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ON THE MODULI OF LOGARITHMIC CONNECTIONS ON ELLIPTIC CURVES

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ABSTRACT. We describe moduli spaces of logarithmic rank 2 connections on elliptic curves with $n \geq 1$ poles and generic residues. In particular, we generalize a previous work by the first and second named authors. Our main approach is to analyze the underlying parabolic bundles; their stability and instability play a major role.

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1. INTRODUCTION

In this paper, we investigate the geometry of certain moduli spaces of connections on complex elliptic curves C . We will consider pairs (E, ∇) where $E \rightarrow C$ is a rank 2 vector bundle and $\nabla: E \rightarrow E \otimes \Omega_C^1(D)$ is a logarithmic connection with (reduced) polar divisor $D = t_1 + \dots + t_n$ with $n \geq 1$. We also prescribe the following data:

- The eigenvalues (ν_i^+, ν_i^-) of $\text{Res}_{t_i}(\nabla)$, for each $i = 1, \dots, n$, such that:
 - (1) $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \notin \mathbb{Z}$, for any $\epsilon_i \in \{+, -\}$;
 - (2) and $\nu_i^+ \neq \nu_i^-$, for $i = 1, \dots, n$;
- A trace connection (L, ζ) , i.e. $\det(E) = L$ and $\text{tr}(\nabla) = \zeta$;

In particular $\sum_i \nu_i^+ + \nu_i^- = -\deg(L)$ which is called the Fuchs relation. Once we have fixed this data, we can define the moduli space \mathbf{Con}^ν of those pairs (E, ∇) up to isomorphism. Due to Inaba's construction [13], \mathbf{Con}^ν is a smooth irreducible quasi-projective variety of dimension $2n$, equipped with an algebraic symplectic structure. These moduli spaces have been constructed much earlier in the analytic setting (see for instance [30, 25]): they are building blocks for the non-abelian Hodge correspondance. They turn out to be hyperkahler manifolds diffeomorphic to tamely ramified Higgs bundle moduli spaces (see also [16, 26, 20]). We also note that these moduli spaces occur in geometric Langlands correspondance (see [1]), Riemann-Hilbert correspondance and isomonodromic deformations (see [14]).

We will consider the forgetful map $\pi: (E, \nabla) \mapsto (E, \mathbf{p})$ which associates to a connection an underlying quasi-parabolic bundle. Given a choice of signs $\epsilon_i \in \{+, -\}$ for each $i = 1, \dots, n$, the parabolic data $\mathbf{p}^\epsilon(\nabla) = (p_1^{\epsilon_1}(\nabla), \dots, p_n^{\epsilon_n}(\nabla))$ consists of the $\nu_i^{\epsilon_i}$ -eigenspace $p_i^{\epsilon_i} \subset E|_{t_i}$ for $\text{Res}_{t_i}(\nabla)$ at each pole; these are well-defined since $\nu_i^+ \neq \nu_i^-$. There exist 2^n underlying quasi-parabolic structures for each connection, according to the choice of $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ giving rise to 2^n forgetful maps $\pi_\epsilon: (E, \nabla) \mapsto (E, \mathbf{p}^\epsilon(\nabla))$.

In Section 2 we study the quasi-parabolic bundles (E, \mathbf{p}) over (C, D) that admit a connection ∇ with prescribed trace and eigenvalues, compatible with parabolic directions. The major difference from the case $n = 2$ investigated in [9] is that, when n is odd, there exist pairs (E, ∇) such that all the underlying quasi-parabolic bundles $(E, \mathbf{p}^\epsilon(\nabla))$ are not μ -semistable for any choice of weights; it occurs for the item 3 of Lemma 2.4. Following the study of stability, we describe a wall-crossing phenomenon in Lemma 2.7 and Lemma 2.8.

In Section 3 we study the logarithmic connections. We investigate \mathbf{Con}^ν via the forgetful map to an underlying quasi-parabolic structure. We are especially concerned with the μ -stability of these quasi-parabolic bundles. It turns out that there exists an open subset of \mathbf{Con}^ν where the underlying vector bundle is E_1 , the unique indecomposable vector bundle of degree one, with given determinant. We call this open subset Con^ν and consider the map

$$\begin{aligned} \text{Par}: \text{Con}^\nu &\longrightarrow (\mathbb{P}^1 \times \mathbb{P}^1) \times \dots \times (\mathbb{P}^1 \times \mathbb{P}^1) \\ (E, \nabla) &\longmapsto (p_1^+(\nabla), p_1^-(\nabla), \dots, p_n^+(\nabla), p_n^-(\nabla)) \end{aligned}$$

that associates to each connection all its residual eigenspaces (i.e. with respect to all eigenvalues ν_i^+ and ν_i^-). The image of Par is contained in S^n , where S is the complement of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. We show that this map is an isomorphism (cf. Theorem 3.3).

We also study the open subset $\mathbf{Con}_{st}^\nu \subset \mathbf{Con}^\nu$ formed by pairs (E, ∇) which admit a μ -stable parabolic bundle $(E, \mathbf{p}^\epsilon(\nabla))$ for some ϵ and some weight vector μ . We call

$$\mathcal{Z}_n = \mathbf{Con}^\nu \setminus \mathbf{Con}_{st}^\nu$$

its complement. We describe \mathcal{Z}_n in Theorem 3.10, it is empty for n even and has four irreducible components which are isomorphic to \mathbb{C}^n for n odd. Assuming $\nu_i^+ - \nu_i^- \notin \{0, 1, -1\}$ for $i \in \{1, \dots, n\}$, we see that \mathfrak{Con}_{st}^ν admits an open covering given by open subsets isomorphic to S^n . It leads to the following characterization of \mathfrak{Con}_{st}^ν (cf. Theorem 3.8):

Theorem A. *Assume that $\nu_i^+ - \nu_i^- \notin \{0, 1, -1\}$ for $i \in \{1, \dots, n\}$. Then \mathfrak{Con}_{st}^ν is obtained by gluing a finite number of copies of S^n via birational maps*

$$\Psi_{J,I}^{\delta,\epsilon}: S^n \dashrightarrow S^n.$$

Moreover, if $\epsilon = \delta$, then $\Psi_{J,I}^{\delta,\epsilon}$ preserves the fibers of π_ϵ .

In the remainder of our work, we do a finer analysis of the open subset Con^ν . Let Bun denote the moduli space of parabolic vector bundles whose underlying vector bundle is E_1 ; it is isomorphic to $(\mathbb{P}^1)^n$. In Section 4, we describe the affine bundle $\text{Con}^\nu \rightarrow \text{Bun}$ via Fuchsian systems; this construction yields a vector bundle \mathcal{E} whose projectivization compactifies Con^ν . Although $\mathbb{P}(\mathcal{E})$ does not depend on the eigenvalues, the boundary divisor is determined by $\nu_i = \nu_i^+ - \nu_i^-$, $i = 1, \dots, n$ (cf. Theorem 4.4):

Theorem B. *The moduli space Con^ν has compactification $\overline{\text{Con}^\nu} = \mathbb{P}(\mathcal{E})$, where the boundary divisor is isomorphic to $\mathbb{P}\text{Higgs}$, the projectivization of the space of Higgs fields on E_1 . Moreover, the inclusion $\mathbb{P}\text{Higgs} \hookrightarrow \mathbb{P}(\mathcal{E})$ is determined, up to automorphisms of $\mathbb{P}(\mathcal{E})$, by (ν_1, \dots, ν_n) .*

In Section 5, we deal with the symplectic structure of the moduli space and compute the explicit expression in the main chart $\text{Con}^\nu \simeq S^n$, see (5.10). We show that (ν_1, \dots, ν_n) is detected by the symplectic structure (cf. Corollary 5.9):

Theorem C. *If there exists a fiber preserving symplectic isomorphism*

$$\begin{array}{ccc} (\text{Con}^\nu, \omega) & \xrightarrow[\sim]{\Phi} & (\text{Con}^{\tilde{\nu}}, \tilde{\omega}) \\ \downarrow \pi_+ & & \downarrow \pi_+ \\ \text{Bun} & \xrightarrow[\sim]{\phi} & \text{Bun} \end{array}$$

then there exists a permutation σ of n elements such that $\tilde{\nu}_k = \nu_{\sigma(k)}$ for every $k \in \{1, \dots, n\}$.

In Theorem C, we have a fixed underlying space of parabolic bundles, Bun , and we can recover (ν_1, \dots, ν_n) from the symplectic structure. The main result of [8] is a Torelli type result which asserts that the moduli space of parabolic bundles determines the punctured curve (C, D) . We wonder if the same is true for Con^ν , or \mathfrak{Con}^ν . This belief is based on some results in the literature, see [28, 6, 5].

In Section 6, we conclude the paper by studying the Apparent map. A global section of E_1 plays the role of a cyclic vector for connection ∇ , which yields a second order ODE on C ; the map App assigns to ∇ the apparent singular points of this equation, see [23]. It leads to an interesting result about the birational geometry of $\overline{\text{Con}^\nu}$ (Theorem 6.2):

Theorem D. *The map $\pi_+ \times \text{App}$ induces a birational map*

$$\pi_+ \times \text{App}: \overline{\text{Con}^\nu} \dashrightarrow \text{Bun} \times |\mathcal{O}_C(w_\infty + D)|$$

whose indeterminacy locus is contained in $\overline{\text{Con}^\nu} \setminus \text{Con}^\nu$. Moreover, given $(E, \mathbf{p}) \in \text{Bun}$, the rank of

$$(\pi_+ \times \text{App})|_{\pi_+^{-1}(E, \mathbf{p})}: \pi_+^{-1}(E, \mathbf{p}) \longrightarrow \{(E, \mathbf{p})\} \times |\mathcal{O}_C(w_\infty + D)|$$

coincides with the cardinality of the set $\{i \mid p_i \notin \mathcal{O}_C\}$.

Logarithmic connections on elliptic curves have been investigated by several authors. First, they were studied in the works of Okamoto and later by Kawai in [18] in relation with isomonodromic deformations. In particular, the symplectic form of [18, Theorem 1], when restricted to a fixed punctured curve (C, D) , must coincide to that one we give in (5.10). However, as [18] is dealing with analytic differential equations written in terms of Weierstrass zeta functions, it is not so easy to relate the computations with ours. More recently, our moduli space has been considered and proved to be related to some moduli spaces of logarithmic connections on \mathbb{P}^1 for the special cases $n = 1$ (see [21]), and $n = 2$ with special eigenvalues (see [22]). We expect that our explicit algebraic approach will allow us in a forthcoming work to relate with Kawai's parameters and provide an algebraic expression of isomonodromy equation in our context.

Remark 1.1 (Notation and convention). Throughout the text C will denote a genus one curve and $D = t_1 + \dots + t_n$ will be a reduced divisor on C . Let $w_\infty \in C$ be a point and let w_0, w_1 and w_λ be the torsion points of the elliptic curve (C, w_∞) . In the construction of \mathbf{Con}^ν , we will assume (without loss of generality) that the determinant is $\mathcal{O}_C(w_\infty)$ and that $D + w_0 + w_1 + w_\lambda$ is reduced.

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2. PARABOLIC VECTOR BUNDLES

Let C be an elliptic curve with $w_\infty \in C$ being its distinguished point. A rank two quasi-parabolic vector bundle (E, \mathbf{p}) on (C, D) , $D = t_1 + \dots + t_n$, consists of a holomorphic vector bundle E of rank two on C and a collection $\mathbf{p} = \{p_1, \dots, p_n\}$ of 1-dimensional linear subspaces $p_i \subset E_{t_i}$. We refer to the points t_i as parabolic points, and to the subspace $p_i \subset E_{t_i}$ as the parabolic direction of E at t_i .

A triple $(E, \mathbf{p}; \mu)$ of a quasi-parabolic vector bundle and an n -tuple $\mu = (\mu_1, \dots, \mu_n)$ of real numbers in the interval $(0, 1)$ is called parabolic vector bundle of rank two. We often write (E, \mathbf{p}) for a parabolic vector bundle when the choice of the weight μ is clear.

Let $(E, \mathbf{p}; \mu)$ be a parabolic vector bundle and let $L \subset E$ be a line subbundle then we define

$$\text{Stab}_\mu(L) := \deg E - 2 \deg L + \sum_{p_k \neq L_{t_k}} \mu_k - \sum_{p_k = L_{t_k}} \mu_k.$$

We say that $(E, \mathbf{p}; \mu)$ is semistable if $\text{Stab}_\mu(L) \geq 0$ holds for every $L \subset E$. It is stable if the strict inequality holds for every line subbundle $L \subset E$. We call $\text{Stab}_\mu(L)$ the parabolic stability of $L \subset E$ with respect to μ .

We denote by $\text{Bun}_{w_\infty}^\mu$ the moduli space of semistable parabolic vector bundles $(E, \mathbf{p}; \mu)$ on (C, D) with $\det E = \mathcal{O}_C(w_\infty)$. In this case, either $E \simeq L \oplus L^{-1}(w_\infty)$ or $E \simeq E_1$, where E_1 is the unique non trivial extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_1 \longrightarrow \mathcal{O}_C(w_\infty) \longrightarrow 0.$$

If there exists $L \subset E$ such that $\text{Stab}_\mu(L)$ is zero then the weights lie on the hyperplane

$$H(d, I) := \left\{ \mu \mid 1 - 2d + \sum_{k \notin I} \mu_k - \sum_{k \in I} \mu_k = 0 \right\}$$

where $d = \deg L$ and $I \subset \{1, \dots, n\}$ denotes the set of indices of those parabolic directions $p_k \subset L_{t_k}$. A connected component of the complement in $(0, 1)^n$ of all these hyperplanes $H(d, I)$ is called a

chamber. If μ and $\tilde{\mu}$ belong to the same chamber then $\text{Bun}_{w_\infty}^\mu = \text{Bun}_{w_\infty}^{\tilde{\mu}}$, see for example [24] or [7, Lemma 2.7].

In the next result, we define an interesting chamber; the underlying vector bundle is fixed and the corresponding moduli space is a product of projective lines.

Proposition 2.1. *The following assertions hold:*

- (1) *The set $\mathfrak{C} := \{\mu \in (0, 1)^n \mid \sum_{k=1}^n \mu_k < 1\}$ is a chamber.*
- (2) *If $\mu \in \mathfrak{C}$ then $\text{Bun}_{w_\infty}^\mu = \{(E, \mathbf{p}) \mid E = E_1\}$. Moreover, it is isomorphic to $(\mathbb{P}^1)^n$.*

Proof. Note that \mathfrak{C} is convex, hence connected. Then (1) follows from proving that \mathfrak{C} does not intersect any wall $H(d, I)$. This is straightforward and we leave it to the reader.

Now we prove (2). Recall that $\det E = \mathcal{O}_C(w_\infty)$ implies that either $E = E_1$ or $E = L \oplus L^{-1}(w_\infty)$ with $\deg L \geq 1$. For the later we have

$$\text{Stab}_\mu(L) = 1 - 2 \deg L + \sum_{p_k \neq L t_k} \mu_k - \sum_{p_k = L t_k} \mu_k \leq 1 - 2 \deg L + \sum_{k=1}^n \mu_k < 2 - 2 \deg L \leq 0$$

for any $\mu \in \mathfrak{C}$, hence E cannot be μ -semistable. Hence $E = E_1$ and each parabolic bundle is completely determined by

$$(p_1, \dots, p_n) \in \mathbb{P}(E_1|_{t_1}) \times \dots \times \mathbb{P}(E_1|_{t_n}) \simeq (\mathbb{P}^1)^n.$$

Thus we get the desired isomorphism. \square

For a weight vector $\mu = (\mu_1, \dots, \mu_n) \in (0, 1)^n$ and a subset $I \subset \{1, \dots, n\}$ of even cardinality, we consider the map $\varphi_I: (0, 1)^n \rightarrow (0, 1)^n$ defined by

$$\varphi_I(\mu) := (\mu'_1, \dots, \mu'_n) \in (0, 1)^n$$

where $\mu'_i = \mu_i$ if $i \notin I$, and $\mu'_i = 1 - \mu_i$ if $i \in I$. Note that φ_I is continuous and preserves the walls $H(d, J)$. Then the image of \mathfrak{C} by φ_I yields a new chamber

$$(2.1) \quad \mathfrak{C}_I := \left\{ \mu \in (0, 1)^n \mid \sum_{k \notin I} \mu_k - \sum_{k \in I} \mu_k + |I| < 1 \right\}$$

where $|I|$ is the cardinality of I . When $I = \emptyset$ then $\mathfrak{C}_I = \mathfrak{C}$.

Each φ_I admits a modular realization as an elementary transformation, which we now describe. Consider the following exact sequence of sheaves

$$0 \rightarrow E' \xrightarrow{\alpha} E \xrightarrow{\beta} \bigoplus_{i \in I} (E_{t_i}/p_i) \rightarrow 0$$

where for each (local) section s of E we define $\beta(s) = (\beta_1(s), \dots, \beta_n(s))$ by $\beta_j(s) = s(t_j) \pmod{p_j}$ if s is defined at t_j , and $\beta_j(s) = 0$ otherwise. Then E' is a vector bundle of rank two such that

$$\det E' = \det E \otimes \mathcal{O}_C \left(- \sum_{i \in I} t_i \right).$$

In particular, E' has degree $1 - |I|$. We define a natural quasi-parabolic structure for E' as follows. If $i \notin I$ then $\alpha_{t_i}: E'_{t_i} \rightarrow E_{t_i}$ is an isomorphism and

$$p'_i = (\alpha_{t_i})^{-1}(p_i) \subset E'_{t_i}$$

is the parabolic direction at t_i . If $i \in I$ we define $p'_i = \ker(\alpha_{t_i})$ as the parabolic direction at t_i . This operation corresponds to the birational transformation of ruled surfaces $\mathbb{P}(E) \dashrightarrow \mathbb{P}(E')$ obtained

by blowing-up the points $p_i \in \mathbb{P}(E_{t_i})$ and then blowing-down the strict transforms of the fibers $\mathbb{P}(E_{t_i})$ to the points $p'_i \in \mathbb{P}(E'_{t_i})$, $i \in I$. This is well-defined since the p_i lie on different fibers.

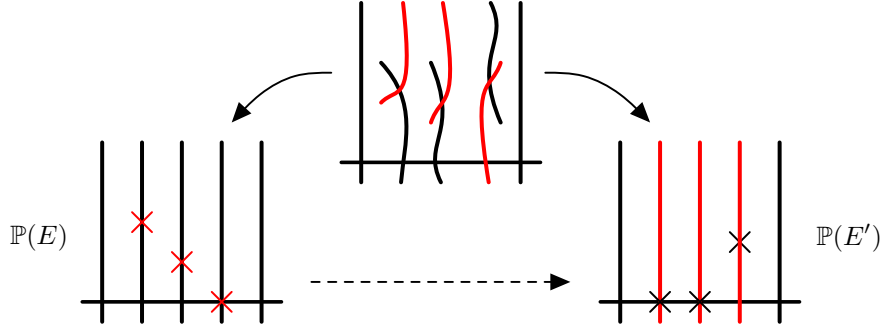


FIGURE 1. Elementary transformation

Since $|I|$ is even, we can fix a square root L_0 of the line bundle $\mathcal{O}_C(\sum_{i \in I} t_i)$, i.e.

$$L_0^2 = \mathcal{O}_C\left(\sum_{i \in I} t_i\right).$$

This gives a correspondence

$$\text{elm}_I: (E, \mathbf{p}) \longmapsto (E' \otimes L_0, \mathbf{p}')$$

between quasi-parabolic vector bundles on (C, D) which have $\mathcal{O}_C(w_\infty)$ as determinant line bundle.

