Fluid dynamic limit of Boltzmann equation for granular hard–spheres in a nearly elastic regime

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FLUID DYNAMIC LIMIT OF BOLTZMANN EQUATION FOR GRANULAR HARD–SPHERES IN A NEARLY ELASTIC REGIME – LIMITE HYDRODYNAMIQUE DE L'ÉQUATION DE BOLTZMANN POUR DES GAZ GRANULAIRES DANS UN RÉGIME QUASI-ÉLASTIQUE

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Abstract. In this paper, we provide the first rigorous derivation of hydrodynamic equations from the Boltzmann equation for inelastic hard spheres with small inelasticity. The hydrodynamic system that we obtain is an incompressible Navier-Stokes-Fourier system with self-consistent forcing terms and, to our knowledge, it is thus the first hydrodynamic system that properly describes rapid granular flows consistently with the kinetic formulation. To this end, we write our Boltzmann equation in a non dimensional form using the dimensionless Knudsen number which is intended to be sent to 0. There are several difficulties in such derivation, the first one coming from the fact that the original Boltzmann equation is free-cooling and, thus, requires a self-similar change of variables to introduce an homogeneous steady state. Such a homogeneous state is not explicit and is heavy-tailed, which is a major obstacle to adapting energy estimates. Additionally, a central challenge is to understand the relation between the restitution coefficient, which quantifies the energy loss at the microscopic level, and the Knudsen number. This is achieved by identifying the correct *nearly elastic regime* to capture nontrivial hydrodynamic behavior. We are, then, able to prove exponential stability uniformly with respect to the Knudsen number for solutions of the rescaled Boltzmann equation in a close to equilibrium regime. Finally, we prove that solutions to the Boltzmann equation converge in a specific weak sense towards a hydrodynamic limit which depends on time and space variables only through macroscopic quantities. Such macroscopic quantities are solutions to a suitable modification of the incompressible Navier-Stokes-Fourier system which appears to be new in this context.

French translation. Dans cet article, nous fournissons la première dérivation rigoureuse d'équations hydrodynamiques à partir de l'équation de Boltzmann pour les sphères dures inélastiques avec faible inélasticité. Le système hydrodynamique que nous obtenons est un système de Navier-Stokes-Fourier incompressible avec des termes de forçage autoconsistants et, à notre connaissance, est donc le premier système hydrodynamique qui décrit correctement les écoulements granulaires rapides conformément à la formulation cinétique. Pour parvenir à ce résultat, nous écrivons l'équation de Boltzmann sous une forme adimensionnelle en utilisant le nombre de Knudsen qui est destiné à tendre vers 0. Cette dérivation présente plusieurs difficultés, la première provenant du fait que l'équation de Boltzmann originale dissipe l'énergie cinétique et le refroidissement libre du gaz nécessite un changement de variables auto-similaire pour introduire un état d'équilibre (spatialement homogène). Un tel état homogène n'est pas explicite et présente une queue lourde, ce qui constitue un obstacle majeur à l'adaptation des estimations d'énergie. En outre, comprendre la relation entre le coefficient de restitution, qui quantifie la perte d'énergie au niveau microscopique, et le nombre de Knudsen représente une importante difficulté. Ceci est réalisé en identifiant le bon *régime quasi élastique* pour capturer un comportement hydrodynamique non trivial. Nous sommes alors en mesure de prouver la stabilité exponentielle uniformément par rapport au nombre de Knudsen pour les solutions de l'équation de Boltzmann remise à l'échelle dans un régime proche de l'équilibre. Enfin, nous prouvons que les solutions de l'équation de Boltzmann convergent dans un sens faible spécifique vers une limite hydrodynamique qui ne dépend des variables de temps et d'espace qu'à travers des quantités macroscopiques. Ces quantités macroscopiques sont solutions d'une modification appropriée du système de Navier-Stokes-Fourier incompressible qui semble être nouveau dans ce contexte.

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Mots-clés: Equation de Boltzmann inélastique; gaz granulaires; régime quasi-élastique; comportement asymptotique; limite hydrodynamique incompressible; nombre de Knudsen.

Shorter Abstract. The paper provides the first rigorous derivation of hydrodynamic equations from the Boltzmann equation for inelastic hard spheres and obtain a new system of hydrodynamic equations describing granular flows. One of the main issue is to identify the correct relation between the restitution coefficient (which quantifies the rate of energy loss at the microscopic level) and the Knudsen number which allows us to obtain non trivial hydrodynamic behavior. In such a regime, we construct strong solutions to the inelastic Boltzmann equation, near thermal equilibrium and prove that such solutions converge, in a specific weak sense, towards some hydrodynamic limit that depends on time and space variables only through macroscopic quantities that satisfy a suitable modification of the incompressible Navier-Stokes-Fourier system.

French translation. Nous obtenons dans ce papier la première dérivation rigoureuse d'équations hydrodynamiques à partir de l'équation de Boltzmann pour des sphères dures inélastiques et en déduisons un nouveau système d'équations hydrodynamiques décrivant les écoulements granulaires. L'une des principales questions est d'identifier la relation correcte – entre le coefficient de restitution (qui quantifie le taux de perte d'énergie au niveau microscopique) et le nombre de Knudsen – qui nous permet d'obtenir un comportement hydrodynamique non trivial. Dans un tel régime, nous construisons des solutions fortes à l'équation de Boltzmann inélastique, près de l'équilibre thermique et prouvons que de telles solutions convergent, dans un sens faible spécifique, vers une certaine limite hydrodynamique qui dépend des variables de temps et d'espace uniquement à travers des quantités macroscopiques qui satisfont une modification appropriée du système incompressible de Navier-Stokes-Fourier.

1. Introduction

The derivation of hydrodynamic models from suitable nonlinear (and possibly non conservative) kinetic equations is a challenging problem which has attracted a lot of attention in the recent years. Besides the well-documented literature dealing with the Boltzmann equation (see Section 1.6 hereafter), a large variety of new kinetic models and limiting processes have been considered, spanning from high friction regimes for kinetic models of swarwing (see e.g. Karper et al. (2015); Figalli & Kang (2019) for the Cucker-Smale model) to the reaction-diffusion limit for

Fitzhugh-Nagumo kinetic equations Crevat et al. (2019). For fluid-kinetic systems, the literature is even more important, we mention simply here the works Goudon et al. (2004a,b) dealing with light or fine particles regimes for the Vlasov-Navier-Stokes system and refer to Han-Kwan & Michel (2021) for the more recent advances on the subject. We also mention the challenging study of gas of charged particles submitted to electro-magnetic forces (Vlasov-Maxwell-Boltzmann system) for which several incompressible fluid limits have been derived recently in the monograph Arsénio and Saint-Raymond (2019).

We consider in the present paper the paradigmatic example of non conservative kinetic equations given by the Boltzmann equation for inelastic hard spheres. In a regime of small inelasticity, we derive in a suitable hydrodynamic limit an incompressible Navier-Stokes-Fourier system with self-consistent forcing terms. This provides, to the best of our knowledge, the first rigorous derivation of hydrodynamic system from kinetic granular flows in physical dimension $d \geqslant 3$.

1.1. **Multiscale descriptions of granular gases.** Granular materials are ubiquitous in nature and understanding the behaviour of granular matter is a relevant challenge from both the physics and mathematics viewpoints. Various descriptions of granular matter have been proposed in the literature, see Garzó (2019). An especially relevant one consists in viewing granular systems as clusters of a large number of discrete macroscopic particles (with size exceeding $1\ \mu\mathrm{m}$, significantly larger than the one of a typical particle described in classical kinetic theory) suffering dissipative interactions. One speaks then of *rapid granular flows* or *gaseous granular matter*. If the number of particles is large enough, it is then common to adopt a kinetic modelling based upon suitable modification of the Boltzmann equation. As usual in kinetic theory, it is then particularly relevant to deduce from this kinetic description the fluid behaviour of the system. This means, roughly speaking, that we look at the granular gas at a scale larger than the mesoscopic one and aim to capture the hydrodynamical features of it through the evolution of macroscopic quantities like *density, bulk velocity* and *temperature* of the gas which satisfy suitable hydrodynamics equations.

One of the main objects of the present work is to make a* first rigorous link *between these two co-existing descriptions by deriving a suitable modification of incompressible Navier-Stokes equation from the Boltzmann equation for inelastic hard-spheres as the Knudsen number goes to zero.

Recall that the Knudsen number ε is proportional to the mean free path between collisions and in order to derive hydrodynamic equations from the Boltzmann equation, the usual strategy consists, roughly speaking, in performing a perturbation analysis in the limit $\varepsilon \to 0$ (meaning that the mean free path is negligible when compared to the typical physical scale length). We point out that these questions are perfectly understood in the elastic case (*molecular gases*) for which rigorous results on the hydrodynamic limits of the Boltzmann equation have been obtained, we refer to the next Section 1.6 for more details and to Saint-Raymond (2009a) for an up-to-date review.

The picture in the context of granular gases is quite different. In fact, a satisfying hydrodynamic equation that properly describes rapid granular flows is still a controversial issue among the physics community. The continuous loss of kinetic energy makes granular gases an *open system* as far as thermodynamics is concerned. Moreover, no non-trivial steady states exist in granular gases without an external energy supply which makes granular gases a prototype of *non-equilibrium* systems. This is an important obstacle in the derivation of hydrodynamical equations from the kinetic description since it is expected that *equilibrium states* play the role of the typical *hydrodynamic solution* where time-space dependence of the single-particle distribution function $F(t,x,v)$ occurs only through suitable hydrodynamic fields like density $\varrho(t,x)$, bulk velocity $u(t,x)$, and temperature $\theta(t,x)$. An additional difficulty is related to the size of particles and scale separation. Recall that granular gases involve *macroscopic particles* whose size is much larger than the one described by the usual Boltzmann equation with elastic interactions referred to as *molecular gases*. As the hydrodynamic description occurs on large time scales (compared to the mean free time) and on large spatial scales (compared to the mean free path) the mesoscopic – continuum scale separation is problematic to justify in full generality for granular gases. We refer to (Garzó, 2019, Section 3.1, p. 102) for more details on this point and observe here that the main concern is related to the time scale induced by the evolution of the temperature (see (1.13) herafter). In particular, as observed in Garzó (2019), this problem can only be answered by ensuring that the $d+2$ hydrodynamic modes associated to density, velocity and temperature decay more slowly than the remaining kinetic excitations at large times. This is the only way that the hydrodynamic excitations emerge as the dominant dynamics. All these physically grounded obstacles make the derivation of hydrodynamic equations from the Boltzmann equation associated to granular gases a reputedly challenging open problem. Quoting Brey & Dufty (2005):

> *"the context of the hydrodynamic equations remains uncertain. What are the relevant space and time scales? How much inelasticity can be described in this way?"*

***The present paper is, to the best of our knowledge, the first rigorous answer to these relevant problems**, at least in dimension $d \geqslant 3$.* We already mentioned that the key point in our analysis is to identify the correct regime which allows to answer these questions: the *nearly elastic* one. In this regime the energy dissipation rate in the systems happens in a controlled fashion since the inelasticity parameter is compensated accordingly to the number of collisions per time unit. This process mimics viscoelasticity as particle collisions become more elastic as the collision dissipation mechanism increases in the limit $\varepsilon \to 0$ (see Assumption 1.1 below). In this way, we are able to consider a re-scaling of the kinetic equation in which a peculiar intermediate asymptotic emerges and prevents the total cooling of the granular gas.

Other regimes can be considered depending on the rate at which kinetic energy is dissipated; for example, an interesting regime is the *mono-kinetic* one which considers the extreme case of infinite energy dissipation rate. In this way, the limit is formally described by enforcing a Dirac mass solution in the kinetic equation yielding the *pressureless Euler system* (corresponding to

sticky particles). Such a regime has been rigorously addressed in the one-dimensional framework in the interesting contribution Jabin & Rey (2017). It is an open question to extend such analysis to higher dimensions since the approach of Jabin & Rey (2017) uses the so-called Bony functional which is a tool specifically tailored for 1D kinetic equations.

1.2. **The Boltzmann equation for granular gases.** We consider here the (freely cooling) Boltzmann equation which provides a statistical description of identical smooth hard spheres suffering binary and *inelastic collisions*:

$$\partial_t F(t,x,v) + v\cdot\nabla_x F(t,x,v) = \mathcal{Q}_\alpha(F,F) \tag{1.1}$$

supplemented with initial condition $F(0,x,v) = F_{\rm in}(x,v)$, where $F(t,x,v)$ is the density of granular particles having position $x \in \mathbb{T}^d$ and velocity $v \in \mathbb{R}^d$ at time $t \geqslant 0$ and $d \geqslant 3$. We consider here for simplicity the case of *flat torus*

$$\mathbb{T}^d_\ell = \mathbb{R}^d/(2\pi\,\ell\,\mathbb{Z})^d \tag{1.2}$$

for some typical length-scale $\ell > 0$. This corresponds to periodic boundary conditions:

$$F(t,x+2\pi\,\ell\boldsymbol{e}_i,v) = F(t,x,v)\,, \qquad \forall\, i = 1,\dots,d$$

where $\boldsymbol{e}_i$ is the i-th vector of the canonical basis of $\mathbb{R}^d$. The collision operator $\mathcal{Q}_\alpha$ is defined in weak form as

$$\int_{\mathbb{R}^d} \mathcal{Q}_\alpha(g,f)(v)\,\psi(v)\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(v)\,g(v_*)\,|v-v_*|\mathcal{A}_\alpha[\psi](v,v_*)\,\mathrm{d}v_*\,\mathrm{d}v\,, \tag{1.3}$$

where

$$\mathcal{A}_\alpha[\psi](v,v_*) = \int_{\mathbb{S}^{d-1}} (\psi(v')+\psi(v'_*)-\psi(v)-\psi(v_*))b(\sigma\cdot\widehat{u})\,\mathrm{d}\sigma\,, \tag{1.4}$$

and the post-collisional velocities (v',v'_*) are given by

$$\begin{aligned} v' = v + \frac{1+\alpha}{4}\,(|u|\sigma - u)\,, \qquad v'_* = v_* - \frac{1+\alpha}{4}\,(|u|\sigma-u)\,,\\ \text{where} \qquad u := v - v_*\,, \qquad \widehat{u} := \frac{u}{|u|}\,. \end{aligned} \tag{1.5}$$

Here, $\mathrm{d}\sigma$ denotes the Lebesgue measure on $\mathbb{S}^{d-1}$ and the angular part b of the collision kernel appearing in (1.4) is a non-negative measurable mapping integrable over $\mathbb{S}^{d-1}$. There is no loss of generality assuming

$$\int_{\mathbb{S}^{d-1}} b(\sigma\cdot\widehat{z})\,\mathrm{d}\sigma = 1\,, \qquad \forall\,\widehat{z}\in\mathbb{S}^{d-1}\,. \tag{1.6}$$

Additional technical assumptions on the angular kernel $b(\,\cdot\,)$ will be needed in the sequel. Namely, in the rest of the paper, we suppose that the two following conditions are satisfied.

- The mapping $\sigma \in \mathbb{S}^{d-1} \mapsto b(\sigma \cdot \widehat{z})$ belongs to $L^2(\mathbb{S}^{d-1})$ for any $\widehat{z} \in \mathbb{S}^{d-1}$, i.e.

$$\int_{-1}^{1} b^2(s) \left(1-s^2\right)^{\frac{d-3}{2}} \mathrm{d}s < \infty \,, \tag{1.7}$$

  which is useful to get estimates on the difference between $\mathcal{Q}_\alpha$ and $\mathcal{Q}_1$ (see Lemma 2.1).
- The following holds

$$\int_{-1}^{1} b(s) \left[(1-s)^{\frac{d-6}{4}} (1+s)^{\frac{d-3}{2}} + (1+s)^{\frac{d-6}{4}} (1-s)^{\frac{d-3}{2}} \right] \mathrm{d}s < \infty \,, \tag{1.8}$$

  we mention that this integral needs to be finite to get bounds on the bilinear operator $\mathcal{Q}_\alpha$ on L^2_v (see Theorem B.1) as well as for deriving smoothness of the so-called Burnett functions (see Lemma A.3).

It is worth mentioning that in the physical case of hard spheres in dimension 3, b is constant and is thus included in our assumptions since (1.7) and (1.8) hold true.

The fundamental distinction between the classical elastic Boltzmann equation and that associated to granular gases lies in the role of the parameter $\alpha \in (0,1)$, the *coefficient of restitution*. This coefficient is given by the ratio between the magnitude of the normal component (along the line of separation between the centers of the two spheres at contact) of the relative velocity after and before the collision (see Section 2 for the detailed microscopic velocities). The case $\alpha = 1$ corresponds to perfectly elastic collisions where kinetic energy is conserved. However, when $\alpha < 1$, part of the kinetic energy of the relative motion is lost since

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\frac{1-\alpha^2}{4} |u|^2 \, (1 - \sigma \cdot \widehat{u}) \leqslant 0 \tag{1.9}$$

where we recall that $u = v - v_*$. It is assumed in this work that α is independent of the relative velocity u (refer to Alonso (2009), Alonso & Lods (2014), and Alonso et al. (2021) for the viscoelastic restitution coefficient case). Notice that the microscopic description (1.5) preserves the momentum

$$v' + v'_* = v + v_* \,.$$

Let us introduce the macroscopic density and bulk velocity respectively defined by

$$\boldsymbol{R}(t) := \int_{\mathbb{R}^d \times \mathbb{T}^d_\ell} F(t,x,v) \, \mathrm{d}v \, \mathrm{d}x \qquad \text{and} \qquad \boldsymbol{U}(t) := \int_{\mathbb{R}^d \times \mathbb{T}^d_\ell} v F(t,x,v) \, \mathrm{d}v \, \mathrm{d}x \,.$$

These quantities are preserved over time, namely:

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{R}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{U}(t) = 0 \,.$$

Consequently, there is no loss of generality in assuming that

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) = 1 \,, \qquad \boldsymbol{U}(t) = \boldsymbol{U}(0) = 0 \,, \qquad \forall \, t \geqslant 0 \,.$$

As already mentioned, the main contrast between elastic and inelastic gases is that in the latter the *granular temperature*

$$\boldsymbol{T}(t) := \frac{1}{|\mathbb{T}^d_\ell|}\int_{\mathbb{R}^d\times\mathbb{T}^d_\ell} |v|^2 F(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$$

is constantly decreasing

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{T}(t) = -(1-\alpha^2)\mathcal{D}(F(t),F(t)) \leqslant 0\,, \qquad \forall\, t\geqslant 0\,.$$

Here $\mathcal{D}(g,g)$ denotes the normalised energy dissipation associated to $\mathcal{Q}_\alpha$, see Mischler & Mouhot (2006), given for suitable f,g by

$$\mathcal{D}(f,g) := \frac{\gamma_b}{4}\int_{\mathbb{T}^d_\ell}\frac{\mathrm{d}x}{|\mathbb{T}^d_\ell|}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,v_*)g(x,v)|v-v_*|^3\,\mathrm{d}v\,\mathrm{d}v_* \tag{1.10}$$

with

$$\gamma_b := \int_{\mathbb{S}^{d-1}}\frac{1-\sigma\cdot\widehat{u}}{2}\,b(\sigma\cdot\widehat{u})\,\mathrm{d}\sigma = |\mathbb{S}^{d-2}|\int_0^\pi b(\cos\theta)\,(\sin\theta)^{d-2}\,\sin^2\left(\frac{\theta}{2}\right)\,\mathrm{d}\theta.$$

In fact, it is possible to show that

$$\boldsymbol{T}(t) \xrightarrow[t\to\infty]{} 0$$

which expresses the *total cooling of granular gases*. Determining the exact dissipation rate of the granular temperature is an important question known as *Haff's law*, see Haff (1983).

1.3. **Navier-Stokes scaling.** To capture some hydrodynamic behaviour of the gas, we need to write the above equation in *nondimensional form* introducing the dimensionless Knudsen number

$$\varepsilon := \frac{\text{mean free path}}{\text{spatial length-scale}}$$

which is assumed to be small. We introduce then a rescaling of time and space to capture the hydrodynamic limit and introduce the particle density

$$F_\varepsilon(t,x,v) = F\left(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon},v\right), \qquad \forall\, t\geqslant 0\,. \tag{1.11}$$

In this case, we choose for simplicity $\ell=\varepsilon$ in (1.2) which ensures now that F_ε is defined on $\mathbb{R}^+\times\mathbb{T}^d\times\mathbb{R}^d$ with $\mathbb{T}^d=\mathbb{T}^d_1$. From now on, we assume for simplicity that the torus $\mathbb{T}^d$ is equipped with the normalized Lebesgue measure, i.e. $|\mathbb{T}^d|=1$. It is well-know that, in the classical elastic case, this scaling leads to the incompressible Navier-Stokes, however, other scalings are possible that yield different hydrodynamic models. Under such a scaling, the typical number of collisions per particle per time unit is ε^{-2}, more specifically, F_ε satisfies the rescaled Boltzmann equation

$$\varepsilon^2\partial_t F_\varepsilon(t,x,v) + \varepsilon\, v\cdot\nabla_x F_\varepsilon(t,x,v) = \mathcal{Q}_\alpha(F_\varepsilon,F_\varepsilon)\,, \qquad (x,v)\in\mathbb{T}^d\times\mathbb{R}^d\,, \tag{1.12a}$$

supplemented with initial condition

$$F_\varepsilon(0,x,v) = F^\varepsilon_{\rm in}(x,v) = F_{\rm in}\left(\frac{x}{\varepsilon},v\right). \tag{1.12b}$$

Conservation of mass and density is preserved under this scaling, consequently, we assume that

$$\boldsymbol{R}_\varepsilon(t) := \int_{\mathbb{R}^d\times\mathbb{T}^d} F_\varepsilon(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = 1\,, \quad \boldsymbol{U}_\varepsilon(t) := \int_{\mathbb{R}^d\times\mathbb{T}^d} F_\varepsilon(t,x,v) v\,\mathrm{d}v\,\mathrm{d}x = 0\,, \quad \forall\, t\geqslant 0\,,$$

and, setting now

$$\boldsymbol{T}_\varepsilon(t) := \int_{\mathbb{R}^d\times\mathbb{T}^d} |v|^2 F_\varepsilon(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,, \qquad \forall\, t\geqslant 0\,, \qquad \forall\,\varepsilon>0\,,$$

the cooling of the granular gas is now given by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{T}_\varepsilon(t) = -\frac{1-\alpha^2}{\varepsilon^2}\mathcal{D}(F_\varepsilon(t),F_\varepsilon(t))\,, \qquad \forall\, t\geqslant 0\,, \tag{1.13}$$

where we recall that $\mathcal{D}$ is defined in (1.10).

1.4. **Self-similar variable and homogeneous cooling state.** Various forcing terms have been added to (1.12a) depending on the underlying physics. Forcing terms prevent the total cooling of the gas (heated bath, thermal bath, see Villani (2006) for details) since they act as an energy supply source to the system and induce the existence of a non-trivial steady state. These are, however, systems different from the free-cooling Boltzmann equation (1.12a) that we aim to investigate here.

To understand better this free-cooling scenario, it is still possible to introduce an intermediate asymptotics and a steady state to work with. This is done by performing a self-similar change of variables

$$F_\varepsilon(t,x,v) = V_\varepsilon(t)^d f_\varepsilon\big(\tau_\varepsilon(t),x,V_\varepsilon(t)v\big)\,, \tag{1.14a}$$

with

$$\tau_\varepsilon(t) := \frac{1}{c_\varepsilon}\log(1+c_\varepsilon\,t)\,, \quad V_\varepsilon(t) := (1+c_\varepsilon\,t)\,, \quad t\geqslant 0, \quad c_\varepsilon>0\,. \tag{1.14b}$$

With the special choice

$$c_\varepsilon := \frac{1-\alpha}{\varepsilon^2}\,, \tag{1.14c}$$

we can prove that f_ε satisfies

$$\varepsilon^2\partial_t f_\varepsilon(t,x,v) + \varepsilon v\cdot\nabla_x f_\varepsilon(t,x,v) + (1-\alpha)\nabla_v\cdot(v f_\varepsilon(t,x,v)) = \mathcal{Q}_\alpha(f_\varepsilon,f_\varepsilon)\,, \tag{1.15}$$

with initial condition

$$f_\varepsilon(0,x,v) = F^\varepsilon_{\rm in}(x,v)\,. \tag{1.16}$$

The underlying drift term $(1-\alpha)\nabla_v\cdot(vf(t,x,v))$ acts as an energy supply which prevents the total cooling down of the gas. Indeed, it has been shown in a series of papers (Mischler et al. (2006); Mischler & Mouhot (2006, 2009)) that there exists a *spatially homogeneous* steady state G_α to (1.15)

which is unique for $\alpha \in (\alpha_0, 1)$ for an explicit threshold value $\alpha_0 \in (0,1)$ [1]. More specifically, for $\alpha \in (\alpha_0, 1)$, there exists a unique solution G_α to the spatially homogeneous steady equation

$$(1-\alpha)\nabla_v \cdot (vG_\alpha(v)) = \mathcal{Q}_\alpha(G_\alpha, G_\alpha) \qquad \text{with} \qquad \int_{\mathbb{R}^d} G_\alpha(v) \begin{pmatrix} 1 \\ v \end{pmatrix} \mathrm{d}v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1.17}$$

Moreover, there exists some constant $C > 0$ independent of α such that

$$\|G_\alpha - \mathcal{M}\|_{L^1_v(\langle v \rangle^2)} \leqslant C(1-\alpha) \tag{1.18}$$

where $\mathcal{M}$ is the Maxwellian distribution

$$\mathcal{M}(v) := (2\pi\vartheta_1)^{-\frac{d}{2}} \exp\left(-\frac{|v|^2}{2\vartheta_1}\right), \qquad v \in \mathbb{R}^d, \tag{1.19}$$

for some explicit temperature $\vartheta_1 > 0$. The Maxwellian distribution $\mathcal{M}(v)$ is a steady solution for $\alpha = 1$ and its prescribed temperature ϑ_1 (which ensures (1.18) to hold) will play a role in the rest of the analysis.

Notice also that the equation in self-similar variables (1.15) preserves mass and *vanishing* momentum. Indeed, a simple computation based on (1.3) gives that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d \times \mathbb{T}^d} f_\varepsilon(t,x,v) v \, \mathrm{d}v \, \mathrm{d}x = \frac{1-\alpha}{\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{T}^d} f_\varepsilon(t,x,v) v \, \mathrm{d}v \, \mathrm{d}x .$$

From now on, we will always assume that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} f_\varepsilon(0,x,v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v \, \mathrm{d}x = \int_{\mathbb{T}^d \times \mathbb{R}^d} F_\varepsilon^{\mathrm{in}}(x,v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v \, \mathrm{d}x = \begin{pmatrix} 1 \\ 0 \\ E_\varepsilon \end{pmatrix} \tag{1.20a}$$

with $E_\varepsilon > 0$ and

$$\frac{E_\varepsilon - d\vartheta_1}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0 . \tag{1.20b}$$

The choice of prescribing as initial energy some constant $E_\varepsilon > 0$ satisfying $\varepsilon^{-1}(E_\varepsilon - d\vartheta_1) \to 0$ as $\varepsilon \to 0$ for our problem is natural because $d\vartheta_1$ is the energy of the Maxwellian $\mathcal{M}$ introduced in (1.19) and as we shall see later on, the restitution coefficient α is intended to tend to 1 as ε goes to 0 in our analysis (see Assumption 1.1).

Using assumption (1.20a) and the fact that G_α has mass 1 and vanishing momentum, it holds

$$\int_{\mathbb{R}^d \times \mathbb{T}^d} f_\varepsilon(t,x,v) \begin{pmatrix} 1 \\ v \end{pmatrix} \mathrm{d}v \, \mathrm{d}x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \forall\, t \geqslant 0 .$$

Three main questions are addressed in this work regarding the solution to (1.15):

[1] Notice that the results of Mischler & Mouhot (2006, 2009) are stated under restrictive assumptions on the collision kernel $b(\,\cdot\,)$ essentially needed for moment control of the solution to the Boltzmann equation for granular gases. Thanks to a general version of the Povzner lemma (see Alonso & Lods (2010, 2013a)), those results are valid under the assumptions (1.7) – (1.8), see Alonso & Lods (2013a) for details.

(Q1) First, we aim to prove the existence and uniqueness of solutions to (1.15) in a close to equilibrium setting, i.e. solutions which are defined *globally* in time and such that

$$\sup_{t\geqslant 0}\|f_\varepsilon(t)-G_\alpha\|\leqslant\delta \tag{1.21}$$

for some positive and explicit $\delta>0$ in a suitable norm $\|\cdot\|$ of a functional space to be identified. The close-to-equilibrium setting is quite relevant for very small Knudsen numbers given the large number of collisions per unit time which keeps the system thermodynamically relaxed.

(Q2) More importantly (though closely related), the scope here is to provide estimates on the constructed solutions f_ε which are *uniform with respect to* ε. This means that, in the previous point, $\delta>0$ is independent of ε. In fact, we are able to prove *exponential time decay* for the difference $\|f_\varepsilon(t)-G_\alpha\|$.

(Q3) Finally, we aim to prove that, as $\varepsilon\to 0$, the solution $f_\varepsilon(t)$ converges towards some hydrodynamic solution which depends on (t,x) only through macroscopic quantities $(\varrho(t,x),u(t,x),\theta(t,x))$ which are solutions to a suitable modification of the incompressible Navier-Stokes system.

The central underlying assumption in the previous program is the following relation between the restitution coefficient and the Knudsen number.

Assumption 1.1. *The restitution coefficient $\alpha(\cdot)$ is a continuously decreasing function of the Knudsen number ε satisfying the scaling behaviour*

$$\alpha(\varepsilon)=1-\varepsilon^2(\lambda_0+\eta(\varepsilon)) \tag{1.22}$$

with $\lambda_0\geqslant 0$ and some function $\eta(\cdot)$ that tends to 0 as ε goes to 0. We also assume that there exists $\varepsilon_0>0$ such that $\eta(\cdot)$ is positive on $(0,\varepsilon_0)$ (which in particular implies that $\varepsilon^{-2}(1-\alpha(\varepsilon))>0$ for $\varepsilon\in(0,\varepsilon_0)$).

As mentioned before, in this regime the energy dissipation rate is controlled along time by mimicking a viscoelastic property in the granular gas which is at contrast to other regimes such as the mono-kinetic limit. In viscoelastic models, nearly elastic regimes emerges naturally on large-time scale, see Bobylev et al. (2000); Alonso & Lods (2014); Alonso et al. (2021) for details.

Because $\varepsilon\to 0$, Assumption 1.1 means that the limit produces a model of the cumulative effect of ***nearly elastic*** collisions in the ***hydrodynamic regime***. Two situations are of interest in our analysis

Case 1: If $\lambda_0=0$ the cumulative effect of the inelasticity is too weak in the hydrodynamic scale and the expected model is the classical Navier-Stokes equation.

Case 2: If $0<\lambda_0<\infty$, the cumulative effect is visible in the hydrodynamic scale and we expect a model different from the Navier-Stokes equation accounting for that. As we mentioned, we require λ_0 to be relatively small compared to some explicit quantities completely

determined by the mass and energy of the initial datum, say, $0 < \lambda_0 \ll 1$ with some explicit upper bounds on λ_0.

We wish to emphasize here that, without Assumption 1.1, it appears hopeless to resort to any kind of linearized technique, which is somehow at the basis of the Navier-Stokes scaling. Indeed, even in the spatially homogeneous case, the asymptotic behaviour of the Boltzmann equation is not clearly understood far from the elastic case (see the discussion in the introduction of Mischler & Mouhot (2009)). We strongly believe that we captured with Assumption 1.1 the correct regime that brings together the delicate balance between inelasticity and Knudsen number adapted to the hydrodynamic asymptotics for the constant restitution coefficient case. We also remark that it is very likely that the more adapted model of viscoelastic hard spheres will display naturally such balance and enjoy the nearly inelastic regime in the long-time dynamic (see Alonso & Lods (2014); Alonso et al. (2021) for more details).

1.5. **Main results.** The main results are both concerned with the solutions to (1.15). The first one is the following Cauchy theorem regarding the existence and uniqueness of close-to-equilibrium solutions to (1.15). A precise statement is given in Theorem 5.1 in Section 5.

Theorem 1.2. *Under Assumption 1.1, one can construct two suitable Banach spaces $\mathcal{E}_1 \subset \mathcal{E}$ such that, for ε, λ_0 and η_0 sufficiently small with respect to the initial mass and energy, if*

$$\|F_{\rm in}^{\varepsilon} - G_{\alpha(\varepsilon)}\|_{\mathcal{E}} \leqslant \varepsilon\, \eta_0$$

then the inelastic Boltzmann equation (1.15) with initial condition (1.16) has a unique solution

$$f_\varepsilon \in \mathcal{C}\big([0,\infty);\mathcal{E}\big) \cap L^1\big([0,\infty);\mathcal{E}_1\big)$$

satisfying, for any $r \in (0,1)$

$$\big\|f_\varepsilon(t) - G_{\alpha(\varepsilon)}\big\|_{\mathcal{E}} \leqslant C(r)\varepsilon\eta_0\, \exp\left(-(1-r)\lambda_\varepsilon\, t\right), \qquad \forall\, t \geqslant 0$$

for some positive constant $C(r)$ depending on r but not on ε and with $\lambda_\varepsilon \underset{\varepsilon\to 0}{\sim} \lambda_0 + \eta(\varepsilon)$ where λ_0 and $\eta(\,\cdot\,)$ have been introduced in Assumption 1.1.

Remark 1.3. *It is worth pointing out that the close-to-equilibrium solutions we construct are shown to decay with an exponential rate as close as we want to $\lambda_\varepsilon \sim \frac{1-\alpha(\varepsilon)}{\varepsilon^2}$. The rate of decay can thus be made uniform with respect to the Knudsen number ε if $\lambda_0 > 0$ in Assumption 1.1 and if $\lambda_0 = 0$, we obtain a rate of decay as close as we want to $\eta(\varepsilon)$, we thus obtain a uniform bound in time but not a uniform rate of decay.*

Theorem 1.2 completely answers queries **(Q1)** and **(Q2)** where the functional spaces $\mathcal{E}_1 \subset \mathcal{E}$ are chosen to be $L^1_v L^2_x$-based Sobolev spaces

$$\mathcal{E} = L^1_v \mathbb{W}^{m,2}_x(\langle v\rangle^q)\,, \qquad \mathcal{E}_1 = L^1_v \mathbb{W}^{m,2}_x(\langle v\rangle^{q+1})$$

for suitable choice of m, q. Exact notations for the functional spaces are introduced in Section 1.8. We already point out here the fact that we do not assume *any kind* of regularity in the velocity

variable v except the mere integrability (no derivative in v are assumed in the spaces $\mathcal{E}_1 \subset \mathcal{E}$). The close-to-equilibrium solutions we construct are shown to decay with a rate that can be made uniform with respect to the Knudsen number ε. Recall here that, since Assumption 1.1 is met, the homogeneous cooling state depends on ε and $G_{\alpha(\varepsilon)} \to \mathcal{M}$ as $\varepsilon \to 0$.

The estimates on the solution f_ε provided by Theorem 1.2 are enough to answer **(Q3)** in the following (we refer to Section 6 for a more accurate statement provided by Theorem 6.1).

Theorem 1.4. *Under the assumptions of Theorem 1.2, set*

$$f_\varepsilon(t,x,v) = G_{\alpha(\varepsilon)} + \varepsilon\, h_\varepsilon(t,x,v)\,,$$

with $h_\varepsilon(0,x,v) = h^\varepsilon_{\rm in}(x,v) = \varepsilon^{-1}\left(F^\varepsilon_{\rm in} - G_{\alpha(\varepsilon)}\right)$. *For a suitable class of "well-prepared" initial datum* $h^\varepsilon_{\rm in}$ *(see Theorem 6.1 for a precise definition) and any* $T>0$, *the family* $\{h_\varepsilon\}_\varepsilon$ *converges in some weak sense to a limit* $\boldsymbol{h} = \boldsymbol{h}(t,x,v)$ *which is such that*

$$\boldsymbol{h}(t,x,v) = \left(\varrho(t,x) + u(t,x)\cdot v + \frac{1}{2}\theta(t,x)(|v|^2 - d\vartheta_1)\right)\mathcal{M}(v)\,, \tag{1.23}$$

where $(\varrho,u,\theta) = (\varrho(t,x), u(t,x), \theta(t,x))$ *are suitable solutions to the following* incompressible Navier-Stokes-Fourier system with forcing

$$\begin{cases} \partial_t u - \frac{\nu}{\vartheta_1}\,\Delta_x u + \vartheta_1\, u\cdot\nabla_x\, u + \nabla_x p = \lambda_0 u\,, \\ \partial_t\,\theta - \frac{\gamma}{\vartheta_1^2}\,\Delta_x\theta + \vartheta_1\, u\cdot\nabla_x\theta = \dfrac{\lambda_0\,\bar c}{2(d+2)}\sqrt{\vartheta_1}\,\theta\,, \\ \mathrm{div}_x u = 0\,, \qquad \varrho + \vartheta_1\,\theta = 0\,, \end{cases} \tag{1.24}$$

subject to initial conditions $(\varrho_{\rm in}, u_{\rm in}, \theta_{\rm in})$ *(entirely determined by the limiting behaviour of* $h^\varepsilon_{\rm in}$ *as* $\varepsilon\to 0$*). The viscosity* $\nu>0$ *and heat conductivity* $\gamma>0$ *are explicit and* $\lambda_0>0$ *is the parameter appearing in* (1.22). *The parameter* $\bar c>0$ *is depending on the collision kernel* $b(\,\cdot\,)$.

Remark 1.5. *The data that we consider here are actually quite general. Indeed, the assumption that we make only tells that the macroscopic projection of* $h^{\rm in}_\varepsilon$ *converges towards some macroscopic distribution and we do not make any assumption on the macroscopic quantities of this distribution (see* (6.1)*). Namely, we do not suppose that the divergence free and the Boussinesq relations are satisfied by* (ϱ_0,u_0,θ_0), *oscillations induced by acoustic waves that could be created by such a lack of assumption is actually absorbed in our notion of weak convergence, the precise notion of which being very peculiar and strongly related to the a priori estimates used for the proof of Theorem 1.2 (see Theorem 6.3 for more details on the type of convergence).*

Remark 1.6. *Whenever Assumption 1.1 is not in force, the well-posedness as well as the hydrodynamic limit obtained in Theorems 1.2 and 1.4 are open questions to the best of our knowledge. Actually, even in the spatially homogeneous case, some small inelasticity assumption is necessary to prove the uniqueness of the homogeneous cooling state as well as its stability (see Mischler & Mouhot (2009)). This is in particular due to the absence of* H*-theorem for granular gases which does provide*

any general nonlinear mechanism driving the solutions towards its equilibrium state like it happens in the elastic case.

It is classical for incompressible Navier-Stokes equations, see (Majda & Bertozzi, 2002, Section 1.8, Chapter I), that the pressure term p acts as a Lagrange multiplier due to the constraint $\mathrm{div}_x u = 0$ and it is recovered (up to a constant) from the knowledge of (ϱ, u, θ).

We point out that the above incompressible Navier-Stokes-Fourier system (6.4) with the self-consistent forcing terms on the right-hand-side is a new system of hydrodynamic equations that, to our knowledge, has never been rigorously derived earlier to describe granular flows. We also notice that the last two identities in (6.4) give respectively the incompressibility condition and a strong Boussinesq relation (see the discussion in Section 6). It is important to point out that in the case $\lambda_0 = 0$, one recovers the classical incompressible Navier-Stokes-Fourier system derived from elastic Boltzmann equation, see Saint-Raymond (2009a). This proves continuity with respect to the restitution coefficient α.

Moreover, in both cases $\lambda_0 = 0$ or $\lambda_0 > 0$, the limiting system (6.4) is *conservative* (for all quantities $\varrho(t,x), u(t,x), \theta(t,x)$ as soon as the initial bulk velocity is vanishing) which illustrates the perfect balance of the self-similar scaling in the hydrodynamic limit.

We finally mention that Theorem 1.4 together with the relations (1.14) provide also a quite precise description of the hydrodynamic behaviour of the original problem (1.12a) in physical variables. In this framework, the aforementioned Case 2 for which $\lambda_0 > 0$ enjoys some special features for which uniform-in-time error estimates can be obtained. Turning back to the original problem (1.12a) not only gives a precise answer to *Haff's law* (with an explicit cooling rate of the granular temperature $\boldsymbol{T}_\varepsilon(t)$) but also describes the cooling rate of the *local temperature* of the gas. Precisely, one can deduce the following

Theorem 1.7 (***Haff's law: local and global***). *Let Assumption 1.1 be in force with*

$$\lambda_0 > 0.$$

We consider a solution $F_\varepsilon(t,x,v)$ to (1.12a) as constructed in Theorem 1.2 for ε_0, λ_0 and η_0 sufficiently small and define the local temperature *of the gas $T_\varepsilon(t,x)$ and the* global temperature *$\boldsymbol{T}_\varepsilon(t)$ as*

$$T_\varepsilon(t,x) = \int_{\mathbb{R}^d} F_\varepsilon(t,x,v)|v|^2 \mathrm{d}v \qquad x \in \mathbb{T}^d, \qquad \boldsymbol{T}_\varepsilon(t) = \frac{1}{\left|\mathbb{T}^d\right|} \int_{\mathbb{T}^d} T_\varepsilon(t,x)\mathrm{d}x, \qquad t \geqslant 0.$$

Then, $\boldsymbol{T}_\varepsilon(t) \approx \frac{d\vartheta_1}{(1+\lambda_0 t)^2}$, for $t \gg \frac{1}{\lambda_0}$, and there exist $C_0, C_1 > 0$ such that

$$\frac{C_0}{(1+\lambda_0 t)^2} \leqslant T_\varepsilon(t,x) \leqslant \frac{C_1}{(1+\lambda_0 t)^2}, \qquad \forall \varepsilon \in (0, \varepsilon_0)$$

for any $t \geqslant 0$ and $x \in \mathbb{T}^d$.

We point out that the time-decay of the temperature for granular gases has been deduced, under some heuristic considerations, in Haff (1983) and an algebraic decay of the order $(1+t)^{-2}$

has been established. This is known as *Haff's law* for the cooling of granular gases. A rigorous proof of such a decay has been established in the spatially homogeneous setting in Mischler & Mouhot (2009) in a regime of relative small inelasticity (i.e. assuming $\alpha \geqslant \alpha_0$ for some explicit value of $\alpha_0 \in (0,1)$ and for an initial datum $F_{\rm in} \in L^p(\mathbb{R}^d)$ $(p>1)$. This result has been extended by the first two authors in Alonso & Lods (2013b) to consider initial datum with finite entropy. For the physically relevant model of visco-elastic hard-spheres, Haff's law predicts a different decay and has been proved rigorously in the spatially homogeneous setting in Alonso & Lods (2010). To our knowledge, the above result is the first result applying to spatially inhomogeneous setting and it shows that the decay of the global temperature predicted by Haff's law is actually accurate also for the *local temperature*: in the perturbative setting considered here, Haff's law occurs somehow *uniformly* with respect to the spatial variable $x \in \mathbb{T}^d$. We refer to Section 7 for a more detailed discussion.

Let us summarize here the main original features of this paper:

- We identify the correct regime of weak inelasticity (Assumption 1.1) which, with a novel use of self-similarity techniques, allows to balance uniformly, in terms of the Knudsen number, the in-and-out fluxes of energy and allows to exploits fully the non Gaussian steady state in the spatially inhomogeneous setting.
- In order to derive exploitable hypocoercivity and energy estimates, we craft a very fine analysis of the collision operator $\mathcal{Q}_\alpha$ and, in particular, provides a sharp quantification of the *nearly elastic* limit $\|\mathcal{Q}_\alpha(f,g) - \mathcal{Q}_1(f,g)\|$ in terms of α. Several existing results have to be refined drastically in order to be able to capture precise smallness estimates of the linearized collision operators.
- We introduce a sophisticated argumentation (including some non standard Gronwall Lemma) exploiting fully the interplay between linear and nonlinear estimates. This approach leads to uniform estimates for the nonlinear spatially inhomogeneous inelastic Boltzmann model in terms of the Knudsen number as well as some long-time decay of the solutions to (1.27).
- We bring a precise quantification of the macroscopic observables in the hydrodynamic limit yielding first to a modified Navier-Stokes-Fourier system and also to a rigorous derivation of both the global and local versions of Haff's law in the spatially inhomogeneous setting.

The reader will experience a self-contained and detailed presentation including the material corresponding to the full derivation of the modified Navier-Stokes-Fourier system and the relevant estimates for the Boltzmann collision operator.

1.6. **Hydrodynamic limits in the elastic case.** The derivation of hydrodynamic limits from the elastic Boltzmann equation is an important problem which received a lot of attention and its origin can be traced back at least to D. Hilbert exposition of its 6th problem at the 1900 International Congress of Mathematicians. We refer the reader to Saint-Raymond (2009a); Golse (2014) for an

up-to-date description of the mathematically relevant results in the field. Roughly speaking three main approaches are adopted for the rigorous derivation of hydrodynamic limits.

A) Many of the early mathematical justifications of hydrodynamic limits of the Boltzmann equation are based on (truncated) asymptotic expansions of the solution around some hydrodynamic solution

$$F_\varepsilon(t,x,v) = F_0(t,x,v)\left(1+\sum_n \varepsilon^n F_n(t,x,v)\right) \tag{1.25}$$

where, typically

$$F_0(t,x,v) = \frac{\varrho(t,x)}{(2\pi\theta(t,x))^{\frac{d}{2}}}\exp\left(-\frac{|v-u(t,x)|^2}{2\theta(t,x)}\right) \tag{1.26}$$

is a *local Maxwellian* associated to the macroscopic fields which is required to satisfy the limiting fluid dynamic equation. This approach (or a variant of it based upon Chapman-Enskog expansion) leads to the first rigorous justification of the compressible Euler limit up to the first singular time for the solution of the Euler system in Caflisch (1980) (see also Lachowicz (1987) for more general initial data and a study of initial layers). In the same way, a justification of the incompressible Navier-Stokes limit has been obtained in De Masi et al. (1989). This approach deals mainly with strong solutions for both the kinetic and fluid equations.

B) Another important line of research concerns weak solutions and a whole program on this topic has been introduced in Bardos et al. (1991, 1993). The goal is to prove the convergence of the renormalized solutions to the Boltzmann equation (as obtained in Di Perna & Lions (1990)) towards weak solutions to the compressible Euler system or to the incompressible Navier-Stokes equations. This program has been continued exhaustively and the convergence have been obtained in several important results (see Golse & Saint-Raymond (2004, 2009); Jiang & Masmoudi (2017); Levermore & Masmoudi (2010); Lions & Masmoudi (2001a,b) to mention just a few). We remark that, in the notion of renormalized solutions for the classical Boltzmann equation, a crucial role is played by the entropy dissipation (H-theorem) which asserts that the entropy of solutions to the Boltzmann equation is non increasing

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{T}^d} F_\varepsilon \log F_\varepsilon(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \leqslant 0\,.$$

This *a priori* estimate is fully exploited in the construction of renormalized solutions to the classical Boltzmann equation and is also fundamental in some justification arguments for the Euler limit, see Saint-Raymond (2009b).

C) A third line of research deals with strong solutions close to equilibrium and exploits a careful spectral analysis of the linearized Boltzmann equation. Strong solutions to the Boltzmann equation close to equilibrium have been obtained in a weighted L^2-framework in the work Ukai (1974) and the *local-in-time* convergence of these solutions towards solution to the compressible

Euler equations have been derived in Nishida (1978). For the limiting incompressible Navier-Stokes solution, a similar result have been carried out in Bardos & Ukai (1991) for smooth global solutions in $\mathbb{R}^3$ with a small initial velocity field. The smallness assumption has been recently removed in Gallagher & Tristani (2020) allowing to treat also non global in time solutions to the Navier-Stokes equation. These results as well as Briant et al. (2019) exploit a very careful description of the spectrum of the linearized Boltzmann equation derived in Ellis & Pinsky (1975). We notice that they are framed in the space $L^2(\mathcal{M}^{-1})$ where the linearized Boltzmann operator is self-adjoint and coercive. The fact that the analysis of Ellis & Pinsky (1975) has been extended recently in Gervais (2021) to larger functional spaces of the type $L^2_v(\langle\cdot\rangle^q)$ opens the gate to some refinements of several of the aforementioned results. We also mention here the work Jiang et al. (2018) which deals with an energy method in $L^2(\mathcal{M}^{-1})$ spaces (see also Guo et al. (2010); Guo (2016) and Rachid (2021)) in order to prove the *strong convergence* of the solutions to the Boltzmann equation towards the incompressible Navier-Stokes equation without resorting to the work of Ellis & Pinsky (1975). We also refer to Gervais & Lods (2023) for a recent unified and spectral approach to the hydrodynamic limits for strong solutions of various kinetic equations.

We mention finally that the works Briant et al. (2019), Carrapatoso et al. (2022) and Gervais (2022) were the main inspirations to answer questions **(Q1)-(Q2)**. Indeed, in Briant et al. (2019) and in Gervais (2022), estimates on the elastic Boltzmann equation in Sobolev spaces with polynomial weight are obtained *uniformly with respect to the Knudsen number* ε. On the other hand, the work Carrapatoso et al. (2022) introduces the main hypocoercivity estimates *without derivative in the velocity variables* which play a fundamental role in our analysis. To answer question **(Q3)**, we will resort to ideas introduced in the theory of renormalized solutions Bardos et al. (1991, 1993); Golse & Saint-Raymond (2004) that we adapt to the notions of solutions we are dealing with here. We notice here already that we cannot resort to the work of Ellis & Pinsky (1975) and need to carefully exploit the properties of the solutions as constructed in Theorem 1.2.

1.7. **The challenge of hydrodynamic limits for granular gases.** There are several reasons which make the derivation of hydrodynamic limits for granular gases a challenging question at the physical level. In regard of the mathematical aspects of the hydrodynamical limit, several hurdles stand on way when trying to adapt the aforementioned approaches:

I) With respect to the strategy given in A), the main difficulty lies in the identification of the typical *hydrodynamic solution*. Such solution is such that the time-space dependence of the one-particle distribution function $F(t,x,v)$ occurs only through suitable hydrodynamic fields like density $\varrho(t,x)$, bulk velocity $u(t,x)$, and temperature $\theta(t,x)$. This is the role played by the Maxwellian F_0 in (1.26) whenever $\alpha = 1$ and one wonders if the homogeneous cooling state G_α plays this role here. This is indeed the case up to first order capturing the fat tails of inelastic distributions, yet surprisingly, a suitable Maxwellian plays the role of the

hydrodynamic solution in the ε-order correction. This Gaussian behaviour emerges in the *hydrodynamic limit* because of the near elastic regime that we treat here.[2]

II) The direction promoted in B) appears for the moment out of reach in the context of granular gases. Renormalized solutions in the context of the inelastic Boltzmann equation (1.27) have not been obtained due to the lack of an H-Theorem for granular gases. It is unclear if the classical entropy (or a suitable modification of it) remains bounded in general for granular gases.

III) Homogeneous cooling states G_α are not explicit, this is a technical difficulty when adapting the approach of Ellis & Pinsky (1975) for the spectral analysis of the linearized inelastic Boltzmann equation in the spatial Fourier variable. Partial interesting results have been obtained in Rey (2013) (devoted to diffusively heated granular gases) but they do not give a complete asymptotic expansion of eigenvalues and eigenfunctions up to the order leading to the Navier-Stokes asymptotic. We mention that obtaining an analogue of the work Ellis & Pinsky (1975) for granular gases would allow, in particular, to quantify the convergence rate towards the limiting model as in the recent work Gallagher & Tristani (2020).

IV) A major obstacle to adapt energy estimates and spectral approach lies in the choice of functional spaces. While the linearized Boltzmann operator associated to elastic interactions is self-adjoint and coercive in the weighted L^2-space $L^2_v(\mathcal{M}^{-1})$, there is no such "self-adjoint" space for the inelastic case. This yields technical difficulties in the study of the spectral analysis of the linearized operator which is still actually missing. Moreover, the energy estimates of Guo et al. (2010); Guo (2016); Jiang & Masmoudi (2017); Jiang et al. (2018) are essentially based upon the coercivity of the linearized operator. For granular gases, it seems that one needs to face the problem *directly* in a L^1_v-setting. Points III) and IV) make the approach C) difficult to directly adapt.

1.8. **Notations and definitions.** We first introduce some useful notations for function spaces. For any nonnegative weight function $m : \mathbb{R}^d \to \mathbb{R}^+$ (notice that all the weights we consider here will depend only on velocity, i.e. $m = m(v)$), we define $L^q_v L^p_x(m)$, $1 \leqslant p, q \leqslant +\infty$, as the Lebesgue space associated to the norm

$$\|h\|_{L^q_v L^p_x(m)} = \big\| \|h(\cdot, v)\|_{L^p_x}\, m(v) \big\|_{L^q_v}\,.$$

We also consider the standard higher-order Sobolev generalizations $\mathbb{W}^{\sigma,q}_v \mathbb{W}^{s,p}_x(m)$ for any $\sigma, s \in \mathbb{N}$ defined by the norm

$$\|h\|_{\mathbb{W}^{\sigma,q}_v \mathbb{W}^{s,p}_x(m)} = \sum_{\substack{0\leqslant s'\leqslant s,\, 0\leqslant \sigma'\leqslant \sigma,\\ s'+\sigma'\leqslant \max(s,\sigma)}} \big\| \|\nabla^{s'}_x \nabla^{\sigma'}_v h(\cdot, v)\|_{L^p_x}\, m(v) \big\|_{L^q_v}\,.$$

[2] See the interesting discussion in Villani (2006), especially the Section 2.8 entitled "What Is the Trouble with Non-Gaussianity"

This definition reduces to the usual weighted Sobolev space $\mathbb{W}^{s,p}_{x,v}(m)$ when $q=p$ and $\sigma=s$. For $m\equiv 1$, we simply denote the associated spaces by $L^q_vL^p_x$ and $\mathbb{W}^{\sigma,q}_v\mathbb{W}^{s,p}_x$.

We consider in the sequel the general weight

$$\varpi_s(v):=(1+|v|^2)^{\frac{s}{2}}\,,\qquad \forall\, v\in\mathbb{R}^d\,,\qquad \forall\, s\geqslant 0\,.$$

For any $z\in\mathbb{R}^d$, $z\neq 0$, we will denote by $\widehat{z}=\frac{z}{|z|}$ the associated unit vector. For two tensors $A=(A_{i,j}),B=(B_{i,j})\in\mathscr{M}_d(\mathbb{R})$, we denote by $A:B$ the scalar $(A:B)=\sum_{i,j}A_{i,j}B_{i,j}\in\mathbb{R}$ as the trace of the matrix product AB whereas, for a vector function $w=w(x)\in\mathbb{R}^d$, the tensor $(\partial_{x_i}w_j)_{i,j}$ is denoted as $\nabla_x w$. We also write $(\mathrm{Div}_x A)^i=\sum_j\partial_{x_j}A_{i,j}(x)$.

Throughout the paper, for $\boldsymbol{A},\boldsymbol{B}>0$, we will indicate $\boldsymbol{A}\lesssim\boldsymbol{B}$ or $\boldsymbol{A}\leqslant C\boldsymbol{B}$ whenever there is a positive constant $C>0$ depending only on fixed numbers (but never on the parameters α and ε) such that $\boldsymbol{A}\leqslant C\boldsymbol{B}$. Notice also that we shall use the same notation C for positive constants that may change from line to line.

1.9. **Strategy of the proof.** The strategy used to prove the main results Theorems 1.2 and 1.4 yields to several intermediate results of independent interest. The approach is perturbative in essence since we are dealing with close-to-equilibrium solutions to (1.15). This means that, in the study of (1.15), we introduce the fluctuation h_ε around the equilibrium G_α defined through

$$f_\varepsilon(t,x,v)=G_\alpha(v)+\varepsilon\, h_\varepsilon(t,x,v)\,,$$

and h_ε satisfies

$$\left\{\begin{array}{l}\partial_t h_\varepsilon(t,x,v)+\dfrac{1}{\varepsilon}v\cdot\nabla_x h_\varepsilon(t,x,v)-\dfrac{1}{\varepsilon^2}\mathscr{L}_\alpha h_\varepsilon(t,x,v)=\dfrac{1}{\varepsilon}\mathcal{Q}_\alpha(h_\varepsilon,h_\varepsilon)(t,x,v)\,,\\ h_\varepsilon(0,x,v)=h_\varepsilon^{\mathrm{in}}(x,v)\,,\end{array}\right. \tag{1.27}$$

where $\mathscr{L}_\alpha$ is the linearized collision operator (local in the x-variable) defined as

$$\mathscr{L}_\alpha h(x,v):=\mathbf{L}_\alpha(h)(x,v)-(1-\alpha)\nabla_v\cdot(vh(x,v))\,, \tag{1.28}$$

with

$$\mathbf{L}_\alpha(h):=2\widetilde{\mathcal{Q}}_\alpha(G_\alpha,h)\,, \tag{1.29}$$

where we set

$$\widetilde{\mathcal{Q}}_\alpha(f,g):=\frac{1}{2}\left\{\mathcal{Q}_\alpha(f,g)+\mathcal{Q}_\alpha(g,f)\right\}\,. \tag{1.30}$$

We also denote by $\mathscr{L}_1$ the linearized operator around $G_1=\mathcal{M}$, that is,

$$\mathscr{L}_1(h)=\mathbf{L}_1(h)=\mathcal{Q}_1(\mathcal{M},h)+\mathcal{Q}_1(h,\mathcal{M})\,. \tag{1.31}$$

It is also worth noticing that assumption (1.20a) and (1.17) result in

$$\int_{\mathbb{T}^d\times\mathbb{R}^d}h_\varepsilon^{\mathrm{in}}(x,v)\begin{pmatrix}1\\ v\end{pmatrix}\,\mathrm{d}v\,\mathrm{d}x=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

Moreover, equation (1.27) preserves mass and *vanishing* momentum since, if h_ε solves (1.27), then one formally has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d\times\mathbb{R}^d} h_\varepsilon(t,x,v)v\,\mathrm{d}v\,\mathrm{d}x = -\int_{\mathbb{T}^d\times\mathbb{R}^d} h_\varepsilon(t,x,v)v\,\mathrm{d}v\,\mathrm{d}x\,. \tag{1.32}$$

Consequently, there is no loss of generality assuming that

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} h_\varepsilon(t,x,v)\begin{pmatrix}1\\ v\end{pmatrix}\mathrm{d}v\,\mathrm{d}x = \begin{pmatrix}0\\ 0\end{pmatrix}, \qquad \forall\, t\geqslant 0\,. \tag{1.33}$$

Notice that under this assumption, the hypothesis made on the energy of the initial data in (1.20a) implies that

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} h_\varepsilon^{\mathrm{in}}(x,v)\,|v|^2\,\mathrm{d}v\,\mathrm{d}x \xrightarrow[\varepsilon\to 0]{} 0\,. \tag{1.34}$$

Indeed, using (1.20a) and Assumption 1.1 combined with (1.18), we obtain

$$\begin{aligned}\int_{\mathbb{T}^d\times\mathbb{R}^d} h_\varepsilon^{\mathrm{in}}(x,v)|v|^2\,\mathrm{d}v\,\mathrm{d}x &= \frac{1}{\varepsilon}\int_{\mathbb{T}^d\times\mathbb{R}^d}\left(F_\varepsilon^{\mathrm{in}}(x,v)-G_{\alpha(\varepsilon)}(v)\right)|v|^2\,\mathrm{d}v\,\mathrm{d}x\\ &= \frac{E_\varepsilon - d\vartheta_1}{\varepsilon}+\frac{1}{\varepsilon}\int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\mathcal{M}(v)-G_{\alpha(\varepsilon)}(v)\right)|v|^2\,\mathrm{d}v\,\mathrm{d}x \xrightarrow[\varepsilon\to 0]{} 0\,.\end{aligned}$$

The above basic estimate and limit is illustrating the two-levels features of the perturbative strategy we adopt to prove Theorem 1.2: $F_\varepsilon^{\mathrm{in}}$ stays close to the equilibrium (close-to-equilibrium perturbation) but this comes from the fact that $G_{\alpha(\varepsilon)}$ is close to $\mathcal{M}$ (nearly elastic regime).

Our approach is indeed perturbative in two aspects: first, as already said, we are considering close-to-equilibrium solutions (i.e. fluctuations around the homogeneous cooling states) and second, we consider a nearly elastic regime (i.e. fluctuations around the classical/elastic Boltzmann equation). This means in particular that we shall enforce in (1.15) the elastic Boltzmann operator (at both the linearized and nonlinear levels) and will treat, up to some extent, the various inelastic operators as source terms which can be controlled in the limit $\varepsilon\to 0$ thanks to Assumption 1.1. Let us try to make this basic idea more precise.

