambda}{\omega} \sum_{k=0}^{q-1} (\frac{q}{k+1})(\frac{s}{\omega})^k]}$; $|\mathcal{P}(W^*)| = q$;

$$w_k = \frac{1}{\omega^k} \sum_{t=1}^k (\frac{\lambda}{\omega-q\lambda})^t \varphi^{(q)}(t, k+t), \quad (k \geq 1), \quad (I.4)$$

where

$$\varphi^{(q)}(1, n) = \binom{n+q-1}{q-1}, \quad (n \geq 2), \quad (I.5a)$$

$$\varphi^{(q)}(m+1, n) = \sum_{p=2m}^{n-2} \binom{q+n-p-1}{q-1} \varphi^{(q)}(m, p) (m = 1, \dots, \lfloor \frac{n-2}{2} \rfloor, n \geq 2). \quad (I.5b)$$

which holds for every $u > 0$ and inherits the piecewise continuity of f .

Now, if $t = 0$, $\delta > 0$ and the system is started from $u = 0$, then $\mathbb{E}[V(u, \delta)] = \lambda \delta \mathbb{E}[f(0)] + o(\delta)$, $\mathbb{E}[v(U(\delta))] = (1 - \lambda \delta) v(0) + \lambda \delta \mathbb{E}[v(X_0)] + o(\delta)$ and, using (H.4) and continuity arguments, we obtain (BCa). Then (BCb) follows by setting $u = 0$ in (H.10) and substituting $v(0)$ with its value computed in (BCa). \square

I Moments of the asymptotic waiting times and rates of growth

In Table I.1 we derive the coefficient sequence $\{w_k\}$ for standard service time distributions (constant, exponential, Erlang), and study its asymptotic growth. The moments of W and their growth rates can be inferred from those of $\{w_k\}$ using the identity $\mathbb{E}[W^k] = k! w_k$.

Proof (Derivation of Table (I.1)) The expression for $\{w_k\}$ were derived either by combination of (PK) and (III.7), or by computation of the moments $\mathbb{E}[X^k]$ and development of (III.3).

(M/D/1) Before showing (I.1), observe from (I.2a)-(I.2b) that the quantity $n! \phi(m, n)$ is in fact the number of possible scenarios that may occur when placing n distinct objects (unordered) into m numbered urns so that each urn contains at least two objects ($n \geq 2m$). For $k = 1$, (I.1) follows directly from (III.3). If we suppose that (I.1) holds for $k = 1, \dots, p$, then, using $\mathbb{E}[X^k] = x^k$,

$$\begin{aligned} w_{p+1} &\stackrel{\text{(III.3)}}{=} \frac{\lambda}{1-\lambda x} \left[\frac{x^{p+2}}{(p+2)!} + \sum_{t=1}^p \frac{x^{p-t+2}}{(p-t+2)!} w_t \right] \\ &\stackrel{\text{(I.1)}}{=} \frac{\lambda}{1-\lambda x} \left[\frac{x^{p+2}}{(p+2)!} + \sum_{t=1}^p \frac{x^{p-t+2}}{(p-t+2)!} \sum_{q=1}^t \left(\frac{\lambda}{1-\lambda x} \right)^q \phi(q, t+q) x^{t+q} \right] \\ &= \frac{\lambda}{1-\lambda x} \frac{x^{p+2}}{(p+2)!} + \sum_{q=1}^p \left(\frac{\lambda}{1-\lambda x} \right)^{q+1} \sum_{t=q}^p \phi(q, t+q) \frac{x^{p+q+2}}{(p-t+2)!} \\ &\stackrel{\text{(I.2a)}}{=} \frac{\lambda}{1-\lambda x} \phi(1, p+2) x^{p+2} + \sum_{q=2}^{p+1} \left(\frac{\lambda}{1-\lambda x} \right)^q \left[\sum_{l=q-1}^p \frac{\phi(q-1, q-1+l)}{(p-l+2)!} \right] x^{p+1+q} \\ &\stackrel{\text{(I.2b)}}{=} \sum_{q=1}^{p+1} \left(\frac{\lambda}{1-\lambda x} \right)^q \phi(q, p+1+q) x^{p+1+q} \end{aligned}$$

and (I.1) holds for all k by induction. We obtain in (I.2a)-(I.2b) a recursive procedure for computing the values of $\{\phi(m, n)\}$. Identity (I.2c), which follows from (I.2a)-(I.2b), tends to simplify and accelerate the process in practice. As for the poles of W^* (cf. Figure 2, they are the (nonzero) solutions of $s - \lambda(1 - e^{-sx}) = 0$ or, equivalently, $(s - \lambda)x e^{(s-\lambda)x} = -\lambda x e^{-\lambda x}$, which is an instance of the equation $z e^z = a$, the solutions of which are given by $W_n(a)$, where W_n is the n th branch of the product logarithm function W_n , [2].

(M/M/1) W^* follows from (PK) and $X^*(s) = \omega/(s + \omega)$. After successive derivations of $W^*(-s)$, which rewrites as $W^*(-s) = [(\omega - \lambda)/\omega] [1 + \lambda/(\omega - \lambda - s)]$, we compute (III.7) and find $w_{a,k} = [(\omega - \lambda)/\omega] [\delta[k] + \lambda(\omega - \lambda - a)^{-k-1}]$, which reduces to (I.3) when $a = 0$.

