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# VOLUME OF THE MINKOWSKI SUMS OF STAR-SHAPED SETS

MATTHIEU FRADELIZI, ZSOLT LÁNGI, AND ARTEM ZVAVITCH

ABSTRACT. For a compact set  $A \subset \mathbb{R}^d$  and an integer  $k \geq 1$ , let us denote by

$$A[k] = \{a_1 + \dots + a_k : a_1, \dots, a_k \in A\} = \sum_{i=1}^k A$$

the Minkowski sum of  $k$  copies of  $A$ . A theorem of Shapley, Folkman and Starr (1969) states that  $\frac{1}{k}A[k]$  converges to the convex hull of  $A$  in Hausdorff distance as  $k$  tends to infinity. Bobkov, Madiman and Wang (2011) conjectured that the volume of  $\frac{1}{k}A[k]$  is non-decreasing in  $k$ , or in other words, in terms of the volume deficit between the convex hull of  $A$  and  $\frac{1}{k}A[k]$ , this convergence is monotone. It was proved by Fradelizi, Madiman, Marsiglietti and Zvavitch (2016) that this conjecture holds true if  $d = 1$  but fails for any  $d \geq 12$ . In this paper we show that the conjecture is true for any star-shaped set  $A \subset \mathbb{R}^d$  for arbitrary dimensions  $d \geq 1$  under the condition  $k \geq d - 1$ . In addition, we investigate the conjecture for connected sets and present a counterexample to a generalization of the conjecture to the Minkowski sum of possibly distinct sets in  $\mathbb{R}^d$ , for any  $d \geq 7$ .

## 1. INTRODUCTION

The Minkowski sum of two sets  $K, L \subset \mathbb{R}^d$  is defined as  $K + L = \{x + y : x \in K, y \in L\}$ , where, for brevity, we set  $A[k] = \sum_{i=1}^k A$ , for any  $k \in \mathbb{N}$  and any compact set  $A \subset \mathbb{R}^d$ . Since Minkowski sum preserves the convexity of the summands and the set  $\frac{1}{k}A[k]$  consists in some particular convex combinations of elements of  $A$ , the containment  $\frac{1}{k}A[k] \subseteq \text{conv } A$ , and, for the special case of convex sets, the equality  $\frac{1}{k}A[k] = \text{conv } A$  trivially holds; here  $\text{conv } A$  denotes the convex hull of  $A$ . These observations suggest that for any compact set  $A$ , the set  $\frac{1}{k}A[k]$  looks “more convex” for larger values of  $k$ . This intuition was formalized by Starr [St1, St2], crediting also Shapley and Folkman, and independently by Emerson and Greenleaf [EG], by proving that the set  $\frac{1}{k}A[k]$  approaches  $\text{conv } A$  in Hausdorff distance as  $k$  approaches infinity and by giving bounds on the speed of this convergence (we refer to [FMMZ2] for more discussion of this fact).

A further step in the investigation of the sequence  $\{\frac{1}{k}A[k]\}$  is to examine the monotonicity of this convergence. Whereas this sequence is clearly not monotonous

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in terms of containment, the main object of this paper is the following conjecture of Bobkov, Madiman, Wang [BMW], relating the volumes of the elements of the sequence, and in which  $\text{vol}(K)$  denotes the Lebesgue measure (volume) of the measurable set  $K \subset \mathbb{R}^d$ .

**Conjecture 1** (Bobkov-Madiman-Wang). Let  $A$  be a compact set in  $\mathbb{R}^d$  for some  $d \in \mathbb{N}$ . Then the sequence

$$\left\{ \text{vol} \left( \frac{1}{k} A[k] \right) \right\}_{k \geq 1}$$

is non-decreasing in  $k$ .

Equivalently, this conjecture asks whether for any integer  $k \geq 1$  and compact set  $A \subset \mathbb{R}^d$ , the following inequality holds

$$(1) \quad \text{vol} \left( \frac{1}{k} A[k] \right) \leq \text{vol} \left( \frac{1}{k+1} A[k+1] \right).$$

This inequality trivially holds for any compact set  $A$  if  $k = 1$  since  $A \subset \frac{1}{2} A[2]$ . In the same way, it is easy to find monotone subsequences of the sequence  $\{\text{vol}(\frac{1}{k} A[k])\}_{k \geq 1}$  by the same argument; one such example is  $\{\text{vol}(\frac{1}{2^m} A[2^m])\}_{m \geq 0}$ . On the other hand, even the first nontrivial case; that is, the inequality  $\text{vol}(\frac{1}{2} A[2]) \leq \text{vol}(\frac{1}{3} A[3])$  seems to require new methods to approach. Conjecture 1 was partially resolved in [FMMZ1, FMMZ2], where, following the approach of [GMR], the authors proved it for any 1-dimensional compact set  $A$ , but constructed counterexamples in  $\mathbb{R}^d$  for any  $d \geq 12$ . More precisely, they showed that for every  $k \geq 2$ , there is  $d_k \in \mathbb{N}$  such that for every  $d \geq d_k$  there is a compact set  $A \subset \mathbb{R}^d$  such that  $\text{vol}(\frac{1}{k} A[k]) > \text{vol}(\frac{1}{k+1} A[k+1])$ . In particular, one has  $d_2 = 12$ , whence Conjecture 1 fails for  $\mathbb{R}^d$  if  $d \geq 12$ .

Our goal is to find additional conditions on  $A$  and  $k$  under which the statement in Conjecture 1, or more precisely when the inequality (1) is satisfied.

In the paper, for any set  $A \subset \mathbb{R}^d$  we denote by  $\dim A$  the dimension of the smallest affine subspace containing  $A$ , and for any  $p, q \in \mathbb{R}^d$ , we denote the closed segment with endpoints  $p, q$  by  $[p, q]$ . To state our main result, let us recall the following well-known concept.

**Definition 1.** A nonempty set  $S \subset \mathbb{R}^d$  is called *star-shaped* with respect to a point  $p$  if for any  $q \in S$ , we have  $[p, q] \subseteq S$ .

Our main result is the following.

**Theorem 1.** *Let  $d \geq 2$  and  $k \geq d - 1$  be positive integers. Then for any compact, star-shaped set  $S \subset \mathbb{R}^d$  we have*

$$\text{vol} \left( \frac{1}{k+1} S[k+1] \right) \geq \text{vol} \left( \frac{1}{k} S[k] \right),$$

*with equality if only if  $\frac{1}{k} S[k] = \text{conv}(S)$ .*

We feel it is worth noting that the compact sets  $A$  constructed in [FMMZ2] as counterexamples to Conjecture 1 are star-shaped, which makes Theorem 1 fairly unexpected.

We prove Theorem 1 in Section 2. In Section 3 we adapt our techniques to investigate connected sets. Our main result in this section is summarized in Theorem 2. Finally, in Section 4 we collect some additional remarks and questions, and, in particular, we construct low dimensional counterexamples to a generalization of Conjecture 1, which also appeared in [BMW].

## 2. CONJECTURE 1 FOR STAR-SHAPED SETS: THE PROOF OF THEOREM 1

We start this section with a couple of Lemmata which are needed for the proof. Throughout this section, we denote  $X_d(t) = \{(x_1, \dots, x_d) \in \mathbb{N}^d : x_1 + \dots + x_d = t\}$  and  $N_d(t) = \text{card } X_d(t)$  to be the number of elements of  $X_d(t)$ .

