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Ricardo J. Alonso, Bertrand Lods, Isabelle Tristani. Fluid dynamic limit of Boltzmann equation for granular hard-spheres in a nearly elastic regime. 2020. hal-02922416v2

HAL Id: hal-02922416

<https://hal.science/hal-02922416v2>

Preprint submitted on 12 Apr 2021

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FLUID DYNAMIC LIMIT OF BOLTZMANN EQUATION FOR GRANULAR HARD-SPHERES IN A NEARLY ELASTIC REGIME

RICARDO ALONSO, BERTRAND LODS, AND ISABELLE TRISTANI

ABSTRACT. In this paper, we provide the first rigorous derivation of hydrodynamic equations from the Boltzmann equation for inelastic hard spheres with small inelasticity. The hydrodynamic system that we obtain is an incompressible Navier-Stokes-Fourier system with self-consistent forcing terms and, to our knowledge, it is thus the first hydrodynamic system that properly describes rapid granular flows consistent with the kinetic formulation. To this end, we write our Boltzmann equation in a non dimensional form using the dimensionless Knudsen number which is intended to be sent to 0. There are several difficulties in such derivation, the first one coming from the fact that the original Boltzmann equation is free-cooling and, thus, requires a self-similar change of variables to introduce an homogeneous steady state. Such a homogeneous state is not explicit and is heavy-tailed, which is a major obstacle to adapting energy estimates and spectral analysis. Additionally, a central challenge is to understand the relation between the restitution coefficient, which quantifies the energy loss at the microscopic level, and the Knudsen number. This is achieved by identifying the correct *nearly elastic regime* to capture nontrivial hydrodynamic behavior. We are, then, able to prove exponential stability uniformly with respect to the Knudsen number for solutions of the rescaled Boltzmann equation in a close to equilibrium regime. Finally, we prove that solutions to the Boltzmann equation converge in a specific weak sense towards a hydrodynamic limit which depends on time and space variables only through macroscopic quantities. Such macroscopic quantities are solutions to a suitable modification of the incompressible Navier-Stokes-Fourier system which appears to be new in this context.

Mathematics Subject Classification (2010): 76P05 Rarefied gas flows, Boltzmann equation [See also 82B40, 82C40, 82D05]; 76T25 Granular flows [See also 74C99, 74E20]; 47H20 Semigroups of nonlinear operators [See also 37L05, 47J35, 54H15, 58D07], 35Q35 PDEs in connection with fluid mechanics; 35Q30 Navier-Stokes equations [See also 76D05, 76D07, 76N10].

Keywords: Inelastic Boltzmann equation; Granular flows; Nearly elastic regime; Long-time asymptotic; Incompressible Navier-Stokes hydrodynamical limit; Knudsen number.

CONTENTS

1. Introduction	3
1.1. Multiscale descriptions of granular gases	3
1.2. The Boltzmann equation for granular gases	5
1.3. Navier-Stokes scaling	7
1.4. Self-similar variable and homogeneous cooling state	8
1.5. Main results	11
1.6. Hydrodynamic limits in the elastic case	13
1.7. The challenge of hydrodynamic limits for granular gases	15
1.8. Notations and definitions	16
1.9. Strategy of the proof	17
1.10. Organization of the paper	21
2. Summary of useful results about the collision operator	21
2.1. The linearized operators \mathcal{L}_α and \mathcal{L}_1	21
2.2. Decomposition of \mathcal{L}_α	26
2.3. The complete linearized operator	27
2.4. The elastic semigroup	32
3. Linear theory in the weakly inelastic regime	34
4. Nonlinear analysis	45
4.1. Estimating h^0	49
4.2. Estimating $\mathbf{P}_0 h^1$	51
4.3. Estimating the complement $(\mathbf{Id} - \mathbf{P}_0)h^1$	54
5. Cauchy Theory	62
5.1. Iteration scheme	62
5.2. Estimating $\ h_{n+1}^0 - h_n^0\ _\mathcal{E}$ and $\ h_{n+1}^1 - h_n^1\ _\mathcal{H}$	64
5.3. Convergence of the iteration scheme	70
6. Hydrodynamic limit	71
6.1. Compactness and convergence	71
6.2. Identification of the limit	79
6.3. About the equations of motion and temperature	81
6.4. Convergence of the nonlinear terms	85
6.5. About the initial conditions	90
6.6. About the original problem in the physical variables	94
Appendix A. About granular gases in the spatial homogeneous setting	95
A.1. The collision operator	95
A.2. Alternative representation of the velocities	96
A.3. Estimates on the collision operator	97
A.4. Dissipativity properties	106

Appendix B. About the original problem in physical variables	109
Appendix C. Tools for the Hydrodynamic limit	111
Appendix D. Proof of Theorem 2.12 and Proposition 2.11	116
References	120

1. INTRODUCTION

The derivation of hydrodynamic models from suitable nonlinear (and possible non conservative) kinetic equations is a challenging problem which has attracted a lot of attention in the recent years. Besides the well-documented literature dealing with the Boltzmann equation (see Section 1.6 hereafter), a large variety of new kinetic models and limiting processes have been considered, spanning from high friction regimes for kinetic models of swarming (see e.g. [Karper et al. \(2015\)](#); [Figalli & Kang \(2019\)](#) for the Cucker-Smale model) to the reaction-diffusion limit for Fitzhugh-Nagumo kinetic equations [Crevat et al. \(2019\)](#). For fluid-kinetic systems, the literature is even more important, we mention simply here the works [Goudon et al. \(2004a,b\)](#) dealing with light or fine particles regimes for the Vlasov-Navier-Stokes system and refer to [Han-Kwan & Michel \(2021\)](#) for the more recent advances on the subject. We also mention the challenging study of gas of charged particles submitted to electro-magnetic forces (Vlasov-Maxwell-Boltzmann system) for which several incompressible fluid limits have been derived recently in the monograph [Arsénio and Saint-Raymond \(2019\)](#).

We consider in the present paper the paradigmatic example of non conservative kinetic equations given by the Boltzmann equation for inelastic hard spheres. In a regime of small inelasticity, we derive in a suitable hydrodynamic limit an incompressible Navier-Stokes-Fourier system with self-consistent forcing terms. This provides, to the best of our knowledge, the first rigorous derivation of hydrodynamic system from kinetic granular flows in physical dimension $d \geq 2$.

1.1. Multiscale descriptions of granular gases. Granular materials are ubiquitous in nature and understanding the behaviour of granular matter is a relevant challenge from both the physics and mathematics viewpoints. Various descriptions of granular matter have been proposed in the literature, see [Garzó \(2019\)](#). An especially relevant one consists in viewing granular systems as clusters of a large number of discrete macroscopic particles (with size exceeding $1 \mu\text{m}$, significantly larger than the one of a typical particle described in classical kinetic theory) suffering dissipative interactions. One speaks then of *rapid granular flows* or *gaseous granular matter*. If the number of particles is large enough, it is then common to adopt a kinetic modelling based upon suitable modification of the Boltzmann equation. As usual in kinetic theory, it is then particularly relevant to deduce from this kinetic description the fluid behaviour of the system. This means, roughly speaking, that we look at the granular gas at a scale larger than the mesoscopic one and aim to

capture the hydrodynamical features of it through the evolution of macroscopic quantities like *density*, *bulk velocity* and *temperature* of the gas which satisfy suitable hydrodynamics equations.

One of the main objects of the present work is to make a first rigorous link between these two co-existing descriptions by deriving a suitable modification of incompressible Navier-Stokes equation from the Boltzmann equation for inelastic hard-spheres as the Knudsen number goes to zero.

Recall that the Knudsen number ε is proportional to the mean free path between collisions and in order to derive hydrodynamic equations from the Boltzmann equation, the usual strategy consists, roughly speaking, in performing a perturbation analysis in the limit $\varepsilon \rightarrow 0$ (meaning that the mean free path is negligible when compared to the typical physical scale length). We point out that these questions are perfectly understood in the elastic case (*molecular gases*) for which rigorous results on the hydrodynamic limits of the Boltzmann equation have been obtained, we refer to the next Section 1.6 for more details and to [Saint-Raymond \(2009a\)](#) for an up-to-date review.