(M/E_q/1) In (I.5a)-(I.5b), $\varphi^{(q)}(m, n)$ can be understood, for $q \geq 1$ and $n \geq 2m$, as the number of possible outcomes when m ordered collections of n_1, n_2, \dots, n_m unordered objects are respectively picked out of m urns $1, 2, \dots, m$ so that $n_1 + \dots + n_m = n$ and $n_p \geq 2$ for $p = 1, \dots, m$, where each urn p is initially assumed to contain $q-1$ distinct objects plus n_p objects randomly drawn (without repetition) from a common set of n additional distinct objects ($p = 1, \dots, m$). We now turn to show (I.4). It is easy to verify that (I.4) is true for $k = 1$. Suppose now that (I.4) holds for $k = 1, \dots, p$, then

$$\begin{aligned} w_{p+1} &\stackrel{\text{(III.3)}}{=} \frac{\lambda \omega}{\omega - q\lambda} \left[\binom{p+q+1}{q-1} \omega^{-(p+2)} + \sum_{t=1}^p \binom{p-t+q+1}{q-1} \omega^{-(p-t+2)} w_t \right] \\ &\stackrel{\text{(I.4)}}{=} \frac{\lambda}{\omega^{p+1}(\omega - q\lambda)} \left[\binom{p+q+1}{q-1} + \sum_{t=1}^p \binom{p-t+q+1}{q-1} \omega^t \left(\sum_{l=1}^t \left(\frac{\lambda}{\omega - q\lambda} \right)^l \frac{\varphi^{(q)}(l, t+l)}{\omega^t} \right) \right] \\ &= \frac{\lambda}{\omega^{p+1}(\omega - q\lambda)} \left[\binom{p+q+1}{q-1} + \sum_{l=1}^p \left(\frac{\lambda}{\omega - q\lambda} \right)^l \sum_{t=l}^p \binom{p-t+q+1}{q-1} \varphi^{(q)}(l, t+l) \right] \\ &\stackrel{\text{(I.5a)}}{=} \frac{\lambda}{\omega^{p+1}(\omega - q\lambda)} \varphi^{(q)}(1, p+2) \\ &\quad + \frac{\lambda}{\omega^{p+1}(\omega - q\lambda)} \sum_{l=1}^p \left(\frac{\lambda}{\omega - q\lambda} \right)^l \left(\sum_{m=2l}^{l+p} \binom{q+(p+l+2)-m-1}{q-1} \right) \varphi^{(q)}(l, m) \\ &\stackrel{\text{(I.5b)}}{=} \frac{1}{\omega^{p+1}} \sum_{l=1}^{p+1} \left(\frac{\lambda}{\omega - q\lambda} \right)^l \varphi^{(q)}(l, p+1+l) \end{aligned}$$

and (I.4) holds for all k by induction. \square

J Proof of Lemma E.1

Lemma E.1 (Coefficients $\{\tilde{\alpha}_k\}$ for quotients of polynomials) *Let g_m and h_n be polynomials of degrees m and n , and consider*

$$f(u) = \frac{g_m(u)}{h_n(u)}, \quad \forall u \in \mathbb{R}_{\geq 0}.$$

For $\tau > 0$, recall (.0) and define $f_k(s) = f(s) p_k(2s/\tau - 1)$ under the assumption $\mathcal{P}(f_k) \cap [0, \infty) = \emptyset$. The Fourier coefficients (.0) of f satisfy, for $k \geq 0$,

$$\tilde{\alpha}_k = \sqrt{\pi} \sum_{q=0}^{l(k)} \frac{\zeta_{-q}(-\tau)^q}{q! \Gamma(\frac{1}{2} - q)} - \sum_{a \in \mathcal{P}(f_k)} \text{Res}_{s=a} \left(f_k(s) s^{-\frac{1}{2}} \right]_{-\pi} (s - \tau)^{-\frac{1}{2}} \Big|_{-\pi}, \quad (\text{E.9})$$

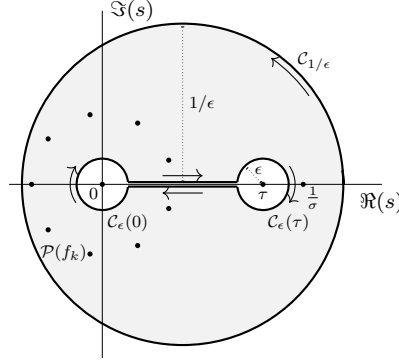


Fig. 6: Singularities of $f_k(s) s^{-1/2}]_{-\pi}(s - \tau)^{-1/2}]_{-\pi}(1 - \sigma s)^{-l(k)}$ and computation of $\hat{\alpha}_k(\sigma)$ by contour integration.

where $l(k) = \max(0, m - n + k)$ is the largest nonnegative integer l such that $\lim_{s \rightarrow 0} s^l f_k\left(\frac{1}{s}\right)$ is finite, and $\{\zeta_q\}$ are the coefficients of the Laurent series at $+\infty$ of the analytic continuation of f_k , i.e.,

$$\zeta_q = \frac{1}{(l(k) + q)!} \lim_{s \rightarrow 0} \frac{d^{l(k)+q}}{ds^{l(k)+q}} \left[s^{l(k)} f_k\left(\frac{1}{s}\right) \right], \quad (q = -l(k), \dots, \infty). \quad (\text{J.1})$$

Proof We would like to compute

$$\tilde{\alpha}_k \stackrel{(\text{J.1})}{=} \frac{1}{\pi} \int_0^\tau \frac{g_m(u) p_k\left(\frac{2u}{\tau} - 1\right)}{h_n(u) \sqrt{u(\tau - u)}} du, \quad (\text{J.2})$$

for any $k \in \mathbb{N}_{\geq 0}$. For $l \in \mathbb{N}_{\geq 0}$ and $\sigma \ll 1/\tau$, we define the altered coefficient

$$\hat{\alpha}_k(\sigma) = \frac{1}{\pi} \int_0^\tau \frac{g_m(u) p_k\left(\frac{2u}{\tau} - 1\right)}{h_n(u) (1 - \sigma u)^l \sqrt{u(\tau - u)}} du,$$