First, our approach requires a very fine analysis of the full linearized operator appearing in (1.27):

$$\mathcal{G}_{\alpha,\varepsilon}h := -\varepsilon^{-1}v\cdot\nabla_x h+\varepsilon^{-2}\mathscr{L}_\alpha h$$

but we wish to insist on the fact that our approach is not directly related to a description of the spectral properties of $\mathcal{G}_{\alpha,\varepsilon}$. We treat $\mathcal{G}_{\alpha,\varepsilon}$ as a *perturbation* of the elastic linearized operator $\mathcal{G}_{1,\varepsilon}$. However, such a perturbation does not fall into the realm of the classical perturbation theory of unbounded operators as described in Kato (1980). Typically, the domain of $\mathcal{G}_{\alpha,\varepsilon}$ is much smaller than the one of $\mathcal{G}_{1,\varepsilon}$ (because of the drift term in velocity) and the relative bound between $\mathcal{G}_{1,\varepsilon}$ and $\mathcal{G}_{\alpha,\varepsilon}$ does not converges to zero in the elastic limit $\alpha\to 1$. It is one of the reasons why it is not easy to deduce the spectral properties of $\mathcal{G}_{\alpha,\varepsilon}$ from those of $\mathcal{G}_{1,\varepsilon}$ (which are well-understood,

see for instance Briant et al. (2019)) and we rather adopt an alternative approach based upon a combination the enlargement methods introduced in Gualdani et al. (2017) and L^2-hypocoercivity methods (see Villani (2009); Carrapatoso et al. (2022)). Indeed, first borrowing ideas from Gualdani et al. (2017); Tristani (2016), we can split $\mathcal{G}_{\alpha,\varepsilon}$ as

$$\mathcal{G}_{\alpha,\varepsilon} = \mathcal{A}_\varepsilon + \mathcal{B}_{\alpha,\varepsilon}$$

where $\mathcal{A}_\varepsilon$ is a regularizing operator in the velocity variable and $\mathcal{B}_{\alpha,\varepsilon}$ is a suitable *dissipative* operator. We refer to Section 2 for details and insist on the fact that, to capture the dissipativity properties of the operator $\mathcal{B}_{\alpha,\varepsilon}$, a very fine analysis of the collision operators $\mathcal{Q}_\alpha$ and $\mathscr{L}_\alpha$ is needed with particular emphasis of the quantification of the elastic limit $\mathcal{Q}_\alpha \to \mathcal{Q}_1$ and $\mathscr{L}_\alpha \to \mathscr{L}_1$ as $\alpha \to 1$ in various functional spaces. Besides this splitting, we also can write

$$\mathcal{G}_{\alpha,\varepsilon} = \mathcal{G}_{1,\varepsilon} + (\mathcal{G}_{\alpha,\varepsilon} - \mathcal{G}_{1,\varepsilon})$$

and exploits the hypocoercivity properties of $\mathcal{G}_{1,\varepsilon}$, hoping the reminder term $\mathcal{G}_{\alpha,\varepsilon} - \mathcal{G}_{1,\varepsilon}$ will not make them degenerate too much. To do so, and because of the additional derivative in the velocity variable (appearing now in the difference $\mathcal{G}_{\alpha,\varepsilon} - \mathcal{G}_{1,\varepsilon}$), we device new *hypocoercivity* estimates for $\mathcal{G}_{1,\alpha}$. We refer to Section 3 for a detailed description of the method and result but just mention here that, even though it seems possible to adapt the result of Briant (2015) in Sobolev spaces in v, we rather adopt a strategy based upon L^2-hypocoercivity working in the space

$$\mathcal{H} := L^2_v \mathbb{W}^{m,2}_x(\mathcal{M}^{-\frac{1}{2}}), \qquad m > \frac{d}{2} \tag{1.35}$$

on which we build a norm $|||\cdot|||_{\mathcal{H}}$ with associated inner product $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle_{\mathcal{H}}$ equivalent to the standard norm $\|\cdot\|_{\mathcal{H}}$ for which

$$\langle\!\langle \mathcal{G}_{1,\varepsilon} h, h \rangle\!\rangle_{\mathcal{H}} \lesssim -\frac{1}{\varepsilon^2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)\, h\|^2_{\mathcal{H}_1} - \|h\|^2_{\mathcal{H}_1}$$

where $\boldsymbol{\pi}_0$ is the spectral projection associated to the zero eigenvalue of $\mathbf{L}_1$ (see Section 3 for details) and $\mathcal{H}_1$ is defined in (3.1) and is such that $\mathcal{H}_1 \hookrightarrow \mathcal{H}$.

With this at hands, in order to prove Theorem 1.2 several *a priori estimates* for the solutions to (1.15) are required. This is done in Section 4. The crucial point in the analysis lies in the splitting of (1.15) into a system of two equations mimicking a spectral enlargement method from a PDE perspective (see the Section 2.3 of Mischler & Mouhot (2016) and Briant et al. (2019) for pioneering ideas on such a splitting). More precisely, the splitting performed in Sections 4 and 5 amounts to look for a solution of (1.27) of the form

$$h_\varepsilon(t) = h^0_\varepsilon(t) + h^1_\varepsilon(t)$$

where $h^0 = h^0_\varepsilon$ and $h^1 = h^1_\varepsilon(t)$ are the solutions to the following system of equations (in order to lighten the notations, in this whole section, we shall omit the dependence on ε for h, h^0 and h^1):

$$\left\{\begin{array}{rcl} \partial_t h^0 & = & \mathcal{B}_{\alpha(\varepsilon),\varepsilon} h^0 + \varepsilon^{-2}\Big[\mathbf{L}_{\alpha(\varepsilon)} h^1 - \mathbf{L}_1 h^1\Big] + (1-\alpha(\varepsilon))\varepsilon^{-2}\mathrm{a}_3\boldsymbol{\varpi}_2\left(\mathbf{Id}-\mathbf{P}_0\right) h^1 \\ & & \quad +\varepsilon^{-1}\Big[\mathcal{Q}_{\alpha(\varepsilon)}(h^0,h^0) + \mathcal{Q}_{\alpha(\varepsilon)}(h^0,h^1) + \mathcal{Q}_{\alpha(\varepsilon)}(h^1,h^0) + \boldsymbol{\pi}_0\mathcal{Q}_{\alpha(\varepsilon)}(h^1,h^1)\Big]\,, \\ h^0(0,x,v) & = & h^\varepsilon_{\mathrm{in}}(x,v) \in \mathcal{E} \end{array}\right. \tag{1.36}$$

where a_3 is some positive constant allowing to control the drift term (see Lemma 3.3) and the projectors $\boldsymbol{\pi}_0$, $\mathbf{P}_0$ are the spectral projectors associated to $\mathbf{L}_1$ and $\mathcal{G}_{1,\alpha}$ respectively (see Eqs. (3.2)-(3.4) for a precise definition) and

$$\left\{\begin{array}{rcl} \partial_t h^1 & = & \mathcal{G}_{1,\varepsilon} h^1 + (1-\alpha(\varepsilon))\varepsilon^{-2}\left[-\mathrm{div}_v\left(v h^1\right) - \mathrm{a}_3\boldsymbol{\varpi}_2\left(\mathbf{Id}-\mathbf{P}_0\right) h^1\right] \\ & & \quad + \varepsilon^{-1}\left(\mathbf{Id}-\boldsymbol{\pi}_0\right)\mathcal{Q}_{\alpha(\varepsilon)}(h^1,h^1) + \mathcal{A}_\varepsilon h^0\,, \\ h^1(0,x,v) & = & 0 \in \mathcal{H}\,. \end{array}\right. \tag{1.37}$$

Tailoring the splitting of (1.27) into two equations (1.36)-(1.37) is actually one of the most difficult part of this work and some comments are in order.

- First, notice that as in Briant et al. (2019), we can analyze the equation on h^1 in the most convenient functional space since we have put 0 as initial data for h^1. It is then natural to study the equation on h^1 in the Hilbert space $\mathcal{H}$ since the elastic linearized operator $\mathcal{G}_{1,\varepsilon}$ is well-understood in this type of space (see Proposition 3.1). It is worth mentioning that in Briant et al. (2019), the equation on h^1 is posed in $\mathbb{W}^{m,2}_{x,v}(\mathcal{M}^{-\frac{1}{2}})$ whereas we study it in the space $\mathcal{H}$ with no derivative in velocity. It is made possible by our above L^2-hypocoercivity results.
- Similarly as in Briant et al. (2019), we have put the nice dissipative part of the linearized equation on h^0 in the equation on h^0 (namely $\mathcal{B}_{\alpha(\varepsilon),\varepsilon}h^0$) while we have put $\mathcal{A}_\varepsilon h^0$ in the equation on h^1. We are able to do this thanks to the regularizing properties of $\mathcal{A}_\varepsilon$ (namely $\mathcal{A}_\varepsilon \in \mathscr{B}(\mathcal{E},\mathcal{H})$).

We already mentioned a first difference with the study by Briant et al. (2019) in the first item concerning the space in which we study the equation on h^1. If we compare our splitting to the one in Briant et al. (2019), it is also much more complicated because of the additional terms coming from the inelasticity of our equation. Let us present and explain those differences.

- First, notice that we can keep the drift term on h^1 in the equation on h^1 because when performing energy estimates in $\mathcal{H}$, it only induces the appearance of bounded terms in h^1 that are small and terms with a loss of weight (see Lemma 3.3) but *no terms with a loss of*

derivative in velocity. This loss of weight is counterbalanced by the introduction of the term $-\mathrm{a}_3\boldsymbol{\varpi}_2\left(\mathbf{Id}-\mathbf{P}_0\right)h^1$.

- The term $\mathrm{a}_3\boldsymbol{\varpi}_2\left(\mathbf{Id}-\mathbf{P}_0\right)h^1$ has thus to be added to the equation on h^0 but it turns out to be harmless when estimated in the functional space $\mathcal{E}$ because of the continuous embedding $\mathcal{H}$ into $\mathcal{E}_2$ where $\mathcal{E}_2$ is defined as

$$\mathcal{E}_2 := L^2_v\mathbb{W}^{m,2}_x(\boldsymbol{\varpi}_{q+2\kappa+3})\,, \qquad \kappa > \frac{d}{2}$$

and acts as an intermediate space which allows somehow to make the link between the L^1_v-space $\mathcal{E}$ and the L^2_v-space $\mathcal{H}$.

- Since we only put the elastic linearized collision operator on h^1 in the equation on h^1, it still remains to deal with the difference $(1-\alpha(\varepsilon))\varepsilon^{-2}(\mathbf{L}_\alpha-\mathbf{L}_1)$. Notice that some crucial point is that we are able to obtain estimates on this term with *no loss of derivative in velocity* in the functional space $\mathcal{E}$ thanks to Lemma 2.4 and we can handle this term still thanks to the embedding $\mathcal{H}\hookrightarrow\mathcal{E}_2$.
- Finally, concerning the nonlinear term $\varepsilon^{-1}\mathcal{Q}_{\alpha(\varepsilon)}(h^1,h^1)$, it is split into two parts. We only keep the microscopic part of this term, namely $\varepsilon^{-1}\left(\mathbf{Id}-\boldsymbol{\pi}_0\right)\mathcal{Q}_{\alpha(\varepsilon)}(h^1,h^1)$ in the equation on h^1 because this term can be handled thanks to the nice hypocoercivity estimates satisfied by $\mathcal{G}_{1,\varepsilon}$ in $\mathcal{H}$ (see Proposition 3.1 and Lemma 3.4) (as the term $\varepsilon^{-1}\mathcal{Q}_1(h^1,h^1)=\varepsilon^{-1}\left(\mathbf{Id}-\boldsymbol{\pi}_0\right)\mathcal{Q}_1(h^1,h^1)$ was treated in Briant et al. (2019)). The remaining part $\varepsilon^{-1}\boldsymbol{\pi}_0\mathcal{Q}_{\alpha(\varepsilon)}(h^1,h^1)$ is then added to the equation on h^0 and does not induce any difficulty because the projector $\boldsymbol{\pi}_0$ regularizes as necessary in velocity and because $\mathcal{B}_{\alpha(\varepsilon),\varepsilon}$ is $-\varepsilon^{-2}$-dissipative in $\mathcal{E}$.

The physical meaning of the above decomposition is not clear to us but appears to be a convenient tool to enforce the energy method we adopt here. Notice that, in the framework of *strong solutions* in the elastic case, some other kinds of splitting have been considered (see Gervais (2022); Gervais & Lods (2023)). They have a clearer physical meaning since they are related to the decomposition of the solution f^ε to the kinetic equation into several pieces (kinetic, macroscopic, dispersive terms) which all have a precise meaning. This is made possible thanks to a suitable mild reformulation of both the Boltzmann equations and Navier-Stokes-Fourier system and a fine spectral analysis of the linearized operator, see Gallagher & Tristani (2020); Bardos & Ukai (1991).

As mentioned earlier, because of the above considerations, we are able to seek $h^1_\varepsilon(t)$ in the above Hilbert space $\mathcal{H}$ and prove bounds of the type

$$\sup_{t\geqslant 0}\left(\|h^1_\varepsilon(t)\|^2_{\mathcal{H}}+\int_0^t\|h^1_\varepsilon(\tau)\|^2_{\mathcal{H}_1}\,\mathrm{d}\tau\right)\lesssim\|h^\varepsilon_{\mathrm{in}}\|_{\mathcal{E}}+\|h^\varepsilon_{\mathrm{in}}\|^2_{\mathcal{E}}$$

where $\mathcal{H}_1$ is defined in (3.1). It is important to point out already that $\mathcal{A}_\varepsilon$ is regularizing only in the velocity variable but not in the x-variable. Therefore, no gain of integrability can be deduced from the action of $\mathcal{A}_\varepsilon$. Therefore, since we look for $h^1_\varepsilon\in\mathcal{H}$, we need to look for h^0_ε *in a space*

based on L^2_x. The velocity regularization properties of $\mathcal{A}_\varepsilon$ allow then to look for

$$h^0_\varepsilon(t) \in L^1_v \mathbb{W}^{m,2}_x(\varpi_q)\,, \qquad \forall\, t \geqslant 0\,.$$

This is the role of Section 4.

In Section 5, we prove Theorem 1.2 introducing a suitable iterative scheme based upon the coupling $\big(h^0_\varepsilon(t),\, h^1_\varepsilon(t)\big)$. We show in practice that the coupled system of kinetic equations satisfied by h^0 and h^1 is well-posed. It is fair to say that the bounds for h^0_ε and h^1_ε given in Sections 4 and 5 play the role of suitable energy estimates as the ones established in the purely Hilbert setting Guo et al. (2010); Guo (2016); Jiang et al. (2018). In particular, these bounds are sufficient to deduce a very peculiar type of weak convergence of $h_\varepsilon(t)$ towards an element in the kernel of the linearized operator $\mathscr{L}_1$, in particular, the limit of h_ε is necessarily of the form (6.3). The notion of weak convergence we use here fully exploits the splitting $h_\varepsilon = h^0_\varepsilon + h^1_\varepsilon$ where we prove that h^0_ε converges to 0 *strongly* in $L^1((0,T); L^1_v \mathbb{W}^{m,2}_x(\varpi_q))$ whereas h^1_ε converges to $\boldsymbol{h}$ *weakly* in $L^2((0,T)\,;\, L^2_v \mathbb{W}^{m,2}_x(\mathcal{M}^{-\frac{1}{2}}))$.

Finally, in Section 6, the regularity of (ϱ, u, θ) obtained via a simple use of Ascoli-Arzela Theorem and the identification of the limiting equations these macroscopic fields satisfy is presented. With the notion of weak convergence at hand presented above, the approach is simpler but reminiscent of the program established in Bardos et al. (1991, 1993). In particular, we can adapt some of the main ideas of Golse & Saint-Raymond (2004) regarding the delicate convergence of nonlinear convection terms. Detailed computations are included to make the paper as much self-contained as possible also because, even in the classical "elastic" case, it is difficult to find a full proof of the convergence towards hydrodynamic limit for the weak solutions we consider here. For such solutions, details of proof are scattered in the literature and full proof of the convergence of nonlinear terms is sometimes only sketched where most of the full detailed proofs are dealing with the more delicate case of renormalized solutions Golse & Saint-Raymond (2004, 2009); Levermore & Masmoudi (2010). In our framework, the terms involving the quadratic operator $\mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon)$ are treated as source terms which converge in distributions to zero whereas the drift term and the dissipation of energy function $\mathcal{D}$ are the objects responsible for the terms in the right-side of the Navier-Stokes system (6.4). We also observe that the derivation of the strong Boussinesq relation is not as straightforward as in the elastic case. Actually, the classical Boussinesq relation

$$\nabla\left(\varrho(t,x) + \vartheta_1 \theta(t,x)\right) = 0$$

is established as in the elastic case. In the elastic case, this relation implies the strong form of Boussinesq relation mainly because the two functions $\varrho(t,x)$ and $\theta(t,x)$ have zero spatial averages. This cannot be deduced directly in the granular context due to the dissipation of energy.

1.10. **Organization of the paper.** The paper is divided into 7 Sections and three Appendices. In the following Section 2, we collect several very fine results regarding the collision operators $\mathscr{L}_\alpha$ and $\mathcal{Q}_\alpha$ quantifying the differences $\|\mathcal{Q}_\alpha - \mathcal{Q}_1\|$ and $\|\mathscr{L}_\alpha - \mathscr{L}_1\|$ in terms of $1-\alpha$ for various norms.

We also introduce in Section 2 the splitting of the operator $\mathcal{G}_{\alpha,\varepsilon} = \mathcal{A}_\varepsilon + \mathcal{B}_{\alpha,\varepsilon}$. As mentioned, even if our final goal is to study the collision operator in spaces built on L^1_v-spaces with polynomial weights, we shall also need to resort to estimates of $\mathscr{L}_\alpha$ in L^2_v-spaces. Section 3 is devoted to the hypocoercivity method for the operator $\mathcal{G}_{1,\varepsilon}$ and its various consequences. In Section 4, we derive the fundamental *a priori* estimates on the close-to-equilibrium solutions to (1.27). It is the most technical part of the work and fully exploits the above splitting (1.36)–(1.37). Section 5 gives the proof of Theorem 1.2 whereas Section 6 gives the full proof of the hydrodynamic limit (Theorem 1.4). In Section 7, we give an informal presentation of the consequences of our analysis on the problem in original physical variables. In Appendix A, we collect some well-known properties useful for the hydrodynamic limit as well as some technical proofs used in Section 6. Finally, Appendix B gives the proof of several technical results of Section 2.

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2. Summary of useful results about the collision operator

2.1. **Strong and weak forms of inelastic Boltzmann collision operators.**

In this section, we collect several results about the Boltzmann collision operator $\mathcal{Q}_\alpha$ for granular gases. Before entering the details of the technical result, we briefly reminds the main physical features of inelastic interactions and the role of the *coefficient of normal restitution* yielding naturally to the n-representation of the collision operator.

As indicated in the Introduction, the Boltzmann equation for granular gases is a well-accepted model that describes collisions in a system composed by a large number of granular particles which are assumed to be hard-spheres with equal unitary mass and that undertake inelastic collisions. If v and v_* denote the velocities of two particles before collision, their respective velocities v' and v'_* after collision are such that the normal relative velocity is dissipative during impact according to the law

$$\big(u' \cdot n\big) = -\alpha\big(u \cdot n\big)\,. \tag{2.1}$$

The unitary vector $n \in \mathbb{S}^{d-1}$ determines the impact direction, that is, n stands for the unit vector that points from the v-particle center to the v_*-particle center at the moment of impact while

$$u = v - v_*\,, \qquad u' = v' - v'_*\,,$$

denote respectively the relative velocity before and after collision. The velocities after collision v' and v'_* are given, in virtue of (2.1) and the conservation of momentum, by

$$v' = v - \frac{1+\alpha}{2}\big(u\cdot n\big)n\,, \qquad v'_* = v_* + \frac{1+\alpha}{2}\big(u\cdot n\big)n\,. \tag{2.2}$$

In particular, the energy relation induced by the collision mechanism can be written as

$$|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2 - \frac{1-\alpha^2}{2}\big(u\cdot n\big)^2 \leqslant |v|^2 + |v_*|^2\ . \tag{2.3}$$

In particular, for $\alpha \in (0,1)$ and in contrast with elastic interactions, the collision mechanism $(v, v_*) \to (v', v'_*)$ is not reversible. This means that the pre-collisional velocities $({}'v, {}'v_*)$ (resulting in (v, v_*) after collision) differ here from the post-collisional ones and can be introduced through the relation

$${}'v = v - \frac{1+\alpha}{2\alpha}\big(u\cdot n\big)n\,, \qquad {}'v_* = v_* + \frac{1+\alpha}{2\alpha}\big(u\cdot n\big)n\,, \tag{2.4}$$

where of course (2.1) reads now $\big(u\cdot n\big) = -\alpha\big({}'u\cdot n\big)$ with ${}'u = {}'v - {}'v_*$. Notice that the Jacobian of the transformation (2.2) is given by

$$\left|\frac{\partial(v', v'_*)}{\partial(v, v_*)}\right| = \alpha\,.$$

With such a representation, for a given pair of distributions $f = f(v)$ and $g = g(v)$ and a given *collision kernel* $B_0(u, n)$, the Boltzmann collision operator associated to B_0 is defined as the difference of two nonnegative operators (gain and loss operators respectively)

$$\mathcal{Q}_{B_0,\alpha}(g, f) = \mathcal{Q}^+_{B_0,\alpha}(g, f) - \mathcal{Q}^-_{B_0,\alpha}(g, f),$$

with

$$\begin{cases} \mathcal{Q}^+_{B_0,\alpha}\big(g, f\big)(v) &= \dfrac{1}{\alpha}\displaystyle\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B_0({}'u, n) f({}'v) g({}'v_*)\,\mathrm{d}v_*\,\mathrm{d}n\,, \\ \mathcal{Q}^-_{B_0,\alpha}\big(g, f\big)(v) &= f(v)\displaystyle\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B_0(u, n) g(v_*)\,\mathrm{d}v_*\,\mathrm{d}n\,, \qquad {}'u = {}'v - {}'v_*\,. \end{cases} \tag{2.5}$$

We will assume that the collision kernel $B_0 = B_0(z, n)$ for $z \in \mathbb{R}^d \setminus \{0\}$, $n \in \mathbb{S}^{d-1}$ is of the form

$$B_0(z, m) = \Phi\big(|z|\big) b_0\big(\widehat{z}\cdot n\big)\,, \qquad \widehat{z} := \frac{z}{|z|}\,,$$

where $\Phi(\,\cdot\,)$ and $b_0(\,\cdot\,)$ are suitable nonnegative functions known as *kinetic potential* and *angular kernel* respectively where $b_0(\,\cdot\,)$ is an even function. For any fixed vector $\widehat{z}$, the angular kernel defines a measure on the sphere through the mapping $n \in \mathbb{S}^{d-1} \mapsto b_0\big(\widehat{z}\cdot n\big) \in [0, \infty]$. Observing that

$$|{}'u|^2 = \frac{|u|^2}{\alpha^2}\left(\alpha^2 + (1-\alpha^2)(\widehat{u}\cdot n)^2\right), \qquad \big(u\cdot n\big) = -\alpha\big({}'u\cdot n\big)\,,$$

we deduce easily that

$$B_0('u,n) = \Phi\left(\frac{|u|}{\alpha}\sqrt{\alpha^2+(1-\alpha^2)(\widehat{u}\cdot n)^2}\right)\, b_0\left(\frac{\widehat{u}\cdot n}{\sqrt{\alpha^2+(1-\alpha^2)(\widehat{u}\cdot n)^2}}\right).$$

In this representation, the weak form of $\mathcal{Q}_{B_0,\alpha}$ is given by

$$\begin{aligned}\int_{\mathbb{R}^d}\mathcal{Q}_{B_0,\alpha}(f,g)(v)\psi(v)\,\mathrm{d}v &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B_0(u,n)f(v)g(v_*)\left[\psi(v')-\psi(v)\right]\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}n\\ &= \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B_0(u,n)f(v)g(v_*)\left[\psi(v')+\psi(v'_*)-\psi(v)-\psi(v_*)\right]\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}n\end{aligned} \tag{2.6}$$

for any test function $\psi=\psi(v)$.

The above representation captures the main physical features of inelastic interactions but, for mathematical purposes, it is more convenient to adopt an equivalent representation (the so-called σ-representation) by setting, for a given pair of velocities (v,v_*) and for $n\in\mathbb{S}^{d-1}$,

$$\sigma = \widehat{u} - 2\,(\widehat{u}\cdot n)n \in \mathbb{S}^{d-1}\,,$$

where we recall that $\widehat{u}=\frac{u}{|u|}$. Such a description provides an alternative parametrization of the unit sphere $\mathbb{S}^{d-1}$ in which the unit vector σ points in the post-collisional relative velocity direction in the case of elastic collisions. In this case, the impact velocity reads

$$|u\cdot n| = |u|\,|\widehat{u}\cdot n| = |u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\,.$$

In this representation, the post-collisional velocities (v',v'_*) are given by

$$v' = v + \frac{1+\alpha}{4}\left(|u|\sigma-u\right), \qquad v'_* = v_* - \frac{1+\alpha}{4}\left(|u|\sigma-u\right). \tag{2.7}$$

Moreover, using the following formula, valid for any continuous function F (see (Bobylev , 2020, Lemma 2.1.1) for a proof in dimension $d=3$),

$$|z|^{d-2}\int_{\mathbb{S}^{d-1}} F\left(\frac{z-|z|\sigma}{2}\right)\mathrm{d}\sigma = 2^{d-1}\int_{\mathbb{S}^{d-1}}|z\cdot n|^{d-2}F((z\cdot n)n)\,\mathrm{d}n\,, \qquad \forall\, z\in\mathbb{R}^d\setminus\{0\}\,, \tag{2.8}$$

one sees that the above weak form (2.6) translates into

$$\begin{aligned}\int_{\mathbb{R}^d}\mathcal{Q}_{B_0,\alpha}(g,f)(v)\psi(v)\,\mathrm{d}v &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} f(v)g(v_*)\left(\psi(v')-\psi(v)\right)B(u,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &= \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} f(v)g(v_*)\Big(\psi(v')+\psi(v'_*)-\psi(v)-\psi(v_*)\Big)B(u,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\end{aligned} \tag{2.9}$$

for any test function $\psi=\psi(v)$ where

$$B(z,\sigma) = \Phi(|z|)b\left(\widehat{z}\cdot\sigma\right), \qquad \forall\, z\in\mathbb{R}^d\setminus\{0\}\,, \quad \forall\,\sigma\in\mathbb{S}^{d-1}\,, \tag{2.10}$$

and $b(\,\cdot\,)$ is related to $b_0(\,\cdot\,)$ through the relation $b_0\big(\widehat{u}\cdot n\big) = 2^{d-1}|\widehat{u}\cdot n|^{d-2} b(\widehat{u}\cdot\sigma)$, i.e.

$$b(s) = 2^{1-d}\left(\frac{1-s}{2}\right)^{\frac{2-d}{2}} b_0\left(\sqrt{\frac{1-s}{2}}\right), \qquad s\in(-1,1)\,. \tag{2.11}$$

Notice that (2.8) implies in particular that the change of variable $n\in\mathbb{S}^{d-1}\mapsto\sigma=\widehat{u}-2(\widehat{u}\cdot n)n\in\mathbb{S}^{d-1}$ is such that

$$\mathrm{d}\sigma = 2^{d-1}|\widehat{u}\cdot n|^{d-2}\,\mathrm{d}n = 2^{d-1}\left(\frac{1-\widehat{u}\cdot\sigma}{2}\right)^{\frac{d-2}{2}}\,\mathrm{d}n$$

and, using the strong n-representation formula (2.5) together with the expression of $B_0('u,n)$ and (2.11), tedious computations show that the strong form of the Boltzmann operator in this σ-representation is given by $\mathcal{Q}_{B_0,\alpha}=\mathcal{Q}^+_{B_0,\alpha}-\mathcal{Q}^-_{B_0,\alpha}$ where

$$\begin{cases} \mathcal{Q}^+_{B_0,\alpha}(g,f)(v) &= \displaystyle\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} f('v)g('v_*)B^+_\alpha(u,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v_* \\ \mathcal{Q}^-_{B_0,\alpha}(g,f)(v) &= f(v)\displaystyle\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} g(v_*)B(u,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v_* \end{cases} \tag{2.12}$$

where, for $\sigma\in\mathbb{S}^{d-1}$, $'v$ and $'v_*$ denote the pre-collisional velocities

$$'v := \frac{v+v_*}{2} - \frac{1-\alpha}{4\alpha}u + \frac{1+\alpha}{4\alpha}|u|\sigma\,, \qquad 'v_* := \frac{v+v_*}{2} + \frac{1-\alpha}{4\alpha}u - \frac{1+\alpha}{4\alpha}|u|\sigma\,,$$

and

$$B^+_\alpha(u,\sigma) := \frac{1}{\alpha}\left(\frac{2}{1+\alpha^2-(1-\alpha^2)(\widehat{u}\cdot\sigma)}\right)^{\frac{d-2}{2}} \Phi\left(\frac{|u|}{\sqrt{2}\alpha}\sqrt{1+\alpha^2-(1-\alpha^2)(\widehat{u}\cdot\sigma)}\right) b\left(\frac{(1+\alpha^2)(\widehat{u}\cdot\sigma)-(1-\alpha^2)}{1+\alpha^2-(1-\alpha^2)(\widehat{u}\cdot\sigma)}\right).$$

We also refer to Appendix A of Carlen et al. (2009) for derivation of such an expression in dimension $d=3$.

A particularly relevant model is the one of hard-spheres corresponding to $\Phi(|u|)=|u|$ which is the model investigated in the core of the paper and, in that case, we simply denote the collision operator $\mathcal{Q}_{B_0,\alpha}$ by $\mathcal{Q}_\alpha$ omitting the dependence with respect to the variable $b(\,\cdot\,)$ which will always be assumed to satisfy (1.6)–(1.7)–(1.8). In such a case

$$B^+_\alpha(u,\sigma) = \frac{|u|}{\alpha^2}b_\alpha(\widehat{u}\cdot\sigma)$$

$$\text{with}\qquad b_\alpha(s) := \left[\frac{2}{1+\alpha^2-(1-\alpha^2)s}\right]^{\frac{d-3}{2}} b\left(\frac{(1+\alpha^2)s-(1-\alpha^2)}{1+\alpha^2-(1-\alpha^2)s}\right), \qquad \forall\, s\in(-1,1)\,. \tag{2.13}$$

2.2. **Estimates on the difference between inelastic and elastic collisions.** Our main technical contribution in this section is the careful (quantitative) study of the differences between $\mathcal{Q}_\alpha - \mathcal{Q}_1$ and the linearized counterpart $\mathbf{L}_\alpha - \mathbf{L}_1$. We improve here significantly previous estimates obtained in Alonso et al. (2010); Alonso & Gamba (2011); Mischler & Mouhot (2009).

2.2.1. *Estimates on $\mathcal{Q}_\alpha - \mathcal{Q}_1$.* Notice first that in all the sequel, we will need quantitative estimates for the bilinear operator $\mathcal{Q}_\alpha(f,g)$ and $\mathcal{Q}_1(f,g)$ in several different functional spaces. We refer to Alonso et al. (2010); Alonso & Gamba (2011); Mischler & Mouhot (2009) for precise statements (see also Theorem B.1, Lemma B.3 and Corollary B.4). A fundamental role in our analysis will be played by the fact that, in some suitable sense, $\mathcal{Q}_\alpha$ is close to the elastic operator $\mathcal{Q}_1$ for $\alpha \simeq 1$.

Let us begin with the following crucial result which justifies the optimal scaling (1.22) and optimise the rate of convergence previously derived in (Mischler & Mouhot, 2009, Proposition 3.1 (iii)) in a different functional framework than ours. It is worth mentioning that due to the lack of symmetry in the two entries of the Boltzmann collision operator for *inelastic collision*, it is not trivial to show that one can choose the entry on which the additional derivative is carried when estimating $\mathcal{Q}_1(f,g) - \mathcal{Q}_\alpha(f,g)$. In the following lemma, we prove that it is possible in a L^2_v- framework thanks to a Bouchut-Desvillettes estimate (which is available only in the L^2_v-framework).

Lemma 2.1. *Let $a = \max\{d-1, 2\}$ and $q \geqslant 0$. For any $\kappa > \frac{d}{2}$ and any $\alpha \in (0,1]$,*

$$\|\mathcal{Q}_\alpha(f,g) - \mathcal{Q}_1(f,g)\|_{L^2_v(\varpi_q)} + \|\mathcal{Q}_\alpha(g,f) - \mathcal{Q}_1(g,f)\|_{L^2_v(\varpi_q)} \\ \lesssim \frac{1-\alpha}{\alpha^a} \|f\|_{L^2_v(\varpi_{q+\kappa+3})} \|g\|_{\mathbb{W}^{1,2}_v(\varpi_{q+\kappa+3})} . \quad (2.14)$$

As a consequence, for any $\ell > \frac{d}{2}$, we have:

$$\|\mathcal{Q}_\alpha(f,g) - \mathcal{Q}_1(f,g)\|_{L^2_v\mathbb{W}^{\ell,2}_x(\varpi_q)} + \|\mathcal{Q}_\alpha(g,f) - \mathcal{Q}_1(g,f)\|_{L^2_v\mathbb{W}^{\ell,2}_x(\varpi_q)} \\ \lesssim \frac{1-\alpha}{\alpha^a} \|f\|_{L^2_v\mathbb{W}^{\ell,2}_x(\varpi_{q+\kappa+3})} \|g\|_{\mathbb{W}^{1,2}_v\mathbb{W}^{\ell,2}_x(\varpi_{q+\kappa+3})} . \quad (2.15)$$

Proof. Step 1. The first part of the proof is dedicated to the estimate on $\mathcal{Q}_\alpha(g,f) - \mathcal{Q}_1(g,f)$. Since

$$\|\mathcal{Q}_\alpha(g,f) - \mathcal{Q}_1(g,f)\|_{L^2_v(\varpi_q)} = \sup_{\|\varphi\|_{L^2_v(\varpi_q)} \leqslant 1} \int_{\mathbb{R}^d} [\mathcal{Q}_\alpha(g,f) - \mathcal{Q}_1(g,f)]\, \varphi \, \mathrm{d}v$$

one needs to estimate, for some test function $\varphi \in L^2_v(\varpi_q)$ the integral

$$I = \int_{\mathbb{R}^d} \big[\mathcal{Q}_\alpha(g,f)(v) - \mathcal{Q}_1(g,f)(v)\big] \varphi(v) \, \mathrm{d}v$$

by some suitable L^2_v norms of φ, f, g and ∇g, the function g being the *only one* to carry the additional derivative. Using the representation (2.9)-(2.7) in the hard-spheres case, one sees that

$$I = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} g(v-u)f(v)|u|\times \\ \times\left(\varphi\left(v+\frac{1+\alpha}{4}\left(|u|\sigma-u\right)\right)-\varphi\left(v+\frac{1}{2}\left(|u|\sigma-u\right)\right)\right) b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}u\,\mathrm{d}v\,.$$

Setting now $w=\frac{1+\alpha}{2}u$, one can split $I=I_1+I_2$ with

$$I_1=\left[\left(\frac{2}{1+\alpha}\right)^{d+1}-1\right]\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} g\left(v-\frac{2}{1+\alpha}u\right) f(v)\varphi(v_1')\,|u|\,b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}u\,\mathrm{d}v\,,$$

$$I_2=\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}}\left(g\left(v-\frac{2}{1+\alpha}u\right)-g(v-u)\right) f(v)\varphi(v_1')\,|u|\,b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}u\,\mathrm{d}v\,.$$

For the first term, thanks to the mean-value theorem, one notices that

$$\left|\left(\frac{2}{1+\alpha}\right)^{d+1}-1\right|\lesssim 1-\alpha\,,\qquad \forall\,\alpha\in(0,1)$$

so that

$$|I_1|\lesssim(1-\alpha)\int_{\mathbb{R}^d\times\mathbb{R}^d\times\times\mathbb{S}^{d-1}}\left|g\left(v-\frac{2}{1+\alpha}u\right)\right||f(v)||\varphi(v_1')|\,|u|\,b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}u\,\mathrm{d}v$$

and, performing back the change of variable $w=\frac{1+\alpha}{2}u$ one sees that

$$|I_1|\lesssim(1-\alpha)\left(\frac{1+\alpha}{2}\right)^{d+1}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}}|g\left(v-u\right)|\,|f(v)||\varphi(v_\alpha')|\,|u|\,b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}u\,\mathrm{d}v \\ \lesssim(1-\alpha)\int_{\mathbb{R}^d}\mathcal{Q}^+_\alpha(|g|,|f|)(v)|\varphi(v)|\,\mathrm{d}v\,.$$

Then, thanks to classical Boltzmann estimates (see Theorem B.1), it holds that

$$\left|I_1\right|\lesssim(1-\alpha)\Big(\|g\|_{L^1_v(\varpi_{q+1})}\|f\|_{L^2_v(\varpi_{q+1})}+\|g\|_{L^2_v(\varpi_{q+1})}\|f\|_{L^1_v(\varpi_{q+1})}\Big)\|\varphi\|_{L^2_v(\varpi_q)}$$

Since $\|\langle\,\cdot\,\rangle^{-2\kappa}\|_{L^1(\mathbb{R}^d)}<\infty$ for $\kappa>\frac{d}{2}$, one can estimate the L^1-norms with L^2 ones as

$$|I_1|\lesssim\Big(\|g\|_{L^2_v(\varpi_{q+\kappa+1})}\|f\|_{L^2_v(\varpi_{q+1})}+\|g\|_{L^2_v(\varpi_{q+\kappa+1})}\|f\|_{L^2_v(\varpi_{q+\kappa+1})}\Big)\|\varphi\|_{L^2_v(\varpi_q)}\tag{2.16}$$

for $\kappa>\frac{d}{2}$.

For the second term one uses Taylor formula:

$$g\left(v-\frac{2}{1+\alpha}u\right)-g(v-u)=-\frac{1-\alpha}{1+\alpha}\,u\cdot\int_0^1\nabla g(\tilde{w}_t)\,\mathrm{d}t$$

where

$$\tilde{w}_t=v-\left(\frac{2}{1+\alpha}-\frac{1-\alpha}{1+\alpha}\,t\right)u\,,\qquad\forall\,t\in(0,1)\,.\tag{2.17}$$

Thus,

$$\big|I_2\big| \leqslant (1-\alpha)\int_0^1 \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \Big|\nabla g\Big(v-\Big(\frac{2}{1+\alpha}-\frac{1-\alpha}{1+\alpha}\,t\Big)\,u\Big)\Big| \\ \big|f(v)\,\varphi(v_1')\big|\,|u|^2\,b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}u\,\mathrm{d}v\,.$$

Since $\frac{2}{1+\alpha}-\frac{1-\alpha}{1+\alpha}\,t \geqslant 1$, it holds that, thanks to classical Boltzmann estimates (see again Theorem B.1),

$$\big|I_2\big| \lesssim (1-\alpha)\Big(\|\nabla g\|_{L^1_v(\varpi_{q+2})}\|f\|_{L^2_v(\varpi_{q+2})} + \|\nabla g\|_{L^2_v(\varpi_{q+2})}\|f\|_{L^1_v(\varpi_{q+2})}\Big)\|\varphi\|_{L^2_v(\varpi_q)}\,.$$

Combining the estimates for I_1 and I_2, we deduce that

$$\|\mathcal{Q}_1(g,f)-\mathcal{Q}_\alpha(g,f)\|_{L^2_v(\varpi_q)} \lesssim (1-\alpha)\|g\|_{\mathbb{W}^{1,2}_v(\varpi_{q+\kappa+2})}\|f\|_{L^2_v(\varpi_{q+\kappa+2})}\,, \qquad \kappa>\frac{d}{2}\,.$$

Step 2. We now deal with $\mathcal{Q}_\alpha(f,g)-\mathcal{Q}_1(f,g)$ and as already mentioned, it is not trivial to show that we can still impose that g is carrying the additional derivative. To prove such an estimate in our $L^2_v(\varpi_q)$ framework, we shall resort to a method reminiscent to Bouchut-Desvillettes estimates and use the weak σ-representation. We need to estimate

$$\begin{aligned} J &= \int_{\mathbb{R}^d}\big[\mathcal{Q}_\alpha(f,g)(v)-\mathcal{Q}_1(f,g)(v)\big]\varphi(v)\,\mathrm{d}v \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} f(v-u)g(v)|u|\times \\ &\qquad \times\left(\varphi\left(v+\frac{1+\alpha}{4}\,(|u|\sigma-u)\right)-\varphi\left(v+\frac{1}{2}\,(|u|\sigma-u)\right)\right)b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}u\,\mathrm{d}v\,. \end{aligned}$$

Here again, thanks to the change of variable $w=\frac{1+\alpha}{2}u$, one splits $J=J_1+J_2$ where, as before (notice that the estimate (2.16) for I_1 does not involve derivatives), one has

$$|J_1| \lesssim \Big(\|f\|_{L^2_v(\varpi_{q+\kappa+1})}\|g\|_{L^2_v(\varpi_{q+1})} + \|f\|_{L^2_v(\varpi_{q+\kappa+1})}\|g\|_{L^2_v(\varpi_{q+\kappa+1})}\Big)\|\varphi\|_{L^2_v(\varpi_q)}$$

whereas

$$J_2 = \int_{\mathbb{R}^d\times\mathbb{R}^d\times S^{d-1}}\left(f\left(v-\frac{2}{1+\alpha}u\right)-f(v-u)\right)g(v)\varphi(v_1')\,|u|\,b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}u\,\mathrm{d}v\,.$$

Again, we use Taylor formula as before to get

$$J_2 = -\frac{1-\alpha}{1+\alpha}\int_0^1 \mathrm{d}t\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} u\cdot\nabla f(\tilde{w}_t)\,g(v)\,\varphi(v_1')\,|u|\,b(\widehat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}u\,\mathrm{d}v$$

where $\tilde{w}_t$ has been defined in (2.17). Integrating by parts in the variable v, we deduce that

$$J_2 = \frac{1-\alpha}{1+\alpha}\int_0^1 \mathrm{d}t\int_{\mathbb{S}^{d-1}}\mathrm{d}\sigma\int_{\mathbb{R}^d}\mathrm{d}v\int_{\mathbb{R}^d} f(\tilde{w}_t)\nabla_v\cdot\big(u|u|g(v)\varphi(v_1')b(\widehat{u}\cdot\sigma)\big)\,\mathrm{d}u$$

i.e.

$$J_2 = \frac{1-\alpha}{1+\alpha}\int_0^1 \mathrm{d}t \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma \int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathbb{R}^d} f(\tilde{w}_t)\,(u\cdot\nabla_v g(v))\,|u|b(\widehat{u}\cdot\sigma)\,\mathrm{d}u$$
$$+\frac{1-\alpha}{1+\alpha}\int_0^1 \mathrm{d}t \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma \int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathbb{R}^d} f(\tilde{w}_t)g(v)\left(u\cdot\nabla_v\varphi(v_1')\right)|u|b(\widehat{u}\cdot\sigma)\,\mathrm{d}u = J_2^1 + J_2^2\,.$$

Clearly, as for I_2, one has

$$\left|J_2^1\right| \lesssim (1-\alpha)\Big(\|\nabla_v g\|_{L^1_v(\varpi_{q+2})}\|f\|_{L^2_v(\varpi_{q+2})} + \|\nabla_v g\|_{L^2_v(\varpi_{q+2})}\|f\|_{L^1_v(\varpi_{q+2})}\Big)\|\varphi\|_{L^2(\varpi_q)}\,.$$

For the second term, one performs, for a fixed $t\in(0,1)$, the change of variable

$$u \mapsto u_* = \beta_t u\,, \qquad \beta_t := \frac{2}{1+\alpha} - \frac{1-\alpha}{1+\alpha}t$$

from which $v_1' = v - \frac{1}{2\beta_t}\left(|u_*|\sigma - u_*\right)$ and

$$J_2^2 = \frac{1-\alpha}{1+\alpha}\int_0^1 \beta_t^{-d-2}\,\mathrm{d}t \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma$$
$$\int_{\mathbb{R}^d}\mathrm{d}v\int_{\mathbb{R}^d} f(v-u_*)g(v)\left(u_*\cdot\nabla_v\varphi\left(v-\frac{1}{2\beta_t}\left(|u_*|\sigma-u_*\right)\right)\right)|u_*|b(\widehat{u_*}\cdot\sigma)\,\mathrm{d}u_*\,.$$

We introduce $e_t\in(0,1)$ such that

$$\frac{1}{2\beta_t} = \frac{1+e_t}{4} \qquad \text{i.e.} \quad e_t := \frac{2\alpha+(1-\alpha)t}{2-(1-\alpha)t}\,, \qquad e_t\in(0,1)$$

and therefore

$$J_2^2 = \frac{1-\alpha}{1+\alpha}\int_0^1 \beta_t^{-d-2}\,\mathrm{d}t \int_{\mathbb{S}^{d-1}}\mathrm{d}\sigma\int_{\mathbb{R}^d}\mathrm{d}v\int_{\mathbb{R}^d} f(v-u)g(v)\left(u\cdot\nabla_v\varphi\left(v_{e_t}'\right)\right)|u|b(\widehat{u}\cdot\sigma)\,\mathrm{d}u$$
$$= \frac{1-\alpha}{1+\alpha}\int_0^1 \beta_t^{-d-2}\,\mathrm{d}t\int_{\mathbb{R}^d}\mathrm{d}v\,\mathcal{Q}^+_{B,e_t}(f,g)\cdot\nabla_v\varphi$$

where, for any $t\in(0,1)$, $\mathcal{Q}_{B,e_t}$ is the (vector valued) Boltzmann collision operator associated with the restitution coefficient e_t and kernel $B(u,\sigma) := u\,|u|\,b(\widehat{u}\cdot\sigma)$. Thus, using an additional integration by parts,

$$J_2^2 = -\frac{1-\alpha}{1+\alpha}\int_0^1 \beta_t^{-d-2}\,\mathrm{d}t\int_{\mathbb{R}^d}\left(\nabla_v\cdot\mathcal{Q}^+_{B,e_t}(f,g)\right)\varphi\,\mathrm{d}v\,.$$

Now, from Bouchut-Desvillettes estimates for $\mathcal{Q}^+_{B,e_t}$, see Bouchut & Desvillettes (1998) and (Mischler & Mouhot, 2006, Theorem 2.5) for a proof in the inelastic case [3], one has, for any

[3]Notice here we need simply to adapt the proof of (Mischler & Mouhot, 2006, Theorem 2.5) with, at the beginning of the proof $F(v,v_*) = f(v)g(v_*)(v-v_*)|v-v_*|$ instead of $F(v,v_*) = f(v)g(v_*)|v-v_*|$ which explains the additional moments ϖ_{q+3} instead of ϖ_{q+2} as in the *op.cit.*

$t \in (0,1)$,

$$\|\mathcal{Q}^+_{B,e_t}(f,g)\|_{\mathbb{W}^{\frac{d-1}{2},2}_v(\varpi_q)} \lesssim \|g\|_{L^2_v(\varpi_{q+3})}\|f\|_{L^2_v(\varpi_{q+3})} + \|g\|_{L^1_v(\varpi_{q+3})}\|f\|_{L^1_v(\varpi_{q+3})}$$

where the multiplicative constant depends on $\|b\|_{L^2(\mathbb{S}^{d-1})}$ but not on the restitution coefficient and, thus, not on t. Since $d \geqslant 3$, this implies that

$$\|\nabla_v \cdot \mathcal{Q}^+_{B,e_t}(f,g)\|_{L^2_v(\varpi_q)} \lesssim \|g\|_{L^2_v(\varpi_{q+3})}\|f\|_{L^2_v(\varpi_{q+3})} + \|g\|_{L^1_v(\varpi_{q+3})}\|f\|_{L^1_v(\varpi_{q+3})}$$

and, since $\sup_{t\in(0,1)} \beta_t^{-1} \leqslant 1$, we deduce that

$$J_2^2 \lesssim (1-\alpha)\|\varphi\|_{L^2_v(\varpi_q)} \left[\|g\|_{L^2_v(\varpi_{q+3})}\|f\|_{L^2_v(\varpi_{q+3})} + \|g\|_{L^1_v(\varpi_{q+3})}\|f\|_{L^1_v(\varpi_{q+3})}\right]$$

which, controlling the L^1 norms with L^2 ones (with higher moments) as in the first part, yields

$$J \lesssim (1-\alpha)\|\varphi\|_{L^2_v(\varpi_q)} \, \|g\|_{\mathbb{W}^{1,2}_v(\varpi_{q+\kappa+3})} \, \|f\|_{L^2_v(\varpi_{q+\kappa+3})}$$

which proves the desired estimate (2.14).

Step 3. The estimate (2.15) can be deduced easily from (2.14) using Fubini theorem and the fact that $\mathbb{W}^{\ell,2}_x$ is an algebra since $\ell > \frac{d}{2}$. □

In our analysis, we will also need an estimate on the difference between $\mathbf{L}_\alpha$ and $\mathbf{L}_1$ (see Lemma 2.5) with no loss of regularity and with a "minimal" loss in weight (i.e. an estimate in the *graph norm*), even if the rate is not anymore optimal (notice that this type of estimate can obviously not being deduced from the previous lemma). To this end, we here state a lemma which is in the spirit of (Mischler & Mouhot, 2009, Proposition 3.2) except from the fact that one of the argument is fixed to be the Maxwellian $\mathcal{M}$. Note that (Mischler & Mouhot, 2009, Proposition 3.2) gives an estimate of this type on the difference between $\mathcal{Q}_\alpha$ and $\mathcal{Q}_1$ for general arguments but the proof heavily relies on the exponential weights they consider. It turns out that we can not adapt easily the proof of (Mischler & Mouhot, 2009, Proposition 3.2) for polynomial weights. However, by using decay properties of $\mathcal{M}$, we are able, to get an estimate on $\mathcal{Q}_\alpha(\mathcal{M},\,\cdot\,) - \mathcal{Q}_1(\mathcal{M},\,\cdot\,)$ and its symmetric, which is enough for our purpose. The proof is reminiscent of the proof of (Mischler & Mouhot, 2009, Proposition 3.2) and (Alonso & Lods , 2013a, Theorem 3.11) but we face some additional difficulties due to the polynomial weight.

Lemma 2.2. *Let $q \geqslant 0$. There exist some explicit $p \in (0,1)$ and $\alpha_1 \in (0,1)$ such that*

$$\begin{aligned}&\|\mathcal{Q}_\alpha(\mathcal{M},f) - \mathcal{Q}_1(\mathcal{M},f)\|_{L^1_v(\varpi_q)}\\ &\quad + \|\mathcal{Q}_\alpha(f,\mathcal{M}) - \mathcal{Q}_1(f,\mathcal{M})\|_{L^1_v(\varpi_q)} \lesssim (1-\alpha)^p \, \|f\|_{L^1_v(\varpi_{q+1})}\,, \qquad \alpha \in [\alpha_1,1]\,.\end{aligned}$$

Proof. We only prove the first estimate on $\mathcal{Q}_1(\,\cdot\,,\mathcal{M}) - \mathcal{Q}_\alpha(\,\cdot\,,\mathcal{M})$, the other one can be treated exactly in the same way. Notice first that

$$\mathcal{Q}_\alpha(f,\mathcal{M}) - \mathcal{Q}_1(f,\mathcal{M}) = \mathcal{Q}^+_\alpha(f,\mathcal{M}) - \mathcal{Q}^+_1(f,\mathcal{M})\,.$$

As in the proofs of (Mischler & Mouhot, 2009, Proposition 3.2) and (Alonso & Lods , 2013a, Theorem 3.11), we set $w := v + v_*$ and $\widehat{w} := w/|w|$ and define $\chi \in [0, \pi/2]$ through $|\cos\chi| := |\widehat{w}\cdot\sigma|$. Let $\delta \in (0,1)$ and $R > 1$ be fixed and let $\eta_\delta \in \mathbb{W}^{1,\infty}(-1,1)$ such that $\eta_\delta(s) = \eta_\delta(-s)$ for any $s \in (0,1)$ and

$$\eta_\delta(s) = \begin{cases} 1 & \text{if} \quad s \in (-1+2\delta, 1-2\delta) \\ 0 & \text{if} \quad s \notin (-1+\delta, 1-\delta) \end{cases}$$

with moreover

$$0 \leqslant \eta_\delta(s) \leqslant 1 \qquad \text{and} \qquad |\eta_\delta'(s)| \leqslant \frac{3}{\delta}\,, \qquad \forall\, s \in (-1,1)\,.$$

Let us define also $\Theta_R(r) = \Theta(r/R)$ with $\Theta(x) = 1$ on $[0,1]$, $\Theta(x) = 1-x$ for $x \in [1,2]$ and $\Theta(x) = 0$ on $[2,\infty)$. We define the set

$$A(\delta) := \{\sigma \in \mathbb{S}^{d-1};\ \sin^2\chi \geqslant \delta\}$$

we split $\mathcal{Q}_\alpha^+$ into

$$\mathcal{Q}_\alpha^+ = \mathcal{Q}_\alpha^{+,r} + \mathcal{Q}_\alpha^{+,l} + \mathcal{Q}_\alpha^{+,a}$$

where the collision operators $\mathcal{Q}_\alpha^{+,i}$ for $i = l, r, a$, are defined in weak form by (2.9) with associated collision kernels defined as

$$B_r(u,\sigma) := \eta_\delta(\widehat{u}\cdot\sigma\,)\,\Theta_R(u)\,|u|b(\widehat{u}\cdot\sigma), \quad B_l(u,\sigma) := \mathbf{1}_{A(\delta)}(\sigma)\,(1-\Theta_R(|u|))\,|u|b(\widehat{u}\cdot\sigma)$$

and

$$B_a(u,\sigma) := |u|b(\widehat{u}\cdot\sigma)\,\left((1-\eta_\delta(\widehat{u}\cdot\sigma\,))\,\Theta_R(|u|) + (1-\Theta_R(|u|))\,\mathbf{1}_{A^c(\delta)}\right)\,.$$

This splitting corresponds to a splitting for *small angles* (corresponding to the kernel B_a), *large velocities* (corresponding to B_l) and the reminder term (corresponding to B_r). The treatment of small angles and of the truncated operator is similar to the one of (Mischler & Mouhot, 2009, Proposition 3.2) and we only recall the results obtained therein:

$$\|\mathcal{Q}_\alpha^{+,a}(f,\mathcal{M})\|_{L^1_v(\varpi_q)} \lesssim \delta\|f\|_{L^1_v(\varpi_{q+1})}\,, \qquad \alpha \in (0,1]\,, \quad R \in (1,\infty)$$

and

$$\|\mathcal{Q}_\alpha^{+,r}(f,\mathcal{M}) - \mathcal{Q}_1^{+,r}(f,\mathcal{M})\|_{L^1_v(\varpi_q)} \lesssim (1-\alpha)\left(\frac{R^2}{\delta} + \frac{R}{\delta^3}\right)\|f\|_{L^1_v(\varpi_q)}\,.$$

Let us now handle the case of large relative velocities. To this end, we estimate $\|\mathcal{Q}_\alpha^{+,l}(f,\mathcal{M})\|_{L^1_v(\varpi_q)}$ by duality and thanks to (2.9)

$$\|\mathcal{Q}_\alpha^{+,l}(f,\mathcal{M})\|_{L^1_v(\varpi_q)} = \sup_{\|\varphi\|_\infty \leqslant 1} \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B_l(u,\sigma)\mathcal{M}(v)f(v_*)\varphi(v')\varpi_q(v')\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\,.$$

Choosing, $\varphi = \varphi(v) \in L^\infty(\mathbb{R}^d)$, one writes

$$I_\varphi := \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B_l(u,\sigma)\mathcal{M}(v)f(v_*)\varphi(v')\varpi_q(v')\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma$$
$$= \int_{\mathbb{R}^d\times\mathbb{R}^d} f(v_*)\left(1-\Theta_R(|u|)\right)|u|\mathcal{M}(v)\,\mathrm{d}v\,\mathrm{d}v_* \int_{A(\delta)} b(\widehat{u}\cdot\sigma)\varphi(v')\varpi_q(v')\,\mathrm{d}\sigma$$

where, by definition of Θ_R,

$$\left(1-\Theta_R(|u|)\right)|u| \leqslant \frac{|u|^2}{2R} \leqslant \frac{|u'|^2}{2\alpha^2 R} \lesssim \frac{1}{\alpha^2 R}\langle v'\rangle^2\langle v'_*\rangle^2$$

noticing that $|u'|^2 = \frac{|u|^2}{2}\left(1+\alpha^2+(1-\alpha^2)\widehat{u}\cdot\sigma\right) \geqslant \alpha^2|u|^2$. Therefore,

$$|I_\varphi| \lesssim \frac{\|\varphi\|_\infty}{\alpha^2 R}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} |f(v_*)|\mathbf{1}_{A(\delta)}(\sigma)b(\widehat{u}\cdot\sigma)\varpi_{q+2}(v')\langle v'_*\rangle^2\mathcal{M}(v)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\,. \tag{2.18}$$

For $\sigma \in A(\delta)$ and $\delta \geqslant 8\frac{1-\alpha}{1+\alpha}$, one has

$$|v|^2 \geqslant \frac{\delta}{16}(|v'|^2+|v'_*|^2)\,. \tag{2.19}$$

It is somehow easier to see this using the pre-collisional velocities. Indeed, observe first that the set $A(\delta)$ is invariant under the change of pre-post collisional change of variables and remark that

$${}'v = v - \frac{1+\alpha}{4\alpha}(u-|u|\sigma) = \frac{w}{2} - \frac{1}{2}\left[\frac{1-\alpha}{2\alpha}u - \frac{1+\alpha}{2\alpha}|u|\sigma\right].$$

Using now that, when $\sin^2\chi \geqslant \delta$, then $|\widehat{w}\cdot\sigma| \leqslant \sqrt{1-\delta} \leqslant 1-\frac{\delta}{2}$, we have

$$|{}'v|^2 \geqslant \frac{1}{4}(|w|^2+|u|^2) - \frac{1+\alpha}{4\alpha}|w||u|\left(1-\frac{\delta}{2}\right) - \frac{1-\alpha}{4\alpha}(|v|^2-|v_*|^2)$$
$$\geqslant |v|^2\left(\frac{1}{2} - \frac{1+\alpha}{4\alpha}\left(1-\frac{\delta}{2}\right) - \frac{1-\alpha}{4\alpha}\right) + |v_*|^2\left(\frac{1}{2} - \frac{1+\alpha}{4\alpha}\left(1-\frac{\delta}{2}\right) + \frac{1-\alpha}{4\alpha}\right)$$
$$\geqslant \delta\frac{1+\alpha}{8\alpha}|v_*|^2 + \left(\delta\frac{1+\alpha}{8\alpha} - \frac{1-\alpha}{2\alpha}\right)|v|^2.$$

Then, if $\delta \geqslant 8\frac{1-\alpha}{1+\alpha}$, we get:

$$|{}'v|^2 \geqslant \delta\frac{1+\alpha}{16\alpha}(|v|^2+|v_*|^2) \geqslant \frac{\delta}{16}(|v|^2+|v_*|^2)\,,$$

which is of course equivalent to (2.19). As a consequence, since $\mathcal{M}(v) \lesssim \exp\left(-\frac{|v|^2}{4\vartheta_1}\right)\mathcal{M}^{\frac12}(v)$, one deduces that

$$\mathcal{M}(v)\langle v'_*\rangle^2\varpi_{q+2}(v') \lesssim e^{-\delta\frac{|v'|^2}{64\vartheta_1}}\varpi_{q+2}(v')e^{-\delta\frac{|v'_*|^2}{64\vartheta_1}}\langle v'_*\rangle^2\mathcal{M}^{\frac12}(v) \lesssim \frac{1}{\delta^{\frac{q+4}{2}}}\mathcal{M}^{\frac12}(v)\,. \tag{2.20}$$

Combining this with (2.18) we deduce that, for $\delta \geqslant 8\frac{1-\alpha}{1+\alpha}$,

$$|I_\varphi| \lesssim \frac{\|\varphi\|_\infty}{\alpha^2 R \delta^{\frac{q+4}{2}}} \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} |f(v_*)| \mathcal{M}^{\frac12}(v) b(\widehat{u}\cdot\sigma) \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma$$

which of course results in

$$\|\mathcal{Q}_\alpha^{+,l}(f,\mathcal{M})\|_{L^1_v(\varpi_q)} = \sup_{\|\varphi\|_\infty\leqslant 1} I_\varphi \lesssim \frac{1}{\alpha^2 R\, \delta^{\frac{q+4}{2}}} \|f\|_{L^1_v}$$

since $b \in L^1(\mathbb{S}^{d-1})$. Gathering the previous estimates, we obtain if $\delta \geqslant 8\frac{1-\alpha}{1+\alpha}$ and α far away from zero:

$$\|\mathcal{Q}_\alpha(f,\mathcal{M}) - \mathcal{Q}_1(f,\mathcal{M})\|_{L^1_v(\varpi_q)} \lesssim \left(\delta + (1-\alpha)\left(\frac{R^2}{\delta} + \frac{R}{\delta^3}\right) + \frac{1}{R\delta^{\frac{q+4}{2}}}\right) \|f\|_{L^1_v(\varpi_{q+1})}\,.$$

Picking now $\delta = (1-\alpha)^p$ for some $p \in (0,1)$ (so that the condition $\delta \geqslant 8\frac{1-\alpha}{1+\alpha}$ will be satisfied for α close enough to 1) and $R = (1-\alpha)^{-p-\frac{q+4}{2}p} = (1-\alpha)^{-p\frac{q+6}{2}}$, we then obtain for α close enough to 1 that

$$\begin{aligned}&\|\mathcal{Q}_\alpha(f,\mathcal{M}) - \mathcal{Q}_1(f,\mathcal{M})\|_{L^1_v(\varpi_q)}\\ &\quad\lesssim \left((1-\alpha)^p + (1-\alpha)^{1-p(q+7)} + (1-\alpha)^{1-p\frac{q+12}{2}}\right)\|f\|_{L^1_v(\varpi_{q+1})}\,,\end{aligned}$$

and, with $p = \frac{1}{q+8}$, $1-p(q+7) = p$ and

$$\begin{aligned}\|\mathcal{Q}_\alpha(f,\mathcal{M}) - \mathcal{Q}_1(f,\mathcal{M})\|_{L^1_v(\varpi_q)} &\lesssim \left((1-\alpha)^p + (1-\alpha)^{\frac{q+4}{2}p}\right)\|f\|_{L^1_v(\varpi_{q+1})}\\ &\lesssim (1-\alpha)^p \|f\|_{L^1_v(\varpi_{q+1})}\,.\end{aligned}$$

This proves the result. □

Estimates on $G_\alpha - \mathcal{M}$. We now investigate the rate of convergence of the equilibrium G_α towards $\mathcal{M}$ as α goes to 1. An optimal convergence rate in L^1-spaces is given in (Mischler & Mouhot, 2009, Step 2, proof of Lemma 4.4): there exists $\alpha_2 > 0$ such that

$$\|G_\alpha - \mathcal{M}\|_{L^1_v(\langle\cdot\rangle m)} \lesssim (1-\alpha)\,, \qquad \alpha \in [\alpha_2, 1]\,, \tag{2.21}$$

for $m(v) = \exp(a\,|v|)$, $a > 0$ small enough. We need to extend this optimal rate of convergence to the Sobolev spaces $L^j_v(\varpi_q)$ for $j = 1, 2$ we shall consider.