**Lemma 1.** *For any integer  $t \geq 1$ , and  $d \geq 2$ , we have  $N_d(t) = \binom{t+d-1}{d-1}$ .*

*Proof.* If  $d = 2$ , then, clearly,  $N_2(t) = t + 1 = \binom{t+2-1}{1}$ . On the other hand, by induction, we have

$$N_d(t) = \sum_{s=0}^t N_{d-1}(s) = \sum_{s=0}^t \binom{s+d-2}{d-2} = \binom{t+d-1}{d-1}.$$

□

**Lemma 2.** *Let  $o$  be the origin of  $\mathbb{R}^d$ ,  $(p_1, \dots, p_d)$  be a basis of  $\mathbb{R}^d$ , and, let  $B = \bigcup_{i=1}^d [o, p_i]$ . Consider  $M \subset \mathbb{R}^d$  such that  $B[k] \subseteq M \subseteq k \text{ conv}(B)$ , then*

$$(2) \quad \text{vol} \left( \frac{1}{k+1} (M + B) \right) \geq \text{vol} \left( \frac{1}{k} M \right),$$

where, equality holds if and only if  $M = k \text{ conv}(B)$ . Furthermore, if  $\text{vol} \left( \frac{1}{k} M \right) \geq \text{vol} \left( \frac{1}{k+1} (M + B) \right) - \delta$  for some  $\delta \geq 0$ , then  $\text{vol}(M) \geq \text{vol}(k \text{ conv}(B)) - C(d, k)\delta$  for some constant  $C(d, k)$  depending only on  $d$  and  $k$ .

*Proof.* Since the inequality (2) is independent of a non-degenerate linear transformation applied to  $B$  and  $M$  simultaneously, we may assume that  $(p_1, \dots, p_d)$  is the canonical basis of  $\mathbb{R}^d$ . Let

$$V(t) = \text{vol}\{(x_1, \dots, x_d) \in [0, 1]^d : x_1 + \dots + x_d \leq t\}.$$

Let  $C_i = i + [0, 1]^d$ ,  $i \in \mathbb{Z}^d$  be the unit cube cells of the lattice  $\mathbb{Z}^d$ , and set  $\mu_i = \text{vol}(C_i \cap M)$ , and  $\lambda_i = \text{vol}(C_i \cap (M + B))$ .

Note that for any  $i \in X_d(t)$ ,  $\text{vol}(C_i \cap k \text{ conv}(B))$  is independent of  $i$ , namely it is equal to 1, if  $t \leq k-d$ , and to  $V(k-t)$  if  $t = k-d+1, \dots, k-1$ . A similar statement holds for  $\text{vol}(C_i \cap (k+1) \text{ conv}(B))$ . The number of unit cells contained in  $k \text{ conv}(B)$  is equal to the number of the solutions of the inequality  $x_1 + x_2 + \dots + x_d \leq k$ , where each variable is a positive integer, and thus, it is  $\binom{k}{d}$ . This yields that the volume of these cells is  $k^d V$ , where  $k^d = k(k-1) \dots (k-d+1)$ , and  $V = \text{vol}(\text{conv } B) = \frac{1}{d!}$ .

Thus,

$$(3) \quad \text{vol}(M) = k^d V + \sum_{t=k-d+1}^{k-1} \sum_{i \in X_t} \mu_i,$$

and

$$\text{vol}(M + B) = (k + 1)^d V + \sum_{t=k-d+2}^k \sum_{i \in X_t} \lambda_i.$$

In the following step, we give a lower bound on the  $\lambda_i$ 's depending on the values of the  $\mu_i$ 's. We say that  $i \in X_d(t)$  and  $i' \in X_d(t+1)$  are *adjacent* if the corresponding cells  $C_i$  and  $C_{i'}$  have a common facet, or in other words, if  $i' - i$  coincides with one of the standard basis vectors  $p_j$ . In this case we write  $ii' \in I$ . Let  $i \in X_d(t)$ , and let  $S = M \cap C_i$ . Then, for every  $j = 1, 2, \dots, d$ ,  $S + p_j \subset (M + B) \cap C_{i'}$  with  $i' = i + p_j$ . Thus, for any  $i \in X_d(t+1)$ ,

$$(4) \quad \lambda_i \geq \max\{\mu_{i'} : i' \in X_d(t) \text{ is adjacent to } i\}.$$

Note that the right-hand side of this inequality is not less than any convex combination of the corresponding  $\mu_{i'}$ s. We show that, using a suitable convex combination for each  $i \in X_d(t+1)$  this inequality implies that

$$(5) \quad \sum_{i \in X_d(t+1)} \lambda_i \geq \frac{t+d}{t+1} \sum_{i \in X_t} \mu_i.$$

Consider some  $i = (i_1, i_2, \dots, i_d) \in X_d(t+1)$ . Then the indices in  $X_d(t)$  adjacent to  $i$  are all of the form  $i - p_j$  for some  $j = 1, 2, \dots, d$ . Furthermore,  $i - p_j$  is adjacent to  $i$  iff  $i_j \geq 1$ , or in other words, iff  $i_j \neq 0$ . Now, for any  $i' \in X_d(t)$  adjacent to  $i$  we set  $\alpha_{ii'} = \frac{i_j}{t+1}$ , where  $i - i' = p_j$  (cf. Figure 1). Then, since  $i \in X_d(t+1)$ , we clearly have  $1 = \sum_{j=1}^d \frac{i_j}{t+1} = \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'}$ . Thus, by (4), we have

$$(6) \quad \lambda_i \geq \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'} \mu_{i'}$$

for all  $i \in X_d(t+1)$ . Now, let  $i' \in X_d(t)$ , and  $i' = (i'_1, i'_2, \dots, i'_d)$ . Then the indices in  $X_d(t+1)$  adjacent to  $i'$  are exactly those of the form  $i' + p_j$  for some  $j = 1, 2, \dots, d$ . Hence,

$$(7) \quad \sum_{i \in X_d(t+1), ii' \in I} \alpha_{ii'} = \sum_{j=1}^d \frac{i'_j + 1}{t+1} = \frac{t+d}{t+1}.$$

Finally, by (6) and (7)

$$\begin{aligned} \sum_{i \in X_d(t+1)} \lambda_i &\geq \sum_{i \in X_d(t+1)} \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'} \mu_{i'} = \\ &= \sum_{i' \in X_d(t)} \left( \sum_{i \in X_d(t+1), ii' \in I} \alpha_{ii'} \right) \mu_{i'} = \frac{t+d}{t+1} \sum_{i' \in X_d(t)} \mu_{i'}. \end{aligned}$$

Note that the sequence  $\left\{ \frac{t+d}{t+1} \right\}$ , where  $t = 0, 1, 2, \dots$ , is strictly decreasing. Hence, using the fact that if  $i \in X_d(t)$ , then  $\mu_i \leq V(k-t)$ , it follows that

$$(8) \quad \text{vol}(M + B) \geq (k+1)^d V + \frac{k+1}{k-d+2} \sum_{t=k-d+1}^{k-1} \sum_{i \in X_d(t)} \mu_i$$