The reader can check that if (E, \mathbf{p}) is semistable with respect to μ , then $\text{elm}_I(E, \mathbf{p})$ is semistable with respect to $\varphi_I(\mu)$. We conclude that the correspondence elm_I defines an isomorphism between moduli spaces

$$\text{elm}_I: \text{Bun}_{w_\infty}^\mu \longrightarrow \text{Bun}_{w_\infty}^{\varphi_I(\mu)}.$$

Definition 2.2. Given $I \subset \{1, \dots, n\}$ of even cardinality, let $\mu \in \mathfrak{C}_I$. We will denote

$$\text{Bun}^I = \text{Bun}_{w_\infty}^\mu.$$

When I is the empty set, we write simply Bun instead of Bun^\emptyset ; it corresponds to the moduli space of parabolic vector bundles whose underlying vector bundle is E_1 .

Remark 2.3. From Proposition 2.1, we conclude that $\text{Bun}^I \simeq (\mathbb{P}^1)^n$ for any $I \subset \{1, \dots, n\}$ of even cardinality.

A quasi-parabolic vector bundle (E, \mathbf{p}) is called decomposable if there exist (L, \mathbf{p}') and (M, \mathbf{p}'') such that $(E, \mathbf{p}) \simeq (L, \mathbf{p}') \oplus (M, \mathbf{p}'')$ as quasi-parabolic vector bundles. Otherwise it is called indecomposable. Note that (E, \mathbf{p}) can be indecomposable with E decomposable as a vector bundle.

Lemma 2.4. *Let (E, \mathbf{p}) be a rank two indecomposable quasi-parabolic bundle, over (C, D) , with $\det E = \mathcal{O}_C(w_\infty)$. Then one of the following holds:*

- (1) E is indecomposable, i.e. $E = E_1$;
- (2) $E = L \oplus L^{-1}(w_\infty)$ and $2 \leq 2 \deg L \leq n$;
- (3) $E = L \oplus L^{-1}(w_\infty)$ with $L^2 = \mathcal{O}_C(D + w_\infty)$, hence $2 \deg L = n + 1$. Moreover, every parabolic direction lies on $L^{-1}(w_\infty)$ except for one that lies outside both subbundles.

Proof. When $E = E_1$ we have nothing to prove. So suppose that $E = L \oplus L^{-1}(w_\infty)$. Since $L \oplus L^{-1}(w_\infty) \simeq M \oplus M^{-1}(w_\infty)$ with $M = L^{-1}(w_\infty)$ we can assume $\deg L = s \geq 1$.

To decompose (E, \mathbf{p}) we need to find an embedding of $L^{-1}(w_\infty)$ in E passing through every direction that does not lie on L . Note that this is the same as finding an automorphism of E that sends every direction outside L to $(0 : 1)$. Let $p_j = (u_j : 1)$ denote the parabolic direction over t_j which is outside L . Recall that

$$\text{End}(E) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \mid \alpha, \delta \in \mathbb{C}, \beta \in H^0(C, L^2(-w_\infty)) \right\}.$$

If $2s \geq n + 2$ then $h^0(L^2(-w_\infty - D + t_j)) = 2s - n \geq 2$ and we are free to choose β_j that vanishes on t_i for $i \neq j$ and such that $\beta_j(t_j) = -u_j$. Thus, choosing $\beta = \sum_{j=1}^n \beta_j$, $\alpha = 1$ and $\delta = 1$, the corresponding automorphism sends any direction p_j outside L to $(0 : 1)$.

Now set $2s = n + 1$. By the same argument as above, to show that (E, \mathbf{p}) is decomposable, we need to find a section β_j of $H^0(C, L^2(-w_\infty))$ that vanishes on t_i for $i \neq j$ and such that $\beta_j(t_j) = -u_j$, for each $j \in \{1, \dots, n\}$. We can find β_j as required if $L^2(-w_\infty - D) \neq \mathcal{O}_C$. Indeed, assume $L^2(-w_\infty - D + t_j) \simeq \mathcal{O}_C(x_j)$ with $x_j \neq t_j$, and take any section α_j of $L^2(-w_\infty - D + t_j)$ with $\alpha_j(t_j) \neq 0$. The desired section is defined as $\beta_j = -\frac{u_j}{\alpha_j(t_j)}\alpha_j$. Hence (E, \mathbf{p}) is decomposable when $L^2(-w_\infty - D) \neq \mathcal{O}_C$.

If $L^2 = \mathcal{O}_C(D + w_\infty)$ we can apply the same argument for $D - t_1$ instead of D to find an embedding of $L^{-1}(w_\infty)$ passing through $n - 1$ parabolic directions outside L . In particular, if (E, \mathbf{p}) is indecomposable then there exists no parabolic direction on L and this finishes the proof. \square

Remark 2.5. The parabolic bundles in the third case of Lemma 2.4 have a peculiar property: they are never μ -semistable, whatever is μ . Indeed, for $E = L \oplus L^{-1}(w_\infty)$ with $2 \deg L = n + 1$, no parabolic direction lying on L , and any weight μ , we have

$$\text{Stab}_\mu(L) = -n + \sum_{j=1}^n \mu_j < 0.$$

We can give a partial converse to this fact. Note that if $E = E_1$ then any quasi-parabolic bundle is stable for $\mu \in \mathbb{C}$.

Lemma 2.6. *Let (E, \mathbf{p}) be an indecomposable quasi-parabolic bundle such that $E = L \oplus L^{-1}(w_\infty)$ and p_j lies outside L , for every j . If $2 \leq 2 \deg L \leq n$ then there exists $I \subset \{1, \dots, n\}$, with $|I| = 2 \deg L$, such that $(E, \mathbf{p}) \in \text{Bun}^I$.*

Proof. Given that every direction p_j lies outside L , we may find an embedding of $L^{-1}(w_\infty)$ that passes through some of these directions. Any subset of directions with cardinality $2 \deg L - 1$ admits at most one embedding of $L^{-1}(w_\infty)$ passing through them. Since (E, \mathbf{p}) is indecomposable, such embedding cannot pass through all p_j . In particular, we can find $k \in \{2 \deg L, \dots, n\}$ such that no embedding of $L^{-1}(w_\infty)$ passes through the directions indexed by $I = \{1, \dots, 2 \deg L - 1, k\}$. It is straightforward to verify that $(E, \mathbf{p}) \in \text{Bun}^I$. \square

Until now we have only considered a rank two E and its line subbundles $L \subset E$. But a more general setting will be suitable for the next results; we may allow subsheaves that are not saturated. We will consider general morphisms $L \rightarrow E$ that do not, necessarily, lead to an embedding of L in E . Recall that, over a curve, being a subbundle means that there exist an injective morphism $L \hookrightarrow E$ whose cokernel is also a line bundle, i.e. L is a saturated subsheaf of E . For a general morphism $\phi: L \rightarrow E$ this does not need to be true. However, we can factor out a divisor Z where ϕ

vanishes, leading to an injective morphism $L(Z) \hookrightarrow E$. For details, see [10, Chapter 2, Proposition 5]. On the other hand, given a subbundle $L \subset E$ we can produce a morphism $L(-Z) \rightarrow E$ that vanishes on the fibers over the support of Z .

Given a morphism $\phi: L \rightarrow E$ we say that its image passes through $p_j \subset E_{t_j}$ if $\phi_{t_j}(L_{t_j}) \subset p_j$.

Lemma 2.7. *Let $I \subset \{1, \dots, n\}$ have cardinality $2k + 2$ with $k \geq 0$ and fix $\mu \in \mathfrak{C}_I$. Then (E_1, \mathbf{p}) is not μ -semistable if and only if there exists a line bundle L of degree $\deg L = -k$ and a morphism $L \rightarrow E_1$ whose image passes through p_j for all $j \in I$.*

Proof. Fix $\mu = (\mu_1, \dots, \mu_n) \in \mathfrak{C}_I$ and recall that $\mu = \varphi_I(\mu')$ for some $\mu' \in \mathfrak{C}$. First assume that (E_1, \mathbf{p}) is not μ -semistable and let $M \subset E_1$ be a subbundle such that $\text{Stab}_\mu(M) < 0$. Denote $A = \{j \in \{1, \dots, n\} \mid p_j = M_{t_j}\}$ so that

$$\begin{aligned} 0 > \text{Stab}_\mu(M) &= 1 - 2 \deg M + \sum_{\substack{j \notin A \\ j \in I}} \mu_j + \sum_{\substack{j \notin A \\ j \notin I}} \mu_j - \sum_{\substack{j \in A \\ j \in I}} \mu_j - \sum_{\substack{j \in A \\ j \notin I}} \mu_j \\ &= 1 - 2 \deg M + \sum_{\substack{j \notin A \\ j \in I}} (1 - \mu'_j) + \sum_{\substack{j \notin A \\ j \notin I}} \mu'_j - \sum_{\substack{j \in A \\ j \in I}} (1 - \mu'_j) - \sum_{\substack{j \in A \\ j \notin I}} \mu'_j \\ &= 1 - 2 \deg M + |I| - 2|A \cap I| - \sum_{\substack{j \notin A \\ j \in I}} \mu'_j + \sum_{\substack{j \notin A \\ j \notin I}} \mu'_j + \sum_{\substack{j \in A \\ j \in I}} \mu'_j - \sum_{\substack{j \in A \\ j \notin I}} \mu'_j. \end{aligned}$$

From $\sum_j \mu'_j < 1$ and $|I| = 2(k + 1)$ we get that

$$- \deg M + k + 1 - |A \cap I| \leq -1.$$

Now let $u := \deg M + k$. The inequality above plus $\deg M \leq 0$ implies that

$$|I \setminus (I \cap A)| = 2k + 2 - |A \cap I| \leq u \leq k \leq |I|,$$

hence there exists $J \subset \{1, \dots, n\}$ such that $|J| = u$ and $I \setminus (I \cap A) \subset J \subset I$. Define $Z = \sum_{j \in J} t_j$. Considering the inclusions $M \hookrightarrow E_1$ and $\mathcal{O}_C(-Z) \hookrightarrow \mathcal{O}_C$ we define a map by the composition

$$\phi: M \otimes \mathcal{O}_C(-Z) \longrightarrow E_1(-Z) \longrightarrow E_1,$$

with the property that it gives the same directions over $D - Z$ and vanishes over Z . Hence the image of ϕ passes through every direction from I and $L := M(-Z)$ is our desired line bundle.

Conversely, suppose that there exists a degree $-k$ line bundle L and a nontrivial morphism $\phi: L \rightarrow E_1$ passing through every p_j , $j \in I$. Let Z be the zero divisor of ϕ and consider the reduction $\phi': L(Z) \rightarrow E_1$. Then ϕ' realizes $L(Z)$ as a subbundle of E_1 and, in particular, $\deg Z \leq k$.

On the other hand, we have that p_j lie on $L(Z)$ for every j such that $t_j \notin \text{Supp} Z$. If A is given as above, we have $|A \cap I| \geq 2k + 2 - \deg Z_{red}$, hence

$$\begin{aligned} 1 - 2 \deg L(Z) + |I| - 2|A \cap I| &= 4k + 3 - 2|A \cap I| - 2 \deg Z \\ &\leq 2(\deg Z_{red} - \deg Z) - 1 \leq -1, \end{aligned}$$

which implies $\text{Stab}_\mu(L(Z)) < 0$ for $\mu \in \mathfrak{C}_I$. □

We now see the real advantage of switching to this slightly more general setting. The previous lemma describes a wall-crossing phenomenon. And the next lemma can be used to describe geometrically the space of quasi-parabolic bundles that become unstable when we cross a wall.

Lemma 2.8. *Let $n = 2k + 2$ for some $k \geq 0$. Let $V \subset (\mathbb{P}^1)^n$ be the locus of points that correspond to quasi-parabolic bundles (E_1, \mathbf{p}) satisfying the following property: there exist a line bundle L of degree $-k$ and a morphism $\phi: L \rightarrow E_1$ whose image passes through \mathbf{p} . Then V is a hypersurface of degree $(2, \dots, 2)$.*

Proof. Let $\pi_j: (\mathbb{P}^1)^n \rightarrow (\mathbb{P}^1)^{n-1}$ be the projection given by forgetting the j th component and let h_j be the class of a fiber of π_j . Then we only need to show that $V \cap h_j = 2$ for every j . Up to permuting indices we only need to consider $j = 2k + 2$.

If $k = 0$ the result follows from [31, Proposition 3.3]. Indeed, for each degree 0 line bundle $L \in \text{Pic}^0(C)$ there exists a unique map $\phi: L \rightarrow E_1$ and the map $L \mapsto \phi_{t_1}(L) \in \mathbb{P}^1$ is generically $2 : 1$. Then, for a generic direction p_1 , there exist two choices for $L \in \text{Pic}^0(C)$ such that $\phi_{t_1}(L) \subset p_1$. Therefore, $(p_1, p_2) \in V$ if and only if p_2 is one of the directions defined by these line bundles, i.e. $V \cap h_2 = 2$.

Now we consider $k \geq 1$. We will show that we can reduce to the previous case. Fix p_1, \dots, p_{2k} generic directions. By generic we mean that there exists no subbundle of degree at least $1 - k$ passing through these directions. Let $L \in \text{Pic}^{-k}(C)$ be any line bundle. To give a map $\phi: L \rightarrow E_1$ passing through p_1, \dots, p_{2k} is equivalent to giving a map $L \rightarrow E'$, where E' is obtained by elementary transformation with respect to p_1, \dots, p_{2k} . Indeed, we have

$$0 \rightarrow E' \xrightarrow{\alpha} E_1 \xrightarrow{\beta} \bigoplus_{j=1}^{2k} (E_1)_{t_j}/p_j \rightarrow 0$$

and $\beta \circ \phi = 0$ if and only if there exists $\phi': L \rightarrow E'$ such that $\phi = \alpha \circ \phi'$. Nonetheless, this is equivalent to giving a map

$$\phi' \otimes 1: L \otimes M \rightarrow E' \otimes M$$

where M is a line bundle such that $M^2 = \mathcal{O}_C(t_1 + \dots + t_{2k})$.

Since p_1, \dots, p_{2k} are generic, E' is indecomposable. In particular, $E' \otimes M = E_1$. We then apply the same argument of the case $k = 0$ to the directions p_{2k+1} and p_{2k} to show that $V \cap h_{2k+2} = 2$. \square

Definition 2.9. Let $n \geq 2$ be an integer and let $I \subset \{1, \dots, n\}$ be a subset of even cardinality. We will denote by $\Gamma_I \subset (\mathbb{P}^1)^n$ the subvariety that parameterizes quasi-parabolic bundles (E_1, \mathbf{p}) that are not μ -semistable for $\mu \in \mathfrak{C}_I$.

Corollary 2.10. *The subvariety $\Gamma_I \subset (\mathbb{P}^1)^n$ is a hypersurface of degree (d_1, \dots, d_n) , where $d_i = 2$ if $i \in I$ and $d_i = 0$ otherwise.*

Proof. Note that the formation of Γ_I depends only on the directions indexed by I . Then it will be a product $\Gamma_I \simeq V \times (\mathbb{P}^1)^{n-|I|}$. Therefore, we may reduce to the case $\Gamma_I = V$, i.e. $|I| = n$ and the conclusion follows from Lemma 2.7 and Lemma 2.8. \square

Remark 2.11. A quasi-parabolic bundle (E_1, \mathbf{p}) is not μ -semistable for $\mu \in \mathfrak{C}_I$ if and only if $\text{elm}_I(E_1, \mathbf{p}) = (E, \mathbf{p}')$ is not μ' -semistable for $\mu' = \varphi_I(\mu) \in \mathfrak{C}$. The later occurs if and only if E splits. Therefore Γ_I corresponds, via elm_I , to the locus in Bun^I of those quasi-parabolic bundles whose underlying vector bundles split.

3. LOGARITHMIC CONNECTIONS

A logarithmic connection on a rank two vector bundle E over C with polar divisor $D = t_1 + \dots + t_n$ is a \mathbb{C} -linear map

$$\nabla: E \rightarrow E \otimes \Omega_C^1(D)$$

satisfying the Leibniz rule

$$\nabla(fs) = s \otimes df + f\nabla(s)$$

for (local) sections s of E and f of \mathcal{O}_C . If $t \in C$ is a pole for ∇ and $U \subset C$ is a small trivializing neighborhood of t , we write $\nabla|_U = d + A$ where $d: \mathcal{O}_C \rightarrow \Omega_C^1$ is the exterior derivative and A is a 2×2 matrix whose coefficients are 1-forms with at most simple poles on t . Note that A depends on the trivialization, but its similarity class does not. Then the residue endomorphism

$$\text{Res}_t(\nabla) := \text{Res}_t(A) \in \text{End}(E_t)$$

is well defined. Let ν_k^+ and ν_k^- be the eigenvalues of $\text{Res}_{t_k}(\nabla)$. The data

$$\nu = (\nu_1^+, \nu_1^-, \dots, \nu_n^+, \nu_n^-) \in \mathbb{C}^{2n}$$

are called the *eigenvalues* of ∇ . The induced trace connection

$$\text{tr}(\nabla): \det(E) \rightarrow \det(E) \otimes \Omega_C^1(D)$$

satisfies $\text{Res}_{t_k}(\text{tr}(\nabla)) = \nu_k^+ + \nu_k^-$ and Residue Theorem yields the Fuchs relation:

$$\deg E + \sum_{k=1}^n (\nu_k^+ + \nu_k^-) = 0.$$

Remark 3.1. Hereafter we will fix the following data:

- (1) A $2n$ -tuple of complex numbers $\nu = (\nu_1^+, \nu_1^-, \dots, \nu_n^+, \nu_n^-)$ satisfying the Fuchs relation

$$1 + \sum_{k=1}^n (\nu_k^+ + \nu_k^-) = 0$$

and the generic condition: $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \notin \mathbb{Z}$ for any $\epsilon_k \in \{+, -\}$, to avoid reducible connections; and $\nu_k^+ \neq \nu_k^-$ for all $k \in \{1, \dots, n\}$, so that the residues have distinguished eigenspaces;

- (2) A fixed trace connection $\zeta: \mathcal{O}_C(w_\infty) \rightarrow \mathcal{O}_C(w_\infty) \otimes \Omega_C^1(D)$ satisfying

$$\text{Res}_{t_k}(\zeta) = \nu_k^+ + \nu_k^-$$

for all $k = 1, \dots, n$.

Then we define the moduli space:

$$\mathfrak{Con}^\nu = \left\{ (E, \nabla) \mid \begin{array}{l} \nabla \text{ has eigenvalue } \nu, \\ \det E = \mathcal{O}_C(w_\infty), \text{tr} \nabla = \zeta \end{array} \right\} / \sim$$

where \sim stands for S -equivalence. In fact, the condition (1) on ν implies that \sim may be thought as equivalence up to isomorphism.