The picture in the context of granular gases is quite different. In fact, a satisfying hydrodynamic equation that properly describes rapid granular flows is still a controversial issue among the physics community. The continuous loss of kinetic energy makes granular gases an *open system* as far as thermodynamics is concerned. Moreover, no non-trivial steady states exist in granular gases without an external energy supply which makes granular gases a prototype of *non-equilibrium* systems. This is an important obstacle in the derivation of hydrodynamical equations from the kinetic description since it is expected that *equilibrium states* play the role of the typical *hydrodynamic solution* where time-space dependence of the single-particle distribution function $F(t, x, v)$ occurs only through suitable hydrodynamic fields like density $\varrho(t, x)$, bulk velocity $u(t, x)$, and temperature $\theta(t, x)$. An additional difficulty is related to the size of particles and scale separation. Recall that granular gases involve *macroscopic particles* whose size is much larger than the one described by the usual Boltzmann equation with elastic interactions referred to as *molecular gases*. As the hydrodynamic description occurs on large time scales (compared to the mean free time) and on large spatial scales (compared to the mean free path) the mesoscopic – continuum scale separation is problematic to justify in full generality for granular gases. We refer to ([Garzó, 2019](#), Section 3.1, p. 102) for more details on this point and observe here that the main concern is related to the time scale induced by the evolution of the temperature (see (1.11) hereafter). In particular, as observed in [Garzó \(2019\)](#), this problem can only be answered with a fine spectral analysis of the linearized Boltzmann equation that ensures that the $d + 2$ hydrodynamic modes associated to density, velocity and temperature decay more slowly than the remaining kinetic excitations at large times. This is the only way that the hydrodynamic excitations emerge as the dominant dynamics. All these physically grounded obstacles make the derivation of hydrodynamic equations from the Boltzmann equation associated to granular gases a reputedly challenging open problem. Quoting [Brey & Dufty \(2005\)](#):

“the context of the hydrodynamic equations remains uncertain. What are the relevant space and time scales? How much inelasticity can be described in this way?”

The present paper is, to the best of our knowledge, the first rigorous answer to these relevant problems, at least in dimension $d \geq 2$. We already mentioned that the key point in our analysis is to identify the correct regime which allows to answer these questions: the *nearly elastic* one. In this regime the energy dissipation rate in the systems happens in a controlled fashion since the inelasticity parameter is compensated accordingly to the number of collisions per time unit. This process mimics viscoelasticity as particle collisions become more elastic as the collision dissipation mechanism increases in the limit $\varepsilon \rightarrow 0$ (see Assumption 1.2 below). In this way, we are able to consider a re-scaling of the kinetic equation in which a peculiar intermediate asymptotic emerges and prevents the total cooling of the granular gas.

Other regimes can be considered depending on the rate at which kinetic energy is dissipated; for example, an interesting regime is the *mono-kinetic* one which considers the extreme case of infinite energy dissipation rate. In this way, the limit is formally described by enforcing a Dirac mass solution in the kinetic equation yielding the *pressureless Euler system* (corresponding to sticky particles). Such a regime has been rigorously addressed in the one-dimensional framework in the interesting contribution Jabin & Rey (2017). It is an open question to extend such analysis to higher dimensions since the approach of Jabin & Rey (2017) uses the so-called Bony functional which is a tool specifically tailored for 1D kinetic equations.

1.2. The Boltzmann equation for granular gases. We consider here the (freely cooling) Boltzmann equation which provides a statistical description of identical smooth hard spheres suffering binary and *inelastic collisions*:

$$\partial_t F(t, x, v) + v \cdot \nabla_x F(t, x, v) = \mathcal{Q}_\alpha(F, F) \quad (1.1)$$

supplemented with initial condition $F(0, x, v) = F_{\text{in}}(x, v)$, where $F(t, x, v)$ is the density of granular particles having position $x \in \mathbb{T}^d$ and velocity $v \in \mathbb{R}^d$ at time $t \geq 0$ and $d \geq 2$. We consider here for simplicity the case of *flat torus*

$$\mathbb{T}_\ell^d = \mathbb{R}^d / (2\pi \ell \mathbb{Z})^d \quad (1.2)$$

for some typical length-scale $\ell > 0$. This corresponds to periodic boundary conditions:

$$F(t, x + 2\pi \ell e_i, v) = F(t, x, v) \quad i = 1, \dots, d$$

where e_i is the i -th vector of the canonical basis of \mathbb{R}^d . The collision operator \mathcal{Q}_α is defined in weak form as

$$\int_{\mathbb{R}^d} \mathcal{Q}_\alpha(g, f)(v) \psi(v) dv = \frac{1}{2} \int_{\mathbb{R}^{2d}} f(v) g(v_*) |v - v_*| \mathcal{A}_\alpha[\psi](v, v_*) dv_* dv, \quad (1.3)$$

where

$$\mathcal{A}_\alpha[\psi](v, v_*) = \int_{\mathbb{S}^{d-1}} (\psi(v') + \psi(v'_*) - \psi(v) - \psi(v_*)) b(\sigma \cdot \hat{u}) d\sigma, \quad (1.4)$$

and the post-collisional velocities (v', v'_*) are given by

$$\begin{aligned} v' &= v + \frac{1+\alpha}{4} (|u|\sigma - u), & v'_* &= v_* - \frac{1+\alpha}{4} (|u|\sigma - u), \\ \text{where } u &= v - v_*, & \hat{u} &= \frac{u}{|u|}. \end{aligned} \quad (1.5)$$

Here, $d\sigma$ denotes the Lebesgue measure on \mathbb{S}^{d-1} and the angular part $b = b(\cos \theta)$ of the collision kernel appearing in (1.4) is a non-negative measurable mapping integrable over \mathbb{S}^{d-1} . There is no loss of generality assuming

$$\int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{u}) d\sigma = 1, \quad \forall \hat{u} \in \mathbb{S}^{d-1}.$$

An additional technical assumption on the angular kernel $b(\cdot)$ will be needed in the sequel, namely, in the rest of the paper, we suppose that there exists $r > 2$ such that

$$\int_{-1}^1 b(s) \left[(1-s)^{\frac{d-3r}{2r}} (1+s)^{\frac{d-3}{2}} + (1+s)^{\frac{d-3r}{2r}} (1-s)^{\frac{d-3}{2}} \right] ds < \infty. \quad (1.6)$$

We just mention here that we need this integral to be finite to get bounds on the bilinear operator \mathcal{Q}_α on L_v^r for $r \geq 2$ (see Theorem A.1). As a consequence, if $b \in L^\infty(\mathbb{S}^{d-1})$ and $d \geq 3$, condition (1.6) holds true for any $r \in [2, 3)$. In particular, our assumption includes the case of hard spheres in dimension $d = 3$.