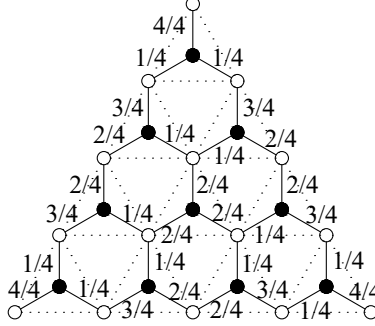


FIGURE 1. Illustration on choosing the weights if  $d = 3$  and  $t = 3$ . The black and empty dots represent the elements of the set  $X_3(3)$  and  $X_3(4)$ , respectively. Dots illustrating adjacent indices are connected by a segment. The weight assigned to the segment connecting the dots representing  $i$  and  $i'$  is equal to  $\alpha_{ii'}$ .

$$-\frac{k+1}{k-d+2} \sum_{t=k-d+1}^{k-1} V(k-t)N_d(t) + \sum_{t=k-d+1}^{k-1} \frac{t+d}{t+1} V(k-t)N_d(t).$$

Observe that  $\sum_{t=k-d+1}^{k-1} V(k-t)N_d(t)$  is the volume of the part of  $k \operatorname{conv}(B)$  belonging to cells that are not contained in it, and thus, it is equal to  $(k^d - k^d)V$ . Similarly,

$$\begin{aligned} \sum_{t=k-d+1}^{k-1} \frac{t+d}{t+1} V(k-t)N_d(t) &= \sum_{t=k-d+1}^{k-1} \frac{t+d}{t+1} V(k-t) \binom{t+d-1}{d-1} = \\ \sum_{t=k-d+1}^{k-1} \sum_{i \in X_d(t)} V(k-t) \binom{t+d}{d-1} &= \sum_{t'=k-d+2}^k \sum_{i \in X_d(t')} V(k+1-t')N_d(t') \end{aligned}$$

is the volume of the part of  $(k+1) \operatorname{conv}(B)$  belonging to cells that are not contained in it. Thus, it is equal to  $((k+1)^d - (k+1)^d)V$ . Substituting these into (8) and by (3), we obtain

$$\begin{aligned} \operatorname{vol}(M+B) &\geq (k+1)^d V + \frac{k+1}{k-d+2} (\operatorname{vol}(M) - k^d V) - \frac{k+1}{k-d+2} (k^d - k^d)V \\ &\quad + ((k+1)^d - (k+1)^d)V \\ &= \frac{k+1}{k-d+2} \operatorname{vol}(M) + \left( (k+1)^d - \frac{k+1}{k-d+2} k^d \right) V. \end{aligned}$$

Thus,

$$(9) \quad \operatorname{vol} \left( \frac{1}{k+1} (M+B) \right) \geq \frac{k^d}{(k-d+2)(k+1)^{d-1}} \operatorname{vol} \left( \frac{1}{k} M \right) + \left( 1 - \frac{k^d}{(k-d+2)(k+1)^{d-1}} \right) V.$$

Since  $\operatorname{vol} \left( \frac{1}{k} M \right) \leq V$ , the first inequality of the lemma readily follows.

Now we prove the equality case. By (9), equality in the lemma implies that  $\text{vol}(\frac{1}{k}M) = V$ , or equivalently,  $\text{vol}(k \text{conv}(B) \setminus M) = 0$ . Note that since  $\text{vol}(k \text{conv}(B)) > 0$ , its interior is not empty. Thus,  $k \text{conv}(B)$  is equal to the closure of its interior. On the other hand,  $\text{vol}(k \text{conv}(B) \setminus M) = 0$  implies that  $\text{int}(k \text{conv} B) \subset M$ , but as  $M$  is compact,  $M = k \text{conv} B$  follows.

Finally, if  $\text{vol}(\frac{1}{k+1}(M+B)) - \delta \leq \text{vol}(\frac{1}{k}M)$ , then in the same way (9) yields the inequality  $\text{vol}(M) \geq \text{vol}(k \text{conv}(B)) - C(d, k)\delta$ , with  $C(d, k) = \frac{k^d}{1 - \frac{k^d}{(k-d+2)(k+1)^{d-1}}}$ .  $\square$

*Proof of Theorem 1.* Without loss of generality, we may assume that  $S$  is star-shaped with respect to the origin. Let  $\varepsilon > 0$  be an arbitrary positive number. By Carathéodory's theorem, we may choose a finite point set  $A_0 \subset S$  such that  $\text{vol}(\text{conv}(S)) - \varepsilon \leq \text{vol}(\text{conv}(A_0))$ , and without loss of generality, we may assume that the points of  $A_0$  are in convex position. Clearly, the star-shaped set  $A = \bigcup_{a \in A_0} [o, a]$  is a subset of  $S$ , satisfying  $\text{vol}(\text{conv}(S)) - \varepsilon \leq \text{vol}(\text{conv}(A))$ . Consider a simplicial decomposition  $\mathcal{F}$  of the boundary of  $\text{conv}(A)$  such that all vertices of  $\mathcal{F}$  are vertices of  $\text{conv}(A)$ . Let the  $(d-1)$ -dimensional faces of  $\mathcal{F}$  be  $F_1, F_2, \dots, F_m$ , and for  $j = 1, 2, \dots, m$ , let  $B_j = \bigcup_{t=1}^d [o, p_t^j]$ , where  $p_1^j, p_2^j, \dots, p_d^j$  are the vertices of  $F_j$ . Then  $B_j \subseteq S$  for all values of  $j$ , the sets  $\text{conv}(B_j)$  are mutually non-overlapping, and  $\text{conv}(A) = \bigcup_{j=1}^m \text{conv}(B_j)$ . Finally, let  $M_j = S[k] \cap (k \text{conv}(B_j))$ . Then, since  $B_j \subseteq S$ , we have  $B_j[k] \subseteq M_j \subseteq (k \text{conv}(B_j))$ . Thus, Lemma 2 implies that  $\text{vol}(\frac{1}{k+1}(M_j + B_j)) \geq \text{vol}(\frac{1}{k}M_j)$ , or in other words,  $\text{vol}(M_j + B_j) \geq \frac{(k+1)^d}{k^d} \text{vol}(M_j)$ . Thus, we have

$$\begin{aligned} \frac{(k+1)^d}{k^d} \text{vol}(S[k] \cap \text{conv}(kA)) &= \sum_{j=1}^m \frac{(k+1)^d}{k^d} \text{vol}(M_j) \\ &\leq \sum_{j=1}^m \text{vol}(M_j + B_j) \leq \text{vol}(S[k+1]). \end{aligned}$$

On the other hand, since  $0 \leq \text{vol}(\text{conv}(S)) - \text{vol}(\text{conv}(A)) \leq \varepsilon$ , we have  $0 < \text{vol}((S[k] \setminus \text{conv}(kA))) \leq k^d \varepsilon$ , implying that

$$\text{vol}\left(\frac{1}{k}S[k]\right) - \varepsilon \leq \text{vol}\left(\frac{1}{k+1}S[k+1]\right).$$

This inequality is satisfied for all positive  $\varepsilon$ , and thus, the inequality part of Theorem 1 holds.

