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# FLAT PARABOLIC VECTOR BUNDLES ON ELLIPTIC CURVES II

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ABSTRACT. We describe the moduli space of logarithmic rank 2 connections on elliptic curves with  $n \geq 2$  poles generalizing a previous work by the first and second named authors.

## 1. INTRODUCTION

In this paper, we investigate the geometry of certain moduli spaces of connections on elliptic curves  $C$ . We consider pairs  $(E, \nabla)$  where  $E \rightarrow C$  is a rank 2 vector bundle and  $\nabla: E \rightarrow E \otimes \Omega_C^1(D)$  is a logarithmic connection with (reduced) polar divisor  $D = t_1 + \dots + t_n$ . We also prescribe the following data:

- The eigenvalues  $(\nu_i^+, \nu_i^-)$  of  $\text{Res}_{t_i}(\nabla)$ , for each  $i = 1, \dots, n$ ;
- A trace connection  $(\det(E), \text{tr}(\nabla))$ .

Once we have fixed this data we can define the moduli space  $\text{Con}^\nu(C, D)$  of those pairs  $(E, \nabla)$  up to isomorphism. For a generic choice of  $\nu$  (compatible with  $\text{tr}(\nabla)$ ) all connections  $(E, \nabla)$  are irreducible and the moduli  $\text{Con}^\nu(C, D)$  can be constructed as a GIT quotient. It follows from [6] that  $\text{Con}^\nu(C, D)$  is a smooth irreducible quasi-projective variety of dimension  $2n$ , equipped with a holomorphic symplectic structure. This genericity condition will be made precise later.

It is natural to consider the forgetful map  $\pi: (E, \nabla) \mapsto (E, \mathbf{p})$  which to a connection associates an underlying quasi-parabolic bundle. Given a choice of signs  $\epsilon_i \in \{+, -\}$  for each  $i = 1, \dots, n$ , the parabolic data  $\mathbf{p} = (p_1^{\epsilon_1}, \dots, p_n^{\epsilon_n})$  consists of the  $\nu_i^{\epsilon_i}$ -eigenspace  $p_i^{\epsilon_i} \subset E|_{t_i}$  for each pole. In particular, we have  $2^n$  underlying quasi-parabolic structures for each connection, depending on the choice of  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ .

The moduli space  $\text{Bun}(C, D)$  of those parabolic bundles admitting a connection is  $n$ -dimensional and the map  $\pi$  above turns to be Lagrangian, i.e. its fibers are Lagrangian submanifolds. However,

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$\text{Bun}(C, D)$  is not a variety, but a non-separated scheme. Over the open subset of simple bundles (i.e. without nontrivial automorphisms), the Lagrangian fibration  $\pi$  is an affine  $\mathbb{A}^n$ -bundle whose linear part is the cotangent bundle  $T^*\text{Bun}(C, D)$ .

**Results.** We fix  $C$  to be the elliptic curve with affine equation  $y^2 = x(x-1)(x-\lambda)$ ,  $\lambda \in \mathbb{C} \setminus \{0, 1\}$ , and denote by  $w_\infty \in C$  the point at infinity. Moreover, we only consider vector bundles  $E$  whose determinant is  $\det E = \mathcal{O}_C(w_\infty)$ .

In Section 2 we study which quasi-parabolic bundles  $(E, \mathbf{p})$  over  $(C, D)$  are  $\nu$ -flat, i.e. admit a connection  $\nabla$  with prescribed trace and exponents, compatible with parabolics. The major difference from the case  $n = 2$  is that when  $n$  is odd there exist  $\nu$ -flat quasi-parabolic vector bundles that are not  $\mu$ -semistable for any choice of weights. It occurs for the item 3 of Lemma 2.3. Following the study of stability we describe a wall-crossing phenomenon in Lemmas 2.4 and 2.5.

In Section 3 we study the logarithmic connections. We investigate  $\text{Con}^\nu(C, D)$  via the forgetful map to an underlying quasi-parabolic structure. We are especially concerned with the  $\mu$ -stability of these quasi-parabolic bundles. It turns out that there exists an open subset of  $\text{Con}^\nu(C, D)$  where the underlying vector bundle is  $E_1$ , the unique nontrivial extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_1 \longrightarrow \mathcal{O}_C(w_\infty) \longrightarrow 0$$

We call this open subset  $\text{Con}^\nu$ . Consider the map

$$\begin{aligned} \text{Par}: \text{Con}^\nu &\longrightarrow (\mathbb{P}^1 \times \mathbb{P}^1) \times \cdots \times (\mathbb{P}^1 \times \mathbb{P}^1) \\ (E, \nabla) &\longmapsto ( (p_1^+, p_1^-), \dots, (p_n^+, p_n^-) ) \end{aligned}$$

that associates to each connection all its residual eigendirections (i.e. with respect to all eigenvalues  $\nu_i^+, \nu_i^-$ ). When  $\nu_i^+ \neq \nu_i^-$  for each  $i = 1, \dots, n$ , we clearly have  $p_i^+ \neq p_i^-$ , so that the image of  $\text{Par}$  is contained in  $S^n$ , where  $S = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \text{diagonal}$ .

**Theorem 3.1.** *Assume  $\nu_i := \nu_i^+ - \nu_i^- \neq 0$  for every  $i \in \{1, \dots, n\}$ . Then the map  $\text{Par}: \text{Con}^\nu \rightarrow S^n$  is an isomorphism.*

For  $n$  odd, we consider  $\Sigma_n$  the space of connections such that any underlying quasi-parabolic bundle falls in item 3 of Lemma 2.3 i.e. are not  $\mu$ -semistable for any  $\mu$ . We describe  $\Sigma_n$  in Proposition 3.4. We also note that any connection of  $\Sigma_n$  can be obtained from a connection on  $E_0$  by performing an elementary transformation centered in  $n$  parabolic directions which lie in the unique maximal sub-bundle  $\mathcal{O}_C$ .

Every  $(E, \nabla) \in \text{Con}^\nu(C, D) \setminus \Sigma_n$  (consider  $\Sigma_n = \emptyset$  if  $n$  is even) has a  $\mu$ -stable underlying quasi-parabolic structure. Moreover, it can be obtained from a connection on  $\text{Con}^\nu$  via elementary transformations.

**Theorem 3.5.** *Assume that  $\nu_1^{\epsilon_1} + \cdots + \nu_n^{\epsilon_n} \notin \mathbb{Z}$ , for any  $\epsilon_i \in \{+, -\}$ , and that  $\nu_i^+ - \nu_i^- \notin \{0, 1, -1\}$  for  $i \in \{1, \dots, n\}$ . Let  $\Sigma_n$  be as (3.5) if  $n$  is odd, and let it be the empty set if  $n$  is even. Then the moduli space  $\text{Con}^\nu(C, D) \setminus \Sigma_n$  can be obtained from  $\text{Con}^\nu \simeq S^n$  by gluing a finite number of copies of  $S^n$  such that each gluing map  $\Psi_{I, \epsilon}: S^n \dashrightarrow \text{Con}^\nu$  is a fiber-preserving isomorphism outside a degree  $(d_1, \dots, d_n)$  divisor  $\Gamma \subset (\mathbb{P}^1)^n$  where  $d_j = 2$  if  $j \in I$  and  $d_j = 0$  otherwise.*

$$\begin{array}{ccc} S^n & \xrightarrow{\quad \Psi_{I, \epsilon} \quad} & \text{Con}^\nu \simeq S^n \\ & \searrow \quad \swarrow & \\ & (\mathbb{P}^1)^n & \end{array}$$

In Section 4 we study logarithmic Fuchsian systems i.e. logarithmic connections for the trivial bundle. The case we are interested in is obtained from  $Con^\nu$  by a suitable elementary transformation. We give an explicit description of such systems via some interesting calculations. In particular, this allows us to study  $\lambda$ -connections to give a compactification  $\overline{Con}^\nu$  of our space  $Con^\nu$ . It has a structure of projective bundle determined by the numbers  $\nu_i = \frac{\nu_i^+ - \nu_i^-}{2}$ . Here  $X \simeq (\mathbb{P}^1)^n$  is the space of underlying quasi-parabolic structures for  $Con^\nu$ , it encodes the  $+$  directions. We also show that the forgetful map extends to the boundary.

**Theorem 4.2.** *We have  $\overline{Con}^\nu = \mathbb{P}(\mathcal{E}^\nu)$ , where  $\mathcal{E}^\nu$  is the extension of  $\mathcal{O}_X$  by  $T^*X$*

$$0 \longrightarrow T^*X \longrightarrow \mathcal{E}^\nu \longrightarrow \mathcal{O}_X \longrightarrow 0$$

determined by

$$(\nu_1, \dots, \nu_n) \in H^1(X, T^*X) \simeq \bigoplus_{j=1}^n H^1(\mathbb{P}_{z_i}^1, T^*\mathbb{P}_{z_i}^1) \simeq \mathbb{C}^n.$$

Following this approach we conclude the paper by studying the Apparent map. It leads to an interesting result about the birational geometry of  $\overline{Con}^\nu$ .

**Theorem 5.2.** *If  $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \neq 0$  for any  $\epsilon_k \in \{+, -\}$  then the map  $\text{Bun} \times \text{App}$  induces a birational map*

$$\text{Bun} \times \text{App}: \overline{Con}^\nu \dashrightarrow X \times |\mathcal{O}_C(w_\infty + D)|$$

whose indeterminacy locus is contained in  $\overline{Con}^\nu \setminus Con^\nu$ . Moreover, given  $(E, \mathbf{p} = \{p_1, \dots, p_n\}) \in X$  the rank of

$$(\text{Bun} \times \text{App})|_{\text{Bun}^{-1}(E, \mathbf{p})}: \text{Bun}^{-1}(E, \mathbf{p}) \longrightarrow |\mathcal{O}_C(w_\infty + D)|$$

coincides with the cardinality of the set  $\{i \mid p_i \notin \mathcal{O}_C\}$ .

## 2. PARABOLIC VECTOR BUNDLES

Let  $C$  be an elliptic curve, and  $\{t_1, \dots, t_n\}$  be a set of distinct points on  $C$  and denote by  $D = t_1 + \dots + t_n$  the divisor defined by them. A rank two *quasi parabolic vector bundle*  $E_{\mathbf{p}} = (E, \mathbf{p})$ ,  $\mathbf{p} = \{p_i\}$ , on  $(C, D)$  consists of a holomorphic vector bundle  $E$  of rank two on  $C$  and for each  $i = 1, \dots, n$ , a 1-dimensional linear subspace  $p_i \subset E_{t_i}$ . We refer to the points  $t_i$ 's as parabolic points, and to the subspace  $p_i \subset E_{t_i}$  as the parabolic direction of  $E$  at  $t_i$ .

A pair  $(E_{\mathbf{p}}; \mu)$  of a quasi parabolic vector bundle and an  $n$ -tuple  $\mu = (\mu_1, \dots, \mu_n)$  of real numbers in the interval  $(0, 1)$  is called *parabolic vector bundle* of rank two. We often write  $E_{\mathbf{p}}$  for a parabolic vector bundle when the choice of the weight  $\mu$  is clear.

Let  $(E_{\mathbf{p}}; \mu)$  be a parabolic vector bundle and let  $L \subset E$  be a line subbundle then we define

$$\text{Stab}_\mu(L) := \deg E - 2 \deg L + \sum_{p_k \neq L_{t_k}} \mu_k - \sum_{p_k = L_{t_k}} \mu_k.$$

We say that  $(E_{\mathbf{p}}; \mu)$  is *semistable* if  $\text{Stab}_\mu(L) \geq 0$  holds for every  $L \subset E$ . It is *stable* if the strict inequality holds for every line subbundle  $L \subset E$ . We call  $\text{Stab}_\mu(L)$  the parabolic stability of  $L \subset E$  with respect to  $\mu$ .

For computations we can assume  $C \subset \mathbb{P}^2$  is the smooth projective cubic curve defined by

$$(2.1) \quad zy^2 = x(x - z)(x - \lambda z)$$

with  $\lambda \in \mathbb{C}$ ,  $\lambda \neq 0, 1$ . We will write  $w_\infty = (0 : 1 : 0) \in C$  for the identity with respect to the group structure, and

$$(2.2) \quad w_0 = (0 : 0 : 1), \quad w_1 = (1 : 0 : 1), \quad w_\lambda = (\lambda : 0 : 1)$$

the 2-torsion points.

We denote by  $\text{Bun}_{w_\infty}^\mu(C, D)$ , or simply  $\text{Bun}_{w_\infty}^\mu$ , the *moduli space* of semistable parabolic vector bundles  $(E_{\mathbf{p}}; \mu)$  on  $(C, D)$  with  $\det E = \mathcal{O}_C(w_\infty)$ . We note that if  $E$  has  $\mathcal{O}_C(w_\infty)$  as determinant line bundle then either  $E \simeq L \oplus L^{-1}(w_\infty)$  or  $E \simeq E_1$ , where  $E_1$  is the unique non trivial extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_1 \longrightarrow \mathcal{O}_C(w_\infty) \longrightarrow 0.$$

If there exists  $L \subset E$  such that  $\text{Stab}_\mu(L)$  is zero then the weights lie on the hyperplane

$$H(d, I) := \left\{ \mu \mid 1 - 2d + \sum_{k \notin I} \mu_k - \sum_{k \in I} \mu_k = 0 \right\}$$

where  $d = \deg L$  and  $I \subset \{1, \dots, n\}$  denotes the set of indices of those parabolic directions  $p_k \subset L_{t_k}$ . A connected component of the complement in  $(0, 1)^n$  of all these hyperplanes  $H(d, I)$  is called a *chamber*. If  $\mu$  and  $\tilde{\mu}$  belong to the same chamber  $\mathcal{C}$  then  $\text{Bun}_{w_\infty}^\mu = \text{Bun}_{w_\infty}^{\tilde{\mu}}$ , see for example [8] or [3, Lemma 2.7]. Hereafter we denote by  $(\mathbb{P}^1)^n$  the product of  $n$  copies of  $\mathbb{P}^1$

$$(\mathbb{P}^1)^n = \underbrace{\mathbb{P}^1 \times \dots \times \mathbb{P}^1}_{n \text{ times}}.$$

We will need the following result.