Algebraic constructions of moduli spaces of connections goes back to the works of Simpson and, in the logarithmic case, Nitsure in [26]. In our setting, it is more convenient to refer to the works of Inaba, Iwasaki and Saito [14], and more precisely Inaba [13]. Indeed, under our generic assumption on ν , each connection ∇ on E defines a unique parabolic structure, by selecting the eigenspace $p_k \subset E|_{t_k}$ associated to ν_k^+ at each pole t_k ; therefore, \mathfrak{Con}^ν can equivalently be viewed as the moduli space of parabolic connections as considered in the work [13] of Inaba. Then it follows from [13, Theorem 2.1, Proposition 5.2] that it is quasi-projective and irreducible of dimension $2n$. Moreover, [13, Theorem 2.2] shows that it is moreover smooth. In fact, in order to fit with the stability condition [13, Definition 2.2], we set $\alpha_1^{(k)} = \frac{1-\mu_k}{2}$ and $\alpha_2^{(k)} = \frac{1+\mu_k}{2}$; our moduli space therefore corresponds to the fiber $\det^{-1}(L, \zeta)$ of the determinant map considered at the beginning

of [13, Section 5]. When $\nu_k^+ = \nu_k^-$ for some k , there exist connections with scalar residue (apparent singular point) which give rise to a singular locus in the moduli space; the role of the parabolic structure in [13] is to get a smooth moduli space even in that case.

Then the moduli space \mathfrak{Con}^ν is a smooth irreducible quasi-projective variety of dimension $2n$, provided that the condition (1) on ν is satisfied. The case $n = 1$ is not covered by [13], but it follows from [21]; we will discuss this case at the end of this section.

There exists also an analytical construction for \mathfrak{Con}^ν following Nakajima [25]. There, moduli spaces of rank two parabolic connections were constructed via hyper-kähler quotients. Thus \mathfrak{Con}^ν has a holomorphic symplectic structure ω .

In the study of \mathfrak{Con}^ν , it is useful to consider the quasi-parabolic bundles underlying a connection. Let us assume $\nu_k^+ \neq \nu_k^-$ for all $k \in \{1, \dots, n\}$. Given a connection (E, ∇) , we associate, for each $k = 1, \dots, n$, a pair of “positive” and “negative” eigenspaces of $\text{Res}_{t_k}(\nabla)$

$$p_k^+(\nabla), p_k^-(\nabla) \in \mathbb{P}(E_{t_k})$$

corresponding to the eigenvalues ν_k^+ and ν_k^- respectively.

Given an n -tuple $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, where each $\epsilon_i \in \{+, -\}$, we denote

$$\mathbf{p}^\epsilon(\nabla) = \{p_1^{\epsilon_1}(\nabla), \dots, p_n^{\epsilon_n}(\nabla)\}$$

and consider $(E, \mathbf{p}^\epsilon(\nabla))$ the quasi-parabolic vector bundle defined by these directions.

Remark 3.2. The hypothesis that $\nu_1^{a_1} + \dots + \nu_n^{a_n} \notin \mathbb{Z}$, for every $a \in \{+,-\}^n$, ensures that $(E, \mathbf{p}^\epsilon(\nabla))$ is an indecomposable quasi-parabolic bundle. Indeed, if (L, \mathbf{p}) is rank one direct summand of $(E, \mathbf{p}^\epsilon(\nabla))$ then the residues of the induced connection on L are either ν_j^+ or ν_j^- ; their sum is $-\deg L$, see [9, Corollary 2.3].

One can then ask for the stability of these quasi-parabolic bundles with respect to some weight. We define

$$\mathfrak{Con}_{st}^\nu = \left\{ (E, \nabla) \in \mathfrak{Con}^\nu \mid \begin{array}{l} \exists \epsilon \in \{+, -\}^n \text{ and } \exists I \subset \{1, \dots, n\} \\ \text{such that } (E, \mathbf{p}^\epsilon(\nabla)) \in \text{Bun}^I \end{array} \right\}.$$

Recall that $|I|$ is always assumed to be even. It follows that

$$\mathfrak{Con}^\nu = \mathfrak{Con}_{st}^\nu \sqcup \mathcal{Z}_n$$

where \mathcal{Z}_n denotes the complement of \mathfrak{Con}_{st}^ν . Our aim in the next subsections is to describe these varieties. We will show that \mathfrak{Con}_{st}^ν can be covered by simple open subsets and that \mathcal{Z}_n falls in two cases: either n is even and $\mathcal{Z}_n = \emptyset$ or n is odd and \mathcal{Z}_n has four connected components, each one is a quotient of \mathbb{C}^{n+1} by a free affine action of the additive group $(\mathbb{C}, +)$, hence isomorphic to \mathbb{C}^n .

3.1. Connections on E_1 . Our main building block in the description of \mathfrak{Con}_{st}^ν is the space defined by

$$\text{Con}^\nu = \{(E, \nabla) \in \mathfrak{Con}^\nu \mid E = E_1\}.$$

Note that every underlying quasi-parabolic bundle lies in $\text{Bun} = \text{Bun}^\emptyset$, see Proposition 2.1 and Definition 2.2. The same proposition shows that $\text{Bun} \simeq (\mathbb{P}^1)^n$. We will see that Con^ν has a similar description.

Let $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal and let $S := (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$ be its complement. Then we define a map

$$\begin{array}{ccc} \text{Par: } \text{Con}^\nu & \longrightarrow & S^n \\ (E_1, \nabla) & \longmapsto & (p_1^+(\nabla), p_1^-(\nabla); \dots; p_n^+(\nabla), p_n^-(\nabla)) \end{array}.$$

This map is in fact an isomorphism.

Theorem 3.3. *The map $\text{Par}: \text{Con}^\nu \rightarrow S^n$ is an isomorphism.*

Proof. We may factor the map Par into two parts:

$$\text{Con}^\nu \xrightarrow{\text{Res}_D} \bigoplus_{j=1}^n \{A \in \text{End}(E_1)_{t_j} \mid A \text{ has eigenvalues } \nu_j^+, \nu_j^-\} \xrightarrow{\pi} S^n$$

where Res_D is the residue map and π sends an endomorphism to the ordered pair of eigenspaces, $+$ then $-$. Both maps are indeed morphisms and we will see that they are both isomorphisms.

First we deal with π . Given an endomorphism A of $(E_1)_{t_j}$ with eigenvalues $\nu_j^+ \neq \nu_j^-$ we can associate the respective eigenspaces (p^+, p^-) . Conversely, fix a local frame for E_1 around t_j giving coordinates $(E_1)_{t_j} \simeq \mathbb{C}^2$. Then to $((z : w), (u : v)) \in S$ we can associate the matrix

$$A = \frac{1}{zv - uw} \begin{pmatrix} z & u \\ w & v \end{pmatrix} \begin{pmatrix} \nu_j^+ & 0 \\ 0 & \nu_j^- \end{pmatrix} \begin{pmatrix} v & -u \\ -w & z \end{pmatrix}$$

defining an element of $\text{End}(E_1)_{t_j}$. Thus π is bijective.

Next we deal with Res_D . If $\nabla_1, \nabla_2 \in \text{Con}^\nu$ have the same residues at t_j , for $j = 1, \dots, n$, then $\nabla_1 - \nabla_2 \in H^0(\mathcal{E}nd(E_1) \otimes \Omega_C)$. Since E_1 is simple and $\Omega_C = \mathcal{O}_C$ we have $\nabla_1 - \nabla_2 = id_E \otimes \eta$ for some holomorphic 1-form η . On the other hand, the definition of Con^ν imposes that ∇_1 and ∇_2 have the same trace. Thus $\eta = 0$ and $\nabla_1 = \nabla_2$, proving that Res_D is injective.

It remains to prove that Res_D is surjective. Fix $A_j \in \text{End}(E_1)_{t_j}$ for $j = 1, \dots, n$ such that A_j has eigenvalues ν_j^+, ν_j^- . Since E_1 is simple and

$$\deg(E_1) + \sum_{j=1}^n \text{tr}(A_j) = 1 + \sum_{j=1}^n \nu_j^+ + \nu_j^- = 0,$$

[4, Lemma 3.2] guarantees the existence of a connection ∇_0 with residue A_j at t_j . Moreover, $\text{tr}(\nabla_0) - \zeta = \theta \in H^0(\Omega_C)$ a global holomorphic 1-form. Hence $\nabla = \nabla_0 - \frac{1}{2} id_{E_1} \theta$ has the prescribed residues and $\text{tr}(\nabla) = \zeta$ so that $\nabla \in \text{Con}^\nu$. \square

We will give an alternative proof of this theorem using explicit computations of Fuchsian systems, see Remark 4.3.

Corollary 3.4. *Con^ν is an affine variety.*

Proof. Since the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ supports an ample divisor, its complement, S , is affine. Therefore $\text{Con}^\nu \simeq S^n$ is also affine. \square

In the next subsection we will see that S^n is a local model for \mathfrak{Con}_{st}^ν .

3.2. Description of \mathfrak{Con}_{st}^ν . From the definition, \mathfrak{Con}_{st}^ν is the space of (isomorphism classes of) connections (E, ∇) for which there exist $I \subset \{1, \dots, n\}$, with $|I|$ even, and $\epsilon \in \{-, +\}^n$ such that $(E, \mathbf{p}^\epsilon(\nabla)) \in \text{Bun}^I$, i.e. $(E, \mathbf{p}^\epsilon(\nabla))$ is μ -stable for any $\mu \in \mathfrak{C}_I$. For each I and ϵ we define

$$\text{Con}_{I, \epsilon}^\nu := \{(E, \nabla) \in \mathfrak{Con}^\nu \mid (E, \mathbf{p}^\epsilon(\nabla)) \in \text{Bun}^I\},$$

hence we get a decomposition

$$(3.1) \quad \mathfrak{Con}_{st}^\nu = \bigcup_{I, \epsilon} \text{Con}_{I, \epsilon}^\nu.$$

Note that $\text{Con}_{\emptyset, \epsilon}^\nu = \text{Con}^\nu$ for any ϵ . Next we will see that, for generic ν , each $\text{Con}_{I, \epsilon}^\nu$ is isomorphic to S^n . More precisely, we will prove that $\text{Con}_{I, \epsilon}^\nu$ is isomorphic to Con^λ , for some eigenvalue λ to be determined. Consider

$$\pi_\epsilon: \text{Con}_{I, \epsilon}^\nu \longrightarrow \text{Bun}^I$$

the forgetful morphism.

Proposition 3.5. *The map elm_I induces a fiber-preserving isomorphism Φ_I^ϵ :*

$$\begin{array}{ccc} \text{Con}_{I, \epsilon}^\nu & \xrightarrow{\Phi_I^\epsilon} & \text{Con}^\lambda \\ \pi_\epsilon \downarrow & & \downarrow \pi_\epsilon \\ \text{Bun}^I & \xrightarrow{\text{elm}_I} & \text{Bun} \end{array}$$

for $\lambda = (\lambda_1^+, \lambda_1^-, \dots, \lambda_n^+, \lambda_n^-)$ defined as follows. If $k \notin I$ then $\lambda_k^+ = \nu_k^+$ and $\lambda_k^- = \nu_k^-$, and if $k \in I$ then

$$\lambda_k^{\epsilon_k} = \nu_k^{-\epsilon_k} + \frac{1}{2}, \text{ and } \lambda_k^{-\epsilon_k} = \nu_k^{\epsilon_k} - \frac{1}{2},$$

where $\{\epsilon_k, -\epsilon_k\} = \{+, -\}$.

Proof. Given $(E, \nabla) \in \text{Con}_{I, \epsilon}^\nu$ we will perform an elementary transformation centered in $\mathbf{p}^\epsilon(\nabla)$. Recall that elm_I sends $(E, \mathbf{p}^\epsilon(\nabla))$ to $(E' \otimes L, \mathbf{p}')$ where E' is obtained from the exact sequence

$$0 \longrightarrow E' \xrightarrow{\alpha} E \longrightarrow \bigoplus_{i \in I} (E_{t_i} / p_i^{\epsilon_i}(\nabla)) \longrightarrow 0,$$

and L is a square root $\mathcal{O}_C(\sum_{i \in I} t_i)$. The pullback connection $\alpha^*(\nabla)$ has the following property, which can be verified in local coordinates. For $k \notin I$, the eigenvalue at t_k are the same $\{\nu_k^+, \nu_k^-\}$. For $k \in I$, the eigenvalues are

$$(\nu_k^{-\epsilon_k} + 1, \nu_k^{\epsilon_k})$$

and $\ker(\alpha_{t_k})$ corresponds to $\nu_k^{-\epsilon_k} + 1$.

Now let $\xi: L \rightarrow L \otimes \Omega_C^1(D)$ be the rank one connection defined as follows. Let $\{U_i\}$ be a trivializing cover for E, E' and L and let $\{G_{ij}\}, \{G'_{ij}\}$ and $\{h_{ij}\}$ be the respective cocycles. Assume further that $\det G_{ij} = h_{ij}^2 \det G'_{ij}$ for every pair (i, j) . Then

$$\xi|_{U_i} = d - \frac{1}{2} \text{tr}(\alpha_i^{-1} d\alpha_i),$$

where α_i is the local expression for α . Note that $\text{tr}(\alpha^*(\nabla) \otimes \xi) = \text{tr}(\nabla)$ and

$$\text{Res}_{t_k} \xi = \begin{cases} -\frac{1}{2}, & k \in I; \\ 0, & k \notin I. \end{cases}$$

The map Φ_I^ϵ is then defined as

$$\Phi_I^\epsilon(E, \nabla) = (E' \otimes L, \alpha^*(\nabla) \otimes \xi).$$

Since it can be reversed by the same process, we have the isomorphism. Moreover, the diagram in the statement commutes from the construction of Φ_I^ϵ . \square

To give an isomorphism $\text{Con}_{I, \epsilon}^\nu \simeq S^n$ we just need to require that λ satisfies the hypothesis of Theorem 3.3, i.e. $\lambda_k^+ \neq \lambda_k^-$ and in Remark 3.1.

Corollary 3.6. *If, for every $k = 1, \dots, n$, $\nu_k^+ - \nu_k^- \notin \{0, 1, -1\}$ then, for every $I \subset \{1, \dots, n\}$, with $|I|$ even, and every $\epsilon \in \{+, -\}^n$,*

$$\text{Con}_{I,\epsilon}^\nu \simeq S^n.$$

Proof. For $k \notin I$ we have $\lambda_k^+ - \lambda_k^- = \nu_k^+ - \nu_k^-$ and for $k \in I$ we have $\lambda_k^+ - \lambda_k^- = \nu_k^- - \nu_k^+ \pm 1$. The result follows from Proposition 3.5 and Theorem 3.3 once we have $\nu_k^+ - \nu_k^- \notin \{0, 1, -1\}$. \square

Hereafter we will also assume that $\nu_k^+ - \nu_k^- \notin \{0, 1, -1\}$ for every $k = 1, \dots, n$. We will also denote by π_ϵ the projection $\pi_\epsilon: S^n \rightarrow (\mathbb{P}^1)^n$ that makes the following diagram commute

$$\begin{array}{ccc} \text{Con}_{I,\epsilon}^\nu & \longrightarrow & S^n \\ \pi_\epsilon \downarrow & & \downarrow \pi_\epsilon \\ \text{Bun}^I & \longrightarrow & (\mathbb{P}^1)^n \end{array}$$

and we define $\tilde{\Gamma}_{I,\epsilon} = \pi_\epsilon^{-1}(\Gamma_I)$, where Γ_I is the hypersurface from Definition 2.9.

Proposition 3.7. *Let $I, J \subset \{1, \dots, n\}$ with even cardinalities and fix $\epsilon \in \{+, -\}^n$. Then*

$$\text{Con}_{I,\epsilon}^\nu \setminus \text{Con}_{J,\epsilon}^\nu \simeq \text{Con}_{I,\epsilon}^\lambda \setminus \text{Con}_{J,\epsilon}^\lambda \simeq \tilde{\Gamma}_{I\Delta J,\epsilon}$$

where $I\Delta J = (I \cup J) \setminus (I \cap J)$ is their symmetric difference.

Proof. Note that $|I\Delta J|$ is even and $\text{elm}_I \circ \text{elm}_J = \text{elm}_{I\Delta J}$; hence Φ_I^ϵ gives an isomorphism

$$\text{Con}_{I,\epsilon}^\nu \setminus \text{Con}_{J,\epsilon}^\nu \simeq \text{Con}^\lambda \setminus \text{Con}_{I\Delta J,\epsilon}^\lambda.$$

An element of $\text{Con}^\lambda \setminus \text{Con}_{I\Delta J,\epsilon}^\lambda$ is a connection whose underlying quasi-parabolic bundle $(E_1, \mathbf{p}^\epsilon(\nabla))$ is stable for the weights $\mu \in \mathcal{C}$ but its image under $\text{elm}_{I\Delta J}$ is unstable or, equivalently, it is μ -stable but $\varphi_{I\Delta J}(\mu)$ -unstable. Therefore, using Theorem 3.3 and Remark 2.11 we get

$$\text{Par}(\text{Con}^\lambda \setminus \text{Con}_{I\Delta J,\epsilon}^\lambda) = \tilde{\Gamma}_{I\Delta J,\epsilon}.$$

\square

Theorem 3.8. *Assume and $\nu_1^{a_1} + \dots + \nu_n^{a_n} \notin \mathbb{Z}$, for any $a_k \in \{+, -\}$, and that $\nu_k^+ - \nu_k^- \notin \{0, 1, -1\}$ for $k \in \{1, \dots, n\}$. Then \mathfrak{Con}_{st}^ν is obtained by gluing a finite number of copies of S^n via birational maps*

$$\Psi_{J,I}^{\delta,\epsilon}: S^n \dashrightarrow S^n.$$

depending on $\epsilon, \delta \in \{+, -\}^n$ and $I, J \subset \{1, \dots, n\}$. Moreover, if $\epsilon = \delta$, then $\Psi_{J,I}^{\delta,\epsilon}$ preserves the fibers of π_ϵ .

Proof. Since $\mathfrak{Con}_{st}^\nu = \cup_{I,\epsilon} \text{Con}_{I,\epsilon}^\nu$, (3.1), we may give local charts $\text{Par} \circ \Phi_I^\epsilon: \text{Con}_{I,\epsilon}^\nu \rightarrow S^n$. Note that $\text{Con}_{I,\epsilon}^\nu \cap \text{Con}_{J,\delta}^\nu$ is Zariski open. Then the maps $\Psi_{J,I}^{\delta,\epsilon}$ are defined by extending the transition maps.

