

Dispatching to Parallel Servers

Solutions of Poisson's Equation for First-Policy Improvement

Supplementary material

Olivier Bilenne

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Olivier Bilenne
Univ. Grenoble Alpes, CNRS, Inria, Grenoble INP, LIG, 38000 Grenoble, France
E-mail: olivier.bilenne@inria.fr

E Proof of (III.2)

Consider

$$K(s) = \frac{e^{s(u+mx-\tau)}}{s + \lambda(1 - e^{sx})}, \quad (\text{E.1})$$

and let $g(n, k) = u + (m - 1 - n + k)x - \tau$ for all $n, k \in \mathbb{N}$. At $-\lambda$, K has the derivatives

$$K^{(n)}(-\lambda) = -(n!/\lambda^{n+1}) \sum_{k=0}^n [(\lambda g(n, k))^k / k!] e^{-\lambda g(n, k)}, \quad (n \in \mathbb{N}). \quad (\text{III.2})$$

Proof (III.2) It is straightforward to verify that (III.2) holds for $n = 0$ and $n = 1$. For $n \geq 2$, we proceed by induction. observe that (E.1) rewrites as $(s + \lambda - \lambda e^{sx})K(s) = e^{sg(n-1, n)}$, which gives, after n differentiations at $-\lambda$:

$$K^{(n)}(-\lambda) = \frac{n}{\lambda} e^{\lambda x} K^{(n-1)}(-\lambda) - \sum_{k=0}^{n-1} \binom{n}{k} x^{n-k} K^{(k)}(-\lambda) - \frac{g(n-1, n)^n}{\lambda} e^{-\lambda g(n, n)}. \quad (\text{E.2})$$

Assuming that (III.2) holds for $n = 0, 1, \dots, p-1$, the second term of the second member of (E.2) reduces for $n = p$ to

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{p}{k} x^{p-k} K^{(k)}(-\lambda) &\stackrel{(\text{III.2})}{=} - \sum_{q=0}^{p-1} \binom{p}{q} x^{p-q} \frac{q!}{\lambda^{q+1}} \sum_{l=0}^q \frac{(\lambda g(q, l))^l}{l!} e^{-\lambda g(q, l)} \\ &= - \sum_{q=0}^{p-1} \frac{p!}{p-q!} x^{p-q} \frac{1}{\lambda^{q+1}} \sum_{l=0}^q \frac{(\lambda g(p, p+l-q))^l}{l!} e^{-\lambda g(p, p+l-q)} \\ &= - \frac{p!}{\lambda^{p+1}} \sum_{t=1}^p \left(\sum_{l=0}^{t-1} \frac{(\lambda x)^{t-l}}{t-l!} \frac{(\lambda g(p, t))^l}{l!} \right) e^{-\lambda g(p, t)} \\ &= - \frac{p!}{\lambda^{p+1}} \sum_{t=1}^p \frac{\lambda^t}{t!} \left(\sum_{l=0}^t \binom{t}{l} x^{t-l} g(p, t)^l - g(p, t)^t \right) e^{-\lambda g(p, t)} \\ &= - \frac{p!}{\lambda^{p+1}} \sum_{t=1}^p \frac{\lambda^t}{t!} ((x + g(p, t))^t - g(p, t)^t) e^{-\lambda g(p, t)} \\ &= - \frac{p!}{\lambda^{p+1}} \sum_{t=1}^p \frac{\lambda^t}{t!} (g(p-1, t)^t - g(p, t)^t) e^{-\lambda g(p, t)}. \end{aligned} \quad (\text{E.3})$$

Inserting (E.3) into (E.2) yields

$$\begin{aligned} K^{(p)}(-\lambda) &\stackrel{(\text{III.2})}{=} - \frac{p!}{\lambda^{p+1}} \sum_{t=0}^{p-1} \frac{(\lambda g(p-1, t))^t}{t!} e^{-\lambda g(p, t)} - \sum_{k=0}^{p-1} \binom{p}{k} x^{p-k} K^{(k)}(-\lambda) - \frac{(\lambda g(p-1, p))^p}{\lambda^{p+1}} e^{-\lambda g(p, p)} \\ &\stackrel{(\text{E.3})}{=} - \frac{p!}{\lambda^{p+1}} \sum_{t=0}^p \frac{(\lambda g(p-1, t))^t}{t!} e^{-\lambda g(p, t)} + \frac{p!}{\lambda^{p+1}} \sum_{t=1}^p \frac{\lambda^t}{t!} (g(p-1, t)^t - g(p, t)^t) e^{-\lambda g(p, t)} \\ &= - \frac{p!}{\lambda^{p+1}} \sum_{t=0}^p \frac{(\lambda g(p, t))^t}{t!} e^{-\lambda g(p, t)} \end{aligned}$$

and (III.2) holds for all n . \square

F Proof of (IV.10)

The coefficients $\beta_{n,0}, \dots, \beta_{n,n}$, defined by

$$\beta_{n,k} = (-\tau)^{-k} \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^l f\left(\frac{l\tau}{n}\right), \quad (k = 0 \dots, n), \quad (\text{F.1})$$

satisfy $\beta_{n,k} = \frac{1}{k!} \tilde{f}_n^{(k)}(0)$, where, for $k = 1, \dots, n$,

$$\tilde{f}_n^{(0)}(l) = f\left(\frac{l\tau}{n}\right), \quad \tilde{f}_n^{(k)}(l) = \frac{n-k+1}{\tau} [\tilde{f}_n^{(k-1)}(l+1) - \tilde{f}_n^{(k-1)}(l)] \quad (l = 0, \dots, n-k). \quad (\text{IV.10})$$