Now, assume that

$$\text{vol}\left(\frac{1}{k}S[k]\right) = \text{vol}\left(\frac{1}{k+1}S[k+1]\right).$$

Then, since  $\text{vol}((S[k] \setminus \text{conv}(kA))) \leq k^d \varepsilon$ , it follows that  $\text{vol}(S[k+1]) - k^d \varepsilon \leq \frac{(k+1)^d}{k^d} \text{vol}(S[k] \cap \text{conv}(kA))$ , and thus,

$$\sum_{j=1}^m \left( \text{vol}\left(\frac{1}{k+1}(M_j + B_j)\right) - \text{vol}\left(\frac{1}{k}M_j\right) \right) \leq \frac{\varepsilon}{(k+1)^d}.$$

For  $j = 1, 2, \dots, m$ , set  $\delta_j = \text{vol}\left(\frac{1}{k+1}(M_j + B_j)\right) - \text{vol}\left(\frac{1}{k}M_j\right)$ . Then, clearly  $\sum \delta_j \leq \frac{\varepsilon}{(k+1)^d}$ . On the other hand, by Lemma 2, for every  $j = 1, 2, \dots, m$ , we have  $\text{vol}(k \text{ conv } B_j) - \text{vol}(M_j) \leq C(k, d)\delta_j$  for some constant depending only on  $k$  and  $d$ . Thus, it follows that

$$\frac{\varepsilon C(k, d)}{(k+1)^d} \geq \text{vol}(\text{conv}(kA)) - \text{vol}(S[k] \cap \text{conv}(kA)),$$

implying that  $\varepsilon\left(k^d + \frac{C(k, d)}{(k+1)^d}\right) \geq \text{vol}(\text{conv}(kS)) - \text{vol}(S[k])$ . This inequality holds for any value  $\varepsilon > 0$ , and hence,  $\text{vol}(\text{conv}(S)) = \text{vol}\left(\frac{1}{k}S[k]\right)$ , or equivalently,  $\text{vol}\left(\text{conv}(S) \setminus \frac{1}{k}S[k]\right) = 0$ . Since  $\text{conv}(S)$  is a compact, convex set with nonempty interior, and  $\frac{1}{k}S[k]$  is compact, to show the equality  $\text{conv}(S) = \frac{1}{k}S[k]$ , we may apply the argument at the end of the proof of Lemma 2.  $\square$

### 3. CONJECTURE 1 FOR CONNECTED SETS

In the first few lemmata we collect some elementary properties of the Minkowski sum of connected sets. Throughout this section,  $e_1, e_2$  denotes the elements of the standard orthonormal basis of  $\mathbb{R}^2$ .

**Lemma 3.** *Let  $A \subset \mathbb{R}^d$  be a compact set with a connected boundary and let  $\partial A \subseteq B \subseteq A$ . Then  $B + B = A + A$ .*

*Proof.* We have  $\partial A + \partial A \subseteq B + B \subseteq A + A$ . Thus it is sufficient to prove that  $\partial A + \partial A = A + A$ . Clearly,  $A + A \supseteq \partial A + \partial A$ . We show that  $\frac{A+A}{2} \subseteq \frac{\partial A + \partial A}{2}$ , which then yields the assertion. Consider a point  $p \in \frac{A+A}{2}$ . Then  $p$  is the midpoint of a segment whose endpoints are points of  $A$ . Let  $\chi_p : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the reflection about  $p$ . To prove that  $p \in \frac{\partial A + \partial A}{2}$  we need to show that for some  $q \in \partial A$ , we have  $\chi_p(q) \in \partial A$ . To do this, let us define  $f_p(x)$  ( $x \in \mathbb{R}^d$ ) as the signed distance of  $\chi_p(x)$  from the boundary of  $A$ , where the sign is positive if  $\chi_p(x) \notin A$ , and not positive if  $\chi_p(x) \in A$ . Here we remark that since  $A$  is compact,  $\partial A$  is compact as well. Let  $x_1$  be a point of  $\partial A$  farthest from  $p$ . If  $\chi_p(x_1) \in A$  then  $\chi_p(x_1) \in \partial A$ , and we are done. Thus, assume that  $\chi_p(x_1) \notin A$ , implying that  $f_p(x_1) > 0$ . Now, since  $p \in \frac{A+A}{2}$ , we have some  $y \in A$  such that  $\chi_p(y) \in A$ . Let  $L$  be the line through  $y, p$  and  $\chi_p(y)$ . Let  $y'$  and  $y''$  be points of  $L \cap \partial A$  closest to  $y$  and  $\chi_p(y)$ , respectively. If  $0 < |y' - y| \leq |y'' - \chi_p(y)|$ , then  $y' \in \partial A$  and  $\chi_p(y') \in A$ . If  $0 < |y'' - \chi_p(y)| \leq |y' - y|$ , then the same holds for  $y''$  in place of  $y'$ . Thus, it follows that for some point  $x_2 \in \partial A$ ,  $\chi_p(x_2) \in A$ . If  $\chi_p(x_2) \in \partial A$ , then we are done, and so we may assume that  $\chi_p(x_2) \in \text{int } A$ , which yields that  $f_p(x_2) < 0$ .

We have shown that  $f_p : \partial A \rightarrow \mathbb{R}$  attains both a positive and a negative value on its domain. On the other hand, since  $f$  is continuous and  $\partial A$  is connected,  $f_p(q) = 0$  for some  $q \in \partial A$ , from which the assertion readily follows.  $\square$

**Remark 1.** Lemma 3 holds also for the boundary of the external connected component of  $\mathbb{R}^d \setminus A$  in place of  $\partial A$ .

**Remark 2.** We note that the equality  $A_1 + A_2 = \partial A_1 + \partial A_2$  does not hold in general for different compact sets  $A_1, A_2$  with connected boundaries. To show it, one may consider the sets  $A_1 = B_2^2$  and  $A_2 = \varepsilon B_2^2$  for some sufficiently small value of  $\varepsilon$ , where  $B_2^d$  be the Euclidean unit ball of dimension  $d$  centered at the origin.



**Remark 3.** Lemma 3 does not hold if we omit the condition that  $\partial A$  is connected. To show it, we may choose  $A$  as the union of  $B_2^2$  and a singleton  $\{p\}$  with  $|p|$  being sufficiently large.

**Corollary 1.** *If  $A$  is a compact set with a connected boundary then  $A + A = A + \partial A = \partial A + \partial A$ . Thus, for any positive integer  $k \geq 2$ , we have  $\sum_{i=1}^k A = \sum_{i=1}^k \partial A$ .*

**Corollary 2.** *For any  $k \geq d - 1$  and compact set  $A$  such that  $\partial S \subseteq A \subseteq S$  for some compact, star-shaped set  $S \subset \mathbb{R}^d$ , we have*

$$\text{vol} \left( \frac{1}{k} A[k] \right) \leq \text{vol} \left( \frac{1}{k+1} A[k+1] \right)$$

*Proof.* Without loss of generality, we may assume that  $S$  is star-shaped with respect to the origin. Set  $S' = S + \varepsilon B_2^d$  for some small value  $\varepsilon > 0$ .