Lemma 2.3. *Let $q \geqslant 0$ be given. There exists some explicit $\alpha_3 \in (0,1)$ such that*

$$\|G_\alpha - \mathcal{M}\|_{L^1_v(\varpi_q)} + \|G_\alpha - \mathcal{M}\|_{L^2_v(\varpi_{q+1})} \lesssim (1-\alpha)\,, \qquad \alpha \in [\alpha_3, 1]\,.$$

Proof. We start with the L^2_v-case. To this end, we slightly modify here a strategy adopted in Alonso & Lods (2013a) which consists in combining a *nonlinear* estimate for $\|G_\alpha - \mathcal{M}\|_{L^2_v(\varpi_q)}$ together with non-quantitative convergence. We fix $q \geqslant 0$, $\kappa > \frac{d}{2}$ and we divide the proof into three steps:

Step 1: non quantitative convergence. Let us prove that

$$\|G_\alpha - \mathcal{M}\|_{L^2_v(\varpi_{q+\kappa+1})} \xrightarrow[\alpha\to 1]{} 0\,. \tag{2.22}$$

We argue here as in (Alonso & Lods , 2013a, Theorem 4.1). We sketch only the main steps. First, from (Mischler & Mouhot, 2009, Proposition 2.1), there is $\overline{\alpha} > 0$ such that for any $k \geqslant 0$ and any $r \geqslant 0$,

$$\sup_{\alpha\in(\overline{\alpha},1)} \|G_\alpha\|_{\mathbb{W}^{k,2}_v(\varpi_r)} < \infty\,.$$

Then, there is a sequence $(\alpha_n)_n$ converging to 1 such that $(G_{\alpha_n})_n$ converges weakly in the space $L^2_v(\varpi_{q+\kappa+1})$ to some limit $\bar{G}$ (notice that, *a priori*, the limit function $\bar{G}$ depends on the choice of q). Using the pointwise decay of $(G_\alpha)_\alpha$ and compact embedding for Sobolev spaces, this convergence is actually strong, i.e. $\lim_n \|G_{\alpha_n} - \bar{G}\|_{L^2_v(\varpi_{q+\kappa+1})} = 0$. According to (2.21), one necessarily has $\bar{G} = \mathcal{M}$ and one deduces easily that whole net $(G_\alpha)_\alpha$ is converging to $\mathcal{M}$. This proves (2.22).

Step 2: nonlinear estimate. We first consider the Maxwellian $\mathcal{M}_\alpha$ with same mass, momentum and energy of G_α and we consider the linearized elastic collision operator around that Maxwellian

$$\mathbf{L} := \mathcal{Q}_1(\,\cdot\,,\mathcal{M}_\alpha) + \mathcal{Q}_1(\mathcal{M}_\alpha,\,\cdot\,)\,.$$

One simply notices that, since $\mathcal{Q}_1(\mathcal{M}_\alpha,\mathcal{M}_\alpha) = 0$,

$$\begin{aligned}\mathbf{L}(G_\alpha) &= \mathcal{Q}_1(G_\alpha - \mathcal{M}_\alpha, \mathcal{M}_\alpha - G_\alpha) + \mathcal{Q}_\alpha(G_\alpha,G_\alpha) + \Big[\mathcal{Q}_1(G_\alpha,G_\alpha) - \mathcal{Q}_\alpha(G_\alpha,G_\alpha)\Big]\\ &= \mathcal{Q}_1(G_\alpha - \mathcal{M}_\alpha, \mathcal{M}_\alpha - G_\alpha) - (1-\alpha)\nabla_v\cdot(vG_\alpha) + \Big[\mathcal{Q}_1(G_\alpha,G_\alpha) - \mathcal{Q}_\alpha(G_\alpha,G_\alpha)\Big]\,.\end{aligned}$$

Therefore, using classical estimates for $\mathcal{Q}_1$ (see Theorem B.1 and Lemma B.3) and considering $\kappa > \frac{d}{2}$, $\alpha > \bar{\alpha}$,

$$\begin{aligned}\|\mathbf{L}(G_\alpha)\|_{L^2_v(\varpi_q)} &\leqslant \|\mathcal{Q}_1(G_\alpha - \mathcal{M}_\alpha, \mathcal{M}_\alpha - G_\alpha)\|_{L^2_v(\varpi_q)}\\ &\quad + (1-\alpha)\,\|G_\alpha\|_{\mathbb{W}^{1,2}_v(\varpi_{q+1})} + \|\mathcal{Q}_1(G_\alpha,G_\alpha) - \mathcal{Q}_\alpha(G_\alpha,G_\alpha)\|_{L^2_v(\varpi_q)}\\ &\lesssim \|G_\alpha - \mathcal{M}_\alpha\|_{L^2_v(\varpi_{q+1})}\|G_\alpha - \mathcal{M}_\alpha\|_{L^2_v(\varpi_{q+\kappa+1})} + (1-\alpha)\|G_\alpha\|_{\mathbb{W}^{1,2}_v(\varpi_{q+1})}\\ &\quad + (1-\alpha)\,\|G_\alpha\|_{L^2_v(\varpi_{q+\kappa+3})}\|G_\alpha\|_{\mathbb{W}^{1,2}_v(\varpi_{q+\kappa+3})}\end{aligned}$$

where we used Lemma 2.1 for estimating the difference $\mathcal{Q}_1(G_\alpha,G_\alpha) - \mathcal{Q}_\alpha(G_\alpha,G_\alpha)$. Since $\sup_\alpha \|G_\alpha\|_{\mathbb{W}^{1,2}_v(\varpi_{q+\kappa+3})} < \infty$, we obtain that there is a positive constant $C > 0$ such that

$$\|\mathbf{L}(G_\alpha)\|_{L^2_v(\varpi_q)} \leqslant C(1-\alpha) + C\|G_\alpha - \mathcal{M}_\alpha\|_{L^2_v(\varpi_{q+1})}\|G_\alpha - \mathcal{M}_\alpha\|_{L^2_v(\varpi_{q+\kappa+1})}\,.$$

We can write $\mathbf{L}(G_\alpha) = \mathbf{L}(G_\alpha - \mathcal{M}_\alpha)$ and, as $G_\alpha - \mathcal{M}_\alpha$ has zero mass, momentum and energy, there is a positive constant $c > 0$ (that can be taken independent of α) such that

$$\|\mathbf{L}(G_\alpha - \mathcal{M}_\alpha)\|_{L^2_v(\varpi_q)} \geqslant c\|G_\alpha - \mathcal{M}_\alpha\|_{L^2_v(\varpi_{q+1})}\,.$$

The constant $c>0$ is actually the norm of the inverse of $\mathbf{L}$ on the subspace of functions with zero mass, momentum and energy; recall that this inverse maps $L^2(\boldsymbol{\varpi}_q)$ into $\mathscr{D}(\mathbf{L})=L^2_v(\boldsymbol{\varpi}_{q+1})$. Therefore, with $\widetilde{C}:=C/c$

$$\begin{aligned}&\|G_\alpha-\mathcal{M}_\alpha\|_{L^2_v(\boldsymbol{\varpi}_{q+1})}\\&\qquad\leqslant \widetilde{C}(1-\alpha)+\widetilde{C}\|G_\alpha-\mathcal{M}_\alpha\|_{L^2_v(\boldsymbol{\varpi}_{q+1})}\|G_\alpha-\mathcal{M}_\alpha\|_{L^2_v(\boldsymbol{\varpi}_{q+\kappa+1})}\,,\qquad \alpha\in[\overline{\alpha},1]\,.\end{aligned}\tag{2.23}$$

Step 4: conclusion in L^2_v. Setting

$$\vartheta_\alpha:=\frac{1}{d}\int_{\mathbb{R}^d}|v|^2\,\mathcal{M}_\alpha(v)\,\mathrm{d}v=\frac{1}{d}\int_{\mathbb{R}^d}|v|^2\,G_\alpha(v)\,\mathrm{d}v\,,$$

one sees easily from (2.21) that $|\vartheta_1-\vartheta_\alpha|\lesssim 1-\alpha$ and then, one can check without difficulty that

$$\|\mathcal{M}_\alpha-\mathcal{M}\|_{L^2_v(\boldsymbol{\varpi}_{q+\kappa+1})}\lesssim 1-\alpha\,,\qquad \alpha\in[\alpha_2,1]\,.\tag{2.24}$$

Thanks to (2.22), there exists $\alpha_3\in[\max(\alpha_2,\overline{\alpha}),1)$ such that

$$\widetilde{C}\|G_\alpha-\mathcal{M}_\alpha\|_{L^2_v(\boldsymbol{\varpi}_{q+\kappa+1})}\leqslant\frac{1}{2}\,,\qquad \alpha\in[\alpha_3,1]$$

where $\widetilde{C}$ is the positive constant in (2.23). Then, (2.23) reads simply as

$$\|G_\alpha-\mathcal{M}_\alpha\|_{L^2_v(\boldsymbol{\varpi}_{q+1})}\leqslant 2\widetilde{C}(1-\alpha)\,,\qquad \alpha\in[\alpha_3,1]\,,$$

and, using (2.24), we end up with

$$\|G_\alpha-\mathcal{M}\|_{L^2_v(\boldsymbol{\varpi}_{q+1})}\lesssim 1-\alpha\,,\qquad \alpha\in[\alpha_3,1]\,,$$

which gives also a quantitative lower bound on α_3.

Step 5: estimate in L^1_v. The estimate on the $L^1_v(\boldsymbol{\varpi}_q)$-norm of $G_\alpha-\mathcal{M}$ can be obtained straightforwardly by using the previous result in L^2_v-spaces and Cauchy-Schwarz inequality. □

2.2.2. *Estimates on $\mathbf{L}_\alpha-\mathbf{L}_1$.* We first provide an estimate on $\mathbf{L}_\alpha-\mathbf{L}_1$ (where we recall that $\mathbf{L}_\alpha$ is defined in (1.29)) in an L^2_v-framework which comes from Lemmas 2.1 and 2.3.

Lemma 2.4. *Consider $q\geqslant 0$. For any $\kappa>\frac{d}{2}$, there exists some explicit $\alpha_4\in(0,1)$ such that for any $h\in L^2_v(\boldsymbol{\varpi}_{q+\kappa+3})$,*

$$\|\mathbf{L}_\alpha h-\mathbf{L}_1 h\|_{L^2_v(\boldsymbol{\varpi}_q)}\lesssim(1-\alpha)\,\|h\|_{L^2_v(\boldsymbol{\varpi}_{q+\kappa+3})}\,,\qquad \alpha\in[\alpha_4,1]\,.\tag{2.25}$$

As a consequence, for any $\ell\geqslant 0$, there holds

$$\|\mathbf{L}_\alpha h-\mathbf{L}_1 h\|_{L^2_v\mathbb{W}^{\ell,2}_x(\boldsymbol{\varpi}_q)}\lesssim(1-\alpha)\,\|h\|_{L^2_v\mathbb{W}^{\ell,2}_x(\boldsymbol{\varpi}_{q+\kappa+3})}\,,\qquad \alpha\in[\alpha_4,1]\,.\tag{2.26}$$

Proof. Notice first that

$$\begin{aligned}\mathbf{L}_\alpha h-\mathbf{L}_1 h=\mathcal{Q}_\alpha(h,G_\alpha-\mathcal{M})+\mathcal{Q}_\alpha(G_\alpha-\mathcal{M},h)\\+\big[\mathcal{Q}_\alpha(h,\mathcal{M})-\mathcal{Q}_1(h,\mathcal{M})\big]+\big[\mathcal{Q}_\alpha(\mathcal{M},h)-\mathcal{Q}_1(\mathcal{M},h)\big]\,.\end{aligned}\tag{2.27}$$

From Theorem B.1,

$$\|\mathcal{Q}_\alpha^+(g,f)\|_{L^2_v(\varpi_q)} + \|\mathcal{Q}_\alpha^+(f,g)\|_{L^2_v(\varpi_q)} \lesssim \|f\|_{L^1_v(\varpi_{q+1})}\, \|g\|_{L^2_v(\varpi_{q+1})}$$

where we recall that $\mathcal{Q}_\alpha^+$ is the gain part of the operator $\mathcal{Q}_\alpha$. On the other hand, using Lemma B.3, we have that for $\kappa > \frac{d}{2}$,

$$\|\mathcal{Q}_\alpha^-(g,f)\|_{L^2_v(\varpi_q)} \lesssim \|f\|_{L^2_v(\varpi_{q+1})} \|g\|_{L^2_v(\varpi_{\kappa+1})}$$

where $\mathcal{Q}_\alpha^-$ is the loss part of the operator $\mathcal{Q}_\alpha$. One deduces that

$$\begin{aligned}
&\|\mathcal{Q}_\alpha(h, G_\alpha - \mathcal{M})\|_{L^2_v(\varpi_q)} + \|\mathcal{Q}_\alpha(G_\alpha - \mathcal{M}, h)\|_{L^2_v(\varpi_q)} \\
&\quad\lesssim \|h\|_{L^2_v(\varpi_{q+1})} \left(\|G_\alpha - \mathcal{M}\|_{L^1_v(\varpi_{q+1})} + \|G_\alpha - \mathcal{M}\|_{L^2_v(\varpi_{\kappa+1})} \right) \\
&\qquad + \|h\|_{L^2_v(\varpi_{\kappa+1})} \|G_\alpha - \mathcal{M}\|_{L^2_v(\varpi_{q+1})}\,.
\end{aligned}$$

Then from Lemma 2.1, for any $\kappa > \frac{d}{2}$, we have:

$$\begin{aligned}
&\|\mathcal{Q}_\alpha(h,\mathcal{M}) - \mathcal{Q}_1(h,\mathcal{M})\|_{L^2_v(\varpi_q)} + \|\mathcal{Q}_\alpha(\mathcal{M},h) - \mathcal{Q}_1(\mathcal{M},h)\|_{L^2_v(\varpi_q)} \\
&\qquad\qquad \lesssim \frac{1-\alpha}{\alpha^a} \|\mathcal{M}\|_{\mathbb{W}^{1,2}_v(\varpi_{q+\kappa+3})}\, \|h\|_{L^2_v(\varpi_{q+\kappa+3})}
\end{aligned}$$

where $a = \max(d-1, 2)$. One can then conclude that (2.25) holds true thanks to Lemma 2.3. □

We will need also to derive an estimate for $\mathbf{L}_\alpha - \mathbf{L}_1$ in its graph norm in a L^1_v-framework, at the price of loosing the sharp convergence rate $(1-\alpha)$.

Lemma 2.5. *For any $q \geqslant 0$, there exists some explicit $\alpha_5 \in (0,1)$ such that*

$$\|\mathbf{L}_\alpha h - \mathbf{L}_1 h\|_{L^1_v(\varpi_q)} \lesssim (1-\alpha)^p\, \|h\|_{L^1_v(\varpi_{q+1})}\,, \qquad \alpha \in [\alpha_5, 1]\,, \tag{2.28}$$

where p is defined in Lemma 2.2. As a consequence, there holds

$$\|\mathbf{L}_\alpha h - \mathbf{L}_1 h\|_{L^1_v L^2_x(\varpi_q)} \lesssim (1-\alpha)^{\frac{p}{2}}\, \|h\|_{L^1_v L^2_x(\varpi_{q+1})}\,, \qquad \alpha \in [\alpha_5, 1]\,. \tag{2.29}$$

Proof. Recall (see Theorem B.1 and Lemma B.3) that for any $\alpha \in (0,1]$,

$$\|\mathcal{Q}_\alpha^\pm(g,f)\|_{L^1_v(\varpi_q)} \lesssim \|g\|_{L^1_v(\varpi_{q+1})}\, \|f\|_{L^1_v(\varpi_{q+1})}\,, \qquad \forall\, f, g \in L^1_v(\varpi_{q+1})\,. \tag{2.30}$$

Using the decomposition (2.27) and choosing $\alpha_5 \in [\max(\alpha_1, \alpha_3), 1)$ (where α_1 is defined in Lemma 2.2 and α_3 in Lemma 2.3), the proof of (2.29) is then a direct consequence of (2.30) and Lemma 2.2 which give

$$\|\mathbf{L}_\alpha h - \mathbf{L}_1 h\|_{L^1_v(\varpi_q)} \lesssim \|h\|_{L^1_v(\varpi_{q+1})}\, \|G_\alpha - \mathcal{M}\|_{L^1_v(\varpi_{q+1})} + (1-\alpha)^p\, \|h\|_{L^1_v(\varpi_{q+1})}\,.$$

Let us prove (2.29). On the one hand, the $L^1_v L^1_x(\varpi_q)$-norm of $\mathbf{L}_\alpha h - \mathbf{L}_1 h$ is estimated using Fubini theorem and (2.28):

$$\|\mathbf{L}_\alpha h - \mathbf{L}_1 h\|_{L^1_v L^1_x(\varpi_q)} \lesssim (1-\alpha)^p\, \|h\|_{L^1_v L^1_x(\varpi_{q+1})}\,.$$

On the other hand, using (2.30) and the fact that $\mathcal{Q}_\alpha^\pm$ are local in x, one can show that

$$\|\mathbf{L}_\alpha h - \mathbf{L}_1 h\|_{L^1_v L^\infty_x(\varpi_q)} \lesssim \|h\|_{L^1_v L^\infty_x(\varpi_{q+1})}\,.$$

We obtain (2.29) by interpolation. □

2.3. **Decomposition of $\mathscr{L}_\alpha$.** Let us now recall the following decomposition of $\mathscr{L}_\alpha$ defined in (1.28) introduced in Gualdani et al. (2017); Tristani (2016) (see also Alonso et al. (2017) for proofs adapted to the case of a general collision kernel b). For any $\delta \in (0,1)$, we consider the cutoff function $0 \leqslant \Theta_\delta = \Theta_\delta(\xi,\xi_*,\sigma) \in \mathcal{C}^\infty(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1})$, assumed to be bounded by 1, which equals 1 on

$$J_\delta := \left\{(\xi,\xi_*,\sigma)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}\,\middle|\,|\xi|\leqslant\delta^{-1}\,,\,2\delta\leqslant|\xi-\xi_*|\leqslant\delta^{-1}\,,\,|\cos\theta|\leqslant 1-2\delta\right\},$$

and whose support is included in $J_{\frac{\delta}{2}}$ (where $\cos\theta = \langle\frac{\xi-\xi_*}{|\xi-\xi_*|},\sigma\rangle$). We then set

$$\begin{aligned}\mathscr{L}_1^{S,\delta}h(\xi) &= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\left[\mathcal{M}(\xi_*')h(\xi')+\mathcal{M}(\xi')h(\xi_*')-\mathcal{M}(\xi)h(\xi_*)\right]\\ &\qquad\qquad\times|\xi-\xi_*|\,b(\cos\theta)\,\Theta_\delta(\xi,\xi_*,\sigma)\,\mathrm{d}\xi_*\,\mathrm{d}\sigma\,,\\ \mathscr{L}_1^{R,\delta}h(\xi) &= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\left[\mathcal{M}(\xi_*')h(\xi')+\mathcal{M}(\xi')h(\xi_*')-\mathcal{M}(\xi)h(\xi_*)\right]\\ &\qquad\qquad\times|\xi-\xi_*|\,b(\cos\theta)\,(1-\Theta_\delta(\xi,\xi,\sigma))\,\mathrm{d}\xi_*\,\mathrm{d}\sigma\,,\end{aligned}$$

so that $\mathscr{L}_1 h = \mathscr{L}_1^{S,\delta}h + \mathscr{L}_1^{R,\delta}h - h\Sigma_{\mathcal{M}}$ where using (1.6), $\Sigma_{\mathcal{M}}$ denotes the mapping

$$\Sigma_{\mathcal{M}}(\xi) := \int_{\mathbb{R}^d}\mathcal{M}(\xi_*)|\xi-\xi_*|\,\mathrm{d}\xi_*\,,\qquad\forall\,\xi\in\mathbb{R}^d\,. \tag{2.31}$$

Recall that there exist $\sigma_0>0$ and $\sigma_1>0$ such that

$$\sigma_0\,\varpi_1(\xi)\leqslant\Sigma_{\mathcal{M}}(\xi)\leqslant\sigma_1\,\varpi_1(\xi)\,,\qquad\forall\,\xi\in\mathbb{R}^d\,. \tag{2.32}$$

Introduce

$$\mathcal{A}^{(\delta)}(h) := \mathscr{L}_1^{S,\delta}(h)\qquad\text{and}\qquad\mathcal{B}_1^{(\delta)}(h) := \mathscr{L}_1^{R,\delta}-\Sigma_{\mathcal{M}}$$

so that $\mathscr{L}_1 = \mathcal{A}^{(\delta)}+\mathcal{B}_1^{(\delta)}$. Let us now recall the known dissipitavity results for the elastic Boltzmann operator in $L^1_vL^2_x$-based Sobolev spaces, see (Gualdani et al., 2017, Lemmas 4.12, 4.14 & Lemma 4.16):

Lemma 2.6. *For any $k\in\mathbb{N}$ and $\delta>0$, there are two positive constants $C_{k,\delta}>0$ and $R_\delta>0$ such that* $\operatorname{supp}\left(\mathcal{A}^{(\delta)}f\right)\subset B(0,R_\delta)$ *and*

$$\|\mathcal{A}^{(\delta)}f\|_{\mathbb{W}^{k,2}_v(\mathbb{R}^d)}\leqslant C_{k,\delta}\|f\|_{L^1_v(\varpi_1)}\,,\qquad\forall\,f\in L^1_v(\varpi_1)\,. \tag{2.33}$$

Moreover, the following holds: for any $q > 2$ and any $\delta \in (0,1)$ it holds

$$\int_{\mathbb{R}^d} \|h(\,\cdot\,,v)\|_{L^2_x}^{-1} \left(\int_{\mathbb{T}^d} \left(\mathcal{B}_1^{(\delta)} h(x,v) \right) h(x,v)\,\mathrm{d}x \right) \varpi_q(v)\,\mathrm{d}v \\ \leqslant (\Lambda_q(\delta) - 1)\, \|h\|_{L^1_v L^2_x(\varpi_q \Sigma_{\mathcal{M}})} \quad (2.34)$$

where $\Lambda_q : (0,1) \to \mathbb{R}^+$ is some explicit function such that $\lim_{\delta \to 0} \Lambda_q(\delta) = \frac{4}{q+2}$.

Remark 2.7. *Notice that this lemma comes from Gualdani et al. (2017) but the constants involved in the final estimates are not the same as in Lemma 4.14 of Gualdani et al. (2017) where it seems that some multiplicative constants coming from (2.32) have been omitted in some computations of their proof.*

This leads to the following decomposition of $\mathscr{L}_\alpha$:

$$\mathscr{L}_\alpha = \mathcal{B}_\alpha^{(\delta)} + \mathcal{A}^{(\delta)} \qquad \text{where} \qquad \mathcal{B}_\alpha^{(\delta)} := \mathcal{B}_1^{(\delta)} + [\mathscr{L}_\alpha - \mathscr{L}_1]\,. \quad (2.35)$$

2.4. **The complete inelastic linearized operator.** The complete linearized operator is given by

$$\mathcal{G}_{\alpha,\varepsilon} h = \varepsilon^{-2} \mathscr{L}_\alpha(h) - \varepsilon^{-1} v \cdot \nabla_x h\,, \qquad \forall\, \alpha \in (0,1]\,.$$

With the previous decomposition of $\mathscr{L}_\alpha$, we have that

$$\mathcal{G}_{\alpha,\varepsilon} = \mathcal{A}_\varepsilon^{(\delta)} + \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$$

where

$$\mathcal{A}_\varepsilon^{(\delta)} := \varepsilon^{-2} \mathcal{A}^{(\delta)}\,, \qquad \mathcal{B}_{\alpha,\varepsilon}^{(\delta)} := \varepsilon^{-2} \mathcal{B}_\alpha^{(\delta)} - \varepsilon^{-1} v \cdot \nabla_x\,.$$

One has the following properties of $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ in $L^1_v L^2_x$-based spaces.

Proposition 2.8. *For any $\ell \geqslant 0$ and $q > 2$, there exist $\alpha_q^\dagger > 0$, $\delta_q^\dagger > 0$ and $\nu_q > 0$ such that for any $\varepsilon \in (0,1]$,*

$$\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2}\nu_q \quad \text{is dissipative in } L^1_v \mathbb{W}^{\ell,2}_x(\varpi_q)$$

for any $\alpha \in (\alpha_q^\dagger, 1)$ and $\delta \in (0, \delta_q^\dagger)$.

Remark 2.9. *Let us be more precise on the estimate of dissipativity we obtain in $L^1_v L^2_x(\varpi_q)$ for further use: for any $\varepsilon \in (0,1]$, any $\alpha \in (\alpha_q^\dagger, 1)$ and any $\delta \in (0, \delta_q^\dagger)$, we have*

$$\int_{\mathbb{R}^d} \|h(\,\cdot\,,v)\|_{L^2_x}^{-1} \left(\int_{\mathbb{T}^d} \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}(h)(x,v) h(x,v)\,\mathrm{d}x \right) \varpi_q(v)\,\mathrm{d}v \leqslant -\varepsilon^{-2}\nu_q \|h\|_{L^1_v L^2_x(\varpi_{q+1})}\,.$$

Proof. Notice that derivatives with respect to the x-variable commute with the operator $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ and this allows to prove the result, without loss of generality, in the special case $\ell = 0$. We first introduce the following notations:

$$\mathcal{P}_\alpha := \mathbf{L}_\alpha - \mathbf{L}_1\,, \qquad T_\alpha := -(1-\alpha)\nabla_v \cdot (v\,\cdot\,)\,.$$

We then write $\mathcal{B}^{(\delta)}_{\alpha,\varepsilon}(h) = \sum_{i=0}^{3} C_i(h)$ with

$$C_0(h) := \varepsilon^{-2}\mathcal{B}^{(\delta)}_1 h\,, \qquad C_1(h) := -\varepsilon^{-1} v\cdot\nabla_x h\,,$$
$$C_2(h) := \varepsilon^{-2}\mathcal{P}_\alpha h\,, \qquad C_3(h) := \varepsilon^{-2} T_\alpha h\,,$$

and correspondingly (with obvious notations),

$$\int_{\mathbb{R}^d} \|h(\,\cdot\,,v)\|^{-1}_{L^2_x}\left(\int_{\mathbb{T}^d} \mathcal{B}^{(\delta)}_{\alpha,\varepsilon}(h)(x,v)h(x,v)\,\mathrm{d}x\right)\varpi_q(v)\,\mathrm{d}v =: \sum_{i=0}^{3} I_i(h)\,.$$

First, $I_1(h) = 0$ since

$$\int_{\mathbb{T}^d} (v\cdot\nabla_x h(x,v))\,h(x,v)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^d} v\cdot\nabla_x h^2(x,v)\,\mathrm{d}x = 0\,.$$

According to (2.34), by taking δ small enough so that $\Lambda_q(\delta) < 1$ (which is possible since $q > 2$), we have

$$I_0(h) \leqslant \varepsilon^{-2}\sigma_0\,(\Lambda_q(\delta) - 1)\,\|h\|_{L^1_v L^2_x(\varpi_{q+1})}\,.$$

Moreover, it follows from Cauchy-Schwarz inequality and (2.29) that there exists $C > 0$ such that for any $\alpha \in [\alpha_5, 1]$,

$$I_2(h) \leqslant \varepsilon^{-2}\int_{\mathbb{R}^d} \|\mathcal{P}_\alpha h(\,\cdot\,,v)\|_{L^2_x}\varpi_q(v)\,\mathrm{d}v \leqslant \varepsilon^{-2} C(1-\alpha)^{\frac{p}{2}}\|h\|_{L^1_v L^2_x(\varpi_{q+1})}\,.$$

Finally, for I_3, one can compute

$$\begin{aligned}
&\int_{\mathbb{R}^d} \|h(\,\cdot\,,v)\|^{-1}_{L^2_x}\int_{\mathbb{T}^d} \nabla_v\cdot(vh(x,v))\,h(x,v)\,\mathrm{d}x\,\varpi_q(v)\,\mathrm{d}v\\
&\quad = d\int_{\mathbb{R}^d}\|h(\,\cdot\,,v)\|_{L^2_x}\varpi_q(v)\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{R}^d}\|h(\,\cdot\,,v)\|^{-1}_{L^2_x}\int_{\mathbb{T}^d} v\cdot\nabla_v h^2(x,v)\,\mathrm{d}x\,\varpi_q(v)\,\mathrm{d}v\\
&\quad = d\int_{\mathbb{R}^d}\|h(\,\cdot\,,v)\|_{L^2_x}\varpi_q(v)\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{R}^d}\|h(\,\cdot\,,v)\|^{-1}_{L^2_x}\,v\cdot\nabla_v\|h(\,\cdot\,,v)\|^2_{L^2_x}\varpi_q(v)\,\mathrm{d}v\\
&\quad = -\int_{\mathbb{R}^d}\|h(\,\cdot\,,v)\|_{L^2_x} v\cdot\nabla_v\varpi_q(v)\,\mathrm{d}v\,.
\end{aligned}$$

Since $v\cdot\nabla_v\varpi_q(v) = q\varpi_q(v) - q\varpi_{q-2}(v)$ we get

$$I_3(h) \leqslant q(1-\alpha)\varepsilon^{-2}\|h\|_{L^1_v L^2_x(\varpi_{q+1})}\,. \tag{2.36}$$

Gathering the previous estimates, one obtains

$$\begin{aligned}
\mathcal{I} &:= \int_{\mathbb{R}^d}\|h(\,\cdot\,,v)\|^{-1}_{L^2_x}\left(\int_{\mathbb{T}^d}\mathcal{B}^{(\delta)}_{\alpha,\varepsilon}(h)(x,v)h(x,v)\,\mathrm{d}x\right)\varpi_q(v)\,\mathrm{d}v\\
&\leqslant \varepsilon^{-2}\left(C(1-\alpha)^{\frac{p}{2}} + \sigma_0(\Lambda_q(\delta)-1) + q(1-\alpha)\right)\|h\|_{L^1_v L^2_x(\varpi_{q+1})}\,.
\end{aligned} \tag{2.37}$$

Recalling that $\lim_{\delta\to 0}(\Lambda_q(\delta)-1) = -\frac{q-2}{q+2} < 0$, we can pick $\delta_q^\dagger$ small enough and then $\alpha_q^\dagger \in (\alpha_5, 1)$ close enough to 1 so that

$$\nu_q := -\inf\left\{ C(1-\alpha)^{\frac{p}{2}} + \sigma_0(\Lambda_q(\delta)-1) + q(1-\alpha)\,;\, \alpha\in(\alpha_q^\dagger,1),\, \delta\in(0,\delta_q^\dagger)\right\} > 0$$

and get the result. □

3. Hypocoercive norm for the elastic problem

We establish here some hypocoercivity result for the elastic linearized operator $\mathcal{G}_{1,\varepsilon}$. For $\varepsilon = 1$, hypocoercivity results have been established notably in Mouhot & Neumann (2006) with the so-called H^1-hypocoercivity method. For $\varepsilon \in (0,1)$, it is important to deduce hypocoercivity results with optimal dependency on ε (see Section 4 where the hypocoercivity estimate below will be crucially used). The results of Briant (2015) extend the ones by Mouhot & Neumann (2006) and provide estimate in Sobolev spaces like $\mathbb{W}^{\ell,2}_{x,v}(\mathcal{M}^{-\frac12})$ for $\varepsilon\in(0,1)$ with $\ell \geqslant 1$ but in our subsequent analysis, it is important to use a space which *does not involve any derivative in the v-variable*, we work therefore in the spaces

$$\mathcal{H} := L^2_v\mathbb{W}^{m,2}_x(\mathcal{M}^{-\frac12}), \qquad \mathcal{H}_1 := L^2_v\mathbb{W}^{m,2}_x(\mathcal{M}^{-\frac12}\langle\cdot\rangle^{\frac12}), \qquad m > \frac{d}{2}. \tag{3.1}$$

Notice that all the results of this section except Lemma 3.4 would be true in $L^2_v\mathbb{W}^{\ell,2}_v$ with $\ell\in\mathbb{N}$, we only state our results in $\mathcal{H}$ because we will use them only in this framework. In order to prove our result in the functional space $\mathcal{H}$, we use the L^2-hypocoercivity method. Let us mention that recent L^2-hypocoercivity results have been obtained in Bernou et al. (2021) (to which we refer for references on hypocoercivity methods in general) for various linearized kinetic equations in bounded domains with general Maxwell boundary conditions. For the Landau equation on the torus, similar results have been obtained in Carrapatoso et al. (2022) and our result is an easy adaptation of (Bernou et al. , 2021, Theorem 5.1) and (Carrapatoso et al., 2022, Proposition 3.2).

Before stating our result, we recall the expression for the spectral projection $\boldsymbol{\pi}_0$ onto the kernel $\mathrm{Ker}(\mathbf{L}_1)$ of the linearized collision operator $\mathbf{L}_1$ *seen as an operator acting in velocity only* on the space $L^2_v(\mathcal{M}^{-\frac12})$:

$$\boldsymbol{\pi}_0(g) := \sum_{i=1}^{d+2}\left(\int_{\mathbb{R}^d} g\,\Psi_i\,\mathrm{d}v\right)\Psi_i\,\mathcal{M}\,, \tag{3.2}$$

where

$$\Psi_1(v) := 1\,, \quad \Psi_i(v) := \frac{v_{i-1}}{\sqrt{\vartheta_1}} \quad (i=2,\dots,d+1)\,, \quad \text{and} \quad \Psi_{d+2}(v) := \frac{1}{\vartheta_1\sqrt{2d}}(|v|^2 - d\vartheta_1)\,. \tag{3.3}$$

Note also (see for example Briant (2015)) that the spectral projection $\mathbf{P}_0$ onto the kernel of $\mathcal{G}_{1,\varepsilon}$ is given by

$$\mathbf{P}_0(g) := \sum_{i=1}^{d+2} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} g\, \Psi_i \, \mathrm{d}x \, \mathrm{d}v \right) \Psi_i \, \mathcal{M} \tag{3.4}$$

and so the difference with respect to the spectral projection $\boldsymbol{\pi}_0$ is that an additional spatial integration is performed.

Proposition 3.1. *On the space $\mathcal{H}$ defined in (3.1), there exists a norm $|||\cdot|||_{\mathcal{H}}$ with associated inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathcal{H}}$ equivalent to the standard norm $\|\cdot\|_{\mathcal{H}}$ for which there exist $\mathrm{a}_1 > 0$ and $\mathrm{a}_2 > 0$ such that*

$$\langle\!\langle \mathcal{G}_{1,\varepsilon} h, h \rangle\!\rangle_{\mathcal{H}} \leqslant -\frac{\mathrm{a}_1}{\varepsilon^2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)\, h\|^2_{\mathcal{H}_1} - \mathrm{a}_1 \|h\|^2_{\mathcal{H}_1} - \mathrm{a}_2 |||h|||^2_{\mathcal{H}} \tag{3.5}$$

holds true for any $h = (\mathbf{Id} - \mathbf{P}_0)h \in \mathscr{D}(\mathcal{G}_{1,\varepsilon}) \subset \mathcal{H}$ where $\boldsymbol{\pi}_0$ (resp. $\mathbf{P}_0$) is defined in (3.2) (resp. (3.4)).

Remark 3.2. *Remark that the equivalent norm $|||\cdot|||_{\mathcal{H}}$ actually depends on ε (see (3.17)) but we do not mention this dependency in our notation because this norm is equivalent to the usual one $\|\cdot\|_{\mathcal{H}}$* uniformly *in $\varepsilon \in (0,1)$. In particular, there exists $C_{\mathcal{H}} > 0$ independent of $\varepsilon \in (0,1)$ such that*

$$C_{\mathcal{H}} \|h\|_{\mathcal{H}} \leqslant |||h|||_{\mathcal{H}} \leqslant C_{\mathcal{H}}^{-1} \|h\|_{\mathcal{H}} \,, \qquad \forall h \in \mathcal{H} \,. \tag{3.6}$$

Proof. The proof follows closely the one in (Carrapatoso et al., 2022, Appendix A) and we provide only the main steps and main changes.

Step 1. In this first step, we provide tools that are useful to carry out our proof. We set

$$L^2_0(\mathbb{T}^d) := \left\{ \phi \in L^2_x(\mathbb{T}^d) \,;\, \int_{\mathbb{T}^d} \phi(x)\, \mathrm{d}x = 0 \right\}.$$

Then, for any $\phi \in L^2_0(\mathbb{T}^d)$ there is a unique solution $f \in \mathbb{W}^{2,2}_x(\mathbb{T}^d) \cap L^2_0(\mathbb{T}^d)$ to the equation

$$-\Delta_x f = \phi \,, \qquad x \in \mathbb{T}^d \,.$$

We denote then by $(-\Delta_x)^{-1}$ the *bounded operator*

$$(-\Delta_x)^{-1} \,:\, \phi \in L^2_0(\mathbb{T}^d) \mapsto f \in \mathbb{W}^{2,2}_x(\mathbb{T}^d) \cap L^2_0(\mathbb{T}^d) \,.$$

In particular, $(-\Delta_x)^{-1} \in \mathscr{B}(L^2_0(\mathbb{T}^d)\,,\, \mathbb{W}^{1,2}_x(\mathbb{T}^d) \cap L^2_0(\mathbb{T}^d))\,.$

For any $f \in L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})$ we introduce the following notation

$$f^{\perp} := (\mathbf{Id} - \boldsymbol{\pi}_0)\, f \tag{3.7}$$

where $\boldsymbol{\pi}_0$ is the spectral projection $\boldsymbol{\pi}_0$ onto the kernel $\mathrm{Ker}(\mathbf{L}_1)$ of the linearized collision operator $\mathbf{L}_1$ defined in (3.2). As in Section 6, it will be more convenient to adopt the equivalent definition of $\boldsymbol{\pi}_0$:

$$\boldsymbol{\pi}_0 f(x,v) = \left(\varrho[f](x) + u[f](x) \cdot v + \frac{1}{2} \theta[f](x)(|v|^2 - d\vartheta_1) \right) \mathcal{M}(v) \,, \tag{3.8}$$

where

$$\varrho[f] := \int_{\mathbb{R}^d} f(\,\cdot\,,v)\,\mathrm{d}v\,, \qquad u[f] = \frac{1}{\vartheta_1}\int_{\mathbb{R}^d} v\, f(\,\cdot\,,v)\,\mathrm{d}v \in \mathbb{R}^d\,,$$

and $\theta[f]$ is defined through

$$\varrho[f] + \vartheta_1\theta[f] = \frac{1}{d\vartheta_1}\int_{\mathbb{R}^d} |v|^2 f(\,\cdot\,,v)\,\mathrm{d}v\,.$$

For general $f \in L^2_{x,v}(\mathcal{M}^{-\frac12})$, the splitting $f = f^\perp + \boldsymbol{\pi}_0 f$ implies that

$$\|f\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac12})} = \|f^\perp\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac12})} + \|\varrho[f]\|^2_{L^2_x} + \|u[f]\|^2_{L^2_x} + \|\theta[f]\|^2_{L^2_x}\,. \tag{3.9}$$

Notice that, for $f \in \mathscr{D}(\mathcal{G}_{1,\varepsilon}) \cap \mathrm{Range}(\mathbf{Id} - \mathbf{P}_0)$, one has $\mathbf{P}_0 f = \int_{\mathbb{T}^d} \boldsymbol{\pi}_0 f\,\mathrm{d}x = 0$ so that

$$\varrho[f],\, u_k[f],\, \theta[f] \in L^2_0(\mathbb{T}^d)\,, \qquad \forall\, k = 1,\dots,d\,.$$

In particular, $(-\Delta_x)^{-1}\,\varrho[f]$, $(-\Delta_x)^{-1}\,u_k[f]$ and $(-\Delta_x)^{-1}\,\theta[f]$ are well-defined and

$$\begin{aligned} \Big\|(-\Delta_x)^{-1}\,\varrho[f]\Big\|_{\mathbb{W}^{2,2}_x} &\lesssim \|\varrho[f]\|_{L^2_x} \lesssim \|f\|_{L^2_{x,v}(\mathcal{M}^{-\frac12})}\,,\\ \Big\|(-\Delta_x)^{-1}\,u_k[f]\Big\|_{\mathbb{W}^{2,2}_x} &\lesssim \|u_k[f]\|_{L^2_x} \lesssim \|f\|_{L^2_{x,v}(\mathcal{M}^{-\frac12})}\\ \text{and}\qquad \Big\|(-\Delta_x)^{-1}\,\theta[f]\Big\|_{\mathbb{W}^{2,2}_x} &\lesssim \|\theta[f]\|_{L^2_x} \lesssim \|f\|_{L^2_{x,v}(\mathcal{M}^{-\frac12})}\,. \end{aligned} \tag{3.10}$$

We introduce here the following notations, for any $k,\ell \in \{1,\dots,d\}$ and any $f \in \mathcal{H}$, we set

$$\boldsymbol{\psi}_k[f](x) := \frac{2}{d\vartheta_1^2}\int_{\mathbb{R}^d} \boldsymbol{b}_k(v) f(x,v)\,\mathrm{d}v = \frac{1}{d\vartheta_1^2}\int_{\mathbb{R}^d} v_k\left(|v|^2 - (d+2)\vartheta_1\right) f(x,v)\,\mathrm{d}v\,, \tag{3.11}$$

where

$$\boldsymbol{b}(v) := \frac12\left(|v|^2 - (d+2)\vartheta_1\right) v\,, \qquad v \in \mathbb{R}^d \tag{3.12}$$

and

$$\boldsymbol{\Theta}_{k\ell}[f](x) := \int_{\mathbb{R}^d} p_{k\ell}(v) f(x,v)\mathcal{M}(v)\,\mathrm{d}v\,, \qquad x \in \mathbb{T}^d \tag{3.13}$$

with

$$p_{k\ell}(v) := \begin{cases} \dfrac{1}{(d+4)\vartheta_1} v_k v_\ell |v|^2 & \text{if } k \neq \ell\,,\\[2ex] \dfrac{d-1}{2}\vartheta_1 + v_k^2 - \dfrac12|v|^2 & \text{if } k = \ell\,. \end{cases}$$

One observes easily that

$$\boldsymbol{\Theta}_{k\ell}[f] = \boldsymbol{\Theta}_{k\ell}[f^\perp] \quad \text{if} \quad k \neq \ell\,, \quad \text{while} \quad \boldsymbol{\Theta}_{kk}[f] = \boldsymbol{\Theta}_{kk}[f^\perp] - \frac{d-1}{2}\vartheta_1^2\theta[f]\,. \tag{3.14}$$

Notice that, since $\boldsymbol{b}_k$ and $p_{k\ell}$ are polynomial function, a simple use of Cauchy-Schwarz inequality shows that

$$\|\boldsymbol{\psi}_k[f]\|_{L^2_x} + \|\boldsymbol{\Theta}_{k\ell}[f]\|_{L^2_x} \lesssim \|f\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}. \tag{3.15}$$

In particular, using (3.14), we deduce that

$$\|\boldsymbol{\Theta}_{k\ell}[f]\|_{L^2_x} \lesssim \|\theta[f]\|_{L^2_x} + \|f^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}. \tag{3.16}$$

We now have the tools needed to develop the proof of Proposition 3.1.
Step 2. Since spatial derivatives commute with $\mathcal{G}_{1,\varepsilon}$, the crucial point is actually to define the equivalent inner product and norm on the space

$$H := L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})$$

with usual inner product $\langle\cdot,\cdot\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}$ and norm $\|\cdot\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}$ while, on the space $L^2_x(\mathbb{T}^d)$ the inner product is denoted $\langle\cdot,\cdot\rangle_{L^2_x}$.

We define the following inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_H$ on $L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})$ as follows: if $\mathbf{P}_0 f = \mathbf{P}_0 g = 0$, then we define

$$\begin{aligned}
\langle\!\langle f,g\rangle\!\rangle_H := {} & \langle f,g\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} \\
& + \varepsilon\eta_1 \sum_{k=1}^d \Big(\langle \partial_{x_k} (-\Delta_x)^{-1} \theta[f], \boldsymbol{\psi}_k[g]\rangle_{L^2_x} + \langle \partial_{x_k} (-\Delta_x)^{-1} \theta[g]\,,\, \boldsymbol{\psi}_k[f]\rangle_{L^2_x} \Big) \\
& + \varepsilon\eta_2 \sum_{k,\ell=1}^d \Big(\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[f]\,,\, \boldsymbol{\Theta}_{k\ell}[g]\rangle_{L^2_x} + \langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[g]\,,\, \boldsymbol{\Theta}_{k\ell}[f]\rangle_{L^2_x} \Big) \\
& + \varepsilon\eta_3 \sum_{k=1}^d \Big(\langle \partial_{x_k} (-\Delta_x)^{-1} \varrho[f]\,,\, u_k[g]\rangle_{L^2_x} + \langle \partial_{x_k} (-\Delta_x)^{-1} \varrho[g]\,,\, u_k[f]\rangle_{L^2_x} \Big)
\end{aligned} \tag{3.17}$$

for some suitable constants $0 < \eta_3 < \eta_2 < \eta_1 < 1$ to be chosen, otherwise, we simply define

$$\langle\!\langle f,g\rangle\!\rangle_H := \langle f,g\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}.$$

It is worth mentioning that in the rest of the paper, the only useful part of the definition is the first one, that is (3.17). The associated norm is denoted by $|\!|\!|\cdot|\!|\!|_H$. Since $\nabla_x (-\Delta_x)^{-1}$ is a bounded operator on L^2_x, one deduces easily from (3.10) and (3.15) that

$$\|f\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} \leqslant |\!|\!|f|\!|\!|^2_H \lesssim \|f\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} + \varepsilon\|f\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} \lesssim \|f\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}$$

for $\varepsilon \in (0,1)$. Therefore, the norms $|\!|\!|\cdot|\!|\!|_H$ and $\|\cdot\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}$ are equivalent *uniformly with respect to* ε.

For $h \in \mathscr{D}(\mathcal{G}_{1,\varepsilon}) \cap \mathrm{Range}(\mathbf{Id} - \mathbf{P}_0) \subset H$ given, we compute $\langle\!\langle \mathcal{G}_{1,\varepsilon} h, h \rangle\!\rangle_H$ as

$$\langle\!\langle \mathcal{G}_{1,\varepsilon} h, h \rangle\!\rangle_H \ = \ \langle \mathcal{G}_{1,\varepsilon} h, h \rangle_{L^2_{x,v}} + \varepsilon \eta_1 (I_1 + I_2) + \varepsilon \eta_2 \sum_{k,\ell=1}^{2} (J_1^{k,\ell} + J_2^{k,\ell}) + \varepsilon \eta_3 (R_1 + R_2)$$

where

$$I_1 := \langle \nabla_x \left(-\Delta_x\right)^{-1} \theta[\mathcal{G}_{1,\varepsilon} h], \boldsymbol{\psi}[h] \rangle_{L^2_x} \,, \quad I_2 := \langle \nabla_x \left(-\Delta_x\right)^{-1} \theta[h] \,, \boldsymbol{\psi}[\mathcal{G}_{1,\varepsilon} h] \rangle_{L^2_x} \,,$$

$$J_1^{k,\ell} := \langle \partial_{x_\ell} \left(-\Delta_x\right)^{-1} u_k[\mathcal{G}_{1,\varepsilon} h] \,, \boldsymbol{\Theta}_{k\ell}[h] \rangle_{L^2_x} \,, \qquad J_2^{k,\ell} := \langle \partial_{x_\ell} \left(-\Delta_x\right)^{-1} u_k[h] \,, \boldsymbol{\Theta}_{k\ell}[\mathcal{G}_{1,\varepsilon} h] \rangle_{L^2_x}$$

and

$$R_1 := \langle \nabla_x \left(-\Delta_x\right)^{-1} \varrho[\mathcal{G}_{1,\varepsilon} h], u[h] \rangle_{L^2_x} \,, \qquad R_2 := \langle \nabla_x \left(-\Delta_x\right)^{-1} \varrho[h], u[\mathcal{G}_{1,\varepsilon} h] \rangle_{L^2_x} \,.$$

We compute and estimate all these terms in succession. First, recalling that

$$\mathcal{G}_{1,\varepsilon} h = \frac{1}{\varepsilon^2} \mathbf{L}_1 h - \frac{1}{\varepsilon} v \cdot \nabla_x h$$

and using that $\mathbf{L}_1$ is self-adjoint and coercive on H: there exists $\mathrm{a}_0 > 0$ such that

$$\langle \mathbf{L}_1 f, f \rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} \leqslant -\mathrm{a}_0 \| \left(\mathbf{Id} - \boldsymbol{\pi}_0\right) f \|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}} \langle \cdot \rangle^{\frac{1}{2}})} \,,$$

for any $f \in \mathscr{D}(\mathcal{G}_{1,\varepsilon}) \subset H$ such that $\mathbf{P}_0 f = 0$ (see Briant (2015) for instance), we deduce first that

$$\langle \mathcal{G}_{1,\varepsilon} h, h \rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} \leqslant -\frac{\mathrm{a}_0}{\varepsilon^2} \| h^\perp \|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}} \langle \cdot \rangle^{\frac{1}{2}})} \tag{3.18}$$

where we recall that $h^\perp = h - \boldsymbol{\pi}_0 h$.

As far as I_1, I_2 are concerned, one has the following technical computations, in the spirit of (Carrapatoso et al., 2022, Proposition 3.2):

$$\theta[\mathcal{G}_{1,\varepsilon} h] = -\frac{1}{\varepsilon} \nabla_x \cdot \boldsymbol{\psi}[h] - \frac{2}{d\varepsilon} \nabla_x \cdot u[h] \,,$$

while

$$\boldsymbol{\psi}_k[\mathcal{G}_{1,\varepsilon} h] = \frac{1}{\varepsilon^2} \boldsymbol{\psi}_k \left[\mathbf{L}_1 h^\perp\right] + \frac{d+2}{d\varepsilon} \vartheta_1 \partial_{x_k} \theta[h] - \frac{1}{\varepsilon} \frac{2}{d\vartheta_1^2} \sum_{j=1}^{d} \partial_{x_j} \int_{\mathbb{R}^d} v_k \boldsymbol{b}_j(v) h^\perp \, \mathrm{d}v$$

With this, arguing as in (Carrapatoso et al., 2022, Proof of Proposition 3.2), we deduce first that

$$\left\| \nabla_x \left(-\Delta_x\right)^{-1} \theta[\mathcal{G}_{1,\varepsilon} h] \right\|_{L^2_x} \lesssim \frac{1}{\varepsilon} \left(\|u[h]\|_{L^2_x} + \|\boldsymbol{\psi}[h]\|_{L^2_x} \right)$$

and then, thanks to (3.15) and the fact that $\boldsymbol{\psi}[h] = \boldsymbol{\psi}[h^\perp]$, we get that

$$|I_1| \lesssim \frac{1}{\varepsilon} \left(\|u[h]\|_{L^2_x} + \|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} \right) \|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} \,.$$

Similarly, using the expression of $\boldsymbol{\psi}_k[\mathcal{G}_{1,\varepsilon}h]$, we can split I_2 as $I_2 = I_{21} + I_{22} + I_{23}$ with

$$I_{21} := \frac{1}{\varepsilon^2}\langle \nabla_x \left(-\Delta_x\right)^{-1} \theta[h]\,,\, \boldsymbol{\psi}[\mathbf{L}_1 h^\perp]\rangle_{L^2_x}\,, \qquad I_{22} := \frac{d+1}{d\varepsilon}\vartheta_1 \langle \nabla_x \left(-\Delta_x\right)^{-1} \theta[h]\,,\, \nabla_x \theta[h]\rangle_{L^2_x}$$

and

$$I_{23} := -\frac{1}{\varepsilon}\frac{2}{d\vartheta_1^2}\sum_{j=1}^d \left\langle \nabla_x \left(-\Delta_x\right)^{-1} \theta[h]\,,\, \partial_{x_j}\int_{\mathbb{R}^d} v_k \boldsymbol{b}_j(v) h^\perp \,\mathrm{d}v \right\rangle_{L^2_x}.$$

Since $\mathbf{L}_1$ is self-adjoint on $L^2_v(\mathcal{M}^{-\frac{1}{2}})$, one has from (3.11)

$$\boldsymbol{\psi}[\mathbf{L}_1 h^\perp] = \frac{2}{d\vartheta_1^2}\int_{\mathbb{R}^d} \boldsymbol{b}(v)\mathcal{M}(v)\mathbf{L}_1(h^\perp)\mathcal{M}^{-1}(v)\,\mathrm{d}v = \frac{2}{d\vartheta_1^2}\int_{\mathbb{R}^d} \mathbf{L}_1(\boldsymbol{b}\mathcal{M})h^\perp \mathcal{M}^{-1}\,\mathrm{d}v$$

and $\|\boldsymbol{\psi}[\mathbf{L}_1 h^\perp]\|_{L^2_x} \lesssim \|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}$ from which

$$|I_{21}| \lesssim \frac{1}{\varepsilon^2}\|\theta[h]\|_{L^2_x}\|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}.$$

By a simple integration by parts, one has

$$I_{22} = -\frac{1}{\varepsilon}\frac{d+2}{d}\vartheta_1\langle \Delta_x \left(-\Delta_x\right)^{-1}\theta[h], \theta[h]\rangle_{L^2_x} = -\frac{1}{\varepsilon}\frac{d+2}{d}\vartheta_1\|\theta[h]\|^2_{L^2_x}.$$

Finally, I_{23} is estimated thanks to Cauchy-Schwarz inequality as in Carrapatoso et al. (2022) to deduce that

$$|I_{23}| \lesssim \frac{1}{\varepsilon}\|\theta[h]\|_{L^2_x}\|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}.$$

Combining all this estimates, we deduce from Young's inequality as in (Carrapatoso et al., 2022, Eq. (A.8)) that there exist $K_0, C_0 > 0$ such that

$$I_1 + I_2 \leqslant -\frac{K_0}{\varepsilon^2}\|\theta[h]\|^2_{L^2_x} + \frac{C_0}{\varepsilon}\|u[h]\|_{L^2_x}\|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} + \frac{C_0}{\varepsilon^3}\|h^\perp\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}. \tag{3.19}$$

To estimate $J_1^{k,\ell}$ and $J_2^{k,\ell}$ we first observe that

$$\begin{aligned} u_k[\mathcal{G}_{1,\varepsilon}h] &= -\frac{1}{\varepsilon}\vartheta_1\partial_{x_k}\left(\varrho[h] + \vartheta_1\theta[h]\right) - \frac{1}{\varepsilon}\sum_{j=1}^d \partial_{x_j}\int_{\mathbb{R}^d} v_k v_j h^\perp \,\mathrm{d}v\,, \\ \boldsymbol{\Theta}_{k\ell}[\mathcal{G}_{1,\varepsilon}h] &= \frac{1}{\varepsilon^2}\boldsymbol{\Theta}_{k\ell}[\mathbf{L}_1 h^\perp] - \frac{1}{\varepsilon}\boldsymbol{\Theta}_{k\ell}[v\cdot\nabla_x \boldsymbol{\pi}_0 h] - \frac{1}{\varepsilon}\boldsymbol{\Theta}_{k\ell}[v\cdot\nabla_x h^\perp] \end{aligned} \tag{3.20}$$

where

$$\boldsymbol{\Theta}_{k\ell}[v\cdot\nabla_x\boldsymbol{\pi}_0 h] = \begin{cases} \vartheta_1^2\left(\partial_{x_k}u_\ell[h] + \partial_{x_\ell}u_k[h]\right) & \text{if } \ell \neq k\,, \\ \vartheta_1^2\left(\partial_{x_k}u_k[h] - \sum_{j\neq k}\partial_{x_j}u_j[h]\right) & \text{if } \ell = k\,. \end{cases}$$

With these technical calculations, we can argue as in Carrapatoso et al. (2022) and first observe that, using an integration by parts and (3.20)

$$\|\nabla_x (-\Delta_x)^{-1} u_k[\mathcal{G}_{1,\varepsilon}h]\|_{L^2_x} = \left\langle u_k[\mathcal{G}_{1,\varepsilon}[h], (-\Delta_x)^{-1} u_k[\mathcal{G}_{1,\varepsilon}h]\right\rangle \\ \lesssim \frac{1}{\varepsilon}\left(\|\varrho[h]\|_{L^2_x} + \|\theta[h]\|_{L^2_x} + \|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac12})}\right)$$

which, thanks to (3.16) easily gives

$$|J_1^{k,\ell}| \lesssim \frac{1}{\varepsilon}\left(\|\varrho[h]\|_{L^2_x} + \|\theta[h]\|_{L^2_x} + \|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac12})}\right)\left(\|\theta[h]\|_{L^2_x} + \|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac12})}\right).$$

For the term $J_2^{k,\ell}$, we use the expression of $\mathbf{\Theta}_{k\ell}[\mathcal{G}_{1,\varepsilon}h]$ in (3.20) to deduce that $J_2^{k,\ell} = J_{21}^{k,\ell} + J_{22}^{k,\ell} + J_{23}^{k,\ell}$ where

$$J_{21}^{k,\ell} := \frac{1}{\varepsilon^2}\left\langle \partial_{x_\ell}(-\Delta_x)^{-1} u_k[h]\,,\, \mathbf{\Theta}_{k\ell}[\mathbf{L}_1 h^\perp]\right\rangle_{L^2_x}, \\ J_{23}^{k,\ell} := \frac{1}{\varepsilon}\left\langle \partial_{x_\ell}(-\Delta_x)^{-1} u_k[h]\,,\, \mathbf{\Theta}_{k\ell}[v\cdot\nabla_x h^\perp]\right\rangle_{L^2_x}$$

and

$$J_{22}^{k,\ell} := \frac{1}{\varepsilon}\left\langle \partial_{x_\ell}(-\Delta_x)^{-1} u_k[h]\,,\, \mathbf{\Theta}_{k\ell}[v\cdot\nabla_x \boldsymbol{\pi}_0 h]\right\rangle_{L^2_x}.$$

One has then easily thanks to Cauchy-Schwarz inequality that

$$J_{21}^{k,\ell} \lesssim \frac{1}{\varepsilon^2}\|u[h]\|_{L^2_x}\|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac12})}, \qquad J_{23}^{k,\ell} \lesssim \frac{1}{\varepsilon}\|u[h]\|_{L^2_x}\|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac12})}$$

while, arguing as in Carrapatoso et al. (2022) ans using integration by parts we see that

$$J_{22} := \sum_{k,\ell=1}^d J_{22}^{k,\ell} = -\frac{\vartheta_1^2}{\varepsilon}\sum_{k=1}^d \|u_k[h]\|^2_{L^2_x} = -\frac{\vartheta_1^2}{\varepsilon}\|u[h]\|_{L^2_x}.$$

Gathering all these estimates, we obtain, as in (Carrapatoso et al., 2022, Eq. (A.16)) and using Young's inequality again that there exist positive constants $K_1, C_1 > 0$ such that

$$\sum_{k,\ell=1}^d \left(J_1^{k,\ell} + J_2^{k,\ell}\right) \leqslant -\frac{K_1}{\varepsilon^2}\|u[h]\|^2_{L^2_x} + \frac{C_1}{\varepsilon}\|\varrho[h]\|_{L^2_x}\|\theta[h]\|_{L^2_x} + \frac{C_1}{\varepsilon}\|\varrho[h]\|_{L^2_x}\|h^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac12})} \\ + \frac{C_1}{\varepsilon}\|\theta[h]\|^2_{L^2_x} + \frac{C_1}{\varepsilon^3}\|h^\perp\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac12})}. \quad (3.21)$$

Let us now estimate the last term $R_1 + R_2$. We begin with observing that

$$\varrho[\mathcal{G}_{1,\varepsilon}h] = -\frac{1}{\varepsilon}\nabla_x \cdot u[h]$$

so that

$$R_1 = \langle \nabla_x \left(-\Delta_x\right)^{-1} \varrho[\mathcal{G}_{1,\varepsilon} h] \, , \, u[h] \rangle_{L^2_x} \lesssim \| \left(-\Delta_x\right)^{-1} \varrho[\mathcal{G}_{1,\varepsilon} h] \|_{L^2_x} \| u[h] \|_{L^2_x} \lesssim \frac{1}{\varepsilon} \| u[h] \|^2_{L^2_x} \, .$$

Using the expression of $u_k[\mathcal{G}_{1,\varepsilon} h]$ in (3.20) again we can split R_2 as $R_2 = R_{21} + R_{22} + R_{23}$ with

$$R_{21} := -\frac{1}{\varepsilon} \vartheta_1 \left\langle \nabla_x \left(-\Delta_x\right)^{-1} \varrho[h], \nabla_x \varrho[h] \right\rangle , \quad R_{22} := -\frac{1}{\varepsilon} \vartheta_1^2 \left\langle \nabla_x \left(-\Delta_x\right)^{-1} \varrho[h], \nabla_x \theta[h] \right\rangle$$

and

$$R_{23} := -\frac{1}{\varepsilon} \sum_{j,k=1}^{d} \left\langle \partial_{x_k} \left(-\Delta_x\right)^{-1} \varrho[h] \, , \, \partial_{x_j} \int_{\mathbb{R}^d} v_k v_j h^\perp \, \mathrm{d}v \right\rangle .$$

Clearly, an integration by parts gives

$$R_{21} = -\frac{1}{\varepsilon} \vartheta_1 \left\langle -\Delta_x \left(-\Delta_x\right)^{-1} \varrho[h], \varrho[h] \right\rangle_{L^2_x} = -\frac{1}{\varepsilon} \vartheta_1 \| \varrho[h] \|^2_{L^2_x}$$

whereas, easily, $R_{22} \lesssim \frac{1}{\varepsilon} \| \varrho[h] \|_{L^2_x} \| \theta[h] \|_{L^2_x}$ and

$$R_{23} \lesssim \frac{1}{\varepsilon} \left\| \nabla_x^2 \left(-\Delta_x\right)^{-1} \varrho[h] \right\|_{L^2_x} \left\| \int_{\mathbb{R}^d} v \otimes v h^\perp \, \mathrm{d}v \right\|_{L^2_x} \lesssim \frac{1}{\varepsilon} \| \varrho[h] \|_{L^2_x} \| h^\perp \|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} .$$

Using Young's inequality after gathering these estimates, we deduce that there are $K_2, C_2 > 0$ such that

$$R_1 + R_2 \leqslant -\frac{K_2}{\varepsilon} \| \varrho[h] \|^2_{L^2_x} + \frac{C_2}{\varepsilon} \| u[h] \|_{L^2_x} + \frac{C_2}{\varepsilon} \| \theta[h] \|^2_{L^2_x} + \frac{C_2}{\varepsilon} \| h^\perp \|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} . \tag{3.22}$$

One sees now that Eqs. (3.18) – (3.19) – (3.21) – (3.22) correspond respectively to Eqs. (A2)–(A8)–(A16)–(A18) of (Carrapatoso et al., 2022, Proof of Prop. 3.2). Therefore, as in Step 5 of the *op. cit.*, we can choose $0 < \eta_3 \ll \eta_2 \ll \eta_1 \ll 1$ small enough and $\mathrm{a}_1, \mathrm{a}_2 > 0$ such that

$$\langle\!\langle \mathcal{G}_{1,\varepsilon} h, h \rangle\!\rangle_H \leqslant -\frac{\mathrm{a}_1}{\varepsilon^2} \| (\mathbf{Id} - \boldsymbol{\pi}_0) \, h \|^2_{\mathcal{H}_1} - \mathrm{a}_1 \| h \|^2_{\mathcal{H}_1} - \mathrm{a}_2 |\!|\!| h |\!|\!|^2_{\mathcal{H}}$$

where we used (3.9). This proves the desired hypocoercivity in $H = L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})$. One extends it without difficulty to $\mathcal{H} = L^2_v \mathbb{W}^{m,2}_x(\mathcal{M}^{-\frac{1}{2}})$ by introducing

$$\langle\!\langle f, g \rangle\!\rangle_{\mathcal{H}} = \sum_{|j| \leqslant m} \langle\!\langle \partial_x^j f, \partial_x^j g \rangle\!\rangle_H \tag{3.23}$$

and by observing that $\mathcal{G}_{1,\varepsilon}$ commutes with x-derivatives. □

The following result ensures that multiplication by polynomials and the drift term behave nicely with respect to this hypocoercive norm:

Lemma 3.3. *The inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathcal{H}}$ associated to the norm $|\!|\!| \cdot |\!|\!|_{\mathcal{H}}$ on $\mathcal{H}$ constructed in Proposition 3.1 is such that there exists $\mathrm{a}_3 > 0, C > 0$ (independent of ε) such that*

$$-\langle\!\langle \operatorname{div}_v (vh) \, , \, h \rangle\!\rangle_{\mathcal{H}} \leqslant \mathrm{a}_3 \langle\!\langle \boldsymbol{\varpi}_2 h, h \rangle\!\rangle_{\mathcal{H}} + C\varepsilon |\!|\!| h |\!|\!|^2_{\mathcal{H}}$$

for any $h \in L^2_v \mathbb{W}^{m,2}_x(\langle \cdot \rangle \mathcal{M}^{-\frac{1}{2}})$.