**Proposition 2.1.** *The following assertions hold:*

- (1) *The set  $\mathfrak{C} := \{\mu \in (0, 1)^n \mid \sum_{k=1}^n \mu_k < 1\}$  is a chamber.*
- (2) *If  $\mu \in \mathfrak{C}$  then  $\text{Bun}_{w_\infty}^\mu = \{(E, \mathbf{p}) \mid E = E_1\}$ . Moreover, it is isomorphic to  $(\mathbb{P}^1)^n$ .*

*Proof.* For the first statement, assume  $\mu \in \mathfrak{C}$ . If  $d \geq 1$  then we see that

$$(2.3) \quad 1 - 2d + \sum_{k \notin I} \mu_k - \sum_{k \in I} \mu_k \leq -1 + \sum_{k=1}^n \mu_k < 0$$

for all subset  $I \subset \{1, \dots, n\}$ . If  $d \leq 0$  then we get

$$(2.4) \quad 1 - 2d + \sum_{k \notin I} \mu_k - \sum_{k \in I} \mu_k = 1 - 2d + 2 \sum_{k \notin I} \mu_k - \sum_{k=1}^n \mu_k > 0$$

for all subset  $I \subset \{1, \dots, n\}$ . Hence (2.3) and (2.4) imply that there is no hyperplane  $H(d, I)$  passing through  $\mu$ . This proves the first part of the statement.

Now we prove (2). Recall that  $\det E = \mathcal{O}_C(w_\infty)$  implies that either  $E = E_1$  or  $E = L \oplus L^{-1}(w_\infty)$  with  $\deg L \geq 1$ . Assume the later by contradiction. Then

$$\text{Stab}_\mu(L) = 1 - 2 \deg L + \sum_{p_k \neq L_{t_k}} \mu_k - \sum_{p_k = L_{t_k}} \mu_k \leq -2 \deg L < 0$$

and  $E$  is not semistable which is absurd. Hence  $E = E_1$ . Each parabolic bundle is then completely determined by

$$(p_1, \dots, p_n) \in \mathbb{P}(E_1|_{t_1}) \times \dots \times \mathbb{P}(E_1|_{t_1}) \simeq (\mathbb{P}^1)^n$$

and we get the desired isomorphism.  $\square$

For a weight vector  $\mu = (\mu_1, \dots, \mu_n) \in (0, 1)^n$  and a subset  $I \subset \{1, \dots, n\}$  of even cardinality, we consider the map  $\varphi_I: (0, 1)^n \rightarrow (0, 1)^n$  defined by

$$\varphi_I(\mu) := (\mu'_1, \dots, \mu'_n) \in (0, 1)^n$$

where  $\mu'_i = \mu_i$  if  $i \notin I$ , and  $\mu'_i = 1 - \mu_i$  if  $i \in I$ . Since it preserves the family of hyperplanes  $H(d, J)$ , the image of  $\mathfrak{C}$  by  $\varphi_I$  yields a new chamber

$$(2.5) \quad \mathfrak{C}_I := \left\{ \mu \in (0, 1)^n \mid \sum_{k \notin I} \mu_k - \sum_{k \in I} \mu_k + |I| < 1 \right\}$$

where  $|I|$  is the cardinality of  $I$ . When  $I = \emptyset$  then  $\mathfrak{C}_I = \mathfrak{C}$ .

Each  $\varphi_I$  admits a modular realization as an elementary transformation, which we now describe. We consider the following exact sequence of sheaves

$$0 \longrightarrow E' \xrightarrow{\alpha} E \xrightarrow{\beta} \bigoplus_{i \in I} (E_{t_i}/p_i) \longrightarrow 0$$

where for each (local) section  $s$  of  $E$  we define  $\beta(s) = (\beta_1(s), \dots, \beta_n(s))$  by  $\beta_j(s) = s(t_j) \pmod{p_j}$  if  $s$  is defined at  $t_j$ , and  $\beta_j(s) = 0$  otherwise. Then  $E'$  is a vector bundle of rank two such that

$$\det E' = \det E \otimes \mathcal{O}_C \left( - \sum_{i \in I} t_i \right).$$

In particular,  $E'$  has degree  $1 - |I|$ . We can see  $E'$  as a quasi parabolic vector bundle on  $(C, D)$  as follows. If  $i \notin I$  then  $\alpha_{t_i}: E'_{t_i} \rightarrow E_{t_i}$  is an isomorphism and

$$p'_i = (\alpha_{t_i})^{-1}(p_i) \subset E'_{t_i}$$

is the parabolic direction at  $t_i$ . If  $i \in I$ , then we define  $p'_i = \ker(\alpha_{t_i})$  as the parabolic direction at  $t_i$ . This operation corresponds to the birational transformation of ruled surfaces  $\mathbb{P}(E) \dashrightarrow \mathbb{P}(E')$  obtained by blowing-up the points  $p_i \in \mathbb{P}(E_{t_i})$  and then blowing-down the strict transforms of the fibers  $\mathbb{P}(E_{t_i})$  to the points  $p'_i \in \mathbb{P}(E'_{t_i})$ ,  $i \in I$ . This is well-defined since the  $p_i$  lie on different fibers. Since  $|I|$  is even, we can fix a square root  $L_0$  of the line bundle  $\mathcal{O}_C(\sum_{i \in I} t_i)$  i.e.

$$L_0^2 = \mathcal{O}_C \left( \sum_{i \in I} t_i \right).$$

This gives a correspondence

$$elem_I: (E, \mathbf{p}) \mapsto (E' \otimes L_0, \mathbf{p}')$$

between quasi parabolic vector bundles on  $(C, D)$  which have  $\mathcal{O}_C(w_\infty)$  as determinant line bundle.

The reader can check that if  $(E, \mathbf{p})$  is semistable with respect to  $\mu$ , then  $elem_I(E, \mathbf{p})$  is semistable with respect to  $\varphi_I(\mu)$ . We conclude that the correspondence  $elem_I$  defines an isomorphism between moduli spaces

$$elem_I: \text{Bun}_{w_\infty}^\mu \longrightarrow \text{Bun}_{w_\infty}^{\varphi_I(\mu)}.$$

**Remark 2.2.** From Proposition 2.1 we conclude that  $\text{Bun}_{w_\infty}^\mu$  is isomorphic to  $(\mathbb{P}^1)^n$  for any  $\mu \in \mathfrak{C}_I$  for any  $I \subset \{1, \dots, n\}$  of even cardinality.

A quasi-parabolic vector bundle  $(E, \mathbf{p})$  is called decomposable if there exist  $(E', \mathbf{p}')$  and  $(E'', \mathbf{p}'')$  such that  $(E, \mathbf{p}) \simeq (E', \mathbf{p}') \oplus (E'', \mathbf{p}'')$  as quasi-parabolic vector bundles. Otherwise it is called indecomposable. Note that  $(E, \mathbf{p})$  can be indecomposable with  $E$  decomposable as a vector bundle.

**Lemma 2.3.** *Let  $(E, \mathbf{p})$  be rank two indecomposable quasi-parabolic bundle, over  $(C, D)$ , with  $\det E = \mathcal{O}_C(w_\infty)$ . Then one of the following holds:*

- (1)  *$E$  is indecomposable i.e.  $E = E_1$ ;*
- (2)  *$E = L \oplus L^{-1}(w_\infty)$  and  $2 \leq 2 \deg L \leq n$ ;*
- (3)  *$E = L \oplus L^{-1}(w_\infty)$  with  $L^2 = \mathcal{O}_C(D + w_\infty)$ , hence  $2 \deg L = n + 1$ . Moreover, every parabolic direction lies on  $L^{-1}(w_\infty)$  except for one that lies outside both subbundles.*

*Proof.* When  $E = E_1$  we have nothing to prove. So suppose that  $E = L \oplus L^{-1}(w_\infty)$ . Since  $L \oplus L^{-1}(w_\infty) \simeq M \oplus M^{-1}(w_\infty)$  with  $M = L^{-1}(w_\infty)$  we can assume  $\deg L = s \geq 1$ .

To decompose  $(E, \mathbf{p})$  we need to find an embedding of  $L^{-1}(w_\infty)$  in  $E$  passing through every direction that does not lie on  $L$ . Note that this is the same as finding an automorphism of  $E$  that sends every direction outside  $L$  to  $(0 : 1)$ . Let  $p_j = (u_j : 1)$  denote the parabolic direction over  $t_j$  which is outside  $L$ . Recall that

$$\text{End}(E) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \mid \alpha, \delta \in \mathbb{C}, \beta \in H^0(C, L^2(-w_\infty)) \right\}.$$

If  $2s \geq n + 2$  then  $h^0(L^2(-w_\infty - D + t_j)) = 2s - n \geq 2$  and we are free to choose  $\beta_j$  that vanishes on  $t_i$  for  $i \neq j$  and such that  $\beta_j(t_j) = -u_j$ . Thus, choosing  $\beta = \sum_{j=1}^n \beta_j$ ,  $\alpha = 1$  and  $\delta = 1$ , the corresponding automorphism sends any direction  $p_j$  outside  $L$  to  $(0 : 1)$ .

Now set  $2s = n + 1$ . By the same argument as above, to show that  $(E, \mathbf{p})$  is decomposable, we need to find a section  $\beta_j$  of  $H^0(C, L^2(-w_\infty))$  that vanishes on  $t_i$  for  $i \neq j$  and such that  $\beta_j(t_j) = -u_j$ , for each  $j \in \{1, \dots, n\}$ . We will show that if  $L^2(-w_\infty - D) \neq \mathcal{O}_C$  then we can find  $\beta_j$  as required. For this, assume  $L^2(-w_\infty - D + t_j) \simeq \mathcal{O}_C(x_j)$  with  $x_j \neq t_j$ , and take any section  $\alpha_j$  of  $L^2(-w_\infty - D + t_j)$  with  $\alpha_j(t_j) \neq 0$ . The desired section  $\beta_j$  is defined as  $\beta_j = -\frac{u_j}{\alpha_j(t_j)} \alpha_j$ . Hence  $(E, \mathbf{p})$  is decomposable when  $L^2(-w_\infty - D) \neq \mathcal{O}_C$ . In addition, if  $L^2 = \mathcal{O}_C(D + w_\infty)$  then we can apply the argument above with  $D - t_1$  in place of  $D$  to find an embedding of  $L^{-1}(w_\infty)$  passing through  $n - 1$  parabolic directions outside  $L$ . In particular, if  $(E, \mathbf{p})$  is indecomposable then there is no parabolic direction on  $L$  and this finishes the proof of the lemma.  $\square$

Until now we have only considered a rank two  $E$  and its line sub-bundles  $L \subset E$ . But a more general setting will be suitable for the next results. We will consider general morphisms  $L \rightarrow E$  that do not, necessarily, lead to an embedding of  $L$  in  $E$ . Recall that being a sub-bundle means that there exist an injective morphism  $L \hookrightarrow E$  whose cokernel is also a line bundle. For a general morphism  $\phi: L \rightarrow E$  this does not need to be true. However, we can factor out a divisor  $Z$  where  $\phi$  vanishes, leading to an injective morphism  $L(Z) \hookrightarrow E$ . For details, see [5, Ch. 2, Proposition 5]. On the other hand, given a sub-bundle  $L \subset E$  we can produce a morphism  $L(-Z) \rightarrow E$  that vanishes on the fibers over the support of  $Z$ .

Given a morphism  $\phi: L \rightarrow E$  we say that its image pass through  $p_j \in E_{t_j}$  if  $\phi_{t_j}(L_{t_j}) \subset p_j$ .

**Lemma 2.4.** *Let  $I \subset \{1, \dots, n\}$  have cardinality  $2k + 2$  with  $k \geq 0$  and fix  $\mu \in \mathfrak{C}_I$ . Then  $(E_1, \mathbf{p})$  is not  $\mu$ -semistable if and only if there exists a line bundle  $L$  of degree  $\deg L = -k$  and a morphism  $L \rightarrow E_1$  whose image pass through  $p_j$  for all  $j \in I$ .*

*Proof.* Fix  $\mu = (\mu_1, \dots, \mu_n) \in \mathfrak{C}_I$ . First recall that  $\mu = \varphi_I(\mu')$  for some  $\mu' \in \mathfrak{C}$ . Then  $(E_1, \mathbf{p})$  is not  $\mu$ -semistable if there exists a sub-bundle  $L \subset E_1$  such that

$$\text{Stab}_\mu(L) = 1 - 2 \deg L + \sum \pm \mu_j = 1 - 2 \deg L + |I| - 2|A| + \sum \pm \mu'_j < 0$$

where  $A \subset I$  is the set of  $j \in I$  such that  $p_j$  lie on  $L$ . As  $\mu' \in \mathfrak{C}$  we have  $|\sum \pm \mu'_j| < 1$  – the actual signs do not matter. Then we need

$$1 - 2 \deg L + |I| - 2|A| \leq -1.$$

Let  $k \geq 0$  be defined by  $|I| = 2k + 2$  and define  $u = \deg L + k$ . Then we rewrite the inequality as  $2k + 2 \leq u + |A|$ . From  $A \subset I$  and  $\deg L \leq 0$  we have that  $0 \leq u \leq k$ .