They fit in the following diagram.

$$\begin{array}{ccccccc}
 & & & & \Psi_{J,I}^{\delta,\epsilon} & & \\
 & & & & \text{-----} & & \\
 S^n & \xleftarrow{\text{Par}} & \text{Con}^\lambda & \xleftarrow{\Phi_I^\epsilon} & \text{Con}_{I,\epsilon}^\nu & \supset & \text{Con}_{I,\epsilon}^\nu \cap \text{Con}_{J,\delta}^\nu & \subset & \text{Con}_{J,\delta}^\nu & \xrightarrow{\Phi_J^\delta} & \text{Con}^\rho & \xrightarrow{\text{Par}} & S^n \\
 \downarrow \pi_\epsilon & & \downarrow \pi_\epsilon & & \downarrow \pi_\epsilon & & & & \downarrow \pi_\delta & & \downarrow \pi_\delta & & \downarrow \pi_\delta \\
 (\mathbb{P}^1)^n & \longleftarrow & \text{Bun} & \longleftarrow & \text{Bun}^I & & & & \text{Bun}^J & \longrightarrow & \text{Bun} & \longrightarrow & (\mathbb{P}^1)^n
 \end{array}$$

If $\delta = \epsilon$ then we can complete the diagram. By abuse of notation, we may write

$$\text{Bun}^{IJ} = \text{Bun}^I \cap \text{Bun}^J$$

the space parabolic bundles in Bun^I that are also μ -stable for $\mu \in \mathfrak{C}_J$ and vice-versa. Thus we get a birational map $\text{Bun}^I \dashrightarrow \text{Bun}^J$ extending the identity on Bun^{IJ} . Hence we get

$$\begin{array}{ccc}
 S^n & \xrightarrow{\Psi_{J,I}^{\epsilon,\epsilon}} & S^n \\
 \pi_\epsilon \downarrow & & \downarrow \pi_\epsilon \\
 (\mathbb{P}^1)^n & \dashrightarrow & (\mathbb{P}^1)^n
 \end{array}$$

□

3.3. Connections with unstable parabolic bundles. Now we describe the space \mathcal{Z}_n of (isomorphism classes of) connections such that every underlying parabolic bundle is not semistable. We will show that it falls in two cases:

- (1) If n is even then $\mathcal{Z}_n = \emptyset$;
- (2) If n is odd then \mathcal{Z}_n has four connected components, each isomorphic to \mathbb{C}^n .

Note that in the last case $\dim \mathcal{Z}_n = n$. Let us make our first reduction.

Lemma 3.9. *If n is even then \mathcal{Z}_n is empty and if n is odd then*

$$(3.2) \quad \mathcal{Z}_n = \{(E, \nabla) \in \mathfrak{Con}^\nu \mid E = L \oplus L^{-1}(w_\infty) \text{ with } L^2 = \mathcal{O}_C(D + w_\infty)\}.$$

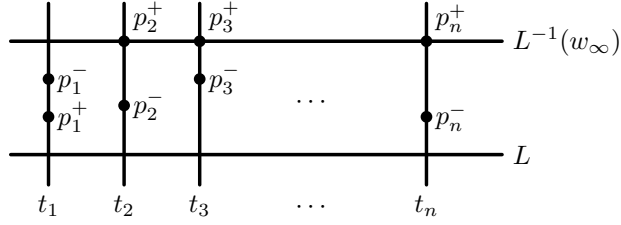
Proof. Suppose that $\mathcal{Z}_n \neq \emptyset$ and let $(E, \nabla) \in \mathcal{Z}_n$. We will show that $E = L \oplus L^{-1}(w_\infty)$ with $L^2 = \mathcal{O}_C(D + w_\infty)$. In particular, n must be odd.

We know that $(E, \mathbf{p}^\epsilon(\nabla))$ is indecomposable for any ϵ , see Remark 3.2. Then we may apply Lemma 2.4. We cannot have $E = E_1$ since it would imply $(E_1, \nabla) \in \text{Con}^\nu$; hence $E = L \oplus L^{-1}(w_\infty)$. In this case, we may take ϵ such that $p_k^{\epsilon k}(\nabla) \notin L_{t_k}$ for every $k = 1, \dots, n$. Note that, by Lemma 2.6, the case $2 \deg L \leq n$ is not possible. Therefore Lemma 2.4 implies that $L^2 = \mathcal{O}_C(D + w_\infty)$.

Conversely, any connection on $L \oplus L^{-1}(w_\infty)$, with $L^2 = \mathcal{O}_C(D + w_\infty)$, represents a point in \mathcal{Z}_n , see Remark 2.5. □

Next we will describe the connections in \mathcal{Z}_n . In order to do so, we compute the logarithmic Atiyah class $\phi_E^A \in \text{End}(E)^\vee$ whose vanishing establishes the existence of a connection with prescribed residues, see [4]. Let $T \in \text{End}(E)$ then ϕ_E^A is defined by

$$\phi_E^A(T) = \phi_E^0(T) + \sum_{j=1}^n \text{tr}(A_j T(t_j))$$

FIGURE 2. Possible configuration of directions for $(E, \nabla) \in \mathcal{Z}_n$.

where A_j is the residue endomorphism over t_j and ϕ_E^0 is the classical Atiyah class, see [3]. In our case,

$$A_j = \begin{pmatrix} u_j & a_j \\ v_j & b_j \end{pmatrix} \begin{pmatrix} \nu_j^+ & 0 \\ 0 & \nu_j^- \end{pmatrix} \begin{pmatrix} b_j & -a_j \\ -v_j & u_j \end{pmatrix}$$

where $u_j b_j - a_j v_j = 1$. Here we may take local coordinates around each t_j such that L and $L^{-1}(w_\infty)$ correspond to $(1 : 0)$ and $(0 : 1)$, respectively.

Note that any direction $p_k^{\epsilon}(\nabla)$ lies outside L , otherwise there would exist a choice of parabolic directions $\mathbf{p}^\epsilon(\nabla)$ such that $(E, \mathbf{p}^\epsilon(\nabla))$ is decomposable and this would force a relation on eigenvalues ν . Indeed, we can find an embedding of $L^{-1}(w_\infty)$ passing through $n - 1$ directions away from L . Then we suppose, without loss of generality, that our directions are as in the Figure 2. In particular $(u_j, v_j) = (0, 1)$ and $a_j = -1$ for $j \geq 2$, and $u_1 v_1 \neq 0$. Up to applying a diagonal automorphism of E we suppose that $u_1 = v_1 = 1$, i.e. $p_1^+(\nabla) = (1 : 1)$.

Note that $\text{End}(E)$ is generated (as a vector space) by the identity, nilpotent endomorphisms and the projection to L . For the identity, $\phi_E^A(1_E)$ gives the Fuchs relation that we already know is valid. Let $\beta \in H^0(C, L^2(-w_\infty))$ and define

$$P(\beta) := \phi_E^A \left(\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \right) = b_1 \beta(t_1) (\nu_1^+ - \nu_1^-) + \sum_{j \geq 2} b_j \beta(t_j) (\nu_j^+ - \nu_j^-).$$

For $j \geq 2$ let $\beta_j \in H^0(C, L^2(-w_\infty))$ with the following property: $\beta_j(t_k) = 0$ if $k \neq 1, j$ and $\beta_j(t_j) = 1$. These sections are unique. In particular, $\beta_j(t_1) \neq 0$ and we have

$$P(\beta_j) = b_1 \beta_j(t_1) (\nu_1^+ - \nu_1^-) + b_j (\nu_j^+ - \nu_j^-)$$

Note that the image of evaluation map $H^0(C, L^2(-w_\infty)) \rightarrow \mathbb{C}^n$, $\beta \mapsto (\beta(t_1), \dots, \beta(t_n))$, has dimension $(n - 1)$; hence the images of the β_j define basis. Therefore $P(\beta) = 0$ for every $\beta \in H^0(C, L^2(-w_\infty))$ if and only if $P(\beta_j) = 0$ for $j \geq 2$, i.e. the ν_j^- direction is

$$(-1 : b_j) = (\nu_j^+ - \nu_j^- : b_1 \beta_j(t_1) (\nu_1^+ - \nu_1^-))$$

For the projection to L we have

$$\phi_E^A \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \deg L + \sum_{j=1}^n u_j b_j \nu_j^+ - a_j v_j \nu_j^- = b_1 (\nu_1^+ - \nu_1^-) + \frac{n+1}{2} + \sum_{j=1}^n \nu_j^- = 0.$$

This implies that the directions over t_1 are $p_1^+(\nabla) = (1 : 1)$ and

$$p_1^-(\nabla) = (b_1 - 1 : b_1) = \left(\frac{n+1}{2} + \nu_1^+ + \sum_{j \geq 2}^n \nu_j^- : \frac{n+1}{2} + \sum_{j=1}^n \nu_j^- \right)$$

Therefore the residues are completely independent of the isomorphism class of (E, ∇) , i.e. the residues of every connection in \mathcal{Z}_n are, up to $\text{Aut}(E)$, in the above configuration. Also note that any two connections with these residues differ by an element of $\text{Hom}(E, E \otimes \Omega_C)$ with vanishing trace. From this discussion we can prove the following result.

Theorem 3.10. *Let n be an odd integer. Then \mathcal{Z}_n has four connected components, each of them being isomorphic to \mathbb{C}^n .*

Proof. First note that there exist precisely four possibilities for the underlying vector bundle of a connection in \mathcal{Z}_n . Indeed, Lemma 3.9 shows that any such vector bundle is $E = L \oplus L^{-1}(w_\infty)$ where L is such that $L^2 = \mathcal{O}_C(D + w_\infty)$. Twisting by 2-torsion line bundles yields four non-isomorphic possibilities for L . Hence four non-isomorphic possibilities for E . Therefore \mathcal{Z}_n has four connected components.

Fix one such E and denote \mathcal{Z}_n^E the corresponding component of \mathcal{Z}_n . Up to the action of $\text{Aut}(E)$, we can fix a configuration of directions as in Figure 2 so that we may only consider connections on E that have this configuration. Note that since $L^2 = \mathcal{O}_C(D + w_\infty)$ the stabilizer of such configuration in $\text{Aut}(E)$ is a copy of the additive group $(\mathbb{C}, +)$ generated by

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

where $\beta \in H^0(L^2(-w_\infty)) \setminus \{0\}$ vanishes on D . On the other hand, if we fix a connection ∇_0 on E , for any other connection ∇ , the difference $\nabla - \nabla_0 \in \text{Hom}(E, E \otimes \Omega_C)$ is a holomorphic Higgs field. Since ∇ and ∇_0 must have the same trace, this Higgs field is traceless. Thus we have an isomorphism

$$\begin{aligned} \mathcal{Z}_n^E &\longrightarrow \text{Higgs}_0(E)/(\mathbb{C}, +) \\ [\nabla] &\longmapsto [\nabla - \nabla_0] \end{aligned}$$

where $\text{Higgs}_0(E)$ is the space of traceless Higgs fields. Note that $\text{Higgs}_0(E) \simeq \mathbb{C}^{n+1}$.

We now explicitly describe the action of $(\mathbb{C}, +)$ on $\text{Higgs}_0(E)$. Locally, we can write

$$\nabla_0 = d + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \text{ and } \varphi = \begin{pmatrix} a_1 & b_1 \\ 0 & -a_1 \end{pmatrix}.$$

Then the action of $t \in \mathbb{C}$ is given by

$$\begin{aligned} t \cdot \varphi &= \begin{pmatrix} 1 & -t\beta \\ 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 0 & td\beta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_0 + a_1 & b_0 + b_1 \\ c_0 & d_0 - a_1 \end{pmatrix} \begin{pmatrix} 1 & t\beta \\ 0 & 1 \end{pmatrix} \right] - \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & -2t\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & -a_1 \end{pmatrix} + \begin{pmatrix} -tc_0\beta & t[(a_0 - d_0)\beta + d\beta] - t^2c_0\beta^2 \\ 0 & tc_0\beta \end{pmatrix}. \end{aligned}$$

In particular, the action is given by affine transformations. Also note that $c_0 = 0$ would force an integer relation $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \in \mathbb{Z}$ which is not possible; hence the action is free. This concludes the proof since the quotient of an affine free action of $(\mathbb{C}, +)$ on \mathbb{C}^{n+1} must be isomorphic to \mathbb{C}^n , see [27, Corollary 7]. \square

In addition, notice that given (E, ∇) representing a point in \mathcal{Z}_n (as in (3.2)) we may perform an elementary transformation centered at all parabolic directions to get (E', ∇') . The subbundle $L \hookrightarrow E = L \oplus L^{-1}(w_\infty)$ becomes $U = L(-D) \otimes M \hookrightarrow E'$ where $M^2 = \mathcal{O}_C(D - w_\infty)$. From $L^2 = \mathcal{O}_C(D + w_\infty)$ we have that $U^2 = \mathcal{O}_C$ and we have an extension

$$(3.3) \quad 0 \longrightarrow U \longrightarrow E' \longrightarrow U \longrightarrow 0.$$

Since L does not pass through any parabolic direction on E , its transformed U passes through every parabolic direction on E' . Thus E' is indecomposable, otherwise we would have a decomposable quasi-parabolic bundle contradicting Remark 3.2. Therefore U is one of the 4 torsion line bundles and E' is the corresponding unique indecomposable extension.

We conclude that any connection representing a point in \mathcal{Z}_n can be obtained from a connection over an indecomposable vector bundle E' as in (3.3), by performing an elementary transformation centered in n parabolic directions which lie in the unique maximal subbundle. Note that there exist n directions to choose and $\dim \text{Higgs}_0(E') = 1$ giving dimension $n + 1$ for the space of connections on E' . Taking the quotient by the action of the automorphism group of the corresponding vector bundle, we get dimension n .

3.4. The classical Painlevé case. Here it is worth to compare the case $n = 1$ with the Painlevé case, i.e. those logarithmic connections over the 4-punctured Riemann sphere. We assume that $D = w_\infty$ and let $(\nu, -\nu - 1)$, $\nu \in \mathbb{C} \setminus \mathbb{Z}$, denote the eigenvalues. The moduli space \mathfrak{Con}^ν parametrizes (E, ∇) , where ∇ is a logarithmic connection over C with a single logarithmic pole over the point w_∞ , i.e. it is a Lamé connection. The whole moduli space writes

$$\mathfrak{Con}^\nu = \text{Con}^\nu \sqcup \mathcal{Z}_1$$

where $\text{Con}^\nu \simeq S$ is formed by (E, ∇) with $E = E_1$. It follows from Theorem 3.10 that \mathcal{Z}_1 is a union of four irreducible curves isomorphic to \mathbb{C} :

$$(3.4) \quad \mathcal{Z}_1^i = \{(E, \nabla) \in \mathfrak{Con}^\nu \mid E = L \oplus L^{-1}(w_\infty) \text{ with } L = \mathcal{O}_C(w_i)\}$$

for $i \in \{0, 1, \lambda, \infty\}$, where w_i are Weierstrass points.

We let $\text{Con}^\theta(\mathbb{P}^1, T)$, $T = 0 + 1 + \lambda + \infty$, denote the moduli space of logarithmic connections over the 4-punctured Riemann sphere (\mathbb{P}^1, T) , with eigenvalue

$$\theta = \left(\pm \frac{1}{4}, \pm \frac{1}{4}, \pm \frac{1}{4}, \theta^\pm \right); \quad \theta^+ = \frac{\nu}{2} - \frac{1}{4}, \quad \theta^- = -\theta^+ - 1.$$

The determinant line bundle and the trace connection on it are fixed, i.e. we fix a rank one connection ζ on $\mathcal{O}_{\mathbb{P}^1}(1)$ with residue -1 at ∞ and for any $(E, \nabla) \in \text{Con}^\theta(\mathbb{P}^1, T)$ we have

$$(\det E, \text{tr} \nabla) = (\mathcal{O}_{\mathbb{P}^1}(1), \zeta).$$

The eigenvalue θ is chosen so that we can construct a map

$$F: \text{Con}^\theta(\mathbb{P}^1, T) \longrightarrow \mathfrak{Con}^\nu$$

which turns out to be an isomorphism (see [21]), and which we now describe. Given

$$(E, \nabla) \in \text{Con}^\theta(\mathbb{P}^1, T)$$

we first take its pullback $(\pi^*E, \pi^*\nabla)$ via the 2-cover $\pi: C \rightarrow \mathbb{P}^1$, the underlying vector bundle has determinant $\det(\pi^*E) = \mathcal{O}_C(2w_\infty)$. After π^* the eigenvalues are multiplied by 2 and by performing an elementary transformation $\text{elm}_{0,1,\lambda}$ over w_0, w_1 and w_λ , centered on directions corresponding to eigenvalues $1/2$, we get

$$(E', \nabla') = \text{elm}_{0,1,\lambda}(\pi^*E, \pi^*\nabla)$$

with $\det E' = \mathcal{O}_C(-w_\infty)$, recall that $w_0 + w_1 + w_\lambda \sim 3w_\infty$. The transformed connection ∇' has apparent singularities over w_0, w_1 and w_λ , thus they disappear after twist by a suitable rank one connection ζ_0 on $\mathcal{O}_C(w_\infty)$. More precisely, we may fix ζ_0 with residues

$$-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \text{ and } \frac{1}{2}$$

over w_0, w_1, w_λ and w_∞ to get

$$F(E, \nabla) := (E', \nabla') \otimes (\mathcal{O}_C(w_\infty), \zeta_0) \in \mathfrak{Con}'.$$

We now describe the preimage of \mathcal{Z}_1 via F . For any $(E, \nabla) \in \text{Con}^\theta(\mathbb{P}^1, T)$ the underlying vector bundle is always $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, see [21]. For each $i \in \{0, 1, \lambda\}$ we let $p_i^+(\nabla)$ denote the parabolic direction with respect to the eigenvalue $1/4$ and $p_\infty^+(\nabla)$ to be the θ^+ direction. Then define $(E, p^+(\nabla))$ the corresponding quasiparabolic bundle. Since $\nu \notin \mathbb{Z}$, we have that $(E, p^+(\nabla))$ is indecomposable and, in particular, there exist at most one direction lying in $\mathcal{O}_{\mathbb{P}^1}(1)$. Indeed, given two parabolic directions outside $\mathcal{O}_{\mathbb{P}^1}(1)$, we can choose an embedding of $\mathcal{O}_{\mathbb{P}^1} \rightarrow E$ passing through them.

Then define

$$A^i = \{(E, \nabla) \in \text{Con}^\theta(\mathbb{P}^1, T) \mid p_i^+(\nabla) \in \mathcal{O}_{\mathbb{P}^1}(1)\}$$

and let A^∞ be the locus of $(E, \nabla) \in \text{Con}^\theta(\mathbb{P}^1, T)$ such that there is an embedding of $\mathcal{O}_{\mathbb{P}^1}$ passing through the three directions $p_0^+(\nabla), p_1^+(\nabla), p_\lambda^+(\nabla)$.

Claim 1. *We claim that F sends A^i to \mathcal{Z}_1^i .*

Let us assume $i = 0$, other cases are similar. Pulling back $(E, \nabla) \in A^0$ to the elliptic curve we get

$$(\pi^*E, \pi^*\nabla) = (\mathcal{O}_C \oplus \mathcal{O}_C(2w_\infty), \pi^*\nabla)$$

with one parabolic direction $p_0^+(\pi^*\nabla)$ lying in $\mathcal{O}_C(2w_\infty)$. We may assume that \mathcal{O}_C pass through $p_1^+(\nabla)$ and $p_\lambda^+(\nabla)$, thus after elementary transformation $\text{elem}_{0,1,\lambda}$ we get

$$(E', \nabla') = (\mathcal{O}_C(-w_0) \oplus \mathcal{O}_C(2w_\infty - w_1 - w_\lambda), \nabla')$$

and twisting by $(\mathcal{O}_C(w_\infty), \zeta_0)$ we obtain

$$F(E, \nabla) = (\tilde{E}, \nabla' \otimes \zeta_0)$$

where $\tilde{E} \simeq L \oplus L^{-1}(w_\infty)$, with $L = \mathcal{O}_C(w_0)$. Consequently, $F(E, \nabla) \in \mathcal{Z}_1^0$. This proves the claim.