which has the property to converge to $\Re(\tilde{\alpha}_k)$ as $\sigma \downarrow 0$. Indeed, since by assumption f and p_k are bounded on $[0, \tau]$, the integrand of (J.2) is absolutely integrable on the interval. As soon as $\sigma \leq 1/(2\tau)$, one has $|(1 - \sigma u)^{-l}| \leq 2^l$ and the conditions of Lebesgue's dominated convergence theorem are met. Now, consider the contour integral in the complex domain

$$\begin{aligned} \gamma_k(\sigma) &= \frac{1}{\pi} \oint_C \left(\frac{g_m(s) p_k\left(\frac{2s}{\tau} - 1\right)}{h_n(s) (1 - \sigma s)^l} \right) s^{-\frac{1}{2}]_{-\pi}} (s - \tau)^{-\frac{1}{2}]_{-\pi}} ds \\ &= \frac{1}{\pi} \left(\oint_{C_{1/\epsilon}} + \oint_{C_\epsilon(0)} + \int_\epsilon^{\tau - \epsilon} + \oint_{C_\epsilon(\tau)} + \int_{\tau - \epsilon}^\epsilon \right) \frac{f_k(s) s^{-\frac{1}{2}]_{-\pi}} (s - \tau)^{-\frac{1}{2}]_{-\pi}}{(1 - \sigma s)^l} ds, \end{aligned} \quad (\text{J.3})$$

where $s^\alpha]_{-\pi} = e^{\alpha(\ln|s| + i \arg s)]_{-\pi}}$ denotes the principal branch of the complex exponentiation, and the circles $C_{1/\epsilon}$, $C_\epsilon(0)$, and $C_\epsilon(\tau)$ are understood as in Figure 6 with $\epsilon > 0$ chosen small enough so that $1/\sigma$ and the poles of f_k all lie between the outer contour C_ϵ and the inner contour.

We proceed to compute $\gamma_k(\sigma)$ term by term. Let $l = \max(0, m - n + k)$. First notice that

$$\lim_{s \rightarrow \infty} \left(s - \frac{\tau}{2} \right) \left(\frac{f_k(s)}{(1 - \sigma s)^l} \right) s^{-\frac{1}{2}]_{-\pi}} (s - \tau)^{-\frac{1}{2}]_{-\pi}} \stackrel{(\text{J.1})}{=} \frac{\zeta_{-l}}{(-\sigma)^l}$$

where we consider that $\zeta_{-l} = 0$ whenever $m - n + k < 0$. By using Jordan's second lemma [9, §3.1.4, Theorem 2] (or, equivalently, by computing the residue at ∞), we find

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \oint_{C_{1/\epsilon}} \frac{f_k(s) s^{-\frac{1}{2}]_{-\pi}} (s - \tau)^{-\frac{1}{2}]_{-\pi}}{(1 - \sigma s)^l} ds = \frac{2i \zeta_{-l}}{(-\sigma)^l}. \quad (\text{J.4})$$

Besides, $\lim_{\epsilon \rightarrow a} (s-a)f_k(s) s^{-\frac{1}{2}}]_{-\pi}(s-\tau)^{-\frac{1}{2}}]_{-\pi} (1-\sigma s)^{-l} = 0$ for $a = 0, \tau$ as a consequence of the assumption $0, \tau \notin \mathcal{P}(f_k)$. It follows from Jordan's first lemma [9, §3.1.4, Theorem 1] that

$$\lim_{s \rightarrow 0, \tau} \frac{1}{\pi} \left(\oint_{\mathcal{C}_\epsilon(0)} + \oint_{\mathcal{C}_\epsilon(\tau)} \right) \frac{f_k(s) s^{-\frac{1}{2}}]_{-\pi}(s-\tau)^{-\frac{1}{2}}]_{-\pi}}{(1-\sigma s)^l} ds = 0. \quad (\text{J.5})$$

Lastly, by inspection of $s^{-\frac{1}{2}}]_{-\pi}(s-\tau)^{-\frac{1}{2}}]_{-\pi}$ right above and below the segment $(0, \tau)$, it can be seen that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \left(\int_\epsilon^{\tau-\epsilon} + \int_{\tau-\epsilon}^\epsilon \right) \frac{f_k(s) s^{-\frac{1}{2}}]_{-\pi}(s-\tau)^{-\frac{1}{2}}]_{-\pi}}{(1-\sigma s)^l} ds = -2i \hat{\alpha}_k(\sigma). \quad (\text{J.6})$$

On the other hand the residue theorem gives

$$\gamma_k(\sigma) = \left(\frac{1}{\pi} \right) 2i\pi \sum_{a \in \mathcal{P}(f_k) \cup \{\frac{1}{\sigma}\}} \text{Res}_{s=a} \left(\frac{f_k(s) s^{-\frac{1}{2}}]_{-\pi}(s-\tau)^{-\frac{1}{2}}]_{-\pi}}{(1-\sigma s)^l} \right) \quad (\text{J.7})$$

We draw our attention to the residue at $1/\sigma$. Using the Taylor development of $(1-x\tau)^{-1/2-j}$ at $x = 0$ we find, for $\sigma < \tau$ and $t \in \mathbb{N}_{\geq 0}$,

$$\begin{aligned} \lim_{s \rightarrow \frac{1}{\sigma}} \frac{d^t}{ds^t} \left[s^{-\frac{1}{2}}]_{-\pi}(s-\tau)^{-\frac{1}{2}}]_{-\pi} \right] &= \sigma^{t+1} \sum_{j=0}^t \binom{t}{j} \frac{\Gamma(\frac{1}{2})^2 (1-\sigma\tau)^{\frac{1}{2}-j}}{\Gamma(\frac{1}{2}-j)\Gamma(\frac{1}{2}-t+j)} \\ &= \sigma^{t+1} \sum_{q=0}^{\infty} \left[\sum_{j=0}^t \binom{t}{j} \frac{\Gamma(\frac{1}{2})^2}{\Gamma(\frac{1}{2}-q-j)\Gamma(\frac{1}{2}-t+j)} \right] \frac{(-\sigma\tau)^q}{q!}. \end{aligned} \quad (\text{J.8})$$