The fundamental distinction between the classical elastic Boltzmann equation and that associated to granular gases lies in the role of the parameter $\alpha \in (0, 1)$, the *coefficient of restitution*. This coefficient is given by the ratio between the magnitude of the normal component (along the line of separation between the centers of the two spheres at contact) of the relative velocity after and before the collision (see Appendix A.2 for the detailed microscopic velocities). The case $\alpha = 1$ corresponds to perfectly elastic collisions where kinetic energy is conserved. However, when $\alpha < 1$, part of the kinetic energy of the relative motion is lost since

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\frac{1-\alpha^2}{4} |u|^2 (1 - \sigma \cdot \hat{u}) \leq 0. \quad (1.7)$$

It is assumed in this work that α is independent of the relative velocity u (refer to Alonso (2009), Alonso & Lods (2014), and Alonso et al. (2021) for the viscoelastic restitution coefficient case). Notice that the microscopic description (1.5) preserves the momentum

$$v' + v'_* = v + v_*$$

and, taking $\psi = 1$ and then $\psi(v) = v$ in (1.3) yields to the following conservation of macroscopic density and bulk velocity

$$\frac{d}{dt} \mathbf{R}(t) := \frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{T}_\ell^d} F(t, x, v) dv dx = 0, \quad \frac{d}{dt} \mathbf{U}(t) := \frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{T}_\ell^d} v F(t, x, v) dv dx = 0.$$

Consequently, there is no loss of generality in assuming that

$$\mathbf{R}(t) = \mathbf{R}(0) = 1, \quad \mathbf{U}(t) = \mathbf{U}(0) = 0 \quad \forall t \geq 0.$$

As mentioned, the main contrast between elastic and inelastic gases is that in the latter the *granular temperature*

$$\mathbf{T}(t) := \frac{1}{|\mathbb{T}_\ell^d|} \int_{\mathbb{R}^d \times \mathbb{T}_\ell^d} |v|^2 F(t, x, v) dv dx$$

is constantly decreasing

$$\frac{d}{dt} \mathbf{T}(t) = -(1 - \alpha^2) \mathcal{D}_\alpha(F(t), F(t)) \leq 0, \quad \forall t \geq 0.$$

Here $\mathcal{D}_\alpha(g, g)$ denotes the normalised energy dissipation associated to \mathcal{Q}_α , see [Mischler & Mouhot \(2006\)](#), given by

$$\mathcal{D}_\alpha(g, g) := \frac{\gamma_b}{4} \int_{\mathbb{T}_\ell^d} \frac{dx}{|\mathbb{T}_\ell^d|} \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, v) g(x, v_*) |v - v_*|^3 dv dv_* \quad (1.8)$$

with

$$\gamma_b := \int_{\mathbb{S}^{d-1}} \frac{1 - \sigma \cdot \hat{u}}{2} b(\sigma \cdot \hat{u}) d\sigma = |\mathbb{S}^{d-2}| \int_0^\pi b(\cos \theta) (\sin \theta)^{d-2} \sin^2 \left(\frac{\theta}{2} \right) d\theta.$$

In fact, it is possible to show that

$$\lim_{t \rightarrow \infty} \mathbf{T}(t) = 0$$

which expresses the *total cooling of granular gases*. Determining the exact dissipation rate of the granular temperature is an important question known as *Haff's law*, see [Haff \(1983\)](#).

1.3. Navier-Stokes scaling. To capture some hydrodynamic behaviour of the gas, we need to write the above equation in *nondimensional form* introducing the dimensionless Knudsen number

$$\varepsilon := \frac{\text{mean free path}}{\text{spatial length-scale}}$$

which is assumed to be small. We introduce then a rescaling of time and space to capture the hydrodynamic limit and introduce the particle density

$$F_\varepsilon(t, x, v) = F \left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, v \right), \quad t \geq 0. \quad (1.9)$$

In this case, we choose for simplicity $\ell = \varepsilon$ in (1.2) which ensures now that F_ε is defined on $\mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d$ with $\mathbb{T}^d = \mathbb{T}_1^d$. From now on, we assume for simplicity that the torus \mathbb{T}^d is equipped with the normalized Lebesgue measure, i.e. $|\mathbb{T}^d| = 1$. It is well-known that, in the classical elastic case, this scaling leads to the incompressible Navier-Stokes, however, other scalings are possible that yield different hydrodynamic models. Under such a scaling, the typical number of collisions per particle per time unit is ε^{-2} , more specifically, F_ε satisfies the rescaled Boltzmann equation

$$\varepsilon^2 \partial_t F_\varepsilon(t, x, v) + \varepsilon v \cdot \nabla_x F_\varepsilon(t, x, v) = \mathcal{Q}_\alpha(F_\varepsilon, F_\varepsilon), \quad (x, v) \in \mathbb{T}^d \times \mathbb{R}^d, \quad (1.10a)$$

supplemented with initial condition

$$F_\varepsilon(0, x, v) = F_{\text{in}}^\varepsilon(x, v) = F_{\text{in}}\left(\frac{x}{\varepsilon}, v\right). \quad (1.10b)$$

Conservation of mass and density is preserved under this scaling, consequently, we assume that

$$\mathbf{R}_\varepsilon(t) = \int_{\mathbb{R}^d \times \mathbb{T}^d} F_\varepsilon(t, x, v) dv dx = 1, \quad \mathbf{U}_\varepsilon(t) = \int_{\mathbb{R}^d \times \mathbb{T}^d} F_\varepsilon(t, x, v) v dv dx = 0, \quad \forall t \geq 0,$$

whereas the cooling of the granular gas is now given by the equation

$$\frac{d}{dt} \mathbf{T}_\varepsilon(t) = -\frac{1-\alpha^2}{\varepsilon^2} \mathcal{D}_\alpha(F_\varepsilon(t), F_\varepsilon(t)), \quad (1.11)$$

where $\mathbf{T}_\varepsilon(t) = \int_{\mathbb{R}^d \times \mathbb{T}^d} |v|^2 F_\varepsilon(t, x, v) dv dx$.

Remark 1.1. *From now on we will always assume that*

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} F_{\text{in}}^\varepsilon(x, v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv dx = \begin{pmatrix} 1 \\ 0 \\ E_{\text{in}} \end{pmatrix}$$

with $E_{\text{in}} > 0$ fixed and independent of ε . It is important to emphasize that, in the sequel, all the threshold values on ε and the various constants involved are actually depending only on this initial choice.

1.4. Self-similar variable and homogeneous cooling state. Various forcing terms have been added to (1.10a) depending on the underlying physics. Forcing terms prevent the total cooling of the gas (heated bath, thermal bath, see Villani (2006) for details) since they act as an energy supply source to the system and induce the existence of a non-trivial steady state. These are, however, systems different from the free-cooling Boltzmann equation (1.10a) that we aim to investigate here.

To understand better this free-cooling scenario, it is still possible to introduce an intermediate asymptotics and a steady state to work with. This is done by performing a self-similar change of variables

$$F_\varepsilon(t, x, v) = V_\varepsilon(t)^d f_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v), \quad (1.12a)$$

with

$$\tau_\varepsilon(t) := \frac{1}{c_\varepsilon} \log(1 + c_\varepsilon t), \quad V_\varepsilon(t) = (1 + c_\varepsilon t), \quad t \geq 0, \quad c_\varepsilon > 0. \quad (1.12b)$$

With the special choice

$$c_\varepsilon = \frac{1-\alpha}{\varepsilon^2}, \quad (1.12c)$$

we can prove that f_ε satisfies

$$\varepsilon^2 \partial_t f_\varepsilon(t, x, v) + \varepsilon v \cdot \nabla_x f_\varepsilon(t, x, v) + \kappa_\alpha \nabla_v \cdot (v f_\varepsilon(t, x, v)) = \mathcal{Q}_\alpha(f_\varepsilon, f_\varepsilon), \quad (1.13)$$

with initial condition

$$f_\varepsilon(0, x, v) = F_{\text{in}}^\varepsilon(x, v).$$

Here

$$\kappa_\alpha = 1 - \alpha > 0, \quad \forall \alpha \in (0, 1).$$

The underlying drift term $\kappa_\alpha \nabla_v \cdot (vf(t, x, v))$ acts as an energy supply which prevents the total cooling down of the gas. Indeed, it has been shown in a series of papers (Mischler et al. (2006); Mischler & Mouhot (2006, 2009)) that there exists a *spatially homogeneous* steady state G_α to (1.13) which is unique for $\alpha \in (\alpha_0, 1)$ for an explicit threshold value $\alpha_0 \in (0, 1)$. More specifically, for $\alpha \in (\alpha_0, 1)$, there exists a unique solution G_α to the spatially homogeneous steady equation

$$\kappa_\alpha \nabla_v \cdot (vG_\alpha(v)) = \mathcal{Q}_\alpha(G_\alpha, G_\alpha),$$

with

$$\int_{\mathbb{R}^d} G_\alpha(v) dv = 1, \quad \int_{\mathbb{R}^d} G_\alpha(v) v dv = 0.$$

Moreover,

$$\lim_{\alpha \rightarrow 1^-} \|G_\alpha - \mathcal{M}\|_{L^1(\langle v \rangle^2)} = 0, \quad (1.14)$$

where \mathcal{M} is the Maxwellian distribution

$$\mathcal{M}(v) = G_1(v) = (2\pi\vartheta_1)^{-\frac{d}{2}} \exp\left(-\frac{|v|^2}{2\vartheta_1}\right), \quad v \in \mathbb{R}^d, \quad (1.15)$$

for some explicit temperature $\vartheta_1 > 0$. The Maxwellian distribution $\mathcal{M}(v)$ is a steady solution for $\alpha = 1$ and its prescribed temperature ϑ_1 (which ensures (1.14) to hold) will play a role in the rest of the analysis. We refer to Appendix A for more details and explanation of the role of ϑ_1 .