Proof (IV.10) Indeed, $\beta_{n,0} = \tilde{f}_n^{(0)}(0)/0!$ is immediate. Suppose now that

$$\tilde{f}_n^{(k)}(l) = \frac{k!}{(-\tau)^k} \binom{n}{k} \sum_{t=0}^k \binom{k}{t} (-1)^t f\left(\frac{t\tau}{n}\right), \quad (l = 0, \dots, n-k), \quad (\text{F.2})$$

holds for $k = 1, \dots, q-1$, where $1 \leq q \leq n-1$. Then, for $l = 0, \dots, n-q$,

$$\begin{aligned} \tilde{f}_n^{(q)}(l) &\stackrel{(\text{IV.10})}{=} \frac{(n-q+1)}{\tau} \left[\tilde{f}_n^{(q-1)}(l+1) - \tilde{f}_n^{(q-1)}(l) \right] \\ &\stackrel{(\text{F.1})}{=} \frac{n-q+1}{\tau} \frac{(q-1)!}{(-\tau)^{q-1}} \binom{n}{q-1} \sum_{t=0}^{q-1} \binom{q-1}{t} (-1)^t \left[f\left(\frac{(t+l+1)\tau}{n}\right) - f\left(\frac{(t+l)\tau}{n}\right) \right] \\ &= \frac{(q!)^2}{(-\tau)^q} \frac{(n-q+1)}{q} \binom{n}{q-1} \left[\sum_{t=1}^q \binom{q-1}{t-1} (-1)^t f\left(\frac{(t+l)\tau}{n}\right) + \sum_{t=0}^{q-1} \binom{q-1}{t} (-1)^t f\left(\frac{(t+l)\tau}{n}\right) \right] \\ &= \frac{(q!)^2}{(-\tau)^q} \binom{n}{q} \left\{ f\left(\frac{l\tau}{n}\right) + (-1)^q f\left(\frac{(q+l)\tau}{n}\right) + \sum_{t=1}^{q-1} \left[\binom{q-1}{t-1} + \binom{q-1}{t} \right] (-1)^t f\left(\frac{(t+l)\tau}{n}\right) \right\} \\ &= \frac{(q!)^2}{(-\tau)^q} \binom{n}{q} \sum_{t=0}^q \binom{q}{t} (-1)^t f\left(\frac{(t+l)\tau}{n}\right) \end{aligned}$$

and (F.2) holds for $k = q$. By induction, (F.2) is true for $0 \leq k \leq n$. By setting $l = 0$ in (F.2), we infer from (F.1) that $\beta_{n,k} = \tilde{f}_n^{(k)}(0)/k!$ for $0 \leq k \leq n$. \square

G Proof of Proposition A.1

Proposition A.1 (Analyticity of W^* and pole location) *Under Assumption 2:*

- (i) The dominant singularity p_W of W^* (i.e., that with largest real value) is a pole with degree 1 lying on the negative real axis $\mathbb{R}_{<0}$. The dominant singularity p_X of X^* is real, negative (possibly infinite) and satisfies $p_X < p_W$. X^* is analytic on $\{s \in \mathbb{C} \mid \Re(s) > p_W\}$.
- (ii) W^* is analytic on $\{s \in \mathbb{C}_0 \mid \Re(s) > p_W\}$, where $\lim_{s \rightarrow \infty} |W^*(s)| \leq 1$.
- (iii) One can find $\epsilon > 0$ such that W^* is analytic on $\{s \in \mathbb{C}_0 \setminus \{p_W\} \mid \Re(s) > p_W - \epsilon\}$.
- (iv) W^* is analytic in a neighborhood of 0, where it rewrites as the series

$$W^*(s) = \sum_{k=0}^{\infty} w_k (-s)^k, \quad \forall s \in \{\sigma \in \mathbb{C}_0 : |\sigma - a| < |p_W|\}, \quad (\text{A.1})$$

in which the coefficients $\{w_k\}$ are given by (III.3) in Table 1, and satisfy $w_k = \mathbb{E}[W^k]/k!$, for $k \in \mathbb{N}$. The series $\{w_k\}$ is asymptotically geometric with asymptotic rate $|p_W|^{-1}$.

- (v) At any point $a \in \mathbb{C}_0$ where W^* is analytic, W^* rewrites as the series

$$W^*(s) = \sum_{k=0}^{\infty} w_{a:k} (a-s)^k, \quad \forall s \in \{\sigma \in \mathbb{C} : |\sigma - a| < r_a\}, \quad (\text{A.2})$$

where r_a denotes the distance from a to the closest singularity of W^* . The coefficients $\{w_{a:k}\}$ are given by (III.4) in Table 1.

Proof (i) First observe in

$$W^*(s) = \frac{(1-\rho)s}{s - \lambda(1 - X^*(s))} \quad (\text{PK})$$

that 0 is a removable singularity of $W^*(s)$. Since X^* and W^* are the Laplace transforms of probability density functions on $\mathbb{R}_{\geq 0}$, their dominant singularities p_X, p_W are real, nonpositive, and possibly infinite $(-\infty)^1$. Besides, $X^*(s)$ is, by definition, continuous and strictly decreasing on $(p_W, +\infty)$ with $\lim_{s \rightarrow \infty} X^*(s) = 0$. It intersects with the straight line $1 - s/\lambda$ at $s = 0$ (removable singularity) with slope strictly larger than $-1/\lambda$ ($\rho < 1$). Since the limit

¹ If F_Y is the probability distribution of a random variable Y on $\mathbb{R}_{\geq 0}$, then

$$\left| \mathcal{L}_{F_Y'}(s) \right| \leq \int_0^{\infty} |e^{-su}| F_Y(du) = \int_0^{\infty} e^{-\Re(s)u} F_Y(du) = \mathcal{L}_{F_Y'}(\Re(s)), \quad \forall s \in \mathbb{C}.$$

It follows that $\mathcal{L}_{F_Y'}$ is absolutely dominated by its expression on the real axis, which is a real, nonnegative function. The dominant singularity p_Y of $\mathcal{L}_{F_Y'}$ lies therefore on the real axis. Since F_Y' is absolutely integrable, p_Y is negative.

value of the derivative of $X^*(s)$ at $-\infty$, given by $\lim_{s \rightarrow -\infty} \mathbb{E}[-Xe^{-sX}]$ is infinite, $X^*(s)$ must necessarily cross $1 - s/\lambda$ at another negative value $p_W > p_X$, which is the dominant (non-removable) singularity of W^* . Moreover, since the slopes of the curves at p_W are different, the derivative of $s - \lambda(1 - X^*(s))$ at p_W is nonzero. It follows that p_W is a pole of degree 1. Because X^* is a Laplace transform, it is analytic on its domain, which includes the half-plane to the right of p_W .

(ii) From the conclusions of (i), we find $\sigma - \lambda(1 - \mathbb{E}[e^{-\sigma X}]) \neq 0$ for every $\sigma \in \{s \in \mathbb{C}_0 \mid \Re(s) > p_W\}$, where $W^*(\sigma)$ is nonzero. Since we also have $\lim_{s \rightarrow \infty} X^*(s) = 0$, it follows from (PK) that $\lim_{s \rightarrow \infty} |W^*(s)| \leq 1$.