For large s , the value of $f_k(s)$ is given by the Laurent series expansion of f_k at $+\infty$, i.e.,

$$f_k(s) = \sum_{q=-l}^{\infty} \zeta_q s^{-q}, \quad (0 \ll |s| < \infty). \quad (\text{J.9})$$

Then,

$$\begin{aligned} \text{Res}_{s=\frac{1}{\sigma}} \left(\frac{f_k(s) s^{-\frac{1}{2}}]_{-\pi}(s-\tau)^{-\frac{1}{2}}]_{-\pi}}{(1-\sigma s)^l} \right) &= \lim_{s \rightarrow \frac{1}{\sigma}} \frac{1}{(l-1)!} \frac{d^{l-1}}{ds^{l-1}} \left[\left(s - \frac{1}{\sigma} \right)^l \left(\frac{f_k(s) s^{-\frac{1}{2}}]_{-\pi}(s-\tau)^{-\frac{1}{2}}]_{-\pi}}{(1-\sigma s)^l} \right) \right] \\ &\stackrel{(\text{J.8})}{=} \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{t=0}^{l-1} \binom{l-1}{t} \frac{f_k^{(l-1-t)}(\frac{1}{\sigma})}{(\frac{1}{\sigma})^{t+1}} \sum_{q=0}^{\infty} \left[\sum_{j=0}^t \frac{\binom{t}{j} \Gamma(\frac{1}{2})^2}{\Gamma(\frac{1}{2}-q-j)\Gamma(\frac{1}{2}-t+j)} \right] \frac{(-\sigma\tau)^q}{q!} \\ &= \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \frac{(-\sigma\tau)^q}{q!} \sum_{t=0}^{l-1} \frac{\binom{l-1}{t} f_k^{(l-1-t)}(\frac{1}{\sigma})}{(\frac{1}{\sigma})^{t+1}} \sum_{j=0}^{l-1} \frac{\binom{t}{j} \Gamma(\frac{1}{2})^2}{\Gamma(\frac{1}{2}-q-j)\Gamma(\frac{1}{2}-t+j)} \\ &= \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \frac{(-\sigma\tau)^q}{q!} \sum_{j=0}^{l-1} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q-j)} \sum_{t=j}^{l-1} \frac{\binom{l-1}{t} \binom{t}{j} \Gamma(\frac{1}{2}) f_k^{(l-1-t)}(\frac{1}{\sigma})}{(\frac{1}{\sigma})^{1+t} \Gamma(\frac{1}{2}-t+j)} \\ &= \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \frac{(-\tau)^q}{q!} \sum_{j=0}^{l-1} \frac{\binom{l-1}{j} \Gamma(\frac{1}{2}) (\frac{1}{\sigma})^{-\frac{1}{2}-q-j}}{\Gamma(\frac{1}{2}-q-j)} \sum_{t=0}^{l-1-j} \frac{\binom{l-1-j}{t} \Gamma(\frac{1}{2}) f_k^{(l-1-j-t)}(\frac{1}{\sigma})}{(\frac{1}{\sigma})^{\frac{1}{2}+t} \Gamma(\frac{1}{2}-t)} \\ &= \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \frac{\Gamma(\frac{1}{2}) (-\tau)^q}{q! \Gamma(\frac{1}{2}-q)} \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{d^{l-1-j}}{dx^{l-1-j}} \left[\frac{x^{-q}}{\sqrt{x}} \right]_{x=\frac{1}{\sigma}} \frac{d^{l-1-j}}{dx^{l-1-j}} \left[\frac{f_k(x)}{\sqrt{x}} \right]_{x=\frac{1}{\sigma}} \\ &= \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \frac{\Gamma(\frac{1}{2}) (-\tau)^q}{q! \Gamma(\frac{1}{2}-q)} \frac{d^{l-1}}{dx^{l-1}} \left[x^{-(q+1)} f_k(x) \right]_{x=\frac{1}{\sigma}}. \end{aligned}$$

For small σ , the function f_k is analytic in a neighborhood of $1/\sigma$, and so is $s^{-(q+1)} f_k(s)$ for any q . It follows that the derivation in (J.10) applies term by term to the Laurent series (J.9), and we find

$$\begin{aligned} \text{Res}_{s=\frac{1}{\sigma}} \left(\frac{f_k(s) s^{-\frac{1}{2}}]_{-\pi}(s-\tau)^{-\frac{1}{2}}]_{-\pi}}{(1-\sigma s)^l} \right) &\stackrel{(\text{J.9})}{=} \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(-\tau)^q}{q!} \frac{d^{l-1}}{dx^{l-1}} \left[\sum_{j=-l}^{\infty} \zeta_j x^{-(j+q+1)} \right]_{x=\frac{1}{\sigma}} \\ &= \frac{\zeta_{-l}}{(-\sigma)^l} - \sum_{q=0}^{\infty} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(-\sigma\tau)^q}{q!} \sum_{j=-\min(q,l)}^{\infty} \binom{l-1+q+j}{l-1} \zeta_j \sigma^j. \end{aligned} \quad (\text{J.10})$$

Observe on the other hand that, for $q + j \geq 0$,

$$\left| \binom{l-1+q+j}{l-1} \right| \leq \frac{(l-1+q+j)^{l-1}}{(l-1)!} \leq \sum_{t=0}^{\infty} \frac{(l-1+q+j)^t}{t!} \leq e^{l-1+q+j}.$$