Next we will produce a degree  $-k$  line bundle and a morphism to  $E_1$  such that its image passes through every direction  $p_j^\epsilon$  with  $j \in I$ . If  $u = 0$  i.e.  $\deg L = -k$ , then  $A = I$  and we are done. If  $u > 0$  or equivalently  $\deg L > -k$  we need to work a little bit.

Fix  $J \subset A$  such that  $|J| = 2k + 2 - u$  and let  $Z = \sum_{j \in I \setminus J} t_j$ . Considering the inclusions  $L \hookrightarrow E_1$  and  $\mathcal{O}_C(-Z) \hookrightarrow \mathcal{O}_C$  we define a map by the composition

$$\phi: L \otimes \mathcal{O}_C(-Z) \longrightarrow E_1(-Z) \longrightarrow E_1,$$

with the property that it gives the same directions over  $D - Z$  and vanishes over  $Z$ . Hence the image of  $\phi$  passes through every direction from  $I$ .

Conversely, suppose that there exists a degree  $-k$  line bundle  $L$  and a nontrivial morphism  $\phi: L \longrightarrow E_1$  passing through every  $p_j$ ,  $j \in I$ . Let  $Z$  be the zero divisor of  $\phi$  and consider the reduction  $\phi': L(Z) \longrightarrow E_1$ . As  $\phi'$  is injective,  $L(Z)$  is a subbundle of  $E_1$ . Hence it has nonpositive degree i.e.  $\deg Z \leq k$ . On the other hand, we have that  $p_j$  lie on  $L(Z)$  for every  $j$  such that  $t_j \notin \text{Supp} Z$ . If  $A$  is given as above, we have  $|A| \geq 2k + 2 - \deg Z_{\text{red}}$ , hence

$$1 - 2 \deg L(Z) + |I| - 2|A| \leq -1 + 2(\deg Z_{\text{red}} - \deg Z) \leq -1$$

This completes the proof.  $\square$

We now see the real advantage of switching to this slightly more general setting. The previous Lemma describes a wall-crossing phenomenon. And the next Lemma can be used to describe geometrically the space of quasi-parabolic bundles that become unstable when we cross a wall.

**Lemma 2.5.** *Let  $n = 2k + 2$  for some  $k \geq 0$ . Let  $V \subset X \simeq (\mathbb{P}^1)^n$  be the locus of points that correspond to quasi-parabolic bundles  $(E_1, \mathbf{p})$  satisfying the following property: there exist a line bundle  $L$  of degree  $\deg L = -k$  and a morphism  $\phi: L \longrightarrow E_1$  whose image passes through  $\mathbf{p}$ . Then  $V$  is a degree  $(2, \dots, 2)$  hypersurface.*

*Proof.* Let  $\pi_j: (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^{n-1}$  be the projection given by forgetting the  $j$ th component and let  $h_j$  be the class of a fiber of  $\pi_j$ . Then we only need to show that  $V \cap h_j = 2$  for every  $j$ . Up to permuting indices we only need to consider  $j = 2k + 2$ .

If  $k = 0$  the result follows from [10, Prop. 3.3]. Indeed, each degree 0 line bundle  $L \in \text{Pic}^0(C)$  has a unique map  $\phi: L \longrightarrow E_1$  and the map  $L \mapsto \phi_{t_1}(L) \in \mathbb{P}^1$  is generically  $2 : 1$ . Then, for a generic direction  $p_1$ , there exist two choices for  $L \in \text{Pic}^0(C)$  such that  $\phi_{t_1}(L) \subset p_1$ . Therefore,  $(p_1, p_2) \in V$  if and only if  $p_2$  one of the directions defined by these line bundles i.e.  $V \cap h_2 = 2$ .

Now we consider  $k \geq 1$ . We will show that we can reduce to the previous case. Fix  $p_1, \dots, p_{2k}$  generic directions. By generic we mean that there is no sub-bundle of degree at least  $1 - k$  passing through these directions. Let  $L \in \text{Pic}^{-k}(C)$  be any line bundle. To give a map  $\phi: L \longrightarrow E_1$  passing through  $p_1, \dots, p_{2k}$  is equivalent to giving a map  $L \longrightarrow E'$ , where  $E'$  is obtained by elementary transformation with respect to  $p_1, \dots, p_{2k}$ . Indeed, we have

$$0 \longrightarrow E' \xrightarrow{\alpha} E_1 \xrightarrow{\beta} \bigoplus_{j=1}^{2k} (E_1)_{t_j}/p_j \longrightarrow 0$$

and  $\beta \circ \phi = 0$  if and only if there exists  $\phi' : L \rightarrow E'$  such that  $\phi = \alpha \circ \phi'$ . Nonetheless, this is equivalent to giving a map

$$\phi' \otimes 1 : L \otimes M \rightarrow E' \otimes M$$

where  $M$  is a line bundle such that  $M^2 = \mathcal{O}_C(t_1 + \dots + t_{2k})$ .

Since  $p_1, \dots, p_{2k}$  are generic,  $E'$  is indecomposable. In particular,  $E' \otimes M = E_1$ . We then apply the same argument of the case  $k = 0$  to the directions  $p_{2k+1}$  and  $p_{2k}$  to show that  $V \cap h_{2k+2} = 2$ .  $\square$

### 3. LOGARITHMIC CONNECTIONS

A *logarithmic connection* on a rank two vector bundle  $E$  over  $C$  with polar divisor  $D = t_1 + \dots + t_n$  is a  $\mathbb{C}$ -linear map

$$\nabla : E \rightarrow E \otimes \Omega_C^1(D)$$

satisfying the Leibniz rule

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

for any local sections  $s$  of  $E$  and  $f$  of  $\mathcal{O}_C$ . If  $t \in C$  is a pole for  $\nabla$  and  $U \subset C$  is a small trivializing neighborhood of  $t$  we write  $\nabla|_U = d + A$  where  $d : \mathcal{O}_C \rightarrow \Omega_C^1$  is the exterior derivative and  $A$  is a  $2 \times 2$  matrix whose coefficients are 1-forms with at most simple poles on  $t$ . Note that  $A$  depends on the trivialization but its conjugacy class does not. Then the residue endomorphism

$$\text{Res}_t(\nabla) := \text{Res}_t(A) \in \text{End}(E_t)$$

is well defined. Let  $\nu_k^+$  and  $\nu_k^-$  be the eigenvalues of  $\text{Res}_{t_k}(\nabla)$ , called the *local exponents* of  $\nabla$  over  $t_k$ . The data

$$\nu = (\nu_1^+, \nu_1^-, \dots, \nu_n^+, \nu_n^-) \in \mathbb{C}^{2n}$$

is called the *local exponent* of  $\nabla$ . Since the connection  $\nabla$  induces a rank one logarithmic connection  $\text{tr}(\nabla) : \det(E) \rightarrow \det(E) \otimes \Omega_C^1(D)$  with

$$\text{Res}_{t_k}(\text{tr}(\nabla)) = \nu_k^+ + \nu_k^-$$

then Residue Theorem yields the Fuchs relation:

$$\deg E + \sum_{k=1}^n (\nu_k^+ + \nu_k^-) = 0.$$

In the definition of our moduli space of connections we need the following data:

- (1) A  $2n$ -tuple of complex numbers  $\nu = (\nu_1^+, \nu_1^-, \dots, \nu_n^+, \nu_n^-)$  satisfying the Fuchs relation

$$1 + \sum_{k=1}^n (\nu_k^+ + \nu_k^-) = 0$$

and the generic condition  $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \notin \mathbb{Z}$  for any  $\epsilon_k \in \{+, -\}$ , to avoid reducible connections;

- (2) A fixed trace connection  $\zeta : \mathcal{O}_C(w_\infty) \rightarrow \mathcal{O}_C(w_\infty) \otimes \Omega_C^1(D)$  satisfying

$$\text{Res}_{t_k}(\zeta) = \nu_k^+ + \nu_k^-$$

for all  $k = 1, \dots, n$ .

We denote by  $\text{Con}^\nu(C, D)$  the *moduli space of rank two connections*  $\nabla: E \rightarrow E \otimes \Omega_C^1(D)$  having local exponent  $\nu$ ,  $\det E = \mathcal{O}_C(w_\infty)$  and  $\text{tr} \nabla = \zeta$ . We write  $(E, \nabla)$  for an element of  $\text{Con}^\nu(C, D)$ .

The moduli space  $\text{Con}^\nu(C, D)$  is a smooth irreducible quasi-projective variety of dimension  $2n$  provided that  $\nu_k^+ \neq \nu_k^-$  for all  $k \in \{1, \dots, n\}$ . Algebraic constructions of moduli spaces of connections goes back to the works of Simpson and, in the logarithmic case, Nitsure in [9]. In our setting, it is more convenient to refer to the works of Inaba, Iwasaki and Saito [7], and more precisely Inaba [6]. Indeed, under our generic assumption on  $\nu$ , each connection  $\nabla$  on  $E$  defines a unique parabolic structure, by selecting the eigendirection  $p_k \subset E|_{t_k}$  associated to  $\nu_k^+$  at each pole  $t_k$ ; therefore,  $\text{Con}^\nu(C, D)$  can equivalently be viewed as the moduli space of parabolic connections as considered in the work [6] of Inaba. Then it follows from [6, Theorem 2.1, Proposition 5.2] that it is quasi-projective and irreducible of dimension  $2n$ . Moreover, [6, Theorem 2.2] shows that it is moreover smooth. In fact, in order to fit with the stability condition [6, Definition 2.2], we set  $\alpha_1^{(k)} = \frac{1-\mu_k}{2}$  and  $\alpha_2^{(k)} = \frac{1+\mu_k}{2}$ ; our moduli space therefore corresponds to the fiber  $\det^{-1}(L, \zeta)$  of the determinant map considered at the beginning of [6, Section 5]. When  $\nu_k^+ = \nu_k^-$  for some  $k$ , there are connections with scalar residue (apparent singular point) which give rise to a singular locus in the moduli space; the role of the parabolic structure in [6] is to get a smooth moduli space even in that case.

Let  $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$  be the diagonal and let  $S := (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$  be its complement. We will to show that, for  $n$  even,  $\text{Con}^\nu(C, D)$  can be written as a union of open subsets which are isomorphic to  $S^n$  – as affine  $\mathbb{C}^n$ -bundle over  $(\mathbb{P}^1)^n$ . When  $n$  is odd, apart from these open subsets, we also have an interesting family of connections that we will call  $\Sigma_n$ . Now we will describe these open sets.

**3.1. Description and glueing of the open charts.** Let us assume  $\nu_k^+ \neq \nu_k^-$  for all  $k \in \{1, \dots, n\}$ . Given a connection  $(E, \nabla) \in \text{Con}^\nu(C, D)$ , one can associate, over each  $t_k$ , a pair of “positive” and “negative” eigendirections of  $\text{Res}_t(\nabla)$

$$p_k^+(\nabla), p_k^-(\nabla) \in \mathbb{P}(E_{t_k})$$

defined by the eigenvalues  $\nu_k^+$  and  $\nu_k^-$  respectively.

Let  $I \subset \{1, \dots, n\}$  have even cardinality and denote by

$$X_I = \text{Bun}_{w_\infty}^\mu, \quad \text{with } \mu \in \mathfrak{C}_I.$$

See (2.5) for the definition of  $\mathfrak{C}_I$ . Proposition 2.1 and Remark 2.2 yield  $X_I \simeq (\mathbb{P}^1)^n$ . Given an  $n$ -tuple  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ , where each  $\epsilon_i \in \{+, -\}$ , we denote

$$\mathbf{p}^\epsilon(\nabla) = \{p_1^{\epsilon_1}(\nabla), \dots, p_n^{\epsilon_n}(\nabla)\}$$

and by  $(E, \mathbf{p}^\epsilon(\nabla))$  the quasi-parabolic vector bundle defined by these directions. Then for each  $I \subset \{1, \dots, n\}$  of even cardinality and  $\epsilon$  as above, we define

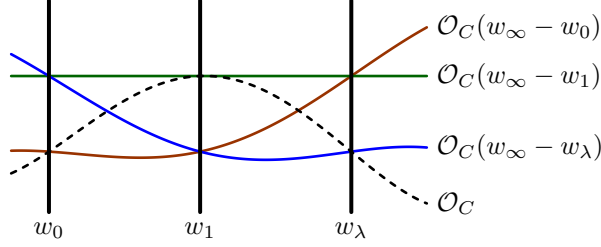
$$\text{Con}_{I, \epsilon}^\nu = \{(E, \nabla) \in \text{Con}^\nu(C, D) \mid (E, \mathbf{p}^\epsilon(\nabla)) \in X_I\}.$$

We note that if  $I = \emptyset$  then  $\text{Con}_{\emptyset, \epsilon}^\nu$  does not depend on  $\epsilon$  and Proposition 2.1 yields

$$\begin{aligned} X &:= X_\emptyset = \{(E, \mathbf{p}) \mid E \simeq E_1\} \\ \text{Con}^\nu &:= \text{Con}_{\emptyset, \epsilon}^\nu = \{(E, \nabla) \mid E \simeq E_1\} \end{aligned}$$

where  $E_1$  is the unique non trivial extension

$$0 \rightarrow \mathcal{O}_C \rightarrow E_1 \rightarrow \mathcal{O}_C(w_\infty) \rightarrow 0.$$

FIGURE 1. Sections of  $\mathbb{P}(E_1)$ 

We want to give an isomorphism between  $Con^\nu_{T,\epsilon}$  and  $S^n$ . To do this we will start with  $Con^\nu$ . Given  $(E_1, \nabla) \in Con^\nu$ , introducing the parabolic structures

$$\begin{aligned} \mathbf{p}^-(\nabla) &= \{p_1^-(\nabla), \dots, p_n^-(\nabla)\} \\ \mathbf{p}^+(\nabla) &= \{p_1^+(\nabla), \dots, p_n^+(\nabla)\} \end{aligned}$$

one has a map

$$\text{Par} : \begin{cases} Con^\nu & \rightarrow X \times X \\ (E_1, \nabla) & \mapsto ((E_1, \mathbf{p}^+), (E_1, \mathbf{p}^-)) \end{cases}$$

We are assuming  $\nu_k^+ \neq \nu_k^-$  for  $k \in \{1, \dots, n\}$ , therefore  $p_k^+(\nabla) \neq p_k^-(\nabla)$ . This implies that the image of  $\text{Par}$  in  $X \times X$  avoid the incidence variety

$$\mathcal{I} := \bigcup_{k=1}^n \{p_k^+ = p_k^-\} \subset X \times X.$$

We can fix a system of coordinates for  $X \times X$  as follows. Let  $L_i \subset E_1$  denote the unique embedding of  $\mathcal{O}_C(w_\infty - w_i)$  for  $i = 0, 1, \lambda$  where  $w_0, w_1, w_\lambda \in C$  are the 2-torsion points. They correspond to sections of  $\mathbb{P}(E_1)$  which have +1 self intersection and intersect each other as in Figure 1.