Proof. As in the proof of Proposition 3.1, it is enough to show the result in the space $H = L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})$ with inner product $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle_H$ defined in (3.17) since here again x-derivatives commutes with the operator $T_0 \,:\, h \mapsto T_0 h = -\nabla_v \cdot (vh)$. From the definition (3.17), we need to estimate carefully $\langle T_0 h, h\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}$ as well as $\|\varrho[T_0 h]\|_{L^2_x}$, $\|u[T_0 h]\|_{L^2_x}$, $\|\theta[T_0 h]\|_{L^2_x}$, $\|\boldsymbol{\psi}_k[T_0 h]\|_{L^2_x}$ and $\|\boldsymbol{\Theta}_{k\ell}[T_0 h]\|_{L^2_x}$. Since

$$\begin{aligned}\langle T_0 h, h\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} &= -\int_{\mathbb{T}^d} \mathrm{d}x \int_{\mathbb{R}^d} \nabla_v \cdot (vh(x,v)) h(x,v) \mathcal{M}^{-1}(v)\,\mathrm{d}v \\ &= -d \int_{\mathbb{T}^d} \mathrm{d}x \int_{\mathbb{R}^d} h^2(x,v)\mathcal{M}^{-1}(v)\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{T}^d} \mathrm{d}x \int_{\mathbb{R}^d} h^2(x,v) \nabla_v \cdot (v\mathcal{M}^{-1}(v))\,\mathrm{d}v \\ &= -\frac{d}{2} \int_{\mathbb{T}^d} \mathrm{d}x \int_{\mathbb{R}^d} h^2(x,v)\mathcal{M}^{-1}(v)\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{T}^d} \mathrm{d}x \int_{\mathbb{R}^d} h^2(x,v) v \cdot \nabla_v \mathcal{M}^{-1}(v)\,\mathrm{d}v\,.\end{aligned}$$

Since $v \cdot \nabla_v \mathcal{M}^{-1}(v) = \frac{1}{\vartheta_1}|v|^2 \mathcal{M}^{-1}(v) = \frac{1}{\vartheta_1}\boldsymbol{\varpi}_2(v)\mathcal{M}^{-1}(v) - \frac{1}{\vartheta_1}\mathcal{M}^{-1}(v)$ we deduce that

$$\langle T_0 h, h\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} = -\frac{1 + d\vartheta_1}{2\vartheta_1}\|h\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} + \frac{1}{2\vartheta_1}\langle \boldsymbol{\varpi}_2 h, h\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}\,. \tag{3.24}$$

Now, by simple integration by parts

$$\varrho[T_0 h] = 0\,, \qquad u[T_0 h] = u[h]\,, \qquad \theta[T_0 h] = \frac{2}{\vartheta_1}\varrho[h] + 2\theta[h]$$

while

$$\boldsymbol{\psi}_k[T_0 h] = 3\boldsymbol{\psi}_k[h] + 2\frac{d+2}{d\vartheta_1^2} u_k[h] \quad \text{and} \quad \boldsymbol{\Theta}_{k\ell}[T_0 h] = \tilde{\boldsymbol{\Theta}}_{k\ell}[h] := \int_{\mathbb{R}^d} \tilde{p}_{k\ell}(v) h(x,v)\mathcal{M}(v)\,\mathrm{d}v$$

with

$$\tilde{p}_{k\ell}(v) := \begin{cases} -\frac{1}{\vartheta_1}|v|^2 p_{k\ell}(v) + 4p_{k\ell}(v)\,, & \text{if} \quad k \neq \ell\,, \\ -\frac{1}{\vartheta_1}|v|^2 p_{kk}(v) + 2p_{kk}(v) - (d-1)\vartheta_1 & \text{if} \quad k = \ell\,. \end{cases}$$

In particular, $\boldsymbol{\Theta}_{k\ell}[T_0 h]$ shares the same properties of $\boldsymbol{\Theta}_{k\ell}[h]$. Since $\nabla_x \left(-\Delta_x\right)^{-1}$ is a bounded operator in H, one thus checks easily that (3.10) and (3.15) yield

$$\langle\!\langle T_0 h, h\rangle\!\rangle_H - \langle T_0 h, h\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} \lesssim \varepsilon \|h\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}\,.$$

Combining this to (3.24) gives the existence of $C > 0$ such that

$$\langle\!\langle T_0 h, h\rangle\!\rangle_H \leqslant \frac{1}{2\vartheta_1}\langle \boldsymbol{\varpi}_2 h, h\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} + C\varepsilon \|h\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}\,.$$

Using the same kind of arguments, we deduce now that

$$\langle \boldsymbol{\varpi}_2 h, h\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} - \langle\!\langle \boldsymbol{\varpi}_2 h, h\rangle\!\rangle_H \lesssim \varepsilon \|h\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}\,.$$

Therefore,

$$\langle\!\langle T_0 h, h\rangle\!\rangle_H \leqslant \frac{1}{2\vartheta_1}\langle\!\langle \boldsymbol{\varpi}_2 h, h\rangle\!\rangle_{\mathcal{H}} + C\varepsilon \|h\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}$$

which is the desired estimate with $\mathrm{a}_3 := \frac{1}{2\vartheta_1}$. □

We end this Section with the following useful estimate, in the spirit of (Briant , 2015, Theorem 2.4) (see also (Briant et al., 2019, Theorem 4.7)):

Lemma 3.4. *The inner product $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle_{\mathcal{H}}$ associated to the norm $|||\cdot|||_{\mathcal{H}}$ on $\mathcal{H}$ constructed in Proposition 3.1 is such that for any $h_1, h_2 \in \mathcal{H}_1$, $g \in \mathcal{H}_1 \cap \mathrm{Range}(\mathbf{Id} - \mathbf{P}_0)$,*

$$\langle\!\langle (\mathbf{Id} - \boldsymbol{\pi}_0)\, \mathcal{Q}_{\alpha(\varepsilon)}(h_1, h_2), g \rangle\!\rangle_{\mathcal{H}} \lesssim (\|h_1\|_{\mathcal{H}_1}\|h_2\|_{\mathcal{H}} + \|h_1\|_{\mathcal{H}}\|h_2\|_{\mathcal{H}_1})\ \|(\mathbf{Id} - \boldsymbol{\pi}_0)\, g\|_{\mathcal{H}_1} + \varepsilon \|h_1\|_{\mathcal{H}}\|h_2\|_{\mathcal{H}}\|\boldsymbol{\pi}_0 g\|_{\mathcal{H}}\,. \quad (3.25)$$

Proof. The proof is based on Lemma B.5 and we recall that, thanks to (3.17) and (3.23), $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle_{\mathcal{H}}$ is defined as:

$$\begin{aligned}
\langle\!\langle f, g \rangle\!\rangle_{\mathcal{H}} &= \langle f, g \rangle_{L^2_v \mathbb{W}^{m,2}_x(\mathcal{M}^{-\frac{1}{2}})} \\
&+ \varepsilon\eta_1 \sum_{k=1}^{d} \Big(\langle \partial_{x_k} (-\Delta_x)^{-1}\, \theta[f], \boldsymbol{\psi}_k[g] \rangle_{\mathbb{W}^{m,2}_x} + \langle \partial_{x_k} (-\Delta_x)^{-1}\, \theta[g]\,,\, \boldsymbol{\psi}_k[f] \rangle_{\mathbb{W}^{m,2}_x} \Big) \\
&+ \varepsilon\eta_2 \sum_{k,\ell=1}^{d} \Big(\langle \partial_{x_\ell} (-\Delta_x)^{-1}\, u_k[f]\,,\, \boldsymbol{\Theta}_{k\ell}[g] \rangle_{\mathbb{W}^{m,2}_x} + \langle \partial_{x_\ell} (-\Delta_x)^{-1}\, u_k[g]\,,\, \boldsymbol{\Theta}_{k\ell}[f] \rangle_{\mathbb{W}^{m,2}_x} \Big) \\
&+ \varepsilon\eta_3 \sum_{k=1}^{d} \Big(\langle \partial_{x_k} (-\Delta_x)^{-1}\, \varrho[f]\,,\, u_k[g] \rangle_{\mathbb{W}^{m,2}_x} + \langle \partial_{x_k} (-\Delta_x)^{-1}\, \varrho[g]\,,\, u_k[f] \rangle_{\mathbb{W}^{m,2}_x} \Big)
\end{aligned} \quad (3.26)$$

for $f, g \in \mathrm{Range}(\mathbf{Id} - \mathbf{P}_0)$. In the rest of the proof, we apply this to

$$f := (\mathbf{Id} - \boldsymbol{\pi}_0)\, \mathcal{Q}_\alpha(h_1, h_2)\,,$$

and one observes that $\theta[f] = u_k[f] = \varrho[f] = 0$. Splitting $g = g^\perp + \boldsymbol{\pi}_0 g$, in the same way $\theta[g^\perp] = u_k[g^\perp] = \varrho[g^\perp] = 0$ so that

$$\langle\!\langle (\mathbf{Id} - \boldsymbol{\pi}_0)\, \mathcal{Q}_\alpha(h_1, h_2), g \rangle\!\rangle_{\mathcal{H}} = \langle\!\langle (\mathbf{Id} - \boldsymbol{\pi}_0)\, \mathcal{Q}_\alpha(h_1, h_2), \boldsymbol{\pi}_0 g \rangle\!\rangle_{\mathcal{H}} + \Big\langle (\mathbf{Id} - \boldsymbol{\pi}_0)\, \mathcal{Q}_\alpha(h_1, h_2), g^\perp \Big\rangle_{\mathcal{H}}$$

where, moreover,

$$\langle (\mathbf{Id} - \boldsymbol{\pi}_0)\, \mathcal{Q}_\alpha(h_1, h_2), \boldsymbol{\pi}_0 g \rangle_{\mathcal{H}} = 0\,, \quad \Big\langle (\mathbf{Id} - \boldsymbol{\pi}_0)\, \mathcal{Q}_\alpha(h_1, h_2), g^\perp \Big\rangle_{\mathcal{H}} = \Big\langle \mathcal{Q}_\alpha(h_1, h_2), g^\perp \Big\rangle_{\mathcal{H}}$$

since $(\mathbf{Id}-\boldsymbol{\pi}_0)$ and $\boldsymbol{\pi}_0$ are orthogonal projections on $L^2_v(\mathcal{M}^{-\frac12})$. With (3.26), we deduce that

$$\langle\!\langle (\mathbf{Id}-\boldsymbol{\pi}_0)\,\mathcal{Q}_\alpha(h_1,h_2),g\rangle\!\rangle_{\mathcal{H}} = \Big\langle \mathcal{Q}_\alpha(h_1,h_2), g^\perp\Big\rangle_{\mathcal{H}} + \varepsilon\eta_1\sum_{k=1}^d \langle \partial_{x_k}\,(-\Delta_x)^{-1}\,\theta[g]\,,\,\boldsymbol{\psi}_k[f]\rangle_{\mathbb{W}^{m,2}_x} + \varepsilon\eta_2\sum_{k,\ell=1}^d \langle \partial_{x_\ell}\,(-\Delta_x)^{-1}\,u_k[g]\,,\,\boldsymbol{\Theta}_{k\ell}[f]\rangle_{\mathbb{W}^{m,2}_x}$$

where we used the fact that $\theta[\boldsymbol{\pi}_0 g]=\theta[g]$, $u_k[\boldsymbol{\pi}_0 g]=u_k[g]$. Notice that we can easily adapt the bounds (3.15) to deduce that

$$\|\boldsymbol{\psi}_k[f]\|_{L^2_x} + \|\boldsymbol{\Theta}_{k\ell}[f]\|_{L^2_x} \lesssim \|f\|_{L^2_{x,v}(\mathcal{M}^{-\frac12}\langle\cdot\rangle^{-1})}\,. \tag{3.27}$$

With such a bound and using and the fact that $\nabla_x\,(-\Delta_x)^{-1}$ is a bounded operator on $\mathbb{W}^{m,2}_x$, we deduce easily that

$$\langle\!\langle (\mathbf{Id}-\boldsymbol{\pi}_0)\,\mathcal{Q}_\alpha(h_1,h_2),g\rangle\!\rangle_{\mathcal{H}} \lesssim \Big\langle \mathcal{Q}_\alpha(h_1,h_2), g^\perp\Big\rangle_{\mathcal{H}} + \varepsilon\left(\|u[g]\|_{\mathbb{W}^{m,2}_x} + \|\theta[g]\|_{\mathbb{W}^{m,2}_x}\right)\|f\|_{\mathcal{H}_{-1}}$$

where we introduced

$$\mathcal{H}_{-1} := L^2_v\mathbb{W}^{m,2}_x(\mathcal{M}^{-\frac12}\langle\cdot\rangle^{-1})\,.$$

According to (B.2) and using that $\mathbf{Id}-\boldsymbol{\pi}_0$ is bounded in $\mathcal{H}_{-1}$, this implies that

$$\langle\!\langle (\mathbf{Id}-\boldsymbol{\pi}_0)\,\mathcal{Q}_\alpha(h_1,h_2),g\rangle\!\rangle_{\mathcal{H}} \lesssim (\|h_1\|_{\mathcal{H}}\|h_2\|_{\mathcal{H}_1} + \|h_1\|_{\mathcal{H}_1}\|h_2\|_{\mathcal{H}})\,\|g^\perp\|_{\mathcal{H}_1} + \varepsilon\left(\|u[g]\|_{\mathbb{W}^{m,2}_x} + \|\theta[g]\|_{\mathbb{W}^{m,2}_x}\right)\|\mathcal{Q}_\alpha(h_1,h_2)\|_{\mathcal{H}_{-1}}\,.$$

Adapting easily the proof of (B.4), one sees that $\|\mathcal{Q}_\alpha(h_1,h_2)\|_{\mathcal{H}_{-1}} \lesssim \|h_1\|_{\mathcal{H}}\|h_2\|_{\mathcal{H}}$ and one deduces that

$$\langle\!\langle (\mathbf{Id}-\boldsymbol{\pi}_0)\,\mathcal{Q}_\alpha(h_1,h_2),g\rangle\!\rangle_{\mathcal{H}} \lesssim (\|h_1\|_{\mathcal{H}}\|h_2\|_{\mathcal{H}_1} + \|h_1\|_{\mathcal{H}_1}\|h_2\|_{\mathcal{H}})\,\|g^\perp\|_{\mathcal{H}_1} + \varepsilon\left(\|u[g]\|_{\mathbb{W}^{m,2}_x} + \|\theta[g]\|_{\mathbb{W}^{m,2}_x}\right)\|h_1\|_{\mathcal{H}}\|h_2\|_{\mathcal{H}}$$

where, obviously, $\|u[g]\|_{\mathbb{W}^{m,2}_x} + \|\theta[g]\|_{\mathbb{W}^{m,2}_x} \lesssim \|\boldsymbol{\pi}_0 g\|_{\mathcal{H}}$. This gives the result. □

4. Nonlinear analysis

We now apply the results obtained so far to the study of Eq. (1.27) under Assumption 1.1. Let us define the functional space $\mathcal{E}$ in which we are going to carry out our analysis:

$$\mathcal{E} := L^1_v\mathbb{W}^{m,2}_x(\boldsymbol{\varpi}_q) \qquad \text{with} \qquad m>\frac{d}{2} \qquad \text{and} \qquad q\geqslant 3 \tag{4.1}$$

and recall that Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_1$ are defined in (3.1). We also recall here that $\mathcal{M}$ is the steady state of $\mathscr{L}_1$ defined in (1.19) while $\mathcal{H}$ is a Hilbert space on which the elastic Boltzmann equation is well-understood (see Subsection 3). We also define the following functional spaces

$$\mathcal{E}_1 := L^1_v \mathbb{W}^{m,2}_x(\varpi_{q+1}), \qquad \mathcal{E}_2 := L^2_v \mathbb{W}^{m,2}_x(\varpi_{q+2\kappa+3}), \tag{4.2}$$

where m and q have been fixed in (4.1) and $\kappa > \frac{d}{2}$ is fixed.

Notice here that we impose here the condition $m > \frac{d}{2}$ in order to ensure that the embedding $\mathbb{W}^{m,2}_x(\mathbb{T}^d) \hookrightarrow L^\infty_x(\mathbb{T}^d)$ is continuous, which allows us to treat nonlinear terms thanks to the underlying Banach algebra structure. Notice also that our analysis is based on the fact that $\mathcal{A}_\varepsilon \in \mathscr{B}(\mathcal{E}, \mathcal{H})$ and on the following continuous embeddings:

$$\mathcal{H} \hookrightarrow \mathcal{E}_2 \hookrightarrow \mathcal{E}_1 \hookrightarrow \mathcal{E}.$$

Since $\mathcal{A}_\varepsilon$ has no regularisation effect on the spatial variable, we are forced to have the same number of spatial derivates in the spaces $\mathcal{E}$ and $\mathcal{H}$. Taking $q \geqslant 3$ allows us to control the dissipation of kinetic energy $\mathcal{D}(f, f)$ defined in (1.10).

For the rest of the paper, we fix $\delta \in (0, \delta_q^\dagger)$ (with q fixed in (4.1) and $\delta_q^\dagger$ defined in Proposition 2.8) and $\varepsilon_1 \in (0, \varepsilon_0)$ (where ε_0 is defined in Assumption 1.1) such that $\alpha(\varepsilon) \in (0, \alpha_q^\dagger)$ for any $\varepsilon \in (0, \varepsilon_1)$ (where $\alpha_q^\dagger$ is defined in Proposition 2.8). It in particular implies that the conclusions of Proposition 2.8 and Remark 2.9 are satisfied for $\mathcal{B}_{\alpha(\varepsilon),\varepsilon} := \mathcal{B}^{(\delta)}_{\alpha(\varepsilon),\varepsilon}$ in the functional space $\mathcal{E}$ defined in (4.1) for any $\varepsilon \in (0, \varepsilon_1)$. In order to lighten the notations, we also denote $\mathcal{G}_\varepsilon := \mathcal{G}_{\alpha(\varepsilon),\varepsilon}$.

4.1. **Splitting of the equation.** As said in the Introduction, we adapt the approach of Briant et al. (2019) and decompose the solution $h = h_\varepsilon$ of (1.27) into

$$h(t,x,v) = h^0(t,x,v) + h^1(t,x,v)$$

where $h^0 = h^0_\varepsilon \in \mathcal{E}$ and $h^1 = h^1_\varepsilon \in \mathcal{H}$ are the solutions to the following system of equations (in order to lighten the notations, in this whole section, we shall omit the dependence on ε for h, h^0 and h^1):

$$\begin{cases} \partial_t h^0 &= \mathcal{B}_{\alpha(\varepsilon),\varepsilon} h^0 + \varepsilon^{-2}\Big[\mathbf{L}_{\alpha(\varepsilon)} h^1 - \mathbf{L}_1 h^1\Big] + (1-\alpha(\varepsilon))\varepsilon^{-2}\mathrm{a}_3 \varpi_2 \left(\mathbf{Id} - \mathbf{P}_0\right) h^1 \\ &\quad + \varepsilon^{-1}\Big[\mathcal{Q}_{\alpha(\varepsilon)}(h^0,h^0) + \mathcal{Q}_{\alpha(\varepsilon)}(h^0,h^1) + \mathcal{Q}_{\alpha(\varepsilon)}(h^1,h^0) + \boldsymbol{\pi}_0 \mathcal{Q}_{\alpha(\varepsilon)}(h^1,h^1)\Big], \\ h^0(0,x,v) &= h^\varepsilon_{\mathrm{in}}(x,v) \in \mathcal{E} \end{cases} \tag{4.3}$$

where a_3 is defined in Lemma 3.3, the projectors $\boldsymbol{\pi}_0$, $\mathbf{P}_0$ in (3.2)-(3.4) and

$$\left\{\begin{array}{rcl} \partial_t h^1 & = & \mathcal{G}_{1,\varepsilon} h^1 + (1-\alpha(\varepsilon))\varepsilon^{-2}\left[-\mathrm{div}_v\left(v h^1\right) - \mathrm{a}_3 \boldsymbol{\varpi}_2 \left(\mathbf{Id} - \mathbf{P}_0\right) h^1\right] \\ & & \qquad + \varepsilon^{-1}\left(\mathbf{Id} - \boldsymbol{\pi}_0\right) \mathcal{Q}_{\alpha(\varepsilon)}(h^1, h^1) + \mathcal{A}_\varepsilon h^0 \,, \\ h^1(0,x,v) & = & 0 \in \mathcal{H} \,. \end{array}\right. \tag{4.4}$$

Since

$$\mathcal{G}_\varepsilon h^1 = \mathcal{G}_{1,\varepsilon} h^1 - (1-\alpha(\varepsilon))\varepsilon^{-2}\mathrm{div}_v\left(v h^1\right) + (1-\alpha(\varepsilon))\varepsilon^{-2}\Big[\mathbf{L}_{\alpha(\varepsilon)} h^1 - \mathbf{L}_1 h^1\Big]$$

while

$$\mathcal{G}_\varepsilon h^0 = \mathcal{B}_{\alpha(\varepsilon),\varepsilon} h^0 + \mathcal{A}_\varepsilon h^0 \,,$$

one checks easily that $h = h^0 + h^1$ satisfies

$$\partial_t h = \mathcal{G}_\varepsilon h + \varepsilon^{-1}\mathcal{Q}_{\alpha(\varepsilon)}(h,h) \,, \qquad h(0,x,v) = h^\varepsilon_{\mathrm{in}}(x,v) \in \mathcal{E} \tag{4.5}$$

which is exactly (1.27). Comments about the splitting of (1.27) into two equations (4.3)-(4.4) are given in the Introduction.

Before starting the analysis of equations (4.3) and (4.4), we recall that h satisfies (see (1.33))

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} h(t,x,v) \begin{pmatrix} 1 \\ v \end{pmatrix} \mathrm{d}v\,\mathrm{d}x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.6}$$

Recalling the definition of $\mathbf{P}_0$ in (3.4), since the part of the projection related to the dissipation of energy will play a particular role in our analysis, we define

$$\mathbb{P}_0 h := \sum_{i=1}^{d+1} \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} h\,\Psi_i\,\mathrm{d}v\,\mathrm{d}x\right)\Psi_i\,\mathcal{M}\,, \quad \Pi_0 h := \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} h\Psi_{d+2}\,\mathrm{d}v\,\mathrm{d}x\right)\Psi_{d+2}\,\mathcal{M}\,. \tag{4.7}$$

Of course, from (3.4), one has $\mathbb{P}_0 = \mathbf{P}_0 - \Pi_0$. Recall also that the eigenfunctions Ψ_j are such that

$$\int_{\mathbb{R}^d} \Psi_i(v)\Psi_j(v)\mathcal{M}(v)\,\mathrm{d}v = \delta_{i,j}\,, \qquad \forall\, i,j = 1,\dots,d+2\,,$$

which in particular implies that, in the Hilbert space $\mathcal{H}$[4], one has $\mathbf{Id} - \mathbf{P}_0 = \mathbf{P}_0^\perp$.

The rest of the section is dedicated to the proof of *a priori estimates* on h^0 and h^1. To this end, during the rest of the section, we assume that $h^0 \in \mathcal{E}$, $h^1 \in \mathcal{H}$ are solutions to (4.3)-(4.4) and that there exists $\Delta_0 \leqslant 1$ such that

$$\sup_{t\geqslant 0}\left(\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{H}}\right) \leqslant \Delta_0\,. \tag{4.8}$$

Mention also that the multiplicative constants involved in the forthcoming estimates of this section may depend on $\Delta_0 \leqslant 1$. We will only mention it when necessary.

[4]Recall here that, on the space $L^2_v(\mathcal{M}^{-\frac{1}{2}})$ the inner product is $\langle f,g\rangle = \int_{\mathbb{R}^d} f(v)g(v)\mathcal{M}^{-1}(v)\,\mathrm{d}v$.

4.2. **Estimating h^0.** For the part of the solution $h^0(t)$ in $\mathcal{E}$, we have the following estimate:

Proposition 4.1. *Let $\mu_0 \in (0, \nu_q)$ (see Proposition 2.8 and Remark 2.9 for the definition of ν_q). Then, there exists an explicit $\varepsilon_2 \in (0, \varepsilon_1)$ such that for any $\varepsilon \in (0, \varepsilon_2)$ and any $t \geqslant 0$,*

$$\|h^0(t)\|_{\mathcal{E}} \lesssim \|h(0)\|_{\mathcal{E}}\, e^{-\frac{\mu_0}{\varepsilon^2}t} + \frac{1-\alpha(\varepsilon)}{\varepsilon^2}\int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)}\|h^1(s)\|_{\mathcal{H}}\,\mathrm{d}s\,. \tag{4.9}$$

As a consequence, for any $\varepsilon \in (0, \varepsilon_2)$ and any $t \geqslant 0$, there holds

$$\|h^0(t)\|^2_{\mathcal{E}} \lesssim \|h(0)\|^2_{\mathcal{E}}\, e^{-\frac{2\mu_0}{\varepsilon^2}t} + \frac{(1-\alpha(\varepsilon))^2}{\varepsilon^2}\int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)}\|h^1(s)\|^2_{\mathcal{H}}\,\mathrm{d}s\,. \tag{4.10}$$

Proof. From the definition of ε_1 and Remark 2.9, for any $\varepsilon \in (0, \varepsilon_1)$, we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|h^0(t)\|_{\mathcal{E}} \leqslant -\frac{\nu_q}{\varepsilon^2}\|h^0(t)\|_{\mathcal{E}_1} + \frac{1}{\varepsilon}\Big(&\|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t),h^0(t))\|_{\mathcal{E}} + \|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t),h^1(t))\|_{\mathcal{E}}\\ &+ \|\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t),h^0(t))\|_{\mathcal{E}} + \big\|\boldsymbol{\pi}_0\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t),h^1(t))\big\|_{\mathcal{E}}\Big)\\ &+ \frac{1}{\varepsilon^2}\big\|\mathbf{L}_{\alpha(\varepsilon)}h^1(t) - \mathbf{L}_1 h^1(t)\big\|_{\mathcal{E}} + \mathrm{a}_3\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\|\boldsymbol{\varpi}_2\left(\mathbf{Id}-\mathbf{P}_0\right)h^1(t)\|_{\mathcal{E}}\,.\end{aligned}$$

Using classical estimates for $\mathcal{Q}_{\alpha(\varepsilon)}$ and $\mathcal{Q}_1$ (see Corollary B.4), there exists $C>0$ such that

$$\begin{aligned}\|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t),h^0(t))\|_{\mathcal{E}} + \|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t),h^1(t))\|_{\mathcal{E}}&\\ + \|\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t),h^0(t))\|_{\mathcal{E}} \leqslant C\Big(\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{E}_1}\Big)&\|h^0(t)\|_{\mathcal{E}_1}\,.\end{aligned}$$

Thanks to Cauchy-Schwarz inequality and Lemma 2.4,

$$\begin{aligned}\frac{1}{\varepsilon^2}\big\|\mathbf{L}_{\alpha(\varepsilon)}h^1(t) - \mathbf{L}_1 h^1(t)\big\|_{\mathcal{E}} &\leqslant \frac{1}{\varepsilon^2}\big\|\mathbf{L}_{\alpha(\varepsilon)}h^1(t) - \mathbf{L}_1 h^1(t)\big\|_{L^2_v\mathbb{W}^{m,2}_x(\boldsymbol{\varpi}_{q+\kappa})}\\ &\leqslant C\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\|h^1(t)\|_{\mathcal{E}_2}\end{aligned}$$

(notice that such estimate is exactly what motivated the definition of $\mathcal{E}_2$). From the conservation of mass and momentum, one deduces from (3.2) that

$$\boldsymbol{\pi}_0\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t),h^1(t)) = \Psi_{d+2}\,\mathcal{M}\int_{\mathbb{R}^d}\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t),h^1(t))(w)\Psi_{d+2}(w)\,\mathrm{d}w.$$

Consequently, as in (1.10),

$$\begin{aligned}\boldsymbol{\pi}_0\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t),h^1(t)) = \Psi_{d+2}\,\mathcal{M}&\\ \frac{\gamma_b(1-\alpha^2(\varepsilon))}{4\vartheta_1\sqrt{2d}}&\int_{\mathbb{R}^d\times\mathbb{R}^d}h^1(t,x,w)h^1(t,x,v_*)|w-v_*|^3\,\mathrm{d}w\,\mathrm{d}v_*\end{aligned}$$

from which one deduces easily that there is $C>0$ such that

$$\begin{aligned}\|\boldsymbol{\pi}_0\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t),h^1(t))\|_{\mathcal{E}} &\leqslant C\,(1-\alpha(\varepsilon))\left(\int_{\mathbb{R}^d}\|h^1(t,\,\cdot\,,v_*)\|_{\mathbb{W}^{m,2}_x}\langle v_*\rangle^3\,\mathrm{d}v_*\right)^2\\ &\leqslant C\,(1-\alpha(\varepsilon))\,\|h^1(t)\|^2_{\mathcal{E}_1}\end{aligned}$$

where we used that $\mathbb{W}^{m,2}_x(\mathbb{T}^d)$ is a Banach algebra since $m>\frac{d}{2}$. Since moreover, we clearly have

$$\|\boldsymbol{\varpi}_2\,(\mathbf{Id}-\mathbf{P}_0)\,h^1(t)\|_{\mathcal{E}}\leqslant C\|h^1(t)\|_{\mathcal{E}_2}\,,$$

using that $\mathcal{H}\hookrightarrow\mathcal{E}_2\hookrightarrow\mathcal{E}_1$, we are able to conclude that there exists $C>0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|h^0(t)\|_{\mathcal{E}}\leqslant-\frac{1}{\varepsilon^2}\Big(\nu_q-\varepsilon\,C\big(\|h^0(t)\|_{\mathcal{E}}+\|h^1(t)\|_{\mathcal{H}}\big)\Big)\|h^0(t)\|_{\mathcal{E}_1}\\ +C\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\|h^1(t)\|_{\mathcal{H}}+C\frac{1-\alpha(\varepsilon)}{\varepsilon}\|h^1(t)\|^2_{\mathcal{H}}\,.\end{aligned}$$

For $\mu_0\in(0,\nu_q)$, we pick $\varepsilon_2\in(0,\varepsilon_1]$ as $\nu_q-\varepsilon_2\,C\,\Delta_0\geqslant\mu_0$. Consequently, using that $\varepsilon_2\|h^1(t)\|^2_{\mathcal{H}}\leqslant\varepsilon_2\Delta_0\|h^1(t)\|_{\mathcal{H}}\lesssim\|h^1(t)\|_{\mathcal{H}}$, we obtain that for any $\varepsilon\in(0,\varepsilon_2)$ and any $t\geqslant0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|h^0(t)\|_{\mathcal{E}}&\leqslant-\frac{\mu_0}{\varepsilon^2}\,\|h^0(t)\|_{\mathcal{E}_1}+C\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\|h^1(t)\|_{\mathcal{H}}+C\frac{1-\alpha(\varepsilon)}{\varepsilon}\|h^1(t)\|^2_{\mathcal{H}}\\ &\leqslant-\frac{\mu_0}{\varepsilon^2}\,\|h^0(t)\|_{\mathcal{E}_1}+C\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\|h^1(t)\|_{\mathcal{H}}\end{aligned}\tag{4.11}$$

which gives (4.9) after integration and using the fact that $h^0(0)=h(0)$. To prove (4.10), we use the fact that by Cauchy-Schwarz inequality, for any nonnegative mapping $t\mapsto\zeta(t)$ and any $\beta>0$, we have that for any $r\in(0,1)$,

$$\begin{aligned}\left(\int_0^t e^{-\beta\,(t-s)}\zeta(s)\,\mathrm{d}s\right)^2&\leqslant\left(\int_0^t e^{-2r\beta\,(t-s)}\,\mathrm{d}s\right)\left(\int_0^t e^{-2(1-r)\beta\,(t-s)}\zeta(s)^2\,\mathrm{d}s\right)\\ &\leqslant\frac{1}{2r\beta}\int_0^t e^{-2(1-r)\beta\,(t-s)}\zeta(s)^2\,\mathrm{d}s\,,\qquad\forall\,t\geqslant0\,.\end{aligned}\tag{4.12}$$

This inequality applied with $r=\frac12$ gives the result. □

4.3. **Estimating $\mathbf{P}_0h^1$.** Let us point out that getting estimates on h^1 is trickier than in Briant et al. (2019), indeed, in the latter paper, the idea was to estimate separately $\mathbf{P}_0h^1$ and $(\mathbf{Id}-\mathbf{P}_0)h^1$ where $\mathbf{P}_0$ is the projector onto $\mathrm{Ker}(\mathcal{G}_{1,\varepsilon})$ defined by (3.4) and thanks to the properties of conservation of mass, momentum and energy of the whole equation, one could write that $\mathbf{P}_0h=0$ so that $\mathbf{P}_0h^1=-\mathbf{P}_0h^0$ and directly get an estimate on $\mathbf{P}_0h^1$ from the one on h^0. In our case, the energy is no longer preserved which induces additional difficulties. However, we keep the same strategy and start by estimating $\mathbf{P}_0h^1$. To this end, we begin with two observations. The first one is related to $\mathbb{P}_0h^1$ where $\mathbb{P}_0$ is defined in (4.7):

Lemma 4.2. *For any $t \geqslant 0$, there holds*

$$\|\mathbb{P}_0 h^1(t)\|_{\mathcal{E}} \lesssim \|h^0(t)\|_{\mathcal{E}} \,.$$

Proof. Note that total mass and vanishing momentum conservation yields for $i = 1, \dots, d+1$,

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} h^0(t,x,v) \Psi_i(v) \, \mathrm{d}v \, \mathrm{d}x + \int_{\mathbb{T}^d \times \mathbb{R}^d} h^1(t,x,v) \Psi_i(v) \, \mathrm{d}v \, \mathrm{d}x = 0 \,.$$

Thus, for any $i = 1, \dots, d+1$,

$$\left| \int_{\mathbb{T}^d \times \mathbb{R}^d} h^1(t,x,v) \Psi_i(v) \, \mathrm{d}v \, \mathrm{d}x \right| = \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} h^0(t,x,v) \Psi_i(v) \, \mathrm{d}v \, \mathrm{d}x \right| \leqslant \max\left(1, \frac{1}{\sqrt{\vartheta_1}}\right) \|h^0(t)\|_{\mathcal{E}}$$

thanks to Cauchy-Schwarz inequality and since $|\Psi_i(v)| \leqslant \max\left(1, \frac{1}{\sqrt{\vartheta_1}}\right) \varpi_q(v)$ for any $i = 1, \dots, d+1$. The final estimate for the projection follows from the previous inequality and (4.7) since $\max_{i=1,\dots,d+1} \|\Psi_i \mathcal{M}\|_{\mathcal{E}} < \infty$. □

A second observation regards the action of Π_0 on the linearized operator $\mathcal{G}_\varepsilon$.

Lemma 4.3. *One has for any $t \geqslant 0$,*

$$\Pi_0 \left[\mathcal{G}_\varepsilon h(t)\right] = \frac{1-\alpha(\varepsilon)}{\varepsilon^2} \left(-1 + r_\varepsilon\right) \Pi_0 h(t) + s_\varepsilon(t) \Psi_{d+2} \mathcal{M} \tag{4.13}$$

where we recall that h is a solution to (4.5) and Ψ_{d+2} is defined in (3.3). Moreover, $r_\varepsilon \in \mathbb{R}$ (independent of t) and $s_\varepsilon(t) \in \mathbb{R}$ are such that for any $t \geqslant 0$,

$$|r_\varepsilon| \lesssim 1 - \alpha(\varepsilon) \,, \qquad |s_\varepsilon(t)| \lesssim \frac{1-\alpha(\varepsilon)}{\varepsilon^2} \left\| \left(\mathbf{Id} - \mathbf{P}_0\right) h(t) \right\|_{L^1_{x,v}(\varpi_3)} \,. \tag{4.14}$$

Proof. The proof is by direct inspection. One first recalls that

$$\begin{aligned} \Pi_0 h &= \frac{1}{\vartheta_1 \sqrt{2d}} \left(\int_{\mathbb{R}^d \times \mathbb{T}^d} h \left(|v|^2 - d\vartheta_1 \right) \, \mathrm{d}v \, \mathrm{d}x \right) \Psi_{d+2} \mathcal{M} \,, \\ &= \frac{1}{\vartheta_1 \sqrt{2d}} \left(\int_{\mathbb{R}^d \times \mathbb{T}^d} h \, |v|^2 \, \mathrm{d}v \, \mathrm{d}x \right) \Psi_{d+2} \mathcal{M} \,. \end{aligned}$$

Notice that since $\displaystyle\int_{\mathbb{T}^d \times \mathbb{R}^d} \mathcal{G}_\varepsilon h \, \mathrm{d}v \, \mathrm{d}x = 0 = \int_{\mathbb{T}^d \times \mathbb{R}^d} \mathcal{G}_{1,\varepsilon} h |v|^2 \, \mathrm{d}v \, \mathrm{d}x$, one has

$$\begin{aligned} \Pi_0 \left[\mathcal{G}_\varepsilon h(t)\right] &= \frac{1}{\varepsilon^2} \frac{1}{\vartheta_1 \sqrt{2d}} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} \mathscr{L}_{\alpha(\varepsilon)} h(t,x,v) |v|^2 \, \mathrm{d}v \, \mathrm{d}x \right) \Psi_{d+2} \mathcal{M} \\ &= \frac{1}{\varepsilon^2} \frac{1}{\vartheta_1 \sqrt{2d}} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\mathbf{L}_{\alpha(\varepsilon)} h(t,x,v) - (1-\alpha(\varepsilon)) \nabla_v \cdot \left(v h(t,x,v) \right) \right) |v|^2 \, \mathrm{d}v \, \mathrm{d}x \right) \Psi_{d+2} \mathcal{M} \\ &= \frac{1}{\varepsilon^2} \frac{1}{\vartheta_1 \sqrt{2d}} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\mathbf{L}_{\alpha(\varepsilon)} h(t,x,v) + 2(1-\alpha(\varepsilon)) h(t,x,v) \right) \right) |v|^2 \, \mathrm{d}v \, \mathrm{d}x \right) \Psi_{d+2} \mathcal{M} \,. \end{aligned}$$

Now, as in (1.10), one can check that

$$\int_{\mathbb{T}^d\times\mathbb{R}^d}\mathbf{L}_{\alpha(\varepsilon)}h(t,x,v)|v|^2\,\mathrm{d}v\,\mathrm{d}x=-\frac{1-\alpha^2(\varepsilon)}{2}\gamma_b\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}h(t,x,v)G_{\alpha(\varepsilon)}(v_*)|v-v_*|^3\,\mathrm{d}v_*\,\mathrm{d}v\,\mathrm{d}x$$

which, writing first $h=\Pi_0 h+(\mathbf{Id}-\Pi_0)\,h$ and then $G_{\alpha(\varepsilon)}=\mathcal{M}+(G_{\alpha(\varepsilon)}-\mathcal{M})$ gives

$$\Pi_0\left[\mathcal{G}_\varepsilon h(t)\right]=a_\varepsilon\Pi_0 h+s_\varepsilon(t)\Psi_{d+2}\mathcal{M}$$

where

$$\begin{aligned}a_\varepsilon:=\frac{1}{\varepsilon^2}\Bigg\{&2(1-\alpha(\varepsilon))-\frac{1-\alpha^2(\varepsilon)}{\vartheta_1\sqrt{2d}}\frac{\gamma_b}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\Psi_{d+2}(v)\mathcal{M}(v)\mathcal{M}(v_*)|v-v_*|^3\,\mathrm{d}v_*\,\mathrm{d}v\\&-\frac{1-\alpha^2(\varepsilon)}{\vartheta_1\sqrt{2d}}\frac{\gamma_b}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\Psi_{d+2}(v)\mathcal{M}(v)\left[G_{\alpha(\varepsilon)}(v_*)-\mathcal{M}(v_*)\right]|v-v_*|^3\,\mathrm{d}v_*\,\mathrm{d}v\Bigg\}\end{aligned}$$

and

$$s_\varepsilon(t):=-\frac{1}{\varepsilon^2}\frac{1-\alpha^2(\varepsilon)}{\vartheta_1\sqrt{2d}}\frac{\gamma_b}{2}\left(\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left[\mathbf{Id}-\Pi_0\right]h(t,x,v)G_{\alpha(\varepsilon)}(v_*)|v-v_*|^3\,\mathrm{d}v_*\,\mathrm{d}v\,\mathrm{d}x\right).$$

One has (see (Mischler & Mouhot, 2009, Lemma 5.19, Eq. (5.10)))

$$\frac{1}{\vartheta_1\sqrt{2d}}\frac{\gamma_b}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\Psi_{d+2}(v)\mathcal{M}(v)\mathcal{M}(v_*)|v-v_*|^3\,\mathrm{d}v\,\mathrm{d}v_*=\frac{3}{2}$$

which results easily in

$$\begin{aligned}a_\varepsilon=\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\Bigg\{&2-\frac{3}{2}(1+\alpha(\varepsilon))\\&-\frac{1+\alpha(\varepsilon)}{\vartheta_1\sqrt{2d}}\frac{\gamma_b}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\Psi_{d+2}(v)\mathcal{M}(v)\left[G_{\alpha(\varepsilon)}(v_*)-\mathcal{M}(v_*)\right]|v-v_*|^3\,\mathrm{d}v_*\,\mathrm{d}v\Bigg\}.\end{aligned}$$

Writing simply $1+\alpha(\varepsilon)=2-(1-\alpha(\varepsilon))$, one sees that

$$a_\varepsilon=\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\left(-1+r_\varepsilon\right),$$

with

$$|r_\varepsilon|\leqslant\frac{3}{2}\left(1-\alpha(\varepsilon)\right)+\frac{\gamma_b}{\vartheta_1\sqrt{2d}}\|\Psi_{d+2}\mathcal{M}\|_{L^1_v(\varpi_3)}\left\|G_{\alpha(\varepsilon)}-\mathcal{M}\right\|_{L^1_v(\varpi_3)}\lesssim 1-\alpha(\varepsilon)$$

thanks to Lemma 2.3. The bound on $s_\varepsilon(t)$ is also obvious since, for solution h to (1.27), conservation of mass and vanishing momentum implies that $\Pi_0 h(t)=\mathbf{P}_0 h(t)$. We thus obtain the desired result. □

In all the sequel, we will denote

$$\lambda_\varepsilon := \frac{1-\alpha(\varepsilon)}{\varepsilon^2}\,(1-r_\varepsilon) \underset{\varepsilon\to 0}{\sim} \frac{1-\alpha(\varepsilon)}{\varepsilon^2} \underset{\varepsilon\to 0}{\sim} \lambda_0 + \eta(\varepsilon) \tag{4.15}$$

where λ_0 and η are defined in Assumption 1.1. Notice that this assumption implies that $\lambda_\varepsilon > 0$ for ε small enough so that there exists $\varepsilon_3 \in (0,\varepsilon_2)$ (where ε_2 is defined in Proposition 4.1) such that

$$0 < \frac{\lambda_\varepsilon}{2} \leqslant \frac{1-\alpha(\varepsilon)}{\varepsilon^2} \leqslant 2\lambda_\varepsilon\,, \qquad \forall\, \varepsilon \in (0,\varepsilon_3)\,. \tag{4.16}$$

We are now able to derive a nice estimate on $\mathbf{P}_0 h^1$:

Lemma 4.4. *We have that for any $\varepsilon \in (0,\varepsilon_3)$ and any $t \geqslant 0$,*

$$\begin{aligned}\|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}} \lesssim \|h^0(t)\|_{\mathcal{E}} + \|h(0)\|_{\mathcal{E}} e^{-\lambda_\varepsilon t} + \varepsilon\lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon (t-s)} \|h^1(s)\|_{\mathcal{H}}\,\mathrm{d}s \\ + \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon (t-s)} \Big(\big\|h^0(s)\big\|_{\mathcal{E}} + \big\|(\mathbf{Id}-\mathbf{P}_0)\,h^1(s)\big\|_{\mathcal{H}} \Big)\,\mathrm{d}s\,.\end{aligned} \tag{4.17}$$

Proof. Due to the properties of preservation of mass and vanishing momentum of our equation, we have $\mathbb{P}_0 h = 0$ (where $\mathbb{P}_0$ has been defined in (4.7)) which implies that $\mathbb{P}_0 h^1 = -\mathbb{P}_0 h^0$. Consequently, we easily get an estimate on $\mathbb{P}_0 h^1$ using that $\mathbb{P}_0 \in \mathscr{B}(\mathcal{E},\mathcal{H})$:

$$\|\mathbb{P}_0 h^1(t)\|_{\mathcal{H}} \lesssim \|h^0(t)\|_{\mathcal{E}}\,. \tag{4.18}$$

It remains to estimate $\Pi_0 h^1$. To this end, we first notice that

$$\Pi_0 h^1 = \mathbf{P}_0 h^1 - \mathbb{P}_0 h^1 = \mathbf{P}_0 h - \mathbf{P}_0 h^0 - \mathbb{P}_0 h^1 = \Pi_0 h - \mathbf{P}_0 h^0 - \mathbb{P}_0 h^1$$

where we used that $\mathbf{P}_0 h = \Pi_0 h$ due to the conservation of mass and vanishing momentum so, using (5.1) with the fact that $\mathbf{P}_0 \in \mathscr{B}(\mathcal{E},\mathcal{H})$ and (4.18), we only need to estimate $\Pi_0 h$ to get an estimate on $\Pi_0 h^1$. To this end, we start by computing the evolution of $\Pi_0 h$. We recall that the equation for h is given by

$$\partial_t h = \mathcal{G}_\varepsilon h + \frac{1}{\varepsilon}\mathcal{Q}_{\alpha(\varepsilon)}(h,h)$$

and that Π_0 has been defined in (4.7). Thus, applying the projection Π_0 and using (4.13)-(4.15)

$$\partial_t\big(\Pi_0 h\big) = -\lambda_\varepsilon \Pi_0 h + s_\varepsilon(t)\Psi_{d+2}\mathcal{M} + \frac{1}{\varepsilon}\Pi_0\mathcal{Q}_{\alpha(\varepsilon)}(h,h)\,,$$

so that

$$\Pi_0 h(t) = \Pi_0 h(0)\, e^{-\lambda_\varepsilon\, t} + \int_0^t e^{-\lambda_\varepsilon\,(t-s)} \left(\frac{1}{\varepsilon}\Pi_0\mathcal{Q}_{\alpha(\varepsilon)}(h(s),h(s)) + s_\varepsilon(s)\Psi_{d+2}\mathcal{M} \right)\,\mathrm{d}s\,. \tag{4.19}$$

Notice that, according to (4.7), $\Pi_0\mathcal{Q}_{\alpha(\varepsilon)}$ is explicit with

$$\big\|\Pi_0\mathcal{Q}_{\alpha(\varepsilon)}(h(s),h(s))\big\|_{\mathcal{H}} = \frac{1}{\vartheta_1\sqrt{2d}}(1-\alpha^2(\varepsilon))\Big|\mathcal{D}(h(s),h(s))\Big|\big\|\Psi_{d+2}\mathcal{M}\big\|_{\mathcal{H}}\,,$$

where we recall that $\mathcal{D}(g,g)$ denotes the normalized energy dissipation associated to $\mathcal{Q}_\alpha$, namely,

$$\mathcal{D}(g,g) = \frac{\gamma_b}{4}\int_{\mathbb{T}^d}\mathrm{d}x\int_{\mathbb{R}^d\times\mathbb{R}^d} g(x,v)g(x,v_*)|v-v_*|^3\,\mathrm{d}v_*\,\mathrm{d}v$$

where γ_b independent of α, see (1.10). Now, one clearly has

$$|\mathcal{D}(h(s),h(s))| \lesssim \int_{\mathbb{T}^d}\left[\int_{\mathbb{R}^d}\varpi_3(v)|h(s,x,v)|\,\mathrm{d}v\right]^2\mathrm{d}x$$

and, using Minkowski's integral inequality, we deduce that

$$|\mathcal{D}(h(s),h(s))| \lesssim \left[\int_{\mathbb{R}^d}\varpi_3(v)\left(\int_{\mathbb{T}^d}|h(s,x,v)|^2\,\mathrm{d}x\right)^{\frac12}\mathrm{d}v\right]^2 \lesssim \|h(s)\|^2_{L^1_vL^2_x(\varpi_3)}\,.$$

Therefore,

$$\big\|\Pi_0\mathcal{Q}_{\alpha(\varepsilon)}(h(s),h(s))\big\|_{\mathcal{H}} \lesssim (1-\alpha(\varepsilon))\|h(s)\|^2_{\mathcal{E}}$$

because $q\geqslant 3$. Thus, applying the $\|\cdot\|_{\mathcal{H}}$-norm in (4.19), one obtains

$$\begin{aligned}\big\|\Pi_0h(t)\big\|_{\mathcal{H}} \lesssim \|\Pi_0h(0)\|_{\mathcal{H}}\,e^{-\lambda_\varepsilon t} + \frac{1-\alpha(\varepsilon)}{\varepsilon}\int_0^t e^{-\lambda_\varepsilon(t-s)}\|h(s)\|^2_{\mathcal{E}}\,\mathrm{d}s\\ + \lambda_\varepsilon\int_0^t e^{-\lambda_\varepsilon(t-s)}\,\|(\mathbf{Id}-\mathbf{P}_0)\,h(s)\|_{\mathcal{E}}\,\mathrm{d}s\end{aligned}\tag{4.20}$$

where we used (4.14) to estimate $\|s_\varepsilon(s)\Psi_{d+2}\mathcal{M}\|_{\mathcal{H}}$. We obtain the desired estimate using that $\Pi_0\in\mathscr{B}(\mathcal{E},\mathcal{H})$, (4.16) and (4.8). □

We make more precise our estimates of $\mathbf{P}_0h^1(t)$ in the following proposition:

Proposition 4.5. *There exists an explicit $\varepsilon_4\in(0,\varepsilon_3)$ such that for any $\varepsilon\in(0,\varepsilon_4)$ and any $t\geqslant 0$, there holds*

$$\begin{aligned}\|\mathbf{P}_0h^1(t)\|_{\mathcal{H}} \lesssim \|h(0)\|_{\mathcal{E}}e^{-\lambda_\varepsilon t} + \lambda_\varepsilon\int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)}\|h^1(s)\|_{\mathcal{H}}\,\mathrm{d}s\\ + \varepsilon\lambda_\varepsilon\int_0^t e^{-\lambda_\varepsilon(t-s)}\|h^1(s)\|_{\mathcal{H}}\,\mathrm{d}s + \lambda_\varepsilon\int_0^t e^{-\lambda_\varepsilon(t-s)}\|(\mathbf{Id}-\mathbf{P}_0)h^1(s)\|_{\mathcal{H}}\,\mathrm{d}s\,.\end{aligned}\tag{4.21}$$

In particular, for any $r\in(0,1)$, any $\varepsilon\in(0,\varepsilon_4)$ and any $t\geqslant 0$, there holds

$$\begin{aligned}\|\mathbf{P}_0h^1(t)\|^2_{\mathcal{H}} \lesssim \|h(0)\|^2_{\mathcal{E}}e^{-2\lambda_\varepsilon t} + \frac{\varepsilon^2\lambda_\varepsilon}{r}\int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)}\|h^1(s)\|^2_{\mathcal{H}}\,\mathrm{d}s\\ + \frac{\lambda_\varepsilon}{r}\int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)}\,\big\|(\mathbf{Id}-\mathbf{P}_0)\,h^1(s)\big\|^2_{\mathcal{H}}\,\mathrm{d}s\,,\end{aligned}\tag{4.22}$$

where the multiplicative constant involved in the previous inequality does not depend on r.

Proof. We insert the bound for $\|h^0(t)\|_{\mathcal{E}}$ obtained in (4.9) combined with (4.16) in the estimate of Lemma 4.4, for any $\varepsilon \in (0,\varepsilon_3)$ and any $t \geqslant 0$, we have:

$$\begin{aligned}\|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}} \lesssim \|h(0)\|_{\mathcal{E}} \left(e^{-\frac{\mu_0}{\varepsilon^2}t} + e^{-\lambda_\varepsilon t}\right) + \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)}\|h^1(s)\|_{\mathcal{H}}\,\mathrm{d}s \\ + \varepsilon\lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)}\|h^1(s)\|_{\mathcal{H}}\,\mathrm{d}s + \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)}\|(\mathbf{Id}-\mathbf{P}_0)h^1(s)\|_{\mathcal{H}}\,\mathrm{d}s \\ + \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)}e^{-\frac{\mu_0}{\varepsilon^2}s}\,\mathrm{d}s\|h(0)\|_{\mathcal{E}} + \lambda_\varepsilon^2 \int_0^t e^{-\lambda_\varepsilon(t-s)}\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)}\|h^1(\tau)\|_{\mathcal{H}}\,\mathrm{d}\tau\,\mathrm{d}s\,. \quad (4.23)\end{aligned}$$

We now choose $\varepsilon_4 \in (0,\varepsilon_3)$ such that $\mu_0 \geqslant 2\varepsilon^2\lambda_\varepsilon$ for $\varepsilon \in (0,\varepsilon_4)$. We also remark that for any $\beta_1 > \beta_2 > 0$ and nonnegative mapping $t \mapsto \zeta(t)$

$$\begin{aligned}\int_0^t e^{-\beta_2(t-s)}\,\mathrm{d}s \int_0^s e^{-\beta_1(s-\tau)}\zeta(\tau)\,\mathrm{d}\tau = e^{-\beta_2 t}\int_0^t e^{\beta_1\tau}\zeta(\tau)\,\mathrm{d}\tau \int_\tau^t e^{-(\beta_1-\beta_2)s}\,\mathrm{d}s \\ \leqslant \frac{1}{\beta_1-\beta_2}\int_0^t e^{-\beta_2(t-\tau)}\zeta(\tau)\,\mathrm{d}\tau\,.\end{aligned} \quad (4.24)$$

Using this estimate with $\beta_1 = \varepsilon^{-2}\mu_0$ and $\beta_2 = \lambda_\varepsilon$ to bound the last term in (4.23) and keeping only the dominant terms, we obtain the wanted estimate (4.21). Concerning (4.22), using (4.12), we get

$$\begin{aligned}\|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}}^2 \lesssim \|h(0)\|_{\mathcal{E}}^2 e^{-2\lambda_\varepsilon t} + (\varepsilon\,\lambda_\varepsilon)^2 \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)}\|h^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s \\ + \frac{\varepsilon^2\lambda_\varepsilon}{r}\int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)}\|h^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s \\ + \frac{\lambda_\varepsilon}{r}\int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)}\left\|(\mathbf{Id}-\mathbf{P}_0)\,h^1(s)\right\|_{\mathcal{H}}^2\,\mathrm{d}s\,,\end{aligned}$$

which provides the wanted result by keeping only the dominant terms. □

4.4. **Estimating the complement $(\mathbf{Id}-\mathbf{P}_0)h^1$.** Let us focus on an estimate on $\mathbf{P}_0^\perp h^1(t)$ with $\mathbf{P}_0^\perp := \mathbf{Id}-\mathbf{P}_0$, the orthogonal projection onto $(\mathrm{Ker}(\mathcal{G}_{1,\varepsilon}))^\perp$ in the Hilbert space $L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})$. The same notation for the operator $\mathcal{G}_{1,\varepsilon}$ in the spaces $\mathcal{E}$ and $\mathcal{H}$ is used. Let us highlight the fact that we can proceed in a similar way as in Briant et al. (2019) to estimate $\mathbf{P}_0^\perp h^1$. More precisely, one crucial point in their estimate of $\mathbf{P}_0^\perp h^1$ was that $\boldsymbol{\pi}_0\mathcal{Q}_1(h^1,h^1)=0$ where $\boldsymbol{\pi}_0$ is defined in (3.2). In order to mimic this approach, we have only put $(\mathbf{Id}-\boldsymbol{\pi}_0)\mathcal{Q}_\alpha(h^1,h^1)$ in our equation (4.4) so that $\boldsymbol{\pi}_0(\mathbf{Id}-\boldsymbol{\pi}_0)\mathcal{Q}_\alpha(h^1,h^1)=0$. Notice that due to the dissipation of energy, we do not have anymore that $\boldsymbol{\pi}_0\mathcal{Q}_\alpha(h^1,h^1)=0$ for $\alpha\neq 1$, which is why the splitting of $\mathcal{Q}_\alpha(h^1,h^1)$ into two parts in (4.3)-(4.4) is so important. Using Proposition 3.1 together with Lemmas 3.3 and 3.4, we are able to obtain some nice estimate on $\mathbf{P}_0^\perp h^1$.