Notice also that the equation in self-similar variables (1.13) preserves mass and *vanishing* momentum. Indeed, a simple computation based on (1.3) gives that

$$\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{T}^d} f_\varepsilon(t, x, v) v dv dx = - \int_{\mathbb{R}^d \times \mathbb{T}^d} f_\varepsilon(t, x, v) v dv dx.$$

Consequently, the assumption made in Remark 1.1 and the fact that G_α has mass 1 and vanishing momentum imply that for any $t \geq 0$, we have

$$\int_{\mathbb{R}^d \times \mathbb{T}^d} f_\varepsilon(t, x, v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv dx = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Three main questions are addressed in this work regarding the solution to (1.13):

(Q1) First, we aim to prove the existence and uniqueness of solutions to (1.13) in a close to equilibrium setting, i.e. solutions which are defined *globally* in time and such that

$$\sup_{t \geq 0} \|f_\varepsilon(t) - G_\alpha\| \leq \delta \quad (1.16)$$

for some positive and explicit $\delta > 0$ in a suitable norm $\|\cdot\|$ of a functional space to be identified. The close-to-equilibrium setting is quite relevant for very small Knudsen numbers given the large number of collisions per unit time which keeps the system thermodynamically relaxed.

- (Q2) More importantly (though closely related), the scope here is to provide estimates on the constructed solutions f_ε which are *uniform with respect to ε* . This means that, in the previous point, $\delta > 0$ is independent of ε . In fact, we are able to prove *exponential time decay* for the difference $\|f_\varepsilon(t) - G_\alpha\|$.
- (Q3) Finally, we aim to prove that, as $\varepsilon \rightarrow 0$, the solution $f_\varepsilon(t)$ converges towards some hydrodynamic solution which depends on (t, x) only through macroscopic quantities $(\varrho(t, x), u(t, x), \theta(t, x))$ which are solutions to a suitable modification of the incompressible Navier-Stokes system.

The central underlying assumption in the previous program is the following relation between the restitution coefficient and the Knudsen number.

Assumption 1.2. *The restitution coefficient $\alpha(\cdot)$ is a continuously decreasing function of the Knudsen number ε satisfying the optimal scaling behaviour*

$$\alpha(\varepsilon) = 1 - \lambda_0 \varepsilon^2 + o(\varepsilon^2) \quad (1.17)$$

with $\lambda_0 \geq 0$.

Indeed, a careful spectral analysis of the linearized collision operator around G_α shows that unless one assumes $1 - \alpha$ comparable to ε^2 the eigenfunction associated to the energy dissipation would explode and prevent (1.16) to hold true. In fact, we require λ_0 to be relatively small with respect to the eigenvalues associated to other kinetic excitations. As mentioned before, in this regime the energy dissipation rate is controlled along time by mimicking a viscoelastic property in the granular gas which is at contrast to other regimes such as the mono-kinetic limit. In viscoelastic models, nearly elastic regimes emerges naturally on large-time scale, see [Bobilev et al. \(2000\)](#); [Alonso & Lods \(2014\)](#); [Alonso et al. \(2021\)](#) for details.

Because $\varepsilon \rightarrow 0$, Assumption 1.2 means that the limit produces a model of the cumulative effect of *nearly elastic* collisions in the *hydrodynamic regime*. Two situations are of interest in our analysis

Case 1: If $\lambda_0 = 0$ the cumulative effect of the inelasticity is too weak in the hydrodynamic scale and the expected model is the classical Navier-Stokes equations. In this case, to ensure that $\bar{\lambda}_\varepsilon > 0$, we need the additional assumption that α satisfies

$$\alpha(\varepsilon) = 1 - \varepsilon^2 \eta(\varepsilon)$$

for some function $\eta(\cdot)$ which is positive on some interval $(0, \bar{\varepsilon})$. This technical assumption will be made in all the sequel.

Case 2: If $0 < \lambda_0 < \infty$, the cumulative effect is visible in the hydrodynamic scale and we expect a model different from the Navier-Stokes equation accounting for that. As we mentioned, we require λ_0 to be relatively small compared to some explicit quantities completely determined by the mass and energy of the initial datum, say, $0 < \lambda_0 \ll 1$ with some explicit upper bounds on λ_0 .

1.5. Main results. The main results are both concerned with the solutions to (1.13). The first one is the following Cauchy theorem regarding the existence and uniqueness of close-to-equilibrium solutions to (1.13). Exact notations for the functional spaces are introduced in Section 1.8.

Theorem 1.3. *Under Assumption 1.2, let*

$$m > d, \quad m - 1 \geq k \geq 0, \quad q \geq 3,$$

be fixed. There exists a triple $(\varepsilon^\dagger, \lambda^\dagger, \mathcal{K}_0^\dagger)$ depending only on the mass and energy of $F_{\text{in}}^\varepsilon$ and m, k, q such that, for $\varepsilon \in (0, \varepsilon^\dagger)$, $\bar{\lambda}_\varepsilon \in (0, \lambda^\dagger)$, and $\mathcal{K}_0 \in (0, \mathcal{K}_0^\dagger)$, if

$$\|F_{\text{in}}^\varepsilon - G_{\alpha(\varepsilon)}\|_{\mathbb{W}_v^{k,1} \mathbb{W}_x^{m,2}(\langle v \rangle^q)} \leq \varepsilon \sqrt{\mathcal{K}_0}$$

then the inelastic Boltzmann equation (1.13) has a unique solution

$$f_\varepsilon \in \mathcal{C}([0, \infty); \mathbb{W}_v^{k,1} \mathbb{W}_x^{m,2}(\langle v \rangle^q))$$

satisfying for $t > 0$

$$\|f_\varepsilon(t) - G_{\alpha(\varepsilon)}\|_{\mathbb{W}_v^{k,1} \mathbb{W}_x^{m,2}(\langle v \rangle^q)} \leq C \varepsilon \sqrt{\mathcal{K}_0} \exp(-\bar{\lambda}_\varepsilon t),$$

$$\text{and } \int_0^t \|f_\varepsilon(\tau) - G_{\alpha(\varepsilon)}\|_{\mathbb{W}_v^{k,1} \mathbb{W}_x^{m,2}(\langle v \rangle^{q+1})} d\tau \leq C \varepsilon \sqrt{\mathcal{K}_0} \min \left\{ 1 + t, 1 + \frac{1}{\bar{\lambda}_\varepsilon} \right\},$$

for some positive constant $C > 0$ independent of ε and where $\bar{\lambda}_\varepsilon \simeq \frac{1-\alpha(\varepsilon)}{\varepsilon^2}$ is the energy eigenvalue of the linearized operator (see Theorem 1.8 hereafter).