In particular, for every  $k \in \{1, \dots, n\}$  we may define a system of coordinates for  $(\mathbb{P}(E_1))_{t_k} \simeq \mathbb{P}^1$  imposing

$$(3.1) \quad (\mathbb{P}(L_0))_{t_k} = (0 : 1), \quad (\mathbb{P}(L_1))_{t_k} = (1 : 1) \text{ and } (\mathbb{P}(L_\lambda))_{t_k} = (1 : 0).$$

Let  $(z_1, \dots, z_n) \in (\mathbb{P}^1)^n$  correspond to  $(E_1, \mathbf{p}^+)$  and  $(\zeta_1, \dots, \zeta_n) \in (\mathbb{P}^1)^n$  to  $(E_1, \mathbf{p}^-)$ . With this system of coordinates we get a map

$$\text{Par} : Con^\nu \longrightarrow S^n$$

where

$$S^n = ((\mathbb{P}^1_{z_1} \times \mathbb{P}^1_{\zeta_1}) \setminus \Delta) \times \dots \times ((\mathbb{P}^1_{z_n} \times \mathbb{P}^1_{\zeta_n}) \setminus \Delta).$$

**Theorem 3.1.** *Assume  $\nu_k^+ - \nu_k^- \neq 0$  for every  $k \in \{1, \dots, n\}$ . Then the map  $\text{Par} : Con^\nu \rightarrow S^n$  is an isomorphism.*

*Proof.* For the injectivity of  $\text{Par}$  we must prove that given  $(E_1, \nabla) \in Con^\nu$ , the  $2n$  eigendirections  $\{p_1^+(\nabla), p_1^-(\nabla), \dots, p_n^+(\nabla), p_n^-(\nabla)\}$  determine  $\nabla$ . Let  $\text{Bun}^+ : Con^\nu \rightarrow \mathbb{P}^1_{z_1} \times \dots \times \mathbb{P}^1_{z_n}$  be the map

which sends  $(E_1, \nabla)$  to  $(E_1, \mathbf{p}^+)$  and  $\tau : S^n \rightarrow \mathbb{P}_{z_1}^1 \times \cdots \times \mathbb{P}_{z_n}^1$  the natural projection

$$\begin{array}{ccc} \text{Con}^\nu & \xrightarrow{\text{Par}} & S^n \\ & \searrow \text{Bun}^+ \quad \swarrow \tau & \\ & \mathbb{P}_{z_1}^1 \times \cdots \times \mathbb{P}_{z_n}^1 & \end{array}$$

We first assume that  $\nabla$  does not have any positive direction in  $L_\lambda$ . Consider the open set

$$U_0 = \{\mathbf{z} = ((z_1 : 1), \dots, (z_n : 1)) \mid z_k \in \mathbb{C}\}.$$

Given  $\mathbf{z} \in U_0$  let  $\nabla_0$  be a connection which has  $\mathbf{z}$  as positive eigendirections and all the negative eigendirections lie in  $L_\lambda$ , which means that

$$\mathbf{p}^-(\nabla_0) = \{(1 : 0), \dots, (1 : 0)\}.$$

In Section 4.1.1, we make  $\nabla_0$  explicit. The residues are

$$(3.2) \quad \text{Res}_{t_k} \nabla_0 = \begin{pmatrix} \nu_k^- & z_k(\nu_k^+ - \nu_k^-) \\ 0 & \nu_k^+ \end{pmatrix}.$$

Now if  $\nabla$  has  $\mathbf{z}$  as positive eigendirection then the difference  $\Theta = \nabla - \nabla_0$  is a Higgs field which is nilpotent with respect to  $\mathbf{z}$ . This means that  $\Theta$  lies in  $\text{Higgs}(E_1, \mathbf{z})$ , which is the moduli space of parabolic Higgs bundles over  $(E_1, \mathbf{z})$ . We may fix a basis  $\{\Theta_1, \dots, \Theta_n\}$  for  $\text{Higgs}(E_1, \mathbf{z})$  such that  $\Theta_k$  is regular over  $t_i$  for  $i \neq k$  and

$$(3.3) \quad \text{Res}_{t_k} \Theta_k = \begin{pmatrix} z_k & -z_k^2 \\ 1 & -z_k \end{pmatrix}.$$

In Section 4.1.1, we make  $\Theta_1, \dots, \Theta_n$  explicit. Since  $\nabla$  can be written as

$$\nabla = \nabla_0 + \sum_{i=1}^n c_i \Theta_i \quad ; \quad c_i \in \mathbb{C},$$

from (3.2) and (3.3) one obtains

$$(3.4) \quad \mathbf{p}^-(\nabla) = \{(c_1 z_1 - \nu_1 : c_1), \dots, (c_n z_n - \nu_n : c_n)\}.$$

The space of connections over  $U_0$  is parametrized by  $(\mathbf{z}, c)$ ,  $c = (c_1, \dots, c_n)$ , and (3.4) shows that  $\text{Par} : \text{Con}^\nu|_{U_0} \rightarrow S^n$  sends  $\nabla_{\mathbf{z}, c}$  to

$$\begin{cases} \mathbf{p}^+(\nabla_{\mathbf{z}, c}) &= ((z_1 : 1), \dots, (z_n : 1)) \\ \mathbf{p}^-(\nabla_{\mathbf{z}, c}) &= ((c_1 z_1 - \nu_1 : c_1), \dots, (c_n z_n - \nu_n : c_n)) \end{cases}$$

In particular  $\nabla_{\mathbf{z}, c}$  is determined by its eigendirections. Also, this proves that given  $\zeta = (\zeta_1, \dots, \zeta_n) \in (\mathbb{P}^1)^n$  such that  $\zeta_i \neq (z_i : 1)$  for  $i \in \{1, \dots, n\}$ , we can find  $\nabla_{\mathbf{z}, c}$  with  $\text{Par}(\nabla_{\mathbf{z}, c}) = (\mathbf{z}, \zeta)$ . We conclude that  $\text{Par}|_{U_0} : \text{Con}^\nu|_{U_0} \rightarrow S^n|_{U_0}$  is an isomorphism between affine bundles. If  $\text{Bun}^+(\nabla) \notin U_0$ , that is, if there is a positive direction  $p_k^+(\nabla)$  lying in  $L_\lambda$  one can fix another system of coordinates by changing  $L_\lambda$  and the same argument above will work.  $\square$

**Corollary 3.2.** *The open subset  $\text{Con}^\nu$  of  $\text{Con}^\nu(C, D)$  given by connections having  $E_1$  as underlying vector bundle is an affine variety.*

*Proof.* Since the diagonal  $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$  supports an ample divisor, its complement,  $S$ , is affine. Therefore  $\text{Con}^\nu \simeq S^n$  is also affine.  $\square$

Now we want to explain how  $elem_I$  induces an isomorphism between  $Con_{I,\epsilon}^\nu$  and  $Con^\lambda$ , where  $\lambda$  will be determined.

**Proposition 3.3.** *The map  $elem_I$  induces a fiber-preserving isomorphism  $\Phi_I$ :*

$$\begin{array}{ccc} Con_{I,\epsilon}^\nu & \xrightarrow[\simeq]{\Phi_I} & Con^\lambda \\ \downarrow & & \downarrow \\ X_I & \xrightarrow{elem_I} & X \end{array}$$

where  $\lambda = (\lambda_1^+, \lambda_1^-, \dots, \lambda_n^+, \lambda_n^-)$  with

$$\begin{cases} \lambda_k^{\epsilon_k} = \nu_k^{\epsilon_k} - 1/2 \\ \lambda_k^{\delta_k} = \nu_k^{\delta_k} + 1/2, \quad \{\delta_k, \epsilon_k\} = \{+, -\} \end{cases}$$

for each  $k \in I$  and  $\lambda_k^\pm$  are left unchanged if  $k \notin I$ .

*Proof.* Given  $(E, \nabla) \in Con_{I,\epsilon}^\nu$  we will perform an elementary transformation centered in  $\mathbf{p}^\epsilon(\nabla)$ . Recall that  $elem_I$  sends  $(E, \mathbf{p})$  to  $(E' \otimes L, \mathbf{p}')$  where  $E'$  is obtained by the exact sequence

$$0 \longrightarrow E' \xrightarrow{\alpha} E \longrightarrow \bigoplus_{i \in I} (E_{t_i}/p_i) \longrightarrow 0$$

and  $L$  is a square root  $\mathcal{O}_C(\sum_{i \in I} t_i)$ . In order to describe  $\Phi_I$ , we fix a rank one connection  $\zeta: L \rightarrow L \otimes \Omega_C^1(D)$  which is regular over  $t_k$  if  $k \notin I$  and

$$\text{Res}_{t_k} \zeta = -\frac{1}{2}; \quad \text{if } k \in I.$$

After an elementary transformation over  $t_k$  centered in  $p_k^{\epsilon_k}$ , the local exponent  $\nu_k^{\epsilon_k}$ , which corresponds to  $p_k^{\epsilon_k}$ , does not change while the other local exponent  $\nu_k^{\delta_k}$  increases by 1. Then the new connection  $\nabla'$  on  $E'$  which is defined as  $\alpha^*(\nabla)$  has local exponents

$$(\nu_k^{\delta_k} + 1, \nu_k^{\epsilon_k}); \quad \text{if } k \in I$$

where  $\{\delta_k, \epsilon_k\} = \{+, -\}$  and the other (when  $k \notin I$ ) are left unchanged.

The map  $\Phi_I$  is defined as

$$\Phi_I(E, \nabla) = (E' \otimes L, \nabla' \otimes \zeta).$$

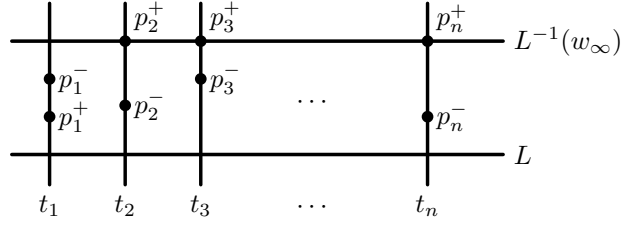
Since it can be reversed by the same process, this concludes the proof of the isomorphism.  $\square$

We will show that the whole moduli space  $Con^\nu(C, D)$  may be covered by those open sets  $Con_{I,\epsilon}^\nu$  and a closed set  $\Sigma_n$  that is nonempty only if  $n$  is odd. We will see that this discrepancy comes from the item (3) in the statement of Lemma 2.3. For now, we define (for  $n$  odd)

$$(3.5) \quad \Sigma_n = \{(E, \nabla) \in Con^\nu(C, D) \mid E = L \oplus L^{-1}(w_\infty) \text{ with } L^2 = \mathcal{O}_C(D + w_\infty)\}.$$

Next we will describe the connections in  $\Sigma_n$ . In order to do so, we compute the logarithmic Atiyah class  $\phi_E^A \in \text{End}(E)^\vee$  whose vanishing establishes the existence of a connection with prescribed residues, see [2]. Let  $T \in \text{End}(E)$  then  $\phi_E^A$  is defined by

$$\phi_E^A(T) = \phi_E^0(T) + \sum_{j=1}^n \text{tr}(A_j T(t_j))$$


 FIGURE 2. Possible configuration of directions for  $(E, \nabla) \in \Sigma_n$ .

where  $A_j$  is the residue endomorphism over  $t_j$  and  $\phi_E^0$  is the classical Atiyah class, see [1]. In our case,

$$A_j = \begin{pmatrix} u_j & a_j \\ v_j & b_j \end{pmatrix} \begin{pmatrix} \nu_j^+ & 0 \\ 0 & \nu_j^- \end{pmatrix} \begin{pmatrix} b_j & -a_j \\ -v_j & u_j \end{pmatrix}$$

where  $u_j b_j - a_j v_j = 1$ . Here we may take local coordinates around each  $t_j$  such that  $L$  and  $L^{-1}(w_\infty)$  correspond to  $(1 : 0)$  and  $(0 : 1)$ , respectively.

Note that any direction  $p_k^{\epsilon_k}(\nabla)$  lies outside  $L$ , otherwise there exists a choice of parabolic directions  $\mathbf{p}^\epsilon(\nabla)$  such that  $(E, \mathbf{p}^\epsilon(\nabla))$  is decomposable and this would force a relation on eigenvalues  $\nu$ . Indeed, we can find an embedding of  $L^{-1}(w_\infty)$  passing through  $n-1$  directions away from  $L$ . Then we may suppose without loss of generality that our directions are as in the Figure 2. In particular  $(u_j, v_j) = (0, 1)$  and  $a_j = -1$  for  $j \geq 2$ , and  $u_1 v_1 \neq 0$ . Up to applying a diagonal automorphism of  $E$  we may suppose that  $u_1 = v_1 = 1$  i.e.  $p_1^+(\nabla) = (1 : 1)$ .