First, we show that  $\partial S'$  is path-connected. Let  $L$  be a ray starting at  $o$ . Since  $o \in \text{int } S'$ ,  $L \cap \partial S' \neq \emptyset$ . Let  $p \in L \cap \partial S'$ . Then there is a point  $q \in S$  such that  $|q - p| = \varepsilon$ . Now, if  $x$  is any relative interior point of  $[o, q]$ , then the line through  $x$  and parallel to  $[p, q]$  intersects  $[o, q]$  at a point at distance less than  $\varepsilon$  from  $x$ . Since  $[o, q] \subseteq S$ , from this it follows that  $x \in S + \varepsilon \text{int } B_2^d \subseteq \text{int } S'$ . In other words, for any  $p \in \partial S'$ , all points of  $[o, p]$  but  $p$  lie in  $\text{int } S'$ . Thus,  $L \cap \partial S'$  is a singleton for any ray  $L$  starting at  $o$ .

Let  $0 < r < R$  such that  $\partial S' \subset H = RB_2^d \setminus (r \text{int } B_2^d)$ . Let  $P : H \rightarrow \mathbb{S}^{d-1}$  be the central projection to  $\mathbb{S}^{d-1}$ . Note that  $P$  is Lipschitz, and thus continuous on  $H$ , and its restriction  $P|_{\partial S'}$  to  $\partial S'$  is bijective. On the other hand, since  $\partial S'$  (as also  $S'$ ) are compact, this implies that the inverse of  $P|_{\partial S'}$  is continuous, that is,  $\partial S'$  and  $\mathbb{S}^{d-1}$  are homeomorphic. Thus,  $\partial S'$  is path-connected.

On the other hand,  $\partial S \subseteq A \subseteq S$  implies that  $A' = A + \varepsilon B_2^d \subseteq S'$ , and  $\partial S' \subseteq \partial S + \varepsilon \mathbb{S}^{d-1} \subseteq \partial S + \varepsilon B_2^d \subseteq A'$ . Now, we may apply Lemma 3 and Corollary 1, and obtain that for any value of  $k \geq d - 1$ ,  $A'[k] = S'[k]$ . Thus, by Theorem 1 it follows that  $\text{vol} \left( \frac{1}{k} A'[k] \right) \leq \text{vol} \left( \frac{1}{k+1} A'[k+1] \right)$ . On the other hand, since volume is continuous with respect to Hausdorff distance, we have  $\lim_{\varepsilon \rightarrow 0^+} \text{vol} \left( \frac{1}{m} A'[m] \right) = \lim_{\varepsilon \rightarrow 0^+} \text{vol} \left( \frac{1}{m} A[m] + \varepsilon B_2^d \right) = \text{vol} \left( \frac{1}{k} A[k] \right)$ , which implies the corollary.  $\square$

Let us denote the closure of a set  $A \subset \mathbb{R}^d$  by  $\text{cl}(A)$ .

**Proposition 1.** *Let  $\gamma \subset \mathbb{R}^2$  be a simple continuous curve connecting  $o$  and  $e_1$  such that its intersection with the  $x$ -axis is  $\{o, e_1\}$ . Let  $D$  be the interior of the closed Jordan curve  $\gamma \cup [o, e_1]$ . For  $i = 0, 1$ , let  $\gamma_i = \frac{i}{2}e_1 + \frac{1}{2}\gamma$ , and  $D_i = \frac{i}{2}e_1 + \frac{1}{2}D$ . Then  $\text{cl}(D \setminus (D_0 \Delta D_1)) \subseteq \frac{1}{2}\gamma[2]$ , where  $\Delta$  denotes symmetric difference.*

*Proof.* For convenience, we assume that  $\gamma$  lies in the half plane  $\{y \leq 0\}$ . As in the proof of Lemma 3, let  $\chi_p : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denote the reflection about  $p \in \mathbb{R}^2$ , and note that  $p \in \frac{1}{2}\gamma[2]$  if and only if there is some point  $q \in \gamma$  such that  $\chi_p(q) \in \gamma$ , or in other words, if  $\gamma \cap \chi_p(\gamma) \neq \emptyset$ . Let  $L$  denote the  $x$ -axis,  $L_p = \chi_p(L)$ , and let  $S_p$  be the infinite strip between  $L$  and  $L_p$  (cf. Figure 2).

First, observe that  $o, e_1 \in \gamma$  yields that  $\gamma_0 \cup \gamma_1 \subset \frac{1}{2}\gamma[2]$ , and  $\gamma \subset \frac{1}{2}\gamma[2]$  trivially holds. Thus, we need to show that if for some point  $p$  we have  $p \in D \setminus \text{cl}(D_0 \cup D_1)$

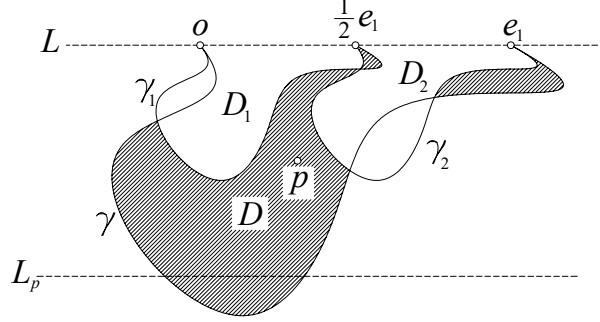


FIGURE 2. An illustration for Proposition 1. The dashed region belongs to  $\frac{1}{2}\gamma[2]$ .

or  $p \in D_0 \cap D_1 \cap D$ , then  $p \in \frac{1}{2}\gamma[2]$ . We do it only for the case  $p \in D \setminus \text{cl}(D_0 \cup D_1)$  since for the second case a similar argument can be applied.

Consider some point  $p \in D \setminus (D_0 \cup D_1)$ . Then  $p \notin \text{cl}(D_0 \cup D_1)$  yields that  $\chi_p(o) = 2p \notin \text{cl} D$ , and the relation  $\chi_p(e_1) \notin \text{cl} D$  follows similarly.

*Case 1:  $\gamma \subset S$ .* Note that in this case  $\chi_p(\gamma) \subset S$ . Since  $p \in D$  and  $\chi_p(o) \notin \text{cl} D$ ,  $\partial D = \gamma \cup [o, e_1]$  and  $[\chi_p(o), p] \cap [o, e_1] = \emptyset$ , it follows by the continuity of  $\gamma$  that  $\gamma \cap [\chi_p(o), p] \neq \emptyset$ . Hence, by the compactness of  $\gamma$ , there is a point  $x \in \gamma \cap [\chi_p(o), p]$  closest to  $p$ . By its choice,  $\chi_p(x) \in D \cup \gamma$ . If  $\chi_p(x) \in \gamma$ , we are done, and thus, we assume that  $\chi_p(x) \in D$ . This implies that  $\chi_p(\gamma)$  contains both interior and exterior points of  $D$ . On the other hand, since  $\chi_p(\gamma) \subset S$ , this implies that  $\chi_p(\gamma) \cap \gamma \neq \emptyset$ .

*Case 2:  $\gamma \not\subset S$ .* Let  $\gamma_p = \gamma \cap S_p$ , and let  $\gamma_1$  and  $\gamma_2$  denote the connected components of  $\gamma_p$  containing  $o$  and  $e_1$ , respectively. For  $i = 0, 1$ , we denote the endpoint of  $\gamma_i$  on  $L_p$  by  $x_i$ . Clearly, since  $\gamma$  is simple and continuous,  $x_1$  is on the left-hand side of  $x_2$ , and the curve  $\gamma_1 \cup [x_1, x_2] \cup \gamma_2 \cup [o, e_1]$  is a Jordan curve. We denote the interior of this curve by  $D_p$ .