Lemma 4.6. *With the notations of Proposition 3.1, assume that Δ_0 defined in (4.8) is small enough so that*

$$\tilde{\mu}_1 := 2\mathrm{a}_1 - c_1\Delta_0^2 > 0 \tag{4.25}$$

where $c_1 > 0$ is a universal constant defined in (4.30). Then, for any $\varepsilon \in (0, \varepsilon_4)$ (where ε_4 is defined in Proposition 4.5) and any $t \geqslant 0$, there holds

$$\begin{aligned}
&\|(\mathbf{Id} - \mathbf{P}_0)h^1(t)\|_{\mathcal{H}}^2 \\
&\quad \lesssim (\lambda_\varepsilon + \Delta_0)\int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s + \frac{1}{\varepsilon^2}\int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}}\,\|h^0(s)\|_{\mathcal{E}}\,\mathrm{d}s
\end{aligned} \tag{4.26}$$

where $\mu_1 := C_{\mathcal{H}}^2\tilde{\mu}_1$ with $C_{\mathcal{H}}$ is the constant appearing in (3.6) related to the equivalence between to $|||\cdot|||_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$.

Proof. Set $\Psi(t) := \mathbf{P}_0^\perp h^1(t)$ for any $t \geqslant 0$. We start by recalling that $h^1(0) = 0$ so that $\Psi(0) = 0$. One checks from (4.4) that

$$\begin{aligned}
\partial_t\Psi = \mathcal{G}_{1,\varepsilon}\Psi + \mathbf{P}_0^\perp\left(\varepsilon^{-1}\left(\mathbf{Id} - \boldsymbol{\pi}_0\right)\mathcal{Q}_{\alpha(\varepsilon)}(h^1, h^1) + \mathcal{A}_\varepsilon h^0\right) \\
- (1-\alpha(\varepsilon))\varepsilon^{-2}\mathbf{P}_0^\perp\left(\nabla_v\cdot\left(vh^1\right)\right) - \mathrm{a}_3(1-\alpha(\varepsilon))\varepsilon^{-2}\mathbf{P}_0^\perp(\boldsymbol{\varpi}_2\Psi)\,.
\end{aligned} \tag{4.27}$$

One observes then that

$$\mathbf{P}_0^\perp\left(\mathbf{Id} - \boldsymbol{\pi}_0\right)\mathcal{Q}_{\alpha(\varepsilon)}(h^1, h^1) = \left(\mathbf{Id} - \boldsymbol{\pi}_0\right)\mathcal{Q}_{\alpha(\varepsilon)}(h^1, h^1)$$

since $\mathbf{P}_0\boldsymbol{\pi}_0 = \mathbf{P}_0$ while,

$$\mathbf{P}_0^\perp\left(\nabla_v\cdot\left(vh^1\right)\right) = \nabla_v\cdot\left(v\Psi\right) - \mathbf{P}_0\nabla_v\cdot\left(vh^1\right) + \nabla_v\cdot\left(v\mathbf{P}_0h^1\right)$$

and $\mathbf{P}_0^\perp(\boldsymbol{\varpi}_2\Psi) = \boldsymbol{\varpi}_2\Psi - \mathbf{P}_0(\boldsymbol{\varpi}_2\Psi)$. We recall that, according to Proposition 3.1, there is a norm $|||\cdot|||_{\mathcal{H}}$ which is equivalent to $\|\cdot\|_{\mathcal{H}}$ independently of ε and which allows us to write nice energy estimates with respect to the associated inner product $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle_{\mathcal{H}}$. From (4.27), we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||\Psi(t)|||_{\mathcal{H}}^2 = \langle\!\langle\mathcal{G}_{1,\varepsilon}\Psi(t), \Psi(t)\rangle\!\rangle_{\mathcal{H}} + \varepsilon^{-1}\langle\!\langle\left(\mathbf{Id} - \boldsymbol{\pi}_0\right)\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t), h^1(t)), \Psi(t)\rangle\!\rangle_{\mathcal{H}} \\
\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\Big(-\langle\!\langle\nabla_v\cdot(v\Psi(t)), \Psi(t)\rangle\!\rangle_{\mathcal{H}} - \mathrm{a}_3\langle\!\langle\boldsymbol{\varpi}_2\Psi(t), \Psi(t)\rangle\!\rangle_{\mathcal{H}}\Big) \\
- \frac{1-\alpha(\varepsilon)}{\varepsilon^2}\langle\!\langle\mathbf{P}_0\nabla_v\cdot\left(vh^1\right) - \nabla_v\cdot\left(v\mathbf{P}_0h^1\right) + \mathrm{a}_3\mathbf{P}_0(\boldsymbol{\varpi}_2\Psi), \Psi(t)\rangle\!\rangle_{\mathcal{H}} \\
+ \langle\!\langle\mathbf{P}_0^\perp\mathcal{A}_\varepsilon h^0(t), \Psi(t)\rangle\!\rangle_{\mathcal{H}} =: I_1 + I_2 + I_3 + I_4 + I_5\,.
\end{aligned}$$

We estimate each of the terms independently. First, according to Proposition 3.1,

$$I_1 \leqslant -\frac{\mathrm{a}_1}{\varepsilon^2}\|\left(\mathbf{Id} - \boldsymbol{\pi}_0\right)\Psi(t)\|_{\mathcal{H}_1}^2 - \mathrm{a}_1\|\Psi(t)\|_{\mathcal{H}_1}^2$$

while, according to Lemma 3.3,

$$I_3 \leqslant C\varepsilon^{-1}(1-\alpha(\varepsilon))\|\Psi(t)\|_{\mathcal{H}}^2 \leqslant C\varepsilon^{-2}(1-\alpha(\varepsilon))\|h^1(t)\|_{\mathcal{H}}^2\,.$$

We deduce from Lemma 3.4 that there is $C>0$ such that

$$I_2 = \varepsilon^{-1}\langle\!\langle (\mathbf{Id}-\boldsymbol{\pi}_0)\,\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t),h^1(t)),\Psi(t)\rangle\!\rangle_{\mathcal{H}} \\ \leqslant C\varepsilon^{-1}\|h^1(t)\|_{\mathcal{H}_1}\|h^1(t)\|_{\mathcal{H}}\,\|\,(\mathbf{Id}-\boldsymbol{\pi}_0)\,\Psi(t)\|_{\mathcal{H}_1} \\ + C\|h^1(t)\|_{\mathcal{H}}^2\,\|\boldsymbol{\pi}_0\Psi(t)\|_{\mathcal{H}}\,.$$

According to Young's inequality,

$$I_2 \leqslant \frac{\eta}{\varepsilon^2}\,\|(\mathbf{Id}-\boldsymbol{\pi}_0)\,\Psi(t)\|_{\mathcal{H}_1}^2 + \frac{C^2}{4\eta}\|h^1(t)\|_{\mathcal{H}}^2\|h^1(t)\|_{\mathcal{H}_1}^2 + C\|h^1(t)\|_{\mathcal{H}}^3\,, \qquad \eta>0\,. \tag{4.28}$$

Obviously, since $|||\cdot|||_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent norms, there is $C>0$ such that

$$I_5 \leqslant C\|\Psi(t)\|_{\mathcal{H}}\,\|\mathbf{P}_0^{\perp}\mathcal{A}_\varepsilon h^0(t)\|_{\mathcal{H}} \leqslant \frac{C}{\varepsilon^2}\|h^1(t)\|_{\mathcal{H}}\|h^0(t)\|_{\mathcal{E}}$$

where we used the regularization properties of $\mathcal{A}$ (see Lemma 2.6) to get the last inequality. Finally, using the regularizing properties of $\mathbf{P}_0$, it is easy to see that

$$I_4 \leqslant C\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\|h^1(t)\|_{\mathcal{H}}^2\,.$$

Therefore, choosing $\eta \leqslant \mathrm{a}_1$ in (4.28), one sees that there is some positive constant $c_0>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}|||\Psi(t)|||_{\mathcal{H}}^2 \leqslant -2\mathrm{a}_1\|\Psi(t)\|_{\mathcal{H}_1}^2 + c_0\|h^1(t)\|_{\mathcal{H}}^2\left(\|h^1(t)\|_{\mathcal{H}_1}^2+\|h^1(t)\|_{\mathcal{H}}\right) \\ + c_0\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\|h^1(t)\|_{\mathcal{H}}^2 + \frac{c_0}{\varepsilon^2}\|h^1(t)\|_{\mathcal{H}}\,\|h^0(t)\|_{\mathcal{E}}\,. \tag{4.29}$$

Writing $h^1=\mathbf{P}_0h^1+\Psi$, we obtain

$$\|h^1(t)\|_{\mathcal{H}}^2\,\left(\|h^1(t)\|_{\mathcal{H}_1}^2+\|h^1(t)\|_{\mathcal{H}}\right) \lesssim \|h^1(t)\|_{\mathcal{H}}^2\Big(\|\mathbf{P}_0h^1(t)\|_{\mathcal{H}_1}^2+\|\Psi(t)\|_{\mathcal{H}_1}^2+\|h^1(t)\|_{\mathcal{H}}\Big) \\ \lesssim \|h^1(t)\|_{\mathcal{H}}^2\Big(\|\Psi(t)\|_{\mathcal{H}_1}^2+\|h^1(t)\|_{\mathcal{H}}^2+\|h^1(t)\|_{\mathcal{H}}\Big)$$

from which we deduce that there exists $c_1>0$ such that

$$c_0\|h^1(t)\|_{\mathcal{H}}^2\,\left(\|h^1(t)\|_{\mathcal{H}_1}^2+\|h^1(t)\|_{\mathcal{H}}\right) \leqslant c_1\Delta_0^2\|\Psi(t)\|_{\mathcal{H}_1}^2 + c_1\Delta_0\|h^1(t)\|_{\mathcal{H}}^2 \tag{4.30}$$

where we used (4.8) and the fact that we assumed $\Delta_0\leqslant 1$. Therefore, assuming that Δ_0 is small enough so that $\widetilde{\mu}_1 := 2\mathrm{a}_1 - c_1\Delta_0^2>0$, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}|||\Psi(t)|||_{\mathcal{H}}^2 \leqslant -\widetilde{\mu}_1\,\|\Psi(t)\|_{\mathcal{H}_1}^2 + c_1\Delta_0\|h^1(t)\|_{\mathcal{H}}^2 \\ + c_0\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\|h^1(t)\|_{\mathcal{H}}^2 + \frac{c_0}{\varepsilon^2}\|h^1(t)\|_{\mathcal{H}}\,\|h^0(t)\|_{\mathcal{E}}\,, \qquad \forall\, t\geqslant 0\,.$$

Using (4.16), we deduce that there exists $c_2 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}|||\Psi(t)|||_{\mathcal{H}}^2 \leqslant -\widetilde{\mu}_1 \, \|\Psi(t)\|_{\mathcal{H}_1}^2 + c_2(\lambda_\varepsilon + \Delta_0)\|h^1(t)\|_{\mathcal{H}}^2 + \frac{c_2}{\varepsilon^2}\|h^1(t)\|_{\mathcal{H}} \, \|h^0(t)\|_{\mathcal{E}} \, ,$$

from which we get the desired estimate (4.26) after integration of the previous differential inequality recalling that $h^1(0) = 0$ and $\|\Psi(t)\|_{\mathcal{H}_1} \geqslant \|\Psi(t)\|_{\mathcal{H}} \geqslant C_{\mathcal{H}}|||\Psi(t)|||_{\mathcal{H}}$. □

To complete the estimate of $\|(\mathbf{Id} - \mathbf{P}_0)h^1(t)\|_{\mathcal{H}}^2$ we need to estimate the last integral in (4.26):

Lemma 4.7. *With the notation of Lemma 4.6, there is an explicit $\varepsilon_5 \in (0, \varepsilon_4)$ such that for any $\varepsilon \in (0, \varepsilon_5)$ and any $t \geqslant 0$,*

$$\frac{1}{\varepsilon^2}\int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}} \, \|h^0(s)\|_{\mathcal{E}} \, \mathrm{d}s \lesssim \|h(0)\|_{\mathcal{E}} e^{-\mu_1 t} + \lambda_\varepsilon \int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}}^2 \, \mathrm{d}s \, .$$

Proof. We use the estimate of $\|h^0(s)\|_{\mathcal{E}}$ provided in (4.9) combined with (4.16) which gives

$$\varepsilon^{-2}\int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}} \, \|h^0(s)\|_{\mathcal{E}} \, \mathrm{d}s \lesssim I_1 + I_2$$

with

$$I_1 = \varepsilon^{-2}\int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}} \, \|h(0)\|_{\mathcal{E}} e^{-\frac{\mu_0}{\varepsilon^2}s} \, \mathrm{d}s \, ,$$

$$I_2 = \varepsilon^{-2}\lambda_\varepsilon \int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}} \, \mathrm{d}s \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)}\|h^1(\tau)\|_{\mathcal{H}} \, \mathrm{d}\tau \, .$$

Choosing ε_5 such that $\mu_0 - \varepsilon_5^2\mu_1 \geqslant \frac{\mu_0}{2}$ and using (4.8), we obtain:

$$I_1 \lesssim \varepsilon^{-2}\|h(0)\|_{\mathcal{E}} e^{-\mu_1 t}\int_0^t e^{-\frac{\mu_0}{2\varepsilon^2}s} \, \mathrm{d}s \lesssim \|h(0)\|_{\mathcal{E}} e^{-\mu_1 t} \, .$$

Concerning I_2, using Young's inequality, we have that

$$I_2 \lesssim \lambda_\varepsilon \int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}}^2 \, \mathrm{d}s + \varepsilon^{-4}\lambda_\varepsilon \int_0^t e^{-\mu_1(t-s)} \, \mathrm{d}s \left(\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)}\|h^1(\tau)\|_{\mathcal{H}} \, \mathrm{d}\tau\right)^2 ,$$

and, using (4.12) with $r = \frac{1}{2}$, we get that

$$I_2 \lesssim \lambda_\varepsilon \int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}}^2 \, \mathrm{d}s + \varepsilon^{-2}\lambda_\varepsilon \int_0^t e^{-\mu_1(t-s)} \, \mathrm{d}s \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)}\|h^1(\tau)\|_{\mathcal{H}}^2 \, \mathrm{d}\tau \, .$$

To estimate the second integral, we remark that

$$\varepsilon^{-2}\lambda_\varepsilon \int_0^t e^{-\mu_1(t-s)} \, \mathrm{d}s \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)}\|h^1(\tau)\|_{\mathcal{H}}^2 \, \mathrm{d}\tau$$

$$= \varepsilon^{-2}\lambda_\varepsilon \int_0^t e^{-\mu_1(t-\tau)}\|h^1(\tau)\|_{\mathcal{H}}^2 \mathrm{d}\tau \int_\tau^t e^{-\left(\frac{\mu_0}{\varepsilon^2}-\mu_1\right)(s-\tau)} \, \mathrm{d}s$$

and using that $\frac{\mu_0}{\varepsilon^2} - \mu_1 \geqslant \frac{\mu_0}{2\varepsilon^2}$ for $\varepsilon \in (0, \varepsilon_5)$, we obtain

$$\varepsilon^{-2}\lambda_\varepsilon \int_0^t e^{-\mu_1(t-s)}\,\mathrm{d}s \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)}\|h^1(\tau)\|_{\mathcal{H}}^2\,\mathrm{d}\tau \lesssim \lambda_\varepsilon \int_0^t e^{-\mu_1(t-\tau)}\|h^1(\tau)\|_{\mathcal{H}}^2\,\mathrm{d}\tau\,.$$

Combining these estimates yields the wanted result. □

We deduce from the previous the following main estimate for $\|(\mathbf{Id} - \mathbf{P}_0)h^1(t)\|_{\mathcal{H}}$:

Proposition 4.8. *Under the Assumptions of Lemma 4.6, for any $\varepsilon \in (0, \varepsilon_5)$ (where ε_5 is defined in Lemma 4.7) and any $t \geqslant 0$,*

$$\|(\mathbf{Id} - \mathbf{P}_0)h^1(t)\|_{\mathcal{H}}^2 \lesssim \|h(0)\|_{\mathcal{E}} e^{-\mu_1 t} + (\lambda_\varepsilon + \Delta_0)\int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s\,. \tag{4.31}$$

In particular, there exist $\varepsilon_6 \in (0, \varepsilon_5)$ and $\lambda_6 > 0$ such that if $\lambda_0 \in [0, \lambda_6)$ (where λ_0 is defined in Asssumption 1.1), for any $r \in (0,1)$, any $\varepsilon \in (0, \varepsilon_6)$ and any $t \geqslant 0$,

$$\begin{aligned}\int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)}\|(\mathbf{Id} - \mathbf{P}_0)h^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s \\ \lesssim \|h(0)\|_{\mathcal{E}} e^{-2(1-r)\lambda_\varepsilon t} + (\lambda_\varepsilon + \Delta_0)\int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)}\|h^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s\end{aligned} \tag{4.32}$$

where the multiplicative constant involved in the previous inequality does not depend on r.

Proof. Inserting the estimate obtained in Lemma 4.7 into (4.26) allows to get directly (4.31). Using (4.24) twice, we deduce then easily (4.32) from (4.31) after integration by choosing λ_6 and ε_6 small enough such that $\mu_1 \geqslant 4\lambda_\varepsilon$ for any $\varepsilon \in (0, \varepsilon_6)$ and any $\lambda_0 \in [0, \lambda_6)$ and thus $\mu_1 - 2(1-r)\lambda_\varepsilon \geqslant \mu_1 - 2\lambda_\varepsilon \geqslant \frac{\mu_1}{2}$ for $\varepsilon \in (0, \varepsilon_6)$. □

4.5. **Final *a priori* estimates.** We deduce from the above the following estimate on h^1:

Proposition 4.9. *Under the assumptions of Lemma 4.6, for any $t \geqslant 0$, $\varepsilon \in (0, \varepsilon_6)$, $\lambda_0 \in [0, \lambda_6)$ (where ε_6 and λ_6 are defined in Proposition 4.8) and $r \in (0,1)$,*

$$\begin{aligned}\|h^1(t)\|_{\mathcal{H}}^2 \lesssim \frac{1}{r}\left(\|h(0)\|_{\mathcal{E}} + \|h(0)\|_{\mathcal{E}}^2\right) e^{-2(1-r)\lambda_\varepsilon t} \\ + \frac{\lambda_\varepsilon}{r}\left(\varepsilon^2 + \lambda_\varepsilon + \Delta_0\right)\int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)}\|h^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s \\ + (\lambda_\varepsilon + \Delta_0)\int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s\,.\end{aligned} \tag{4.33}$$

Proof. We gather estimates (4.31) and (4.22) (combined with (4.32)) to obtain

$$\begin{aligned}\|h^1(t)\|_{\mathcal{H}}^2 \lesssim \|h(0)\|_{\mathcal{E}} e^{-\mu_1 t} + (\lambda_\varepsilon + \Delta_0)\int_0^t e^{-\mu_1(t-s)}\|h^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s \\ + \|h(0)\|_{\mathcal{E}}^2 e^{-2\lambda_\varepsilon t} + \frac{\varepsilon^2\lambda_\varepsilon}{r}\int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)}\|h^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s \\ + \frac{\lambda_\varepsilon}{r}\|h(0)\|_{\mathcal{E}} e^{-2(1-r)\lambda_\varepsilon t} + \frac{\lambda_\varepsilon}{r}(\lambda_\varepsilon+\Delta_0)\int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)}\|h^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s\,. \quad (4.34)\end{aligned}$$

For $\varepsilon \in (0,\varepsilon_6)$, $\mu_1 \geqslant 2\lambda_\varepsilon$ (and thus $\lambda_\varepsilon \lesssim 1$). Then, using that $r<1$, we are able to obtain the wanted result. □

By a refined Gronwall type argument, we are able to derive from this the following decay rate for $\|h^1(t)\|_{\mathcal{H}}$:

Corollary 4.10. *Let $r \in (0,1)$. There exist $\varepsilon_r \in (0,\varepsilon_6)$, $\lambda_r \in (0,\lambda_6)$ (where ε_6 and λ_6 are defined in Proposition 4.9), $\Delta_{0,r}>0$ and $C_r>0$ depending on r such that for any $\varepsilon \in (0,\varepsilon_r)$, $\lambda_0 \in [0,\lambda_r)$ (where λ_0 is defined in Assumption 1.1), $\Delta_0 \in (0,\Delta_{0,r})$ (where Δ_0 is defined in (4.8)) and any $t \geqslant 0$, there holds*

$$\|h^1(t)\|_{\mathcal{H}}^2 \leqslant C_r\left(\|h(0)\|_{\mathcal{E}} + \|h(0)\|_{\mathcal{E}}^2\right)\exp\left(-2(1-r)\lambda_\varepsilon\, t\right)\,. \quad (4.35)$$

Proof. Let $r \in (0,1)$, set $r' := \frac{r}{2}$ and

$$\boldsymbol{x}(t) := e^{2(1-r')\lambda_\varepsilon t}\|h^1(t)\|_{\mathcal{H}}^2\,, \qquad t \geqslant 0\,.$$

In order to lighten the notations of the coming proof, we also introduce the following notations:

$$\mathcal{K}_0 := \|h(0)\|_{\mathcal{E}} + \|h(0)\|_{\mathcal{E}}^2 \quad (4.36)$$

and

$$\mathcal{K}_1 := \mathcal{K}_1(r',\varepsilon,\lambda_\varepsilon,\Delta_0) = \frac{\lambda_\varepsilon}{r'}\left(\varepsilon^2+\lambda_\varepsilon+\Delta_0\right)\,, \qquad \mathcal{K}_2 := \mathcal{K}_2(\lambda_\varepsilon,\Delta_0) = \lambda_\varepsilon+\Delta_0\,,$$

and

$$\mu_2 := \mu_2(r,\varepsilon) = \mu_1 - 2(1-r')\lambda_\varepsilon > 0.$$

Inequality (4.33) applied with r' instead of r implies that there exists a universal constant $C>0$ such that

$$\boldsymbol{x}(t) \leqslant C\frac{\mathcal{K}_0}{r'} + C\mathcal{K}_1\int_0^t \boldsymbol{x}(s)\,\mathrm{d}s + C\mathcal{K}_2 e^{-\mu_2 t}\int_0^t e^{\mu_2 s}\boldsymbol{x}(s)\,\mathrm{d}s =: \boldsymbol{y}(t)\,.$$

It implies that

$$\begin{aligned}\boldsymbol{y}'(t) &= C(\mathcal{K}_1+\mathcal{K}_2)\boldsymbol{x}(t) - \mu_2 C\mathcal{K}_2 e^{-\mu_2 t}\int_0^t e^{\mu_2 s}\boldsymbol{x}(s)\,\mathrm{d}s \\ &= C(\mathcal{K}_1+\mathcal{K}_2)\boldsymbol{x}(t) - \mu_2\left(\boldsymbol{y}(t) - C\frac{\mathcal{K}_0}{r'} - C\mathcal{K}_1\int_0^t \boldsymbol{x}(s)\,\mathrm{d}s\right) \\ &\leqslant -\left(\mu_2 - C(\mathcal{K}_1+\mathcal{K}_2)\right)\boldsymbol{y}(t) + \mu_2 C\frac{\mathcal{K}_0}{r'} + \mu_2 C\mathcal{K}_1\int_0^t \boldsymbol{y}(s)\,\mathrm{d}s\,.\end{aligned}$$

Taking now ε_r, λ_r and $\Delta_{0,r}$ small enough so that $\mu_2 - C(\mathcal{K}_1+\mathcal{K}_2) \geqslant \frac{\mu_1}{2}$ for any $\varepsilon \in (0,\varepsilon_r)$, $\lambda_0 \in [0,\lambda_r)$ and $\Delta_0 \in (0,\Delta_{0,r})$, we can deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{\mu_1}{2}t}\boldsymbol{y}(t)\right) \leqslant e^{\frac{\mu_1}{2}t}\left(\mu_2 C\frac{\mathcal{K}_0}{r'} + \mu_2 C\mathcal{K}_1\int_0^t \boldsymbol{y}(s)\,\mathrm{d}s\right)\,.$$

Integrating in time the last inequality (notice that $\boldsymbol{y}(0) = C\frac{\mathcal{K}_0}{r'}$) yields

$$e^{\frac{\mu_1}{2}t}\boldsymbol{y}(t) \leqslant C\frac{\mathcal{K}_0}{r'} + \mu_2 C\frac{\mathcal{K}_0}{r'}\int_0^t e^{\frac{\mu_1}{2}s}\,\mathrm{d}s + \mu_2 C\mathcal{K}_1\int_0^t e^{\frac{\mu_1}{2}s}\mathrm{d}s\int_0^s \boldsymbol{y}(\tau)\,\mathrm{d}\tau\,.$$

Using that

$$\int_0^t e^{\frac{\mu_1}{2}s}\int_0^s \boldsymbol{y}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \int_0^t \boldsymbol{y}(\tau)\mathrm{d}\tau\int_\tau^t e^{\frac{\mu_1}{2}s}\,\mathrm{d}s \leqslant \frac{2}{\mu_1}e^{\frac{\mu_1}{2}t}\int_0^t \boldsymbol{y}(\tau)\,\mathrm{d}\tau\,,$$

we can conclude that

$$\boldsymbol{y}(t) \lesssim \frac{\mathcal{K}_0}{r'} + \mathcal{K}_1\int_0^t \boldsymbol{y}(s)\,\mathrm{d}s\,.$$

From the standard Gronwall inequality, we deduce that for any $t \geqslant 0$,

$$\boldsymbol{x}(t) = e^{2(1-r')\lambda_\varepsilon t}\|h^1(t)\|_{\mathcal{H}}^2 \leqslant \boldsymbol{y}(t) \lesssim \frac{\mathcal{K}_0}{r'}e^{\mathcal{K}_1 t}\,.$$

Up to reducing the values of ε_r, λ_r, $\Delta_{0,r}$, we can furthermore assume that $\varepsilon^2 + \lambda_\varepsilon + \Delta_0 \leqslant \frac{r^2}{2}$ which in particular implies that $\mathcal{K}_1 - 2(1-r')\lambda_\varepsilon \leqslant -2(1-r)\lambda_\varepsilon$ so that

$$\|h^1(t)\|_{\mathcal{H}}^2 \lesssim \frac{\mathcal{K}_0}{r}e^{-2(1-r)\lambda_\varepsilon t}\,, \qquad \forall\, t \geqslant 0$$

which is the desired estimate. □

We are now able to state the main result of this section which provides a result of decay for the solution h to (1.27):

Theorem 4.11. *Let $r \in (0,1)$. There exist $\varepsilon_r > 0$, $\lambda_r > 0$, $\Delta_{0,r} > 0$ and $C_r > 0$ such that for any $\varepsilon \in (0,\varepsilon_r)$, $\lambda_0 \in [0,\lambda_r)$, $\Delta_0 \in (0,\Delta_{0,r})$ and any $t \geqslant 0$,*

$$\|h(t)\|_{\mathcal{E}}^2 \leqslant C_r\left(\|h(0)\|_{\mathcal{E}} + \|h(0)\|_{\mathcal{E}}^2\right)\exp\left(-2(1-r)\lambda_\varepsilon t\right)\,.$$

Proof. The result is obtained by gathering estimates (4.10) (combined with (4.16)) and (4.35) and keeping only the dominant terms. □

We also point out the gain of decay in h in the following lemma.

Lemma 4.12. *Under the same conditions of Theorem 4.11, it follows that*

$$\int_0^t \|h(\tau)\|_{\mathcal{E}_1}\,\mathrm{d}\tau \leqslant C_r\left(\sqrt{\|h(0)\|_{\mathcal{E}}}+\|h(0)\|_{\mathcal{E}}\right)\min\left\{1+t,1+\frac{1}{\lambda_\varepsilon}\right\}, \qquad \forall\, t\geqslant 0\,.$$

In particular, $\|h(\,\cdot\,)\|_{\mathcal{E}_1}$ *is integrable and exists a.e. in* $(0,T)$ *for any* $T>0$.

Proof. After performing time integration of equation (4.11) in $[0,t]$ one finds that

$$\begin{aligned}\|h^0(t)\|_{\mathcal{E}}+\frac{\mu_0}{\varepsilon^2}\int_0^t \|h^0(s)\|_{\mathcal{E}_1}\,\mathrm{d}s&\\ \lesssim \|h(0)\|_{\mathcal{E}}+\lambda_\varepsilon\int_0^t\|h^1(s)\|_{\mathcal{H}}\,\mathrm{d}s &\lesssim \sqrt{\|h(0)\|_{\mathcal{E}}}+\|h(0)\|_{\mathcal{E}}\,, \qquad \forall\, t\geqslant 0\,, \quad (4.37)\end{aligned}$$

where we used (4.35) in the latter inequality. Using the continuous embedding $\mathcal{H}\hookrightarrow\mathcal{E}_1$ and (4.35) once more, we obtain

$$\begin{aligned}\int_0^t\|h(s)\|_{\mathcal{E}_1}\,\mathrm{d}s &\lesssim \int_0^t\|h^0(s)\|_{\mathcal{E}_1}\,\mathrm{d}s+\int_0^t\|h^1(s)\|_{\mathcal{H}}\,\mathrm{d}s\\ &\lesssim \varepsilon^2\left(\sqrt{\|h(0)\|_{\mathcal{E}}}+\|h(0)\|_{\mathcal{E}}\right)+C_r\left(\sqrt{\|h(0)\|_{\mathcal{E}}}+\|h(0)\|_{\mathcal{E}}\right)\int_0^t e^{-\frac{1-r}{2}\lambda_\varepsilon s}\,\mathrm{d}s\,,\end{aligned}$$

which gives the result. □

Remark 4.13. *Of course, for a fixed* $\varepsilon>0$*, one can replace* $\min\left\{1+t,1+\frac{1}{\lambda_\varepsilon}\right\}$ *by* $1+\frac{1}{\lambda_\varepsilon}$ *and the above estimate shows that* $h(t)=h_\varepsilon(t)\in L^1([0,\infty),\mathcal{E}_1)$*. However, in the case in which* $\lambda_0=0$ *then the bound is not uniform with respect to* ε*. In practice, two situations occur according to the value of* λ_0 *in Assumption 1.1:*

a) If $\lambda_0>0$*, then the family* $\{h_\varepsilon(t)\}_{\varepsilon\geqslant 0}$ *is bounded in* $L^1([0,\infty),\mathcal{E}_1)$*,*

b) If $\lambda_0=0$ *then for any* $T>0$*, the family* $\{h_\varepsilon(t)\}_{\varepsilon\geqslant 0}$ *is bounded in* $L^1([0,T],\mathcal{E}_1)$*.*

5. Cauchy Theory

We assume that Assumption 1.1 is satisfied. The scope of this Section is to prove the well-posedness of the system (4.3)-(4.4) thanks to the *a priori* estimates derived in the previous section. We namely aim to prove the following precise version of Theorem 1.2 (reformulated in terms of the variable h_ε) and we shall use the functional spaces introduced in (3.1), (4.1), (4.2).

Theorem 5.1. *Let $r \in (0,1)$. There exists a triple $(\varepsilon_r, \lambda_r, \eta_0)$ of positive constants that depend on the mass and energy of $F_{\rm in}^\varepsilon$, m and q, (ε_r and λ_r additionally depending on r whereas η_0 is not) such that, for $\varepsilon \in (0, \varepsilon_r)$, $\lambda_0 \in [0, \lambda_r)$, if*

$$\|h_{\rm in}^\varepsilon\|_{\mathcal{E}} \leqslant \eta_0$$

then the inelastic Boltzmann equation (1.27) has a unique solution $h_\varepsilon \in \mathcal{C}\big([0,\infty);\mathcal{E}\big) \cap L^1\big([0,\infty);\mathcal{E}_1\big)$ satisfying

$$\|h_\varepsilon(t)\|_{\mathcal{E}} \leqslant C(r,\eta_0)\, \exp\left(-(1-r)\lambda_\varepsilon t\right),$$
$$\text{and} \qquad \int_0^t \|h_\varepsilon(t)\|_{\mathcal{E}_1}\, {\rm d}\tau \leqslant C(r,\eta_0) \min\left\{1+t, 1+\frac{1}{\lambda_\varepsilon}\right\}, \qquad \forall\, t \geqslant 0\,,$$

for some positive constant $C(r,\eta_0) > 0$ independent of ε and where we recall that $\lambda_\varepsilon \underset{\varepsilon\to 0}{\sim} \frac{1-\alpha(\varepsilon)}{\varepsilon^2}$ is defined in (4.15).

Remark 5.2. *Under the same assumptions, we can actually prove the following estimates (which will be useful in what follows) on h_ε^0 and h_ε^1 that are respectively solutions to (4.3) and (4.4). Let $T > 0$, then*

$$\|h_\varepsilon^0\|_{L^\infty((0,T)\,;\mathcal{E})} \lesssim 1 \quad \text{and} \quad \|h_\varepsilon^0\|_{L^1((0,T)\,;\mathcal{E}_1)} \lesssim \varepsilon^2 \tag{5.1}$$

as well as

$$\|h_\varepsilon^1\|_{L^\infty((0,T)\,;\mathcal{H})} \lesssim 1 \quad \text{and} \quad \|h_\varepsilon^1\|_{L^2((0,T)\,;\mathcal{H}_1)} \lesssim 1 \tag{5.2}$$

where we recall that the spaces $\mathcal{H}$ and $\mathcal{H}_1$ are defined in (3.1). Notice that in the previous inequalities, the multiplicative constants only involve quantities related to the initial data of the problem and are independent of ε.

As in Section 4, we shall consider δ and ε small enough so that the conclusions of Proposition 2.8 and Remark 2.9 are satisfied in the functional space $\mathcal{E}$ and we denote $\mathcal{B}_\varepsilon = \mathcal{B}^{(\delta)}_{\alpha(\varepsilon),\varepsilon}$ as well as $\mathcal{G}_\varepsilon = \mathcal{G}_{\alpha(\varepsilon),\varepsilon}$.

5.1. **Iteration scheme.** Let us follow the iteration scheme of (Tristani, 2016, Section 3) with suitable modifications. We are seeking to approximate the solution to the inelastic Boltzmann equation using the iteration scheme

$$\begin{cases} \partial_t h_{n+1}(t) &= \mathcal{G}_\varepsilon h_{n+1}(t) + \varepsilon^{-1}\mathcal{Q}_{\alpha(\varepsilon)}(h_n(t), h_n(t))\,, \quad n \geqslant 1\,, \\ \partial_t h_1(t) &= \mathcal{G}_\varepsilon h_1(t)\,, \\ h_n(0) &= h(0) \in \mathcal{E}\,, \qquad n \geqslant 1\,, \end{cases} \tag{5.3}$$

where the initial perturbation $h(0)$ has zero mass and momentum. This is done using the decomposition of previous section. More precisely, writing $h_n = h_n^0 + h_n^1$ we consider solutions with

the coupled system

$$\left\{\begin{array}{rcl} \partial_t h^0_{n+1} &=& \mathcal{B}_\varepsilon h^0_{n+1} + \varepsilon^{-1}\mathcal{Q}_{\alpha(\varepsilon)}(h^0_n,h^0_n) + \varepsilon^{-1}\Big[\mathcal{Q}_{\alpha(\varepsilon)}(h^0_n,h^1_n) + \mathcal{Q}_{\alpha(\varepsilon)}(h^1_n,h^0_n)\Big] \\ && +\varepsilon^{-2}\Big[\mathbf{L}_{\alpha(\varepsilon)}h^1_{n+1} - \mathbf{L}_1 h^1_{n+1}\Big] + \varepsilon^{-1}\boldsymbol{\pi}_0\mathcal{Q}_{\alpha(\varepsilon)}(h^1_n,h^1_n) \\ && + (1-\alpha)\varepsilon^{-2}\mathrm{a}_3\boldsymbol{\varpi}_2\left(\mathbf{Id}-\mathbf{P}_0\right)h^1_{n+1}\,, \\ h^0_{n+1}(0) &=& h^0(0) = h^\varepsilon_{\mathrm{in}} \in \mathcal{E}\,, \end{array}\right. \tag{5.4}$$

and

$$\left\{\begin{array}{rcl} \partial_t h^1_{n+1} &=& \mathcal{G}_{1,\varepsilon}h^1_{n+1} + \varepsilon^{-1}\left(\mathbf{Id}-\boldsymbol{\pi}_0\right)\mathcal{Q}_{\alpha(\varepsilon)}(h^1_n,h^1_n) \\ && -(1-\alpha(\varepsilon))\varepsilon^{-2}\left[\mathrm{div}_v\left(vh^1_{n+1}\right) + \mathrm{a}_3\boldsymbol{\varpi}_2\left(\mathbf{Id}-\mathbf{P}_0\right)h^1_{n+1}\right] + \mathcal{A}_\varepsilon h^0_{n+1}\,, \\ h^1_{n+1}(0) &=& h^1(0) = 0 \in \mathcal{H}\,, \end{array}\right. \tag{5.5}$$

where $\mathrm{a}_3 > 0$ is defined in Lemma 3.3. Motivated by the *a priori* estimates of Section 4, we introduce the following norms

$$|\!|\!| g |\!|\!|_0 := \sup_{t\geqslant 0}\left(\|g(t)\|_{\mathcal{E}} + \varepsilon^{-2}\int_0^t \|g(\tau)\|_{\mathcal{E}_1}\,\mathrm{d}\tau\right), \qquad g\in\mathcal{C}([0,\infty),\mathcal{E})\,,$$

and

$$|\!|\!| g |\!|\!|_1 := \sup_{t\geqslant 0}\left(\|g(t)\|^2_{\mathcal{H}} + \int_0^t \|g(\tau)\|^2_{\mathcal{H}_1}\,\mathrm{d}\tau\right)^{\frac12}, \qquad g\in\mathcal{C}([0,\infty),\mathcal{H})\,,$$

where we recall that $\mathcal{E},\mathcal{E}_1,\mathcal{H}$ and $\mathcal{H}_1$ are defined in (4.1), (3.1) and (4.2).

Notice that $\big(\mathcal{C}([0,\infty),\mathcal{E})\,;\,|\!|\!|\cdot|\!|\!|_0\big)$ and $\big(\mathcal{C}([0,\infty),\mathcal{H})\,;\,|\!|\!|\cdot|\!|\!|_1\big)$ are Banach spaces. In particular, the space

$$\mathbb{B} := \mathcal{C}([0,\infty),\mathcal{E})\times\mathcal{C}([0,\infty),\mathcal{H})$$

endowed with the norm

$$|\!|\!|(g,h)|\!|\!| := |\!|\!| g |\!|\!|_0 + |\!|\!| h |\!|\!|_1 \qquad \text{for} \quad (g,h)\in\mathbb{B}\,,$$

is a Banach space. Define then

$$\mathcal{X}_0 := \Big\{h^0\in\mathcal{C}\big([0,\infty);\mathcal{E}\big)\;\Big|\;|\!|\!|h^0|\!|\!|_0 \leqslant C\sqrt{\mathcal{K}_0}\Big\}\,,$$
$$\mathcal{X}_1 := \Big\{h^1\in\mathcal{C}\big([0,\infty);\mathcal{H}\big)\;\Big|\;|\!|\!|h^1|\!|\!|_1 \leqslant C\sqrt{\mathcal{K}_0}\Big\}\,, \tag{5.6}$$

for some positive constant $C>0$ which can be explicitly estimated from the subsequent computations and where we recall that $\mathcal{K}_0$ has been defined in (4.36) by $\mathcal{K}_0 = \|h(0)\|_{\mathcal{E}} + \|h(0)\|^2_{\mathcal{E}}$. The system (5.4)-(5.5) is a simplified coupled version of the system (4.3)-(4.4) with all nonlinear terms as sources. Notice however that the coupling between h^0_{n+1} and h^1_{n+1} in the system makes

it *nonlinear*. However, because $\mathcal{G}_\varepsilon$ is the generator of a C_0-semigroup in $\mathcal{E}$, equation (5.3) is well-posed and

$$h_{n+1}(t) = \mathcal{V}_\varepsilon(t)h(0) + \varepsilon^{-1}\int_0^t \mathcal{V}_\varepsilon(t-s)\mathcal{Q}_{\alpha(\varepsilon)}(h_n(s),h_n(s))\,\mathrm{d}s$$

where $\{\mathcal{V}_\varepsilon(t)\,;\,t\geqslant 0\}$ is the C_0-semigroup in $\mathcal{E}$ generated by $\mathcal{G}_\varepsilon$. With this at hands, substitute in (5.4) the term h^1_{n+1} by $h_{n+1}-h^0_{n+1}$ and look at $h_{n+1}(t)$ as an additional source term. In the same way for (5.5), the system (5.4)-(5.5) becomes linear (in terms of h^0_{n+1} and h^1_{n+1}) and admits, for any $n\in\mathbb{N}$, a unique solution. One can use a slight modification of the ideas of Section 4 to check that the iteration scheme is stable, that is, the mapping

$$\left(h^0_n,h^1_n\right)\in\mathcal{X}_0\times\mathcal{X}_1\mapsto\left(h^0_{n+1},h^1_{n+1}\right)\in\mathcal{X}_0\times\mathcal{X}_1$$

is well defined. Indeed, existence of the scheme is guaranteed by the linear theory as the iteration scheme is based on the linear equation. Moreover, note that (5.3) preserves the conservation laws: mass conservation and vanishing momentum, which were essential for the *a priori* estimates related to $\mathbf{P}_0h^1$. Thus, proceeding as in Theorem 4.11 with $r=\frac{1}{2}$ for example, stability holds true under the conditions of the *a priori* estimates, that is, for $\varepsilon\in\left(0,\varepsilon_{\frac{1}{2}}\right)$, $\lambda_0\in\left[0,\lambda_{\frac{1}{2}}\right)$ and

$$\sup_{t\geqslant 0}\left(\|h^1_n(t)\|_{\mathcal{H}}+\|h^0_n(t)\|_{\mathcal{E}}\right)\leqslant C\sqrt{\mathcal{K}_0}\leqslant\Delta_{0,\frac{1}{2}}\,,\qquad n\in\mathbb{N}\,.$$

This latter condition is possible by taking $\mathcal{K}_0$ smaller than a threshold depending only on the initial mass and energy E_ε

$$\mathcal{K}_0\leqslant\left(\Delta_{0,\frac{1}{2}}/C\right)^2=:\mathcal{K}_0^\dagger\,.$$

We leave the details to the reader and focus in the next subsections on the convergence of the scheme. For the rest of the section, we also set $\lambda^\dagger:=\lambda_{\frac{1}{2}}$ and choose $\varepsilon^\dagger\in(0,\varepsilon_{\frac{1}{2}})$ such that $\nu_q\geqslant 2\varepsilon^2C^2_{\mathcal{H}}\mathrm{a}_1$ where ν_q, $C_{\mathcal{H}}$ and a_1 are respectively defined in Proposition 2.8, (3.6) and Proposition 3.5 (this condition will be useful in the proof of Lemma 5.5).

5.2. **Estimating $\|h^0_{n+1}-h^0_n\|_{\mathcal{E}}$ and $\|h^1_{n+1}-h^1_n\|_{\mathcal{H}}$.** To prove the convergence of the scheme, we define for $n\in\mathbb{N}$

$$d^0_{n+1}:=h^0_{n+1}-h^0_n\,,\qquad\qquad d^1_{n+1}:=h^1_{n+1}-h^1_n\,.$$

Then, one deduces from (5.4) and (5.5)

$$\left\{\begin{array}{rcl}\partial_t d^0_{n+1} &=& \mathcal{B}_\varepsilon d^0_{n+1}+\varepsilon^{-2}\Big[\mathbf{L}_{\alpha(\varepsilon)}d^1_{n+1}-\mathbf{L}_1d^1_{n+1}\Big]\\ &&\qquad+(1-\alpha)\varepsilon^{-2}\mathrm{a}_3\boldsymbol{\varpi}_2\left(\mathbf{Id}-\mathbf{P}_0\right)d^1_{n+1}+\varepsilon^{-1}\mathcal{F}^0_n\,,\\ d^0_{n+1}(0) &=& 0\,,\end{array}\right.\tag{5.7}$$

and

$$\left\{\begin{array}{rcl} \partial_t d^1_{n+1} & = & \mathcal{G}_{1,\varepsilon} d^1_{n+1} - (1-\alpha(\varepsilon))\varepsilon^{-2}\left[\mathrm{div}_v\left(v d^1_{n+1}\right) + \mathrm{a}_3 \boldsymbol{\varpi}_2 \left(\mathbf{Id} - \mathbf{P}_0\right) d^1_{n+1}\right] \\ & & \qquad + \mathcal{A}_\varepsilon d^0_{n+1} + \varepsilon^{-1}\mathcal{F}^1_n\,, \\ d^1_{n+1}(0) & = & 0\,. \end{array}\right. \tag{5.8}$$

The sources $\mathcal{F}^i_n$, for $i \in \{0,1\}$, correspond to the bilinear terms and depend only on the previous iterations $\{h^i_n, h^i_{n-1}\}$, for $i \in \{0,1\}$ and $n \geqslant 2$ (see (5.11) and (5.13) for the precise expression). We introduce

$$\begin{aligned} \Psi^1_n(t) &:= \|h^0_n(t)\|_{\mathcal{E}_1} + \|h^0_{n-1}(t)\|_{\mathcal{E}_1}\,, \\ \Psi^\infty_n(t) &:= \|h^0_n(t)\|_{\mathcal{E}} + \|h^0_{n-1}(t)\|_{\mathcal{E}} + \|h^1_n(t)\|_{\mathcal{H}} + \|h^1_{n-1}(t)\|_{\mathcal{H}}\,, \end{aligned}$$

which satisfy

$$\sup_{t\geqslant 0}\left(\Psi^\infty_n(t) + \varepsilon^{-2}\int_0^t \Psi^1_n(\tau)\,\mathrm{d}\tau\right) \leqslant C\sqrt{\mathcal{K}_0}\,, \qquad n \geqslant 2\,, \tag{5.9}$$

for $h^0_n, h^0_{n-1} \in \mathcal{X}_0$, and $h^1_n, h^1_{n-1} \in \mathcal{X}_1$. Consequently, the following estimate for d^0_{n+1} follows under suitable modifications of the arguments leading to Proposition 4.1.

Lemma 5.3. *Let $\varepsilon \in (0, \varepsilon^\dagger)$, $\lambda_0 \in [0, \lambda^\dagger)$ and $\mathcal{K}_0 \leqslant \mathcal{K}^\dagger_0$. For any $t \geqslant 0$, we have that*

$$\begin{aligned} \|d^0_{n+1}(t)\|_{\mathcal{E}} \lesssim \lambda_\varepsilon \int_0^t e^{-\frac{\nu_q}{\varepsilon^2}(t-s)} \|d^1_{n+1}(s)\|_{\mathcal{H}}\,\mathrm{d}s \\ + \varepsilon^{-1}\int_0^t e^{-\frac{\nu_q}{\varepsilon^2}(t-s)} \Psi^1_n(s)\left(\|d^0_n(s)\|_{\mathcal{E}} + \|d^1_n(s)\|_{\mathcal{H}}\right)\mathrm{d}s \\ + \int_0^t e^{-\frac{\nu_q}{\varepsilon^2}(t-s)} \Psi^\infty_n(s)\left(\varepsilon^{-1}\|d^0_n(s)\|_{\mathcal{E}_1} + \varepsilon\lambda_\varepsilon \|d^1_n(s)\|_{\mathcal{H}}\right)\mathrm{d}s \end{aligned} \tag{5.10}$$

where we recall that ν_q is defined in Proposition 2.8 and Remark 2.9.

Proof. As in the proof of Proposition 4.1, we use the fact here that $\varepsilon^{-2}\nu_q + \mathcal{B}_\varepsilon$ is *dissipative* so that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|d^0_{n+1}(t)\|_{\mathcal{E}} &\leqslant -\frac{\nu_q}{\varepsilon^2}\|d^0_{n+1}(t)\|_{\mathcal{E}_1} + \varepsilon^{-1}\|\mathcal{F}^0_n(t)\|_{\mathcal{E}} + \varepsilon^{-2}\left\|\mathbf{L}_{\alpha(\varepsilon)} d^1_{n+1}(t) - \mathbf{L}_1 d^1_{n+1}(t)\right\|_{\mathcal{E}} \\ &\qquad + \frac{1-\alpha(\varepsilon)}{\varepsilon^2}\mathrm{a}_3\left\|\boldsymbol{\varpi}_2\left(\mathbf{Id} - \mathbf{P}_0\right) d^1_{n+1}(t)\right\|_{\mathcal{E}} \\ &\leqslant -\frac{\nu_q}{\varepsilon^2}\|d^0_{n+1}(t)\|_{\mathcal{E}_1} + \varepsilon^{-1}\|\mathcal{F}^0_n(t)\|_{\mathcal{E}} + C\lambda_\varepsilon\|d^1_{n+1}(t)\|_{\mathcal{H}}\,. \end{aligned}$$

We need to estimate $\|\mathcal{F}^0_n(t)\|_{\mathcal{E}}$. One has,

$$\begin{aligned} \mathcal{F}^0_n = \mathcal{Q}_{\alpha(\varepsilon)}(d^0_n, h^0_n) + \mathcal{Q}_{\alpha(\varepsilon)}(h^0_{n-1}, d^0_n) + 2\widetilde{\mathcal{Q}}_{\alpha(\varepsilon)}(d^0_n, h^1_n) + 2\widetilde{\mathcal{Q}}_{\alpha(\varepsilon)}(h^0_{n-1}, d^1_n) \\ + \boldsymbol{\pi}_0\mathcal{Q}_{\alpha(\varepsilon)}(d^1_n, h^1_n) + \boldsymbol{\pi}_0\mathcal{Q}_{\alpha(\varepsilon)}(h^1_{n-1}, d^1_n) \end{aligned} \tag{5.11}$$

where we used the notation $\widetilde{\mathcal{Q}}_\alpha$ introduced in (1.30). Using (1.10) and using that $\mathbb{W}^{m,2}_x$ is a Banach algebra, we have

$$\|\boldsymbol{\pi}_0\mathcal{Q}_{\alpha(\varepsilon)}(f,g)\|_{\mathcal{E}} \lesssim (1-\alpha(\varepsilon))\|f\|_{L^1_v\mathbb{W}^{m,2}_x(\boldsymbol{\varpi}_3)}\|g\|_{L^1_v\mathbb{W}^{m,2}_x(\boldsymbol{\varpi}_3)}\,.$$

Therefore, using that $q \geqslant 3$ and using Corollary B.4:

$$\begin{aligned}\|\mathcal{F}^0_n\|_{\mathcal{E}} \lesssim \|d^0_n\|_{\mathcal{E}_1}\left(\|h^0_n\|_{\mathcal{E}}+\|h^0_{n-1}\|_{\mathcal{E}}\right)+\|d^0_n\|_{\mathcal{E}}\left(\|h^0_n\|_{\mathcal{E}_1}+\|h^0_{n-1}\|_{\mathcal{E}_1}\right)\\ +\|d^0_n\|_{\mathcal{E}}\|h^1_n\|_{\mathcal{E}_1}+\|d^0_n\|_{\mathcal{E}_1}\|h^1_n\|_{\mathcal{E}}+\|d^1_n\|_{\mathcal{E}_1}\|h^0_{n-1}\|_{\mathcal{E}}\\ +\|d^1_n\|_{\mathcal{E}}\|h^0_{n-1}\|_{\mathcal{E}_1}+\varepsilon^2\lambda_\varepsilon\|d^1_n\|_{\mathcal{E}_1}\left(\|h^1_n\|_{\mathcal{E}_1}+\|h^1_{n-1}\|_{\mathcal{E}_1}\right)\end{aligned}$$

where we used that $1-\alpha(\varepsilon)\lesssim\varepsilon^2\lambda_\varepsilon$. Using that $\|\cdot\|_{\mathcal{E}_1}\lesssim\|\cdot\|_{\mathcal{H}}$, we get

$$\|\mathcal{F}^0_n\|_{\mathcal{E}} \lesssim \|d^0_n\|_{\mathcal{E}_1}\Psi^\infty_n+\left(\|d^0_n\|_{\mathcal{E}}+\|d^1_n\|_{\mathcal{H}}\right)\Psi^1_n+\varepsilon^2\lambda_\varepsilon\|d^1_n\|_{\mathcal{H}}\Psi^\infty_n\,.$$

This leads to the desired estimate. □

We now focus on $\mathbf{P}_0 d^1_{n+1}$ in the following lemma:

Lemma 5.4. *Let $\varepsilon\in(0,\varepsilon^\dagger)$, $\lambda_0\in[0,\lambda^\dagger)$ and $\mathcal{K}_0\leqslant\mathcal{K}_0^\dagger$. For any $t\geqslant 0$, we have that*

$$\begin{aligned}\|\mathbf{P}_0 d^1_{n+1}(t)\|_{\mathcal{H}} \lesssim \lambda_\varepsilon\int_0^t e^{-\frac{\nu_q}{\varepsilon^2}(t-s)}\|d^1_{n+1}(s)\|_{\mathcal{H}}\,\mathrm{d}s\\ +\varepsilon^{-1}\int_0^t e^{-\frac{\nu_q}{\varepsilon^2}(t-s)}\Psi^1_n(s)\Big(\|d^0_n(s)\|_{\mathcal{E}}+\|d^1_n(s)\|_{\mathcal{H}}\Big)\,\mathrm{d}s\\ +\int_0^t e^{-\frac{\nu_q}{\varepsilon^2}(t-s)}\Psi^\infty_n(s)\Big(\varepsilon^{-1}\|d^0_n(s)\|_{\mathcal{E}_1}+\varepsilon\lambda_\varepsilon\|d^1_n(s)\|_{\mathcal{H}}\Big)\,\mathrm{d}s\\ +\varepsilon\lambda_\varepsilon\int_0^t e^{-\lambda_\varepsilon(t-s)}\Psi^\infty_n(s)\Big(\|d^0_n(s)\|_{\mathcal{E}}+\|d^1_n(s)\|_{\mathcal{H}}\Big)\,\mathrm{d}s\\ +\lambda_\varepsilon\int_0^t e^{-\lambda_\varepsilon(t-s)}\left(\|d^0_{n+1}(s)\|_{\mathcal{E}}+\|\mathbf{P}_0^\perp d^1_{n+1}(s)\|_{\mathcal{H}}\right)\,\mathrm{d}s\,.\end{aligned}$$

Proof. Since the difference $h_{n+1}-h_n=d^0_{n+1}+d^1_{n+1}$ has zero mass and momentum, one can follow the line of proof of Lemma 4.4 to deduce that

$$\begin{aligned}\|\mathbf{P}_0 d^1_{n+1}(t)\|_{\mathcal{H}} \lesssim \|d^0_{n+1}(t)\|_{\mathcal{E}}+\varepsilon\lambda_\varepsilon\int_0^t e^{-\lambda_\varepsilon(t-s)}\Psi^\infty_n(s)\Big(\|d^0_n(s)\|_{\mathcal{E}}+\|d^1_n(s)\|_{\mathcal{H}}\Big)\,\mathrm{d}s\\ +\lambda_\varepsilon\int_0^t e^{-\lambda_\varepsilon(t-s)}\left(\|d^0_{n+1}(s)\|_{\mathcal{E}}+\|\mathbf{P}_0^\perp d^1_{n+1}(s)\|_{\mathcal{E}}\right)\,\mathrm{d}s\,.\end{aligned}$$

Consequently, plugging (5.10) in the right side yields the wanted result. □

Let us focus on estimating $\mathbf{P}_0^\perp d^1_{n+1}(t)$. To do so, we introduce the functions Φ^1_n and Φ^∞_n defined by

$$\Phi^1_n(t):=\|h^1_n(t)\|^2_{\mathcal{H}_1}+\|h^1_{n-1}(t)\|^2_{\mathcal{H}_1}\quad\text{and}\quad\Phi^\infty_n(t):=\|h^1_n(t)\|^2_{\mathcal{H}}+\|h^1_{n-1}(t)\|^2_{\mathcal{H}}$$

which satisfy

$$\sup_{t\geqslant 0}\left(\Phi_n^\infty(t)+\int_0^t\Phi_n^1(\tau)\,\mathrm{d}\tau\right)\leqslant C\mathcal{K}_0\,,\quad n\geqslant 2\,. \tag{5.12}$$

One has the following lemma.

Lemma 5.5. *Let $\varepsilon\in(0,\varepsilon^\dagger)$, $\lambda_0\in[0,\lambda^\dagger)$ and $\mathcal{K}_0\leqslant\mathcal{K}_0^\dagger$. For any $t\geqslant 0$, we have that*

$$\begin{aligned}\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2\lesssim&\int_0^t e^{-\nu(t-s)}\Phi_n^1(s)\|d_n^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s\\&+\int_0^t e^{-\nu(t-s)}\Phi_n^\infty(s)\|d_n^1(s)\|_{\mathcal{H}_1}^2\,\mathrm{d}s+\lambda_\varepsilon\int_0^t e^{-\nu(t-s)}\|d_{n+1}^1(s)\|_{\mathcal{H}}^2\,\mathrm{d}s\\&+\varepsilon^{-1}\left(\sup_{s\geqslant 0}\|d_{n+1}^1(s)\|_{\mathcal{H}}\right)\int_0^t e^{-\nu(t-\tau)}\,\Psi_n^1(\tau)\left(\|d_n^0(\tau)\|_{\mathcal{E}}+\|d_n^1(\tau)\|_{\mathcal{H}}\right)\mathrm{d}\tau\\&+\left(\sup_{s\geqslant 0}\|d_{n+1}^1(s)\|_{\mathcal{H}}\right)\int_0^t e^{-\nu(t-\tau)}\,\Psi_n^\infty(\tau)\left(\varepsilon^{-1}\|d_n^0(\tau)\|_{\mathcal{E}}+\varepsilon\,\lambda_\varepsilon\|d_n^1(\tau)\|_{\mathcal{H}}\right)\mathrm{d}\tau\end{aligned}$$

where $\nu>0$ is defined in the proof.