Note that  $\text{End}(E)$  is generated (as a vector space) by the identity, nilpotent endomorphisms and the projection to  $L$ . For the identity,  $\phi_E^A(1_E)$  gives the Fuchs relation that we already know is valid. Let  $\beta \in H^0(C, L^2(-w_\infty))$ , then define

$$P(\beta) := \phi_E^A \left( \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \right) = b_1 \beta(t_1) (\nu_1^+ - \nu_1^-) + \sum_{j \geq 2} b_j \beta(t_j) (\nu_j^+ - \nu_j^-).$$

For  $j \geq 2$  let  $\beta_j \in H^0(C, L^2(-w_\infty))$  with the following property:  $\beta_j(t_k) = 0$  if  $k \neq 1, j$  and  $\beta_j(t_j) = 1$ . These sections are unique. In particular,  $\beta_j(t_1) \neq 0$  and we have

$$P(\beta_j) = b_1 \beta_j(t_1) (\nu_1^+ - \nu_1^-) + b_j (\nu_j^+ - \nu_j^-)$$

Note that the image of  $\beta \mapsto (\beta(t_1), \dots, \beta(t_n))$  is a  $(n-1)$ -dimensional vector space, hence the images of the  $\beta_j$  give a basis. Therefore  $P(\beta) = 0$  for every  $\beta \in H^0(C, L^2(-w_\infty))$  if and only if  $P(\beta_j) = 0$  for  $j \geq 2$  i.e. the  $\nu_j^-$  direction is

$$(-1 : b_j) = (\nu_j^+ - \nu_j^- : b_1 \beta_j(t_1) (\nu_1^+ - \nu_1^-))$$

For the projection to  $L$  we have

$$\phi_E^A \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \deg L + \sum_{j=1}^n u_j b_j \nu_j^+ - a_j v_j \nu_j^- = b_1 (\nu_1^+ - \nu_1^-) + \frac{n+1}{2} + \sum_{j=1}^n \nu_j^- = 0.$$

This implies that the directions over  $t_1$  are  $p_1^+(\nabla) = (1 : 1)$  and

$$p_1^-(\nabla) = (b_1 - 1 : b_1) = \left( \frac{n+1}{2} + \sum_{j \geq 2}^n \nu_j^- + \nu_j^+ : \frac{n+1}{2} + \sum_{j=1}^n \nu_j^- \right)$$

Therefore the residues are completely independent of the isomorphism class of  $(E, \nabla)$  i.e. the residues of every connection in  $\Sigma_n$  are, up to  $\text{Aut}(E)$ , in the above configuration. Also note that any two connections with these residues differ by an element of  $\text{Hom}(E, E \otimes \Omega_C)$  with vanishing trace. From this discussion we can prove the following.

**Proposition 3.4.** *Let  $n$  be an odd integer and let  $\nu$  be local exponents such that  $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \notin \mathbb{Z}$  and  $\nu_j^+ \neq \nu_j^-$ . Then  $\Sigma_n$  has four connected components, each isomorphic to  $\text{Higgs}_0(E)$ , the space of traceless holomorphic Higgs fields for the underlying vector bundle. Moreover,*

$$\dim \Sigma_n = n + 1.$$

*Proof.* First note that there are precisely four possibilities for the underlying vector bundle of a connection in  $\Sigma_n$ . Indeed, any such vector bundle is  $E = L \oplus L^{-1}(w_\infty)$  where  $L$  is such that  $L^2 = \mathcal{O}_C(D + w_\infty)$ . Twisting by 2-torsion line bundles leads to four nonisomorphic possibilities for  $L$ . Hence four nonisomorphic possibilities for  $E$ . Therefore  $\Sigma_n$  has four connected components.

Fix one such  $E$  and denote  $\Sigma_n^E$  the corresponding component of  $\Sigma_n$ . Now fix  $(E, \nabla_0) \in \Sigma_n^E$ . From the discussion above we know that for any other  $(E, \nabla) \in \Sigma_n^E$ , the difference  $\nabla - \nabla_0$  is a holomorphic Higgs field. Since  $\nabla$  and  $\nabla_0$  must have the same trace, this Higgs field is traceless. Thus we have an isomorphism

$$\begin{aligned} \Sigma_n^E &\longrightarrow \text{Higgs}_0(E) \\ \nabla &\longmapsto \nabla - \nabla_0 \end{aligned}$$

To conclude we note that its dimension is  $1 + h^0(L^2(-w_\infty)) = n + 1$ .  $\square$

In addition, notice that by performing an elementary transformation centered at  $n$  parabolic directions outside  $L$ , then  $E$  can be transformed into  $E_0$  where  $E_0$  corresponds to the unique, up to isomorphism, nontrivial extension

$$0 \rightarrow \mathcal{O}_C \rightarrow E_0 \rightarrow \mathcal{O}_C \rightarrow 0.$$

We conclude that any connection of  $\Sigma_n$  can be obtained from a connection on  $E_0$  by performing an elementary transformation centered in  $n$  parabolic directions which lie in the unique maximal sub-bundle  $\mathcal{O}_C$ . Note that there are  $n$  directions to choose and  $\dim \text{Higgs}_0(E) = 1$  giving (naively) dimension  $n + 1$ .

**Proposition 3.5.** *Assume the generic condition  $\nu_1^{a_1} + \dots + \nu_n^{a_n} \notin \mathbb{Z}$  for any  $a_k \in \{+, -\}$  and  $\nu_k^+ \neq \nu_k^-$  for every  $k \in \{1, \dots, n\}$ . The moduli space  $\text{Con}^\nu(C, D)$  can be written as a union*

$$\text{Con}^\nu(C, D) = \begin{cases} \text{Con}^\nu \cup_{I, \epsilon} \text{Con}_{I, \epsilon}^\nu, & \text{if } n \text{ is even} \\ \text{Con}^\nu \cup_{I, \epsilon} \text{Con}_{I, \epsilon}^\nu \cup \Sigma_n, & \text{if } n \text{ is odd} \end{cases}$$

where  $I$  runs over all the nonempty subsets of  $\{1, \dots, n\}$  of even cardinality and  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  with  $\epsilon_k \in \{+, -\}$ .

*Proof.* Let  $(E, \nabla) \in \text{Con}^\nu(C, D)$ . If  $E = E_1$  then  $(E, \nabla) \in \text{Con}^\nu$ , so we can assume  $E = L \oplus L^{-1}(w_\infty)$  with  $\deg L = s \geq 1$ . The generic condition on  $\nu$  yields  $(E, \mathbf{p}^\epsilon(\nabla))$  indecomposable for any choice of  $\epsilon$ , see [4, Corollary 2.3]. Then Lemma 2.3 implies that  $2s \leq n + 1$ .

Let us first assume  $2s \leq n$ . We will show that  $(E, \nabla) \in \text{Con}_{I, \epsilon}^\nu$  for some  $I$  and  $\epsilon$  as above. For this, we choose  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  in such a way that  $p_k^{\epsilon_k}(\nabla) \notin L_{t_k}$  for every  $k$ , it can be done because  $\nu_k^+ \neq \nu_k^-$ . Since  $(E, \mathbf{p}^\epsilon(\nabla))$  is indecomposable there is a parabolic direction which does not lie in  $L$  and there is no embedding of  $L^{-1}(w_\infty)$  containing this direction. Then for sake of simplicity one can suppose  $p_1^{\epsilon_1}(\nabla) \notin L_{t_1}$  and  $p_1^{\epsilon_1}(\nabla) \notin L^{-1}(w_\infty)_{t_1}$  without loss of generality. Now we choose  $I = \{1, \dots, 2s\}$  and fix  $\mu_0 \in \mathfrak{C}_I$  where

$$\mathfrak{C}_I = \left\{ \mu \in (0, 1)^n \mid \sum_{k=2s+1}^n \mu_k + \sum_{k=1}^{2s} (1 - \mu_k) < 1 \right\}.$$

We see that  $(E, \mathbf{p}^\epsilon(\nabla))$  is  $\mu_0$ -stable, because  $E$  can be transformed into  $E_1$  by performing an elementary transformation centered over parabolic directions  $p_k^\epsilon(\nabla)$  over  $I$ . This shows that  $(E, \mathbf{p}^\epsilon(\nabla)) \in X_I$  and thus we conclude that  $(E, \nabla) \in \text{Con}_{I, \epsilon}^\nu$ .

If we now assume  $2s = n + 1$  then Lemma 2.3 implies that  $(E, \nabla) \in \Sigma_n$  and this finishes the proof of the proposition.  $\square$

Let  $\text{Bun}^\epsilon : \text{Con}^\nu \rightarrow X \simeq (\mathbb{P}^1)^n$  be the forgetful map which sends  $(E_1, \nabla)$  to  $(E_1, \mathbf{p}^\epsilon(\nabla))$ .

**Proposition 3.6.** *Let  $I \subset \{1, \dots, n\}$  with even cardinality, and fix  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  with  $\epsilon_k \in \{+, -\}$ . The set  $\text{Con}^\nu \setminus \text{Con}_{I, \epsilon}^\nu$  coincides with the pre-image  $(\text{Bun}^\epsilon)^{-1}(V)$  of a hypersurface  $V \subset X$  whose degree is  $(d_1, \dots, d_n)$ , where  $d_j = 2$  if  $j \in I$  and  $d_j = 0$  otherwise.*

*Proof.* An element of  $\text{Con}^\nu \setminus \text{Con}_{I, \epsilon}^\nu$  is a connection whose underlying quasi-parabolic bundle  $(E_1, \mathbf{p}^\epsilon(\nabla))$  is stable for the weights  $\mu \in \mathfrak{C}$  but its image under  $\text{elem}_I$  is unstable or, equivalently, it is  $\mu$ -stable but  $\varphi_I(\mu)$ -unstable. This already shows that the boundary is fibered over some subset  $V \subset X$ . We need to describe  $V$ .

Note that no information on the directions not indexed by  $I$  goes to the formation of  $V$ . Then it will be a product  $V \simeq V' \times (\mathbb{P}^1)^{n-|I|}$ . Therefore, we may reduce to the case  $V = V'$  i.e.  $|I| = n$  and the conclusion follows from Lemmas 2.4 and 2.5.  $\square$

**Corollary 3.7.** *Assume that  $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \notin \mathbb{Z}$ , for any  $\epsilon_k \in \{+, -\}$ , and that  $\nu_k^+ - \nu_k^- \notin \{0, 1, -1\}$  for  $k \in \{1, \dots, n\}$ . The complement  $\text{Con}_{I, \epsilon}^\nu \setminus \text{Con}^\nu$  is isomorphic to a  $\mathbb{C}^n$ -bundle over  $V \subset (\mathbb{P}^1)^n$ , where  $V$  is a hypersurface of degree  $(d_1, \dots, d_n)$ , with  $d_j = 2$  if  $j \in I$  and  $d_j = 0$  otherwise.*

*Proof.* By performing an elementary transformation centered in  $\mathbf{p}^\epsilon(\nabla)$  we switch both open sets, taking into account the change of eigenvalues

$$\text{Con}_{I, \epsilon}^\nu \rightarrow \text{Con}^\lambda \quad \text{and} \quad \text{Con}^\nu \rightarrow \text{Con}_{I, \epsilon}^\lambda.$$

The map  $\text{Con}_{I, \epsilon}^\nu \rightarrow \text{Con}^\lambda$  sends the complement  $\text{Con}_{I, \epsilon}^\nu \setminus \text{Con}^\nu$  to the complement  $\text{Con}^\lambda \setminus \text{Con}_{I, \epsilon}^\lambda$  and the conclusion follows from Proposition 3.6.  $\square$

We have shown in Theorem 3.1 that  $\text{Con}^\nu \simeq S^n$ . In the next result we obtain a big open subset of  $\text{Con}^\nu(C, D)$ . It coincides with the whole moduli space if  $n$  is even, by attaching finite number of copies of  $S^n$ , and such that each gluing map is a fiberwise isomorphism.

**Theorem 3.8.** *Assume that  $\nu_1^{a_1} + \dots + \nu_n^{a_n} \notin \mathbb{Z}$ , for any  $a_k \in \{+, -\}$ , and that  $\nu_k^+ - \nu_k^- \notin \{0, 1, -1\}$  for  $k \in \{1, \dots, n\}$ . Let  $\Sigma_n$  be as (3.5) if  $n$  is odd, and let it be the empty set if  $n$  is even. Then the moduli space  $\text{Con}^\nu(C, D) \setminus \Sigma_n$  can be obtained from  $\text{Con}^\nu \simeq S^n$  by gluing a finite number of*

copies of  $S^n$  such that each gluing map  $\Psi_{I,\epsilon}: S^n \dashrightarrow \text{Con}^\nu$  is a fiber-preserving isomorphism outside a degree  $(d_1, \dots, d_n)$  divisor  $\Gamma \subset (\mathbb{P}^1)^n$  where  $d_j = 2$  if  $j \in I$  and  $d_j = 0$  otherwise.

$$\begin{array}{ccc} S^n & \xrightarrow{\quad \Psi_{I,\epsilon} \quad} & \text{Con}^\nu \simeq S^n \\ & \searrow \quad \swarrow & \\ & (\mathbb{P}^1)^n & \end{array}$$

*Proof.* Proposition 3.5 yields

$$\text{Con}^\nu(C, D) \setminus \Sigma_n = \text{Con}^\nu \cup_{I,\epsilon} \text{Con}_{I,\epsilon}^\nu.$$

From Proposition 3.3 we obtain a fiber-preserving isomorphism  $\Phi_{I,\epsilon}: \text{Con}_{I,\epsilon}^\nu \rightarrow \text{Con}^\lambda$  for each  $I$  and  $\epsilon$ . The hypothesis  $\nu_k^+ - \nu_k^- \notin \{0, 1, -1\}$  and Theorem 3.1 yield  $\text{Con}^\lambda \simeq S^n$ . The remaining part of the proof follows from Proposition 3.6.  $\square$

#### 4. FUCHSIAN SYSTEMS WITH $n + 3$ POLES

Given  $(E_1, \nabla) \in \text{Con}^\nu$  we can associate a  $\mathfrak{sl}_2$ -connection on the trivial bundle  $\mathcal{O}_C \oplus \mathcal{O}_C$  by performing an elementary transformation over the 2-torsion points  $w_0, w_1$  and  $w_\lambda$ . This process will create new singularities which are apparent.