Consider the case where  $p \notin D_p$ . Then  $p$  is an exterior point of  $D_p$ , and there is a connected component  $\gamma^*$  of  $\gamma_p$ , with endpoints on  $L_p$ , that separates  $p$  from  $L$ . Since the reflections of the endpoints of  $\gamma^*$  about  $p$  lie on  $L$ , we may apply the argument in Case 1, and obtain that  $\emptyset \neq \gamma^* \cap \chi_p(\gamma^*) \subseteq \gamma \cap \chi_p(\gamma)$ . Thus, we may assume that  $p \in D_p$ .

If  $\chi_p(x_1) \in [o, e_1]$ , then the continuity of  $\gamma_1$  and  $\chi_p(o) \notin \text{cl} D$  implies that  $\emptyset \neq \gamma_1 \cap \chi_p(\gamma_1) \subseteq \gamma \cap \chi_p(\gamma)$ . If  $\chi_p(x_2) \in [o, e_1]$ , then we may apply a similar argument, and thus we may assume that  $\chi_p(x_1), \chi_p(x_2) \notin [o, e_1]$ . This implies that either  $[\chi_p(x_1), \chi_p(x_2)]$  and  $[o, p_1]$  are disjoint, or  $[o, p_1] \subset [\chi_p(x_1), \chi_p(x_2)]$ .

Consider the case where  $[\chi_p(x_1), \chi_p(x_2)]$  and  $[o, p_1]$  are disjoint; without loss of generality we may assume that  $\chi_p(x_1), \chi_p(x_2), o$  and  $e_1$  are in this consecutive order on  $L$ . Let  $U$  be the closure of the connected component of  $S \setminus \gamma_1$  containing  $\gamma_2$ . Then  $\chi_p(p) = p \in \text{int} U \cap \chi_p(U)$ , implying that  $\emptyset \neq \gamma_1 \cap \chi_p(\gamma_1) \subseteq \gamma \cap \chi_p(\gamma)$ . Thus, we may assume that  $[o, p_1] \subset [\chi_p(x_1), \chi_p(x_2)]$ . Since from this it follows that  $[\chi_p(o), \chi_p(e_1)] \subset [x_1, x_2]$ ,  $\chi_p(o) \notin \text{cl} D$  yields that there is a connected component

$\gamma'$ , with endpoints on  $L_p$ , that separates  $\chi_p(o)$  from  $L$ . Thus,  $\gamma'$  separates  $\chi_p(o)$  also from  $\chi_p(x_1) \in L$ , which yields that  $\emptyset \neq \gamma' \cap \chi_p(\gamma_1) \subseteq \gamma \cap \chi_p(\gamma)$ .  $\square$

The proof of Lemma 4 is based on the idea of the proof of Proposition 1, with some necessary modifications.

**Lemma 4.** *Let  $k \geq 2$ , and let  $\gamma \subset \mathbb{R}^2$  be a convex, continuous curve connecting  $o$  and  $e_1$  such that its intersection with the  $x$ -axis is  $\{o, e_1\}$ . Let  $D$  be the interior of the closed Jordan curve  $\gamma \cup [o, e_1]$ . For  $i = 0, 1, \dots, k-1$ , let  $\gamma_i = \frac{i}{k}e_1 + \frac{1}{k}\gamma$ , and  $D_i = \frac{i}{k}e_1 + \frac{1}{k}D$ . Then  $\text{cl}\left(D \setminus \left(\bigcup_{i=1}^k D_i\right)\right) \subseteq \frac{1}{k}\gamma[k]$ , and for any  $i \neq j$ ,  $D_i \cap D_j \subseteq \frac{1}{k}\gamma[k]$ .*

*Proof.* First observe that  $D$  is convex, hence  $D_i$  is contained in  $D$  for all values of  $i$ . In the proof, we denote the  $x$ -axis by  $L$ , for any  $p \in \mathbb{R}^2$  the homothety with center  $p$  and ratio  $-\frac{1}{k-1}$  by  $\chi_p^k : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . Furthermore, we set  $L_p^k = \chi_p^k(L)$ , and denote the infinite strip between  $L$  and  $L_p^k$  by  $S$ . The assertion for  $k = 2$  is a special case of Proposition 1. To prove it for  $k \geq 3$ , we apply induction on  $k$ , and assume that the lemma holds for  $\gamma[k-1]$ .

Let  $p \in \text{cl}(D) \setminus \left(\bigcup_{i=1}^k D_i\right)$ . Clearly, since  $\partial D = \gamma \subseteq \gamma[k]$ , we may assume that  $p \in D$ . By the induction hypothesis for  $\frac{k-1}{k}\gamma$ , if  $p \in X_1 = \frac{k-1}{k}\text{cl}D$ , then  $p \in \frac{k-1}{k} \cdot \frac{1}{k-1}\gamma[k-1] = \frac{1}{k}\gamma[k-1] \subseteq \frac{1}{k}\gamma[k]$ . Similarly, if  $p \in X_2 = \frac{1}{k}e_1 + \frac{k-1}{k}\text{cl}D$ , then  $p \in \frac{1}{k}e_1 + \frac{1}{k}\gamma[k-1] \subseteq \frac{1}{k}\gamma[k]$ . Thus, assume that  $p \notin X_1 \cup X_2$ , which yields that  $\chi_p^k(o)$  and  $\chi_p^k(e_1)$  are in the exterior of  $D$ . Let the (unique) intersection point of  $[p, \chi_p^k(o)]$  and  $\gamma$  be  $q_1$  and the (unique) intersection point of  $[p, \chi_p^k(e_1)]$  and  $\gamma$  be  $q_2$ . As  $\chi_p^k(q_1) \in [o, p]$ , the convexity of  $D$  implies that  $\chi_p^k(q_1) \in D$ , and the containment  $\chi_p^k(q_2) \in D$  follows similarly.

Similarly like in Proposition 1, if  $\gamma \subset S$ , then by continuity,  $\gamma \cap \chi_p^k(\gamma) \neq \emptyset$ , which implies the containment  $p \in \frac{1}{k}\gamma[k]$ . Assume that  $\gamma \not\subset S$ . Then  $S \cap \gamma$  has two connected components  $\gamma_1, \gamma_2$ , where we choose the indices such that  $o \in \gamma_1$ , and  $e_1 \in \gamma_2$ . Clearly, we have either  $q_1 \in \gamma_2, q_2 \in \gamma_1$ , or both. If  $q_1 \in \gamma_2$ , then the containment relations  $\chi(q_1) \in D, \chi(e_1) \notin \text{cl}D$ , and  $\chi_p^k(\gamma_2) \subset S$  yield that  $\emptyset \neq \gamma_1 \cap \chi_p^k(\gamma_2) \subset \gamma \cap \chi_p^k(\gamma)$ . If  $q_2 \in \gamma_1$ , then the assertion follows by a similar argument.