4. FUCHSIAN SYSTEMS WITH $n + 3$ POLES

Given $(E_1, \nabla) \in \text{Con}'$, we can associate a \mathfrak{sl}_2 -connection on the trivial bundle $\mathcal{O}_C \oplus \mathcal{O}_C$ by performing an elementary transformation for a particular choice of directions over the 2-torsion points w_0, w_1 and w_λ . We begin the section establishing this correspondence. The process creates new singularities which are apparent, i.e. they become regular points after one elementary transformation.

We have that $t \in C$ is an apparent singular point for ∇ if the residual part $\text{Res}_t \nabla$ has $\{\frac{1}{2}, -\frac{1}{2}\}$ as eigenvalues and the $\frac{1}{2}$ -eigenspace of $\text{Res}_t \nabla$ is also invariant by the constant part of the connection matrix.

Recall that $D' = w_0 + w_1 + w_\lambda + D$ is also reduced, see Remark 1.1. Let $(\nu_1, \dots, \nu_n) \in (\mathbb{C}^*)^n$ and fix an eigenvalue σ by setting $(\frac{1}{2}, -\frac{1}{2})$ over w_0, w_1 and w_λ , and $(\frac{\nu_j}{2}, -\frac{\nu_j}{2})$ over t_j , $j = 1, \dots, n$, i.e.

$$(4.1) \quad \sigma = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{\nu_1}{2}, -\frac{\nu_1}{2}, \dots, \frac{\nu_n}{2}, -\frac{\nu_n}{2} \right).$$

We denote by $\text{Syst}^\sigma(C, D')$ the moduli space of Fuchsian systems (i.e. logarithmic \mathfrak{sl}_2 -connections on the trivial bundle $\mathcal{O}_C \oplus \mathcal{O}_C$) having pole divisor D' , eigenvalue σ and such that:

- the three singular points w_0, w_1 and w_λ are apparent;
- over w_0, w_1 and w_λ the corresponding $\frac{1}{2}$ -eigenspaces are $(1 : 0)$, $(1 : 1)$ and $(0 : 1)$ respectively.

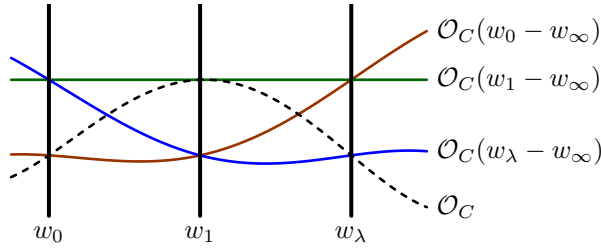


FIGURE 3. Sections of $\mathbb{P}(E_1)$

Proposition 4.1. *There is an isomorphism of moduli spaces*

$$\text{Con}^\nu \xrightarrow{\sim} \text{Syst}^\sigma(C, D')$$

where $\nu_j = \nu_j^+ - \nu_j^-$ for $j = 1, \dots, n$.

Proof. Let $(E_1, \nabla) \in \text{Con}^\nu$. The subbundles $\mathcal{O}_C(w_0 - w_\infty), \mathcal{O}_C(w_1 - w_\infty), \mathcal{O}_C(w_\lambda - w_\infty) \subset E_1$ intersect as in Figure 3 and we apply elementary transformation on these directions so that they become three disjoint copies of $\mathcal{O}_C(-w_\infty)$. Hence the elementary transformed of (E_1, ∇) is (E, ∇') where $E = \mathcal{O}_C(-w_\infty) \oplus \mathcal{O}_C(-w_\infty)$ and the eigenvalues of ∇' are the same as ∇ at the t_k , $k = 1, \dots, n$, and the eigenvalues at w_0, w_1 and w_λ are equal to $(1, 0)$. Consider the connection $(\mathcal{O}_C(w_\infty), \xi)$ such that

$$\text{Res}_{w_0} \xi = \text{Res}_{w_1} \xi = \text{Res}_{w_\lambda} \xi = -\frac{1}{2} \quad \text{and} \quad \text{Res}_{t_k} \xi = -\frac{\nu_k^+ + \nu_k^-}{2}.$$

It exists since $\sum_{k=1}^n \nu_k^+ + \nu_k^- = -1$ by Fuchs relation. Therefore

$$(E_1, \nabla) \mapsto (\mathcal{O}_C \oplus \mathcal{O}_C, \nabla' \otimes \xi) \in \text{Syst}^\sigma(C, D')$$

is our desired isomorphism. \square

Consider the space Bun_0 of parabolic bundles (E, \mathbf{p}) over (C, D') such that E is the trivial bundle, the parabolic directions over w_0, w_1 and w_λ are $(1 : 0)$, $(1 : 1)$ and $(0 : 1)$, respectively, and we let the parabolic directions over $D = t_1 + \dots + t_n$ vary. We have a natural identification $\text{Bun}_0 = (\mathbb{P}^1)^n$ where each copy of \mathbb{P}^1 parametrizes parabolic directions over t_j .

The elementary transformation used in the proof yields an alternative proof for the isomorphism $\text{Bun} \simeq (\mathbb{P}^1)^n$ in Proposition 2.1. Indeed, (E_1, \mathbf{p}) corresponds to $(\mathcal{O}_C \oplus \mathcal{O}_C, \mathbf{p}')$ and, up to automorphism, we can fix the directions over w_0, w_1 and w_λ so that it defines an element of Bun_0 ; in particular, this gives us the identification

$$(4.2) \quad \text{Bun} \simeq \text{Bun}_0.$$

4.1. The affine bundle of Fuchsian systems. The moduli space $\text{Syst}^\sigma(C, D')$ is an affine bundle of rank n over Bun_0 . We will describe its trivializations over the Zariski open sets

$$U_0 = \prod_{j=1}^n \{(z_j : w_j) \in \mathbb{P}^1 \mid w_j = 1\} \quad \text{and} \quad U_\infty = \prod_{j=1}^n \{(z_j : w_j) \in \mathbb{P}^1 \mid z_j = 1\},$$

and then give the affine transition map.

In order to make explicit computations, we will consider that C is given by the affine equation $y^2 = x(x-1)(x-\lambda)$, with $\lambda \in \mathbb{C} \setminus \{0, 1\}$, and $w_\infty \in C$ will be the point at infinity. In particular, the torsion points are $w_0 = (0, 0)$, $w_1 = (1, 0)$ and $w_\lambda = (\lambda, 0)$. To begin with, let us fix a basis of meromorphic one-forms with at most simple poles on D' . Consider the holomorphic one-form $\omega = \frac{dx}{2y} = \frac{dy}{3x^2 - 2(1+\lambda)x + \lambda}$ and define

$$\phi_0 = \frac{(1-\lambda)x}{y}\omega, \quad \phi_1 = \frac{-\lambda(x-1)}{y}\omega \quad \text{and} \quad \theta_j = \frac{x_j(x_jx - \lambda)}{x_jy - y_jx}\omega,$$

where $t_j = (x_j, y_j)$. These $n+3$ one-forms give us the desired basis, and their residues (and constant term at w_i) are given as follows in terms of local coordinate y :

- at w_0 , we have $x = O(y^2)$, and:

$$\omega = \frac{dy}{\lambda} + O(y), \quad \phi_0 = O(y), \quad \phi_1 = \frac{dy}{y} + O(y), \quad \theta_j = -\frac{dy}{y} - \frac{y_j}{\lambda x_j} dy + O(y);$$

- at w_1 , we have $x = 1 + O(y^2)$, and:

$$\omega = \frac{dy}{1-\lambda} + O(y), \quad \phi_0 = \frac{dy}{y} + O(y), \quad \phi_1 = O(y), \quad \theta_j = \frac{x_j(x_j - \lambda)}{y_j} dy + O(y);$$

- at w_λ , we have $x = \lambda + O(y^2)$, and:

$$\omega = \frac{dy}{\lambda(\lambda-1)} + O(y), \quad \phi_0 = -\frac{dy}{y} + O(y), \quad \phi_1 = -\frac{dy}{y} + O(y), \quad \theta_j = -\frac{x_j(x_j - 1)}{y_j} dy + O(y);$$

- at t_j , the 1-forms are holomorphic except for θ_j which has a simple pole with $\text{Res}_{t_j} \theta_j = 1$.

Therefore, any Fuchsian system $\nabla \in \text{Syst}^\sigma(C, D')$ can be written as

$$\nabla = d + \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix} \phi_0 + \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} \phi_1 + \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix} \omega + \sum_{j=1}^n \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & -\alpha_j \end{pmatrix} \theta_j,$$

and, as we impose apparent singularities at w_0 , w_1 and w_λ , we get

$$\begin{aligned} a_0 &= -\sum_{j=1}^n \alpha_j, & b_0 &= \frac{1}{2} + \sum_{j=1}^n \alpha_j, & c_0 &= \frac{1}{2} - \sum_{j=1}^n \alpha_j, \\ a_1 &= \frac{1}{2} + \sum_{j=1}^n \alpha_j, & b_1 &= -\frac{1}{2} - \sum_{j=1}^n \alpha_j, & c_1 &= \sum_{j=1}^n \gamma_j, \\ a_2 &= \sum_{j=1}^n \frac{y_j}{2} \left[\frac{2\alpha_j + \beta_j - \gamma_j}{x_j - 1} + \frac{\gamma_j}{x_j} - \frac{\beta_j}{x_j - \lambda} \right], & b_2 &= \sum_{j=1}^n \frac{y_j}{(x_j - \lambda)} \beta_j, & c_2 &= \sum_{j=1}^n \frac{y_j}{x_j} \gamma_j. \end{aligned}$$

In particular, ∇ is completely determined by the residues

$$\text{Res}_{t_j}(\nabla) = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & -\alpha_j \end{pmatrix}.$$

Proposition 4.2. *If $\nu_j \neq 0$ for $j = 1, \dots, n$ then the map $\text{Par}: \text{Syst}^\sigma(C, D') \rightarrow S^n$ that associates to ∇ the eigenspaces of $\text{Res}_{t_j}(\nabla)$, $j = 1, \dots, n$, is an isomorphism.*

Proof. We know that ∇ is determined by $\text{Res}_{t_j}(\nabla)$, $j = 1, \dots, n$. On the other hand,

$$\begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & -\alpha_j \end{pmatrix} = \frac{1}{zv - wu} \begin{pmatrix} z & u \\ w & v \end{pmatrix} \cdot \begin{pmatrix} \frac{\nu_j}{2} & 0 \\ 0 & -\frac{\nu_j}{2} \end{pmatrix} \cdot \begin{pmatrix} v & -u \\ -w & z \end{pmatrix}$$

where $(z : w)$ and $(u : v)$ are the eigenspaces associated to $\frac{\nu_j}{2}$ and $-\frac{\nu_j}{2}$, respectively. Note that $S = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{zv - wu = 0\}$, hence the isomorphism is clear. \square

Remark 4.3. Note that we can give an alternative proof of Theorem 3.3 using that Proposition 4.1 and Proposition 4.2 combined give an isomorphism by the composition

$$\text{Con}^\nu \longrightarrow \text{Syst}^\sigma(C, D') \longrightarrow S^n.$$

4.1.1. *Trivialization over U_0 .* Given $\mathbf{z} = (z_1, \dots, z_n) \in U_0$ we define ∇_0 as the connection which has $(z_j : 1)$ as eigenspaces corresponding to $\frac{\nu_j}{2}$ and has $(1 : 0)$ as eigenspaces corresponding to $-\frac{\nu_j}{2}$. It can be written as

$$\nabla_0 = d + \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix} \phi_0 + \begin{pmatrix} a_1 & b_1 \\ 0 & -a_1 \end{pmatrix} \phi_1 + \begin{pmatrix} a_2 & b_2 \\ 0 & -a_2 \end{pmatrix} \omega + \sum_{j=1}^n \frac{\nu_j}{2} \begin{pmatrix} -1 & 2z_j \\ 0 & 1 \end{pmatrix} \theta_j.$$

with

$$\begin{aligned} a_0 &= \sum_{j=1}^n \frac{\nu_j}{2} & b_0 &= \frac{1}{2} - \sum_{j=1}^n \frac{\nu_j}{2} & c_0 &= \frac{1}{2} + \sum_{j=1}^n \frac{\nu_j}{2} \\ a_1 &= \frac{1}{2} - \sum_{j=1}^n \frac{\nu_j}{2} & b_1 &= -\frac{1}{2} + \sum_{j=1}^n \frac{\nu_j}{2} \\ a_2 &= \sum_{j=1}^n \frac{\nu_j y_j}{2} \left[\frac{z_j - 1}{x_j - 1} - \frac{z_j}{x_j - \lambda} \right] & b_2 &= \sum_{j=1}^n \nu_j y_j \frac{z_j}{(x_j - \lambda)} \end{aligned}$$

Now if ∇ is any connection which has \mathbf{z} as positive eigenspaces then the difference $\Theta = \nabla - \nabla_0$ is a Higgs field which is nilpotent with respect to \mathbf{z} . This means that Θ is a strongly parabolic

Higgs field over $(\mathcal{O}_C \oplus \mathcal{O}_C, \mathbf{z})$. We shall fix a basis $\{\Theta_1^0, \dots, \Theta_n^0\}$ for the space of strongly parabolic Higgs fields such that

$$\text{Res}_{t_j} \Theta_j^0 = \begin{pmatrix} z_j & -z_j^2 \\ 1 & -z_j \end{pmatrix}$$

and $\text{Res}_{t_i} \Theta_j^0 = 0$ for $i \neq j$. So, we define

$$\Theta_j^0 = \begin{pmatrix} -z_j & z_j \\ -z_j & z_j \end{pmatrix} \phi_0 + \begin{pmatrix} z_j & -z_j \\ 1 & -z_j \end{pmatrix} \phi_1 + A_j \cdot \omega + \begin{pmatrix} z_j & -z_j^2 \\ 1 & -z_j \end{pmatrix} \theta_j$$

where

$$A_j = \begin{pmatrix} \frac{y_j}{2} \left[\frac{-(z_j-1)^2}{x_j-1} + \frac{1}{x_j} + \frac{z_j^2}{x_j-\lambda} \right] & -\frac{y_j z_j^2}{x_j-\lambda} \\ \frac{y_j}{x_j} & -\frac{y_j}{2} \left[\frac{-(z_j-1)^2}{x_j-1} + \frac{1}{x_j} + \frac{z_j^2}{x_j-\lambda} \right] \end{pmatrix}.$$

This matrix A_j has been chosen to assure apparent singularities over w_0, w_1 and w_λ .

Any \mathfrak{sl}_2 -connection $(E, \nabla) \in \text{Syst}^\sigma(C, D')$ having $(z_j : 1)$ as parabolic direction (over t_j) corresponding to $\frac{\nu_j}{2}$ can be written as

$$\nabla = \nabla_0 + r_j \Theta_j^0$$

for suitable $r_j \in \mathbb{C}$. It has $(z_j r_j - \nu_j : r_j)$ as the complementary direction corresponding to $-\frac{\nu_j}{2}$.

4.1.2. *Trivialization over U_∞ .* Similarly, we define ∇_∞ having $(1 : w_j)$ as eigenspaces corresponding to $\frac{\nu_j}{2}$ and $(0 : 1)$ as eigenspaces corresponding to $-\frac{\nu_j}{2}$:

$$\nabla_\infty = d + \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix} \phi_0 + \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} \phi_1 + \begin{pmatrix} a_2 & 0 \\ c_2 & -a_2 \end{pmatrix} \omega + \sum_{j=1}^n \frac{\nu_j}{2} \begin{pmatrix} 1 & 0 \\ 2w_j & -1 \end{pmatrix} \theta_j$$

with

$$\begin{aligned} a_0 &= -\sum_{j=1}^n \frac{\nu_j}{2} & b_0 &= \frac{1}{2} + \sum_{j=1}^n \frac{\nu_j}{2} & c_0 &= \frac{1}{2} - \sum_{j=1}^n \frac{\nu_j}{2} \\ a_1 &= \frac{1}{2} + \sum_{j=1}^n \frac{\nu_j}{2} & b_1 &= -\frac{1}{2} - \sum_{j=1}^n \frac{\nu_j}{2} & c_1 &= \sum_{j=1}^n \nu_j w_j \\ a_2 &= \sum_{j=1}^n \frac{\nu_j y_j}{2} \left[\frac{(1-w_j)}{x_j-1} + \frac{w_j}{x_j} \right] & c_2 &= \sum_{j=1}^n \frac{y_j}{x_j} \nu_j w_j \end{aligned}$$

And the strongly parabolic Higgs fields are defined by

$$\Theta_j^\infty = \begin{pmatrix} -w_j & w_j \\ -w_j & w_j \end{pmatrix} \phi_0 + \begin{pmatrix} w_j & -w_j \\ w_j^2 & -w_j \end{pmatrix} \phi_1 + B_j \cdot \omega + \begin{pmatrix} w_j & -1 \\ w_j^2 & -w_j \end{pmatrix} \theta_j$$

where

$$B_j = \begin{pmatrix} \frac{y_j}{2} \left[\frac{-(1-w_j)^2}{x_j-1} + \frac{w_j^2}{x_j} + \frac{1}{x_j-\lambda} \right] & \frac{-y_j}{(x_j-\lambda)} \\ \frac{y_j w_j^2}{x_j} & -\frac{y_j}{2} \left[\frac{-(1-w_j)^2}{x_j-1} + \frac{w_j^2}{x_j} + \frac{1}{x_j-\lambda} \right] \end{pmatrix}.$$

Given a Fuchsian system $\nabla \in \text{Syst}^\sigma(C, D')$ with $(1 : w_j)$ as parabolic direction (over t_j) corresponding to $\frac{\nu_j}{2}$, we can write

$$\nabla = \nabla_\infty + s_j \Theta_j^\infty$$

for suitable $s_j \in \mathbb{C}$. The complementary direction corresponding to $-\frac{\nu_j}{2}$ is $(s_j : w_j s_j + \nu_j)$.

4.1.3. *Transition matrix.* From the trivializations on U_0 and U_∞ we may compute a transition affine transformation for the affine bundle $\text{Syst}^\sigma(C, D')$. Since $U_0 \cup U_\infty$ covers the basis $\text{Bun}_0 \simeq (\mathbb{P}^1)^n$ minus a subvariety of codimension two, then the bundle structure of $\text{Syst}^\sigma(C, D')$ is determined by this affine transformation.