We say that  $t \in C$  is an *apparent singular point* for  $\nabla$  if the residual part  $\text{Res}_t \nabla$  has  $\{\frac{1}{2}, -\frac{1}{2}\}$  as eigenvalues and the  $\frac{1}{2}$ -eigendirection of  $\text{Res}_t \nabla$  is also invariant by the constant part of the connection matrix.

Suppose that  $D' = D + w_0 + w_1 + w_\lambda$  is also reduced. Let  $(\nu_1, \dots, \nu_n) \in \mathbb{C}^n$  and fix a local exponent

$$(4.1) \quad \theta = \left( \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \nu_1, \dots, \pm \nu_n \right).$$

We denote by  $\text{Syst}^\theta(C, D')$  the moduli space of Fuchsian systems (i.e. logarithmic  $\mathfrak{sl}_2$ -connections on the trivial bundle  $\mathcal{O}_C \oplus \mathcal{O}_C$ ) having  $D'$  as divisor of poles,  $\theta$  as local exponents and such that:

- the three singular points  $w_0, w_1$  and  $w_\lambda$  are apparent singular points;
- over  $w_0, w_1$  and  $w_\lambda$  the corresponding  $\frac{1}{2}$ -eigendirections are  $(1 : 0)$ ,  $(1 : 1)$  and  $(0 : 1)$  respectively.

**Proposition 4.1.** *There is an isomorphism of moduli spaces*

$$(4.2) \quad \text{Con}^\nu \xrightarrow{\sim} \text{Syst}^\theta(C, D')$$

where  $\nu_j = \frac{\nu_j^+ - \nu_j^-}{2}$  for  $j = 1, \dots, n$ .

*Proof.* Let  $(E_1, \nabla) \in \text{Con}^\nu$ . First note that the intersection configuration as in Figure 1 tells us that, after the elementary transformation,  $\mathcal{O}_C(w_0 - w_\infty), \mathcal{O}_C(w_1 - w_\infty), \mathcal{O}_C(w_\lambda - w_\infty) \subset E_1$  become three disjoint copies of  $\mathcal{O}_C(-w_\infty)$ . Hence the elementary transformed of  $(E_1, \nabla)$  is  $(E, \nabla')$  where  $E = \mathcal{O}_C(-w_\infty) \oplus \mathcal{O}_C(-w_\infty)$  and the local exponents of  $\nabla'$  are the same as  $\nabla$  at the  $t_k, k = 1, \dots, n$ , and the exponents at  $w_0, w_1$  and  $w_\lambda$  are equal to  $(1, 0)$ . Consider the connection  $(\mathcal{O}_C(w_\infty), \zeta)$  such that

$$\text{Res}_{w_0} \zeta = \text{Res}_{w_1} \zeta = \text{Res}_{w_\lambda} \zeta = -\frac{1}{2} \quad \text{and} \quad \text{Res}_{t_k} \zeta = -\frac{\nu_k^+ + \nu_k^-}{2}.$$

This is possible since  $\sum_{k=1}^n \nu_k^+ + \nu_k^- = -1$  by Fuchs' relation. It follows that

$$(E_1, \nabla) \mapsto (\mathcal{O}_C \oplus \mathcal{O}_C, \nabla' \otimes \zeta) \in \text{Syst}^\theta(C, D')$$

is our desired isomorphism.  $\square$

**4.1. The affine bundle of Fuchsian systems.** Consider the space  $X$  of parabolic bundles  $(E, \mathbf{p})$  over  $(C, D')$  such that  $E$  is the trivial bundle, the parabolic directions over  $w_0, w_1$  and  $w_\lambda$  are  $(1 : 0), (1 : 1)$  and  $(0 : 1)$ , respectively, and we let the parabolic directions over  $D = t_1 + \dots + t_n$  vary. Then  $X = (\mathbb{P}^1)^n$  where each copy of  $\mathbb{P}^1$  parametrizes parabolic directions over  $t_j$ .

The moduli space  $\text{Syst}^\theta(C, D')$  is an affine bundle of rank  $n$  over  $X$ . We will describe the trivializations of this bundle over the Zariski open sets

$$U_0 = \prod_{j=1}^n \{(z_j : w_j) \in \mathbb{P}^1 \mid w_j = 1\} \quad \text{and} \quad U_\infty = \prod_{j=1}^n \{(z_j : w_j) \in \mathbb{P}^1 \mid z_j = 1\},$$

and then give the affine transition map.

To begin with let us fix a basis of meromorphic one-forms with at most simple poles on  $D'$ . Consider the holomorphic one-form  $\omega = \frac{dx}{2y}$  and define

$$\phi_0 = \frac{(1-\lambda)x}{y}\omega, \quad \phi_1 = \frac{-\lambda(x-1)}{y}\omega \quad \text{and} \quad \theta_j = \frac{x_j(x_jx - \lambda)}{x_jy - y_jx}\omega,$$

where  $t_j = (x_j, y_j)$ . These  $n+3$  one-forms give us the desired basis. It follows that each  $\phi_i$  has a zero on  $w_i$  and a pole on  $w_j$ ,  $j \neq i$ , and each  $\theta_j$  has poles on  $w_0$  and  $t_j$ . The residues have been chosen in such a way that

$$\begin{aligned} \text{Res}_{t_j} \theta_j &= 1, & \text{Res}_{w_0} \theta_j &= -1 \\ \text{Res}_{w_1} \phi_0 &= 1, & \text{Res}_{w_\lambda} \phi_0 &= -1 \\ \text{Res}_{w_0} \phi_1 &= 1, & \text{Res}_{w_\lambda} \phi_1 &= -1 \end{aligned}$$

**4.1.1. Trivialization over  $U_0$ .** Given  $\mathbf{z} = (z_1, \dots, z_n) \in U_0$  we define

$$\nabla_0 : \mathcal{O}_C \oplus \mathcal{O}_C \rightarrow (\mathcal{O}_C \oplus \mathcal{O}_C) \otimes \Omega_C^1(D')$$

as the connection which has  $(z_j : 1)$  as eigendirections corresponding to  $\nu_j$  and has  $(1 : 0)$  as eigendirections corresponding to  $-\nu_j$ . It can be written as

$$\nabla_0 = d + \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix} \phi_0 + \begin{pmatrix} a_1 & b_1 \\ 0 & -a_1 \end{pmatrix} \phi_1 + \begin{pmatrix} a_2 & b_2 \\ 0 & -a_2 \end{pmatrix} \omega + \sum_{j=1}^n \nu_j \begin{pmatrix} -1 & 2z_j \\ 0 & 1 \end{pmatrix} \theta_j.$$

with

$$\begin{aligned} a_0 &= \sum_{j=1}^n \nu_j & b_0 &= \frac{1}{2} - \sum_{j=1}^n \nu_j & c_0 &= \frac{1}{2} + \sum_{j=1}^n \nu_j \\ a_1 &= \frac{1}{2} - \sum_{j=1}^n \nu_j & b_1 &= -\frac{1}{2} + \sum_{j=1}^n \nu_j \\ a_2 &= \sum_{j=1}^n \frac{y_j}{2} \left[ \frac{2(-\nu_j) + 2\nu_j z_j}{x_j - 1} - \frac{2\nu_j z_j}{x_j - \lambda} \right] & b_2 &= \sum_{j=1}^n \frac{y_j}{(x_j - \lambda)} 2\nu_j z_j \end{aligned}$$

Now if  $\nabla$  is any connection which has  $\mathbf{z}$  as positive eigendirections then the difference  $\Theta = \nabla - \nabla_0$  is a Higgs field which is nilpotent with respect to  $\mathbf{z}$ . This means that  $\Theta$  is a strongly parabolic Higgs field over  $(\mathcal{O}_C \oplus \mathcal{O}_C, \mathbf{z})$ . We shall fix a basis  $\{\Theta_1^0, \dots, \Theta_n^0\}$  for the space of strongly parabolic Higgs fields such that

$$\text{Res}_{t_j} \Theta_j^0 = \begin{pmatrix} z_j & -z_j^2 \\ 1 & -z_j \end{pmatrix}$$

and  $\Theta_j^0$  is regular elsewhere. So, we define

$$\Theta_j^0 = \begin{pmatrix} -z_j & z_j \\ -z_j & z_j \end{pmatrix} \phi_0 + \begin{pmatrix} z_j & -z_j \\ 1 & -z_j \end{pmatrix} \phi_1 + A_j \cdot \omega + \begin{pmatrix} z_j & -z_j^2 \\ 1 & -z_j \end{pmatrix} \theta_j$$

where

$$A_j = \begin{pmatrix} \frac{y_j}{2} \left[ \frac{-(z_j-1)^2}{x_j-1} + \frac{1}{x_j} + \frac{z_j^2}{x_j-\lambda} \right] & -\frac{y_j z_j^2}{x_j-\lambda} \\ \frac{y_j}{x_j} & -\frac{y_j}{2} \left[ \frac{-(z_j-1)^2}{x_j-1} + \frac{1}{x_j} + \frac{z_j^2}{x_j-\lambda} \right] \end{pmatrix}.$$

This matrix  $A_j$  has been chosen to assure apparent singularities over  $w_0, w_1$  and  $w_\lambda$ .

Any  $\mathfrak{sl}_2$ -connection  $(E, \nabla) \in \text{Syst}^\theta(C, D')$  having  $(z_j : 1)$  as parabolic direction (over  $t_j$ ) corresponding to  $\nu_j$  can be write as

$$\nabla = \nabla_0 + r_j \Theta_j^0$$

for suitable  $r_j \in \mathbb{C}$ . It has  $(z_j r_j - 2\nu_j : r_j)$  as the complementary direction corresponding to  $-\nu_j$ .

4.1.2. *Trivialization over  $U_\infty$ .* Similarly, we define  $\nabla_\infty$  having  $(1 : w_j)$  as eigendirections corresponding to  $\nu_j$  and  $(0 : 1)$  as eigendirections corresponding to  $-\nu_j$ :

$$\nabla_\infty = d + \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix} \phi_0 + \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} \phi_1 + \begin{pmatrix} a_2 & 0 \\ c_2 & -a_2 \end{pmatrix} \omega + \sum_{j=1}^n \nu_j \begin{pmatrix} 1 & 0 \\ 2w_j & -1 \end{pmatrix} \theta_j$$

with

$$\begin{aligned} a_0 &= -\sum_{j=1}^n \nu_j & b_0 &= \frac{1}{2} + \sum_{j=1}^n \nu_j & c_0 &= \frac{1}{2} - \sum_{j=1}^n \nu_j \\ a_1 &= \frac{1}{2} + \sum_{j=1}^n \nu_j & b_1 &= -\frac{1}{2} - \sum_{j=1}^n \nu_j & c_1 &= \sum_{j=1}^n \nu_j 2w_j \\ a_2 &= \sum_{j=1}^n y_j \nu_j \left[ \frac{(1-w_j)}{x_j-1} + \frac{w_j}{x_j} \right] & c_2 &= \sum_{j=1}^n \frac{y_j}{x_j} \nu_j 2w_j \end{aligned}$$

And the strongly parabolic Higgs fields are defined by

$$\Theta_j^\infty = \begin{pmatrix} -w_j & w_j \\ -w_j & w_j \end{pmatrix} \phi_0 + \begin{pmatrix} w_j & -w_j \\ w_j^2 & -w_j \end{pmatrix} \phi_1 + B_j \cdot \omega + \begin{pmatrix} w_j & -1 \\ w_j^2 & -w_j \end{pmatrix} \theta_j$$

where

$$B_j = \begin{pmatrix} \frac{y_j}{2} \left[ \frac{-(1-w_j)^2}{x_j-1} + \frac{w_j^2}{x_j} + \frac{1}{x_j-\lambda} \right] & \frac{-y_j}{(x_j-\lambda)} \\ \frac{y_j w_j^2}{x_j} & -\frac{y_j}{2} \left[ \frac{-(1-w_j)^2}{x_j-1} + \frac{w_j^2}{x_j} + \frac{1}{x_j-\lambda} \right] \end{pmatrix}.$$

Given a  $\mathfrak{sl}_2$ -connection  $(E, \nabla) \in \text{Syst}^\theta(C, D')$  with  $(1 : w_j)$  as parabolic direction (over  $t_j$ ) corresponding to  $\nu_j$ , we can write

$$\nabla = \nabla_\infty + s_j \Theta_j^\infty$$

for suitable  $s_j \in \mathbb{C}$ . The complementary direction corresponding to  $-\nu_j$  is  $(s_j : w_j s_j + 2\nu_j)$ .

4.1.3. *Transition matrix.* From the trivializations on  $U_0$  and  $U_\infty$  we may compute a transition affine transformation for the affine bundle  $\text{Syst}^\theta(C, D')$ . Since  $U_0 \cup U_\infty$  covers the basis  $X \simeq (\mathbb{P}^1)^n$  minus a subvariety of codimension two, then the bundle structure of  $\text{Syst}^\theta(C, D')$  is determined by this affine transformation.