(iii) On the vertical axis $s = p_W + it$, the denominator of $W^*(s)$ is given by

$$\{p_W - \lambda(1 - \mathbb{E}[e^{-p_W X} \cos(tX)])\} + i\{t - \lambda\mathbb{E}[e^{-p_W X} \sin(tX)]\}. \quad (\text{G.1})$$

The real part of (G.1) can only be 0 if $\cos(tX) = 1$ almost everywhere with respect to F_X , where $\sin(tX) = 0$ and the imaginary part of (G.1) reduces to t . Hence, the only singularity on the axis $s = p_W + it$ is p_W . Next, we show that it is not possible to find a sequence $\{\hat{s}_k\}$ of singularities of W^* such that $\Re(\hat{s}_k) \rightarrow 0$. Suppose it is: the sequence is either bounded or not. If it is bounded, then there exists a subsequence of $\{\hat{s}_k\}$ of poles converging towards a point of the imaginary axis, which can only be p_W , and consequently $\hat{s}_k - \lambda(1 - X^*(\hat{s}_k)) = 0$ for the subsequence converging towards p_W . By analytic continuation, $s - \lambda(1 - X^*(s)) = 0$ in a neighborhood of p_W , which is impossible. Suppose now that $\{\hat{s}_k\}$ is unbounded and converges to $i\infty$. If $\hat{s}_k = p_W - \delta_k + it_k$, the imaginary part of $s - \lambda(1 - X^*(s))$ is given by

$$t_k - \lambda\mathbb{E}[e^{-p_W X} \sin(t_k X)] - \lambda\mathbb{E}[Xe^{-p_W X} \sin(t_k X)]\delta_k + o(\delta_k), \quad (\text{G.2})$$

where $|\mathbb{E}[e^{-p_W X} \sin(t_k X)]| \leq \mathbb{E}[e^{-p_W X}] = 1 - p_W/\lambda$ is a finite quantity, while the deviation $|\mathbb{E}[Xe^{-p_W X} \sin(t_k X)]| \leq \mathbb{E}[Xe^{-p_W X}]$ is dominated by the (finite) slope of $X^*(s)$ at p_W . Since $t_k \rightarrow \infty$, (G.2) diverges, and the sequence does not exist. It follows that there is no singularity with imaginary value arbitrarily close to that of p_W .

(iv) The series expansion of W^* at 0 is $W^*(s) = \sum_{k=0}^{\infty} (\mathbb{E}[W^k]/k!)(-s)^k$, where $\mathbb{E}[W^k]$ is the k^{th} moment of the waiting time distribution. These moments have known expressions which satisfy $\mathbb{E}[W^k] = k! w_k$ for all k —see e.g. [2, §5.1.5]). Hence (A.1) is the (unique) Taylor expansion of W^* at 0. Now, the ratio test for this Taylor series tells us that $\{\mathbb{E}[W^k]/k!\}$ grows asymptotically exponentially with asymptotic rate \underline{a} if and only if the Taylor series converges on the interior of a disc with radius \underline{a} centered at the origin, and diverges outside the disc, thus betraying the presence of a singularity of W^* on the circle. Besides, since $\{\mathbb{E}[W^k]/k!\}$ has only real, nonnegative values, the series takes its largest absolute value on the circle at the intersection with the negative branch of the real axis. It follows in that case that $-\underline{a} = p_W$ is the dominant singularity of W^* .

(v) For $n \in \mathbb{N}$ and the specified point a , consider the cost function $f(u) = (-u)^n e^{-au} = (\partial^n / \partial a^n)[e^{-au}]$. It follows from Table 1 and the Leibniz integral rule that

$$c'(u) = \frac{\lambda}{1-\rho} \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k} W^*(a)}{da^{n-k}} (-u)^k e^{-au}, \quad (\text{G.3})$$

and from Table 1 that

$$\begin{aligned} c'(u) &= (-1)^n \frac{\lambda n!}{1-\rho} \sum_{k=0}^n w_{a:n-k} \frac{u^k e^{-au}}{k!} \\ &= \frac{\lambda}{1-\rho} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (n-k)! w_{a:n-k} (-u^k) e^{-au}. \end{aligned} \quad (\text{G.4})$$

Inspection of (G.3) and (G.4) then yields $(d^k / da^k)[W^*(a)]/k! = (-1)^k w_{a:k}$ for all k , and (A.2) follows from the Taylor series of W^* at a . \square

H Proof of Proposition 2 (variant)

This appendix contains an alternate proof for Proposition 2. First, we recall basic properties of the M/G/1 queue.

Lemma H.1 *Let $0 \leq u_2 \leq u_1$ and suppose that the M/G/1 queue, initially at state u_1 , reaches state u_2 for the first time after a random period of time T , during which N jobs have arrived. Then,*

$$(a) \quad \mathbb{E}[T] = \frac{1}{1-\rho} (u_1 - u_2), \quad (b) \quad \mathbb{E}[N] = \frac{\lambda}{1-\rho} (u_1 - u_2). \quad (\text{H.1})$$

Proof Lemma H.1 is a consequence of the law of large numbers. Consider n realizations of the setting, and denote by T_1, \dots, T_n the random values observed for the variable T , by N_1, \dots, N_n those observed for the variable N , and by $\{X_{k,l}\}_{l=1}^{N_k}, \dots, \{X_{k,l}\}_{l=1}^{N_n}$. Since the rate of the Poisson process is equal to the density of arrivals per unit of time,

$$\lambda = \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n N_k}{\sum_{k=1}^n T_k} = \left(\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n N_k}{n} \right) \left(\lim_{n \rightarrow \infty} \frac{n}{\sum_{k=1}^n T_k} \right) = \frac{\mathbb{E}[N]}{\mathbb{E}[T]}. \quad (\text{H.2})$$

Similarly,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{l=1}^{N_k} X_{k,l} = \left(\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \sum_{l=1}^{N_k} X_{k,l}}{\sum_{k=1}^n N_k} \right) \left(\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n N_k}{n} \right) = \mathbb{E}[X] \mathbb{E}[N]. \quad (\text{H.3})$$

By definition of the variables, we also have $u_2 = u_1 + \sum_{l=1}^{N_k} X_{k,l} - T_k$ for $k = 1, \dots, n$, and it follows that

$$\mathbb{E}[T] = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n T_k = u_1 - u_2 + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{l=1}^{N_k} X_{k,l} \stackrel{(\text{H.3})}{=} u_1 - u_2 + \mathbb{E}[X] \mathbb{E}[N] \stackrel{(\text{H.2})}{=} u_1 - u_2 + \rho \mathbb{E}[T]$$

which yields (H.1a), and (H.1b) follows from (H.2). \square

Proposition 2 (Value function) *Let f satisfy Assumption 2 and be piecewise continuous.*