Proof. One deduces from (5.8) that $\mathbf{P}_0^\perp d_{n+1}^1(t)$ is such that

$$\begin{aligned}\partial_t\mathbf{P}_0^\perp d_{n+1}^1(t)=\mathcal{G}_{1,\varepsilon}\mathbf{P}_0^\perp\,d_{n+1}^1(t)-\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\left[\mathbf{P}_0^\perp\left(\mathrm{div}_v\left(vd_{n+1}^1\right)\right)+\mathrm{a}_3\mathbf{P}_0^\perp\left(\boldsymbol{\varpi}_2\mathbf{P}_0^\perp d_{n+1}^1\right)\right]\\+\mathbf{P}_0^\perp\mathcal{A}_\varepsilon d_{n+1}^0(t)+\varepsilon^{-1}\mathbf{P}_0^\perp\mathcal{F}_n^1\end{aligned}$$

where

$$\mathcal{F}_n^1:=(\mathbf{Id}-\boldsymbol{\pi}_0)\left[\mathcal{Q}_1(d_n^1,h_n^1)+\mathcal{Q}_1(h_{n-1}^1,d_n^1)\right]. \tag{5.13}$$

Following the argument leading to inequality (4.29), one deduces first that, recalling that $|\!|\!|\cdot|\!|\!|_{\mathcal{H}}$ is the hypocoercivity norm introduced in Proposition 3.1 and $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle_{\mathcal{H}}$ is the associated inner product that there exists a positive constant $c_0>0$ such that

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\left|\!\left|\!\left|\mathbf{P}_0^\perp d_{n+1}^1(t)\right|\!\right|\!\right|_{\mathcal{H}}^2\leqslant&-\frac{\mathrm{a}_1}{\varepsilon^2}\left\|(\mathbf{Id}-\boldsymbol{\pi}_0)\,\mathbf{P}_0^\perp d_{n+1}^1(t)\right\|_{\mathcal{H}_1}^2-\mathrm{a}_1\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}_1}^2\\&+\frac{c_0}{\varepsilon^2}\left\|d_{n+1}^1(t)\right\|_{\mathcal{H}}\|d_{n+1}^0(t)\|_{\mathcal{E}}+c_0\frac{1-\alpha(\varepsilon)}{\varepsilon^2}\|d_{n+1}^1(t)\|_{\mathcal{H}}^2\\&+c_0\varepsilon^{-1}\|d_n^1(t)\|_{\mathcal{H}_1}\left(\|h_n^1(t)\|_{\mathcal{H}}+\|h_{n-1}^1(t)\|_{\mathcal{H}}\right)\left\|(\mathbf{Id}-\boldsymbol{\pi}_0)\,\mathbf{P}_0^\perp d_{n+1}^1(t)\right\|_{\mathcal{H}_1}\\&+c_0\varepsilon^{-1}\|d_n^1(t)\|_{\mathcal{H}}\left(\|h_n^1(t)\|_{\mathcal{H}_1}+\|h_{n-1}^1(t)\|_{\mathcal{H}_1}\right)\left\|(\mathbf{Id}-\boldsymbol{\pi}_0)\,\mathbf{P}_0^\perp d_{n+1}^1(t)\right\|_{\mathcal{H}_1}\\&+c_0\left\|\boldsymbol{\pi}_0\mathbf{P}_0^\perp d_{n+1}^1(t)\right\|_{\mathcal{H}}\|d_n^1(t)\|_{\mathcal{H}}\left(\|h_n^1(t)\|_{\mathcal{H}}+\|h_{n-1}^1(t)\|_{\mathcal{H}}\right).\end{aligned}$$

There exists $c_1 > 0$ such that

$$
\begin{aligned}
c_0 \left\| \boldsymbol{\pi}_0 \mathbf{P}_0^\perp d_{n+1}^1(t) \right\|_{\mathcal{H}} \|d_n^1(t)\|_{\mathcal{H}} \left(\|h_n^1(t)\|_{\mathcal{H}} + \|h_{n-1}^1(t)\|_{\mathcal{H}} \right) \\
\leqslant c_1 \|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}_1} \|d_n^1(t)\|_{\mathcal{H}} \left(\|h_n^1(t)\|_{\mathcal{H}} + \|h_{n-1}^1(t)\|_{\mathcal{H}} \right) \\
\leqslant \frac{\mathrm{a}_1}{2} \|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}_1}^2 + \frac{2c_1^2}{\mathrm{a}_1} \Phi_n^\infty(t) \|d_n^1(t)\|_{\mathcal{H}}^2
\end{aligned}
$$

while, using again Young's inequality as in (4.28), one deduces that, for any $\eta > 0$,

$$
\begin{aligned}
c_0 \varepsilon^{-1} \|d_n^1(t)\|_{\mathcal{H}_1} \left(\|h_n^1(t)\|_{\mathcal{H}} + \|h_{n-1}^1(t)\|_{\mathcal{H}} \right) \left\| (\mathbf{Id} - \boldsymbol{\pi}_0) \mathbf{P}_0^\perp d_{n+1}^1(t) \right\|_{\mathcal{H}_1} \\
+ c_0 \varepsilon^{-1} \|d_n^1(t)\|_{\mathcal{H}} \left(\|h_n^1(t)\|_{\mathcal{H}_1} + \|h_{n-1}^1(t)\|_{\mathcal{H}_1} \right) \left\| (\mathbf{Id} - \boldsymbol{\pi}_0) \mathbf{P}_0^\perp d_{n+1}^1(t) \right\|_{\mathcal{H}_1} \\
\leqslant c_0 \frac{\eta}{\varepsilon^2} \left\| (\mathbf{Id} - \boldsymbol{\pi}_0) \mathbf{P}_0^\perp d_{n+1}^1(t) \right\|_{\mathcal{H}_1}^2 \\
+ \frac{c_0}{4\eta} \left(\|d_n^1(t)\|_{\mathcal{H}_1}^2 \Phi_n^\infty(t) + \|d_n^1(t)\|_{\mathcal{H}}^2 \Phi_n^1(t) \right)^2 .
\end{aligned}
$$

Picking η small enough so that $c_0 \eta \leqslant \mathrm{a}_1$ and, arguing as in the estimates leading to (4.29), this provides the existence of $c_2 > 0$ such that

$$
\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \mathbf{P}_0^\perp d_{n+1}^1(t) \right\|_{\mathcal{H}}^2 \leqslant -\frac{\mathrm{a}_1}{2} \|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}_1}^2 + c_2 \Phi_n^\infty(t) \|d_n^1(t)\|_{\mathcal{H}_1}^2 + c_2 \Phi_n^1(t) \|d_n^1(t)\|_{\mathcal{H}}^2 \\
+ c_2 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \|d_{n+1}^1(t)\|_{\mathcal{H}}^2 + \frac{c_2}{\varepsilon^2} \|d_{n+1}^1(t)\|_{\mathcal{H}} \|d_{n+1}^0(t)\|_{\mathcal{E}} . \quad (5.14)
\end{aligned}
$$

Setting $\nu := C_{\mathcal{H}}^2 \mathrm{a}_1$ (similarly as in Lemma 4.6), we deduce after integration that

$$
\begin{aligned}
\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 \lesssim \int_0^t e^{-\nu(t-s)} \Phi_n^1(s) \|d_n^1(s)\|_{\mathcal{H}}^2 \, \mathrm{d}s + \int_0^t e^{-\nu(t-s)} \Phi_n^\infty(s) \|d_n^1(s)\|_{\mathcal{H}_1}^2 \, \mathrm{d}s \\
+ \lambda_\varepsilon \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 \, \mathrm{d}s \\
+ \frac{1}{\varepsilon^2} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \|d_{n+1}^0(s)\|_{\mathcal{E}} \, \mathrm{d}s . \quad (5.15)
\end{aligned}
$$

The latter term in the right side of (5.15) can be estimated using (5.10):

$$
\frac{1}{\varepsilon^2} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \|d_{n+1}^0(s)\|_{\mathcal{E}} \, \mathrm{d}s \lesssim \sum_{i=1}^3 \mathcal{T}_i ,
$$

with

$$\mathcal{T}_1 := \frac{\lambda_\varepsilon}{\varepsilon^2}\int_0^t e^{-\nu(t-s)}\|d^1_{n+1}(s)\|_{\mathcal{H}}\left(\int_0^s e^{-\frac{\nu_q}{\varepsilon^2}(s-\tau)}\|d^1_{n+1}(\tau)\|_{\mathcal{H}}\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$\mathcal{T}_2 := \varepsilon^{-3}\int_0^t e^{-\nu(t-s)}\|d^1_{n+1}(s)\|_{\mathcal{H}}\left[\int_0^s e^{-\frac{\nu_q}{\varepsilon^2}(s-\tau)}\Psi^1_n(\tau)\Big(\|d^0_n(\tau)\|_{\mathcal{E}}+\|d^1_n(\tau)\|_{\mathcal{H}}\Big)\,\mathrm{d}\tau\right]\mathrm{d}s$$

$$\mathcal{T}_3 := \varepsilon^{-2}\int_0^t e^{-\nu(t-s)}\|d^1_{n+1}(s)\|_{\mathcal{H}}\left[\int_0^s e^{-\frac{\nu_q}{\varepsilon^2}(s-\tau)}\Psi^\infty_n(\tau)\Big(\varepsilon^{-1}\|d^0_n(\tau)\|_{\mathcal{E}_1}+\varepsilon\lambda_\varepsilon\|d^1_n(\tau)\|_{\mathcal{H}}\Big)\,\mathrm{d}\tau\right]\mathrm{d}s\,.$$

It is easy to check, using (4.24) and the fact that $\nu_q \geqslant 2\varepsilon^2\nu$ for $\varepsilon < \varepsilon^\dagger$, that

$$\mathcal{T}_2 \lesssim \varepsilon^{-1}\left(\sup_{s\geqslant 0}\|d^1_{n+1}(s)\|_{\mathcal{H}}\right)\int_0^t e^{-\nu(t-\tau)}\,\Psi^1_n(\tau)\,\Big(\|d^0_n(\tau)\|_{\mathcal{E}}+\|d^1_n(\tau)\|_{\mathcal{H}}\Big)\,\mathrm{d}\tau$$

and

$$\mathcal{T}_3 \lesssim \left(\sup_{s\geqslant 0}\|d^1_{n+1}(s)\|_{\mathcal{H}}\right)\int_0^t e^{-\nu(t-\tau)}\,\Psi^\infty_n(\tau)\,\Big(\varepsilon^{-1}\|d^0_n(\tau)\|_{\mathcal{E}_1}+\varepsilon\,\lambda_\varepsilon\|d^1_n(\tau)\|_{\mathcal{H}}\Big)\,\mathrm{d}\tau.$$

The estimate for $\mathcal{T}_1$ is a bit more involved. Thanks to Cauchy-Schwarz inequality one first has

$$\mathcal{T}_1 \leqslant \frac{\lambda_\varepsilon}{\varepsilon^2}\left(\int_0^t e^{-\nu(t-s)}\|d^1_{n+1}(s)\|^2_{\mathcal{H}}\,\mathrm{d}s\right)^{\frac12}\left(\int_0^t e^{-\nu(t-s)}Y^2(s)\,\mathrm{d}s\right)^{\frac12}$$

where

$$Y(s) := \int_0^s e^{-\frac{\nu_q}{\varepsilon^2}(s-\tau)}\|d^1_{n+1}(\tau)\|_{\mathcal{H}}\,\mathrm{d}\tau, \qquad s\in(0,t).$$

Thanks to (4.12) applied with $r=\frac12$,

$$Y^2(s) \lesssim \varepsilon^2\int_0^s e^{-\frac{\nu_q}{\varepsilon^2}(s-\tau)}\|d^1_{n+1}(\tau)\|^2_{\mathcal{H}}\,\mathrm{d}\tau$$

and, using now (4.24) and the fact that $\nu_q \geqslant 2\varepsilon^2\nu$,

$$\int_0^t e^{-\nu(t-s)}Y^2(s)\,\mathrm{d}s \lesssim \varepsilon^2\int_0^t e^{-\nu(t-s)}\,\mathrm{d}s\int_0^s e^{-\frac{\nu_q}{\varepsilon^2}(s-\tau)}\|d^1_{n+1}(\tau)\|^2_{\mathcal{H}}\,\mathrm{d}\tau$$
$$\lesssim \varepsilon^4\int_0^t e^{-\nu(t-s)}\|d^1_{n+1}(s)\|^2_{\mathcal{H}}\,\mathrm{d}s\,.$$

We deduce finally that

$$\mathcal{T}_1 \lesssim \lambda_\varepsilon\int_0^t e^{-\nu(t-s)}\|d^1_{n+1}(s)\|^2_{\mathcal{H}}\,\mathrm{d}s$$

and this, together with the estimates for $\mathcal{T}_2$ and $\mathcal{T}_3$, gives the desired conclusion. □

Introducing now the quantities

$$\Xi_n^0 := \sup_{t\geqslant 0}\left(\|d_n^0(t)\|_{\mathcal{E}} + \varepsilon^{-2}\int_0^t \|d_n^0(\tau)\|_{\mathcal{E}_1}\,\mathrm{d}\tau\right),$$
$$\Xi_n^1 := \sup_{t\geqslant 0}\left(\|d_n^1(t)\|_{\mathcal{H}}^2 + \int_0^t \|d_n^1(\tau)\|_{\mathcal{H}_1}^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}, \qquad n\geqslant 2,$$

we can gather the three previous lemmas and use (5.9) to obtain the following result.

Proposition 5.6. *Let $\varepsilon\in(0,\varepsilon^\dagger)$, $\lambda_0\in[0,\lambda^\dagger)$ and $\mathcal{K}_0\leqslant\mathcal{K}_0^\dagger$. For any $t\geqslant 0$, we have that*

$$\|d_{n+1}^0(t)\|_{\mathcal{E}} \lesssim \varepsilon^2\lambda_\varepsilon\,\Xi_{n+1}^1 + \varepsilon\sqrt{\mathcal{K}_0}\left(\Xi_n^0+\Xi_n^1\right), \tag{5.16}$$

while

$$\|d_{n+1}^1(t)\|_{\mathcal{H}} \lesssim \sqrt{\lambda_\varepsilon+\varepsilon}\,\Xi_{n+1}^1 + \sqrt{\varepsilon\mathcal{K}_0}\,\Xi_n^0 + \sqrt{\mathcal{K}_0}\,\Xi_n^1. \tag{5.17}$$

Proof. First, we claim that

$$\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}} \lesssim \sqrt{\lambda_\varepsilon+\varepsilon}\,\Xi_{n+1}^1 + \sqrt{\varepsilon\mathcal{K}_0}\,\Xi_n^0 + \sqrt{\mathcal{K}_0}\,\Xi_n^1, \tag{5.18}$$

Indeed, from Lemma 5.5, we have that

$$\begin{aligned}\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 \lesssim \lambda_\varepsilon\left[\Xi_{n+1}^1\right]^2 + \left[\Xi_n^1\right]^2\left(\int_0^t\Phi_n^1(s)\,\mathrm{d}s + \sup_{s\geqslant 0}\Phi_n^\infty(s)\right)\\ + \varepsilon^{-1}\Xi_{n+1}^1\left(\Xi_n^0+\Xi_n^1\right)\int_0^t\Psi_n^1(\tau)\,\mathrm{d}\tau\\ + \Xi_{n+1}^1\left(\sup_{s\geqslant 0}\Psi_n^\infty(s)\right)\left(\varepsilon\,\Xi_n^0+\varepsilon\lambda_\varepsilon\Xi_n^1\right).\end{aligned}$$

We can thus invoke (5.9) and (5.12) to deduce that

$$\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 \lesssim \mathcal{K}_0\left[\Xi_n^1\right]^2 + \lambda_\varepsilon\left[\Xi_{n+1}^1\right]^2 + \varepsilon\sqrt{\mathcal{K}_0}\left(\Xi_n^0+\Xi_n^1\right)\Xi_{n+1}^1$$

where we used that $\varepsilon\lambda_\varepsilon\lesssim\varepsilon$. From Young's inequality, we deduce that

$$\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 \lesssim (\lambda_\varepsilon+\varepsilon)\left[\Xi_{n+1}^1\right]^2 + \mathcal{K}_0\left[\Xi_n^1\right]^2 + \varepsilon\mathcal{K}_0\left[\Xi_n^0\right]^2,$$

which proves (5.18). Then, we have that

$$\|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{H}} \lesssim \sqrt{\lambda_\varepsilon+\varepsilon}\,\Xi_{n+1}^1 + \sqrt{\varepsilon\,\mathcal{K}_0}\,\Xi_n^0 + \sqrt{\mathcal{K}_0}\Xi_n^1. \tag{5.19}$$

This inequality is a consequence of Lemma 5.4 combined with (5.9), (5.16) and (5.18). In the same way, the estimate (5.16) is easily deduced from Lemma 5.3. To end the proof, it remains to prove that □

We slightly modify here the proof of Lemma 5.3 and Lemma 5.5 to get the following:

Proposition 5.7. *Let $\varepsilon \in (0, \varepsilon^\dagger)$, $\lambda_0 \in [0, \lambda^\dagger)$ and $\mathcal{K}_0 \leqslant \mathcal{K}_0^\dagger$. For any $t \geqslant 0$, we have that*

$$\frac{1}{\varepsilon^2} \int_0^t \|d_{n+1}^0(s)\|_{\mathcal{E}_1} \, \mathrm{d}s \lesssim \lambda_\varepsilon \, \mathbf{\Xi}_{n+1}^1 + \varepsilon \sqrt{\mathcal{K}_0} \left(\mathbf{\Xi}_n^0 + \mathbf{\Xi}_n^1 \right), \tag{5.20}$$

and

$$\left(\int_0^t \|d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \lesssim \sqrt{\lambda_\varepsilon + \varepsilon} \, \mathbf{\Xi}_{n+1}^1 + \sqrt{\varepsilon \, \mathcal{K}_0} \, \mathbf{\Xi}_n^0 + \sqrt{\mathcal{K}_0} \mathbf{\Xi}_n^1 . \tag{5.21}$$

Proof. To prove (5.20), we follow the argument that led to Lemma 5.3 and recall that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|d_{n+1}^0(t)\|_{\mathcal{E}} \leqslant -\frac{\nu_q}{\varepsilon^2} \|d_{n+1}^0(t)\|_{\mathcal{E}_1} + \varepsilon^{-1} \|\mathcal{F}_n^0(t)\|_{\mathcal{E}} + C\lambda_\varepsilon \|d_{n+1}^1(t)\|_{\mathcal{H}} .$$

After integration over $[0, t]$, using that $d_{n+1}^0(0) = 0$, we get that

$$\|d_{n+1}^0(t)\|_{\mathcal{E}} + \frac{1}{\varepsilon^2} \int_0^t \|d_{n+1}^0(s)\|_{\mathcal{E}_1} \, \mathrm{d}s \lesssim \varepsilon^{-1} \int_0^t \|\mathcal{F}_n^0(s)\|_{\mathcal{E}} \, \mathrm{d}s + \lambda_\varepsilon \int_0^t \|d_{n+1}^1(s)\|_{\mathcal{H}} \, \mathrm{d}s ,$$

and, recalling that $\mathcal{F}_n^0$ is given by (5.11), we estimate $\|\mathcal{F}_n^0(s)\|_{\mathcal{E}}$ as in the proof of Lemma 5.3 to obtain that

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_0^t \|d_{n+1}^0(s)\|_{\mathcal{E}_1} \, \mathrm{d}s \lesssim \lambda_\varepsilon \int_0^t \|d_{n+1}^1(s)\|_{\mathcal{H}} \, \mathrm{d}s + \varepsilon^{-1} \int_0^t \Psi_n^1(s) \Big(\|d_n^0(s)\|_{\mathcal{E}} + \|d_n^1(s)\|_{\mathcal{H}} \Big) \, \mathrm{d}s \\ + \int_0^t \Psi_n^\infty(s) \Big(\varepsilon^{-1} \|d_n^0(s)\|_{\mathcal{E}_1} + \varepsilon \lambda_\varepsilon \|d_n^1(s)\|_{\mathcal{H}} \Big) \, \mathrm{d}s . \end{aligned}$$

This yields (5.20). In the same way, we adapt the proof of Lemma 5.5 and, according to (5.14), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left| \left| \left| \mathbf{P}_0^\perp d_{n+1}^1(t) \right| \right| \right|_{\mathcal{H}}^2 + \frac{\nu}{2} \left| \left| \left| \mathbf{P}_0^\perp d_{n+1}^1(t) \right| \right| \right|_{\mathcal{H}}^2 \\ \leqslant -\frac{\mathrm{a}_1}{2} \|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}_1}^2 + C\Phi_n^\infty(t) \|d_n^1(t)\|_{\mathcal{H}_1}^2 + C\Phi_n^1(t) \|d_n^1(t)\|_{\mathcal{H}}^2 , \\ + C\lambda_\varepsilon \|d_{n+1}^1(t)\|_{\mathcal{H}}^2 + \frac{C}{\varepsilon^2} \|d_{n+1}^1(t)\|_{\mathcal{H}} \|d_{n+1}^0(t)\|_{\mathcal{E}} , \end{aligned}$$

which, after integration, gives

$$\begin{aligned} \int_0^t e^{-\frac{\nu}{2}(t-\tau)} \|\mathbf{P}_0^\perp d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 \, \mathrm{d}\tau \lesssim \int_0^t e^{-\frac{\nu}{2}(t-s)} \Phi_n^1(s) \|d_n^1(s)\|_{\mathcal{H}}^2 \, \mathrm{d}s \\ + \lambda_\varepsilon \int_0^t e^{-\frac{\nu}{2}(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 \mathrm{d}s \\ + \int_0^t e^{-\frac{\nu}{2}(t-s)} \Phi_n^\infty(s) \|d_n^1(s)\|_{\mathcal{H}_1}^2 \, \mathrm{d}s + \frac{1}{\varepsilon^2} \int_0^t e^{-\frac{\nu}{2}(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \, \|d_{n+1}^0(s)\|_{\mathcal{E}} \, \mathrm{d}s . \end{aligned}$$

This estimate is similar to (5.15) and therefore we can resume both the proofs of Lemma 5.5 and Proposition 5.6 to obtain that

$$\left(\int_0^t \|\mathbf{P}_0^\perp d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \lesssim \sqrt{\lambda_\varepsilon + \varepsilon}\, \mathbf{\Xi}_{n+1}^1 + \sqrt{\varepsilon\, \mathcal{K}_0}\, \mathbf{\Xi}_n^0 + \sqrt{\mathcal{K}_0} \mathbf{\Xi}_n^1 \,.$$

Using Lemma 5.4 combined with (5.9), (5.16) and (5.17), we also obtain that

$$\left(\int_0^t \|\mathbf{P}_0 d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \lesssim \sqrt{\lambda_\varepsilon + \varepsilon}\, \mathbf{\Xi}_{n+1}^1 + \sqrt{\varepsilon\, \mathcal{K}_0}\, \mathbf{\Xi}_n^0 + \sqrt{\mathcal{K}_0} \mathbf{\Xi}_n^1 \,.$$

Adding these two estimates, one deduces (5.21). □

5.3. **Convergence of the iteration scheme.** We are now in position to conclude our analysis by proving the convergence of the iteration scheme. Suitably adding (5.16) and (5.20) and taking the supremum in time, one has that

$$\mathbf{\Xi}_{n+1}^0 \lesssim \lambda_\varepsilon\, \mathbf{\Xi}_{n+1}^1 + \varepsilon \sqrt{\mathcal{K}_0} \left(\mathbf{\Xi}_n^0 + \mathbf{\Xi}_n^1 \right). \tag{5.22}$$

Similarly, adding (5.17) and (5.21) and taking the supremum in time it holds that

$$\mathbf{\Xi}_{n+1}^1 \lesssim \sqrt{\lambda_\varepsilon + \varepsilon}\, \mathbf{\Xi}_{n+1}^1 + \sqrt{\varepsilon \mathcal{K}_0}\, \mathbf{\Xi}_n^0 + \sqrt{\mathcal{K}_0}\, \mathbf{\Xi}_n^1 \,. \tag{5.23}$$

Let us define $\mathscr{E}_n := \mathbf{\Xi}_n^0 + \mathbf{\Xi}_n^1$, for $n \geqslant 2$. Adding the estimates (5.22) and (5.23), we conclude that there exists $C > 0$ such that $\mathscr{E}_{n+1} \leqslant C\sqrt{\lambda_\varepsilon + \varepsilon}\, \mathscr{E}_{n+1} + C\sqrt{\mathcal{K}_0}\, \mathscr{E}_n$. Thus, choosing ε sufficiently small such that $C\sqrt{\lambda_\varepsilon + \varepsilon} \leqslant \frac{1}{2}$, we get that $\mathscr{E}_{n+1} \leqslant \frac{1}{2} C \sqrt{\mathcal{K}_0}\, \mathscr{E}_n$ from which

$$\mathscr{E}_{n+1} \leqslant \left(\frac{C}{2} \sqrt{\mathcal{K}_0} \right)^{n-1} \mathscr{E}_2 \,, \qquad \forall\, n \geqslant 2 \,.$$

Choosing $\mathcal{K}_0 \leqslant \mathcal{K}_0^\dagger < 4C^{-2}$ so that

$$\theta := \frac{C}{2} \sqrt{\mathcal{K}_0^\dagger} < 1$$

we deduce that, in the Banach space $(\mathbb{B}, |\!|\!| \cdot |\!|\!|)$, one has for $m > n \geqslant 1$,

$$|\!|\!| (h_m^0, h_m^1) - (h_n^0, h_n^1) |\!|\!| \leqslant \sum_{i=n}^{m-1} \mathscr{E}_{i+1} \leqslant \mathscr{E}_2\, \frac{\theta^{n-1}}{1-\theta} \,.$$

Whence the sequence $\left\{ (h_n^0, h_n^1) \right\}_n \subset \mathcal{X}_0 \times \mathcal{X}_1 \subset \mathbb{B}$ is a Cauchy sequence and it converges in $(\mathbb{B}, |\!|\!| \cdot |\!|\!|)$ to a limit $(h^0, h^1) \in \mathcal{X}_0 \times \mathcal{X}_1$. Of course, such limit satisfies equations (4.3) and (4.4). Thus, $h = h^0 + h^1$ is a solution to the inelastic Boltzmann problem (1.27). Such solution is unique in the class of functions that we consider since, at essence, we proved that the problem is a contraction on $\mathcal{X}_0 \times \mathcal{X}_1$. This completes the proof of the existence part of Theorem 5.1 recalling that $f_\varepsilon = G_{\alpha(\varepsilon)} + \varepsilon h_\varepsilon$. If one wants to obtain the rates that are stated in the Theorem, one has to be more careful in the previous proof and keep track of every rate at each step. We chose not to do it in order to lighten the proof and also because the proof would be completely similar.

Remark 5.8. *We point out that, as observed by an anonymous referee, our construction and fixed point argument give the* uniqueness *of solutions to* (1.27) only *for the class of solutions $h \in \mathcal{X}_0$ that can be written as a sum $h = h^0 + h^1$ with $(h^0, h^1) \in \mathcal{X}_0 \times \mathcal{X}_1$. Strictly speaking, other solutions in $\mathcal{X}_0$ could exist. However, uniqueness of general solutions to the spatially inhomogeneous Boltzmann equation (with a forcing term given by a Laplace operator) has been established in Tristani (2016) (for any choice of $\varepsilon > 0$). The argument of Tristani (2016) can be easily adapted to deal with the forcing term given by the drift term as considered in* (1.27). *Therefore, in the functional space $\mathcal{X}_0$, uniqueness of solutions to* (1.27) *holds true.*

6. Hydrodynamic limit

In this section, we modify a bit the assumptions made on the functional spaces $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ defined in (4.1)-(4.2), we shall consider m and q satisfying

$$m > \frac{d}{2} \qquad \text{and} \qquad q \geqslant 5$$

but keep the same notations $\mathcal{E}, \mathcal{E}_1$ and $\mathcal{E}_2$. We also denote by $(\varepsilon^\dagger, \lambda^\dagger, \eta_0)$ the threshold values such that the conclusions of Theorem 5.1 and Remark 5.2 are satisfied in those functional spaces. The estimates on the solution h_ε provided by Theorem 5.1 and Remark 5.2 are enough to prove that the solution $h_\varepsilon(t)$ converges towards some hydrodynamic solution $\boldsymbol{h}$ which depends on (t, x) only through macroscopic quantities $(\varrho(t,x), u(t,x), \theta(t,x))$ which are solutions to a suitable modification of the incompressible Navier-Stokes system. This is done under an additional assumption on the initial datum, namely (6.1), that is lightly restrictive as explained in Remark 1.5. Before stating our main convergence result, we introduce the notation

$$\mathscr{W}_\ell := \left(\mathbb{W}^{\ell,2}_x \left(\mathbb{T}^d \right) \right)^{d+2}, \quad \ell \in \mathbb{N}\,.$$

We prove here the following precised version of Theorem 1.4 presented in the Introduction:

Theorem 6.1. *We suppose that the assumptions of Theorem 5.1 are satisfied. We assume furthermore that there exists $(\varrho_0, u_0, \theta_0) \in \mathscr{W}_m$ such that*

$$\| \boldsymbol{\pi}_0 h^\varepsilon_{\rm in} - h_0 \|_{\mathcal{E}} \xrightarrow[\varepsilon \to 0]{} 0\,, \tag{6.1}$$

where we recall that $\boldsymbol{\pi}_0$ is the projection onto the kernel of $\mathbf{L}_1$ defined in (3.2) *and*

$$h_0(x,v) := \left(\varrho_0(x) + u_0(x) \cdot v + \frac{1}{2} \theta_0(x) (|v|^2 - d\vartheta_1) \right) \mathcal{M}(v)\,. \tag{6.2}$$

Then, for any $T > 0$, the family of solutions $\{h_\varepsilon\}_\varepsilon$ constructed in Theorem 5.1 converges in some weak sense to a limit $\boldsymbol{h} = \boldsymbol{h}(t,x,v)$ which is such that

$$\boldsymbol{h}(t,x,v) = \left(\varrho(t,x) + u(t,x) \cdot v + \frac{1}{2} \theta(t,x) (|v|^2 - d\vartheta_1) \right) \mathcal{M}(v)\,, \tag{6.3}$$

where

$$(\varrho, u, \theta) \in \mathcal{C}\left([0,T]\,;\mathscr{W}_{m-1}\right) \cap L^2\left((0,T)\,;\mathscr{W}_m\right)$$

is solution to the following incompressible Navier-Stokes-Fourier system with forcing

$$\begin{cases} \partial_t u - \frac{\nu}{\vartheta_1}\,\Delta_x u + \vartheta_1\, u\cdot\nabla_x\, u + \nabla_x p = \lambda_0 u\,, \\ \partial_t\,\theta - \frac{\gamma}{\vartheta_1^2}\,\Delta_x\theta + \vartheta_1\, u\cdot\nabla_x\theta = \dfrac{\lambda_0\,\bar{c}}{2(d+2)}\sqrt{\vartheta_1}\,\theta\,, \\ \mathrm{div}_x u = 0\,, \qquad \varrho + \vartheta_1\,\theta = 0\,, \end{cases} \tag{6.4}$$

subject to initial conditions $(\varrho_{\rm in}, u_{\rm in}, \theta_{\rm in})$ *defined by*

$$u_{\rm in} := \mathcal{P} u_0\,, \quad \theta_{\rm in} := \frac{d}{d+2}\theta_0 - \frac{2}{(d+2)\vartheta_1}\varrho_0\,, \quad \varrho_{\rm in} := -\vartheta_1\theta_{\rm in}\,, \tag{6.5}$$

where $\mathcal{P}$ *is the Leray projection on divergence-free vector fields and* $(\varrho_0, u_0, \theta_0)$ *have been introduced in* (6.2)*. The viscosity* $\nu > 0$ *and heat conductivity* $\gamma > 0$ *are explicit and* $\lambda_0 \geqslant 0$ *is the parameter appearing in Assumption 1.1. The parameter* $\bar{c} > 0$ *is depending on the collision kernel* $b(\,\cdot\,)$*.*

In what follows, we shall consider $\{h_\varepsilon\}_\varepsilon$ a family of solutions to (1.27) constructed in this theorem that splits as $h_\varepsilon = h^0_\varepsilon + h^1_\varepsilon$ with h^0_ε and h^1_ε defined in Section 4 (notice that in this last section, we will once again specify that $h = h_\varepsilon$, $h^0 = h^0_\varepsilon$ and $h^1 = h^1_\varepsilon$ depend on ε). We also fix $T > 0$ for the rest of the section. Finally, mention that to lighten notations, we will write α for $\alpha(\varepsilon)$ but recall that $\alpha = \alpha(\varepsilon)$ satisfies Assumption 1.1.

6.1. **Compactness and convergence.** One can prove the following estimate for time-averages of the microscopic part of h_ε, namely on $(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon$, which in particular tells that this microscopic part vanishes in the limit $\varepsilon \to 0$:

Lemma 6.2. *For any* $0 \leqslant t_1 \leqslant t_2 \leqslant T$*, there holds:*

$$\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon(\tau)\|_{\mathcal{E}}\,\mathrm{d}\tau \lesssim \varepsilon\sqrt{t_2 - t_1}\,, \tag{6.6}$$

where we recall that $\boldsymbol{\pi}_0$ *is the projection onto the kernel of* $\mathbf{L}_1$ *defined in* (3.2)*.*

Proof. We first remark that

$$\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon(\tau)\|_{\mathcal{E}}\,\mathrm{d}\tau \lesssim \left(\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h^0_\varepsilon(\tau)\|^2_{\mathcal{E}}\,\mathrm{d}\tau\right)^{\frac12}\sqrt{t_2 - t_1} + \left(\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h^1_\varepsilon(\tau)\|^2_{\mathcal{H}_1}\,\mathrm{d}\tau\right)^{\frac12}\sqrt{t_2 - t_1}.$$

The first term is estimated thanks to (5.1), which gives:

$$\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h^0_\varepsilon(\tau)\|^2_{\mathcal{E}}\,\mathrm{d}\tau \lesssim \|(\mathbf{Id} - \boldsymbol{\pi}_0)h^0_\varepsilon\|_{L^\infty((0,T)\,;\mathcal{E})}\|(\mathbf{Id} - \boldsymbol{\pi}_0)h^0_\varepsilon\|_{L^1((0,T)\,;\mathcal{E}_1)} \lesssim \varepsilon^2.$$

Concerning the second one, we perform similar computations as in the proof of Lemma 4.6. We recall that h_ε^1 solves (4.4) and consider $|||\cdot|||_{\mathcal{H}}$ an hypocoercive norm on $\mathcal{H}$ (see Proposition 3.1). We then have, with $\Psi(t) = (\mathbf{Id} - \mathbf{P}_0)\, h_\varepsilon^1(t)$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||\Psi(t)|||_{\mathcal{H}}^2 \leqslant -\frac{\mathrm{a}_1}{2\varepsilon^2}\|(\mathbf{Id} - \boldsymbol{\pi}_0)\Psi(t)\|_{\mathcal{H}_1}^2 - \mathrm{a}_1\|\Psi(t)\|_{\mathcal{H}_1}^2 + C\|h_\varepsilon^1(t)\|_{\mathcal{H}}^2\left(\|h_\varepsilon^1(t)\|_{\mathcal{H}_1}^2 + \|h_\varepsilon^1(t)\|_{\mathcal{H}} + 1\right) + \frac{C}{\varepsilon^2}\|h_\varepsilon^0(t)\|_{\mathcal{E}}\|h_\varepsilon^1(t)\|_{\mathcal{H}}$$

from which we deduce that

$$\frac{1}{\varepsilon^2}\int_{t_1}^{t_2}\|(\mathbf{Id} - \boldsymbol{\pi}_0)\Psi(\tau)\|_{\mathcal{H}_1}^2\,\mathrm{d}\tau \lesssim \|h_\varepsilon^1(t_1)\|_{\mathcal{H}}^2 + \int_{t_1}^{t_2}\|h_\varepsilon^1(\tau)\|_{\mathcal{H}}^2\left(\|h_\varepsilon^1(\tau)\|_{\mathcal{H}_1}^2 + \|h^1(\tau)\|_{\mathcal{H}} + 1\right)\mathrm{d}\tau + \frac{1}{\varepsilon^2}\int_{t_1}^{t_2}\|h_\varepsilon^0(\tau)\|_{\mathcal{E}}\|h_\varepsilon^1(\tau)\|_{\mathcal{H}}\,\mathrm{d}\tau \lesssim 1$$

where we used the fact that the norm $|||\cdot|||_{\mathcal{H}}$ is equivalent to the usual one $\|\cdot\|_{\mathcal{H}}$ uniformly in ε as well as (5.1) and (5.2) to get the last estimate. Now, since $\boldsymbol{\pi}_0\mathbf{P}_0 = \mathbf{P}_0$, one has

$$(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1 = (\mathbf{Id} - \boldsymbol{\pi}_0)\Psi(t)$$

and one deduces that

$$\int_{t_1}^{t_2}\|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(\tau)\|_{\mathcal{H}_1}^2\,\mathrm{d}\tau \lesssim \varepsilon^2$$

and this allows to conclude to the wanted estimate. □

We deduce the following convergence result (whose proof is immediate using estimates (5.1), (5.2) together with (6.6)):

Theorem 6.3. *There exists $\boldsymbol{h} = \boldsymbol{\pi}_0(\boldsymbol{h}) \in L^2\left((0,T);\mathcal{H}\right)$ such that up to extraction of a subsequence, one has*

$$\begin{cases} \left\{h_\varepsilon^0\right\}_\varepsilon \text{ converges to } 0 \text{ strongly in } L^1((0,T)\,;\mathcal{E}_1)\,, \\ \left\{h_\varepsilon^1\right\}_\varepsilon \text{ converges to } \boldsymbol{h} \text{ weakly in } L^2\left((0,T)\,;\mathcal{H}\right)\,. \end{cases} \tag{6.7}$$

In particular, there exist

$$\varrho \in L^2\left((0,T)\,;\mathbb{W}_x^{m,2}(\mathbb{T}^d)\right)\,, \qquad u \in L^2\left((0,T)\,;\left(\mathbb{W}_x^{m,2}(\mathbb{T}^d)\right)^d\right)\,, \qquad \theta \in L^2\left((0,T)\,;\mathbb{W}_x^{m,2}(\mathbb{T}^d)\right)\,,$$

such that

$$\boldsymbol{h}(t,x,v) = \left(\varrho(t,x) + u(t,x)\cdot v + \frac{1}{2}\theta(t,x)(|v|^2 - d\vartheta_1)\right)\mathcal{M}(v) \tag{6.8}$$

where $\mathcal{M}$ is the Maxwellian distribution introduced in (1.19).

Remark 6.4. *The convergence (6.7) can be made even more precise since from Lemma 6.2, we also have*

$$\left\{(\mathbf{Id}-\boldsymbol{\pi}_0)\, h_\varepsilon^1\right\}_\varepsilon \text{ converges strongly to } 0 \text{ in } L^2\left((0,T)\,;\mathcal{H}\right).$$

This means somehow that the only part of h_ε which prevents the strong convergence towards $\boldsymbol{h}$ is $\left\{\boldsymbol{\pi}_0 h_\varepsilon^1\right\}_\varepsilon$.

Remark 6.5. *Notice that the hydrodynamic quantites (ϱ,u,θ) in (6.8) can be expressed in terms of $\boldsymbol{h}$ through the following equalities:*

$$\varrho(t,x)=\int_{\mathbb{R}^d}\boldsymbol{h}(t,x,v)\,\mathrm{d}v\,,\quad u(t,x)=\frac{1}{\vartheta_1}\int_{\mathbb{R}^d}\boldsymbol{h}(t,x,v)v\,\mathrm{d}v\,,$$
$$\theta(t,x)=\int_{\mathbb{R}^d}\boldsymbol{h}(t,x,v)\frac{|v|^2-d\vartheta_1}{\vartheta_1^2 d}\,\mathrm{d}v\,. \tag{6.9}$$

Because of Theorem 6.3 and for simplicity sake, from here on, we will write that our sequences converge even if it is true up to an extraction. We now aim to fully characterise the limit $\boldsymbol{h}$ obtained in Theorem 6.3. To do so, we are going to identify the limit equations satisfied by the macroscopic quantities (ϱ,u,θ) in (6.8) following the same lines as in the elastic case and more precisely the same path of Bardos et al. (1993); Golse & Saint-Raymond (2004) exploiting the fact that the mode of convergence in Theorem 6.3 is stronger than the one of Bardos et al. (1993); Golse & Saint-Raymond (2004). The regime of weak inelasticity is central in the analysis. The main idea is to write equations satisfied by averages in velocity of h_ε and to study the convergence of each term. To this end, we begin by a result about convergence of velocity averages of h_ε and in what follows, we will use the following notation: for $g=g(x,v)$,

$$\langle g\rangle:=\int_{\mathbb{R}^d} g(\,\cdot\,,v)\,\mathrm{d}v$$

which is now a function of the spatial variable only.

Lemma 6.6. *Let $\{h_\varepsilon\}_\varepsilon$ be converging to $\boldsymbol{h}$ in the sense of Theorem 6.3. Then, for any function $\psi=\psi(v)$ such that $|\psi(v)|\lesssim\boldsymbol{\varpi}_q(v)$, one has*

$$\left\langle\psi\, h_\varepsilon\right\rangle\xrightarrow[\varepsilon\to 0]{}\left\langle\psi\,\boldsymbol{h}\right\rangle\qquad\text{in}\qquad\mathscr{D}'_{t,x}\,. \tag{6.10}$$

Proof. Let ψ be such that $|\psi(v)|\lesssim\boldsymbol{\varpi}_q(v)$ and let $\varphi=\varphi(t,x)\in\mathcal{C}_c^\infty((0,T)\times\mathbb{T}^d)$ be given. One computes

$$I_\varepsilon:=\int_0^T\mathrm{d}t\int_{\mathbb{T}^d}\varphi(t,x)\left(\left\langle\psi\, h_\varepsilon\right\rangle-\left\langle\psi\,\boldsymbol{h}\right\rangle\right)\mathrm{d}x=I_\varepsilon^0+I_\varepsilon^1$$

where

$$I_\varepsilon^0:=\int_0^T\mathrm{d}t\int_{\mathbb{T}^d}\varphi(t,x)\left\langle\psi\, h_\varepsilon^0\right\rangle\mathrm{d}x\,,\qquad I_\varepsilon^1:=\int_0^T\mathrm{d}t\int_{\mathbb{T}^d}\varphi(t,x)\left(\left\langle\psi\, h_\varepsilon^1\right\rangle-\left\langle\psi\,\boldsymbol{h}\right\rangle\right)\mathrm{d}x\,.$$

Because $|I^0_\varepsilon| \lesssim \|\varphi\|_{L^\infty_{t,x}} \int_0^T \|h^0_\varepsilon(t)\|_{L^1_{x,v}(\varpi_q)}\,\mathrm{d}t \lesssim \|\varphi\|_{L^\infty_{t,x}} \int_0^T \|h^0_\varepsilon(t)\|_{L^1_v L^2_x(\varpi_q)}\,\mathrm{d}t$, we deduce from (6.7) that $\lim_{\varepsilon\to 0} I^0_\varepsilon = 0$. In the same way, one has

$$I^1_\varepsilon = \int_0^T \mathrm{d}t \int_{\mathbb{T}^d\times\mathbb{R}^d} (\psi(v)\,\mathcal{M}(v)\,\varphi(t,x))\,\left(h^1_\varepsilon(t,x,v) - \boldsymbol{h}(t,x,v)\right)\,\mathrm{d}x\,\mathcal{M}^{-1}(v)\,\mathrm{d}v$$

and, since we have

$$(t,x,v) \longmapsto \psi(v)\,\mathcal{M}(v)\,\varphi(t,x) \in L^2((0,T);\mathcal{H})\,, \tag{6.11}$$

we deduce from (6.7) that $\lim_{\varepsilon\to 0} I^1_\varepsilon = 0$. This proves (6.10). □

6.2. **Incompressibility condition and Boussinesq relation.** Using Lemma 6.6, we are able to obtain a first result about the incompressibility of u and to give a first version of Boussinesq relation on ϱ and θ.

Lemma 6.7. *With the notations of Theorem 6.3, the limit $\boldsymbol{h}$ given by (6.8) satisfies on $(0,T)\times\mathbb{T}^d$ the* incompressibility condition

$$\mathrm{div}_x u = 0\,, \tag{6.12}$$

as well as Boussinesq relation

$$\nabla_x\left(\varrho + \vartheta_1\theta\right) = 0\,. \tag{6.13}$$

As a consequence, introducing for almost every $t\in(0,T)$,

$$E(t) := \int_{\mathbb{T}^d} \theta(t,x)\,\mathrm{d}x\,, \tag{6.14}$$

one has strengthened Boussinesq relation*: for almost every $(t,x)\in(0,T)\times\mathbb{T}^d$,*

$$\varrho(t,x) + \vartheta_1\left(\theta(t,x) - E(t)\right) = 0\,. \tag{6.15}$$

Proof. Set

$$\varrho_\varepsilon(t,x) := \int_{\mathbb{R}^d} h_\varepsilon(t,x,v)\,\mathrm{d}v\,, \qquad u_\varepsilon(t,x) := \frac{1}{\vartheta_1}\int_{\mathbb{R}^d} v\,h_\varepsilon(t,x,v)\,\mathrm{d}v\,,$$

and, multiplying (1.27) with 1 and v and integrating in velocity, we get

$$\varepsilon\partial_t\varrho_\varepsilon + \vartheta_1\mathrm{div}_x\left(u_\varepsilon\right) = 0\,, \tag{6.16}$$

$$\varepsilon\,\partial_t u_\varepsilon + \mathrm{Div}_x\left(\boldsymbol{J}_\varepsilon\right) = \frac{1-\alpha}{\varepsilon}u_\varepsilon\,, \tag{6.17}$$

where $\boldsymbol{J}_\varepsilon = \boldsymbol{J}_\varepsilon(t,x)$ denotes the tensor

$$\boldsymbol{J}_\varepsilon(t,x) := \frac{1}{\vartheta_1}\int_{\mathbb{R}^d} v\otimes v\,h_\varepsilon(t,x,v)\,\mathrm{d}v\,,$$

since both $\mathbf{L}_\alpha$ and $\mathcal{Q}_\alpha$ conserve mass and momentum. The proof of (6.12) is straightforward since $\varepsilon\partial_t\varrho_\varepsilon \to 0$ and $\mathrm{div}_x(u_\varepsilon)\to\mathrm{div}_x u$ in the distribution sense from Lemma 6.6. Let us give

the detail for the sake of completeness. Multiplying (6.16) with a function $\varphi \in \mathcal{C}_c^\infty((0,T)\times\mathbb{T}^d)$ and integrating over $(0,T)\times\mathbb{T}^d$, we get that

$$-\vartheta_1 \int_0^T \mathrm{d}t \int_{\mathbb{T}^d} \nabla_x \varphi(t,x)\cdot u_\varepsilon(t,x)\,\mathrm{d}x = \varepsilon \int_0^T \mathrm{d}t \int_{\mathbb{T}^d} \varrho_\varepsilon(t,x)\partial_t\varphi(t,x)\,\mathrm{d}x\,,$$

which, taking the limit $\varepsilon\to 0$ and because $\varrho_\varepsilon \to \varrho$ and $u_\varepsilon \to u$ in $\mathscr{D}'_{t,x}$, yields

$$\int_0^T \mathrm{d}t \int_{\mathbb{T}^d} \nabla_x \varphi(t,x)\cdot u(t,x)\,\mathrm{d}x = 0\,, \qquad \forall\,\varphi\in\mathcal{C}_c^\infty((0,T)\times\mathbb{T}^d)\,.$$

Since $u(t,x)\in L^2((0,T)\,;(\mathbb{W}^{m,2}_x(\mathbb{T}^d)^d))$, the incompressibility condition (6.12) holds true. In the same way, for any $i=1,\dots,d$ and $\varphi\in\mathcal{C}_c^\infty((0,T)\times\mathbb{T}^d)$, noticing that

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^T u_\varepsilon^i\,\partial_t\varphi(t,x)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\frac{1-\alpha}{\varepsilon}\int_0^T \mathrm{d}t\int_{\mathbb{T}^d} u_\varepsilon^i(t,x)\varphi(t,x)\,\mathrm{d}x = 0\,,$$

we get that

$$0 = \lim_{\varepsilon\to 0}\sum_{j=1}^d \int_0^T \mathrm{d}t\int_{\mathbb{T}^d} \boldsymbol{J}_\varepsilon^{i,j}(t,x)\partial_{x_j}\varphi(t,x)\,\mathrm{d}x = \sum_{j=1}^d \int_0^T \mathrm{d}t\int_{\mathbb{T}^d}\boldsymbol{J}_0^{i,j}(t,x)\partial_{x_j}\varphi(t,x)\,\mathrm{d}x\,,$$

where

$$\boldsymbol{J}_0^{i,j}(t,x) := \frac{1}{\vartheta_1}\int_{\mathbb{R}^d} v_i\,v_j\,\boldsymbol{h}(t,x,v)\,\mathrm{d}v = \left(\varrho(t,x)+\vartheta_1\theta(t,x)\right)\delta_{ij}\,, \qquad \forall\, i,j=1,\dots,d\,.$$

Therefore, for any $i=1,\dots,d$,

$$\int_0^T \mathrm{d}t\int_{\mathbb{T}^d}\left(\varrho(t,x)+\vartheta_1\theta(t,x)\right)\,\partial_{x_i}\varphi(t,x)\,\mathrm{d}x = 0\,, \qquad \forall\,\varphi\in\mathcal{C}_c^\infty((0,T)\times\mathbb{T}^d)\,.$$

As before, this gives the Boussinesq relation (6.13). To show that Boussinesq relation can be strengthened, one notices that

$$\int_{\mathbb{T}^d}\varrho_\varepsilon(t,x)\,\mathrm{d}x \xrightarrow[\varepsilon\to 0]{} \int_{\mathbb{T}^d}\varrho(t,x)\,\mathrm{d}x \qquad \text{in} \qquad \mathscr{D}'_t$$

from which we deduce, from the conservation of mass for (6.18), that for almost every $t>0$,

$$\int_{\mathbb{T}^d}\varrho(t,x)\,\mathrm{d}x = 0\,.$$

With the definition of E in (4.1), this implies that for almost every $t>0$,

$$\int_{\mathbb{T}^d}\left(\varrho(t,x)+\vartheta_1\left(\theta(t,x)-E(t)\right)\right)\mathrm{d}x = 0\,,$$

and, this combined with (6.13) yields the strengthened form (6.15). □

Remark 6.8. *Notice here that the derivation of the strong Boussinesq relation $\varrho + \vartheta_1\theta = 0$ is not as straightforward as in the elastic case. In the elastic case, the classical Boussinesq relation $\nabla_x(\varrho + \vartheta_1\theta) = 0$ implies the strong form of Boussinesq because the two functions ϱ and θ have zero spatial averages. This cannot be deduced directly in the granular context due to the dissipation of energy and we will see later on how to obtain it (see Proposition 6.19).*

6.3. **Local conservation laws.** We are now going to write a system of local conservation laws and the first step is to study the limit of some of the terms of this system thanks to Lemma 6.6.

Recall (1.27):

$$\varepsilon\partial_t h_\varepsilon + v\cdot\nabla_x h_\varepsilon + \varepsilon^{-1}(1-\alpha)\nabla_v\cdot(vh_\varepsilon) = \varepsilon^{-1}\mathbf{L}_\alpha h_\varepsilon + \mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon)\,, \tag{6.18}$$

under the scaling hypothesis that $\alpha = 1 - \lambda_0\varepsilon^2 + o(\varepsilon^2)$, $\lambda_0 \geqslant 0$ (see Assumption 1.1). Multiplying (6.18) respectively with 1, v, $|v|^2/2$, we observe that the quantities

$$\Big\langle h_\varepsilon \Big\rangle\,, \quad \Big\langle v h_\varepsilon \Big\rangle\,, \quad \Big\langle \frac12|v|^2 \Big\rangle\,, \quad \Big\langle \frac12|v|^2 v\, h_\varepsilon \Big\rangle \quad \text{and} \quad \Big\langle v\otimes v\, h_\varepsilon \Big\rangle$$

are important. As in the classical elastic case, we write

$$\Big\langle v\otimes v\, h_\varepsilon \Big\rangle = \Big\langle \boldsymbol{A}\, h_\varepsilon \Big\rangle + p_\varepsilon \mathbf{Id}$$

where we define p_ε and the traceless tensor $\boldsymbol{A}$ as

$$p_\varepsilon := \Big\langle \frac1d|v|^2\, h_\varepsilon \Big\rangle \qquad \text{and} \qquad \boldsymbol{A} = \boldsymbol{A}(v) := v\otimes v - \frac1d|v|^2\mathbf{Id}\,. \tag{6.19}$$

Properties of this tensor are established in Appendix A. In a more precise way, one obtains, after integrating (6.18) against 1, v_i, $\frac{|v|^2}{2}$,

$$\partial_t\Big\langle h_\varepsilon \Big\rangle + \frac1\varepsilon \mathrm{div}_x\Big\langle v\, h_\varepsilon \Big\rangle = 0\,, \tag{6.20a}$$

$$\partial_t\Big\langle v\, h_\varepsilon \Big\rangle + \frac1\varepsilon \mathrm{Div}_x\Big\langle \boldsymbol{A}\, h_\varepsilon \Big\rangle + \frac1\varepsilon\nabla_x p_\varepsilon = \frac{1-\alpha}{\varepsilon^2}\Big\langle v\, h_\varepsilon \Big\rangle\,, \tag{6.20b}$$

$$\partial_t\Big\langle \frac12|v|^2 h_\varepsilon \Big\rangle + \frac1\varepsilon \mathrm{div}_x\Big\langle \frac12|v|^2 v\, h_\varepsilon \Big\rangle = \frac{1}{\varepsilon^3}\mathscr{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2(1-\alpha)}{\varepsilon^2}\Big\langle \frac12|v|^2 h_\varepsilon \Big\rangle\,, \tag{6.20c}$$

where we recall that $f_\varepsilon = G_\alpha + \varepsilon h_\varepsilon$ and where

$$\mathscr{J}_\alpha(f,f) := \int_{\mathbb{R}^d} \left[\mathcal{Q}_\alpha(f,f) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha)\right] |v|^2\,\mathrm{d}v\,.$$

Notice that, using (6.9) as well as Lemma 6.6 and Lemma 6.7,

$$\mathrm{div}_x\Big\langle v\, h_\varepsilon\Big\rangle \xrightarrow[\varepsilon\to 0]{} \vartheta_1 \mathrm{div}_x u = 0\,, \qquad \Big\langle \frac{1}{2}|v|^2 h_\varepsilon\Big\rangle \xrightarrow[\varepsilon\to 0]{} \frac{d\vartheta_1}{2}\,(\varrho+\vartheta_1\theta)\,,$$
$$\nabla_x p_\varepsilon \xrightarrow[\varepsilon\to 0]{} \frac{1}{d}\nabla_x\Big\langle |v|^2 \boldsymbol{h}\Big\rangle = \vartheta_1\nabla_x(\varrho+\vartheta_1\theta) = 0\,,$$
$$\Big\langle \boldsymbol{A}\, h_\varepsilon\Big\rangle \xrightarrow[\varepsilon\to 0]{} \Big\langle \boldsymbol{A}\,\boldsymbol{h}\Big\rangle = 0\,,$$
$$\Big\langle \frac{1}{2}|v|^2 v_j\, h_\varepsilon\Big\rangle \xrightarrow[\varepsilon\to 0]{} \Big\langle \frac{1}{2}|v|^2 v_j\,\boldsymbol{h}\Big\rangle = \frac{1}{2}u_j\Big\langle |v|^2 v_j^2 \mathcal{M}\Big\rangle = \frac{d+2}{2}\vartheta_1^2 u_j\,, \qquad \forall\, j=1,\dots,d\,,$$

where all the limits hold in $\mathscr{D}'_{t,x}$ and where $\Big\langle \boldsymbol{A}\boldsymbol{h}\Big\rangle = 0$ because $\boldsymbol{h}\in \mathrm{Ker}(\mathbf{L}_1)$ (see Lemma A.1). Moreover, under the scaling of Assumption 1.1,

$$\frac{1-\alpha}{\varepsilon^2}\Big\langle v\, h_\varepsilon\Big\rangle \xrightarrow[\varepsilon\to 0]{} \vartheta_1\lambda_0 u \qquad \text{in} \qquad \mathscr{D}'_{t,x}\,.$$

The limit of $\varepsilon^{-3}\mathscr{J}_\alpha(f_\varepsilon,f_\varepsilon)$ is handled in the following lemma.

Lemma 6.9. *It holds that*

$$\frac{1}{\varepsilon^3}\mathscr{J}_\alpha(f_\varepsilon,f_\varepsilon) \xrightarrow[\varepsilon\to 0]{} \mathcal{J}_0 \qquad \textit{in} \qquad \mathscr{D}'_{t,x}\,,$$

where

$$\mathcal{J}_0(t,x) := -\lambda_0\,\bar{c}\,\vartheta_1^{\frac{3}{2}}\left(\varrho(t,x)+\frac{3}{4}\vartheta_1\,\theta(t,x)\right)\,, \qquad (t,x)\in(0,T)\times\mathbb{T}^d$$

for some positive constant $\bar{c}$ depending only on the angular kernel $b(\,\cdot\,)$ and d. In particular,

$$\mathcal{J}_0(t,x) = -\lambda_0\,\bar{c}\,\vartheta_1^{\frac{5}{2}}\left(E(t)-\frac{1}{4}\theta(t,x)\right)\,, \qquad (t,x)\in(0,T)\times\mathbb{T}^d$$

where E is defined in (4.1).

Proof. We recall, see (1.10), that

$$\int_{\mathbb{R}^d}|v|^2\mathcal{Q}_\alpha(g,f)\,\mathrm{d}v = -(1-\alpha^2)\frac{\gamma_b}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(v)g(v_*)\,|v-v_*|^3\,\mathrm{d}v\,\mathrm{d}v_*\,.$$

Thus, for $f_\varepsilon = G_\alpha + \varepsilon\, h_\varepsilon$, we obtain

$$\frac{1}{\varepsilon^3}\mathscr{J}_\alpha(f_\varepsilon,f_\varepsilon) = -\frac{\gamma_b}{4}\frac{1-\alpha^2}{\varepsilon^2}\Bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d}\left[h_\varepsilon(v)G_\alpha(v_*)+h_\varepsilon(v_*)G_\alpha(v)\right]|v-v_*|^3\,\mathrm{d}v\,\mathrm{d}v_*$$
$$+\,\varepsilon\int_{\mathbb{R}^d\times\mathbb{R}^d} h_\varepsilon(v)h_\varepsilon(v_*)|v-v_*|^3\,\mathrm{d}v\,\mathrm{d}v_*\Bigg)\,. \tag{6.21}$$

Recall that $\lim_{\varepsilon\to 0}\frac{1-\alpha}{\varepsilon^2}=\lambda_0$. It is clear from Minkowski's integral inequality that the $\mathbb{W}^{m,2}_x(\mathbb{T}^d)$ norm of the last term in the right-side is controlled by $\|h_\varepsilon\|^2_{\mathcal{E}}$. Theorem 5.1 implies that the last term in (6.21) is converging to 0 in $L^1((0,T);\mathbb{W}^{m,2}_x(\mathbb{T}^d))$. One handles the first term in the

right-side using Theorem 6.3 and the fact that $G_\alpha \to \mathcal{M}$ strongly from Lemma 2.3. Details are left to the reader. We then easily obtain the convergence of $\varepsilon^{-3}\mathscr{J}_\alpha(f_\varepsilon, f_\varepsilon)$ towards

$$\mathcal{J}_0 := -\lambda_0\,\gamma_b \int_{\mathbb{R}^d\times\mathbb{R}^d} \boldsymbol{h}(t,x,v)\mathcal{M}(v_*)|v-v_*|^3\,\mathrm{d}v\,\mathrm{d}v_*\,.$$

The expression of $\mathcal{J}_0$ is then obtained by direct inspection from (6.8) with

$$\bar{c} := \gamma_b\, a, \qquad a := \frac{2\sqrt{2}}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}|v|^2\right)|v|^3\,\mathrm{d}v\,,$$

where

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{M}(v)\mathcal{M}(v_*)|v-v_*|^3\,\mathrm{d}v\,\mathrm{d}v_* = \vartheta_1^{\frac{3}{2}}\,a\,,$$

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{M}(v)\mathcal{M}(v_*)|v|^2|v-v_*|^3\,\mathrm{d}v\,\mathrm{d}v_* = \frac{2d+3}{2}\,\vartheta_1^{\frac{5}{2}}\,a\,.$$

We refer to (Mischler & Mouhot, 2009, Lemma A.1) for these identities. The second part of the lemma follows from the strengthened Boussinesq relation (6.15). □

6.4. **About the equations of motion and temperature.** As in Bardos et al. (1993); Golse & Saint-Raymond (2004), in order to investigate the limiting behaviour of the system (6.20) as $\varepsilon \to 0$, we need to investigate the limit in the distributional sense of

$$\frac{1}{\varepsilon}\mathrm{Div}_x\Big\langle \boldsymbol{A}\,h_\varepsilon\Big\rangle = -\frac{1}{\varepsilon}\mathrm{Div}_x\Big\langle \phi\,\mathbf{L}_1 h_\varepsilon\Big\rangle \tag{6.22}$$

and

$$\frac{1}{\varepsilon}\mathrm{div}_x\Big\langle \boldsymbol{b}\,h_\varepsilon\Big\rangle = -\frac{1}{\varepsilon}\mathrm{div}_x\Big\langle \psi\,\mathbf{L}_1 h_\varepsilon\Big\rangle \tag{6.23}$$

where we recall that $\boldsymbol{b}$ is defined through (3.12):

$$\boldsymbol{b}(v) = \frac{1}{2}\left(|v|^2 - (d+2)\vartheta_1\right)v\,, \qquad v\in\mathbb{R}^d$$

and where the *Burnett functions* ϕ and ψ are defined in Lemma A.1 and we used that $\mathbf{L}_1$ is selfadjoint in $L^2_v(\mathcal{M}^{-\frac{1}{2}})$.