In order to make this affine transformation explicit we note that

$$\Theta_j^\infty = w_j^2 \Theta_j^0 \left( \frac{1}{w_j} \right),$$

and

$$\nabla_0 \left( \frac{1}{w} \right) = d + \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix} \phi_0 + \begin{pmatrix} a_1 & b_1 \\ 0 & -a_1 \end{pmatrix} \phi_1 + \begin{pmatrix} a_2 & b_2 \\ 0 & -a_2 \end{pmatrix} \omega + \sum_{j=1}^n \nu_j \begin{pmatrix} -1 & \frac{2}{w_j} \\ 0 & 1 \end{pmatrix} \theta_j.$$

with

$$\begin{aligned} a_0 &= \sum_{j=1}^n \nu_j & b_0 &= \frac{1}{2} - \sum_{j=1}^n \nu_j & c_0 &= \frac{1}{2} + \sum_{j=1}^n \nu_j \\ a_1 &= \frac{1}{2} - \sum_{j=1}^n \nu_j & b_1 &= -\frac{1}{2} + \sum_{j=1}^n \nu_j \\ a_2 &= \sum_{j=1}^n \frac{y_j \nu_j}{w_j} \left[ \frac{1-w_j}{x_j-1} - \frac{1}{x_j-\lambda} \right] & b_2 &= \sum_{j=1}^n \frac{y_j}{(x_j-\lambda)} 2 \frac{\nu_j}{w_j} \end{aligned}$$

Hence

$$\nabla_\infty = \nabla_0 \left( \frac{1}{w_1}, \dots, \frac{1}{w_n} \right) + \sum_{j=1}^n 2\nu_j w_j \Theta_j^0 \left( \frac{1}{w_j} \right)$$

Over the intersection  $U_0 \cap U_\infty$  we have

$$\begin{aligned} (s_1, \dots, s_n) &\mapsto \nabla_\infty(w_1, \dots, w_n) + \sum_{j=1}^n s_j \Theta_j^\infty(w_j) = \\ &= \nabla_0 \left( \frac{1}{w_1}, \dots, \frac{1}{w_n} \right) + \sum_{j=1}^n (2\nu_j w_j + s_j w_j^2) \Theta_j^0 \left( \frac{1}{w_j} \right) = \\ &= \nabla_0(z_1, \dots, z_n) + \sum_{j=1}^n (2\nu_j w_j + s_j w_j^2) \Theta_j^0(z_j) = \\ &= \nabla_0(z_1, \dots, z_n) + \sum_{j=1}^n r_j \Theta_j^0(z_j) \end{aligned}$$

which gives us the transition affine transformation

$$\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} w_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & w_n^2 \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} + \begin{pmatrix} 2\nu_1 w_1 \\ \vdots \\ 2\nu_n w_n \end{pmatrix}$$

that can also be written as

$$(4.3) \quad \begin{pmatrix} 1 \\ r_1 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 2\nu_1 w_1 & w_1^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2\nu_n w_n & 0 & \dots & w_n^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ s_1 \\ \vdots \\ s_n \end{pmatrix}$$

This last matrix can be seen as the transition matrix of the compactification of the moduli space  $Con^\nu$  obtained by allowing  $\lambda$ -connections:  $\nabla_\lambda = \lambda \cdot \nabla_0 + \Theta$ . When  $\lambda \neq 0$  then  $\nabla_\lambda$  is equivalent to  $\nabla_0 + \lambda^{-1}\Theta$  up to homothety, when  $\lambda = 0$  then  $\nabla_\lambda$  coincides with the Higgs field  $\Theta$ . On the boundary we get the projectivized moduli space of Higgs fields

$$\mathbb{P}Higgs^\nu := \overline{Con}^\nu \setminus Con^\nu.$$

Since  $U_0 \cup U_\infty$  covers  $X \simeq (\mathbb{P}^1)^n$  minus a subvariety of codimension two then the transition matrix given in (4.3) determines the affine bundle  $Con^\nu$  and this yields the following result.

**Theorem 4.2.** *We have  $\overline{Con}^\nu = \mathbb{P}(\mathcal{E}^\nu)$ , where  $\mathcal{E}^\nu$  is the extension of  $\mathcal{O}_X$  by  $T^*X$*

$$0 \longrightarrow T^*X \longrightarrow \mathcal{E}^\nu \longrightarrow \mathcal{O}_X \longrightarrow 0$$

determined by

$$(\nu_1, \dots, \nu_n) \in H^1(X, T^*X) \simeq \bigoplus_{j=1}^n H^1(\mathbb{P}_{z_j}^1, T^*\mathbb{P}_{z_j}^1) \simeq \mathbb{C}^n.$$

## 5. APPARENT MAP

Given a connection  $\nabla: E_1 \rightarrow E_1 \otimes \Omega_C^1(D)$  we can define an  $\mathcal{O}_C$ -linear map:

$$\varphi_\nabla: \mathcal{O}_C \xrightarrow{s} E_1 \xrightarrow{\nabla} E_1 \otimes \Omega_C^1(D) \longrightarrow (E_1/\mathcal{O}_C) \otimes \Omega_C^1(D)$$

where the last arrow is defined by the quotient map from  $E_1$  to  $E_1/s(\mathcal{O}_C) \simeq \mathcal{O}_C(w_\infty)$  (denoted  $E_1/\mathcal{O}_C$ ) for short. The zero divisor of  $\varphi_\nabla$  defines an element of the linear system

$$Z(\varphi_\nabla) \in |\mathcal{O}_C(w_\infty + D)|.$$

Since  $\deg D = n$  and  $\Omega_C^1 \simeq \mathcal{O}_C$  ( $C$  is elliptic) then

$$|\mathcal{O}_C(w_\infty + D)| = \mathbb{P}(H^0(C, (E_1/\mathcal{O}_C) \otimes \Omega_C^1(D))) \simeq \mathbb{P}^n.$$

Hence we may define a rational map

$$\text{App}: Con^\nu \dashrightarrow |\mathcal{O}_C(w_\infty + D)|$$

which associates  $(E_1, \nabla)$  to  $Z(\varphi_\nabla)$ . Under a generic hypothesis on the spectral data we show that App is in fact a morphism. It turns out that this hypothesis is also necessary as we prove in the following lemma.

**Lemma 5.1.** *The rational map  $\text{App}: Con^\nu \dashrightarrow |\mathcal{O}_C(w_\infty + D)|$  is in fact a morphism if and only if  $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \neq 0$  for any  $\epsilon_k \in \{+, -\}$ .*

*Proof.* The indeterminacy locus of App is composed by the connections  $\nabla$  mapped to zero in  $(E_1/\mathcal{O}_C) \otimes \Omega_C^1(D)$  which means that there exists a meromorphic 1-form  $\xi$  with poles at most on  $D$  such that

$$\nabla(s) = s\xi.$$

In such case we have for each  $j = 1, \dots, n$  that  $s(t_j) \in p_j^{\epsilon_j}(\nabla)$  with  $\epsilon_j \in \{+, -\}$ . And we also have that  $\nabla$  restricts to a logarithmic connection on  $\mathcal{O}_C$  whence

$$\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} = 0.$$

Conversely, suppose that there exist  $\delta_j \in \{+, -\}$ ,  $j = 1, \dots, n$ , such that  $\nu_1^{\delta_1} + \dots + \nu_n^{\delta_n} = 0$ . On the one hand there exists a meromorphic 1-form  $\xi$  with simple poles only on  $D$  and prescribed residues

$$\text{Res}_{t_j} \xi = \nu_j^{\delta_j}.$$

On the other hand let  $\nabla$  be a logarithmic connection such that  $s(t_j) \in p_j^{\delta_j}(\nabla)$ . It follows that  $\nabla(s) - s\xi$  is a holomorphic section of  $E_1 \otimes \Omega_C^1$ . Since  $H^0(C, E_1 \otimes \Omega_C^1)$  is generated by  $s\omega$  (recall that  $\omega$  is a holomorphic one-form on  $C$ ) it follows that

$$\nabla(s) = s(\xi + c\omega)$$

for some  $c \in \mathbb{C}$ . Hence  $\varphi_\nabla = 0$ .  $\square$

We may also consider the map

$$\text{Bun}: \text{Con}^\nu \longrightarrow X \simeq (\mathbb{P}^1)^n$$

which sends  $(E_1, \nabla)$  to the parabolic vector bundle  $(E_1, \mathbf{p}^+(\nabla))$ . We also denote  $\text{Bun}$  the natural extension of this map to  $\overline{\text{Con}}^\nu$ . Combining these two maps we can give a birational model for  $\overline{\text{Con}}^\nu$  as we prove in the following result.

**Theorem 5.2.** *If  $\nu_1^{\epsilon_1} + \dots + \nu_n^{\epsilon_n} \neq 0$  for any  $\epsilon_k \in \{+, -\}$  then the map  $\text{Bun} \times \text{App}$  induces a birational map*

$$\text{Bun} \times \text{App}: \overline{\text{Con}}^\nu \dashrightarrow X \times |\mathcal{O}_C(w_\infty + D)|$$

whose indeterminacy locus is contained in  $\overline{\text{Con}}^\nu \setminus \text{Con}^\nu$ . Moreover, given  $(E, \mathbf{p} = \{p_1, \dots, p_n\}) \in X$  the rank of

$$(\text{Bun} \times \text{App})|_{\text{Bun}^{-1}(E, \mathbf{p})}: \text{Bun}^{-1}(E, \mathbf{p}) \longrightarrow |\mathcal{O}_C(w_\infty + D)|$$

coincides with the cardinality of the set  $\{j \mid p_j \notin \mathcal{O}_C\}$ .

*Proof.* Let  $s$  denote the section  $\mathcal{O}_C \hookrightarrow E_1$ . For any logarithmic connection  $\nabla$  for  $E_1$  with poles on  $D = t_1 + \dots + t_n$  we have that  $\text{App}(\nabla)$  is given by the zero divisor of

$$\varphi_\nabla = \nabla(s) \wedge s \in H^0(C, \det(E_1) \otimes \Omega_C(D))$$

which is the same as defining

$$\text{App}(\nabla) = [\varphi_\nabla] \in \mathbb{P}(H^0(C, \det(E_1) \otimes \Omega_C(D))).$$

By Lemma 5.1 the condition on the local exponents implies that  $\text{App}$ , hence  $\text{Bun} \times \text{App}$ , is regular on  $\text{Con}^\nu$ . Therefore the indeterminacy locus must lie only in the boundary divisor.

Now fix  $(E_1, \mathbf{p} = \{p_1, \dots, p_n\}) \in X$ . We will compute the rank of  $\text{App}|_{\text{Bun}^{-1}(E_1, \mathbf{p})}$ . Consider  $H^0(\mathcal{E}nd_0(E_1) \otimes \Omega_C(t_j))$  the space of traceless Higgs fields with simple pole on  $t_j$ . Since its dimension is three, there exists a unique (up to scalar multiplication) strongly parabolic Higgs field  $\Theta_j^p$  with respect to  $\mathbf{p}$  i.e.  $\text{Res}_{t_j}(\Theta_j^p)$  is nilpotent with image  $p_j$  and has no other poles. Let  $\nabla_0$  be a connection in  $\text{Bun}^{-1}(E_1, \mathbf{p})$ . Then any other  $\lambda$ -connection  $\nabla \in \text{Bun}^{-1}(E_1, \mathbf{p})$  can be written in a unique way as

$$\nabla = c_0 \nabla_0 + \sum_{j=1}^n c_j \Theta_j^p,$$

for some  $(c_0 : \dots : c_n) \in \mathbb{P}^n$ . Hence

$$\varphi_\nabla = c_0 \nabla_0(s) \wedge s + \sum_{j=1}^n c_j \Theta_j^p(s) \wedge s.$$

Suppose that  $\Theta_j^p(s) \wedge s \neq 0$  for each  $j$ . Since they have one pole each and at different points these sections are linearly independent. Since  $H^0(C, \mathcal{O}_C(w_\infty + D))$  has dimension  $n + 1$  they form a basis together with  $\nabla_0(s) \wedge s$  and the Apparent map restricted to this fiber is an isomorphism. Indeed, if

$$c_0 \nabla_0(s) \wedge s + \sum_{j=1}^n c_j \Theta_j^p(s) \wedge s = \left( c_0 \nabla_0(s) + \sum_{j=1}^n c_j \Theta_j^p(s) \right) \wedge s = 0$$

then, by Lemma 5.1,  $c_0 = 0$  and it follows that  $c_1 = \dots = c_n = 0$  since the  $\Theta_j^p(s) \wedge s$  are linearly independent.

Now suppose that there exists  $j \in \{1, \dots, n\}$  such that  $\Theta_j^p(s) \wedge s = 0$ . This occurs if and only if  $\Theta_j^p(s)$  is holomorphic at  $t_j$  which in turn means that  $s(t_j) \in p_j$ . Indeed, since  $H^0(C, E_1 \otimes \Omega_C)$  is generated by  $s\omega$ , we know that  $\Theta_j^p(s)$  is holomorphic at  $t_j$  if and only if  $\Theta_j^p(s) = c s\omega$  for some constant  $c \in \mathbb{C}$ . On the other hand, it is clear that  $\text{Res}_{t_j}(\Theta_j^p(s)) \in p_j$  is zero if and only if  $s(t_j) \in p_j$ .

Henceforth we conclude that, in general, the image of  $\text{App}|_{\text{Bun}^{-1}(p)}$  is the linear space spanned by  $\nabla_0(s) \wedge s$  and  $\Theta_j^p(s) \wedge s$  such that  $s(t_j) \notin p_j$ . In particular, the (projective) rank is given by

$$\text{rk App}|_{\text{Bun}^{-1}(E_1, \mathbf{p})} = \#\{j \mid s(t_j) \notin p_j\}.$$

□

**5.1. Explicit computation.** The Apparent map may be easily computed in an appropriate setting that we will present here. We note that this explicit description that allowed us to derive the results in this section. We hope it may provide more insight to the reader.