(i) *The value function (VF) is continuous, almost everywhere continuously differentiable, and semi-differentiable with right derivative*

$$\partial_+ v(u) = \lambda (f^+(u) - \bar{f} + \mathbb{E}[v(u+X) - v(u)]), \quad \forall u \in \mathbb{R}_{>0}, \quad (\text{DE})$$

where $f^+(u) := \lim_{t \rightarrow u^+} f(t)$. At $u = 0$, one has

$$v(0) = f(0) - \bar{f} + \mathbb{E}[v(X_0)], \quad (\text{BCa})$$

$$v'(0) = \lambda (f^+(0) - f(0) + \mathbb{E}[v(X) - v(X_0)]). \quad (\text{BCb})$$

(ii) *The value function is given by*

$$v(u) = v(0) + c(u) - \frac{\lambda \bar{f}}{1-\rho} u, \quad \forall u \in \mathbb{R}_{\geq 0}, \quad (\text{S})$$

where $c : \mathbb{R} \mapsto \mathbb{R}$ is continuous, almost everywhere continuously differentiable, and semi-differentiable with right-derivative

$$\partial_+ c(u) = \frac{\lambda}{1-\rho} \mathbb{E}[f(u+W)], \quad \forall u \in \mathbb{R}. \quad (\text{CVF})$$

Proof (Proposition 2) Start the queue at state u . The quantity $V(u, t)$ appearing in (VF) rewrites, for any $T \geq 0$ and for t large enough, as $V(u, t) = V(u, T) + V(U(T), t - T)$, where $U(T)$ denotes the backlog observed after time T . It follows from the Markov property of the system and from the definition VF of the value function that

$$v(u) = \mathbb{E}[V(u, T) - \lambda \bar{f} T] + \mathbb{E}[v(U(T))]. \quad (\text{H.4})$$

We now turn to show the claims, starting with (ii), from which useful properties of v (existence, continuity) can be inferred.

(ii) If in (H.4) u is taken to be the initial backlog of the queue and T is defined as the time when the queueing system reaches the empty state ($u = 0$) for the first time, we get (S), with $c(u) = \mathbb{E}[V(u, T)]$ defined as the expected total cost incurred until the queue is first empty. In order to compute $\mathbb{E}[V(u_1, T)]$, we introduce the following notion: given any random set S containing a random number of stochastic real variables, we define the measure $\Psi_S : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ as $\Psi_S((-\infty, t]) = \mathbb{E}[\sum_{t' \in S} \theta(t - t')]$ for all $t \in \mathbb{R}$, where θ denotes the step function with the convention $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ otherwise. If a function f is defined on the same domain as S and measurable with respect to Ψ_S , then

$$\Psi_S f = \int_{-\infty}^{\infty} f(\xi) \Psi_S(d\xi) = \mathbb{E}[\sum_{t' \in S} f(t')] \quad (\text{H.5})$$

provided that f is integrable on \mathbb{R} with respect to Ψ_S . In particular, if S denotes the set of the times of the successive arrivals of a Poisson process with density λ initiated at a time t_0 , we have $\Psi_S(dt) = \lambda dt$ for $t \geq t_0$, and find $\Psi_S((-\infty, t]) = \lambda(t - t_0)\theta(t - t_0)$.

To compute $c(u)$, we report the arrival times of all the jobs landed during the random period of time T , and classify the coinciding backlog values as follows:

$$Y_1, Y_1 + U_{11}, \dots, Y_1 + U_{1M_1}, Y_2, Y_2 + U_{21}, \dots, Y_2 + U_{2M_2}, Y_3, \dots, Y_N, Y_N + U_{N1}, \dots, Y_N + U_{NM_N},$$

in which $B = \{Y_1, \dots, Y_N\}$ is the longest nonincreasing sequence of backlogs and, for $p = 1, \dots, N$, $A_p = \{U_{p0}, U_{p1}, \dots, U_{pM_p}\}$ is the sequence of relative backlog values interposed in-between, where we have used the convention $U_{p0} = 0$ for $p = 1, \dots, N$. First observe that Y_1, \dots, Y_N rewrite as $Y_p = u - T_p$ for $p = 1, \dots, N$, where the variables T_1, T_2, \dots form a Poisson process with rate λ . It follows that Ψ_B is given by

$$\Psi_B(dt) = \lambda \mathbf{1}_{[0, u]}(t) dt. \quad (\text{H.6})$$

We compute the quantities $\Psi_{A_1}, \dots, \Psi_{A_N}$, by applying the law of large numbers. First observe, for $p \in \{1, \dots, N\}$, that $U_{p0}, \dots, U_{pM_p}, U_{p0}, \dots, U_{pM_p}, U_{p0}, \dots$ form the successive waiting times of an analogous M/G/1 queue with service time convention (X, X) . Taking n samples of the process, we find, for $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} \Psi_{A_p}([0, t]) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\sum_{q=0}^{M_p^{(k)}} \theta(t - U_{pq}^{(k)}) \right) \\ &= \left(\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n (M_p^{(k)} + 1)}{n} \right) \left(\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \sum_{q=0}^{M_p^{(k)}} \theta(t - U_{pq}^{(k)})}{\sum_{k=1}^n (M_p^{(k)} + 1)} \right) \\ &= (1 + \mathbb{E}[M_p]) \mathbb{E}[\theta(t - W)] \stackrel{(\text{H.1b})}{=} \left(1 + \frac{\lambda \mathbb{E}[X_p]}{1 - \rho} \right) \mu_W([0, t]) = \frac{\mu_W([0, t])}{1 - \rho}. \end{aligned} \quad (\text{H.7})$$

We are now able to compute $c(u)$, and we find

$$\begin{aligned} c(u) &= \mathbb{E} \left[\sum_{p=1}^N \sum_{q=0}^{M_p} f(Y_p + U_{pq}) \right] \\ &= \mathbb{E} \left[\sum_{p=1}^N \mathbb{E} \left[\sum_{q=0}^{M_p} f(Y_p + U_{pq}) | Y_p \right] \right] \\ &\stackrel{(\text{H.5})}{=} \mathbb{E} \left[\sum_{p=1}^N \int_{-\infty}^{\infty} f(Y_p + t) \Psi_{A_p}(dt) \right] \\ &\stackrel{(\text{H.7})}{=} \frac{1}{1 - \rho} \mathbb{E} \left[\sum_{p=1}^N \int_0^{\infty} f(Y_p + t) \mu_W(dt) \right] \\ &\stackrel{(\text{H.5})}{=} \frac{1}{1 - \rho} \int_0^{\infty} \left(\int_0^{\infty} f(\xi + t) \mu_W(dt) \right) \Psi_B(d\xi) \\ &\stackrel{(\text{H.6})}{=} \frac{\lambda}{1 - \rho} \int_0^u \left(\int_0^{\infty} f(\xi + t) \mu_W(dt) \right) d\xi \\ &= \frac{\lambda}{1 - \rho} \int_0^u \mathbb{E} [f(\xi + W)] d\xi. \end{aligned} \quad (\text{H.8})$$

Then, (CVF) follows by right differentiation of (H.8).