Hence, for small σ ,

$$\sum_{j=-\min(q,l)}^{\infty} \left| \binom{l-1+q+j}{l-1} \zeta_j \sigma^j \right| \leq e^{l-1+q} \sum_{j=-l}^{\infty} |\zeta_j| (\sigma e)^j \leq \kappa(\sigma, l) e^{l-1+q}, \quad (\text{J.11})$$

with $\kappa(\sigma, l) < \infty$ by absolute convergence of the above series. Consequently,

$$\sum_{q=0}^{\infty} \sum_{j=-\min(q,l)}^{\infty} \left| \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(-\sigma\tau)^q}{q!} \binom{l-1+q+j}{l-1} \zeta_j \sigma^j \right| \stackrel{(\text{J.11})}{\leq} \kappa(\sigma, l) e^{l-1} \sum_{q=0}^{\infty} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(\sigma\tau)^q}{q!} e^q$$

is a finite quantity as it passes the ratio test for $\sigma < 1/\tau$ —this can be shown using Stirling's formula. It follows from Fubini's theorem that the summation order in (J.10) can be permuted. By setting $t = j + q$ we find

$$\begin{aligned} \text{Res}_{s=\frac{1}{\sigma}} & \left(\frac{f_k(s) s^{-\frac{1}{2}} \lfloor_{-\pi} (s-\tau)^{-\frac{1}{2}} \rfloor_{-\pi}}{(1-\sigma s)^l} \right) \\ &= \frac{\zeta_{-l}}{(-\sigma)^l} - \sum_{q=0}^{\infty} \sum_{t=\max(0, q-l)}^{\infty} \left[\zeta_{t-q} \binom{l-1+t}{l-1} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(-\tau)^q}{q!} \right] \sigma^t \\ &= \frac{\zeta_{-l}}{(-\sigma)^l} - \sum_{q=0}^l \zeta_{-q} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(-\tau)^q}{q!} + O(\sigma). \end{aligned} \quad (\text{J.12})$$

We eventually obtain (E.9) by combining (J.3) and (J.7), together with the intermediate results (J.4), (J.5), (J.6), and (J.12), and letting $\sigma \rightarrow 0$. \square

K Computation of core functions: examples

Example K.1 (Step cost function and identical service times) The cost function $f = \mathbf{1}_{[\tau, \infty)}$ is considered with constant service times $X = x > 0$. The relative value function to this problem was derived in [4] as a solution of (DE). We have $f_0 = 0$, $f_1 = 1$, $\mathcal{L}_f(s) = (1/s) e^{-s\tau}$, $\zeta(s, \tau) = 1/s$, $\mathcal{P}(\mathcal{L}_f) = \{0\}$, and $p_W = -\lambda [1 + (1/\lambda x) W_{-1}(-\lambda x e^{-\lambda x})] < 0$, as detailed in Appendix I.

For $u \in (\tau, \infty)$, we find,

$$\partial_+ c(u) \stackrel{(\text{III.10})}{=} \frac{\lambda}{1-\lambda x} \text{Res}_{s=0} \left(\frac{1-\lambda x}{s+\lambda(1-e^{sx})} e^{s(u-\tau)} \right) \stackrel{(\text{III.6})}{=} \frac{\lambda}{1-\lambda x}.$$

For $u \in (0, \tau)$, we inspect the positions of the poles and set $\gamma \in (0, -p_W)$. Decomposing $W^*(-s)$ as in (III.1), we find $\mathcal{P}(\zeta(\cdot, \tau)) = \{0\}$, $\mathcal{P}_u^c = \{-\lambda\}$, and $\mathcal{P}_u^p = \emptyset$. Since $f_0 = 0$ and \mathcal{P}_u^p is empty, the first and third terms in (III.13) both vanish and W_u^p needs not be considered. We find,

$$c'(u) \stackrel{(\text{III.13})}{=} \sum_{p \in \{-\lambda, 0\}} \text{Res}_{s=p} \left(\frac{\lambda^{m+1} e^{s(u+mx-\tau)}}{(s+\lambda)^m [s+\lambda(1-e^{sx})]} \right) \quad (\text{K.1})$$

where $m = \tilde{m}_1(u) = \lceil (\tau - u)/x \rceil$. We let $g(n, k) = u + (m-1-n+k)x - \tau$ for all $n, k \in \mathbb{N}$, and set $K(s) = e^{s(u+mx-\tau)} / [s + \lambda(1 - e^{sx})]$, the derivatives of which can be computed by induction (see Lemma K.1 at the end of the section). At $-\lambda$, we find the derivatives

$$K^{(n)}(-\lambda) = -(n!/\lambda^{n+1}) \sum_{k=0}^n [(\lambda g(n, k))^k / k!] e^{-\lambda g(n, k)}, \quad (n \in \mathbb{N}), \quad (\text{K.2})$$

and (K.1) reduces, for $u \in [0, \tau)$, to

$$\begin{aligned} c'(u) & \stackrel{(\text{III.6})}{=} \lim_{s \rightarrow 0} s \left(\frac{\lambda^{m+1}}{(s+\lambda)^m} K(s) \right) + \frac{1}{(m-1)!} \lim_{s \rightarrow -\lambda} (\lambda^{m+1} K^{(m-1)}(s)) \\ & \stackrel{(\text{K.2})}{=} \frac{\lambda}{1-\lambda x} - \lambda \sum_{k=0}^{m-1} \frac{(\lambda(u+kx-\tau))^k}{k!} e^{-\lambda(u+kx-\tau)}. \end{aligned}$$