Under the transformation  $E_1 \rightarrow \mathcal{O}_C \oplus \mathcal{O}_C$  – see Proposition 4.1 – the unique (up to scalar multiplication) section  $\mathcal{O}_C \hookrightarrow E_1$  becomes  $L^{-1} \hookrightarrow \mathcal{O}_C \oplus \mathcal{O}_C$  where  $L = \mathcal{O}_C(w_0 + w_1 + w_\lambda - w_\infty)$ . This map is defined by

$$h \mapsto \begin{pmatrix} fh \\ gh \end{pmatrix}$$

where  $f, g$  are rational functions with poles on  $w_0 + w_1 + w_\lambda$  and zero on  $w_\infty$ . The space of such rational functions has dimension four and imposing that the principal parts of  $(f, g)$  are in  $(1 : 0)$ ,  $(1 : 1)$  and  $(0 : 1)$  at  $w_0, w_1$  and  $w_\lambda$ , respectively, leaves us with a unique choice, up to scalar multiplication. We then fix

$$(f, g) = \left( \frac{x - \lambda}{y}, \frac{(1 - \lambda)x}{y} \right).$$

**Remark 5.3.** In this system of coordinates,  $z_j = \frac{x_j - \lambda}{x_j(1 - \lambda)}$  is the direction determined by  $\mathcal{O}_C \subset E_1$  over  $t_j = (x_j, y_j)$ .

Given a Fuchsian system  $\nabla$ , the Apparent map is induced by the composition

$$L^{-1} \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} \mathcal{O}_C \oplus \mathcal{O}_C \xrightarrow{\nabla} (\mathcal{O}_C \oplus \mathcal{O}_C) \otimes \Omega_C^1(D') \xrightarrow{\begin{pmatrix} g & -f \end{pmatrix}} L \otimes \Omega_C^1(D').$$

If we write  $\nabla = d + \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$  then  $\varphi_\nabla = [gdf - fdg + 2fg\alpha + g^2\beta - f^2\gamma]$  which is a global section of

$$L^{\otimes 2} \otimes \Omega_C^1(D') = \mathcal{O}_C(3w_0 + 3w_1 + 3w_\lambda - 2w_\infty + D).$$

One can check, by direct inspection on the principal parts, that the choice of  $f$  and  $g$  implies that  $\varphi_\nabla$  has simple poles. Hence we have

$$\text{App}: \text{Syst}^\nu(C, D) \rightarrow |w_0 + w_1 + w_\lambda - 2w_\infty + D|.$$

Consider the a basis for  $H^0(C, \Omega_C^1(w_0 + w_1 + w_\lambda - 2w_\infty + D))$  given by the meromorphic one-forms

$$\eta_0 = (fg^2 - gf^2)\omega \quad \text{and} \quad \eta_j = fg \left( \frac{y_j}{x_j - 1} \omega + \theta_j \right), \quad j = 1, \dots, n.$$

Note that  $gdf - fdg = -2\eta_0$ . We will use our universal family to describe this map.

5.1.1. *Computation over  $U_0$ .* Note that for a Fuchsian system  $\nabla = \nabla_0 + \sum_{j=1}^n r_j \Theta_j^0$ ,

$$\varphi_\nabla = \varphi_{\nabla_0} + \sum_{j=1}^n r_j \left( \Theta_j^0 \cdot \begin{pmatrix} f \\ g \end{pmatrix} \right) \wedge \begin{pmatrix} f \\ g \end{pmatrix}.$$

By direct computation with the given basis, we have that

$$\begin{aligned} \varphi_{\nabla_0} &= -2\eta_0 + \frac{3}{2}\eta_0 - \sum_{j=1}^n \nu_j [\eta_0 + 2\eta_j] + \sum_{j=1}^n \frac{2\nu_j z_j x_j (1 - \lambda)}{x_j - \lambda} [\eta_0 + \eta_j] \\ &= \left( \frac{-1}{2} + \sum_{j=1}^n \nu_j \frac{2z_j(1 - \lambda)x_j - x_j + \lambda}{x_j - \lambda} \right) \eta_0 - \sum_{j=1}^n 2\nu_j \frac{x_j z_j (\lambda - 1) + x_j - \lambda}{x_j - \lambda} \eta_j \end{aligned}$$

Again by direct computation with the chosen basis we have

$$\begin{aligned} \left( \Theta_j^0 \cdot \begin{pmatrix} f \\ g \end{pmatrix} \right) \wedge \begin{pmatrix} f \\ g \end{pmatrix} &= z_j [\eta_0 + 2\eta_j] + \frac{-z_j^2 x_j (1 - \lambda)}{x_j - \lambda} [\eta_0 + \eta_j] + \frac{x_j - \lambda}{x_j (\lambda - 1)} \eta_j = \\ &= \left( \frac{z_j (x_j z_j (\lambda - 1) + x_j - \lambda)}{x_j - \lambda} \right) \eta_0 + \frac{(x_j z_j (\lambda - 1) + x_j - \lambda)^2}{(x_j - \lambda) x_j (\lambda - 1)} \eta_j. \end{aligned}$$

From the computations above we have  $\text{App}(\nabla) = a_0 \eta_0 + \sum_{j=1}^n a_j \eta_j$  where

$$\begin{aligned} a_0 &= \left( \frac{-1}{2} + \sum_{j=1}^n \nu_j \frac{2z_j(1 - \lambda)x_j - x_j + \lambda}{x_j - \lambda} \right) + \sum_{j=1}^n r_j \left( \frac{z_j (x_j z_j (\lambda - 1) + x_j - \lambda)}{x_j - \lambda} \right), \\ a_j &= 2\nu_j \frac{x_j z_j (\lambda - 1) + x_j - \lambda}{x_j - \lambda} + r_j \frac{(x_j z_j (\lambda - 1) + x_j - \lambda)^2}{(x_j - \lambda) x_j (\lambda - 1)}, \quad j = 1, \dots, n. \end{aligned}$$

It may be better written in matrix form:

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} + \sum_{j=1}^n \nu_j \frac{2z_j(1 - \lambda)x_j - x_j + \lambda}{x_j - \lambda} & \frac{z_1(x_1 z_1(\lambda - 1) + x_1 - \lambda)}{x_1 - \lambda} & \dots & \frac{z_n(x_n z_n(\lambda - 1) + x_n - \lambda)}{x_n - \lambda} \\ -2\nu_1 \frac{x_1 z_1(\lambda - 1) + x_1 - \lambda}{x_1 - \lambda} & \frac{(x_1 z_1(\lambda - 1) + x_1 - \lambda)^2}{(x_1 - \lambda) x_1 (\lambda - 1)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -2\nu_n \frac{x_n z_n(\lambda - 1) + x_n - \lambda}{x_n - \lambda} & 0 & \dots & \frac{(x_n z_n(\lambda - 1) + x_n - \lambda)^2}{(x_n - \lambda) x_n (\lambda - 1)} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ r_1 \\ \vdots \\ r_n \end{pmatrix}.$$

Recall that the determinant of such a bordered matrix is given by

$$\det \begin{pmatrix} a & u_1 & \cdots & u_n \\ v_1 & X_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_n & 0 & \cdots & X_n \end{pmatrix} = \left( \prod_{j=1}^n X_j \right) \left( a - \sum_{j=1}^n \frac{u_j v_j}{X_j} \right)$$

Then the determinant of the matrix of the apparent map is

$$\det(\text{App})(z_1, \dots, z_n) = - \left( \frac{1}{2} + \sum_{j=1}^n \nu_j \right) \prod_{j=1}^n \frac{(x_j z_j (\lambda - 1) + x_j - \lambda)^2}{(x_j - \lambda) x_j (\lambda - 1)}.$$

Therefore the degeneracy locus is supported on the union of the  $n$  hyperplanes  $\left\{ z_j = \frac{x_j - \lambda}{x_j (\lambda - 1)} \right\}$  if  $\frac{1}{2} + \sum_{j=1}^n \nu_j \neq 0$ . The condition  $z_j = \frac{x_j - \lambda}{x_j (\lambda - 1)}$  is equivalent to say that  $p_j^+(\nabla) \subset \mathcal{O}_C$ , see Remark 5.3. Note that the rank of App drops by  $k$  over a point  $z$  in the intersection of  $k$  hyperplanes. If  $z_j = \frac{x_j - \lambda}{x_j (\lambda - 1)}$  for all  $j$  then for all  $\nabla$ ,

$$\text{App}(\nabla) = \left( -\frac{1}{2} + \sum_{j=1}^n \nu_j \right) \eta_0.$$

Then either  $-\frac{1}{2} + \sum_{j=1}^n \nu_j = 0$  and the map is not defined or  $-\frac{1}{2} + \sum_{j=1}^n \nu_j \neq 0$  and the image is a single point.

If  $\frac{1}{2} + \sum_{j=1}^n \nu_j = 0$  then the matrix of the apparent map becomes

$$\begin{pmatrix} \sum_{j=1}^n 2\nu_j \frac{z_j (1-\lambda) x_j}{x_j - \lambda} & \frac{z_1 (x_1 z_1 (\lambda - 1) + x_1 - \lambda)}{x_1 - \lambda} & \cdots & \frac{z_n (x_n z_n (\lambda - 1) + x_n - \lambda)}{x_n - \lambda} \\ -2\nu_1 \frac{x_1 z_1 (\lambda - 1) + x_1 - \lambda}{x_1 - \lambda} & \frac{(x_1 z_1 (\lambda - 1) + x_1 - \lambda)^2}{(x_1 - \lambda) x_1 (\lambda - 1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -2\nu_n \frac{x_n z_n (\lambda - 1) + x_n - \lambda}{x_n - \lambda} & 0 & \cdots & \frac{(x_n z_n (\lambda - 1) + x_n - \lambda)^2}{(x_n - \lambda) x_n (\lambda - 1)} \end{pmatrix}$$

and it follows that

$$\text{App}(\nabla_0) = \sum_{j=1}^n \frac{-2\nu_j (\lambda - 1) x_j}{x_j z_j (\lambda - 1) + x_j - \lambda} \text{App}(\Theta_j^0)$$

whence the image of the Apparent map is spanned by the functions  $\text{App}(\Theta_j^0)$ .

**Remark 5.4.** In Theorem 5.2 we made the hypothesis that  $\nu_1^{\epsilon_1} + \cdots + \nu_n^{\epsilon_n} \neq 0$  for any  $\epsilon_k \in \{+, -\}$ .

Now recall that  $\nu_j = \frac{\nu_j^+ - \nu_j^-}{2}$  hence, by Fuchs relation, we have  $\frac{1}{2} + \sum_{j=1}^n \nu_j = -\sum_{j=1}^n \nu_j^-$  and  $-\frac{1}{2} + \sum_{j=1}^n \nu_j = \sum_{j=1}^n \nu_j^+$ . Thus hypothesis avoids the vanishing of these numbers.

5.1.2. *Explicit calculation over  $U_\infty$ .* In the same fashion one can compute

$$\varphi_{\nabla_\infty} = \left( \frac{-1}{2} + \sum_{j=1}^n \nu_j \right) \eta_0 + \sum_{j=1}^n 2\nu_j \left( \frac{x_j (1 - \lambda) - w_j (x_j - \lambda)}{x_j (1 - \lambda)} \right) \eta_j$$

and

$$\left( \Theta_j^\infty \cdot \begin{pmatrix} f \\ g \end{pmatrix} \right) \wedge \begin{pmatrix} f \\ g \end{pmatrix} = -\frac{x_j (1 - \lambda) - w_j (x_j - \lambda)}{x_j - \lambda} \eta_0 - \frac{(x_j (1 - \lambda) - w_j (x_j - \lambda))^2}{(1 - \lambda) x_j (x_j - \lambda)} \eta_j.$$

Therefore, for  $\nabla = \nabla_\infty + \sum_{j=1}^n s_j \Theta_j^\infty$  we have

$$\text{App}(\nabla) = b_0 \eta_0 + \sum_{j=1}^n b_j \eta_j$$

with

$$b_0 = \left( \frac{-1}{2} + \sum_{j=1}^n \nu_j \right) - \sum_{j=1}^n s_j \frac{x_j(1-\lambda) - w_j(x_j - \lambda)}{x_j - \lambda}$$

$$b_j = 2\nu_j \left( \frac{x_j(1-\lambda) - w_j(x_j - \lambda)}{x_j(1-\lambda)} \right) - s_j \frac{(x_j(1-\lambda) - w_j(x_j - \lambda))^2}{(1-\lambda)x_j(x_j - \lambda)}, \quad j = 1, \dots, n.$$

In matrix form we have

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} + \sum_{j=1}^n \nu_j & -\frac{x_1(1-\lambda) - w_1(x_1 - \lambda)}{x_1 - \lambda} & \cdots & -\frac{x_n(1-\lambda) - w_n(x_n - \lambda)}{x_n - \lambda} \\ 2\nu_1 \frac{x_1(1-\lambda) - w_1(x_1 - \lambda)}{x_1(1-\lambda)} & -\frac{(x_1(1-\lambda) - w_1(x_1 - \lambda))^2}{(1-\lambda)x_1(x_1 - \lambda)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2\nu_n \frac{x_n(1-\lambda) - w_n(x_n - \lambda)}{x_n(1-\lambda)} & 0 & \cdots & -\frac{(x_n(1-\lambda) - w_n(x_n - \lambda))^2}{(1-\lambda)x_n(x_n - \lambda)} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ s_1 \\ \vdots \\ s_n \end{pmatrix}$$

Using the change of coordinates given by  $z_j = \frac{1}{w_j}$  and the transition matrix 4.3 one readily sees that  $a_j = b_j$ ,  $j = 1, \dots, n$  for any  $\nabla \in U_0 \cap U_\infty$ .

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