(i) Let $T = \delta$, where $\delta > 0$ is a small time step. Since the inter-arrival times are exponentially distributed, $\text{Prob}(N = 0) = 1 - \lambda\delta + o(\delta)$, $\text{Prob}(N = 1) = \lambda\delta + o(\delta)$, and $\text{Prob}(N \geq 2) = o(\delta)$.

First consider $t > 0$ and initialize the system at time $u = t + \delta$. By considering separately the three events $N = 0$, $N = 1$ and $N > 1$, we find that $\mathbb{E}[V(u, \delta)] = \lambda\delta \mathbb{E}[f(\dot{U})] + o(\delta)$,

with \hat{U} uniformly distributed over $(t, u]$, and $\mathbb{E}[v(U(\delta))] = (1 - \lambda\delta)v(t) + \lambda\delta\mathbb{E}[v(t+X)] + o(\delta)$. Introducing these two results into (H.4) gives

$$v(t + \delta) = v(t) + \lambda\delta\mathbb{E}[f(\hat{U}) - \bar{f}] + \lambda\delta\mathbb{E}[v(t+X) - v(t)] + o(\delta), \quad (\text{H.9})$$

where under Assumption 2 the expectations are necessarily finite (cf. (iii)). By letting $\delta \downarrow 0$ in (H.9), we find that v is continuous, and right differentiable at t with right derivative

$$\partial_+ v(u) = \lambda(f^+(u) - \bar{f} + \mathbb{E}[v(u+X)] - v(u)), \quad (\text{H.10})$$

which holds for every $u > 0$ and inherits the piecewise continuity of f .

Now, if $t = 0$, $\delta > 0$ and the system is started from $u = 0$, then $\mathbb{E}[V(u, \delta)] = \lambda\delta\mathbb{E}[f(0)] + o(\delta)$, $\mathbb{E}[v(U(\delta))] = (1 - \lambda\delta)v(0) + \lambda\delta\mathbb{E}[v(X_0)] + o(\delta)$ and, using (H.4) and continuity arguments, we obtain (BCa). Then (BCb) follows by setting $u = 0$ in (H.10) and substituting $v(0)$ with its value computed in (BCa). \square

I Derivation of Table B.1

Proof (Derivation of Table B.1) The expression for $\{w_k\}$ were derived either by combination of (PK) and (III.7), or by computation of the moments $\mathbb{E}[X^k]$ and development of (III.3).

(M/D/1) Before showing (B.1), observe from (B.2a)-(B.2b) that the quantity $n!\phi(m, n)$ is in fact the number of possible scenarios that may occur when placing n distinct objects (unordered) into m numbered urns so that each urn contains at least two objects ($n \geq 2m$). For $k = 1$, (B.1) follows directly from (III.3). If we suppose that (B.1) holds for $k = 1, \dots, p$, then, using $\mathbb{E}[X^k] = x^k$,

$$\begin{aligned} w_{p+1} &\stackrel{(\text{III.3})}{=} \frac{\lambda}{1-\lambda x} \left[\frac{x^{p+2}}{(p+2)!} + \sum_{t=1}^p \frac{x^{p-t+2}}{(p-t+2)!} w_t \right] \\ &\stackrel{(\text{B.1})}{=} \frac{\lambda}{1-\lambda x} \left[\frac{x^{p+2}}{(p+2)!} + \sum_{t=1}^p \frac{x^{p-t+2}}{(p-t+2)!} \sum_{q=1}^t \left(\frac{\lambda}{1-\lambda x} \right)^q \phi(q, t+q) x^{t+q} \right] \\ &= \frac{\lambda}{1-\lambda x} \frac{x^{p+2}}{(p+2)!} + \sum_{q=1}^p \left(\frac{\lambda}{1-\lambda x} \right)^{q+1} \sum_{t=q}^p \phi(q, t+q) \frac{x^{p+q+2}}{(p-t+2)!} \\ &\stackrel{(\text{II.2a})}{=} \frac{\lambda}{1-\lambda x} \phi(1, p+2) x^{p+2} + \sum_{q=2}^{p+1} \left(\frac{\lambda}{1-\lambda x} \right)^q \left[\sum_{l=q-1}^p \frac{\phi(q-1, q-1+l)}{(p-l+2)!} \right] x^{p+1+q} \\ &\stackrel{(\text{B.2b})}{=} \sum_{q=1}^{p+1} \left(\frac{\lambda}{1-\lambda x} \right)^q \phi(q, p+1+q) x^{p+1+q} \end{aligned}$$

and (B.1) holds for all k by induction. We obtain in (B.2a)-(B.2b) a recursive procedure for computing the values of $\{\phi(m, n)\}$. Identity (B.2c), which follows from (B.2a)-(B.2b), tends to simplify and accelerate the process in practice. As for the poles of W^* (cf. Figure 2, they are the (nonzero) solutions of $s - \lambda(1 - e^{-sx}) = 0$ or, equivalently, $(s - \lambda)xe^{(s-\lambda)x} = -\lambda xe^{-\lambda x}$, which is an instance of the equation $ze^z = a$, the solutions of which are given by $W_n(a)$, where W_n is the n th branch of the product logarithm function W_n , [1].

(M/M/1) W^* follows from (PK) and $X^*(s) = \omega/(s + \omega)$. After successive derivations of $W^*(-s)$, which rewrites as $W^*(-s) = [(\omega - \lambda)/\omega] [1 + \lambda/(\omega - \lambda - s)]$, we compute (III.7) and find $w_{a:k} = [(\omega - \lambda)/\omega] [\delta[k] + \lambda(\omega - \lambda - a)^{-k-1}]$, which reduces to (B.3) when $a = 0$.