In order to make this affine transformation explicit we note that

$$\Theta_j^\infty = w_j^2 \Theta_j^0 \left(\frac{1}{w_j} \right),$$

and

$$\nabla_0 \left(\frac{1}{w} \right) = d + \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix} \phi_0 + \begin{pmatrix} a_1 & b_1 \\ 0 & -a_1 \end{pmatrix} \phi_1 + \begin{pmatrix} a_2 & b_2 \\ 0 & -a_2 \end{pmatrix} \omega + \sum_{j=1}^n \frac{\nu_j}{2} \begin{pmatrix} -1 & \frac{2}{w_j} \\ 0 & 1 \end{pmatrix} \theta_j.$$

with

$$\begin{aligned} a_0 &= \sum_{j=1}^n \frac{\nu_j}{2} & b_0 &= \frac{1}{2} - \sum_{j=1}^n \frac{\nu_j}{2} & c_0 &= \frac{1}{2} + \sum_{j=1}^n \frac{\nu_j}{2} \\ a_1 &= \frac{1}{2} - \sum_{j=1}^n \frac{\nu_j}{2} & b_1 &= -\frac{1}{2} + \sum_{j=1}^n \frac{\nu_j}{2} \\ a_2 &= \sum_{j=1}^n \frac{y_j \nu_j}{2w_j} \left[\frac{1-w_j}{x_j-1} - \frac{1}{x_j-\lambda} \right] & b_2 &= \sum_{j=1}^n \frac{y_j}{(x_j-\lambda)} \frac{\nu_j}{w_j} \end{aligned}$$

Hence

$$(4.3) \quad \nabla_\infty = \nabla_0 \left(\frac{1}{w_1}, \dots, \frac{1}{w_n} \right) + \sum_{j=1}^n \nu_j w_j \Theta_j^0 \left(\frac{1}{w_j} \right).$$

Over the intersection $U_0 \cap U_\infty$ we have

$$\begin{aligned} (s_1, \dots, s_n) &\mapsto \nabla_\infty(w_1, \dots, w_n) + \sum_{j=1}^n s_j \Theta_j^\infty(w_j) = \\ &= \nabla_0 \left(\frac{1}{w_1}, \dots, \frac{1}{w_n} \right) + \sum_{j=1}^n (\nu_j w_j + s_j w_j^2) \Theta_j^0 \left(\frac{1}{w_j} \right) = \\ &= \nabla_0(z_1, \dots, z_n) + \sum_{j=1}^n (\nu_j w_j + s_j w_j^2) \Theta_j^0(z_j) = \\ &= \nabla_0(z_1, \dots, z_n) + \sum_{j=1}^n r_j \Theta_j^0(z_j) \end{aligned}$$

which gives us the transition affine transformation

$$(4.4) \quad \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} w_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & w_n^2 \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} + \begin{pmatrix} \nu_1 w_1 \\ \vdots \\ \nu_n w_n \end{pmatrix}.$$

Note that if we choose new trivializations such that $r'_j = \frac{\rho_j}{\nu_j} r_j$ and $s'_j = \frac{\rho_j}{\nu_j} s_j$, this transition map becomes

$$\begin{pmatrix} r'_1 \\ \vdots \\ r'_n \end{pmatrix} = \begin{pmatrix} w_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & w_n^2 \end{pmatrix} \cdot \begin{pmatrix} s'_1 \\ \vdots \\ s'_n \end{pmatrix} + \begin{pmatrix} \rho_1 w_1 \\ \vdots \\ \rho_n w_n \end{pmatrix}.$$

Hence the affine bundles $\text{Con}^\nu \xrightarrow{\pi_+} \text{Bun}$ are all isomorphic, not depending on ν ; this agrees with Theorem 3.3. The transition map (4.4) can also be written as

$$(4.5) \quad \begin{pmatrix} \lambda \\ r_1 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \nu_1 w_1 & w_1^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \nu_n w_n & 0 & \dots & w_n^2 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ s_1 \\ \vdots \\ s_n \end{pmatrix}$$

and can be thought, owing to (4.2), as the transition matrix of a vector bundle \mathcal{E} over Bun that parameterizes logarithmic λ -connections: triples (E_1, λ, ∇) where $\nabla: E_1 \rightarrow E_1 \otimes \Omega_C(D)$ and $\lambda \in \mathbb{C}$ such that

$$\nabla(fs) = \lambda s \otimes df + f \nabla(s),$$

see [29, Section 4]. If $\lambda \neq 0$ then $\frac{1}{\lambda} \nabla$ is an usual logarithmic connection and if $\lambda = 0$ then ∇ is a Higgs field. We note that the complement of $U_0 \cup U_\infty$ is a subvariety of codimension two, hence the transition matrix given in (4.5) determines \mathcal{E} .

The projectivization of \mathcal{E} provides a compactification $\overline{\text{Con}}^\nu$ for the moduli space Con^ν ; in the boundary we get the projectivized moduli space of Higgs fields

$$\mathbb{P}\text{Higgs} \simeq \overline{\text{Con}}^\nu \setminus \text{Con}^\nu.$$

The inclusion $\mathbb{P}\text{Higgs} \hookrightarrow \mathbb{P}(\mathcal{E})$ comes from the natural extension

$$(4.6) \quad 0 \longrightarrow T^*\text{Bun} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\text{Bun}} \longrightarrow 0$$

where the first map is the inclusion of Higgs fields and the last is the projection $(E_1, \lambda, \nabla) \mapsto \lambda$. We saw that \mathcal{E} does not depend on ν . However, this extension, hence the inclusion $\mathbb{P}\text{Higgs} \hookrightarrow \mathbb{P}(\mathcal{E})$, is determined by the exponents ν .

Theorem 4.4. *The moduli space Con^ν has compactification $\overline{\text{Con}}^\nu = \mathbb{P}(\mathcal{E})$, where the boundary divisor is isomorphic to $\mathbb{P}\text{Higgs}$, the projectivization of the space of Higgs fields on E_1 . Moreover, the inclusion $\mathbb{P}\text{Higgs} \hookrightarrow \mathbb{P}(\mathcal{E})$ is determined, up to automorphisms of $\mathbb{P}(\mathcal{E})$, by (ν_1, \dots, ν_n) .*

Proof. The inclusion $\phi: \mathbb{P}\text{Higgs} \hookrightarrow \mathbb{P}(\mathcal{E})$ comes from the projectivization of (4.6). We will show that the isomorphism class of this extension is determined by (ν_1, \dots, ν_n) . This is equivalent to determining ϕ , up to composing with an automorphism of $\mathbb{P}(\mathcal{E})$ as a \mathbb{P}^n -bundle over Bun . The proof will follow by computing the extension class $e^\nu \in H^1(\text{Bun}, T^*\text{Bun})$, see [3, p. 185].

Over U_0 , the map $\mathcal{O}_{\text{Bun}} \rightarrow \mathcal{E}^\nu$, given by $h \mapsto h \nabla_0$, defines a splitting; the same is true for ∇_∞ over U_∞ . Therefore e^ν is represented, on $U_0 \cap U_\infty$, by

$$\nabla_\infty - \nabla_0 = \sum_{j=1}^n \nu_j w_j \Theta_j^0 \left(\frac{1}{w_j} \right),$$

see the equation in display (4.3). Any cohomologous cocycle must be

$$\begin{aligned} \sum_{j=1}^n \nu_j w_j \Theta_j^0 \left(\frac{1}{w_j} \right) + \sum_{j=1}^n b_j(w_1, \dots, w_n) \Theta_j^\infty(w_j) - \sum_{j=1}^n a_j \left(\frac{1}{w_1}, \dots, \frac{1}{w_n} \right) \Theta_j^0 \left(\frac{1}{w_j} \right) = \\ = \sum_{i=1}^n \left[\nu_j w_j + b_j(w_1, \dots, w_n) w_j^2 - a_j \left(\frac{1}{w_1}, \dots, \frac{1}{w_n} \right) \right] \Theta_j^0 \left(\frac{1}{w_j} \right) \end{aligned}$$

for some holomorphic functions a_i and b_i . In particular, the linear terms $\nu_j w_j$ are left unchanged, whence the extension is completely determined by (ν_1, \dots, ν_n) . \square

Returning to the affine bundle $\text{Con}^\nu \xrightarrow{\pi^+} \text{Bun}$, we have seen that it does not distinguish ν . However, if we consider Con^ν with the inherited symplectic structure from \mathfrak{Con}^ν , the picture is different; this will be the topic of the next section. The trivializations in (4.4) will be important to bring the symplectic form to the Darboux normal form.

5. SYMPLECTIC STRUCTURE

It follows from the work of Iwasaki [16] that moduli spaces of logarithmic connections on curves carry a natural symplectic structure. In fact, there exists a 2-form defined on a larger moduli space where the poles are allowed to move on C inducing a Poisson structure whose kernel defines the isomonodromic deformations, therefore a Hamiltonian system. Moreover, through the Riemann-Hilbert correspondence, Iwasaki proved in [17] that this 2-form coincides with the symplectic structure on moduli spaces of representations that originated through the works of Atiyah-Bott and Goldman. In fact, Iwasaki considers moduli spaces of Sturm-Liouville operators rather than connections. In order to define the symplectic structure globally on our moduli space, the approach of Arinkin-Lysenko in [2] and Iwasaki in [17] is more convenient for us. We briefly recall how the tangent space and the symplectic structure are constructed there. Later we will give an explicit formula for the 2-form on an open set of our moduli space.

First we recall the following result from [12, Section 15].

Lemma 5.1. *Assume that the connection (E, ∇) has a simple pole at $t \in C$ with eigenvalues $\{\nu^+, \nu^-\}$, and let $z: (C, t) \rightarrow (\mathbb{C}, 0)$ be a local coordinate. Then there exists a local trivialization of the vector bundle $\Phi: E|_{(C, t)} \rightarrow (\mathbb{C}, 0) \times \mathbb{C}^2$ such that $\Phi_* \nabla$ is one of the following models:*

$$(5.1) \quad d + \begin{pmatrix} \nu^+ & 0 \\ 0 & \nu^- \end{pmatrix} \frac{dx}{x},$$

or, in the resonant case $\{\nu^+, \nu^-\} = \{\nu, \nu + n\}$, $n \in \mathbb{Z}_{\geq 0}$

$$(5.2) \quad d + \begin{pmatrix} \nu & x^n \\ 0 & \nu + n \end{pmatrix} \frac{dx}{x}.$$

Moreover, in the resonant case $\nu^+ - \nu^- \in \mathbb{Z}$, the model (5.2) is generic among the two, i.e. occurs for an open set of the deformations of (E, ∇) .

Next we want to describe the tangent space of \mathfrak{Con}^ν at a (generic) connection $(E, \nabla) \in \mathfrak{Con}^\nu$. Consider a first order deformation $(E_\epsilon, \nabla_\epsilon)$ of (E, ∇) ; we briefly recall the definition to set up notation. Denote $R = \mathbb{C}[\epsilon]$, with $\epsilon^2 = 0$, the ring of dual numbers. Then E_ϵ is a vector bundle

over $C \times \text{Spec}(R)$, flat over $\text{Spec}(R)$, such that the restriction induced by $R \twoheadrightarrow R/(\epsilon) = \mathbb{C}$ gives $E_\epsilon \otimes_R \mathbb{C} = E$. Moreover there exist a covering $\{U_i\}$ of C and isomorphisms

$$(5.3) \quad \Phi_i: E_\epsilon|_{U_i \times \text{Spec}(R)} \xrightarrow{\sim} E|_{U_i} \otimes_{\mathbb{C}} R,$$

such that $\Phi_i \otimes 1: E_\epsilon|_{U_i \times \text{Spec}(R)} \otimes_R \mathbb{C} = E|_{U_i} \rightarrow E|_{U_i}$ is the identity. On the other hand, ∇_ϵ is a relative connection, i.e. an R -linear map $\nabla_\epsilon: E_\epsilon \rightarrow E_\epsilon \otimes_{\mathbb{C}} \Omega_C(D)$ satisfying the relative Leibniz rule, such that its restriction to $E_\epsilon \otimes_R \mathbb{C} = E$ is ∇ . Up to refining $\{U_i\}$, the isomorphisms Φ_i plus trivializations of E amount to writing

$$(5.4) \quad \nabla_\epsilon|_{U_i \times \text{Spec}(R)} = d + A_i^0 + \epsilon A_i^1$$

where $\nabla|_{U_i} = d + A_i^0$ and $A_i^1 \in \text{Hom}(E|_{U_i}, E \otimes \Omega_C(D)|_{U_i})$. We are specially interested in deformations that preserve the trace and residual eigenvalues, which impose conditions on A_i^1 .

We can refine $\{U_i\}$ further so that the restriction of ∇ is either trivial or can be normalized like in Lemma 5.1. Since the local models (5.1) with $\nu^+ - \nu^- \notin \mathbb{Z}$, and (5.2) are stable under small deformations with fixed eigenvalues, we deduce that there exist bundle automorphisms

$$\Psi_i: E|_{U_i} \otimes_{\mathbb{C}} R \xrightarrow{\sim} E|_{U_i} \otimes_{\mathbb{C}} R$$

such that $\nabla_\epsilon|_{U_i \times \text{Spec}(R)} = \Psi_i^*(\nabla \otimes 1)$. Therefore we may assume that the maps Φ_i of (5.3) are such that $A_i^1 = 0$ in (5.4).

Over $U_i \cap U_j$ the comparison $\Phi_{ij} = \Phi_i \circ \Phi_j^{-1}$ defines an automorphism of $E|_{U_i \cap U_j} \otimes_{\mathbb{C}} R$ preserving the connection $\nabla|_{U_i \cap U_j} \otimes 1$. By construction $\Phi_{ij} = \text{id} + \epsilon \phi_{ij}$ and we get

$$(\nabla \otimes 1)(\Phi_{ij}(s_0 + \epsilon s_1)) = (\nabla \otimes 1)(s_0 + \epsilon(s_1 + \phi_{ij}(s_0))) = \nabla(s_0) + \epsilon[\nabla(s_1) + \nabla(\phi_{ij}(s_0))]$$

||

$$\Phi_{ij}((\nabla \otimes 1)(s_0 + \epsilon s_1)) = \Phi_{ij}(\nabla(s_0) + \epsilon \nabla(s_1)) = \nabla(s_0) + \epsilon[\nabla(s_1) + \phi_{ij}(\nabla(s_0))].$$

Hence $\nabla(\phi_{ij}(s_0)) = \phi_{ij}(\nabla(s_0))$, i.e., ϕ_{ij} preserves $\nabla|_{U_i \cap U_j}$.

Since we are assuming that the trace connection is fixed, $\det(\Phi_i)$ comes from a global isomorphism $\rho: \det(E_\epsilon) \xrightarrow{\sim} \det(E) \otimes_{\mathbb{C}} R$. Hence we have that $1 = \det \Phi_{ij} = 1 + \epsilon \text{tr} \phi_{ij}$. Then $\text{tr} \phi_{ij} = 0$ and we get an element $\{\phi_{ij}\}$ of $H^1(\mathfrak{sl}(E, \nabla))$, the sheaf of trace-free endomorphisms preserving ∇ . Therefore

$$T_{(E, \nabla)} \mathfrak{Con}^\nu \simeq H^1(\mathfrak{sl}(E, \nabla)).$$

Observe that $\mathfrak{sl}(E, \nabla)$ is a subsheaf of $\mathfrak{sl}(E, \mathbf{p})$ the sheaf of endomorphisms of the quasi-parabolic bundle (E, \mathbf{p}) with trace zero. Also note that ∇ induces a natural (logarithmic) connection on the vector bundle $\mathfrak{sl}(E)$, namely $\nabla^1: \phi \mapsto \nabla \circ \phi - \phi \circ \nabla$. We then construct a complex of sheaves of abelian groups

$$(5.5) \quad 0 \longrightarrow \mathfrak{sl}(E, \nabla) \longrightarrow \mathfrak{sl}(E, \mathbf{p}) \xrightarrow{\nabla^1} \mathfrak{sl}(E, \mathbf{p}) \otimes \Omega_C^1(D)^{\text{nil}} \longrightarrow 0$$

where the rightmost member corresponds to parabolic Higgs fields with nilpotent residues. The two sheaves on the right correspond to \mathcal{F}^0 and \mathcal{F}^1 in [14, 13, 15, 19]. Note that ∇^1 is not \mathcal{O}_C -linear and $\mathfrak{sl}(E, \nabla)$ is not an \mathcal{O}_C -module. Next we show that for a generic connection the deformation complex (5.5) is indeed exact.

Lemma 5.2. *For a generic connection (E, ∇) in the moduli space, the sequence in display (5.5) is exact.*

Proof. First note that the kernel of ∇^1 is precisely $\mathfrak{sl}(E, \nabla)$ and the leftmost map is the inclusion. Thus we only need to show that ∇^1 is surjective, for a generic ∇ . It will follow from a direct computation with local models.

Outside the polar locus, (E, ∇) is locally trivial and the map ∇^1 may be written as

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} da & db \\ dc & -da \end{pmatrix}$$

so that local surjectivity of ∇^1 at a regular point follows from local integration of holomorphic 1-forms. On the other hand, at a non resonant pole, i.e. $\nu^+ - \nu^- \notin \mathbb{Z}$, the connection (E, ∇) is locally like (5.1) and the map ∇^1 may be written as

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} da & db + (\nu^+ - \nu^-)b \frac{dx}{x} \\ dc + (\nu^- - \nu^+)c \frac{dx}{x} & -da \end{pmatrix}$$

where $c(0) = 0$ (parabolic condition); for the local surjectivity, we just have to check that a matrix of 1-forms

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

is in the image if and only if α and γ are holomorphic (parabolic and nilpotent condition), and β has a simple pole, which is straightforward. Finally, consider a resonant pole of the form (5.2). Then the map ∇^1 may be written as

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} da + cx^n \frac{dx}{x} & db - (nb + 2ax^n) \frac{dx}{x} \\ dc + nc \frac{dx}{x} & -da - cx^n \frac{dx}{x} \end{pmatrix}.$$

To prove the surjectivity in that case, we have to successively integrate

$$\begin{cases} dc + nc \frac{dx}{x} & = \gamma \\ da + cx^n \frac{dx}{x} & = \alpha \\ db - (nb + 2ax^n) \frac{dx}{x} & = \beta \end{cases}$$

In the first equation, $c(x)$ can be derived such that $c(0) = 0$ (parabolicity); therefore we can integrate the second one and find $a(x)$ up to a constant; finally, for a good choice of $a(0)$, the last equation admits a solution $b(x)$. \square

Hereafter we will assume that (E, \mathbf{p}) is μ -stable for some weight μ and we will denote by \mathcal{B} the corresponding moduli space of parabolic bundles.

Note that (E, \mathbf{p}) is simple, i.e. $H^0(\mathfrak{sl}(E, \mathbf{p})) = 0$, then we get $H^1(\mathfrak{sl}(E, \mathbf{p}) \otimes \Omega_C^1(D)^{\text{nil}}) = 0$ by Serre duality. The long exact sequence of cohomology of (5.5) gives

$$(5.6) \quad 0 \longrightarrow H^0(\mathfrak{sl}(E, \mathbf{p}) \otimes \Omega_C^1(D)^{\text{nil}}) \longrightarrow H^1(\mathfrak{sl}(E, \nabla)) \rightarrow H^1(\mathfrak{sl}(E, \mathbf{p})) \longrightarrow 0$$

In the middle we get the tangent space to the moduli space of connections at (E, ∇) . On the left-hand side, we get the space of Higgs fields, which does not depend on ∇ . And on the right-hand side we have $H^1(\mathfrak{sl}(E, \mathbf{p}))$ that is the tangent space to \mathcal{B} at (E, \mathbf{p}) . In particular, the image of $\{\phi_{ij}\}$ in $H^1(\mathfrak{sl}(E, \mathbf{p}))$, where we omit the connection and just keep track of the parabolic data, corresponds to the underlying first order deformation of the parabolic bundle.