Remark 1.4. *We wish to clarify here the role of the parameter λ^\dagger in the above result (and in several similar others in the sequel). Of course, for any choice of ε , the parameter $\bar{\lambda}_\varepsilon$ is fixed as it is the energy eigenvalue. Asking it to be smaller than the threshold value λ^\dagger has to be however understood as a constraint on the parameter λ_0 appearing in (1.17). The above result applies to any $\lambda_0 \geq 0$ such that $\bar{\lambda}_\varepsilon < \lambda^\dagger$.*

Theorem 1.3 completely answers queries (Q1) and (Q2) where the functional space is chosen to be a $L_v^1 L_x^2$ -based Sobolev space

$$\mathbb{W}_v^{k,1} \mathbb{W}_x^{m,2}(\langle v \rangle^q)$$

and the close-to-equilibrium solutions are shown to decay with a rate that can be made uniform with respect to the Knudsen number ε . Recall here that, since Assumption 1.2 is met, the homogeneous cooling state depends on ε and $G_{\alpha(\varepsilon)} \rightarrow \mathcal{M}$ as $\varepsilon \rightarrow 0$.

The estimates on the solution f_ε provided by Theorem 1.3 are enough to answer (Q3). This is done under some additional assumption on the initial datum. Namely

Theorem 1.5. *Under the Assumptions of Theorem 1.3, set*

$$f_\varepsilon(t, x, v) = G_{\alpha(\varepsilon)} + \varepsilon h_\varepsilon(t, x, v),$$

with $h_\varepsilon(0, x, v) = h_{\text{in}}^\varepsilon(x, v) = \varepsilon^{-1} (F_{\text{in}}^\varepsilon - G_{\alpha(\varepsilon)})$ such that

$$\lim_{\varepsilon \rightarrow 0} \|\pi_0 h_{\text{in}}^\varepsilon - h_0\|_{L_v^1 \mathbb{W}_x^{m,2}} = 0,$$

where π_0 stands for the projection over the elastic linearized Boltzmann operator (see Section 6 for a precise definition)

$$h_0(x, v) = (\varrho_0(x) + u_0(x) \cdot v + \frac{1}{2} \theta_0(x) (|v|^2 - d\vartheta_1)) \mathcal{M}(v),$$

with \mathcal{M} being the Maxwellian distribution introduced in (1.15) and

$$(\varrho_0, u_0, \theta_0) \in \mathscr{W}_m,$$

where we set $\mathscr{W}_\ell := \mathbb{W}_x^{\ell,2}(\mathbb{T}^d) \times \left(\mathbb{W}_x^{\ell,2}(\mathbb{T}^d) \right)^d \times \mathbb{W}_x^{\ell,2}(\mathbb{T}^d)$ for any $\ell \in \mathbb{N}$.

Then, for any $T > 0$, $\{h_\varepsilon\}_\varepsilon$ converges in some weak sense to a limit $\mathbf{h} = \mathbf{h}(t, x, v)$ which is such that

$$\mathbf{h}(t, x, v) = \left(\varrho(t, x) + u(t, x) \cdot v + \frac{1}{2} \theta(t, x) (|v|^2 - d\vartheta_1) \right) \mathcal{M}(v), \quad (1.18)$$

where

$$(\varrho, u, \theta) \in \mathcal{C}([0, T]; \mathscr{W}_{m-2}) \cap L^1((0, T); \mathscr{W}_m),$$

is solution to the following incompressible Navier-Stokes-Fourier system with forcing

$$\begin{cases} \partial_t u - \frac{\nu}{\vartheta_1} \Delta_x u + \vartheta_1 u \cdot \nabla_x u + \nabla_x p = \lambda_0 u, \\ \partial_t \theta - \frac{\gamma}{\vartheta_1^2} \Delta_x \theta + \vartheta_1 u \cdot \nabla_x \theta = \frac{\lambda_0 \bar{c}}{2(d+2)} \sqrt{\vartheta_1} \theta, \\ \operatorname{div}_x u = 0, \quad \varrho + \vartheta_1 \theta = 0, \end{cases} \quad (1.19)$$

subject to initial conditions $(\varrho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}})$ given by

$$u_{\text{in}} = u(0) = \mathcal{P}u_0, \quad \theta_{\text{in}} = \theta(0) = \frac{d}{d+2} \theta_0 - \frac{2}{(d+2)\vartheta_1} \varrho_0, \quad \varrho_{\text{in}} = \varrho(0) = -\vartheta_1 \theta_{\text{in}},$$

where $\mathcal{P}u_0$ is the Leray projection of u_0 on divergence-free vector fields. The viscosity $\nu > 0$ and heat conductivity $\gamma > 0$ are explicit and $\lambda_0 > 0$ is the parameter appearing in (1.17). The parameter $\bar{c} > 0$ is depending on the collision kernel $b(\cdot)$.

The precise notion of weak convergence in the above Theorem 1.5 is very peculiar and strongly related to the *a priori* estimates used for the proof of Theorem 1.3. The mode of convergence is detailed in Theorem 6.3, see also Section 6.2 for more details.

It is classical for incompressible Navier-Stokes equations, see (Majda & Bertozzi, 2002, Section 1.8, Chapter I), that the pressure term p acts as a Lagrange multiplier due to the constraint $\operatorname{div}_x u = 0$ and it is recovered (up to a constant) from the knowledge of (ϱ, u, θ) .

We point out that the above incompressible Navier-Stokes-Fourier system (1.19) with the self-consistent forcing terms on the right-hand-side is a new system of hydrodynamic equations that, to our knowledge, has never been rigorously derived earlier to describe granular flows. We also notice that the last two identities in (1.19) give respectively the incompressibility condition and a strong Boussinesq relation (see the discussion in Section 6). It is important to point out that in the case $\lambda_0 = 0$, one recovers the classical incompressible Navier-Stokes-Fourier system derived from elastic Boltzmann equation, see Saint-Raymond (2009a). This proves continuity with respect to the restitution coefficient α .

We finally mention that the above Theorem 1.5 together with the relations (1.12) provide also a quite precise description of the hydrodynamic behaviour of the original problem (1.10a) in physical variables. In this framework, the above mentioned Case 2 for which $\lambda_0 > 0$ enjoys some special features for which uniform-in-time error estimates can be obtained. Turning back to the original problem (1.10a) not only gives a precise answer to *Haff's law* (with an explicit cooling rate of the granular temperature $T_\varepsilon(t)$) but also describes the cooling rate of the *local temperature* $\int_{\mathbb{R}^d} F_\varepsilon(t, x, v) |v|^2 dv$. We refer to Section 6.6 and Appendix B for a more detailed discussion.

1.6. Hydrodynamic limits in the elastic case. The derivation of hydrodynamic limits from the elastic Boltzmann equation is an important problem which received a lot of attention and its origin can be traced back at least to D. Hilbert exposition of its 6th problem at the 1900 International Congress of Mathematicians. We refer the reader to Saint-Raymond (2009a); Golse (2014) for an up-to-date description of the mathematically relevant results in the field. Roughly speaking three main approaches are adopted for the rigorous derivation of hydrodynamic limits.

A) Many of the early mathematical justifications of hydrodynamic limits of the Boltzmann equation are based on (truncated) asymptotic expansions of the solution around some hydrodynamic solution

$$F_\varepsilon(t, x, v) = F_0(t, x, v) \left(1 + \sum_n \varepsilon^n F_n(t, x, v) \right) \quad (1.20)$$

where, typically

$$F_0(t, x, v) = \frac{\varrho(t, x)}{(2\pi\theta(t, x))^{\frac{d}{2}}} \exp\left(-\frac{|v - u(t, x)|^2}{2\theta(t, x)}\right) \quad (1.21)$$

is a *local Maxwellian* associated to the macroscopic fields which is required to satisfy the limiting fluid dynamic equation. This approach (or a variant of it based upon Chapman-Enskog

expansion) leads to the first rigorous justification of the compressible Euler limit up to the first singular time for the solution of the Euler system in Caffisch (1980) (see also Lachowicz (1987) for more general initial data and a study of initial layers). In the same way, a justification of the incompressible Navier-Stokes limit has been obtained in De Masi et al. (1989). This approach deals mainly with strong solutions for both the kinetic and fluid equations.