(M/E_q/1) In (B.5a)-(B.5b), $\varphi^{(q)}(m, n)$ can be understood, for $q \geq 1$ and $n \geq 2m$, as the number of possible outcomes when m ordered collections of n_1, n_2, \dots, n_m unordered objects are respectively picked out of m urns $1, 2, \dots, m$ so that $n_1 + \dots + n_m = n$ and $n_p \geq 2$ for $p = 1, \dots, m$, where each urn p is initially assumed to contain $q-1$ distinct objects plus n_p objects randomly drawn (without repetition) from a common set of n additional distinct objects ($p = 1, \dots, m$). We now turn to show (B.4). It is easy to verify that (B.4) is

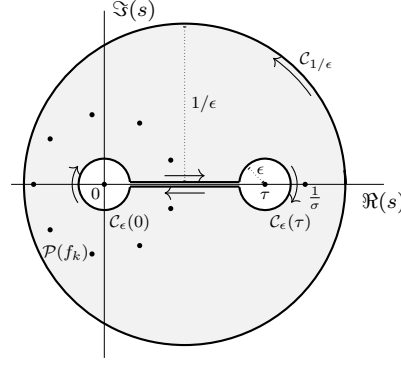


Fig. 6: Singularities of $f_k(s) s^{-1/2}]_{-\pi}(s - \tau)^{-1/2}]_{-\pi}(1 - \sigma s)^{-l(k)}$ and computation of $\hat{\alpha}_k(\sigma)$ by contour integration.

true for $k = 1$. Suppose now that (B.4) holds for $k = 1, \dots, p$, then

$$\begin{aligned}
w_{p+1} &\stackrel{(II.3)}{=} \frac{\lambda \omega}{\omega - q\lambda} \left[\binom{p+q+1}{q-1} \omega^{-(p+2)} + \sum_{t=1}^p \binom{p-t+q+1}{q-1} \omega^{-(p-t+2)} w_t \right] \\
&\stackrel{(B.4)}{=} \frac{\lambda}{\omega^{p+1}(\omega - q\lambda)} \left[\binom{p+q+1}{q-1} + \sum_{t=1}^p \binom{p-t+q+1}{q-1} \omega^t \left(\sum_{l=1}^t \left(\frac{\lambda}{\omega - q\lambda} \right)^l \frac{\varphi^{(q)}(l, t+l)}{\omega^t} \right) \right] \\
&= \frac{\lambda}{\omega^{p+1}(\omega - q\lambda)} \left[\binom{p+q+1}{q-1} + \sum_{l=1}^p \left(\frac{\lambda}{\omega - q\lambda} \right)^l \sum_{t=l}^p \binom{p-t+q+1}{q-1} \varphi^{(q)}(l, t+l) \right] \\
&\stackrel{(B.5a)}{=} \frac{\lambda}{\omega^{p+1}(\omega - q\lambda)} \varphi^{(q)}(1, p+2) \\
&\quad + \frac{\lambda}{\omega^{p+1}(\omega - q\lambda)} \sum_{l=1}^p \left(\frac{\lambda}{\omega - q\lambda} \right)^l \left(\sum_{m=2l}^{l+p} \binom{q+(p+l+2)-m-1}{q-1} \varphi^{(q)}(l, m) \right) \\
&\stackrel{(B.5b)}{=} \frac{1}{\omega^{p+1}} \sum_{l=1}^{p+1} \left(\frac{\lambda}{\omega - q\lambda} \right)^l \varphi^{(q)}(l, p+1+l)
\end{aligned}$$

and (B.4) holds for all k by induction. \square

J Proof of Lemma D.1

Lemma D.1 (Coefficients $\{\hat{\alpha}_k\}$ for quotients of polynomials) *Let g_m and h_n be polynomials of degrees m and n , and consider*

$$f(u) = \frac{g_m(u)}{h_n(u)}, \quad \forall u \in \mathbb{R}_{\geq 0}.$$

For $\tau > 0$, recall (IV.19) and define $f_k(s) = f(s) p_k(2s/\tau - 1)$ under the assumption $\mathcal{P}(f_k) \cap [0, \infty) = \emptyset$. The Fourier coefficients (IV.21) of f satisfy, for $k \geq 0$,

$$\hat{\alpha}_k = \sqrt{\pi} \sum_{q=0}^{l(k)} \frac{\zeta_{-q} (-\tau)^q}{q! \Gamma(\frac{1}{2} - q)} - \sum_{a \in \mathcal{P}(f_k)} \text{Res}_{s=a} \left(f_k(s) s^{-\frac{1}{2}}]_{-\pi}(s - \tau)^{-\frac{1}{2}}]_{-\pi} \right), \quad (\text{D.9})$$

where $l(k) = \max(0, m - n + k)$ is the largest nonnegative integer l such that $\lim_{s \rightarrow 0} s^l f_k(\frac{1}{s})$ is finite, and $\{\zeta_q\}$ are the coefficients of the Laurent series at $+\infty$ of the analytic continuation of f_k , i.e.,

$$\zeta_q = \frac{1}{(l(k) + q)!} \lim_{s \rightarrow 0} \frac{d^{l(k)+q}}{ds^{l(k)+q}} \left[s^{l(k)} f_k \left(\frac{1}{s} \right) \right], \quad (q = -l(k), \dots, \infty). \quad (\text{J.1})$$

Proof We would like to compute

$$\tilde{\alpha}_k \stackrel{(IV.21)}{=} \frac{1}{\pi} \int_0^\tau \frac{g_m(u) p_k \left(\frac{2u}{\tau} - 1 \right)}{h_n(u) \sqrt{u(\tau-u)}} du, \quad (J.2)$$

for any $k \in \mathbb{N}_{\geq 0}$. For $l \in \mathbb{N}_{\geq 0}$ and $\sigma \ll 1/\tau$, we define the altered coefficient

$$\hat{\alpha}_k(\sigma) = \frac{1}{\pi} \int_0^\tau \frac{g_m(u) p_k \left(\frac{2u}{\tau} - 1 \right)}{h_n(u) (1-\sigma u)^l \sqrt{u(\tau-u)}} du,$$

which has the property to converge to $\Re(\tilde{\alpha}_k)$ as $\sigma \downarrow 0$. Indeed, since by assumption f and p_k are bounded on $[0, \tau]$, the integrand of (J.2) is absolutely integrable on the interval. As soon as $\sigma \leq 1/(2\tau)$, one has $|(1-\sigma u)^{-l}| \leq 2^l$ and the conditions of Lebesgue's dominated convergence theorem are met. Now, consider the contour integral in the complex domain