We can check that the map $H^0(\mathfrak{sl}(E, \mathbf{p}) \otimes \Omega_C^1(D)^{\text{nil}}) \hookrightarrow H^1(\mathfrak{sl}(E, \nabla))$ of (5.6) is the derivative of the natural action $\nabla \mapsto \nabla + \theta$ for a global parabolic Higgs field on the affine space of parabolic

connections on (E, \mathbf{p}) . Indeed, if we write $\nabla_\epsilon = \nabla + \epsilon\theta$ and $\Phi_i = \text{id} + \epsilon\phi_i$, then we get

$$\begin{aligned} (\nabla + \epsilon\theta)|_{U_i} &= (\Phi_i)^*\nabla = (\text{id} - \epsilon\phi_i) \circ \nabla \circ (\text{id} + \epsilon\phi_i) = \\ &= \nabla + \epsilon(\nabla \circ \phi_i - \phi_i \circ \nabla) = \nabla + \epsilon\nabla^1(\phi_i) \end{aligned}$$

The first order deformation is therefore encoded in $\{\phi_j - \phi_i\}$, which, by construction, has zero image in $H^1(\mathfrak{sl}(E, \mathbf{p}))$. But the image of θ in $H^1(\mathfrak{sl}(E, \nabla))$ given by (5.6) gives exactly the same formula: we first integrate $\nabla^1\phi_i = \theta|_{U_i}$ and then associate $\{\phi_j - \phi_i\}$.

One can define a bilinear map on $T_{(E, \nabla)}\mathfrak{Con}^\nu$ as follows:

$$(5.7) \quad \begin{aligned} H^1(\mathfrak{sl}(E, \nabla)) \times H^1(\mathfrak{sl}(E, \nabla)) &\xrightarrow{\omega_{\text{Con}}} H^2(\mathbb{C}) \simeq \mathbb{C}; \\ (\{\phi_{ij}\}, \{\psi_{ij}\}) &\longmapsto \{u_{ijk} = \text{tr}(\phi_{ij}\psi_{jk})\}. \end{aligned}$$

One can choose the covering $\{U_i\}$ such that (E, ∇) is trivial over $U_i \cap U_j$. Observe that $\phi_{ij}\psi_{jk}$ defines a section of $\mathfrak{gl}(E, \nabla)$ for which trace and determinant are constant; indeed, in a trivialization of (E, ∇) on $U_i \cap U_j \cap U_k$, the section $\phi_{ij}\psi_{jk}$ is constant. This is why u_{ijk} is a section of the constant sheaf \mathbb{C} . One can show that this bilinear form is non degenerate, and defines a closed 2-form ω_{Con} on \mathfrak{Con}^ν which is symplectic:

$$d\omega_{\text{Con}} \equiv 0 \text{ and } \omega_{\text{Con}}^n = \underbrace{\omega_{\text{Con}} \wedge \cdots \wedge \omega_{\text{Con}}}_{n \text{ times}} \neq 0,$$

i.e. ω_{Con}^n defines a holomorphic volume form, a trivialization of the canonical bundle. We refer to [16] for details and proofs.

Remark 5.3. For the non generic resonant case

$$d + \begin{pmatrix} \nu & 0 \\ 0 & \nu + n \end{pmatrix} \frac{dx}{x}, \quad n \in \mathbb{Z}_{\geq 0}$$

the sequence (5.5) is not exact anymore: ∇^1 is not surjective. This is why the description of the symplectic structure is more complicated in [2, 14, 13, 15, 19]. They use hypercohomology to overcome this difficulty. In our case, we want to understand the symplectic structure at a generic point of the moduli space, so it is not necessary to consider this kind of models.

In order to describe the symplectic structure ω_{Con} , we will make a reduction step. We will see that, locally, we can split $T_{(E, \nabla)}\mathfrak{Con}^\nu$ as a direct sum of two distinguished subspaces: one concerning Higgs fields and the other being the tangent space to a Lagrangian submanifold. This will allow us to compare ω_{Con} with the natural symplectic form for Higgs fields.

Consider

$$\text{Higgs}(E, \mathbf{p}) = H^0(\mathfrak{sl}(E, \mathbf{p}) \otimes \Omega_C^1(D)^{\text{nil}})$$

the space of Higgs fields for E with nilpotent residues preserving \mathbf{p} . We define a pairing

$$(5.8) \quad \begin{aligned} \text{Higgs}(E, \mathbf{p}) \times H^1(\mathfrak{sl}(E, \mathbf{p})) &\xrightarrow{\eta} H^1(\Omega_C^1) \simeq \mathbb{C} \\ (\Theta, \{\phi_{ij}\}) &\longmapsto \{\text{tr}(\Theta \cdot \phi_{ij})\} \end{aligned}$$

One proves, see [16], that η is a perfect pairing via identifying $H^1(\mathfrak{sl}(E, \mathbf{p}))^\vee \simeq T_{(E, \mathbf{p})}^*\mathcal{B}$ and $\text{Higgs}(E, \mathbf{p})$. In other words, pairing η corresponds to the Liouville form on $T^*\mathcal{B}$.

Consider the following diagram

$$(5.9) \quad \begin{array}{ccc} & H^1(\mathfrak{sl}(E, \nabla)) \times H^1(\mathfrak{sl}(E, \nabla)) & \xrightarrow{\omega_{\text{Con}}} H^2(\mathbb{C}) \\ \text{Higgs}(E, \mathbf{p}) \times H^1(\mathfrak{sl}(E, \nabla)) & \nearrow & \uparrow \delta \\ & \text{Higgs}(E, \mathbf{p}) \times H^1(\mathfrak{sl}(E, \mathbf{p})) & \xrightarrow{\eta} H^1(\Omega^1) \end{array}$$

where we have used the maps from (5.6) and δ is the connecting morphism coming from the long exact sequence associated to

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_C \xrightarrow{d} \Omega_C^1 \longrightarrow 0.$$

Then we reconcile ω_{Con} and the pairing η .

Lemma 5.4. *The diagram (5.9) is commutative.*

Proof. Let $(\theta, \{\psi_{ij}\})$ be an element of $\text{Higgs}(E, \mathbf{p}) \times H^1(\mathfrak{sl}(E, \nabla))$. As θ is a parabolic Higgs field, one can locally write $\theta|_{U_i} = \nabla^1(\varphi_i)$ and the upper map yields $(\{\varphi_j - \varphi_i\}, \{\psi_{ij}\}) \in H^1(\mathfrak{sl}(E, \nabla))^2$, which is sent to $\{\text{tr}((\varphi_j - \varphi_i) \circ \psi_{jk})\} \in H^2(\mathbb{C})$. On the other hand, by first going down, the pairing η from (5.8) gives $\{\text{tr}(\theta \circ \psi_{ij})\} \in H^1(\Omega^1)$. To compute the image by δ , we first remark that we have

$$\begin{aligned} \text{tr}(\theta \circ \psi_{ij}) &= \text{tr}(\nabla^1(\varphi_i) \circ \psi_{ij}) = \\ &= \text{tr}(\nabla^1(\varphi_i \circ \psi_{ij})) - \text{tr}(\varphi_i \circ \nabla^1(\psi_{ij})) = \\ &= \text{tr}(\nabla^1(\varphi_i \circ \psi_{ij})) = \\ &= d(\text{tr}(\varphi_i \circ \psi_{ij})) \end{aligned}$$

Here, we have used that $\nabla^1(\psi_{ij}) = 0$ and the properties $\nabla^1(F \circ G) = \nabla^1(F) \circ G + F \circ \nabla^1(G)$ and $\text{tr}(\nabla^1(F)) = d(\text{tr}(F))$, which can be deduced from a local computation in matrix form. Therefore, we have

$$\begin{aligned} \delta(\{\text{tr}(\theta \circ \psi_{ij})\}) &= \{\text{tr}(\varphi_i \circ \psi_{ij}) + \text{tr}(\varphi_j \circ \psi_{jk}) - \text{tr}(\varphi_i \circ \psi_{ik})\} = \\ &= \{\text{tr}(\varphi_i \circ (\psi_{ij} - \psi_{ik})) + \text{tr}(\varphi_j \circ \psi_{jk})\} = \\ &= \{\text{tr}((\varphi_j - \varphi_i) \circ \psi_{jk})\} \end{aligned}$$

and get the same result. \square

Given a choice of directions $\epsilon \in \{+, -\}^n$, let \mathcal{C} denote the open subset of \mathfrak{Con}^ν composed by the connections (E, ∇) such that $(E, \mathbf{p}^\epsilon(\nabla))$ belongs to \mathcal{B} . Recall that a subvariety is called Lagrangian the restriction of symplectic form vanishes identically.

Assume now that we are given a local section $\nabla_0: B \rightarrow \mathcal{C}$ on an open subset $B \subset \mathcal{B}$. Then we can define an isomorphism to the moduli space of parabolic Higgs bundles

$$\Psi_{\nabla_0}: \mathcal{C}|_B \longrightarrow \text{Higgs}|_B$$

by setting $(E, \nabla) \mapsto (E, \nabla - \nabla_0(E, \mathbf{p}))$; the inverse being given by $(E, \varphi) \mapsto (E, \nabla_0(E, \mathbf{p}) + \varphi)$. Recall that (5.8) identifies Higgs fields and the cotangent space to \mathcal{B} . This yields an identification between the moduli space of parabolic Higgs bundles and the cotangent bundle $T^*\mathcal{B}$ which, itself, carries the Liouville symplectic structure that we will call ω_{Higgs} .

Recall that Ψ_{∇_0} is a symplectomorphism if $\Psi_{\nabla_0}^* \omega_{\text{Higgs}} = \omega_{\text{Con}}$. On the other hand, we say that the section ∇_0 is Lagrangian if so is its image.

Proposition 5.5. *The isomorphism Ψ_{∇_0} is a symplectomorphism if and only if the section $\nabla_0: B \rightarrow \mathcal{C}$ is Lagrangian.*

Before proving this proposition, let us recall some basic property of the Liouville form $\omega_{\text{Liouville}}$ on T^*M for an arbitrary manifold M . If (p_1, \dots, p_n) are coordinates on M , and if we denote by (q_1, \dots, q_n) the coordinates on the fibers such that dp_i corresponds to the section $q_i \equiv 1$ and $q_j \equiv 0$ for $j \neq i$, then $\omega_{\text{Liouville}} = \sum_{i=1}^n dp_i \wedge dq_i$. Given a section $\alpha: M \rightarrow T^*M$, we can define an affine bundle transformation by

$$\psi_\alpha: T^*M \rightarrow T^*M; (p, q) \mapsto (p, q + \alpha(p)),$$

where T^*M is viewed as an affine bundle.

Lemma 5.6. *Let M , α and ψ_α as above. Then the following are equivalent:*

- ψ_α is a symplectomorphism,
- $\alpha(M)$ is Lagrangian (as a submanifold),
- $d\alpha = 0$ (i.e. closed as a 1-form).

Proof of Proposition 5.5. We want to prove that Ψ_{∇_0} is a symplectomorphism; then we only need to show that $\Psi_{\nabla_0}^* \omega_{\text{Higgs}} = \omega_{\text{Con}}$ at an arbitrary point $(E, \varphi) \in \text{Higgs}$. As the vector bundle Higgs identifies by (5.8) to $T^*\mathcal{B}$, we can use Lemma 5.6 to compose with an affine transformation sending φ to the zero Higgs field. Now, the tangent space decomposes as

$$T_{(E,0)}\text{Higgs} = \text{Higgs}(E, \mathbf{p}) \oplus T_{(E,\mathbf{p})}\mathcal{B}$$

and the symplectic form ω_{Higgs} may be written as

$$\omega_{\text{Higgs}}(u_1 + v_1, u_2 + v_2) = \eta(u_1, v_2) - \eta(u_2, v_1)$$

where η is the pairing given by (5.8). Now, the tangent space at (E, ∇) decomposes as

$$T_{(E,\nabla)}\mathcal{C} = \text{Higgs}(E, \mathbf{p}) \oplus T_{(E,\nabla)}\nabla_0(B) = \text{Higgs}(E, \mathbf{p}) \oplus D\nabla_0(T_{(E,\mathbf{p})}\mathcal{B})$$

and note that the differential of $\Psi_{\nabla_0}^{-1}$ is the block diagonal $D\Psi_{\nabla_0}^{-1} = id \oplus D\nabla_0$. Now ω_{Con} may be written as

$$\begin{aligned} \omega_{\text{Con}}(u_1 + D\nabla_0 v_1, u_2 + D\nabla_0 v_2) &= \omega_{\text{Con}}(u_1, D\nabla_0 v_2) - \omega_{\text{Con}}(u_2, D\nabla_0 v_1) + \\ &\quad + \omega_{\text{Con}}(u_1, u_2) + \omega_{\text{Con}}(D\nabla_0 v_1, D\nabla_0 v_2) \\ &= \eta(u_1, D\nabla_0 v_2) - \eta(u_2, D\nabla_0 v_1) + \omega_{\text{Con}}(D\nabla_0 v_1, D\nabla_0 v_2) \\ &= \Psi_{\nabla_0}^* \omega_{\text{Higgs}}(u_1 + v_1, u_2 + v_2) + \omega_{\text{Con}}(D\nabla_0 v_1, D\nabla_0 v_2) \end{aligned}$$

Indeed, it follows from the diagram (5.9). To prove the vanishing of $\omega_{\text{Con}}(u_1, u_2)$, note that by (5.6) the image of $\text{Higgs}(E, \mathbf{p})$ in $H^1(\mathfrak{sl}(E, \mathbf{p}))$ is zero. Hence by taking the lower path in (5.9) the pairing η yields $0 \in H^1(\Omega^1)$.

Therefore $\Psi_{\nabla_0}^* \omega_{\text{Higgs}} = \omega_{\text{Con}}$ if and only if $\omega_{\text{Con}}(D\nabla_0 v_1, D\nabla_0 v_2) = 0$, i.e. $\nabla_0(B)$ is Lagrangian. \square

In our situation, we can compute this symplectic structure explicitly on a Zariski open set. Indeed, let $\mathcal{C} \rightarrow \mathcal{B}$ be given by the choice of directions $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{-, +\}^n$ and let $\delta = -\epsilon$ be the opposite choice. One can verify that a fiber of $(E, \nabla) \mapsto (E, \mathbf{p}^\delta(\nabla))$ is a rational section of $(E, \nabla) \mapsto (E, \mathbf{p}^\epsilon(\nabla))$. The reduction is symplectic if the chosen section is Lagrangian, wherever it is defined. This is ensured if the projection is Lagrangian, which is covered by the following result.

Lemma 5.7. *The forgetful map $\mathcal{C} \rightarrow \mathcal{B}; (E, \nabla) \mapsto (E, \mathbf{p}^\epsilon(\nabla))$ is Lagrangian, i.e. its fibers are Lagrangian subvarieties.*

Proof. The differential of the forgetful map at (E, ∇) determines the dual to the map $\text{Higgs}(E, \mathbf{p}) \rightarrow H^1(\mathfrak{sl}(E, \nabla))$ from (5.6). Then we want to prove that the pairing ω_{Con} of (5.7) is zero on a pair $(\{\phi_{ij}\}, \{\psi_{ij}\})$ where each entry is in the image of $\text{Higgs}(E, \mathbf{p})$. We showed that this is true in the proof of the last proposition. \square

We now apply this construction to exhibit the symplectic form on Con^ν via explicit computations with Fuchsian systems.

5.1. Computation of the symplectic structure. We have seen that choosing a Lagrangian section ∇_0 of $\pi_\epsilon: \text{Con}^\nu \rightarrow \text{Bun}$ allows us to reduce the computation of the symplectic structure of Con^ν to that of Higgs moduli space (see Proposition 5.5). We can apply this to the setting of Fuchsian systems. The section ∇_0 constructed in section 4.1.1 corresponds, via the isomorphism of Proposition 4.1, to the fiber of $\pi_{-\epsilon}$ over the point $(\infty, \dots, \infty) \in \text{Bun} \simeq (\mathbb{P}^1)^n$; it is therefore Lagrangian. Let us compute the symplectic structure on the moduli space of Higgs bundles. For this, we consider the chart:

$$\mathbb{C}^{2n} \rightarrow \text{Higgs}(C, D); (z_1, \dots, z_n, r_1, \dots, r_n) \mapsto (E_1, \Theta)$$

where Θ is obtained by applying elementary transformations at w_0, w_1, w_λ to

$$r_1 \Theta_1^0 + \dots + r_n \Theta_n^0$$

as defined in section 4.1.1. The image coincides with the restriction $\text{Higgs}(C, D)|_{U_0}$ of Higgs fields on parabolic bundles belonging to the chart $U_0 \subset \text{Bun}$ of section 4.1.1. We want now to compute the isomorphism $\text{Higgs}(C, D)|_{U_0} \simeq T^*U_0$ so that the symplectic form can be easily deduced from the Liouville form.

Here we follow the ideas of the proof of [11, Proposition 6.1]. Given a parabolic vector bundle (E_1, \mathbf{p}) in U_0 , or equivalently $(\mathcal{O}_C \oplus \mathcal{O}_C, \mathbf{p}')$ defined by $(z_1, \dots, z_n) \in \mathbb{C}^n$, Serre Duality (5.8) provides a perfect pairing

$$\langle \cdot, \cdot \rangle : \text{Higgs}(E, \mathbf{p}) \times H^1(\mathfrak{sl}(E, \mathbf{p})) \rightarrow \mathbb{C}$$

which allows us to identify the fiber of Higgs at (E_1, \mathbf{p}) with the dual of the tangent space

$$T_{(E_1, \mathbf{p})}U_0 \simeq H^1(\mathfrak{sl}(E_1, \mathbf{p})).$$

Let us compute first this latter identification.