Since the limiting vector-field u is divergence-free from Lemma 6.7, it turns out enough to investigate only the limit of $\mathcal{P}\mathrm{Div}_x\Big\langle \varepsilon^{-1}\boldsymbol{A}\,h_\varepsilon\Big\rangle$ where we recall that $\mathcal{P}$ is the Leray projection on divergence-free vector fields[5] . We begin with a strong compactness result:

[5]Recall that, for a vector field $\boldsymbol{u}$, $\mathcal{P}\boldsymbol{u} = \boldsymbol{u} - \nabla\,\Delta^{-1}(\nabla\cdot\boldsymbol{u})$. On the torus, it can be defined via Fourier expansion, if $\boldsymbol{u} = \sum_{k\in\mathbb{Z}^d} \mathrm{a}_k e^{ik\cdot x}$, $\mathrm{a}_k\in\mathbb{C}^d$, then $\mathcal{P}\boldsymbol{u} = \sum_{k\in\mathbb{Z}^d}\left(\mathbf{Id}_d - \frac{k\otimes k}{|k|^2}\right)\mathrm{a}_k e^{ik\cdot x}$.

Lemma 6.10. *Introduce for* $(t,x)\in(0,T)\times\mathbb{T}^d$*:*

$$\boldsymbol{u}_\varepsilon(t,x):=\exp\left(-t\frac{1-\alpha}{\varepsilon^2}\right)\mathcal{P}u_\varepsilon(t,x)\qquad\text{and}\qquad\boldsymbol{\theta}_\varepsilon(t,x):=\left\langle\frac{1}{2}\big(|v|^2-(d+2)\vartheta_1\big)h_\varepsilon\right\rangle. \tag{6.24}$$

Then, $\{\partial_t\boldsymbol{u}_\varepsilon\}_\varepsilon$ *and* $\{\partial_t\boldsymbol{\theta}_\varepsilon\}_\varepsilon$ *are bounded in* $L^1\left((0,T)\,;\mathbb{W}^{m,2}_x(\mathbb{T}^d)\right)$*. Consequently, up to the extraction of a subsequence,*

$$\int_0^T\|\mathcal{P}u_\varepsilon(t)-u(t)\|_{\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)}\,\mathrm{d}t\xrightarrow[\varepsilon\to0]{}0 \tag{6.25}$$

and

$$\int_0^T\|\boldsymbol{\theta}_\varepsilon(t,\,\cdot\,)-\boldsymbol{\theta}_0(t,\,\cdot\,)\|_{\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)}\,\mathrm{d}t\xrightarrow[\varepsilon\to0]{}0 \tag{6.26}$$

where

$$\boldsymbol{\theta}_0(t,x):=\left\langle\frac{1}{2}(|v|^2-(d+2)\vartheta_1)\boldsymbol{h}\right\rangle=\frac{d\vartheta_1}{2}\left(\varrho(t,x)+\vartheta_1\theta(t,x)\right)-\frac{d+2}{2}\vartheta_1\varrho(t,x)\,. \tag{6.27}$$

In other words, $\{\mathcal{P}u_\varepsilon\}_\varepsilon$ *(resp.* $\{\boldsymbol{\theta}_\varepsilon\}_\varepsilon$*) converges strongly to* $u=\mathcal{P}u$ *(resp.* $\boldsymbol{\theta}_0$*) in the space* $L^1\left((0,T)\,;\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)\right)$*.*

Proof. We begin with the proof of (6.25). We apply the Leray projection $\mathcal{P}$ to (6.20b) to eliminate the pressure gradient term. Then, we have that

$$\partial_t\boldsymbol{u}_\varepsilon=-\exp\left(-t\frac{1-\alpha}{\varepsilon^2}\right)\mathcal{P}\left(\frac{1}{\vartheta_1}\mathrm{Div}_x\left\langle\frac{1}{\varepsilon}\boldsymbol{A}\,h_\varepsilon\right\rangle\right).$$

Notice that, since $\{h_\varepsilon\}_\varepsilon$ is bounded in $L^1((0,T)\,;\mathcal{E})$ by Minkowski's integral inequality, one has that

$$\{\boldsymbol{u}_\varepsilon\}_\varepsilon\text{ is bounded in }L^1\left((0,T)\,;\mathbb{W}^{m,2}_x(\mathbb{T}^d)\right).$$

Moreover, since $\langle\boldsymbol{A}\,h_\varepsilon\rangle=\langle\boldsymbol{A}\,(\mathbf{Id}-\boldsymbol{\pi}_0)\,h_\varepsilon\rangle$ (see Lemma A.1), we deduce from Lemma 6.2 and Minkowski's integral inequality that

$$\sup_\varepsilon\int_0^T\left\|\mathcal{P}\left(\mathrm{Div}_x\left\langle\frac{1}{\varepsilon}\boldsymbol{A}\,h_\varepsilon\right\rangle\right)\right\|_{\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)}\mathrm{d}t<\infty\,.$$

In particular

$$\{\partial_t\boldsymbol{u}_\varepsilon\}_\varepsilon\text{ is bounded in }L^1\left((0,T)\,;\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)\right).$$

Applying (Simon, 1987, Corollary 4) with $X=\mathbb{W}^{m,2}_x(\mathbb{T}^d)$ and $B=Y=\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)$ (so that the embedding of X into B is compact by Rellich-Kondrachov Theorem (Taylor, 1996, Proposition 3.4, p. 330)), we deduce that $\{\boldsymbol{u}_\varepsilon\}_\varepsilon$ is relatively compact in $L^1\left((0,T);\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)\right)$. The result of strong convergence follows easily since we already now that $\mathcal{P}u_\varepsilon$ converges to u in $\mathscr{D}'_{t,x}$ (see Lemma 6.6 and recall $u=\mathcal{P}u$ since u is divergence-free).

The proof of (6.26) is similar. We begin with observing that, multiplying (6.20a) with $-\frac{d+2}{2}\vartheta_1$ and add it to (6.20c), we obtain the evolution of $\boldsymbol{\theta}_\varepsilon$:

$$\partial_t \boldsymbol{\theta}_\varepsilon + \frac{1}{\varepsilon}\mathrm{div}_x\Big\langle \boldsymbol{b}\, h_\varepsilon\Big\rangle = \frac{1}{\varepsilon^3}\mathscr{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2(1-\alpha)}{\varepsilon^2}\Big\langle \frac{1}{2}|v|^2 h_\varepsilon\Big\rangle\,. \tag{6.28}$$

Notice that $\{\boldsymbol{\theta}_\varepsilon\}_\varepsilon$ is bounded in $L^1\Big((0,T)\,;\mathbb{W}^{m,2}_x(\mathbb{T}^d)\Big)$ while, because $\langle \boldsymbol{b}\, h_\varepsilon\rangle = \langle \boldsymbol{b}\,(\mathbf{Id}-\boldsymbol{\pi}_0)\, h_\varepsilon\rangle$ (see Lemma A.1), we deduce from Lemma 6.2 by Minkowski's integral inequality that

$$\sup_\varepsilon \int_0^T \left\|\mathrm{div}_x\Big\langle \frac{1}{\varepsilon}\boldsymbol{b}\, h_\varepsilon\Big\rangle\right\|_{\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)} \mathrm{d}t < \infty\,.$$

It is easy to see that the right-hand side of (6.28) is also bounded in $L^1\Big((0,T)\,;\mathbb{W}^{m,2}_x(\mathbb{T}^d)\Big)$ so that $\{\partial_t\boldsymbol{\theta}_\varepsilon\}_\varepsilon$ is bounded in $L^1\Big((0,T)\,;\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)\Big)$. Using again (Simon, 1987, Corollary 4) together with Rellich-Kondrachov Theorem, we deduce as before that $\{\boldsymbol{\theta}_\varepsilon\}_\varepsilon$ is relatively compact in $L^1\Big((0,T)\,;\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)\Big)$. Since we already know that $\boldsymbol{\theta}_\varepsilon$ converges in the distributional sense to $\boldsymbol{\theta}_0$ (see Lemma 6.6), we get the result of strong convergence. □

Remark 6.11. *Notice that if we compare our approach to the elastic case, we have added the exponential term in the definition of $\boldsymbol{u}_\varepsilon$ in order to absorbe the term in the RHS in (6.20b). Notice also that as in the elastic case, the study of the limit $\varepsilon\to 0$ in the equations satisfied by $\boldsymbol{u}_\varepsilon$ and $\boldsymbol{\theta}_\varepsilon$ is more favorable than the direct study of convergence of (6.20a)-(6.20b)-(6.20c) because compared to (6.20a)-(6.20b)-(6.20c), the gradient term in (6.20b) has been eliminated thanks to the Leray projector and also because $\Big\langle \boldsymbol{A}\boldsymbol{\pi}_0\Big\rangle = 0$ and $\Big\langle \boldsymbol{b}\boldsymbol{\pi}_0\Big\rangle = 0$ from Lemma A.1 so that thanks to Lemma 6.2, we know that the quantities $\varepsilon^{-1}\mathrm{Div}_x\Big\langle \boldsymbol{A}\, h_\varepsilon\Big\rangle$ and $\varepsilon^{-1}\mathrm{div}_x\Big\langle \boldsymbol{b}\, h_\varepsilon\Big\rangle$ are bounded in $L^1\Big((0,T)\,;\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)\Big)$.*

We can now give a preliminary result about the problem of convergence for (6.22):

Lemma 6.12. *In the distributional sense,*

$$\mathcal{P}\mathrm{Div}_x\left(\Big\langle \frac{1}{\varepsilon}\boldsymbol{A}\, h_\varepsilon\Big\rangle - \Big\langle \phi\,\mathcal{Q}_1\left(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon\right)\Big\rangle\right) \xrightarrow[\varepsilon\to 0]{} -\nu\,\Delta_x u \tag{6.29}$$

where ν is defined in Lemma A.1.

Proof. When compared to the elastic case, $\mathbf{L}_1 h_\varepsilon$ does not appear in (6.18). We add it, as well as the quadratic elastic Boltzmann operator when applied to the macroscopic part of h_ε, by force and rewrite the latter as

$$\begin{aligned}\varepsilon\partial_t h_\varepsilon + \; v\cdot\nabla_x h_\varepsilon - \varepsilon^{-1}\mathbf{L}_1 h_\varepsilon &= \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon)\\ &\quad - \varepsilon^{-1}(1-\alpha)\mathrm{div}_v(v h_\varepsilon) + \varepsilon^{-1}\left(\mathbf{L}_\alpha h_\varepsilon - \mathbf{L}_1 h_\varepsilon\right) + (\mathcal{Q}_\alpha - \mathcal{Q}_1)(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon)\\ &\quad + 2\widetilde{\mathcal{Q}}_\alpha((\mathbf{Id}-\boldsymbol{\pi}_0)h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) + \mathcal{Q}_\alpha((\mathbf{Id}-\boldsymbol{\pi}_0)h_\varepsilon, (\mathbf{Id}-\boldsymbol{\pi}_0)h_\varepsilon)\,.\end{aligned} \tag{6.30}$$

We interpret the last five terms as a source term

$$\boldsymbol{S}_\varepsilon := -\varepsilon^{-1}(1-\alpha)\mathrm{div}_v(vh_\varepsilon) + \varepsilon^{-1}\left(\mathbf{L}_\alpha h_\varepsilon - \mathbf{L}_1 h_\varepsilon\right) + (\mathcal{Q}_\alpha - \mathcal{Q}_1)(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon)$$
$$+ 2\widetilde{\mathcal{Q}}_\alpha((\mathbf{Id}-\boldsymbol{\pi}_0)h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) + \mathcal{Q}_\alpha((\mathbf{Id}-\boldsymbol{\pi}_0)h_\varepsilon, (\mathbf{Id}-\boldsymbol{\pi}_0)h_\varepsilon) =: \sum_{j=1}^{5} \boldsymbol{S}_\varepsilon^j \,. \quad (6.31)$$

We first remark that

$$\left\langle \phi^{i,j}\, \boldsymbol{S}_\varepsilon^1 \right\rangle = \varepsilon^{-1}(1-\alpha)\left\langle h_\varepsilon \nabla_v \phi^{i,j} \cdot v \right\rangle .$$

Using the estimates on $\nabla_v \phi$ provided in Lemma A.3 as well as Assumption 1.1, we have that

$$\left|\left\langle \phi^{i,j}\, \boldsymbol{S}_\varepsilon^1 \right\rangle\right| \lesssim \varepsilon \left(\|h_\varepsilon^0\|_{L^\infty((0,T)\,;\,\mathcal{E})} + \|h_\varepsilon^1\|_{L^\infty((0,T)\,;\,\mathcal{H})} \right) \lesssim \varepsilon \quad (6.32)$$

where we used (5.1) and (5.2) to get the last inequality.

Let us now prove that

$$\left\| \boldsymbol{S}_\varepsilon^2 + \ldots + \boldsymbol{S}_\varepsilon^5 \right\|_{L^1((0,T)\,;\,L^1_v L^2_x(\boldsymbol{\varpi}_{q-1}))} \lesssim \varepsilon \,. \quad (6.33)$$

Regarding $\boldsymbol{S}_\varepsilon^2$, we have that

$$\frac{1}{\varepsilon}\left\|\mathbf{L}_\alpha h_\varepsilon - \mathbf{L}_1 h_\varepsilon\right\|_{L^1((0,T)\,;\,L^1_v L^2_x(\boldsymbol{\varpi}_{q-1}))}$$
$$\leqslant \frac{1}{\varepsilon}\left(\left\|\mathbf{L}_\alpha h_\varepsilon^0\right\|_{L^1((0,T)\,;\,L^1_v L^2_x(\boldsymbol{\varpi}_{q-1}))} + \left\|\mathbf{L}_1 h_\varepsilon^0\right\|_{L^1((0,T)\,;\,L^1_v L^2_x(\boldsymbol{\varpi}_{q-1}))} \right)$$
$$+ \frac{1}{\varepsilon}\left\|\mathbf{L}_\alpha h_\varepsilon^1 - \mathbf{L}_1 h_\varepsilon^1\right\|_{L^1((0,T)\,;\,L^1_v L^2_x(\boldsymbol{\varpi}_{q-1}))} .$$

Using Corollary B.4, we have that

$$\left\|\mathbf{L}_\alpha h_\varepsilon^0\right\|_{L^1((0,T)\,;\,L^1_v L^2_x(\boldsymbol{\varpi}_{q-1}))} + \left\|\mathbf{L}_1 h_\varepsilon^0\right\|_{L^1((0,T)\,;\,L^1_v L^2_x(\boldsymbol{\varpi}_{q-1}))}$$
$$\lesssim \|h_\varepsilon^0\|_{L^1((0,T)\,;\,L^1_v L^2_x(\boldsymbol{\varpi}_q))} \lesssim \|h_\varepsilon^0\|_{L^1((0,T)\,;\,\mathcal{E})} \lesssim \varepsilon^2$$

where we used (5.1) to get the last inequality. Combining Lemma 2.4 with Cauchy-Schwarz inequality, Assumption 1.1 and (5.2), we have that

$$\left\|\mathbf{L}_\alpha h_\varepsilon^1 - \mathbf{L}_1 h_\varepsilon^1\right\|_{L^1((0,T)\,;\,L^1_v L^2_x(\boldsymbol{\varpi}_{q-1}))} \lesssim \varepsilon^2 \|h_\varepsilon^1\|_{L^1((0,T)\,;\,\mathcal{H})} \lesssim \varepsilon^2 \,.$$

Gathering the two previous inequalities, we obtain that

$$\left\|\boldsymbol{S}_\varepsilon^2\right\|_{L^1((0,T)\,;\,L^1_v L^2_x(\boldsymbol{\varpi}_{q-1}))} \lesssim \varepsilon \,.$$

We can handle $\boldsymbol{S}_\varepsilon^3$ in a similar but simpler way. Using Cauchy-Schwarz inequality, Lemma 2.1, the regularizing properties of $\boldsymbol{\pi}_0$ in velocity and Asusmption 1.1, we get that

$$\left\|\boldsymbol{S}_\varepsilon^3\right\|_{L^1((0,T)\,;\,L^1_v L^2_x(\boldsymbol{\varpi}_{q-1}))} \lesssim \varepsilon^2 \|h_\varepsilon\|_{L^1((0,T)\,;\,\mathcal{E})} \lesssim \varepsilon$$

where we used (5.1)-(5.2) to conclude. The last terms $\boldsymbol{S}_\varepsilon^4$ and $\boldsymbol{S}_\varepsilon^5$ are treated similarly. Using Corollary B.4 and Lemma 6.2 with (5.1)-(5.2), we have that

$$\left\|\boldsymbol{S}_\varepsilon^4+\boldsymbol{S}_\varepsilon^5\right\|_{L^1((0,T)\,;\,L^1_vL^2_x(\boldsymbol{\varpi}_{q-1}))}\lesssim\|(\mathbf{Id}-\boldsymbol{\pi}_0)h_\varepsilon\|_{L^1((0,T)\,;\,\mathcal{E})}\|h_\varepsilon\|_{L^\infty((0,T)\,;\,\mathcal{E})}\lesssim\varepsilon\,.$$

This ends the proof of (6.33).

Then, multiplying (6.30) by ϕ and integrating over $\mathbb{R}^d$, we get using (6.22) that, for any $i,\ j=1,\dots,d$,

$$\begin{aligned}\varepsilon\partial_t\Big\langle\phi^{i,j}h_\varepsilon\Big\rangle+\mathrm{div}_x\Big\langle v\,\phi^{i,j}h_\varepsilon\Big\rangle-\varepsilon^{-1}\Big\langle\phi^{i,j}\mathbf{L}_1h_\varepsilon\Big\rangle&\\=\Big\langle\,\phi^{i,j}\,\mathcal{Q}_1(\boldsymbol{\pi}_0h_\varepsilon,\boldsymbol{\pi}_0h_\varepsilon)\Big\rangle+\Big\langle\phi^{i,j}\,\boldsymbol{S}_\varepsilon\Big\rangle\,.&\quad(6.34)\end{aligned}$$

According to Lemma 6.6, (6.32) and (6.33), we have that

$$\varepsilon\partial_t\Big\langle\phi^{i,j}\,h_\varepsilon\Big\rangle\xrightarrow[\varepsilon\to0]{}0\,,\qquad\mathrm{div}_x\Big\langle v\,\phi^{i,j}\,h_\varepsilon\Big\rangle\xrightarrow[\varepsilon\to0]{}\mathrm{div}_x\Big\langle v\,\phi^{i,j}\,\boldsymbol{h}\Big\rangle\,,\qquad\Big\langle\,\phi^{i,j}\,\boldsymbol{S}_\varepsilon\Big\rangle\xrightarrow[\varepsilon\to0]{}0\,,$$

where the limits are all meant in the distributional sense. From Lemma A.6 in Appendix A, one has

$$\Big\langle v_\ell\,\phi^{i,j}\,\boldsymbol{h}\Big\rangle=\begin{cases}\nu\,u_j & \text{if } i\neq j\,,\ \ell=i\,,\\ \nu\,u_i & \text{if } i\neq j\,,\ \ell=j\,,\\ -\frac{2}{d}\nu\,u_\ell+2\nu\,u_i\delta_{i\ell} & \text{if } i=j\,,\\ 0 & \text{else}\,.\end{cases}$$

Therefore, using the incompressibility condition,

$$\mathrm{div}_x\Big\langle v\,\phi^{i,j}\,\boldsymbol{h}\Big\rangle=\nu\left(\partial_{x_j}u_i+\partial_{x_i}u_j\right).$$

We deduce that in $\mathscr{D}'_{t,x}$

$$\left(\frac{1}{\varepsilon}\Big\langle\phi^{i,j}\,\mathbf{L}_1h_\varepsilon\Big\rangle+\Big\langle\phi^{i,j}\,\mathcal{Q}_1\left(\boldsymbol{\pi}_0h_\varepsilon,\boldsymbol{\pi}_0h_\varepsilon\right)\Big\rangle\right)\xrightarrow[\varepsilon\to0]{}\nu(\partial_{x_j}u_i+\partial_{x_i}u_j)\,.$$

Applying the Div_x operator, one deduces that in $\mathscr{D}'_{t,x}$

$$\mathrm{Div}^i_x\left(\frac{1}{\varepsilon}\Big\langle\phi\,\mathbf{L}_1h_\varepsilon\Big\rangle+\Big\langle\phi\,\mathcal{Q}_1\left(\boldsymbol{\pi}_0h_\varepsilon,\boldsymbol{\pi}_0h_\varepsilon\right)\Big\rangle\right)\xrightarrow[\varepsilon\to0]{}\nu\Delta_xu_i\,,$$

where we use the incompressibility condition to deduce that $\mathrm{Div}^i_x\left(\partial_{x_j}u_i+\partial_{x_i}u_j\right)=\Delta_xu_i$. This proves the result. □

In the same spirit, we have the following which now regards (6.23).

Lemma 6.13. *In the distributional sense,*

$$\frac{1}{\varepsilon}\mathrm{div}_x\Big\langle\boldsymbol{b}\,h_\varepsilon\Big\rangle+\mathrm{div}_x\Big\langle\psi\,\mathcal{Q}_1(\boldsymbol{\pi}_0h_\varepsilon,\boldsymbol{\pi}_0h_\varepsilon\Big\rangle\xrightarrow[\varepsilon\to0]{}-\frac{d+2}{2}\,\gamma\,\Delta_x\theta\qquad(6.35)$$

where γ is defined in Lemma A.1.

Proof. The proof is similar to the one of Lemma 6.12, remark that multiplying (6.30) by ψ_i (recall that ψ is defined by (A.1)), it holds that

$$\varepsilon\partial_t\Big\langle\psi_i h_\varepsilon\Big\rangle+\mathrm{div}_x\Big\langle v\,\psi_i h_\varepsilon\Big\rangle-\varepsilon^{-1}\Big\langle\psi_i\mathbf{L}_1 h_\varepsilon\Big\rangle=\Big\langle\,\psi_i\,\mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon,\boldsymbol{\pi}_0 h_\varepsilon)\Big\rangle+\Big\langle\psi_i\,\boldsymbol{S}_\varepsilon\Big\rangle\,.$$

It then follows that

$$\left(\frac{1}{\varepsilon}\Big\langle\psi_i\mathbf{L}_1 h_\varepsilon\Big\rangle+\Big\langle\psi_i\mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon,\boldsymbol{\pi}_0 h_\varepsilon\Big\rangle\right)\xrightarrow[\varepsilon\to 0]{}\mathrm{div}_x\Big\langle v\,\psi_i\boldsymbol{h}\Big\rangle=\frac{d+2}{2}\,\gamma\,\partial_{x_i}\theta$$

thanks to Lemma A.7 in Appendix A, which gives the result. □

To determine the distributional limit of (6.22) and (6.23), it remains to explicit the limit of the nonlinear terms

$$\mathcal{P}\mathrm{Div}_x\Big\langle\phi\,\mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon,\boldsymbol{\pi}_0 h_\varepsilon)\Big\rangle\quad\text{ and }\quad\mathrm{div}_x\Big\langle\psi\mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon,\boldsymbol{\pi}_0 h_\varepsilon)\Big\rangle$$

respectively. One has the following whose proof is adapted from (Golse & Saint-Raymond , 2004, Corollary 5.7).

Lemma 6.14. *We have*

$$\mathcal{P}\mathrm{Div}_x\Big\langle\phi\,\mathcal{Q}_1\left(\boldsymbol{\pi}_0 h_\varepsilon,\boldsymbol{\pi}_0 h_\varepsilon\right)\Big\rangle\xrightarrow[\varepsilon\to 0]{}\vartheta_1^2\mathcal{P}\mathrm{Div}_x(u\otimes u)\qquad\textit{in}\qquad\mathscr{D}'_{t,x}$$

and

$$\mathrm{div}_x\Big\langle\psi\,\mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon,\boldsymbol{\pi}_0 h_\varepsilon)\Big\rangle\xrightarrow[\varepsilon\to 0]{}\frac{d+2}{2}\vartheta_1^3 u\cdot\nabla_x\theta\qquad\textit{in}\qquad\mathscr{D}'_{t,x}\,.$$

In particular,

$$\mathcal{P}\mathrm{Div}_x\Big\langle\frac{1}{\varepsilon}\boldsymbol{A}\,h_\varepsilon\Big\rangle\xrightarrow[\varepsilon\to 0]{}-\nu\Delta_x u+\vartheta_1^2\mathcal{P}\mathrm{Div}_x(u\otimes u)\qquad\textit{in}\qquad\mathscr{D}'_{t,x}\tag{6.36}$$

while

$$\mathrm{div}_x\Big\langle\frac{1}{\varepsilon}\boldsymbol{b}\,h_\varepsilon\Big\rangle\xrightarrow[\varepsilon\to 0]{}-\frac{d+2}{2}\left(\gamma\,\Delta_x\theta-\vartheta_1^3 u\cdot\nabla_x\theta\right)\qquad\textit{in}\qquad\mathscr{D}'_{t,x}\tag{6.37}$$

where ν and γ are defined in Lemma A.1.

Proof. Writing

$$\boldsymbol{\pi}_0 h_\varepsilon=\left(\varrho_\varepsilon(t,x)+u_\varepsilon(t,x)\cdot v+\frac{1}{2}\theta_\varepsilon(t,x)\left(|v|^2-d\vartheta_1\right)\right)\mathcal{M}(v)$$

we first observe that, according to Lemma A.5 and Lemma A.7 in Appendix A,

$$\Big\langle\phi\,\mathcal{Q}_1\left(\boldsymbol{\pi}_0 h_\varepsilon,\boldsymbol{\pi}_0 h_\varepsilon\right)\Big\rangle=\vartheta_1^2\left[u_\varepsilon\otimes u_\varepsilon-\frac{2}{d}|u_\varepsilon|^2\mathbf{Id}\right]$$

and

$$\Big\langle\psi\,\mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon,\boldsymbol{\pi}_0 h_\varepsilon)\Big\rangle=\frac{d+2}{2}\vartheta_1^3\left(\theta_\varepsilon\,u_\varepsilon\right).$$

Therefore,

$$\mathcal{P}\mathrm{Div}_x\Big\langle\phi\,\mathcal{Q}_1\left(\boldsymbol{\pi}_0 h_\varepsilon,\boldsymbol{\pi}_0 h_\varepsilon\right)\Big\rangle=\vartheta_1^2\mathcal{P}\mathrm{Div}_x\left(u_\varepsilon\otimes u_\varepsilon\right)$$

since $\mathrm{Div}_x\left(|u_\varepsilon|^2\mathbf{Id}\right)$ is a gradient term and

$$\mathrm{div}_x\Big\langle \psi\, \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon)\Big\rangle = \frac{d+2}{2}\vartheta_1^3\,\mathrm{div}_x\left(\theta_\varepsilon\, u_\varepsilon\right)\,.$$

We then write $u_\varepsilon = \mathcal{P}u_\varepsilon + (\mathbf{Id}-\mathcal{P})u_\varepsilon$. Due to the strong convergence of $\mathcal{P}u_\varepsilon$ towards u in $L^1\left((0,T)\,;\mathbb{W}_x^{m-1,2}(\mathbb{T}^d)\right)$ (see Lemma 6.10) and the weak convergence of u_ε (see Lemma 6.6), we see that

$$\mathcal{P}\mathrm{Div}_x\left(u_\varepsilon\otimes u_\varepsilon - (\mathbf{Id}-\mathcal{P})u_\varepsilon\otimes(\mathbf{Id}-\mathcal{P})u_\varepsilon\right)\xrightarrow[\varepsilon\to 0]{}\mathcal{P}\mathrm{Div}_x\left(u\otimes u\right)\qquad\text{in}\qquad\mathscr{D}'_{t,x}\,.$$

So, to prove the first part of the Lemma, it remains to prove that

$$\mathcal{P}\mathrm{Div}_x\left((\mathbf{Id}-\mathcal{P})u_\varepsilon\otimes(\mathbf{Id}-\mathcal{P})u_\varepsilon\right)\xrightarrow[\varepsilon\to 0]{}0\qquad\text{in}\qquad\mathscr{D}'_{t,x}\,.\tag{6.38}$$

Moreover, as in (Golse & Saint-Raymond , 2004, Corollary 5.7), we set

$$\boldsymbol{\beta}_\varepsilon := \frac{1}{d\,\vartheta_1}\Big\langle |v|^2 h_\varepsilon\Big\rangle = \varrho_\varepsilon + \vartheta_1\theta_\varepsilon$$

which is such that $\theta_\varepsilon = \frac{2}{(d+2)\vartheta_1}\left(\boldsymbol{\beta}_\varepsilon + \frac{1}{\vartheta_1}\boldsymbol{\theta}_\varepsilon\right)$ and

$$\begin{aligned}\mathrm{div}_x(\theta_\varepsilon u_\varepsilon) &= \frac{2}{(d+2)\vartheta_1}\mathrm{div}_x\left(\boldsymbol{\beta}_\varepsilon u_\varepsilon + \frac{1}{\vartheta_1}u_\varepsilon\boldsymbol{\theta}_\varepsilon\right)\\ &= \frac{2}{(d+2)\vartheta_1}\mathrm{div}_x\left(\boldsymbol{\beta}_\varepsilon\left(\mathbf{Id}-\mathcal{P}\right)u_\varepsilon\right) + \frac{2}{(d+2)\vartheta_1}\left[\mathrm{div}_x\left(\boldsymbol{\beta}_\varepsilon\mathcal{P}u_\varepsilon + \frac{1}{\vartheta_1}u_\varepsilon\boldsymbol{\theta}_\varepsilon\right)\right].\end{aligned}$$

Therefore, using the strong convergence of $\boldsymbol{\theta}_\varepsilon$ towards $\boldsymbol{\theta}_0$ in $L^1\left((0,T)\,;\mathbb{W}_x^{m-1,2}(\mathbb{T}^d)\right)$ given by Lemma 6.10 together with the weak convergence of u_ε to u from Lemma 6.6, we get

$$\frac{2}{(d+2)\vartheta_1^2}\mathrm{div}_x(u_\varepsilon\boldsymbol{\theta}_\varepsilon)\xrightarrow[\varepsilon\to 0]{}\frac{2}{(d+2)\vartheta_1^2}\mathrm{div}_x(u\,\boldsymbol{\theta}_0)\qquad\text{in}\qquad\mathscr{D}'_{t,x}$$

whereas from the strong convergence of $\mathcal{P}u_\varepsilon$ to u with the weak convergence of $\boldsymbol{\beta}_\varepsilon$ towards $\varrho+\vartheta_1\theta$ we get

$$\mathrm{div}_x(\boldsymbol{\beta}_\varepsilon\mathcal{P}u_\varepsilon)\xrightarrow[\varepsilon\to 0]{}\mathrm{div}_x\left(u\left(\varrho+\vartheta_1\theta\right)\right)=0\qquad\text{in}\qquad\mathscr{D}'_{t,x}$$

where we used both the incompressiblity condition (6.12) together with Boussinesq relation (6.13). Notice that, thanks to (6.12), it holds

$$\frac{2}{(d+2)\vartheta_1^2}\mathrm{div}_x(u\boldsymbol{\theta}_0) = \frac{2}{(d+2)\vartheta_1^2}u\cdot\nabla_x\boldsymbol{\theta}_0 = u\cdot\nabla_x\theta$$

where we used the expression of $\boldsymbol{\theta}_0$ together with Bousinesq relation (6.13). This shows that

$$\mathrm{div}_x(\theta_\varepsilon u_\varepsilon) - \frac{2}{(d+2)\vartheta_1}\mathrm{div}_x\left(\boldsymbol{\beta}_\varepsilon\left(\mathbf{Id}-\mathcal{P}\right)u_\varepsilon\right)\xrightarrow[\varepsilon\to 0]{}u\cdot\nabla_x\theta\qquad\text{in}\qquad\mathscr{D}'_{t,x}$$

and, to get the second part of the result, we need to prove that

$$\mathrm{div}_x\left(\boldsymbol{\beta}_\varepsilon\left(\mathbf{Id}-\mathcal{P}\right)u_\varepsilon\right)\xrightarrow[\varepsilon\to 0]{}0\qquad\text{in}\qquad\mathscr{D}'_{t,x}\,. \tag{6.39}$$

Let us now focus on the proof of (6.38) and (6.39). One observes that, Equation (6.20b) reads

$$\varepsilon\,\partial_t u_\varepsilon+\nabla_x\boldsymbol{\beta}_\varepsilon=\frac{1-\alpha}{\varepsilon}u_\varepsilon-\frac{1}{\vartheta_1}\mathrm{Div}_x\Big\langle\boldsymbol{A}\,h_\varepsilon\Big\rangle \tag{6.40}$$

whereas (6.20c) can be reformulated as

$$\varepsilon\partial_t\boldsymbol{\beta}_\varepsilon+\mathrm{div}_x\Big\langle\frac{1}{d\vartheta_1}|v|^2v\,h_\varepsilon\Big\rangle=\frac{2}{d\vartheta_1\varepsilon^2}\mathscr{J}_\alpha(f_\varepsilon,f_\varepsilon)+\frac{2(1-\alpha)}{\varepsilon}\boldsymbol{\beta}_\varepsilon \tag{6.41}$$

where we check easily that

$$\begin{aligned}\mathrm{div}_x\Big\langle\frac{1}{d\vartheta_1}|v|^2v\,h_\varepsilon\Big\rangle&=\frac{2}{d\vartheta_1}\mathrm{div}_x\Big\langle\boldsymbol{b}\,h_\varepsilon\Big\rangle+\frac{d+2}{d}\vartheta_1\mathrm{div}_x u_\varepsilon\\&=\frac{2}{d\vartheta_1}\mathrm{div}_x\Big\langle\boldsymbol{b}\,h_\varepsilon\Big\rangle+\frac{d+2}{d}\vartheta_1\mathrm{div}_x\left(\mathbf{Id}-\mathcal{P}\right)u_\varepsilon\,.\end{aligned}$$

Recall that from Theorem 5.1, $h_\varepsilon\in L^\infty\left((0,T)\,;\mathcal{E}\right)$ so that by Minkowski's integral inequality, $\boldsymbol{\beta}_\varepsilon\in L^\infty\Big((0,T)\,;\mathbb{W}^{m,2}_x(\mathbb{T}^d)\Big)$ and using (Majda & Bertozzi, 2002, Proposition 1.6, p. 33)), we can write

$$(\mathbf{Id}-\mathcal{P})u_\varepsilon=\nabla_x\boldsymbol{U}_\varepsilon$$

with $\boldsymbol{U}_\varepsilon\in L^\infty\left((0,T)\,;\Big(\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)\Big)^d\right)$. After applying $(\mathbf{Id}-\mathcal{P})$ to (6.40) and reformulating (6.41), we obtain that $\boldsymbol{U}_\varepsilon$ and $\boldsymbol{\beta}_\varepsilon$ satisfy

$$\begin{cases}\varepsilon\partial_t\nabla_x\boldsymbol{U}_\varepsilon+\nabla_x\boldsymbol{\beta}_\varepsilon=\boldsymbol{F}_\varepsilon\\[2ex]\varepsilon\partial_t\boldsymbol{\beta}_\varepsilon+\frac{d+2}{d}\vartheta_1\Delta_x\boldsymbol{U}_\varepsilon=\boldsymbol{G}_\varepsilon\end{cases} \tag{6.42}$$

with

$$\begin{aligned}\boldsymbol{F}_\varepsilon&:=\frac{1-\alpha}{\varepsilon}\nabla_x\boldsymbol{U}_\varepsilon-\frac{1}{\vartheta_1}(\mathbf{Id}-\mathcal{P})\mathrm{Div}_x\Big\langle\boldsymbol{A}\,h_\varepsilon\Big\rangle\\\boldsymbol{G}_\varepsilon&:=-\frac{2}{d\vartheta_1}\mathrm{div}_x\Big\langle\boldsymbol{b}\,h_\varepsilon\Big\rangle+\frac{2}{d\vartheta_1\varepsilon^2}\mathscr{J}_\alpha(f_\varepsilon,f_\varepsilon)+\frac{2(1-\alpha)}{\varepsilon}\boldsymbol{\beta}_\varepsilon\,.\end{aligned}$$

From (5.1)-(5.2), Lemmas 6.2, 6.9 and Asssumption 1.1, it is easy to see that

$$\|\boldsymbol{F}_\varepsilon\|_{L^1((0,T)\,;\,\mathbb{W}^{m-1,2}_x(\mathbb{T}^d))}\lesssim\varepsilon\qquad\text{and}\qquad\|\boldsymbol{G}_\varepsilon\|_{L^1((0,T)\,;\,\mathbb{W}^{m-1,2}_x(\mathbb{T}^d))}\lesssim\varepsilon\,.$$

so that both $\boldsymbol{F}_\varepsilon$ and $\boldsymbol{G}_\varepsilon$ converge *strongly* to 0 in $L^1((0,T)\,;L^2_x(\mathbb{T}^d))$ and

$$\boldsymbol{U}_\varepsilon\in L^\infty((0,T)\,;(\mathbb{W}^{1,2}_x(\mathbb{T}^d))^d)\,,\qquad\boldsymbol{\beta}_\varepsilon\in L^\infty((0,T)\,;L^2_x(\mathbb{T}^d))\,.$$

Then, according to the *compensated compactness* argument of Lions & Masmoudi (1999) recalled in Proposition A.8 in Appendix A, we deduce that (6.38) and (6.39) hold true and this achieves the proof. The proofs of (6.36) and (6.37) follow then from an application of Lemmas 6.12 and 6.13. □

Coming back to the system of equations (6.20) and with the preliminary results of Section 6.3, we get the following where we wrote $\mathcal{P}\mathrm{Div}_x(u\otimes u)=\mathrm{Div}_x(u\otimes u)+\vartheta_1^{-1}\nabla_x p$, see (Majda & Bertozzi, 2002, Proposition 1.6).

Proposition 6.15. *The limit velocity u in (6.8) satisfies*

$$\partial_t u-\frac{\nu}{\vartheta_1}\,\Delta_x u+\vartheta_1\mathrm{Div}_x\,(u\otimes u)+\nabla_x p=\lambda_0 u \tag{6.43}$$

while the limit temperature θ in (6.8) satisfies

$$\partial_t\theta-\frac{\gamma}{\vartheta_1^2}\,\Delta_x\theta+\vartheta_1\,u\cdot\nabla_x\theta=\frac{2}{(d+2)\vartheta_1^2}\mathcal{J}_0+\frac{2d\lambda_0}{d+2}E+\frac{2}{d+2}\frac{\mathrm{d}}{\mathrm{d}t}E \tag{6.44}$$

where we recall that $\mathcal{J}_0$ is defined in Lemma 6.9 and E is defined in (4.1).

Remark 6.16. *The viscosity and heat conductivity coefficients ν and γ are explicit and fully determined by the elastic linearized collision operator $\boldsymbol{L}_1$ (see Lemma A.1).*

Remark 6.17. *Notice also that, due to (6.12), $\mathrm{Div}_x(u\otimes u)=(u\cdot\nabla_x)\,u$ and (6.43) is nothing but a* reinforced *Navier-Stokes equation associated to a divergence-free source term given by $\lambda_0 u$ which can be interpreted as an energy supply/self-consistent force acting on the hydrodynamical system because of the self-similar rescaling.*

Proof of Proposition 6.15. The proof of (6.43) is a straightforward consequence of the previous lemma. To investigate the evolution of θ, we recall that $\boldsymbol{\theta}_\varepsilon$ satisfies (6.28). We notice that

$$\frac{1}{\varepsilon^3}\mathscr{J}_\alpha(f_\varepsilon,f_\varepsilon)+\frac{2(1-\alpha)}{\varepsilon^2}\left\langle\frac{1}{2}|v|^2h_\varepsilon\right\rangle\xrightarrow[\varepsilon\to0]{}\mathcal{J}_0+d\vartheta_1\lambda_0\,(\varrho+\vartheta_1\theta)\;,$$

whereas

$$\boldsymbol{\theta}_\varepsilon\xrightarrow[\varepsilon\to0]{}\left\langle\frac{1}{2}(|v|^2-(d+2)\vartheta_1)\boldsymbol{h}\right\rangle=\frac{d\vartheta_1}{2}\,(\varrho+\vartheta_1\theta)-\frac{d+2}{2}\vartheta_1\varrho\qquad\text{in}\qquad\mathscr{D}'_{t,x}\,.$$

We deduce from (6.37), performing the distributional limit of (6.28), that

$$\frac{d\vartheta_1}{2}\partial_t\,(\varrho+\vartheta_1\theta)-\frac{d+2}{2}\vartheta_1\partial_t\varrho-\frac{d+2}{2}\gamma\,\Delta_x\theta+\frac{d+2}{2}\vartheta_1^3\,u\cdot\nabla_x\theta$$
$$=\mathcal{J}_0+d\vartheta_1\lambda_0\,(\varrho+\vartheta_1\theta)\,. \tag{6.45}$$

Using the strengthened Boussinesq relation (6.15), we see that

$$\partial_t\,(\varrho+\vartheta_1\theta)=\vartheta_1\frac{\mathrm{d}}{\mathrm{d}t}E\qquad\text{and}\qquad\partial_t\varrho=-\vartheta_1\left(\partial_t\theta-\frac{\mathrm{d}}{\mathrm{d}t}E\right)\,,$$

and get the result. □

6.5. **About the initial conditions.** Recall that in Theorem 6.3, the convergence of h_ε to $\boldsymbol{h}$ given by (6.8) is known to hold only for a subsequence and, in particular, at initial time, different subsequences could converge towards different initial datum. In such a case, (ϱ, u, θ) could be different solutions to the same system. In Theorem 6.1, the initial datum is prescribed by ensuring the convergence of $\boldsymbol{\pi}_0 h_{\rm in}^\varepsilon$ towards a *single* possible limit where $\boldsymbol{\pi}_0$ is defined in (3.2) (recall that the initial data for (ϱ, u, θ) is defined in (6.5)). Using the additional assumption (6.1), we can use Arzelà-Ascoli theorem to deduce some regularity results (in time) on our hydrodynamic quantities.

Lemma 6.18. *Consider the sequences $\{\boldsymbol{u}_\varepsilon\}_\varepsilon$ and $\{\boldsymbol{\theta}_\varepsilon\}_\varepsilon$ defined in Lemma 6.10. The time-depending mappings*

$$t \in [0,T] \longmapsto \|\boldsymbol{u}_\varepsilon(t)\|_{\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)} \quad \textit{and} \quad t \in [0,T] \longmapsto \|\boldsymbol{\theta}_\varepsilon(t)\|_{\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)}$$

are Hölder continuous uniformly in ε. As a consequence, the limiting quantities u and $\boldsymbol{\theta}_0$ belong to $\mathcal{C}([0,T]\,;\mathbb{W}^{m-1,2}_x(\mathbb{T}^d))$.

Proof. Recall that $\boldsymbol{\theta}_\varepsilon$ is defined in (6.24). For any test-function $\varphi = \varphi(x) \in \mathcal{C}^\infty_c(\mathbb{T}^d)$ and multi-index β with $|\beta| \leqslant m-1$, multiplying (6.28) with $\partial^\beta_x \varphi$ and integrating in time and space, one has that for any $0 \leqslant t_1 \leqslant t_2$,

$$\begin{aligned}\int_{\mathbb{T}^d} \Big[\partial^\beta_x \boldsymbol{\theta}_\varepsilon(t_2,x) - \partial^\beta_x \boldsymbol{\theta}_\varepsilon(t_1,x)\Big]\,\varphi\,\mathrm{d}x \\ = \int_{t_1}^{t_2} \mathrm{d}t \int_{\mathbb{T}^d} \mathrm{div}_x \Big\langle \frac{1}{\varepsilon}\boldsymbol{b}\,\partial^\beta_x h_\varepsilon \Big\rangle \varphi\,\mathrm{d}x + \frac{1}{\varepsilon^3}\int_{t_1}^{t_2}\mathrm{d}t\int_{\mathbb{T}^d} \partial^\beta_x \mathscr{J}_\alpha(f_\varepsilon,f_\varepsilon)\varphi\,\mathrm{d}x \\ + \frac{2(1-\alpha)}{\varepsilon^2}\int_{t_1}^{t_2}\mathrm{d}t\int_{\mathbb{T}^d}\Big\langle \frac{1}{2}|v|^2\partial^\beta_x h_\varepsilon\Big\rangle\varphi\,\mathrm{d}x\,. \end{aligned} \tag{6.46}$$

Notice that

$$\left|\int_{\mathbb{T}^d}\Big\langle \frac{1}{2}|v|^2\partial^\beta_x h_\varepsilon\Big\rangle\varphi\,\mathrm{d}x\right| \leqslant \|\varphi\|_{L^2_x}\left\|\Big\langle \frac{1}{2}|v|^2\partial^\beta_x h_\varepsilon\Big\rangle\right\|_{L^2_x} \leqslant \|\varphi\|_{L^2_x}\left\|\frac{1}{2}|v|^2\partial^\beta_x h_\varepsilon\right\|_{L^1_v L^2_x}$$

thanks to Minkowski's integral inequality. Clearly, since $\varepsilon^{-2}(1-\alpha)\to\lambda_0$,

$$\frac{2(1-\alpha)}{\varepsilon^2}\left|\int_{t_1}^{t_2}\mathrm{d}t\int_{\mathbb{T}^d}\Big\langle\frac{1}{2}|v|^2\partial^\beta_x h_\varepsilon\Big\rangle\varphi\,\mathrm{d}x\right| \lesssim \|\varphi\|_{L^2}\int_{t_1}^{t_2}\|h_\varepsilon\|_{L^1_v\mathbb{W}^{m-1,2}_x(\boldsymbol{\varpi}_2)}\,\mathrm{d}t \lesssim t_2 - t_1$$

from the general estimate in Theorem 5.1. In the same way, since

$$\partial^\beta_x \mathscr{J}_\alpha(f_\varepsilon,f_\varepsilon) = \mathscr{J}_\alpha(\partial^\beta_x f_\varepsilon, f_\varepsilon) + \mathscr{J}_\alpha(f_\varepsilon,\partial^\beta_x f_\varepsilon)\,,$$

with $f_\varepsilon = G_\alpha + \varepsilon\, h_\varepsilon$, one deduces again from Theorem 5.1 that

$$\left| \int_{t_1}^{t_2} \mathrm{d}t \int_{\mathbb{T}^d} \frac{1}{\varepsilon^3} \partial_x^\beta \mathscr{J}_\alpha(f_\varepsilon, f_\varepsilon) \varphi \, \mathrm{d}x \right| \\ \lesssim \|\varphi\|_{L^2_x} \int_{t_1}^{t_2} \|h_\varepsilon\|_{L^1_v \mathbb{W}^{m-1,2}_x(\boldsymbol{\varpi}_3)} \left(1 + \|h_\varepsilon\|_{L^1_v \mathbb{W}^{m-1,2}_x(\boldsymbol{\varpi}_3)} \right) \mathrm{d}t \lesssim t_2 - t_1 \,.$$

Moreover, recalling that $\left\langle \boldsymbol{b} h_\varepsilon \right\rangle = \left\langle \boldsymbol{b}(\mathbf{Id} - \boldsymbol{\pi}_0) h_\varepsilon \right\rangle$ (see Lemma A.1) for any $t \geqslant 0$, one deduces easily from Lemma 6.2 that

$$\left| \int_{t_1}^{t_2} \frac{1}{\varepsilon} \mathrm{div}_x \left\langle \boldsymbol{b}\, h_\varepsilon \right\rangle \mathrm{d}t \right| \lesssim \sqrt{t_2 - t_1} \,.$$

Since ∂_x^β commutes with $\boldsymbol{\pi}_0$ we deduce easily that for any $0 \leqslant |\beta| \leqslant m-1$,

$$\left| \int_{t_1}^{t_2} \frac{1}{\varepsilon} \mathrm{div}_x \left\langle \boldsymbol{b}\, \partial_x^\beta h_\varepsilon \right\rangle \mathrm{d}t \right| \lesssim \sqrt{t_2 - t_1} \,. \tag{6.47}$$

We conclude with (6.46) that

$$\left| \int_{\mathbb{T}^d} \left[\partial_x^\beta \boldsymbol{\theta}_\varepsilon(t_2, x) - \partial_x^\beta \boldsymbol{\theta}_\varepsilon(t_1, x) \right] \varphi \, \mathrm{d}x \right| \lesssim \|\varphi\|_{L^2_x} \sqrt{t_2 - t_1} \,.$$

Since $\mathscr{C}^\infty_c(\mathbb{T}^d)$ is dense in $L^2(\mathbb{T}^d)$, the previous estimate is true for any $\varphi \in L^2(\mathbb{T}^d)$ and, taking the supremum over all $\varphi \in L^2(\mathbb{T}^d)$, we deduce that for any $0 \leqslant |\beta| \leqslant m-1$,

$$\left\| \partial_x^\beta \boldsymbol{\theta}_\varepsilon(t_2) - \partial_x^\beta \boldsymbol{\theta}_\varepsilon(t_1) \right\|_{L^2_x} \lesssim \sqrt{t_2 - t_1}$$

and, the time-depending mappings $t \in [0,T] \longmapsto \|\boldsymbol{\theta}_\varepsilon(t)\|_{\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)}$ are thus Hölder continuous uniformly in ε. One can thus apply Arzelà-Ascoli theorem to get that $\boldsymbol{\theta}_\varepsilon$ converge strongly in $\mathcal{C}\big([0,T]\,; \mathbb{W}^{m-1,2}_x(\mathbb{T}^d)\big)$ towards $\boldsymbol{\theta}_0$ defined in (6.27) that also belong to $\mathcal{C}\big([0,T]\,; \mathbb{W}^{m-1,2}_x(\mathbb{T}^d)\big)$.

The proof about $\boldsymbol{u}_\varepsilon$ is similar and we thus skip it. □

6.6. **Limit equations.** To end the identification of the limit equations, we go back to the strong Boussinesq equation (6.15) and prove the following result:

Proposition 6.19. *It holds that*

$$E(t) = 0 \,, \qquad \forall\, t \in [0,T]$$

where $E = E(t)$ is defined in (4.1). Consequently, the limiting temperature θ in (6.8) satisfies

$$\partial_t \theta - \frac{\gamma}{\vartheta_1^2} \Delta_x \theta + \vartheta_1\, u \cdot \nabla_x \theta = \frac{\lambda_0\, \bar{c}}{2(d+2)} \sqrt{\vartheta_1}\, \theta \,. \tag{6.48}$$

where γ is defined in Lemma A.1, λ_0 in Assumption 1.1 and $\bar{c}$ in Lemma 6.9. Moreover, the strong Boussinesq relation holds true:

$$\varrho + \vartheta_1 \theta = 0 \,, \qquad \forall\, (t,x) \in [0,T] \times \mathbb{T}^d \,. \tag{6.49}$$

Proof. Using Lemma 6.9 and averaging in position the equation (6.44), it is easy to prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \bar{c}_0\, E(t)$$

for some some constant $\bar{c}_0 \in \mathbb{R}$. Moreover, on the one hand, from (6.5), we have

$$E(0) = \int_{\mathbb{T}^d} \theta(0,x)\,\mathrm{d}x = -\frac{1}{\vartheta_1}\int_{\mathbb{T}^d} \varrho(0,x)\,\mathrm{d}x\,. \tag{6.50}$$

On the other hand, from the definition of $\boldsymbol{\theta}_0$ in (6.27), we also have

$$E(0) = \frac{2}{\vartheta_1^2 d}\int_{\mathbb{T}^d} \boldsymbol{\theta}_0(0,x)\,\mathrm{d}x + \frac{2}{\vartheta_1 d}\int_{\mathbb{T}^d} \varrho(0,x)\,\mathrm{d}x\,. \tag{6.51}$$

From Lemma 6.18, we also know that $\boldsymbol{\theta}_\varepsilon$ converges towards $\boldsymbol{\theta}_0$ in $\mathcal{C}\big([0,T]\,;\,\mathbb{W}^{m-1,2}_x(\mathbb{T}^d)\big)$. Consequently, we deduce that

$$\int_{\mathbb{T}^d} \boldsymbol{\theta}_0(0,x)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{\mathbb{T}^d}\Big\langle \frac{|v|^2-(d+2)\vartheta_1}{2}h_\varepsilon(0,x)\Big\rangle\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{\mathbb{T}^d}\Big\langle \frac{1}{2}|v|^2 h_\varepsilon(0,x)\Big\rangle\,\mathrm{d}x$$

where we used (1.33) to get the last equality. From (1.34), we deduce that

$$\int_{\mathbb{T}^d} \boldsymbol{\theta}_0(0,x)\,\mathrm{d}x = 0\,.$$

Coming back to (6.50)-(6.51), we deduce that

$$E(0) = -\frac{1}{\vartheta_1}\int_{\mathbb{T}^d}\varrho(0,x)\,\mathrm{d}x = \frac{2}{\vartheta_1 d}\int_{\mathbb{T}^d}\varrho(0,x)\,\mathrm{d}x$$

which implies that $E(0)=0$. This concludes the proof. □

Gathering the results we obtained in Propositions 6.15 and 6.19, we are able to end the proof of Theorem 6.1.

7. About the original problem in the physical variables

The above considerations allow us to get a quite precise description of the asymptotic behaviour for the original physical problem (1.12a) in the case $\lambda_0 > 0$ by carefully estimating the error between the solution $F_\varepsilon(t,x,v)$ to (1.12a) and the solution $\boldsymbol{h}(t,x,v)$ defined in (6.3). We keep the presentation informal, explaining only the main idea allowing to derive both *global and local* versions of Haff's law for granular gases. We recall here that Haff's law as predicted in the seminal paper Haff (1983) asserts that the temperature of a freely cooling granular gases of hard-spheres decays like $(1+t)^{-2}$ as $t\to\infty$. It has been proven rigorously in the spatially homogeneous case in Mischler & Mouhot (2006) (see also Alonso & Lods (2010) for a version of Haff's law for viscoelastic gases in the spatially homogeneous case). We derive here a version of Haff's law valid for small values of ε in the spatially inhomogeneous framework we adopted.

In this section, we work under the conditions of Theorem 5.1. Recall that, if $F_\varepsilon(t,x,v)$ denotes the solution to the Boltzmann equation (1.12a) with associated Knudsen number ε, the time-scale

functions $\tau_\varepsilon(t), V_\varepsilon(t)$ that relate the problem in original (physical) variables to its self-similar counterpart

$$F_\varepsilon(t,x,v) = V_\varepsilon(t)^d f_\varepsilon\big(\tau_\varepsilon(t), x, V_\varepsilon(t)v\big)$$

are given by

$$\tau_\varepsilon(t) = \frac{1}{c_\varepsilon}\ln(1+c_\varepsilon\, t)\,, \quad V_\varepsilon(t) = 1 + c_\varepsilon\, t\,, \qquad \forall\, t \geqslant 0\,,$$

where $c_\varepsilon = \frac{1-\alpha(\varepsilon)}{\varepsilon^2}$ (see (1.14)). Recalling that $\boldsymbol{h}$ as defined in (6.3) is the weak limit of h_ε (up to a suitable subsequence), we introduce the error term $\boldsymbol{e}_\varepsilon$ is given by

$$\boldsymbol{e}_\varepsilon(t,x,v) := V_\varepsilon(t)^d\left(h_\varepsilon(\tau_\varepsilon(t),x,V_\varepsilon(t)v) - \boldsymbol{h}(\tau_\varepsilon(t),x,V_\varepsilon(t)v)\right).$$

Under Assumption 1.1, a relevant phenomenon occurs when considering the purely dissipative case $\lambda_0 > 0$. In such a case, the term $\boldsymbol{e}_\varepsilon(t,x,v)$ becomes a *uniform in time* error term. More precisely, the following error estimate holds.

Lemma 7.1. *We suppose $\lambda_0 > 0$ in Assumption 1.1. Let $a \in (0,1)$. For ε small enough (depending on a), the following estimate holds up to possibly extracting a subsequence,*

$$\left|\langle \boldsymbol{e}_\varepsilon(t), |v|^\kappa\varphi\rangle\right| \leqslant C(\varphi, a, \eta_0)\,(1+\lambda_0 t)^{-\kappa-a}\,, \qquad \varphi \in \mathcal{C}^1_{v,b}L^\infty_x\,, \qquad 0 \leqslant \kappa \leqslant q-1\,, \quad (7.1)$$

where we denoted by $\mathcal{C}^1_{v,b}$ the set of $\mathcal{C}^1$ functions in v that are bounded as well as their first derivatives and recall $q \geqslant 3$ is defined in $\mathcal{E}$

Proof. Let $a \in (0,1)$. After a change of variables it follows that, for any test-function φ,

$$\begin{aligned}
&\langle \boldsymbol{e}_\varepsilon(t), |v|^\kappa\varphi\rangle \\
&\quad = V_\varepsilon(t)^{-\kappa}\int_{\mathbb{T}^d\times\mathbb{R}^d}\Big(h_\varepsilon(\tau_\varepsilon(t),x,v) - \boldsymbol{h}(\tau_\varepsilon(t),x,v)\Big)|v|^\kappa\big(\varphi(x,V_\varepsilon(t)^{-1}v) - \varphi(x,0)\big)\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad + V_\varepsilon(t)^{-\kappa}\int_{\mathbb{T}^d\times\mathbb{R}^d}\Big(h_\varepsilon(\tau_\varepsilon(t),x,v) - \boldsymbol{h}(\tau_\varepsilon(t),x,v)\Big)|v|^\kappa\varphi(x,0)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad =: \mathcal{I}_1(t) + \mathcal{I}_2(t)\,.
\end{aligned}$$

Note that, up to a subsequence, $\boldsymbol{h}$ is the weak$-\star$ limit of $\{h_\varepsilon\}_\varepsilon$ in $L^\infty\big((0,\infty);\mathcal{E}\big)$. Thus, for any $t>0$, $\|\boldsymbol{h}\|_{L^\infty((t,\infty);\mathcal{E})} \leqslant \liminf_{\varepsilon\searrow 0}\|h_\varepsilon\|_{L^\infty((t,\infty);\mathcal{E})}$. Consequently, for $r \in (0,1)$ such that $1-r > a$ thanks to Theorem 5.1, it holds that

$$\|\boldsymbol{h}\|_{L^\infty((t,\infty);\mathcal{E})} \leqslant C(r,\eta_0)\,e^{-(1-r)\lambda_0 t}\,, \qquad \forall\, t>0\,. \quad (7.2)$$

In regard of $\mathcal{I}_1(t)$, note that

$$\big|\varphi(x,V_\varepsilon(t)^{-1}v) - \varphi(x,0)\big| \leqslant V_\varepsilon(t)^{-1}|v|\sup_v\sup_x\big|\partial_v\varphi(x,v)\big| = C(\varphi)V_\varepsilon(t)^{-1}|v|\,,$$

so that the following holds:

$$\begin{aligned}\big|\mathcal{I}_1(t)\big| &\leqslant C(\varphi)V_\varepsilon(t)^{-\kappa-1}\|h_\varepsilon(\tau_\varepsilon(t)) - \boldsymbol{h}(\tau_\varepsilon(t))\|_{L^1_{v,x}(\boldsymbol{\varpi}_{\kappa+1})}\\ &\leqslant C(\varphi)V_\varepsilon(t)^{-\kappa-1}\|h_\varepsilon(\tau_\varepsilon(t)) - \boldsymbol{h}(\tau_\varepsilon(t))\|_{\mathcal{E}}\\ &\leqslant C(\varphi,r,\eta_0)\,V_\varepsilon(t)^{-\kappa-1}\left(e^{-(1-r)\lambda_\varepsilon\tau_\varepsilon(t)} + e^{-(1-r)\lambda_0\tau_\varepsilon(t)}\right).\end{aligned}$$

Recalling that $\lambda_\varepsilon = c_\varepsilon(1-r_\varepsilon)$ with $r_\varepsilon \to 0$ as $\varepsilon \to 0$ (see (4.15)), we deduce that for ε small enough,

$$e^{-(1-r)\lambda_\varepsilon\tau_\varepsilon(t)} \lesssim (1+c_\varepsilon t)^{-a} \lesssim (1+\lambda_0 t)^{-a}.$$

Similarly, for ε small enough, $e^{-(1-r)\lambda_0\tau_\varepsilon(t)} \lesssim (1+\lambda_0 t)^{-a}$ and $V_\varepsilon(t)^{-\kappa-1} \lesssim (1+\lambda_0 t)^{-\kappa-1}$. In conclusion, we obtain that

$$\Big|\mathcal{I}_1(t)\Big| \leqslant C(\varphi,r,\eta_0)(1+\lambda_0 t)^{-\kappa-1-a}\,.$$

The term $\mathcal{I}_2(t)$ is treated similarly and we obtain easily that

$$\Big|\mathcal{I}_2(t)\Big| \leqslant C(\varphi,r,\eta_0)\,(1+\lambda_0 t)^{-\kappa-a}$$

which ends the proof. □

Let us explain how Lemma 7.1 allows to deduce the large time behaviour of F_ε in the weak sense defined through (7.1).