$$\begin{aligned} \gamma_k(\sigma) &= \frac{1}{\pi} \oint_{\mathcal{C}} \left(\frac{g_m(s) p_k \left(\frac{2s}{\tau} - 1 \right)}{h_n(s) (1-\sigma s)^l} \right) s^{-\frac{1}{2}]_{-\pi}} (s-\tau)^{-\frac{1}{2}]_{-\pi}} ds \\ &= \frac{1}{\pi} \left(\oint_{\mathcal{C}_{1/\epsilon}} + \oint_{\mathcal{C}_\epsilon(0)} + \int_\epsilon^{\tau-\epsilon} + \oint_{\mathcal{C}_\epsilon(\tau)} + \int_{\tau-\epsilon}^\epsilon \right) \frac{f_k(s) s^{-\frac{1}{2}]_{-\pi}} (s-\tau)^{-\frac{1}{2}]_{-\pi}}{(1-\sigma s)^l} ds, \end{aligned} \quad (J.3)$$

where $s^\alpha]_{-\pi} = e^{\alpha(\ln|s| + i \arg s]_{-\pi})}$ denotes the principal branch of the complex exponentiation, and the circles $\mathcal{C}_{1/\epsilon}$, $\mathcal{C}_\epsilon(0)$, and $\mathcal{C}_\epsilon(\tau)$ are understood as in Figure 6 with $\epsilon > 0$ chosen small enough so that $1/\sigma$ and the poles of f_k all lie between the outer contour \mathcal{C}_ϵ and the inner contour.

We proceed to compute $\gamma_k(\sigma)$ term by term. Let $l = \max(0, m - n + k)$. First notice that

$$\lim_{s \rightarrow \infty} (s - \frac{\tau}{2}) \left(\frac{f_k(s)}{(1-\sigma s)^l} \right) s^{-\frac{1}{2}]_{-\pi}} (s-\tau)^{-\frac{1}{2}]_{-\pi}} \stackrel{(J.1)}{=} \frac{\zeta_{-l}}{(-\sigma)^l}$$

where we consider that $\zeta_{-l} = 0$ whenever $m - n + k < 0$. By using Jordan's second lemma [3, §3.1.4, Theorem 2] (or, equivalently, by computing the residue at ∞), we find

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \oint_{\mathcal{C}_{1/\epsilon}} \frac{f_k(s) s^{-\frac{1}{2}]_{-\pi}} (s-\tau)^{-\frac{1}{2}]_{-\pi}}{(1-\sigma s)^l} ds = \frac{2i \zeta_{-l}}{(-\sigma)^l}. \quad (J.4)$$

Besides, $\lim_{\epsilon \rightarrow a} (s-a) f_k(s) s^{-\frac{1}{2}]_{-\pi}} (s-\tau)^{-\frac{1}{2}]_{-\pi}} (1-\sigma s)^{-l} = 0$ for $a = 0, \tau$ as a consequence of the assumption $0, \tau \notin \mathcal{P}(f_k)$. It follows from Jordan's first lemma [3, §3.1.4, Theorem 1] that

$$\lim_{s \rightarrow 0, \tau} \frac{1}{\pi} \left(\oint_{\mathcal{C}_\epsilon(0)} + \oint_{\mathcal{C}_\epsilon(\tau)} \right) \frac{f_k(s) s^{-\frac{1}{2}]_{-\pi}} (s-\tau)^{-\frac{1}{2}]_{-\pi}}{(1-\sigma s)^l} ds = 0. \quad (J.5)$$

Lastly, by inspection of $s^{-\frac{1}{2}]_{-\pi}} (s-\tau)^{-\frac{1}{2}]_{-\pi}}$ right above and below the segment $(0, \tau)$, it can be seen that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \left(\int_\epsilon^{\tau-\epsilon} + \int_{\tau-\epsilon}^\epsilon \right) \frac{f_k(s) s^{-\frac{1}{2}]_{-\pi}} (s-\tau)^{-\frac{1}{2}]_{-\pi}}{(1-\sigma s)^l} ds = -2i \hat{\alpha}_k(\sigma). \quad (J.6)$$

On the other hand the residue theorem gives

$$\gamma_k(\sigma) = \left(\frac{1}{\pi} \right) 2i\pi \sum_{a \in \mathcal{P}(f_k) \cup \{ \frac{1}{\sigma} \}} \text{Res}_{s=a} \left(\frac{f_k(s) s^{-\frac{1}{2}]_{-\pi}} (s-\tau)^{-\frac{1}{2}]_{-\pi}}{(1-\sigma s)^l} \right) \quad (J.7)$$

We draw our attention to the residue at $1/\sigma$. Using the Taylor development of $(1-x\tau)^{-1/2-j}$ at $x = 0$ we find, for $\sigma < \tau$ and $t \in \mathbb{N}_{\geq 0}$,

$$\begin{aligned} \lim_{s \rightarrow \frac{1}{\sigma}} \frac{d^t}{ds^t} \left[s^{-\frac{1}{2}]_{-\pi}} (s-\tau)^{-\frac{1}{2}]_{-\pi}} \right] &= \sigma^{t+1} \sum_{j=0}^t \binom{t}{j} \frac{\Gamma(\frac{1}{2})^2 (1-\sigma\tau)^{\frac{1}{2}-j}}{\Gamma(\frac{1}{2}-j) \Gamma(\frac{1}{2}-t+j)} \\ &= \sigma^{t+1} \sum_{q=0}^\infty \left[\sum_{j=0}^t \binom{t}{j} \frac{\Gamma(\frac{1}{2})^2}{\Gamma(\frac{1}{2}-q-j) \Gamma(\frac{1}{2}-t+j)} \right] \frac{(-\sigma\tau)^q}{q!}. \end{aligned} \quad (J.8)$$

For large s , the value of $f_k(s)$ is given by the Laurent series expansion of f_k at $+\infty$, i.e.,

$$f_k(s) = \sum_{q=-l}^{\infty} \zeta_q s^{-q}, \quad (0 \ll |s| < \infty). \quad (\text{J.9})$$