- B) Another important line of research concerns weak solutions and a whole program on this topic has been introduced in Bardos et al. (1991, 1993). The goal is to prove the convergence of the renormalized solutions to the Boltzmann equation (as obtained in Di Perna & Lions (1990)) towards weak solutions to the compressible Euler system or to the incompressible Navier-Stokes equations. This program has been continued exhaustively and the convergence have been obtained in several important results (see Golse & Saint-Raymond (2004, 2009); Jiang & Masmoudi (2017); Levermore & Masmoudi (2010); Lions & Masmoudi (2001a,b) to mention just a few). We remark that, in the notion of renormalized solutions for the classical Boltzmann equation, a crucial role is played by the entropy dissipation (H -theorem) which asserts that the entropy of solutions to the Boltzmann equation is non increasing

$$\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{T}^d} F_\varepsilon \log F_\varepsilon(t, x, v) dv dx \leq 0.$$

This *a priori* estimate is fully exploited in the construction of renormalized solutions to the classical Boltzmann equation and is also fundamental in some justification arguments for the Euler limit, see Saint-Raymond (2009b).

- C) A third line of research deals with strong solutions close to equilibrium and exploits a careful spectral analysis of the linearized Boltzmann equation. Strong solutions to the Boltzmann equation close to equilibrium have been obtained in a weighted L^2 -framework in the work Ukai (1974) and the *local-in-time* convergence of these solutions towards solution to the compressible Euler equations have been derived in Nishida (1978). For the limiting incompressible Navier-Stokes solution, a similar result have been carried out in Bardos & Ukai (1991) for smooth global solutions in \mathbb{R}^3 with a small initial velocity field. The smallness assumption has been recently removed in Gallagher & Tristani (2020) allowing to treat also non global in time solutions to the Navier-Stokes equation. These results as well as Briant (2015) exploit a very careful description of the spectrum of the linearized Boltzmann equation derived in Ellis & Pinsky (1975). We notice that they are framed in the space $L^2(\mathcal{M}^{-1})$ where the linearized Boltzmann operator is self-adjoint and coercive. The fact that the analysis of Ellis & Pinsky (1975) has been extended recently in Gervais (2021) to larger functional spaces of the type $L_v^2(\langle \cdot \rangle^q)$ opens the gate to some refinements of several of the aforementioned results.

We mention in particular two papers whose approaches are the closest to the one adopted here. The work Briant et al. (2019) was the main inspiration to answer questions (Q1)-(Q2). Indeed, in Briant et al. (2019), the first estimates on the elastic Boltzmann equation in Sobolev spaces with polynomial weight (based on L^1) are obtained *uniformly with respect to the Knudsen number*

ε . Also, the work [Jiang et al. \(2018\)](#) deals with an energy method in $L^2(\mathcal{M}^{-1})$ spaces (see also [Guo et al. \(2010\)](#); [Guo \(2016\)](#)) in order to prove the *strong convergence* of the solutions to the Boltzmann equation towards the incompressible Navier-Stokes equation without resorting to the work of [Ellis & Pinsky \(1975\)](#). We adopt a similar strategy to answer **(Q3)**.

1.7. The challenge of hydrodynamic limits for granular gases. There are several reasons which make the derivation of hydrodynamic limits for granular gases a challenging question at the physical level. In regard of the mathematical aspects of the hydrodynamical limit, several hurdles stand on way when trying to adapt the aforementioned approaches:

- I) With respect to the strategy given in A), the main difficulty lies in the identification of the typical *hydrodynamic solution*. Such solution is such that the time-space dependence of the one-particle distribution function $F(t, x, v)$ occurs only through suitable hydrodynamic fields like density $\varrho(t, x)$, bulk velocity $u(t, x)$, and temperature $\theta(t, x)$. This is the role played by the Maxwellian F_0 in (1.21) whenever $\alpha = 1$ and one wonders if the homogeneous cooling state G_α plays this role here. This is indeed the case up to first order capturing the fat tails of inelastic distributions, yet surprisingly, a suitable Maxwellian plays the role of the hydrodynamic solution in the ε -order correction. This Gaussian behaviour emerges in the *hydrodynamic limit* because of the near elastic regime that we treat here.¹
- II) The direction promoted in B) appears for the moment out of reach in the context of granular gases. Renormalized solutions in the context of the inelastic Boltzmann equation (1.22) have not been obtained due to the lack of an H -Theorem for granular gases. It is unclear if the classical entropy (or a suitable modification of it) remains bounded in general for granular gases.
- III) Homogeneous cooling states G_α are not explicit, this is a technical difficulty when adapting the approach of [Ellis & Pinsky \(1975\)](#) for the spectral analysis of the linearized inelastic Boltzmann equation in the spatial Fourier variable. Partial interesting results have been obtained in [Rey \(2013\)](#) (devoted to diffusively heated granular gases) but they do not give a complete asymptotic expansion of eigenvalues and eigenfunctions up to the order leading to the Navier-Stokes asymptotic. We mention that obtaining an analogue of the work [Ellis & Pinsky \(1975\)](#) for granular gases would allow, in particular, to quantify the convergence rate towards the limiting model as in the recent work [Gallagher & Tristani \(2020\)](#).
- IV) A major obstacle to adapt energy estimates and spectral approach lies in the choice of functional spaces. While the linearized Boltzmann operator associated to elastic interactions is self-adjoint and coercive in the weighted L^2 -space $L^2_v(\mathcal{M}^{-1})$, there is no such “self-adjoint” space for the inelastic case. This yields technical difficulties in the study of the spectral

¹See the interesting discussion in [Villani \(2006\)](#), especially the Section 2.8 entitled “What Is the Trouble with Non-Gaussianity”

analysis of the linearized operator ². Moreover, the energy estimates of Guo et al. (2010); Guo (2016); Jiang & Masmoudi (2017); Jiang et al. (2018) are essentially based upon the coercivity of the linearized operator. For granular gases, it seems that one needs to face the problem *directly* in a L_v^1 -setting. Points III) and IV) make the approach C) difficult to directly adapt.

1.8. Notations and definitions. We first introduce some useful notations for function spaces. For any nonnegative weight function $m : \mathbb{R}^d \rightarrow \mathbb{R}^+$ (notice that all the weights we consider here will depend only on velocity, i.e. $m = m(v)$), we define $L_v^q L_x^p(m)$, $1 \leq p, q \leq +\infty$, as the Lebesgue space associated to the norm

$$\|h\|_{L_v^q L_x^p(m)} = \| \|h(\cdot, v)\|_{L_x^p} m(v) \|_{L_v^q}.$$

We also consider the standard higher-order Sobolev generalizations $\mathbb{W}_v^{\sigma, q} \mathbb{W}_x^{s, p}(m)$ for any $\sigma, s \in \mathbb{N}$ defined by the norm

$$\|h\|_{\mathbb{W}_v^{\sigma, q} \mathbb{W}_x^{s, p}(m)} = \sum_{\substack{0 \leq s' \leq s, 0 \leq \sigma' \leq \sigma, \\ s' + \sigma' \leq \max(s, \sigma)}} \| \| \nabla_x^{s'} \nabla_v^{\sigma'} h(\cdot, v) \|_{L_x^p} m(v) \|_{L_v^q}.$$

This definition reduces to the usual weighted Sobolev space $\mathbb{W}_{v, x}^{s, p}(m)$ when $q = p$ and $\sigma = s$. For $m \equiv 1$, we simply denote the associated spaces by $L_v^q L_x^p$ and $\mathbb{W}_v^{\sigma, q} \mathbb{W}_x^{s, p}$.

We consider in the sequel the general weight

$$\varpi_s(v) = (1 + |v|^2)^{\frac{s}{2}}, \quad v \in \mathbb{R}^d, \quad s \geq 0.$$

On the complex plane, for any $a \in \mathbb{R}$, we set

$$\mathbb{C}_a := \{z \in \mathbb{C} ; \operatorname{Re} z > -a\}, \quad \mathbb{C}_a^* := \mathbb{C}_a \setminus \{0\}$$

and, for any $r > 0$, we set

$$\mathbb{D}(r) = \{z \in \mathbb{C} ; |z| \leq r\}.$$

We also introduce the following notion of hypo-dissipativity in a general Banach space.