Proof of Theorem 1.7. Indeed, recalling the relations (1.14) together with Theorem 1.4 one has

$$\begin{aligned}F_\varepsilon(t,x,v) &= V_\varepsilon(t)^d f_\varepsilon\big(\tau_\varepsilon(t),x,V_\varepsilon(t)v\big)\\ &= V_\varepsilon(t)^d\Big(G_{\alpha(\varepsilon)}(V_\varepsilon(t)v) + \varepsilon\, h_\varepsilon(\tau_\varepsilon(t),x,V_\varepsilon(t)v)\Big)\\ &= V_\varepsilon(t)^d\Big(G_{\alpha(\varepsilon)}(V_\varepsilon(t)v) + \varepsilon\,\boldsymbol{h}(\tau_\varepsilon(t),x,V_\varepsilon(t)v)\Big) + \varepsilon\,\boldsymbol{e}_\varepsilon(t,x,v)\,,\end{aligned}$$

From Lemma 7.1, one can deduce that

$$\begin{aligned}F_\varepsilon(t,x,v) = V_\varepsilon(t)^d\Big(&G_{\alpha(\varepsilon)}(V_\varepsilon(t)v) + \varepsilon\,\big(\varrho(\tau_\varepsilon(t),x) + u(\tau_\varepsilon(t),x)\cdot(V_\varepsilon(t)v)\\ &+ \frac{1}{2}\theta(\tau_\varepsilon(t),x)(|V_\varepsilon(t)v|^2 - d\vartheta_1)\big)\mathcal{M}(V_\varepsilon(t)v)\Big) + \varepsilon\mathrm{O}\Big((1+\lambda_0 t)^{-\kappa-a}\Big)\,,\end{aligned} \tag{7.3}$$

in the *weak sense* described in (7.1). In particular, if $\varphi = 1$ and $\kappa = 2$, one finds from (7.3) an explicit expression for *Haff's law*. Indeed, the optimal cooling rate of the temperature is described

by

$$\begin{aligned}\boldsymbol{T}_\varepsilon(t) &= \frac{1}{\left|\mathbb{T}^d\right|}\int_{\mathbb{T}^d\times\mathbb{R}^d} F_\varepsilon(t,x,v)|v|^2\,\mathrm{d}v\,\mathrm{d}x \\ &= \frac{1}{V_\varepsilon(t)^2}\bigg(\int_{\mathbb{R}^d} G_{\alpha(\varepsilon)}(v)|v|^2\,\mathrm{d}v + \frac{\varepsilon}{2\left|\mathbb{T}^d\right|}\int_{\mathbb{R}^d}\big(|v|^2-d\vartheta_1\big)|v|^2\mathcal{M}(v)\,\mathrm{d}v\int_{\mathbb{T}^d}\theta(\tau_\varepsilon(t),x)\,\mathrm{d}x \\ &\qquad + \frac{\varepsilon}{\left|\mathbb{T}^d\right|}\int_{\mathbb{R}^d}|v|^2\mathcal{M}(v)\,\mathrm{d}v\int_{\mathbb{T}^d}\varrho(\tau_\varepsilon(t),x)\,\mathrm{d}x\bigg) + \varepsilon\mathrm{O}\Big((1+\lambda_0 t)^{-2-a}\Big) \\ &\approx \frac{d\vartheta_1}{(1+\lambda_0 t)^2}\bigg(1+\frac{\varepsilon}{\left|\mathbb{T}^d\right|}\bigg(d\vartheta_1\int_{\mathbb{T}^d}\theta(\tau_\varepsilon(t),x)\,\mathrm{d}x + 2\int_{\mathbb{T}^d}\varrho(\tau_\varepsilon(t),x)\,\mathrm{d}x\bigg)\bigg)\,, \qquad t\gg\frac{1}{\lambda_0}\,.\end{aligned}$$

Recalling that the fluctuation h_ε is such that the average mass and temperature both vanish at all times, we deduce the precised Haff's law

$$\boldsymbol{T}_\varepsilon(t)\approx\frac{d\vartheta_1}{(1+\lambda_0 t)^2}\,,\qquad t\gg\frac{1}{\lambda_0}\,.$$

We can actually show that the Haff's law holds uniformly *locally* in space due to the boundedness of the solutions that we treat here. This is not expected in a general context where more general solutions are considered. Consider $0\leqslant\kappa\leqslant q$. Note that

$$\int_{\mathbb{R}^d} f_\varepsilon(\tau_\varepsilon(t),x,w)|w|^\kappa\,\mathrm{d}w = \int_{\mathbb{R}^d} G_{\alpha(\varepsilon)}(w)|w|^\kappa\,\mathrm{d}w + \varepsilon\int_{\mathbb{R}^d} h_\varepsilon(\tau_\varepsilon(t),x,w)|w|^\kappa\,\mathrm{d}w\,.$$

Thanks to Sobolev embedding it holds that

$$\bigg|\sup_{x\in\mathbb{T}^d}\int_{\mathbb{R}^d} h_\varepsilon(\tau_\varepsilon(t),x,w)|w|^\kappa\,\mathrm{d}v\bigg|\leqslant C\|h_\varepsilon(\tau_\varepsilon(t))\|_{\mathcal{E}}\leqslant C(\eta_0)\,.$$

Therefore, for sufficiently small $\varepsilon>0$, there exist two positive constants $c(\eta_0)$ and $C(\eta_0)$ such that

$$c(\eta_0)\leqslant\int_{\mathbb{R}^d} f_\varepsilon(\tau_\varepsilon(t),x,w)|w|^\kappa\,\mathrm{d}w\leqslant C(\eta_0)\,,\qquad\forall\,t\geqslant 0\,,$$

which leads, for the physical problem, to

$$V_\varepsilon(t)^{-\kappa}c(\eta_0)\leqslant\int_{\mathbb{R}^d} F_\varepsilon(t,x,v)|v|^\kappa\,\mathrm{d}v\leqslant V_\varepsilon(t)^{-\kappa}C(\eta_0)\,,\qquad\forall\,t\geqslant 0\,.$$

In particular, this estimate renders a local version of Haff's law

$$\int_{\mathbb{R}^d} F_\varepsilon(t,x,v)|v|^2\,\mathrm{d}v\sim\big(1+\lambda_0 t\big)^{-2}\,,\qquad\forall\,t\geqslant 0\,,\qquad\forall\,x\in\mathbb{T}^d\,.$$

□

Appendix A. Tools for the Hydrodynamic limit

We collect several tools that are used in Section 6.3 to derive the modified incompressible Navier-Stokes system. Various known computations regarding the elastic Boltzmann operator are needed but we also need some new estimates on the so-called *Burnett functions* ϕ and ψ defined in the following Lemma (recall that $\mathbf{L}_1$ is defined in (1.31)):

Lemma A.1. *Let $\boldsymbol{A}$ be the traceless tensor defined in (6.19) and let $\boldsymbol{b}$ be the vector defined in (3.12)). One has that $\boldsymbol{A}\mathcal{M}$, $\boldsymbol{b}\mathcal{M} \in (\mathrm{Ker}(\mathbf{L}_1))^\perp$ in $L^2_v(\mathcal{M}^{-\frac{1}{2}})$ and there exists two radial functions $\chi_i = \chi_i(|v|)$, $i = 1, 2$, such that*

$$\phi(v) = \chi_1(|v|)\boldsymbol{A}(v) \in \mathscr{M}_d(\mathbb{R}) \qquad \text{and} \qquad \psi(v) = \chi_2(|v|)\boldsymbol{b}(v) \in \mathbb{R}^d\,,$$

satisfy

$$\mathbf{L}_1(\phi\,\mathcal{M}) = -\boldsymbol{A}\,\mathcal{M}\,, \qquad \mathbf{L}_1(\psi\,\mathcal{M}) = -\boldsymbol{b}\,\mathcal{M}\,. \tag{A.1}$$

Moreover,

$$\Big\langle \phi^{i,j}\mathbf{L}_1(\phi^{k,\ell}\mathcal{M})\Big\rangle = -\nu\left(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\right)$$
$$\Big\langle \psi_i\mathbf{L}_1(\psi_j\mathcal{M})\Big\rangle = -\frac{d+2}{2}\gamma\,\delta_{ij}\,, \qquad \forall\, i,j,k,\ell \in \{1,\dots,d\}\,, \tag{A.2}$$

with

$$\nu := -\frac{1}{(d-1)(d+2)}\Big\langle \phi\,:\,\mathbf{L}_1(\phi\mathcal{M})\Big\rangle \geqslant 0\,, \qquad \gamma := -\frac{2}{d(d+2)}\Big\langle \psi\cdot\mathbf{L}_1(\psi\mathcal{M})\Big\rangle \geqslant 0\,.$$

Finally,

$$|\phi^{i,j}(v)| \lesssim \varpi_3(v)\,, \qquad |\psi_i(v)| \lesssim \varpi_4(v)\,, \qquad \forall\, i,j \in \{1,\dots,d\}\,.$$

Proof. The tensor $\boldsymbol{A}$ and the vector $\boldsymbol{b}$ satisfy

$$\Big\langle \boldsymbol{A}^{k,\ell}\Psi_i\,\mathcal{M}\Big\rangle = 0\,, \quad \Big\langle \boldsymbol{b}\Psi_i\,\mathcal{M}\Big\rangle = 0\,, \quad \forall\, i = 1,\dots,d+2\,, \qquad \forall\, k,\ell \in \{1,\dots,d\}\,, \tag{A.3}$$

from which we get the first part of the result. We refer to Desvillettes & Golse (1994) and Bardos et al. (1993) for the proof of the second part of the Lemma, just mind that the linearized Boltzmann operator considered in such references is defined as $Lg = -\mathcal{M}^{-1}\mathbf{L}_1(\mathcal{M}\,g)$. We refer to (Bardos et al. , 1993, Lemma 4.4) for the proof of (A.2). We refer to (Golse & Saint-Raymond , 2005, Proposition 6.5) for the last estimates on $\phi^{i,j}$ and ψ. □

Remark A.2. *Notice that if $\zeta = \zeta(|v|)$ is radially symmetric, then*

$$\Big\langle \zeta\,\boldsymbol{A}^{i,j}\,\mathcal{M}\Big\rangle = \Big\langle \zeta\,\mathbf{L}_1(\phi\,\mathcal{M})\Big\rangle = 0\,, \qquad \forall\, i,j = 1,\dots,d\,.$$

In the case of Maxwell interactions, the above Burnett functions ϕ and ψ are actually explicit (see Cercignani (1988) and (Bobylev , 2020, Eqs. (7.3.9) & (7.3.15))) and turn out to be smooth with a growth of $\nabla_v\phi$ and $\nabla_v\psi$ similar to that of ϕ and ψ. In order to evaluate the contribution

of the drift term in the hydrodynamical limit (see Lemma 6.12), we need also pointwise estimates on $\nabla_v \phi^{i,j}$ and $\nabla_v \psi_i$ that we did not find in the literature. It is the purpose of the next lemma.

Lemma A.3. *If $b(\,\cdot\,)$ satisfies (1.8), then the* Burnett functions $\phi^{i,j}$ *and* ψ_i, $(i,j=1,\dots,d)$ *satisfy*

$$|\nabla_v \phi^{i,j}(v)| \lesssim \varpi_a(v)\,, \qquad |\nabla_v \psi_i(v)| \lesssim \varpi_a(v)\,, \qquad \forall\, i,j \in \{1,\dots,d\}$$

for any $a > 3 + \frac{d}{2}$.

Proof. We only give the proof of $\phi^{i,j}$, the other one being exactly the same. For simplicity, we *fix* $i,j=1,\dots,d$ and only write $\phi = \phi^{i,j}$ and $\boldsymbol{A} = \boldsymbol{A}^{i,j}$. The equation satisfied by ϕ is

$$\mathbf{L}_1(\phi\mathcal{M}) = -\boldsymbol{A}\mathcal{M}\,,$$

where $\mathbf{L}_1$ is the linealized elastic collision operator. Recall that, since we are dealing here with *elastic interactions*, by a simple symmetry argument and replacing $b(s)$ with

$$\tilde{b}(s) = b(s) + b(-s)\,, \qquad \forall\, s \in (0,1)\,,$$

there is no loss of generality in assuming that b is supported in the upper half sphere, i.e. $b(s)=0$ for $s<0$. Before proving that for $a > 3+\frac{d}{2}$, $\varpi_{-a}\nabla_v\phi \in L^\infty$, we first prove that $\varpi_{-a}\nabla_v\phi \in L^2$ for $a > 3+\frac{d}{2}$. One has

$$\mathcal{Q}_1^-(\mathcal{M},\phi\mathcal{M}) = \boldsymbol{A}\mathcal{M} + \mathcal{Q}_1^+(\mathcal{M},\phi\mathcal{M}) + \mathcal{Q}_1^+(\phi\mathcal{M},\mathcal{M}) - \mathcal{Q}_1^-(\phi\mathcal{M},\mathcal{M})\,.$$

Since we are dealing here with elastic interactions, it holds $\mathcal{M}'\mathcal{M}'_* = \mathcal{M}\mathcal{M}_*$ and one can rephrase the above identity as

$$\phi(v)\Sigma_{\mathcal{M}}(v) = \boldsymbol{A}(v) + \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \left(\phi' + \phi'_* - \phi_*\right)\mathcal{M}_*\,|u|\,b(\hat{u}\cdot\sigma)\,\mathrm{d}v_*\,\mathrm{d}\sigma\,,$$

where we recall that $u = v - v_*$ and $\Sigma_{\mathcal{M}}$ is defined in (2.31). Taking the gradient of the above identity, we see that

$$\nabla_v\phi(v) = \Sigma_{\mathcal{M}}^{-1}(v)\Big(\nabla_v\boldsymbol{A}(v) + \nabla_v\Gamma_1[\phi](v) + \nabla_v\Gamma_2[\phi](v) - \Gamma_0[\phi](v) - \phi(v)\nabla_v\Sigma_{\mathcal{M}}(v)\Big) \tag{A.4}$$

where one introduced

$$\Gamma_1[\phi](v) := \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \phi'\,\mathcal{M}_*|u|\,b(\hat{u}\cdot\sigma)\,\mathrm{d}v_*\,\mathrm{d}\sigma, \qquad \Gamma_2[\phi](v) := \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \phi'_*\mathcal{M}_*\,|u|\,b(\hat{u}\cdot\sigma)\,\mathrm{d}v_*\,\mathrm{d}\sigma\,,$$

and

$$\Gamma_0[\phi](v) := \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \phi_*\mathcal{M}_*\nabla_v|u|b(\hat{u}\cdot\sigma)\,\mathrm{d}v_*\,\mathrm{d}\sigma = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \phi_*\mathcal{M}_*\frac{u}{|u|}b(\hat{u}\cdot\sigma)\,\mathrm{d}v_*\,\mathrm{d}\sigma\,.$$

Now, recalling that $\Sigma_{\mathcal{M}}$ is such that $\varpi_1 \lesssim \Sigma_{\mathcal{M}} \lesssim \varpi_1$ and $|\nabla_v\Sigma_{\mathcal{M}}| \lesssim 1$, one has

$$\begin{aligned}\|\varpi_{-a}\nabla_v\phi\|_{L^2} \lesssim\ & \|\varpi_{-a-1}\nabla_v\boldsymbol{A}\|_{L^2} + \|\varpi_{-a-1}\nabla_v\Gamma_1[\phi]\|_{L^2} \\ & + \|\varpi_{-a-1}\nabla_v\Gamma_2[\phi]\|_{L^2} + \|\varpi_{-a-1}\Gamma_0[\phi]\|_{L^2} + \|\varpi_{-a-1}\phi\|_{L^2}\,.\end{aligned}$$

To estimate $\nabla_v \Gamma_1[f]$, we transfer the gradient to the Maxwellian $\mathcal{M}$ by a suitable change of variables. More precisely, we express $\Gamma_1[\phi]$ as follows, performing the singular change of variables $u \mapsto z = \frac{u-|u|\sigma}{2}$,

$$\Gamma_1[\phi](v) = 2^{d-1} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \phi(v-z) \mathcal{M}\left(v - 2z + \frac{|z|}{\widehat{z}\cdot\sigma}\sigma\right) \frac{|z|}{|\widehat{z}\cdot\sigma|} b\big(1-2(\widehat{z}\cdot\sigma)^2\big) \frac{\mathrm{d}z}{|\widehat{z}\cdot\sigma|^2}\,\mathrm{d}\sigma\,,$$

where we note that $v' = v - z$, $v_* = v - 2z - |u|\sigma$, and $|z|^2 = \frac{|u|^2}{2}(1-\widehat{u}\cdot\sigma) = -|u| z\cdot\sigma$ so that

$$\hat{u}\cdot\sigma = 1 + \frac{2}{|u|} z\cdot\sigma = 1 - 2(\hat{z}\cdot\sigma)^2\,, \quad \text{and} \quad |u| = -\frac{|z|}{\widehat{z}\cdot\sigma}\,,$$

so that the Jacobian of the transformation $u \mapsto z$ is

$$\det\left(\frac{\partial z}{\partial u}\right) = 2^{-d}\,(1-\widehat{u}\cdot\sigma) = 2^{1-d}(\widehat{z}\cdot\sigma)^2\,.$$

Now, perform the change of variables $z \mapsto w = v - z$, we have

$$\Gamma_1[\phi](v) = 2^{d-1} \int_{\mathbb{R}^d} \phi(w)|v-w|\,\mathrm{d}w \int_{\mathbb{S}^{d-1}} \mathcal{M}\left(-v + 2w + \frac{|v-w|}{\widehat{(v-w)}\cdot\sigma}\sigma\right) b\big(1-2(\widehat{(v-w)}\cdot\sigma)^2\big) \frac{\mathrm{d}\sigma}{|\widehat{(v-w)}\cdot\sigma|^3}\,.$$

We compute the integral with respect to σ using polar coordinates. For fixed vectors w and v we set $\cos\chi = \widehat{(v-w)}\cdot\sigma$ so that

$$\sigma = \cos\chi\,\widehat{(v-w)} + \sin\chi\,\widehat{\xi}\,,$$

with $\widehat{\xi} = \widehat{\xi}_{v-w} \in \mathbb{S}^{d-2}$ is orthogonal to $v-w$, so that

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \mathcal{M}\left(-v + 2w + \frac{|v-w|}{\widehat{(v-w)}\cdot\sigma}\sigma\right) b\big(1-2(\widehat{(v-w)}\cdot\sigma)^2\big) \frac{\mathrm{d}\sigma}{|\widehat{(v-w)}\cdot\sigma|^3} \\ = \int_{\mathbb{S}^{d-2}} \mathrm{d}\widehat{\xi} \int_0^{\frac{\pi}{2}} \mathcal{M}\left(w + |v-w|\frac{\sin\chi}{\cos\chi}\widehat{\xi}\right) \frac{b\big(1-2\cos^2\chi\big)}{\cos^3\chi} \sin\chi\,\mathrm{d}\chi\,,\end{aligned}$$

where we notice that

$$-v + 2w + \frac{|v-w|}{\widehat{(v-w)}\cdot\sigma}\sigma = w + |v-w|\frac{\sin\chi}{\cos\chi}\widehat{\xi}\,.$$

This gives the equivalent formulation of $\Gamma_1[\phi]$

$$\begin{aligned}\Gamma_1[\phi](v) = 2^{d-1} \int_{\mathbb{R}^d} \phi(w)|v-w|\,\mathrm{d}w \int_{\mathbb{S}^{d-2}} \mathrm{d}\widehat{\xi} \times \\ \times \int_0^{\frac{\pi}{2}} \mathcal{M}\left(w + |v-w|\frac{\sin\chi}{\cos\chi}\widehat{\xi}\right) \frac{b\big(1-2\cos^2\chi\big)}{\cos^3\chi} \sin\chi\,\mathrm{d}\chi\,.\end{aligned}$$

One checks without difficulty that

$$\left|\nabla_v\big(|v-w|\,\widehat{\xi}_{v-w}\big)\right| \lesssim 1\,,$$

while, since $|\nabla_v|v-w|\,| \leqslant 1$,

$$|\nabla_v\Gamma_1[\phi](v)| \lesssim \int_{\mathbb{R}^d}|\phi(w)|\,\mathrm{d}w\int_{\mathbb{S}^{d-2}}\mathrm{d}\widehat{\xi}\int_0^{\frac{\pi}{2}}\mathcal{M}\left(w+|v-w|\frac{\sin\chi}{\cos\chi}\widehat{\xi}\right)\frac{b\big(1-2\cos^2\chi\big)}{\cos^3\chi}\sin\chi\,\mathrm{d}\chi$$
$$+\int_{\mathbb{R}^d}|\phi(w)|\,|v-w|\,\mathrm{d}w\int_{\mathbb{S}^{d-2}}\mathrm{d}\widehat{\xi}\int_0^{\frac{\pi}{2}}|\nabla\mathcal{M}|\left(w+|v-w|\frac{\sin\chi}{\cos\chi}\widehat{\xi}\right)\frac{b\big(1-2\cos^2\chi\big)}{\cos^4\chi}\sin\chi\,\mathrm{d}\chi\,.$$

Performing backwards all the change of variables, one sees that

$$|\nabla_v\Gamma_1[\phi](v)| \lesssim \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\big|\phi'\big|\;\mathcal{M}_*\,b(\hat{u}\cdot\sigma)\,\mathrm{d}v_*\,\mathrm{d}\sigma$$
$$+\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}|\phi'|\,|(\nabla\mathcal{M})_*|\,|u|\,\frac{b(\widehat{u}\cdot\sigma)}{\sqrt{1-\widehat{u}\cdot\sigma}}\,\mathrm{d}v_*\,\mathrm{d}\sigma\,.$$

Therefore, since $\varpi_{-\beta}(v)\lesssim\varpi_\beta(v_*)\varpi_{-\beta}(v')$ for any $\beta\geqslant 0$ and $|u|\lesssim\varpi_1(v)\varpi_1(v_*)$,

$$|\varpi_{-a-1}\nabla_v\Gamma_1[\phi](v)| \lesssim \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\big(|\phi|\varpi_{-a-1}\big)'\,\big(\mathcal{M}\varpi_{a+1}\big)_*\;b(\hat{u}\cdot\sigma)\,\mathrm{d}v_*\,\mathrm{d}\sigma$$
$$+\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\big(|\phi|\varpi_{-a}\big)'\,\big(|\nabla\mathcal{M}|\varpi_{a+1}\big)_*\;\frac{b(\widehat{u}\cdot\sigma)}{\sqrt{1-\widehat{u}\cdot\sigma}}\,\mathrm{d}v_*\,\mathrm{d}\sigma =: F_1(v)+F_2(v)\,.$$

With the notation of Alonso et al. (2010) and with $v'=v_*+u^+$, one sees for instance that the last integral can be written as

$$F_2(v) := \int_{\mathbb{R}^d}G(v_*)\mathcal{P}(1,\tau_{-v_*}\varphi)(v)\,\mathrm{d}v_*\,,$$

where $G:=|\nabla\mathcal{M}|\varpi_{a+1}$, $\varphi:=\varpi_{-a}|\phi|$ and

$$\mathcal{P}(f,g)(v) := \int_{\mathbb{S}^{d-1}}f(u^-)g(u^+)\boldsymbol{b}(\hat{u}\cdot\sigma)\,\mathrm{d}\sigma\,,$$

with $\boldsymbol{b}(s):=\frac{b(s)}{\sqrt{1-s}}$, $u^-:=\frac12(u-|u|\sigma)$, $u^+=u-u^-$. Then, by Minkoskwi's integral inequality

$$\|F_2\|_{L^2}\leqslant\|G\|_{L^1}\sup_{v_*}\|\mathcal{P}(1,\tau_{-v_*}\varphi)\|_{L^2}$$

and, according to (Alonso et al. , 2010, Theorem 5) with $p=\infty, q=r=2$ and $\alpha=0$, one has

$$\|\mathcal{P}(1,\tau_{-v_*}\varphi)\|_{L^2}\leqslant C\,\|\tau_{-v_*}\varphi\|_{L^2}\qquad\text{with}\qquad C\lesssim\int_0^1\boldsymbol{b}(s)\,\big(1-s^2\big)^{\frac{d-3}{2}}\,(1+s)^{-\frac{d}{4}}\,\mathrm{d}s\,.$$

Recalling that $b(\,\cdot\,)$ is supported on the upper half-sphere, one sees then that

$$\|F_2\|_{L^2}\lesssim\left(\int_0^1 b(s)(1-s)^{\frac{d-4}{2}}\,\mathrm{d}s\right)\||\nabla\mathcal{M}|\varpi_{a+1}\|_{L^1}\;\|\varpi_{-a}\phi\|_{L^2}\,.$$

Arguing in the same way with F_1, we deduce that

$$\|\varpi_{-a-1}\nabla_v\Gamma_1[\phi]\|_{L^2} \lesssim \left(\int_0^1 b(s)(1-s)^{\frac{d-4}{2}}\,\mathrm{d}s\right)\left(\||\nabla\mathcal{M}|\varpi_{a+1}\|_{L^1}+\|\mathcal{M}\varpi_{a+1}\|_{L^1}\right)\|\varpi_{-a}\phi\|_{L^2}\,.$$

For the second term $\nabla_v\Gamma_2[\phi]$ we use the regular change of variables $u\mapsto z=\frac{u+|u|\sigma}{2}$ and proceed identically. We end up with the estimate

$$|\varpi_{-a-1}\nabla_v\Gamma_2[\phi](v)| \lesssim \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(|\phi|\varpi_{-a-1})'_*\,(\mathcal{M}\varpi_{a+1})_*\,b(\hat{u}\cdot\sigma)\,\mathrm{d}v_*\,\mathrm{d}\sigma + \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(|\phi|\varpi_{-a})'_*\,(|\nabla\mathcal{M}|\varpi_{a+1})_*\,\frac{b(\widehat{u}\cdot\sigma)}{\sqrt{1+\widehat{u}\cdot\sigma}}\,\mathrm{d}v_*\,\mathrm{d}\sigma =: \tilde{F}_1(v)+\tilde{F}_2(v)\,.$$

As before, writing now $v'_*=v_*+u^-$, one has for instance

$$\tilde{F}_2(v)=\int_{\mathbb{R}^d}G(v_*)\mathcal{P}(\tau_{-v_*}\varphi,1)(v)\,\mathrm{d}v_*\,,$$

with now

$$\mathcal{P}(f,g)(v)=\int_{\mathbb{S}^{d-1}}f(u^-)g(u^+)\tilde{\boldsymbol{b}}(\hat{u}\cdot\sigma)\,\mathrm{d}\sigma\,,\qquad \tilde{\boldsymbol{b}}(s):=\frac{b(s)}{\sqrt{1+s}}\,.$$

According to (Alonso et al. , 2010, Theorem 5) with now $p=r=2$ and $q=\infty, \alpha=0$, one has

$$\|\mathcal{P}(\tau_{-v_*}\varphi,1)\|_{L^2}\leqslant\tilde{C}\,\|\tau_{-v_*}\varphi\|_{L^2}\qquad\text{with}\qquad \tilde{C}\lesssim\int_0^1\tilde{\boldsymbol{b}}(s)\,(1-s)^{\frac{d-3}{2}}\,(1-s)^{-\frac{d}{4}}\,\mathrm{d}s\,.$$

Therefore, as before using Minkowski's integral inequality and recalling that $b(\,\cdot\,)$ is supported on $(0,1)$, we get

$$\|\tilde{F}_2\|_{L^2}\lesssim\left(\int_0^1 b(s)\,(1-s)^{\frac{d-6}{4}}\,\mathrm{d}s\right)\||\nabla\mathcal{M}|\varpi_{a+1}\|_{L^1}\|\varpi_{-a}\phi\|_{L^2}\,.$$

Consequently,

$$\|\varpi_{-a-1}\nabla_v\Gamma_2[\phi]\|_{L^2} \lesssim \left(\int_0^1 b(s)\,(1-s)^{\frac{d-6}{4}}\,\mathrm{d}s\right)\left(\||\nabla\mathcal{M}|\varpi_{a+1}\|_{L^1}+\|\mathcal{M}\varpi_{a+1}\|_{L^1}\right)\|\varpi_{-a}\phi\|_{L^2}\,.$$

Collecting the above arguments and estimating $\Gamma_0[\phi]$ in a similar fashion, we end up with

$$\|\varpi_{-a}\nabla_v\phi\|_{L^2}\lesssim C_a(b)\,\|\varpi_{-a}\phi\|_{L^2}+\|\varpi_{-a-1}\nabla_v\boldsymbol{A}\|_{L^2}+\|\varpi_{-a-1}\phi\|_{L^2}\,,\qquad\forall a>0\,,\quad\text{(A.5)}$$

where

$$C_a(b)\lesssim\int_0^1 b(s)\left[(1-s)^{\frac{d-6}{4}}+(1-s)^{\frac{d-4}{2}}\right]\mathrm{d}s\,.$$

Since $|\nabla_v \boldsymbol{A}| \lesssim \varpi_1$ and $|\phi| \lesssim \varpi_3$, one sees that $\max\left(\|\varpi_{-a}\phi\|_{L^2}, \|\varpi_{-a-1}\nabla_v \boldsymbol{A}\|_{L^2}\right) < \infty$ for $a > 3 + \frac{d}{2}$ and

$$\varpi_{-a}\nabla_v\phi \in L^2(\mathbb{R}^d)\,, \qquad \forall\, a > 3 + \frac{d}{2}\,.$$

Now, this control in L^2 of the gradient of ϕ implies a control in L^∞ of that same gradient. Indeed, coming back to (A.4) one has

$$\begin{aligned}\|\varpi_{-a}\nabla_v\phi\|_{L^\infty} \lesssim \|\varpi_{-a-1}\nabla_v \boldsymbol{A}\|_{L^\infty} + \|\varpi_{-a-1}\nabla_v\Gamma_1[\phi]\|_{L^\infty} \\ + \|\varpi_{-a-1}\nabla_v\Gamma_2[\phi]\|_{L^\infty} + \|\varpi_{-a-1}\Gamma_0[\phi]\|_{L^\infty} + \|\varpi_{-a-1}\phi\|_{L^\infty}\end{aligned}$$

where one can write

$$\nabla_v\Gamma_1[\phi](v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \phi'\, \mathcal{M}_*\nabla_v|u|\, b(\hat{u}\cdot\sigma)\,\mathrm{d}v_*\,\mathrm{d}\sigma + \Gamma_1[\nabla_v\phi](v)\,,$$

so that

$$\|\varpi_{-a-1}\nabla_v\Gamma_1[\phi]\|_{L^\infty} \lesssim \|\varpi_{-a}\Gamma_0[\phi]\|_{L^\infty} + \|\varpi_{-a}\Gamma_1[|\nabla_v\phi|]\|_{L^\infty}\,,$$

where

$$|\varpi_{-a-1}\Gamma_1[\nabla_v\phi](v)| \lesssim \int_{\mathbb{R}^d} (\mathcal{M}\varpi_{a+1})_*\, \mathcal{P}(1, \varpi_{-a}|\nabla_v\phi|)\,\mathrm{d}v_*\,.$$

Invoking again (Alonso et al. , 2010, Theorem 5) and Cauchy-Schwarz inequality one has then

$$\|\varpi_{-a-1}\Gamma_1[|\nabla_v\phi|]\|_{L^\infty} \leqslant \|\mathcal{M}\,\varpi_{a+1}\|_{L^2}\|\mathcal{P}(1, \varpi_{-a}|\nabla_v\phi|)\|_{L^2} \lesssim C_a(b)\|\varpi_{-a}|\nabla_v\phi|\,\|_{L^2}\,,$$

where $C_a(b)$ is as before. Using (A.5) for the other terms and sees that, provided

$$\int_0^1 b(s)\left[(1-s)^{\frac{d-6}{4}} + (1-s)^{\frac{d-4}{2}}\right]\mathrm{d}s < \infty \tag{A.6}$$

one has $\varpi_{-a}\nabla_v\phi \in L^\infty(\mathbb{R}^d)$ for any $a > 3 + \frac{d}{2}$. Notice that, for $d \geqslant 2$, $\frac{d-4}{2} \geqslant \frac{d-6}{4}$ so that the above assumption amounts simply to

$$\int_0^1 b(s)\,(1-s)^{\frac{d-6}{4}}\,\mathrm{d}s < \infty$$

which holds true since $b(\,\cdot\,)$ satisfies (1.8). □

Remark A.4. *It is very likely that the above growth of $|\nabla_v\phi| \lesssim \varpi_a$ with $a > 3 + \frac{d}{2}$ is not optimal. In the case of Maxwell interactions the growth is actually*

$$|v\cdot\nabla_v\phi(v)| \lesssim |\phi(v)|\,, \qquad |v\cdot\nabla_v\psi(v)| \lesssim |\psi(v)|\,.$$

We refer to Bobylev (2020) for more details.

Lemma A.5. *For $\boldsymbol{h}$ given by (6.8), it holds that*

$$\left\langle \phi\, \mathcal{Q}_1(\boldsymbol{h}, \boldsymbol{h})\right\rangle = \vartheta_1^2\left(u\otimes u - \frac{2}{d}|u|^2\mathbf{Id}\right)\,, \qquad \forall\, i,j = 1,\dots,d\,.$$

Proof. As observed in (Cercignani , 1970, Eq. (60)), if $g\mathcal{M} \in \mathrm{Ker}(\mathbf{L}_1)$ then $\mathcal{Q}_1(g\mathcal{M}, g\mathcal{M}) = -\frac{1}{2}\mathbf{L}_1(g^2\mathcal{M})$. Therefore, with $g = \varrho + u\cdot v + \frac{1}{2}(|v|^2 - 2\vartheta_1)$,

$$\mathcal{Q}_1(\boldsymbol{h},\boldsymbol{h}) = -\frac{1}{2}\mathbf{L}_1((u\cdot v)^2\mathcal{M}) - \frac{1}{8}\theta^2\mathbf{L}_1(|v|^4\mathcal{M}) + \theta\, u\cdot\mathbf{L}_1(\frac{1}{2}|v|^2 v\mathcal{M}) \tag{A.7}$$

One checks that

$$\Big\langle \phi^{i,j}\mathbf{L}_1(|v|^4\mathcal{M})\Big\rangle = 0\,,$$

whereas $\mathbf{L}_1(\frac{1}{2}|v|^2 v\mathcal{M}) = \mathbf{L}_1(\boldsymbol{b}\mathcal{M})$, from which

$$\Big\langle \phi^{i,j}\mathbf{L}_1(\frac{1}{2}|v|^2 v\mathcal{M})\Big\rangle = \Big\langle \boldsymbol{b}\mathbf{L}_1(\phi^{i,j}\mathcal{M})\Big\rangle = -\Big\langle \boldsymbol{b}\,\boldsymbol{A}^{i,j}\mathcal{M}\Big\rangle = 0\,,$$

since $\boldsymbol{b}\,\boldsymbol{A}^{i,j}$ is an even function. Therefore, we obtain that

$$\Big\langle \phi^{i,j}\mathcal{Q}_1(\boldsymbol{h},\boldsymbol{h})\Big\rangle = -\frac{1}{2}\sum_{k,\ell} u_k u_\ell \Big\langle \phi^{i,j}\mathbf{L}_1(v_k v_\ell\mathcal{M})\Big\rangle = \frac{1}{2}\sum_{k,\ell} u_k u_\ell \Big\langle v_k v_\ell \boldsymbol{A}^{i,j}\mathcal{M}\Big\rangle\,. \tag{A.8}$$

As for (A.2), one checks that if $i \neq j$

$$\sum_{k,\ell} u_k u_\ell \Big\langle v_k v_\ell \boldsymbol{A}^{i,j}\mathcal{M}\Big\rangle = \sum_{\{k,\ell\}=\{i,j\}} u_k u_\ell \Big\langle v_i^2 v_j^2\mathcal{M}\Big\rangle = 2u_i u_j\Big\langle v_i^2 v_j^2\mathcal{M}\Big\rangle\,,$$

whereas, for $i = j$,

$$\sum_{k,\ell} u_k u_\ell \Big\langle v_k v_\ell \boldsymbol{A}^{i,i}\mathcal{M}\Big\rangle = \sum_{k=1}^d u_k^2\left(\Big\langle v_i^2 v_k^2\mathcal{M}\Big\rangle - \frac{1}{d}\Big\langle v_k^2|v|^2\mathcal{M}\Big\rangle\right)\,.$$

Notice that $a := \Big\langle v_i^2 v_j^2\mathcal{M}\Big\rangle$ is independent of i, j, thus, it is not difficult to check that

$$(d-1)a = \frac{1}{d}\int_{\mathbb{R}^d}|v|^4\mathcal{M}\,\mathrm{d}v - \int_{\mathbb{R}^d} v_1^4\mathcal{M}(v)\,\mathrm{d}v = (d-1)\vartheta_1^2\,,$$

that is, $a = \vartheta_1^2$. In the same way, for any $k \in \{1,\dots,d\}$

$$\Big\langle v_k^2|v|^2\mathcal{M}\Big\rangle = \frac{1}{d}\Big\langle |v|^4\mathcal{M}\Big\rangle = (d+2)\vartheta_1^2\,,$$

whereas

$$\Big\langle v_k^2 v_i^2\mathcal{M}\Big\rangle = \begin{cases} a = \vartheta_1^2 & \text{if } k \neq i\,,\\ \Big\langle v_i^4\mathcal{M}\Big\rangle = 3\vartheta_1^2 & \text{if } k = i\,,\end{cases}$$

so that,

$$\sum_{k,\ell} u_k u_\ell \Big\langle v_k v_\ell \boldsymbol{A}^{i,i}\mathcal{M}\Big\rangle = \vartheta_1^2\sum_{k\neq i} u_k^2 + 3\vartheta_1^2 u_i^2 - \frac{d+2}{d}|u|^2\vartheta_1^2 = 2\vartheta_1^2 u_i^2 - \frac{2}{d}\vartheta_1^2|u|^2\,.$$

Gathering these last computations, we get

$$\Big\langle \phi^{i,j}\mathcal{Q}_1((u\cdot v)\mathcal{M},(u\cdot v)\mathcal{M})\Big\rangle = \vartheta_1^2\left(u_i u_j - \frac{2}{d}|u|^2\delta_{i,j}\right)\,,$$

which, combined with (A.8) gives the result. □

Lemma A.6. *Let $\boldsymbol{h}$ be given by (6.8). For any $i,j=1,\dots,d$ it holds that*

$$\Big\langle v_\ell\,\phi^{i,j}\,\boldsymbol{h}\Big\rangle=\begin{cases}\nu\,u_j & \text{if } i\neq j\,,\ \ell=i\,,\\ \nu\,u_i & \text{if } i\neq j\,,\ \ell=j\,,\\ -\frac{2}{d}\nu\,u_\ell+2\nu\,u_i\delta_{i\ell} & \text{if } i=j\,,\\ 0 & \text{else.}\end{cases}$$

Proof. Using the fact that χ_1 is radial, similar computations to that of Lemma A.5 imply that for $\ell\in\{1,\dots,d\}$,

$$\begin{aligned}\Big\langle v_\ell\,\phi^{i,j}\,\boldsymbol{h}\Big\rangle&=\sum_{k=1}^d u_k\Big\langle v_\ell v_k\,\phi^{i,j}\,\mathcal{M}\Big\rangle=\sum_{k=1}^d u_k\Big\langle v_\ell v_k\,\phi^{i,j}\,\mathcal{M}\Big\rangle\\&=\sum_{k=1}^d u_k\left(\Big\langle \boldsymbol{A}^{k,\ell}\,\phi^{i,j}\,\mathcal{M}\Big\rangle+\frac{1}{d}\Big\langle|v|^2\,\phi^{i,j}\,\mathcal{M}\Big\rangle\delta_{k\ell}\right)\\&=-\sum_{k=1}^d u_k\Big\langle\phi^{i,j}\mathbf{L}_1(\phi^{k,\ell}\mathcal{M})\Big\rangle\,,\end{aligned}$$

where we used that $\mathbf{L}_1(\phi\mathcal{M})=-\boldsymbol{A}\mathcal{M}$ and $\Big\langle|v|^2\phi^{i,j}\mathcal{M}\Big\rangle=0$. This gives the result thanks to (A.2). □

Lemma A.7. *Let $\boldsymbol{h}$ be given by (6.8). For any $i=1,\dots,d$, it holds that*

$$\Big\langle\psi_i\,\mathcal{Q}_1(\boldsymbol{h},\boldsymbol{h})\Big\rangle=\frac{d+2}{2}\vartheta_1^3\,(\theta\,u_i)\,,$$

and, if ϱ and θ satisfies Boussinesq relation (6.13), then

$$\mathrm{div}_x\Big\langle\psi_i\,\boldsymbol{h}\,v\Big\rangle=\gamma\frac{d+2}{2}\partial_{x_i}\theta\,.$$

Proof. On the one hand, using (A.7) it holds that

$$\Big\langle\psi_i\mathcal{Q}_1(\boldsymbol{h},\boldsymbol{h})\Big\rangle=\theta\,u\cdot\Big\langle\psi_i\mathbf{L}_1(\frac{1}{2}|v|^2v\mathcal{M})\Big\rangle=\theta\,u\cdot\Big\langle\psi_i\,\mathbf{L}_1(\boldsymbol{b}\mathcal{M})\Big\rangle\,,$$

since, ψ_i being odd, one has $\Big\langle\psi_i\mathbf{L}_1((u\cdot v)^2\mathcal{M})\Big\rangle=\Big\langle\psi_i\mathbf{L}_1(|v|^4\mathcal{M})\Big\rangle=0$. Now,

$$\Big\langle\psi_i\,\mathbf{L}_1(\boldsymbol{b}\mathcal{M})\Big\rangle=\Big\langle\boldsymbol{b}\,\mathbf{L}_1(\psi_i\mathcal{M})\Big\rangle=-\Big\langle\boldsymbol{b}\mathcal{M}\,\boldsymbol{b}_i\Big\rangle\,,$$

and a direct computations show that

$$\Big\langle\boldsymbol{b}_j\boldsymbol{b}_i\,\mathcal{M}\Big\rangle=-\frac{1}{4d}\Big\langle\left(|v|^2-(d+2)\vartheta_1\right)^2|v|^2\mathcal{M}\Big\rangle\,\delta_{ij}=-\frac{d+2}{2}\vartheta_1^3\,\delta_{ij}\,,$$

which gives the expression for $\Big\langle \psi_i \, \mathcal{Q}_1(\boldsymbol{h},\boldsymbol{h})\Big\rangle$. On the other hand, using symmetry properties, one checks that

$$\Big\langle \psi_i \boldsymbol{h}\, v_\ell \Big\rangle = \varrho \Big\langle \psi_i v_i \mathcal{M}\Big\rangle \delta_{i\ell} + \frac{1}{2}\theta \Big\langle \psi_i (|v|^2 - d\vartheta_1) v_i \mathcal{M}\Big\rangle \, \delta_{i\ell}\,,$$

from which

$$\mathrm{div}_x \Big\langle \psi_i \, \boldsymbol{h}\, v\Big\rangle = \Big\langle \psi_i v_i \mathcal{M}\Big\rangle \, \partial_{x_i}\varrho + \frac{1}{2}\Big\langle \psi_i \big(|v|^2 - d\vartheta_1\big) v_i \mathcal{M}\Big\rangle \, \partial_{x_i}\theta\,.$$

Writing $\frac{1}{2}\Big\langle \psi_i (|v|^2 - d\vartheta_1) v_i \mathcal{M}\Big\rangle = \Big\langle \psi_i \, \boldsymbol{b}_i \mathcal{M}\Big\rangle + \vartheta_1 \Big\langle \psi_i \, v_i \mathcal{M}\Big\rangle$ and using Boussinesq relation (6.13), one gets that

$$\mathrm{div}_x \Big\langle \psi_i \, \boldsymbol{h}\, v\Big\rangle = \Big\langle \psi_i \, \boldsymbol{b}_i \mathcal{M}\Big\rangle \, \partial_{x_i}\theta = \gamma \frac{d+2}{2} \partial_{x_i}\theta\,,$$

where the identity $\Big\langle \psi_i \, \boldsymbol{b}_i \mathcal{M}\Big\rangle = -\Big\langle \psi_i \mathbf{L}_1(\psi_i \mathcal{M})\Big\rangle$ was used together with (A.2). □

To handle the convergence of nonlinear terms, we resort to the following compensated compactness result extracted from Lions & Masmoudi (1999) (see also (Golse & Saint-Raymond , 2004, Lemma 13.1, Appendix D). The original result in Lions & Masmoudi (1999) is proven in the whole space but is easily adapted to the case of the torus.

Proposition A.8. *Let $c \neq 0$ and $T > 0$. Consider two families $\{\phi_\varepsilon\}_\varepsilon$ and $\{\psi_\varepsilon\}$ bounded in $L^\infty((0,T)\,; L^2_x(\mathbb{T}^d))$ and in $L^\infty((0,T)\,; \mathbb{W}^{1,2}_x(\mathbb{T}^d))$ respectively, such that*

$$\begin{cases} \partial_t \nabla_x \psi_\varepsilon + \dfrac{c^2}{\varepsilon}\nabla_x \phi_\varepsilon = \dfrac{1}{\varepsilon} F_\varepsilon \\ \\ \partial_t \phi_\varepsilon + \dfrac{1}{\varepsilon}\Delta_x \psi_\varepsilon = \dfrac{1}{\varepsilon} G_\varepsilon \end{cases}$$

where F_ε and G_ε converge strongly to 0 in $L^1((0,T)\,; L^2_x(\mathbb{T}^d))$. Then,

$$\mathcal{P}\mathrm{Div}_x \left(\nabla_x \psi_\varepsilon \otimes \nabla_x \psi_\varepsilon\right) \xrightarrow[\varepsilon\to 0]{} 0\,, \qquad \mathrm{div}_x \left(\phi_\varepsilon \nabla_x \psi_\varepsilon\right) \xrightarrow[\varepsilon\to 0]{} 0$$

in the sense of distributions on $(0,T)\times \mathbb{T}^d$.

Appendix B. Estimates on the collision operator

We first recall a crucial estimate on the Boltzmann collision operator established in Theorem 1 of Alonso et al. (2010) (see also Alonso & Gamba (2011)):

Theorem B.1 (Theorem 1, Alonso et al. (2010))**.** *Consider $q \geqslant 0$, $r \in [1,\infty)$. For any $f \in L^r_v(\boldsymbol{\varpi}_{q+1})$ and $g \in L^1_v(\boldsymbol{\varpi}_{q+1})$,*

$$\|\mathcal{Q}^+_\alpha(f,g)\|_{L^r_v(\boldsymbol{\varpi}_q)} \leqslant C_1 \|f\|_{L^r_v(\boldsymbol{\varpi}_{q+1})} \|g\|_{L^1_v(\boldsymbol{\varpi}_{q+1})}\,,$$

and

$$\|\mathcal{Q}^+_\alpha(g,f)\|_{L^r_v(\boldsymbol{\varpi}_q)} \leqslant C_2 \|f\|_{L^r_v(\boldsymbol{\varpi}_{q+1})} \, \|g\|_{L^1_v(\boldsymbol{\varpi}_{q+1})}$$

with

$$C_1 = C_1(r) = 2^{\frac{q+1}{2}+\frac{1}{r}+\frac{d}{r'}}|\mathbb{S}^{d-2}|\int_{-1}^{1}\left(\frac{1-s}{2}\right)^{-\frac{d}{2r'}} b(s)\left(1-s^2\right)^{\frac{d-3}{2}}\,\mathrm{d}s$$

and

$$C_2 = C_2(r) = 2^{\frac{q+1}{2}+\frac{1}{r}}|\mathbb{S}^{d-2}|\int_{-1}^{1}\left(\frac{1+s}{2}+\frac{(1-\alpha)^2}{4}\frac{1+s}{2}\right)^{-\frac{d}{2r'}} b(s)\left(1-s^2\right)^{\frac{d-3}{2}}\,\mathrm{d}s$$

where $\frac{1}{r}+\frac{1}{r'}=1$.

Remark B.2. *Throughout the paper, we assume* $b(\,\cdot\,)$ *satisfies assumption* (1.8) *so that* $C_1(r)<\infty$ *and* $C_2(r)<\infty$ *for any* $r\in[1,2]$.

We also recall a classical estimate (whose proof is immediate using Hölder inequality) for the loss part of the collision operator.

Lemma B.3. *Consider* $q\geqslant 0$, $r\in[1,2]$ *and* b *such that condition* (1.8) *is satisfied. Then, for any* $\kappa'>\frac{d(r-1)}{r}$, *any* $f\in L^r_v(\varpi_{q+1})$ *and* $g\in L^r_v(\varpi_{\kappa'+1})$, *we have:*

$$\|\mathcal{Q}^-_\alpha(g,f)\|_{L^r_v(\varpi_q)}\lesssim\|g\|_{L^r_v(\varpi_{\kappa'+1})}\|f\|_{L^r_v(\varpi_{q+1})}\,.$$

From the two previous results, using that the operators $\mathcal{Q}^\pm_\alpha$ are local in x and that $\mathbb{W}^{\ell,2}_x$ is an algebra for $\ell>\frac{d}{2}$, we can deduce the following :

Corollary B.4. *Consider* $q\geqslant 0$, $r\in[1,2)$, b *such that condition* (1.8) *is satisfied and* $\ell>\frac{d}{2}$. *Then, for any* $\kappa'>\frac{d(r-1)}{r}$, *any* $f\in L^r_v\mathbb{W}^{\ell,2}_x(\varpi_{q+1})$ *and* $g\in L^r_v\mathbb{W}^{\ell,2}_x(\varpi_{\kappa'+1})\cap L^1_v\mathbb{W}^{\ell,2}_x(\varpi_{q+1})$, *we have:*

$$\|\mathcal{Q}_\alpha(g,f)\|_{L^r_v\mathbb{W}^{\ell,2}_x(\varpi_q)}\lesssim\|f\|_{L^r_v\mathbb{W}^{\ell,2}_x(\varpi_{q+1})}\left(\|g\|_{L^r_v\mathbb{W}^{\ell,2}_x(\varpi_{\kappa'+1})}+\|g\|_{L^1_v\mathbb{W}^{\ell,2}_x(\varpi_{q+1})}\right)$$

The proof of Lemma 3.4 is based on the following technical lemma:

Lemma B.5. *Consider* b *such that assumption* (1.7) *is satisfied. Then, there exists* $C>0$ *such that for any* f, g *and* $\varphi\in L^2_v(\mathcal{M}^{-\frac12}\langle\,\cdot\,\rangle^{\frac12})$,

$$\begin{aligned}\left\langle\mathcal{Q}^\pm_\alpha(g,f),\varphi\right\rangle_{L^2_v(\mathcal{M}^{-\frac12})}\leqslant C\|\varphi\|_{L^2_v(\mathcal{M}^{-\frac12}\langle\,\cdot\,\rangle^{\frac12})}\Big(&\|f\|_{L^2_v(\mathcal{M}^{-\frac12}\langle\,\cdot\,\rangle^{\frac12})}\|g\|_{L^2_v(\mathcal{M}^{-\frac12})}\\&+\|f\|_{L^2_v(\mathcal{M}^{-\frac12})}\|g\|_{L^2_v(\mathcal{M}^{-\frac12}\langle\,\cdot\,\rangle^{\frac12})}\Big)\,.\end{aligned}\tag{B.1}$$

In particular, if f, g *and* $\varphi\in\mathcal{H}_1$, *it holds*

$$\left\langle\mathcal{Q}^\pm_\alpha(g,f),\varphi\right\rangle_{\mathcal{H}}\leqslant C\|\varphi\|_{\mathcal{H}_1}\left(\|f\|_{\mathcal{H}_1}\|g\|_{\mathcal{H}}+\|f\|_{\mathcal{H}}\|g\|_{\mathcal{H}_1}\right)\tag{B.2}$$

where the functional spaces $\mathcal{H}$ *and* $\mathcal{H}_1$ *are defined in* (3.1).

Proof. We use the strong form of $\mathcal{Q}^{\pm}_{\alpha}(g,f)$ given in (2.12) and prove the result only for $\mathcal{Q}^{+}_{\alpha}(g,f)$, the proof for $\mathcal{Q}^{-}_{\alpha}(g,f) = \mathcal{Q}^{-}_{1}(g,f)$ being simpler and well-known. Notice that (B.1) is dealing only with spatially homogeneous functions g, f and φ. We set

$$F := \mathcal{M}^{-\frac{1}{2}} f\,, \qquad G := \mathcal{M}^{-\frac{1}{2}} g\,, \qquad \Phi := \mathcal{M}^{-\frac{1}{2}} \varphi\,. \tag{B.3}$$

Given $v \in \mathbb{R}^d$, one has

$$\begin{aligned}\left|\mathcal{Q}^{+}_{\alpha}\big(g,f\big)(v)\right| &\leqslant \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B^{+}_{\alpha}(u,\sigma)\,|f(v')|\,|g(v'_*)|\,\mathrm{d}\sigma\,\mathrm{d}v_*\,,\\ &\leqslant \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B^{+}_{\alpha}(u,\sigma)\mathcal{M}^{\frac{1}{2}}(v')\mathcal{M}^{\frac{1}{2}}(v'_*)\,|F(v')|\,|G(v'_*)|\,\mathrm{d}\sigma\,\mathrm{d}v_*\,,\end{aligned}$$

where, according to (2.13), $B^{+}_{\alpha}(u,\sigma) = \frac{|u|}{\alpha^2} b_\alpha(\widehat{u}\cdot\sigma)$ with

$$b_\alpha(s) = \left[\frac{2}{1+\alpha^2-(1-\alpha^2)s}\right]^{\frac{d-3}{2}} b\left(\frac{(1+\alpha^2)s-(1-\alpha^2)}{1+\alpha^2-(1-\alpha^2)s}\right)\,, \qquad s\in(-1,1)\,.$$

Due to the dissipation of kinetic energy, $|v'|^2+|v'_*|^2 \geqslant |v|^2+|v_*|^2$ which translates into

$$\mathcal{M}^{\frac{1}{2}}(v')\mathcal{M}^{\frac{1}{2}}(v'_*) \leqslant \mathcal{M}^{\frac{1}{2}}(v)\mathcal{M}^{\frac{1}{2}}(v_*)$$

so that

$$\left|\mathcal{Q}^{+}_{\alpha}\big(g,f\big)(v)\right| \leqslant \mathcal{M}^{\frac{1}{2}}(v)\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B^{+}_{\alpha}(u,\sigma)\mathcal{M}^{\frac{1}{2}}(v_*)\,|F(v')|\,|G(v'_*)|\,\mathrm{d}\sigma\,\mathrm{d}v_*\,.$$

Thanks to Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}\left|\mathcal{Q}^{+}_{\alpha}\big(g,f\big)(v)\right| \leqslant \mathcal{M}^{\frac{1}{2}}(v)&\left(\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |F(v')|^2\,|G(v'_*)|^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\right)^{\frac{1}{2}}\\ &\left(\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \big(B^{+}_{\alpha}(u,\sigma)\big)^2\,\mathcal{M}(v_*)\,\mathrm{d}v_*\,\mathrm{d}\sigma\right)^{\frac{1}{2}}\end{aligned}$$

Now, for fixed $(v,v_*)\in\mathbb{R}^d\times\mathbb{R}^d$, we compute

$$\int_{\mathbb{S}^{d-1}} (B^{+}_{\alpha}(u,\sigma))^2\,\mathrm{d}\sigma = \frac{|u|^2}{\alpha^4}|\mathbb{S}^{d-2}|\int_{-1}^{1}\big(1-s^2\big)^{\frac{d-3}{2}}\,b^2_\alpha(s)\,\mathrm{d}s$$

and, performing the change of variable $t = \frac{(1+\alpha^2)s-(1-\alpha^2)}{1+\alpha^2-(1-\alpha^2)s}$ so that $s = \frac{(1+\alpha^2)t+(1-\alpha^2)}{1+\alpha^2+(1-\alpha^2)t}$ and $\mathrm{d}s = \frac{4\alpha^2}{(1+\alpha^2+(1-\alpha^2)t)^2}\,\mathrm{d}t$, we get

$$\begin{aligned}
&\int_{-1}^1 \left(1-s^2\right)^{\frac{d-3}{2}} b_\alpha^2(s)\,\mathrm{d}s \\
&\quad = 4\alpha^2 \int_{-1}^1 \left(\frac{4\alpha^2(1-t^2)}{(1+\alpha^2+(1-\alpha^2)t)^2}\right)^{\frac{d-3}{2}} \left[\frac{1}{4\alpha^2}\left(1+\alpha^2+(1-\alpha^2)t\right)\right]^{d-3} \\
&\qquad\qquad b^2(t)\frac{\mathrm{d}t}{(1+\alpha^2+(1-\alpha^2)t)^2} \\
&\quad = (2\alpha)^{5-d}\int_{-1}^1 \left(1-t^2\right)^{\frac{d-3}{2}} \frac{b^2(t)\,\mathrm{d}t}{(1+\alpha^2+(1-\alpha^2)t)^2}\,.
\end{aligned}$$

In particular, according to assumption (1.7) and since $1+\alpha^2+(1-\alpha^2)t \geqslant 2\alpha^2$, one deduces that

$$\int_{\mathbb{S}^{d-1}} (B_\alpha^+(u,\sigma))^2\,\mathrm{d}\sigma \leqslant 2^{d+4}\frac{|u|^2}{\alpha^{7+d}}|\mathbb{S}^{d-2}|\int_{-1}^1 \left(1-t^2\right)^{\frac{d-3}{2}} b^2(t)\,\mathrm{d}t = C_b\frac{|u|^2}{\alpha^{7+d}}$$

from which

$$\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \left(B_\alpha^+(u,\sigma)\right)^2 \mathcal{M}(v_*)\,\mathrm{d}v_*\,\mathrm{d}\sigma \leqslant C_b\,\alpha^{-7-d}\int_{\mathbb{R}^d}|v-v_*|^2\mathcal{M}(v_*)\,\mathrm{d}v_* \lesssim \langle v\rangle^2\,.$$

Therefore

$$\left|\mathcal{Q}_\alpha^+\big(g,f\big)(v)\right| \lesssim \mathcal{M}^{\frac12}(v)\langle v\rangle\,\left(\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |F(v')|^2\,|G(v'_*)|^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\right)^{\frac12}.$$

Mutiplying by $\varphi(v)\mathcal{M}^{-1}(v)$ and integrating over $\mathbb{R}^d$, we get

$$\left\langle \mathcal{Q}_\alpha^+(g,f),\varphi\right\rangle_{L^2_v(\mathcal{M}^{-\frac12})} \lesssim \int_{\mathbb{R}^d}\Phi(v)\langle v\rangle\,\left(\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |F(v')|^2\,|G(v'_*)|^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\right)^{\frac12}\mathrm{d}v\,.$$

Using Cauchy-Schwarz inequality again, we obtain

$$\begin{aligned}
\left\langle \mathcal{Q}_\alpha^+(g,f),\varphi\right\rangle_{L^2_v(\mathcal{M}^{-\frac12})} &\lesssim \|\langle\cdot\rangle^{\frac12}\Phi\|_{L^2_v}\left(\int_{\mathbb{R}^d}\langle v\rangle\,\mathrm{d}v\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |F(v')|^2\,|G(v'_*)|^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\right)^{\frac12} \\
&\lesssim \|\langle\cdot\rangle^{\frac12}\Phi\|_{L^2_v}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}}\langle v'\rangle\,|F(v)|^2\,|G(v_*)|^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\mathrm{d}v\right)^{\frac12},
\end{aligned}$$

where we used the pre-post collisional change of variable in the last estimate. Since

$$\langle v'\rangle \lesssim \langle v\rangle + \langle v_*\rangle$$

we deduce that

$$\left\langle \mathcal{Q}_\alpha^+(g,f),\varphi\right\rangle_{L^2_v(\mathcal{M}^{-\frac12})} \lesssim \|\langle\cdot\rangle^{\frac12}\Phi\|_{L^2_v}\left(\|F\|_{L^2_v}\|\langle\cdot\rangle^{\frac12}G\|_{L^2_v} + \|\langle\cdot\rangle^{\frac12}F\|_{L^2_v}\|G\|_{L^2_v}\right),$$

which is exactly (B.1) thanks to (B.3). Notice that (B.1) can be reformulated equivalently as

$$\|\mathcal{Q}_\alpha^\pm(g,f)\|_{L^2_v(\mathcal{M}^{-\frac12}\langle\cdot\rangle^{-\frac12})} \leqslant C\bigg(\|f\|_{L^2_v(\mathcal{M}^{-\frac12}\langle\cdot\rangle^{\frac12})}\|g\|_{L^2_v(\mathcal{M}^{-\frac12})} + \|f\|_{L^2_v(\mathcal{M}^{-\frac12})}\|g\|_{L^2_v(\mathcal{M}^{-\frac12}\langle\cdot\rangle^{\frac12})}\bigg).$$

For functions f, g depending on x, since $\mathbb{W}^{\ell,2}_x(\mathbb{T}^d)$ is a Banach algebra for $\ell > \frac{d}{2}$ and $\mathcal{Q}_\alpha^\pm$ is local in x, we deduce that (B.2) holds true, or equivalently

$$\|\mathcal{Q}_\alpha^\pm(g,f)\|_{L^2_v\mathbb{W}^{\ell,2}_x(\mathcal{M}^{-\frac12}\langle\cdot\rangle^{-\frac12})} \leqslant C\bigg(\|f\|_{\mathcal{H}_1}\|g\|_{\mathcal{H}} + \|f\|_{\mathcal{H}}\|g\|_{\mathcal{H}_1}\bigg). \tag{B.4}$$

This achieves the proof. □

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