Finally, we consider the case that  $p \in D_i \cap D_j$  for some  $i < j$ . In this case the convexity of  $D$  implies that  $p \in D_s$  for any  $i \leq s \leq j$ . This yields that there are some distinct values  $i, j \leq k-1$  or  $i, j \geq 2$  such that  $p \in D_i \cap D_j$ . Thus, the assertion readily follows from the induction hypothesis.  $\square$

Lemma 5 is a variant of Lemma 2 for some path-connected sets in  $\mathbb{R}^2$ .

**Lemma 5.** *Let  $k \geq 2$  and  $\gamma$  be a bounded convex curve in  $\mathbb{R}^2$ , and let  $\gamma[k] \subseteq M \subseteq k \text{ conv } \gamma$ . Then*

$$\text{area}\left(\frac{1}{k}M\right) \leq \text{area}\left(\frac{1}{k+1}(M + \gamma)\right).$$

*Proof.* If  $\gamma$  is closed, then Lemma 3 yields that  $\frac{1}{k}\gamma[k] = \text{conv } \gamma$  for all  $k \geq 2$ , which clearly implies the statement. Assume that  $\gamma$  is not closed. Since the inequalities

in Lemma 5 do not change under affine transformations, we may assume that the endpoints of  $\gamma$  are  $o$  and  $e_1$ , and the  $x$ -axis is a supporting line of  $\text{conv } \gamma$ .

Let us define

$$D = \text{conv } \gamma, \alpha = \text{area}(D \cap (e_1 + D)), \text{ and } \beta = \text{area}(D \cap ((e_1 + D) \cup (-e_1 + D))).$$

Note that  $0 \leq \alpha \leq \beta \leq 2\alpha$ . Let  $D_i = ie_1 + D$  for  $i = 0, 1, \dots, k$ . For  $0 \leq i \leq k-1$ , let  $\mu_i$  be the area of the region of  $M$  in  $D_i$  that do not belong to any  $D_j$ ,  $j \neq i$ , where we note that since  $k \geq 2$ , by Lemma 4 we have that all other points of  $D_i$  belong to  $M$ . Similarly, for  $0 \leq i \leq k$ , let  $\lambda_i$  be the area of the region of  $M + \gamma$  in  $D_i$  that do not belong to any  $D_j$ ,  $j \neq i$ . An elementary computation shows that

$$\begin{aligned} \text{area}(M) &= k^2 \text{area}(D) - 2(\text{area}(D) - \alpha) - (k-2)(\text{area}(D) - \beta) + \sum_{i=0}^{k-1} \mu_i \\ (10) \quad &= (k^2 - k) \text{area}(D) + 2\alpha + (k-2)\beta + \sum_{i=0}^{k-1} \mu_i, \end{aligned}$$

and similarly,

$$(11) \quad \text{area}(M + \gamma) = (k^2 + k) \text{area}(D) + 2\alpha + (k-1)\beta + \sum_{i=0}^k \lambda_i.$$

Since  $o, e_1 \in \gamma$ , we have  $M, e_1 + M \subseteq M + \gamma$ . Thus,  $\lambda_0 \geq \mu_0$ ,  $\lambda_k \geq \mu_{k-1}$ ,  $\lambda_1 \geq \max\{\mu_0 - (\beta - \alpha), \mu_1\}$ ,  $\lambda_{k-1} \geq \max\{\mu_{k-2}, \mu_{k-1} - (\beta - \alpha)\}$ , and for  $2 \leq i \leq k-2$ ,  $\lambda_i \geq \max\{\mu_{i-1}, \mu_i\}$ . Since  $\lambda_i \geq \frac{i}{k}\mu_{i-1} + \frac{k-i}{k}\mu_i$  if  $2 \leq i \leq k-2$ , and  $\lambda_i \geq \frac{i}{k}\mu_{i-1} + \frac{k-i}{k}\mu_i - \frac{1}{k}(\beta - \alpha)$  if  $i = 1$  or  $i = k-1$ , it follows that  $\sum_{i=0}^k \lambda_i \frac{k+1}{k} \geq \sum_{i=1}^{k-1} \mu_i - \frac{2}{k}(\beta - \alpha)$ . Thus, by (10),

$$\sum_{i=0}^k \lambda_i \geq \frac{k+1}{k} (\text{area}(M) - (k^2 - k) \text{area}(D) - 2\alpha - (k-2)\beta) - \frac{2}{k}(\beta - \alpha).$$

After substituting this into (11) and simplifying, we obtain

$$\text{area}(M + \gamma) \geq \frac{k+1}{k} \text{area}(M) + (k+1) \text{area}(D),$$

which yields

$$\text{area}\left(\frac{1}{k+1}(M + \gamma)\right) \geq \frac{k}{k+1} \text{area}\left(\frac{1}{k}M\right) + \frac{1}{k+1} \text{area}(D).$$

Thus, the inequality  $\text{area}\left(\frac{1}{k}M\right) \leq \text{area}(D)$  yields the assertion.  $\square$

In Theorem 2, by an open topological disc we mean the bounded connected component defined by a Jordan curve.

**Theorem 2.** *Let  $k \geq 2$ . Let  $K$  be a plane convex body, and let  $\mathcal{F} = \{F_i : i \in I\}$  be a family of pairwise disjoint topological discs open in  $K$  such that if  $F_i \cap \partial K \neq \emptyset$  then  $F_i \cap \partial K$  is a connected curve and  $F_i$  is convex. Let  $X = K \setminus (\bigcup_{i \in I} F_i)$ . Then*

$$\text{area}\left(\frac{1}{k}X[k]\right) \leq \text{area}\left(\frac{1}{k+1}X[k+1]\right).$$

*Proof.* First, note that since each member of  $\mathcal{F}$  has positive area, it has countably many elements; indeed, for any  $\delta > 0$  there are only finitely many elements  $F_i$  of  $\mathcal{F}$  for which  $\text{area}(F_i \cap K) \geq \delta$ , and thus, we may list the elements according to area. Furthermore, since  $X$  is compact,  $\text{area}(X)$  exists.

By Lemma 3, we may assume that every member of  $\mathcal{F}$  intersects  $\partial K$ . For any  $i \in I$ , let  $\gamma_i$  denote the part of  $\partial F_i$  in  $K$ . Clearly,  $\gamma_i$  is a convex curve, and the line through two of its endpoints supports  $K \setminus F_i$ . Choose some finite subfamily  $I_\varepsilon \subseteq I$  such that  $\text{area}(X_\varepsilon \setminus X) \leq \varepsilon$ , where  $X_\varepsilon = K \setminus (\bigcup_{i \in I_\varepsilon} F_i)$ . This is possible, since for any ordering of the elements,  $\sum_{i \in I} \text{area}(K \cap F_i)$  is a bounded series with positive elements, and hence, it is absolute convergent, and convex sets with small area and bounded diameter are contained in a small neighborhood of their boundary.