Integrating the last expression from τ to u gives, for $u \in [0, \tau)$,

$$\begin{aligned} c(u) &= c(\tau) + \frac{\lambda(u-\tau)}{1-\lambda x} - \lambda \sum_{k=0}^{m-1} \int_{\tau-kx}^u \frac{(\lambda(t+kx-\tau))^k}{k!} e^{-\lambda(t+kx-\tau)} dt \\ &= c(\tau) + \frac{\lambda(u-\tau)}{1-\lambda x} - \lambda \sum_{k=0}^{m-1} \int_0^{u+kx-\tau} \frac{(\lambda\xi)^k}{k!} e^{-\lambda\xi} d\xi \\ &= c(\tau) + \frac{\lambda(u-\tau)}{1-\lambda x} + \sum_{k=0}^{m-1} \left(e^{-\lambda(u+kx-\tau)} \sum_{q=0}^k \frac{(\lambda(u+kx-\tau))^q}{q!} - 1 \right) \\ &= c(\tau) + \frac{\lambda(u-\tau)}{1-\lambda x} - \tilde{m}_1(u) + \sum_{k=0}^{\tilde{m}_1(u)-1} e^{-\lambda(u+kx-\tau)} \sum_{q=0}^k \frac{(\lambda(u+kx-\tau))^q}{q!}, \end{aligned}$$

where $c(\tau) = \frac{\lambda\tau}{1-\lambda x} + \tilde{m}_1(0) - \sum_{k=0}^{\tilde{m}_1(0)-1} e^{-\lambda(kx-\tau)} \sum_{q=0}^k (\lambda(kx-\tau))^q / q!$, and $\tilde{m}_1(t) = \lceil (\tau - t)/x \rceil$. Our result is coherent with [4, Theorem 2]. \circ

Example K.2 (Core function from a Taylor series) Assume that the service times for $u > 0$ follow the exponential distribution $F_X(x) = 1 - e^{-\omega x}$ discussed in Appendix I, where $\omega > \lambda$ in order to satisfy Assumption 1, and $p_W = \lambda - \omega$. Consider the cost function $f(u) = 1 - e^{-au}$, with $\Re(a) < \omega - \lambda$ (Assumption 2) and $a \neq 0$. This cost function, which is given much attention in [5], is entire ($\rho = 1$) of exponential type $\sigma = |a|$. Theorem 2 claims that the derivation of the relative value function from a Taylor series at 0 is possible if $|a| < |p_W|$. This can be verified. Using the notations of Section IV.1, we find $\tilde{f}_n = \delta[n] - (-a)^n$ for $n \in \mathbb{N}$ and, with the help of Appendix I,

$$\mathcal{Z}_{\tilde{f}(\infty)}(z) = \frac{a}{z+a}, \quad \mathcal{Z}_{\tilde{h}}\left(\frac{1}{z}\right) \stackrel{(I.3)}{=} \frac{\lambda(z-\omega)}{z-(\omega-\lambda)}, \quad \mathcal{Z}_{\tilde{c}(\infty)}(z) \stackrel{(D.1)}{=} \frac{\lambda a}{a+\omega-\lambda} \left(\frac{a+\omega}{z+a} - \frac{\lambda}{z-(\omega-\lambda)} \right),$$

with $\text{ROC}(\mathcal{Z}_{\tilde{f}(\infty)}) = \{z \in \mathbb{C} \mid |z| > |a|\}$, $\text{ROC}(\mathcal{Z}_{\tilde{h}}) = \{z \in \mathbb{C} \mid |z| > \omega - \lambda\}$ and, in consequence, $\text{ROC}(\mathcal{Z}_{\tilde{c}(\infty)}) = \{z \in \mathbb{C} \mid |a| < |z| < \omega - \lambda\}$, which, as predicted, is nonempty if $|a| < \omega - \lambda$ and empty if $|a| > \omega - \lambda$. Picking W^* from Table I.1, the inverse Z-transform of $\mathcal{Z}_{\tilde{c}(\infty)}$ then gives, for $n \in \mathbb{N}$,

$$\tilde{c}_n = \frac{\lambda(a+\omega)}{a+\omega-\lambda} (\delta[n] - (-a)^n) + \frac{\lambda^2 a}{(\omega-\lambda)(a+\omega-\lambda)} \delta[n] = \frac{\lambda}{1-\rho} (\delta[n] - W^*(a)(-a)^n),$$

which is the n -th derivative at 0 of $\lambda/(1-\rho)(1 - W^*(a)e^{-au})$. It follows that (IV.3a) converges for $u \in \mathbb{R}_{\geq 0}$, and we find, in accordance with Table 1,

$$c(u) \stackrel{(IV.3a)}{=} [\lambda/(1-\rho)] [u - W^*(a)(1 - e^{-au})/a].$$

Interval bounds. Notice that $f \in \hat{f}^{(n)} + [r^{(n)}]$ holds if we set $[r^{(n)}](u) = [0, a^{n+1}/(n+1)!] u^{n+1}$ for n even, and $[r^{(n)}] = [-a^{n+1}/(n+1)!, 0] u^{n+1}$ for n odd. The resulting interval $[\rho^{(n)}]$ follows by inspection of Table 1. Figure 3 displays the interval bounds $[c^{(n)}] = \tilde{c}^{(n)} + [\rho^{(n)}]$ obtained for c for various real values of a . The sequence $\{c^{(n)}\}$ shows to converge towards c for as long as $a < \omega - \lambda$. The generation of such a sequence is, however, impossible when $a \geq \omega - \lambda$, as the limit coefficients \tilde{c}_k are then infinite. \circ

In the next two examples, we consider the piecewise cost function

$$f(u) = \sum_{j=0}^n \varsigma_j u^j \mathbf{1}_{[0,\tau)}(u) + \xi(u) \mathbf{1}_{[\tau,\infty)}(u), \quad \forall u \in \mathbb{R}_{\geq 0},$$

where $\xi(u) = \bar{\zeta} u^k e^{-au}$, $n, k \in \mathbb{N}$, and $a \in \mathbb{C}$.