Definition 1.6. *Let $(X, \|\cdot\|)$ be a given Banach space. A closed (unbounded) linear operator $A : \mathcal{D}(A) \subset X \rightarrow X$ is said to be hypo-dissipative on X if there exists a norm, denoted by $\| \cdot \|$, equivalent to the $\|\cdot\|$ -norm such that A is dissipative on the space $(X, \| \cdot \|)$, that is,*

$$\|(\lambda - A)h\| \geq \lambda \|h\|, \quad \forall \lambda > 0, \quad h \in \mathcal{D}(A).$$

²Recall that the powerful enlargement techniques for the elastic Boltzmann equation are based on the *knowledge* of the spectral structure in the space $L^2(\mathcal{M}^{-1})$ (and Sobolev spaces built on it) which can be extended to the more natural L^1 -setting.

Remark 1.7. This is equivalent to the following (see Proposition 3.23, p. 88 in Engel & Nagel (1999)): if $\|\cdot\|_\star$ denotes the norm on the dual space X^\star , for all $h \in \mathcal{D}(A)$, there exists $\mathbf{u}_h \in X^\star$ such that

$$[\mathbf{u}_h, h] = \|h\|^2 = \|\mathbf{u}_h\|_\star^2 \quad \text{and} \quad \text{Re}[\mathbf{u}_h, Ah] \leq 0,$$

where $[\cdot, \cdot]$ denotes the duality bracket between $(X^\star, \|\cdot\|_\star)$ and $(X, \|\cdot\|)$.

For two tensors $A = (A_{i,j}), B = (B_{i,j}) \in \mathcal{M}_d(\mathbb{R})$, we denote by $A : B$ the scalar $(A : B) = \sum_{i,j} A_{i,j} B_{i,j} \in \mathbb{R}$ as the trace of the matrix product AB whereas, for a vector function $w = w(x) \in \mathbb{R}^d$, the tensor $(\partial_{x_i} w_j)_{i,j}$ is denoted as $\nabla_x w$. We also write $(\text{Div}_x A)^i = \sum_j \partial_{x_j} A_{i,j}(x)$.

1.9. Strategy of the proof. The strategy used to prove the main results Theorems 1.3 and 1.5 yields to several intermediate results of independent interest. The approach is perturbative in essence since we are dealing with close-to-equilibrium solutions to (1.13). This means that, in the study of (1.13), we introduce the fluctuation h_ε around the equilibrium G_α defined through

$$f_\varepsilon(t, x, v) = G_\alpha(v) + \varepsilon h_\varepsilon(t, x, v),$$

and h_ε satisfies

$$\begin{cases} \partial_t h_\varepsilon(t, x, v) + \frac{1}{\varepsilon} v \cdot \nabla_x h_\varepsilon(t, x, v) - \frac{1}{\varepsilon^2} \mathcal{L}_\alpha h_\varepsilon(t, x, v) = \frac{1}{\varepsilon} \mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon)(t, x, v), \\ h_\varepsilon(0, x, v) = h_\varepsilon^{\text{in}}(x, v), \end{cases} \quad (1.22)$$

where \mathcal{L}_α is the linearized collision operator (local in the x -variable) defined as

$$\mathcal{L}_\alpha h(x, v) = \mathbf{L}_\alpha(h)(x, v) - \kappa_\alpha \nabla_v \cdot (vh(x, v)),$$

with

$$\mathbf{L}_\alpha(h) = 2\tilde{\mathcal{Q}}_\alpha(G_\alpha, h),$$

where we set

$$\tilde{\mathcal{Q}}_\alpha(f, g) = \frac{1}{2} \{ \mathcal{Q}_\alpha(f, g) + \mathcal{Q}_\alpha(g, f) \}. \quad (1.23)$$

We also denote by \mathcal{L}_1 the linearized operator around $G_1 = \mathcal{M}$, that is,

$$\mathcal{L}_1(h) = \mathbf{L}_1(h) = \mathcal{Q}_1(\mathcal{M}, h) + \mathcal{Q}_1(h, \mathcal{M}).$$

The method of proof requires first a careful spectral analysis of the full linearized operator appearing in (1.22):

$$\mathcal{G}_{\alpha,\varepsilon} h := -\varepsilon^{-1} v \cdot \nabla_x h + \varepsilon^{-2} \mathcal{L}_\alpha h.$$

Such a spectral analysis has to be performed in a suitable Sobolev space, our starting point is an enlargement argument developed in Gualdani et al. (2017) (that has then extended to the case $\varepsilon \neq 1$ in Briant et al. (2019)) to study the spectrum of the elastic operator in a large class of Sobolev spaces.

More precisely, in our approach, we treat $\mathcal{G}_{\alpha,\varepsilon}$ as a *perturbation*³ of the elastic linearized operator $\mathcal{G}_{1,\varepsilon}$. The spectrum of $\mathcal{G}_{1,\varepsilon}$ in $\mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\langle v \rangle^q)$ is well-understood [Briant et al. \(2019\)](#), so, it is possible to deduce from this characterisation the spectrum of $\mathcal{G}_{\alpha,\varepsilon}$ using ideas from [Tristani \(2016\)](#). We *only* study the spectrum of $\mathcal{G}_{\alpha,\varepsilon}$ without requiring knowledge of the decay of the semigroup associated to $\mathcal{G}_{\alpha,\varepsilon}$. This simplifies the technicalities of the spectral analysis performed in [Section 3](#) related to Dyson-Phillips iterates which leads to the spectral mapping theorem [Gualdani et al. \(2017\)](#); [Tristani \(2016\)](#). Most notably, in this simplified approach one is able to identify the optimal scaling [\(1.17\)](#) of the restitution coefficient. It is worth mentioning that capturing the optimal scaling [\(1.17\)](#) for this linear and spectral approach yields to rather involved and technical analysis with the introduction of several kinds of function spaces of type $\mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\langle v \rangle^q)$ and $\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\langle v \rangle^q)$ and the properties of the linearized operator on each of those spaces.

The scaling [\(1.17\)](#) is precisely the one which allows to preserve exactly $d + 2$ eigenvalues in the neighbourhood of zero (recall that 0 is an eigenvalue of multiplicity $d + 2$ in the elastic case). Recalling that, in any reasonable space, the elastic operator has a spectral gap of size $\mu_\star > 0$, i.e.

$$\mathfrak{S}(\mathcal{G}_{1,\varepsilon}) \cap \{z \in \mathbb{C} ; \operatorname{Re} z > -\mu_\star\} = \{0\}$$

where 0 is an eigenvalue of algebraic multiplicity $d + 2$ which is associated to the eigenfunctions $\{\mathcal{M}, v_j \mathcal{M}, |v|^2 \mathcal{M}, j = 1, \dots, d\}$, one can prove the following theorem

Theorem 1.8. *Assume that Assumption 1.2 is met and consider the Banach space*

$$\mathcal{E} := \begin{cases} \mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q), & \ell \in \mathbb{N}, s \geq 0, \ell \geq s, \quad q > q^\star, \\ \text{or} \\ \mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\varpi_q), & \ell \in \mathbb{N}, s \geq 0, \ell \geq s, \quad q > 2, \end{cases}$$

where q^\star is defined in [\(2.17\)](#). There exists some explicit $\nu_\star > 0$ such that, if $\mu \in (\mu_\star - \nu_\star, \mu_\star)$, there is some explicit $\bar{\varepsilon} > 0$ depending only on $\mu_\star - \mu$ and such that, for all $\varepsilon \in (0, \bar{\varepsilon})$, the linearized operator

$$\mathcal{G}_{\alpha,\varepsilon} : \mathcal{D}(\mathcal{G}_{\alpha,\varepsilon}) \subset \mathcal{E} \rightarrow \mathcal{E}$$

has the spectral property:

$$\mathfrak{S}(\mathcal{G}_{\alpha,\varepsilon}) \cap \{z \in \mathbb{C} ; \operatorname{Re} z \geq -\mu\} = \{\lambda_1(\varepsilon), \dots, \lambda_{d+2}(\varepsilon)\}, \quad (1.24)$$

where $\lambda_1(\varepsilon), \dots, \lambda_{d+2}(\varepsilon)$ are eigenvalues of \mathcal{G}_ε (not necessarily distinct) with

$$|\lambda_j(\varepsilon)| \leq \mu_\star - \mu \quad \text{for } j = 1, \dots, d + 2.$$

³This perturbation does not fall into the realm of the classical perturbation theory of unbounded operators as described in [Kato \(1980\)](#). Typically, the domain of $\mathcal{G}_{\alpha,\varepsilon}$ is much smaller than the one of $\mathcal{G}_{1,\varepsilon}$ (because of the drift term in velocity) and the relative bound between $\mathcal{G}_{1,\varepsilon}$ and $\mathcal{G}_{\alpha,\varepsilon}$ does not converges to zero in the elastic limit $\alpha \rightarrow 1$.