For any  $i \in I_\varepsilon$ , we set  $D_i = F_i \cap K$ , and observe that  $D_i$  is a convex set separated from  $X_\varepsilon$  by the convex curve  $\gamma_i$ . Clearly, for  $k \geq 2$ , only points in the  $D_i$ 's may not belong to  $\frac{1}{k}X_\varepsilon[k]$ , and by Lemma 4, only points contained in exactly one homothetic copy  $\frac{1}{k}D_{ij}$ ,  $j = 1, 2, \dots, k$  of  $\frac{1}{k}D_i$  in  $D_i$ . Let  $M_i = (X[k] \cap (kD_i))$ . Then  $M_i \subseteq \text{conv}(kD_i)$ , and thus, Lemma 5 yields that

$$\text{area}\left(\frac{1}{k}M_i\right) \leq \text{area}\left(\frac{1}{k+1}(M_i + \gamma_i)\right).$$

On the other hand, with the notation  $D_\varepsilon = \bigcup_{i \in I_\varepsilon} D_i$ , we have

$$\text{area}\left(\frac{1}{k}X[k] \cap D\right) = \sum_{i \in I_\varepsilon} \text{area}\left(\frac{1}{k}M_i\right),$$

and

$$\text{area}\left(\frac{1}{k}X[k+1] \cap D\right) \geq \sum_{i \in I_\varepsilon} \text{area}\left(\frac{1}{k+1}(M_i + \gamma_i)\right),$$

and thus, we have  $\text{area}\left(\frac{1}{k}X[k] \cap D\right) \leq \text{area}\left(\frac{1}{k+1}X[k+1] \cap D\right)$ . On the other hand, since  $\text{area}(X_\varepsilon \setminus X) < \varepsilon$ ,  $X_\varepsilon \cup D = \text{conv} X$ , and  $X \subseteq X_\varepsilon$ , we have that  $\text{area}\left(\frac{1}{m}X[m] \setminus D\right) \leq \varepsilon$  for all  $m \geq 1$ . This implies that

$$\text{area}\left(\frac{1}{k}X[k]\right) \leq \text{area}\left(\frac{1}{k+1}X[k+1]\right) - \varepsilon.$$

This holds for all  $\varepsilon > 0$ , which yields the assertion.  $\square$

#### 4. ADDITIONAL REMARKS AND QUESTIONS

**Remark 4.** One can ask if the statement of Theorem 1 holds for arbitrary measure instead of volume. The answer to this question is negative. Indeed, consider the measure  $\mu(K) = \text{vol}(K \cap C)$ , where  $C = [-\frac{1}{d}, \frac{1}{d}]^d$  and  $S = \bigcup_{i=1}^d [o, e_i]$ , where  $e_1, e_2, \dots, e_d$  are the vectors of the standard orthonormal basis. Then, clearly, we have

$$\mu\left(\frac{1}{2k}S[2k]\right) = \frac{1}{2d} \text{vol}(C) > \mu\left(\frac{1}{2k+1}S[2k+1]\right).$$

**Remark 5.** The statement of Theorem 1 does not hold for arbitrary measures even for rotationally invariant measures in the plane: for any value of  $k$  there is a compact, star-shaped set  $S \subset \mathbb{R}^2$  such that  $\text{vol}\left(\frac{1}{k}S[k] \cap B_2^2\right) > \text{vol}\left(\frac{1}{k+1}S[k+1] \cap B_2^2\right)$ .

To prove this, set  $S = [o, e_1] \cup [o, e_2]$ , and let  $E$  denote the ellipse centered at  $o$  and containing the points  $(1 - 1/k, 0)$  and  $(1 - 2/k, 1/k)$ . It is an elementary computation to check that in this case  $\text{vol}(\frac{1}{k}S[k] \cap E) = \frac{1}{4} \text{vol}(E)$ . On the other hand, the boundary point  $(1 - 2/(k+1), 1/(k+1))$  of  $\frac{1}{k+1}S[k+1]$  lies in  $\text{int}(E)$ , which implies that  $\text{vol}(\frac{1}{k+1}S[k+1] \cap E) < \frac{1}{4} \text{vol}(E)$ . Now, if  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined as the linear transformation mapping  $E$  into  $B_2^2$ , then  $f(S)$  satisfies the required conditions.

One can use star-shaped sets together with ideas from [FMMZ2] to give a negative answer to a more general version of Conjecture 1, also from [BMW].

**Conjecture 2** (Bobkov-Madiman-Wang). For any  $k \geq 2$ , star-shaped compact sets  $A_1, A_2, \dots, A_{k+1}$  in  $\mathbb{R}^d$ , we have

$$\text{vol}\left(\sum_{i=1}^{k+1} A_i\right)^{1/d} \geq \frac{1}{k} \sum_{i=1}^{k+1} \text{vol}\left(\sum_{j \neq i} A_j\right)^{1/d}.$$

in particular, for  $k = 2$ ,

$$(12) \quad \begin{aligned} & \text{vol}(A_1 + A_2 + A_3)^{1/d} \\ & \geq \frac{1}{2} \left( \text{vol}_n(A_1 + A_2)^{1/d} + \text{vol}(A_1 + A_3)^{1/d} + \text{vol}(A_2 + A_3)^{1/d} \right). \end{aligned}$$

The above conjecture is trivial for convex sets. Moreover, (12) is true when  $A_1 = A_2$  and  $A_1$  is convex. Indeed, in this case (12) is equivalent to

$$\begin{aligned} \text{vol}(A_1 + A_1 + A_3)^{1/d} & \geq \frac{1}{2} \left( \text{vol}(2A_1)^{1/d} + 2 \text{vol}_n(A_1 + A_3)^{1/d} \right), \\ \text{vol}(A_1 + A_1 + A_3)^{1/d} & \geq \text{vol}(A_1)^{1/d} + \text{vol}(A_1 + A_3)^{1/d}, \end{aligned}$$

where the last inequality follows from the Brunn-Minkowski inequality [Sch].

It was proved in [FMMZ2] that Conjecture 2 is true in  $\mathbb{R}$ . Since an affirmative answer to Conjecture 2 implies also Conjecture 1, the former is also false for  $d \geq 12$  by [FMMZ1, FMMZ2]. Here we show that Conjecture 2 is false in  $\mathbb{R}^d$  even for  $d \geq 7$ .

**Proposition 2.** For any  $d \geq 7$ , there are compact, star-shaped sets  $A_1, A_2, A_3 \subset \mathbb{R}^d$  satisfying

$$\text{vol}(A_1 + A_2 + A_3)^{1/d} < \frac{1}{2} \left( \text{vol}(A_1 + A_2)^{1/d} + \text{vol}(A_1 + A_3)^{1/d} + \text{vol}(A_2 + A_3)^{1/d} \right).$$

*Proof.* We give the proof for  $d = 7$  and the result follows for  $d > 7$  by taking direct products with a cube. consider the sets

$$A_1 = [0, 1]^4 \times \{0\}^3; A_2 = \{0\}^4 \times [0, 1]^3 \text{ and } A_3 = ([0, a]^4 \times \{0\}^3) \cup (\{0\}^4 \times [0, b]^3),$$

where we select  $a, b > 0$  later. An elementary consideration shows that

$$\text{vol}(A_1 + A_3) = b^3, \text{vol}(A_2 + A_3) = a^4 \text{ and } \text{vol}(A_1 + A_2) = 1,$$

and

$$\text{vol}(A_1 + A_2 + A_3) = (a + 1)^4 + (b + 1)^3 - 1.$$

The last step is to show that, with  $a = 3$  and  $b = 6$ , the quantity

$$((a + 1)^4 + (b + 1)^3 - 1)^{1/7} - \frac{1}{2} \left( a^{4/7} + b^{3/7} + 1 \right)$$

is negative, which gives a counterexample to (12).  $\square$

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