Example K.3 (Polynomial cost in an interval) Consider service times exponentially distributed with parameter $\omega > \lambda$, i.e $F_X(x) = 1 - e^{-\omega x}$, and the cost function (IV.4) with $\xi \equiv 0$. For this problem we have $f_0(u) = \sum_{j=0}^n \varsigma_j u^j$, $f_1 = 0$ and $\Delta = -f_0$. Besides,

$$\mathcal{L}_{f_0}(s) = \sum_{j=0}^n \varsigma_j \frac{j!}{s^{j+1}}, \quad \zeta(s, \tau) \stackrel{(III.9)}{=} - \sum_{j=0}^n \varsigma_j j! \sum_{q=0}^j \frac{\tau^q}{q! s^{j-q+1}}. \quad (\text{K.3})$$

Since $\mathcal{P}(\mathcal{L}_f) = \emptyset$, (III.10) gives $\partial_+ c(u) = 0$ for $u \in (\tau, \infty)$.

For $u \in (0, \tau)$: using (K.3), $\mathcal{P}(\mathcal{L}_{f_0}) = \{0\}$, $\rho = \lambda/\omega$, and the expression for W^* given in Table I.1, (III.11) reduces, after straightforward computations, to

$$\begin{aligned} \partial_+ c(u) &= \frac{\lambda\omega}{\omega-\lambda} \sum_{j=0}^n j! \varsigma_j [\text{Res}_{s=0} (W^*(-s) \frac{e^{su}}{s^{j+1}}) + \sum_{q=0}^j \frac{\tau^q}{q!} \text{Res}_{s=\omega-\lambda} (W^*(-s) \frac{e^{s(u-\tau)}}{s^{j-q+1}})] \\ &\stackrel{\text{(III.6)}}{=} \lambda \sum_{j=0}^n \varsigma_j u^j + \lambda^2 \sum_{j=0}^n \frac{j! \varsigma_j}{(\omega-\lambda)^{j+1}} \left(\sum_{q=0}^j \frac{[(\omega-\lambda)u]^q}{q!} - e^{-(\omega-\lambda)(\tau-u)} \sum_{q=0}^j \frac{[(\omega-\lambda)\tau]^q}{q!} \right). \end{aligned} \quad (\text{K.4})$$

Integration of (K.4) yields, for $u \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} c(u) &= \lambda \sum_{j=0}^n \varsigma_j \frac{[\min(u, \tau)]^{j+1}}{j+1} \\ &\quad + \lambda^2 \sum_{j=0}^n \frac{j! \varsigma_j}{(\omega-\lambda)^{j+2}} \sum_{q=0}^j \left\{ \frac{[(\omega-\lambda) \min(u, \tau)]^{q+1}}{(q+1)!} - \frac{e^{(\omega-\lambda) \min(u, \tau) - 1} [(\omega-\lambda)\tau]^q}{e^{(\omega-\lambda)\tau}} \right\}. \end{aligned}$$

Example K.4 Consider service times exponentially distributed with parameter $\omega > \lambda$, and the cost function $f(u) = u^k e^{-au} \mathbf{1}_{[\tau, \infty)}(u)$, i.e. (IV.4) with $n = 0$, $\varsigma_0 = 0$. We have $f_0(u) = 0$ and $f_1 = \Delta = u^k e^{-au}$, so that

$$\mathcal{L}_{f_0}(s) = 0, \quad \zeta(s, \tau) \stackrel{\text{(III.9)}}{=} k! e^{-a\tau} \sum_{q=0}^k \frac{\tau^q}{q!(s+a)^{k-q+1}}. \quad (\text{K.5})$$

For $u \in (\tau, \infty)$, we use Table I.1 and $\mathcal{P}(\mathcal{L}_f) = \{-a\}$, and get

$$\begin{aligned} \partial_+ c(u) &\stackrel{\text{(III.10)}}{=} \frac{k! \lambda \omega e^{-a\tau}}{\omega-\lambda} \sum_{q=0}^k \frac{\tau^{k-q}}{(k-q)!} \text{Res}_{s=-a} (W^*(-s) \frac{e^{s(u-\tau)}}{(s+a)^{q+1}}) \\ &\stackrel{\text{(III.6)}}{=} \lambda u^k e^{-au} + \frac{k! \lambda^2}{(\omega-\lambda+a)^{k+1}} \left\{ \sum_{q=0}^k \frac{1}{q!} [(\omega-\lambda+a)u]^q \right\} e^{-au}. \end{aligned} \quad (\text{K.6})$$

Alternatively, (K.6) can be derived from (III.7) with cost function f_1 , or by inspection of Table 1 for f_1 via computation of (III.4).

For $u \in (0, \tau)$, we combine (III.11) with (K.5), $\mathcal{P}(\mathcal{L}_{f_0}) = \emptyset$, and W^* (Table I.1) to get

$$\begin{aligned} \partial_+ c(u) &= -\frac{\lambda\omega}{\omega-\lambda} k! e^{-a\tau} \sum_{q=0}^k \text{Res}_{s=\omega-\lambda} (W^*(-s) \frac{\tau^q}{q!(s+a)^{k-q+1}} e^{s(u-\tau)}) \\ &= \frac{k! \lambda^2 e^{-a\tau}}{(\omega-\lambda+a)^{k+1}} \left\{ \sum_{q=0}^k \frac{1}{q!} [(\omega-\lambda+a)\tau]^q \right\} e^{(\omega-\lambda)(u-\tau)}. \end{aligned}$$