Then,

$$\begin{aligned} & \text{Res}_{s=\frac{1}{\sigma}} \left(\frac{f_k(s) s^{-\frac{1}{2}}]_{-\pi} (s-\tau)^{-\frac{1}{2}}]_{-\pi}}{(1-\sigma s)^l} \right) \\ &= \lim_{s \rightarrow \frac{1}{\sigma}} \frac{1}{(l-1)!} \frac{d^{l-1}}{ds^{l-1}} \left[\left(s - \frac{1}{\sigma} \right)^l \left(\frac{f_k(s) s^{-\frac{1}{2}}]_{-\pi} (s-\tau)^{-\frac{1}{2}}]_{-\pi}}{(1-\sigma s)^l} \right) \right] \\ &\stackrel{(\text{J.8})}{=} \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{t=0}^{l-1} \binom{l-1}{t} \frac{f_k^{(l-1-t)}(\frac{1}{\sigma})}{(\frac{1}{\sigma})^{t+1}} \sum_{q=0}^{\infty} \left[\sum_{j=0}^t \frac{\binom{t}{j} \Gamma(\frac{1}{2})^2}{\Gamma(\frac{1}{2}-q-j) \Gamma(\frac{1}{2}-t+j)} \right] \frac{(-\sigma\tau)^q}{q!} \\ &= \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \frac{(-\sigma\tau)^q}{q!} \sum_{t=0}^{l-1} \frac{\binom{l-1}{t} f_k^{(l-1-t)}(\frac{1}{\sigma})}{(\frac{1}{\sigma})^{t+1}} \sum_{j=0}^{l-1} \frac{\binom{t}{j} \Gamma(\frac{1}{2})^2}{\Gamma(\frac{1}{2}-q-j) \Gamma(\frac{1}{2}-t+j)} \\ &= \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \frac{(-\sigma\tau)^q}{q!} \sum_{j=0}^{l-1} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q-j)} \sum_{t=j}^{l-1} \frac{\binom{l-1}{t} \binom{t}{j} \Gamma(\frac{1}{2}) f_k^{(l-1-t)}(\frac{1}{\sigma})}{(\frac{1}{\sigma})^{1+t} \Gamma(\frac{1}{2}-t+j)} \\ &= \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \frac{(-\tau)^q}{q!} \sum_{j=0}^{l-1} \frac{\binom{l-1}{j} \Gamma(\frac{1}{2}) (\frac{1}{\sigma})^{-\frac{1}{2}-q-j}}{\Gamma(\frac{1}{2}-q-j)} \sum_{t=0}^{l-1-j} \frac{\binom{l-1-j}{t} \Gamma(\frac{1}{2}) f_k^{(l-1-j-t)}(\frac{1}{\sigma})}{(\frac{1}{\sigma})^{\frac{1}{2}+t} \Gamma(\frac{1}{2}-t)} \\ &= \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \frac{\Gamma(\frac{1}{2}) (-\tau)^q}{q! \Gamma(\frac{1}{2}-q)} \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{d^{l-1-j}}{dx^{l-1-j}} \left[\frac{x^{-q}}{\sqrt{x}} \right]_{x=\frac{1}{\sigma}} \frac{d^{l-1-j}}{dx^{l-1-j}} \left[\frac{f_k(x)}{\sqrt{x}} \right]_{x=\frac{1}{\sigma}} \\ &= \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \frac{\Gamma(\frac{1}{2}) (-\tau)^q}{q! \Gamma(\frac{1}{2}-q)} \frac{d^{l-1}}{dx^{l-1}} \left[x^{-(q+1)} f_k(x) \right]_{x=\frac{1}{\sigma}}. \end{aligned}$$

For small σ , the function f_k is analytic in a neighborhood of $1/\sigma$, and so is $s^{-(q+1)} f_k(s)$ for any q . It follows that the derivation in (J.10) applies term by term to the Laurent series (J.9), and we find

$$\begin{aligned} & \text{Res}_{s=\frac{1}{\sigma}} \left(\frac{f_k(s) s^{-\frac{1}{2}}]_{-\pi} (s-\tau)^{-\frac{1}{2}}]_{-\pi}}{(1-\sigma s)^l} \right) \\ &\stackrel{(\text{J.9})}{=} \frac{(-\frac{1}{\sigma})^l}{(l-1)!} \sum_{q=0}^{\infty} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(-\tau)^q}{q!} \frac{d^{l-1}}{dx^{l-1}} \left[\sum_{j=-l}^{\infty} \zeta_j x^{-(j+q+1)} \right]_{x=\frac{1}{\sigma}} \\ &= \frac{\zeta_{-l}}{(-\sigma)^l} - \sum_{q=0}^{\infty} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(-\sigma\tau)^q}{q!} \sum_{j=-\min(q,l)}^{\infty} \binom{l-1+q+j}{l-1} \zeta_j \sigma^j. \end{aligned} \quad (\text{J.10})$$

Observe on the other hand that, for $q+j \geq 0$,

$$\left| \binom{l-1+q+j}{l-1} \right| \leq \frac{(l-1+q+j)^{l-1}}{(l-1)!} \leq \sum_{t=0}^{\infty} \frac{(l-1+q+j)^t}{t!} \leq e^{l-1+q+j}.$$

Hence, for small σ ,

$$\sum_{j=-\min(q,l)}^{\infty} \binom{l-1+q+j}{l-1} \zeta_j \sigma^j \leq e^{l-1+q} \sum_{j=-l}^{\infty} |\zeta_j| (\sigma e)^j \leq \kappa(\sigma, l) e^{l-1+q}, \quad (\text{J.11})$$

with $\kappa(\sigma, l) < \infty$ by absolute convergence of the above series. Consequently,

$$\sum_{q=0}^{\infty} \sum_{j=-\min(q,l)}^{\infty} \left| \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(-\sigma\tau)^q}{q!} \binom{l-1+q+j}{l-1} \zeta_j \sigma^j \right| \stackrel{(\text{J.11})}{\leq} \kappa(\sigma, l) e^{l-1} \sum_{q=0}^{\infty} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(\sigma\tau)^q}{q!} e^q$$

is a finite quantity as it passes the ratio test for $\sigma < 1/\tau$ —this can be shown using Stirling's formula. It follows from Fubini's theorem that the summation order in (J.10) can be permuted. By setting $t = j + q$ we find

$$\begin{aligned} & \text{Res}_{s=\frac{1}{\sigma}} \left(\frac{f_k(s) s^{-\frac{1}{2}}]_{-\pi} (s-\tau)^{-\frac{1}{2}}]_{-\pi}}{(1-\sigma s)^l} \right) \\ &= \frac{\zeta_{-l}}{(-\sigma)^l} - \sum_{q=0}^{\infty} \sum_{t=\max(0, q-l)}^{\infty} \left[\zeta_{t-q} \binom{l-1+t}{l-1} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(-\tau)^q}{q!} \right] \sigma^t \\ &= \frac{\zeta_{-l}}{(-\sigma)^l} - \sum_{q=0}^l \zeta_{-q} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)} \right) \frac{(-\tau)^q}{q!} + O(\sigma). \end{aligned} \quad (\text{J.12})$$

We eventually obtain (D.9) by combining (J.3) and (J.7), together with the intermediate results (J.4), (J.5), (J.6), and (J.12), and letting $\sigma \rightarrow 0$. \square

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