The deformation $(z_1, \dots, z_j + s, \dots, z_n)$ of the j^{th} component of the parabolic structure can be realized as follows. Cut out the curve C into two open sets $U_j = C - \{t_j\}$ and V_j a small disk around t_j . Then we can paste the restrictions of $(\mathcal{O}_C \oplus \mathcal{O}_C, \mathbf{p}')$ to these open sets by means of the following gauge transformation over the restriction

$$(\mathcal{O}_C \oplus \mathcal{O}_C, \mathbf{p}')|_{U_j \cap V_j} \rightarrow (\mathcal{O}_C \oplus \mathcal{O}_C, \mathbf{p}')|_{U_j \cap V_j}; \begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}.$$

This clearly shifts the parabolic direction $(z_j : 1) \mapsto (z_j + s : 1)$ without moving other parabolic directions. After derivating, we obtain that the infinitesimal deformation $\frac{\partial}{\partial z_j}$ corresponds to the gauge transformation $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in H^0(U_j \cup V_j, \mathfrak{sl}(\mathcal{O}_C \oplus \mathcal{O}_C, \mathbf{p}'))$, which can better be viewed as a

cocycle with respect to the covering (U_j, V_j) . We can now compute the Serre Duality: the pairing of the Higgs field Θ_i^0 with the above cocycle is defined by

$$\mathrm{tr} \left(\Theta_k^0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = -z_k \phi_0 + \phi_1 + \frac{y_k}{x_k} \omega + \theta_k$$

a holomorphic 1-form on $U_j \cap V_j$. Viewed as a cocycle in $H^1(\Omega)$, one can give a meromorphic resolution by considering it as the difference between the zero 1-form on U_j and the meromorphic 1-form $-z_k \phi_0 + \phi_1 + \frac{y_k}{x_k} \omega + \theta_k$ on V_j . The residue at t_j gives the pairing:

$$\left\langle \Theta_k^0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle = \mathrm{Res}_{t_j} \left(-z_k \phi_0 + \phi_1 + \frac{y_k}{x_k} \omega + \theta_k \right) = \delta_j^k$$

where δ_j^k is the Kronecker symbol. We deduce that Θ_j^0 is dual to $\frac{\partial}{\partial z_j}$, and the symplectic structure on $\mathrm{Higgs} \ni \sum_j r_j \Theta_j^0$ is given by

$$\omega_{\mathrm{Higgs}} = dz_1 \wedge dr_1 + \cdots + dz_n \wedge dr_n.$$

Now, because our section $\nabla_0: U_0 \rightarrow \mathrm{Con}$ is Lagrangian, we can apply Proposition 5.5 and deduce the symplectic structure on $\mathrm{Con} \ni \nabla_0 + \sum_j r_j \Theta_j^0$, namely:

$$(5.10) \quad \omega_{\mathrm{Con}} = dz_1 \wedge dr_1 + \cdots + dz_n \wedge dr_n.$$

The same construction shows, mutatis mutandis, that the symplectic form on $\nabla_\infty + \sum_j s_j \Theta_j^\infty$ is

$$(5.11) \quad \omega_{\mathrm{Con}} = -(dw_1 \wedge ds_1 + \cdots + dw_n \wedge ds_n).$$

Next we will see how ω_{Con} encodes the eigenvalue ν . First we consider only maps that preserves the affine bundle structure.

Theorem 5.8. *If there exists a bundle symplectic isomorphism*

$$\begin{array}{ccc} (\mathrm{Con}^\nu, \omega) & \xrightarrow[\sim]{\Phi} & (\mathrm{Con}^{\tilde{\nu}}, \tilde{\omega}) \\ \searrow \pi_+ & & \swarrow \pi_+ \\ & \mathrm{Bun} & \end{array}$$

then $\tilde{\nu} = \nu$ and Φ is the identity.

Proof. Since $\pi_+ \circ \Phi = \pi_+$, it preserves $\pi_+^{-1}(U_0)$ and $\pi_+^{-1}(U_\infty)$. Then we may write the map in coordinates. For Con^ν we will consider the two coordinate patches:

$$(\pi_+^{-1}(U_0), (z, r)) \text{ and } (\pi_+^{-1}(U_\infty), (w, s))$$

where $z_j = \frac{1}{w_j}$ and $r_j = s_j w_j^2 + \nu_j w_j$, see (4.4). For $\mathrm{Con}^{\tilde{\nu}}$ we consider the corresponding open sets with coordinates (z, \tilde{r}) and (w, \tilde{s}) , respectively, where $z_j = \frac{1}{w_j}$ and $\tilde{r}_j = \tilde{s}_j w_j^2 + \tilde{\nu}_j w_j$. In both cases, π_+ is the projection onto the first n coordinates.

Therefore we can express Φ as

$$\begin{aligned} \Phi|_{\pi_+^{-1}(U_0)} &= (z_1, \dots, z_n, \phi_1^0(z, r), \dots, \phi_n^0(z, r)) = (z, \tilde{r}), \\ \Phi|_{\pi_+^{-1}(U_\infty)} &= (w_1, \dots, w_n, \phi_1^\infty(w, s), \dots, \phi_n^\infty(w, s)) = (w, \tilde{s}), \end{aligned}$$

where $\phi_j^0(z, r)$ and $\phi_j^\infty(w, s)$ are holomorphic functions on \mathbb{C}^{2n} satisfying the compatibility conditions:

$$(5.12) \quad \phi_j^\infty(\cdots, w_k, \cdots, s_k, \cdots) w_j^2 + \tilde{\nu}_j w_j = \phi_j^0\left(\cdots, \frac{1}{w_k}, \cdots, s_k w_k^2 + \nu_k w_k, \cdots\right), \quad \forall w \in (\mathbb{C}^*)^n.$$

The condition that Φ is symplectic, $\Phi^* \tilde{\omega} = \omega$, translates to the coordinate charts via the 2-forms in display (5.10) and (5.11). In particular, ϕ_j^0 and ϕ_j^∞ must satisfy

$$\frac{\partial \phi_j^0}{\partial r_k} = \frac{\partial \phi_j^\infty}{\partial s_k} = \delta_k^j,$$

where δ_k^j is the Kronecker symbol. Therefore, there exist holomorphic functions $c_j^0(z)$ and $c_j^\infty(w)$ such that

$$\phi_j^0(z, r) = r_j + c_j^0(z) \text{ and } \phi_j^\infty(w, s) = s_j + c_j^\infty(w).$$

Substituting these expressions in (5.12) yields

$$(5.13) \quad c_j^0\left(\frac{1}{w_1}, \cdots, \frac{1}{w_n}\right) = c_j^\infty(w) w_j^2 + (\tilde{\nu}_j - \nu_j) w_j, \quad \forall w \in (\mathbb{C}^*)^n.$$

It follows that c_j^0 extends as a holomorphic function on $U_0 \cup U_\infty$. On the other hand, the complement of $U_0 \cup U_\infty$ in $(\mathbb{P}^1)^n$ has codimension 2, which implies that c_j^0 extends to the whole space by Hartogs's Theorem. Thus c_j^0 is constant and so is c_j^∞ , by the same argument. Thus, the equation (5.13) implies that

$$c_j^0 = c_j^\infty = 0 \text{ and } \tilde{\nu}_j = \nu_j.$$

□

We can use the map $\text{Par}: \text{Con}^\nu \rightarrow S^n$ given by Theorem 3.3 to push ω_{Con} forward to S^n . The opposite parabolic structure of our universal family $\nabla = \nabla_0 + \sum_j r_j \Theta_j^0$ is given by

$$(\zeta_j : 1) := (z_j r_j - \nu_j : r_j).$$

Therefore, if we denote by (z_j, ζ_j) the coordinates on the j^{th} factor $S \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{z_j = \zeta_j\}$, then we have

$$\zeta_j = z_j - \frac{\nu_j}{r_j} \Leftrightarrow r_j = \frac{\nu_j}{z_j - \zeta_j}$$

which yields after substitution:

$$\omega_\nu = \sum_{i=1}^n \nu_i \frac{dz_i \wedge d\zeta_i}{(z_i - \zeta_i)^2}.$$

Using this identification we can prove the following corollary.

Corollary 5.9. *If there exists fiber preserving symplectic isomorphism*

$$\begin{array}{ccc} (\text{Con}^\nu, \omega) & \xrightarrow{\sim} \Phi & (\text{Con}^{\tilde{\nu}}, \tilde{\omega}) \\ \downarrow \pi_+ & & \downarrow \pi_+ \\ \text{Bun} & \xrightarrow{\sim} \phi & \text{Bun} \end{array}$$

then there exists a permutation σ of n elements such that $\tilde{\nu}_k = \nu_{\sigma(k)}$ for every $k \in \{1, \dots, n\}$.

Proof. Note that the case where ϕ is the identity is ensured by Theorem 5.8. In particular, the identity on Bun has a unique lifting which is the identity on Con^ν . It follows that, a general Φ is determined by the map ϕ on the base. Indeed, if Φ_1 and Φ_2 lift ϕ , then $\Phi_2 \circ \Phi_1^{-1}$ lifts the identity, hence $\Phi_2 = \Phi_1$. The proof of the statement will follow from a reduction to this case.

Up to composing with Par, we may prove the result for

$$\Phi: (S^n, \omega_\nu) \longrightarrow (S^n, \omega_{\bar{\nu}}).$$

Any automorphism ϕ of $(\mathbb{P}^1)^n$ has the form

$$\phi(z) = (\varphi_1(z_{\sigma(1)}), \dots, \varphi_n(z_{\sigma(n)}))$$

where σ is a permutation of n elements and φ_i are Möbius transformations. Now define

$$\Xi(z, \zeta) = (\varphi_1(z_{\sigma(1)}), \dots, \varphi_n(z_{\sigma(n)}), \varphi_1(\zeta_{\sigma(1)}), \dots, \varphi_n(\zeta_{\sigma(n)})).$$

We will show that Ξ is an automorphism of S^n such that $\Xi^* \omega_{\nu^\sigma} = \omega_\nu$, where $\nu_j^\sigma = \nu_{\sigma(j)}$. Recall that any Möbius transformation is a composition of basic transformations $t \mapsto \alpha t$, $t \mapsto t + 1$ and $t \mapsto 1/t$. Therefore we can factorize each φ_j to show that

$$\left(\frac{d\varphi_j(z_{\sigma(j)}) \wedge d\varphi_j(\zeta_{\sigma(j)})}{(\varphi_j(z_{\sigma(j)}) - \varphi_j(\zeta_{\sigma(j)}))^2} \right) = \frac{dz_{\sigma(j)} \wedge d\zeta_{\sigma(j)}}{(z_{\sigma(j)} - \zeta_{\sigma(j)})^2}$$

for all $j \in \{1, \dots, n\}$, and this shows that $\Xi^*(\omega_{\nu^\sigma}) = \omega_\nu$.

Now consider the Φ the unique extension of ϕ such that $\Phi^* \omega_{\bar{\nu}} = \omega_\nu$. We have that $\Phi \circ \Xi^{-1}$ lifts the identity and $(\Phi \circ \Xi^{-1})^* \omega_{\bar{\nu}} = \omega_{\nu^\sigma}$. Thus the result follows from Theorem 5.8. \square

6. APPARENT MAP

Given a connection $\nabla: E_1 \rightarrow E_1 \otimes \Omega_C^1(D)$ we can define an \mathcal{O}_C -linear map:

$$\varphi_\nabla: \mathcal{O}_C \xrightarrow{s} E_1 \xrightarrow{\nabla} E_1 \otimes \Omega_C^1(D) \longrightarrow (E_1/\mathcal{O}_C) \otimes \Omega_C^1(D)$$

where the last arrow is defined by the quotient map from E_1 to $E_1/s(\mathcal{O}_C) \simeq \mathcal{O}_C(w_\infty)$ (denoted E_1/\mathcal{O}_C for short). The zero divisor of φ_∇ defines an element of the linear system

$$Z(\varphi_\nabla) \in |\mathcal{O}_C(w_\infty + D)|.$$

Since $\deg D = n$ and $\Omega_C^1 \simeq \mathcal{O}_C$, as C has genus one, then

$$|\mathcal{O}_C(w_\infty + D)| = \mathbb{P}(H^0(C, (E_1/\mathcal{O}_C) \otimes \Omega_C^1(D))) \simeq \mathbb{P}^n.$$

Hence we may define a rational map

$$\text{App}: \text{Con}^\nu \dashrightarrow |\mathcal{O}_C(w_\infty + D)|$$

which associates (E_1, ∇) to $Z(\varphi_\nabla)$. Under a generic hypothesis on the eigenvalues we show that App is in fact a morphism. It turns out that this hypothesis is also necessary as we prove in the following lemma.

Lemma 6.1. *The rational map $\text{App}: \text{Con}^\nu \dashrightarrow |\mathcal{O}_C(w_\infty + D)|$ is a morphism if and only if $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \neq 0$ for any $\epsilon_k \in \{+, -\}$.*

Proof. The indeterminacy locus of App is composed by the connections ∇ mapped to zero in $(E_1/\mathcal{O}_C) \otimes \Omega_C^1(D)$ which means that there exists a meromorphic 1-form ξ with poles at most on D such that

$$\nabla(s) = s\xi.$$

In such case we have for each $j = 1, \dots, n$ that $s(t_j) \in p_j^{\epsilon_j}(\nabla)$ with $\epsilon_j \in \{+, -\}$. And we also have that ∇ restricts to a logarithmic connection on \mathcal{O}_C whence

$$\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} = 0.$$

Conversely, suppose that there exist $\delta_j \in \{+, -\}$, $j = 1, \dots, n$, such that $\nu_1^{\delta_1} + \dots + \nu_n^{\delta_n} = 0$. On the one hand there exists a meromorphic 1-form ξ with simple poles only on D and prescribed residues

$$\text{Res}_{t_j} \xi = \nu_j^{\delta_j}.$$

On the other hand let ∇ be a logarithmic connection such that $s(t_j) \in p_j^{\delta_j}(\nabla)$. It follows that $\nabla(s) - s\xi$ is a holomorphic section of $E_1 \otimes \Omega_C^1$. Since $H^0(C, E_1 \otimes \Omega_C^1)$ is generated by $s\omega$ (recall that ω is a holomorphic one-form on C) it follows that

$$\nabla(s) = s(\xi + c\omega)$$

for some $c \in \mathbb{C}$. Hence $\varphi_\nabla = 0$. □

We may also consider the forgetful map

$$\pi_+ : \text{Con}^\nu \longrightarrow \text{Bun} \simeq (\mathbb{P}^1)^n$$

which sends (E_1, ∇) to the parabolic vector bundle $(E_1, \mathbf{p}^+(\nabla))$. We also denote π_+ the natural extension of this map to $\overline{\text{Con}}^\nu$. Combining these two maps we can give a birational model for $\overline{\text{Con}}^\nu$ as we prove in the following result.

Theorem 6.2. *If $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \neq 0$ for any $\epsilon_k \in \{+, -\}$ then the map $\pi_+ \times \text{App}$ induces a birational map*

$$\pi_+ \times \text{App} : \overline{\text{Con}}^\nu \dashrightarrow \text{Bun} \times |\mathcal{O}_C(w_\infty + D)|$$

whose indeterminacy locus is contained in $\overline{\text{Con}}^\nu \setminus \text{Con}^\nu$. Moreover, given $(E, \mathbf{p}) \in \text{Bun}_0$ the rank of

$$(\pi_+ \times \text{App})|_{\pi_+^{-1}(E, \mathbf{p})} : \pi_+^{-1}(E, \mathbf{p}) \longrightarrow |\mathcal{O}_C(w_\infty + D)|$$

coincides with the cardinality of the set $\{j \mid p_j \notin \mathcal{O}_C\}$.

Proof. Let s denote the section $\mathcal{O}_C \hookrightarrow E_1$. For any logarithmic connection ∇ on E_1 with poles on $D = t_1 + \dots + t_n$ we have that $\text{App}(\nabla)$ is given by the zero divisor of

$$\varphi_\nabla = \nabla(s) \wedge s \in H^0(C, \det(E_1) \otimes \Omega_C(D))$$

which is the same as defining

$$\text{App}(\nabla) = [\varphi_\nabla] \in \mathbb{P}(H^0(C, \det(E_1) \otimes \Omega_C(D))).$$

By Lemma 6.1 the condition on the eigenvalues implies that App , hence $\text{Bun} \times \text{App}$, is regular on Con^ν . Therefore the indeterminacy locus must lie only in the boundary divisor.

Now fix $(E_1, \mathbf{p} = \{p_1, \dots, p_n\}) \in \text{Bun}$. We will compute the rank of $\text{App}|_{\pi_+^{-1}(E_1, \mathbf{p})}$. Consider $H^0(\mathfrak{sl}(E_1) \otimes \Omega_C(t_j))$ the space of traceless Higgs fields with simple pole on t_j . Since its dimension is three, there exists a unique (up to scalar multiplication) strongly parabolic Higgs field Θ_j^p with respect to \mathbf{p} , i.e. $\text{Res}_{t_j}(\Theta_j^p)$ is nilpotent with image p_j and has no other poles. Let ∇_0 be a connection in $\pi_+^{-1}(E_1, \mathbf{p})$. Then any other λ -connection $\nabla \in \pi_+^{-1}(E_1, \mathbf{p})$ can be written in a unique way as

$$\nabla = c_0 \nabla_0 + \sum_{j=1}^n c_j \Theta_j^p,$$

for some $(c_0 : \cdots : c_n) \in \mathbb{P}^n$. Hence

$$\varphi_{\nabla} = c_0 \nabla_0(s) \wedge s + \sum_{j=1}^n c_j \Theta_j^p(s) \wedge s.$$

Suppose that $\Theta_j^p(s) \wedge s \neq 0$ for each j . Since they have one pole each and at different points these sections are linearly independent. Since $H^0(C, \mathcal{O}_C(w_{\infty} + D))$ has dimension $n + 1$ they form a basis together with $\nabla_0(s) \wedge s$ and the Apparent map restricted to this fiber is an isomorphism. Indeed, if

$$c_0 \nabla_0(s) \wedge s + \sum_{j=1}^n c_j \Theta_j^p(s) \wedge s = \left(c_0 \nabla_0(s) + \sum_{j=1}^n c_j \Theta_j^p(s) \right) \wedge s = 0$$

then, by Lemma 6.1, $c_0 = 0$ and it follows that $c_1 = \cdots = c_n = 0$ since the $\Theta_j^p(s) \wedge s$ are linearly independent.

Now suppose that there exists $j \in \{1, \dots, n\}$ such that $\Theta_j^p(s) \wedge s = 0$. This occurs if and only if $\Theta_j^p(s)$ is holomorphic at t_j which in turn means that $s(t_j) \in p_j$. Indeed, since $H^0(C, E_1 \otimes \Omega_C)$ is generated by $s\omega$, we know that $\Theta_j^p(s)$ is holomorphic at t_j if and only if $\Theta_j^p(s) = c s\omega$ for some constant $c \in \mathbb{C}$. On the other hand, it is clear that $\text{Res}_{t_j}(\Theta_j^p(s)) \in p_j$ is zero if and only if $s(t_j) \in p_j$.

Henceforth we conclude that, in general, the image of $\text{App}|_{\pi_+^{-1}(p)}$ is the linear space spanned by $\nabla_0(s) \wedge s$ and $\Theta_j^p(s) \wedge s$ such that $s(t_j) \notin p_j$. In particular, the (projective) rank is given by

$$\text{rk App}|_{\pi_+^{-1}(E_1, \mathbf{p})} = \#\{j \mid s(t_j) \notin p_j\}.$$

□

Remark 6.3. The Apparent map may be computed, explicitly, via the identification with Fuchsian systems. In this case, the section $\mathcal{O}_C \hookrightarrow E_1$ becomes

$$\mathcal{O}_C(w_{\infty} - w_0 - w_1 - w_{\lambda}) \hookrightarrow \mathcal{O}_C \oplus \mathcal{O}_C,$$

given by multiplication with $\left(\frac{x-\lambda}{y}, \frac{(1-\lambda)x}{y}\right)$, and we can use the local universal connections over U_0 and U_{∞} to express App. Following this path, one can give an alternative proof of Theorem 6.2.

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