Hence, if $a \neq 0$,

$$\begin{aligned} c(u) &= \frac{k! \lambda^2 e^{-(\omega-\lambda+a)\tau}}{(\omega-\lambda)(\omega-\lambda+a)^{k+1}} \left\{ \sum_{q=0}^k \frac{1}{q!} [(\omega-\lambda+a)\tau]^q \right\} (e^{(\omega-\lambda) \min(u, \tau)} - 1) \\ &\quad + \frac{k! \lambda^2}{\omega-\lambda} \left\{ \sum_{q=0}^k \frac{1}{q!} [a^{-(k-q+1)} - (\omega-\lambda+a)^{-(k-q+1)}] [\tau^q e^{-a\tau} - \max(u, \tau)^q e^{-a \max(u, \tau)}] \right\}, \end{aligned}$$

and, if $a = 0$,

$$\begin{aligned} c(u) &= \frac{k! \lambda^2 e^{-(\omega-\lambda)\tau}}{(\omega-\lambda)^{k+2}} \left\{ \sum_{q=0}^k \frac{1}{q!} [(\omega-\lambda)\tau]^q \right\} (e^{(\omega-\lambda) \min(u, \tau)} - 1) \\ &\quad + \frac{\lambda \tau^{k+1}}{k+1} ([\max(\frac{u}{\tau}, 1)]^{k+1} - 1) + \frac{k! \lambda^2}{(\omega-\lambda)^{k+2}} \left\{ \sum_{q=0}^k \frac{[(\omega-\lambda)\tau]^{q+1}}{(q+1)!} ([\max(\frac{u}{\tau}, 1)]^{q+1} - 1) \right\}. \end{aligned}$$

Lemma K.1 *Proof of (K.2) Consider*

$$K(s) = \frac{e^{s(u+mx-\tau)}}{s + \lambda(1 - e^{sx})}, \quad (\text{K.7})$$

and let $g(n, k) = u + (m-1-n+k)x - \tau$ for all $n, k \in \mathbb{N}$. At $-\lambda$, K has the derivatives

$$K^{(n)}(-\lambda) = -(n!/\lambda^{n+1}) \sum_{k=0}^n [(\lambda g(n, k))^k / k!] e^{-\lambda g(n, k)}, \quad (n \in \mathbb{N}). \quad (\text{K.2})$$

Proof It is straightforward to verify that (K.2) holds for $n = 0$ and $n = 1$. For $n \geq 2$, we proceed by induction. observe that (K.7) rewrites as $(s + \lambda - \lambda e^{sx})K(s) = e^{sg(n-1, n)}$, which gives, after n differentiations at $-\lambda$:

$$K^{(n)}(-\lambda) = \frac{n}{\lambda} e^{\lambda x} K^{(n-1)}(-\lambda) - \sum_{k=0}^{n-1} \binom{n}{k} x^{n-k} K^{(k)}(-\lambda) - \frac{g(n-1, n)^n}{\lambda} e^{-\lambda g(n, n)}. \quad (\text{K.8})$$

Assuming that (K.2) holds for $n = 0, 1, \dots, p-1$, the second term of the second member of (K.8) reduces for $n = p$ to

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{p}{k} x^{p-k} K^{(k)}(-\lambda) &\stackrel{(\text{K.2})}{=} - \sum_{q=0}^{p-1} \binom{p}{q} x^{p-q} \frac{q!}{\lambda^{q+1}} \sum_{l=0}^q \frac{(\lambda g(q, l))^l}{l!} e^{-\lambda g(q, l)} \\ &= - \sum_{q=0}^{p-1} \frac{p!}{p-q!} x^{p-q} \frac{1}{\lambda^{q+1}} \sum_{l=0}^q \frac{(\lambda g(p, p+l-q))^l}{l!} e^{-\lambda g(p, p+l-q)} \\ &= - \frac{p!}{\lambda^{p+1}} \sum_{t=1}^p \left(\sum_{l=0}^{t-1} \frac{(\lambda x)^{t-l}}{t-l!} \frac{(\lambda g(p, t))^l}{l!} \right) e^{-\lambda g(p, t)} \\ &= - \frac{p!}{\lambda^{p+1}} \sum_{t=1}^p \frac{\lambda^t}{t!} \left(\sum_{l=0}^t \binom{t}{l} x^{t-l} g(p, t)^l - g(p, t)^t \right) e^{-\lambda g(p, t)} \\ &= - \frac{p!}{\lambda^{p+1}} \sum_{t=1}^p \frac{\lambda^t}{t!} \left((x + g(p, t))^t - g(p, t)^t \right) e^{-\lambda g(p, t)} \\ &= - \frac{p!}{\lambda^{p+1}} \sum_{t=1}^p \frac{\lambda^t}{t!} \left(g(p-1, t)^t - g(p, t)^t \right) e^{-\lambda g(p, t)}. \end{aligned} \quad (\text{K.9})$$

Inserting (K.9) into (K.8) yields

$$\begin{aligned} K^{(p)}(-\lambda) &\stackrel{(\text{K.2})}{=} - \frac{p!}{\lambda^{p+1}} \sum_{t=0}^{p-1} \frac{(\lambda g(p-1, t))^t}{t!} e^{-\lambda g(p, t)} - \sum_{k=0}^{p-1} \binom{p}{k} x^{p-k} K^{(k)}(-\lambda) - \frac{(\lambda g(p-1, p))^p}{\lambda^{p+1}} e^{-\lambda g(p, p)} \\ &\stackrel{(\text{K.9})}{=} - \frac{p!}{\lambda^{p+1}} \sum_{t=0}^p \frac{(\lambda g(p-1, t))^t}{t!} e^{-\lambda g(p, t)} + \frac{p!}{\lambda^{p+1}} \sum_{t=1}^p \frac{\lambda^t}{t!} \left(g(p-1, t)^t - g(p, t)^t \right) e^{-\lambda g(p, t)} \\ &= - \frac{p!}{\lambda^{p+1}} \sum_{t=0}^p \frac{(\lambda g(p, t))^t}{t!} e^{-\lambda g(p, t)} \end{aligned}$$

and (K.2) holds for all n . \square

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