More precisely, it follows that

$$\begin{aligned} \mathfrak{S}(\mathcal{G}_{\alpha,\varepsilon}) \cap \{z \in \mathbb{C} ; \operatorname{Re} z \geq -\mu\} &= \mathfrak{S}(\varepsilon^{-2} \mathcal{L}_\alpha) \cap \{z \in \mathbb{C} ; \operatorname{Re} z \geq -\mu\} \\ &= \{\lambda_1(\varepsilon), \dots, \lambda_{d+2}(\varepsilon)\}, \end{aligned}$$

with

$$\lambda_1(\varepsilon) = 0, \quad \lambda_j(\varepsilon) = \varepsilon^{-2} \kappa_{\alpha(\varepsilon)}, \quad j = 2, \dots, d+1,$$

and

$$\lambda_{d+2}(\varepsilon) = -\bar{\lambda}_\varepsilon = -\frac{1 - \alpha(\varepsilon)}{\varepsilon^2} + \mathcal{O}(\varepsilon^2), \quad \text{for } \varepsilon \simeq 0. \quad (1.25)$$

To prove Theorem 1.8, it is necessary to strengthen several results of Mischler & Mouhot (2009) and obtain *sharp* convergence rate in the elastic limit for the linearized operator. Typically, one needs to prove that, for suitable topology

$$\mathcal{L}_\alpha - \mathcal{L}_1 \simeq (1 - \alpha) \quad (1.26)$$

which gives an estimate of the type

$$(\mathcal{G}_{\alpha,\varepsilon} - \mathcal{G}_{1,\varepsilon}) \simeq \frac{1 - \alpha}{\varepsilon^2}.$$

Our aim is of course to capture the properties of $\mathcal{G}_{\alpha,\varepsilon}$ in space built on L_v^1 but, for technical reasons, we will also need to investigate the above ansatz (1.26) in some L_v^2 -spaces with polynomial moments. This is done in Sections 2 and 3.

After the spectral analysis is performed, in order to prove Theorem 1.3 several *a priori estimates* for the solutions to (1.13) are required. This is done in Section 4. The crucial point in the analysis lies in the splitting of (1.13) into a system of two equations mimicking a spectral enlargement method from a PDE perspective (see the Section 2.3 of Mischler & Mouhot (2016) and Briant et al. (2019) for pioneering ideas on such a splitting). More precisely, the splitting performed in Sections 4 and 5 amounts to look for a solution of (1.22) of the form

$$h_\varepsilon(t) = h_\varepsilon^0(t) + h_\varepsilon^1(t)$$

where $h_\varepsilon^1(t)$ is solution to the *linearized elastic equation* with a source term involving the reminder $h_\varepsilon^0(t)$, namely,

$$\partial_t h_\varepsilon^1(t) = \mathcal{G}_{1,\varepsilon} h_\varepsilon^1 + \varepsilon^{-1} \mathcal{Q}_1(h_\varepsilon^1, h_\varepsilon^1) + \mathcal{A}_\varepsilon h_\varepsilon^0 \quad (1.27)$$

having zero initial datum and where \mathcal{A}_ε is a regularizing operator (see Section 3 for a precise definition). In this way we seek $h_\varepsilon^1(t)$ in the Hilbert space

$$h_\varepsilon^1(t) \in \mathbb{W}_{v,x}^{m,2} \left(\mathcal{M}^{-1/2} \right) =: \mathcal{H}$$

with $m > d$ and prove bounds of the type

$$\sup_{t \geq 0} \left(\|h_\varepsilon^1(t)\|_{\mathcal{H}}^2 + \int_0^t \|h_\varepsilon^1(\tau)\|_{\mathcal{H}_1}^2 d\tau \right) \leq CK_0$$

where \mathcal{H}_1 is the domain of $\mathcal{G}_{1,\varepsilon}$ in \mathcal{H} , \mathcal{K}_0 depends only on the initial datum $h_{\text{in}}^\varepsilon$. With such a splitting, it is possible to fully exploit the elastic problem and treat h_ε as a perturbation of this solution. It is important to point out already that \mathcal{A}_ε is regularizing only in the velocity variable but not in the x -variable. Therefore, no gain of integrability can be deduced from the action of \mathcal{A}_ε . Therefore, since we look for $h_\varepsilon^1 \in \mathcal{H}$ in (1.27), we need to look for h_ε^0 in a space based on L_x^2 . The velocity regularization properties of \mathcal{A}_ε allow then to look for

$$h_\varepsilon^0(t) \in \mathbb{W}_v^{k,1} \mathbb{W}_x^{m,2}(\varpi_q), \quad t \geq 0.$$

This is the role of Section 4. Of course, to study the equation solved then by $h_\varepsilon^0(t)$, a careful study of the linearized operator on spaces $\mathbb{W}_v^{k,1} \mathbb{W}_x^{m,2}(\varpi_q)$ is necessary (see Sections 2 and 3) yielding highly technical additional difficulties.

In Section 5, we prove Theorem 1.3 introducing a suitable iterative scheme based upon the coupling $(h_\varepsilon^0(t), h_\varepsilon^1(t))$. We show in practice that the coupled system of kinetic equations satisfied by h^0 and h^1 is well-posed. It is fair to say that the bounds for h_ε^0 and h_ε^1 given in Sections 4 and 5 play the role of suitable energy estimates as the ones established in the purely Hilbert setting Guo et al. (2010); Guo (2016); Jiang et al. (2018). In particular, these bounds are sufficient to deduce a very peculiar type of weak convergence of $h_\varepsilon(t)$ towards an element in the kernel of the linearized operator \mathcal{L}_1 , in particular, the limit of h_ε is necessarily of the form (1.18). The notion of weak convergence we use here fully exploits the splitting $h_\varepsilon = h_\varepsilon^0 + h_\varepsilon^1$ where we prove that h_ε^0 converges to 0 *strongly* in $L^1((0, T); L_v^1 \mathbb{W}_x^{m,2}(\varpi_q))$ whereas h_ε^1 converges to h *weakly* in $L^2((0, T); L_v^2 \mathbb{W}_x^{m,2}(\mathcal{M}^{-\frac{1}{2}}))$.

Finally, in Section 6, the regularity of (ϱ, u, θ) obtained via a simple use of Ascoli-Arzelà Theorem and the identification of the limiting equations these macroscopic fields satisfy is presented. With the notion of weak convergence at hand presented above, the approach is simpler but reminiscent of the program established in Bardos et al. (1991, 1993). In particular, we can adapt some of the main ideas of Golse & Saint-Raymond (2004) regarding the delicate convergence of nonlinear convection terms. Detailed computations are included to make the paper as much self-contained as possible also because, even in the classical “elastic” case, it is difficult to find a full proof of the convergence towards hydrodynamic limit for the weak solutions we consider here. For such solutions, details of proof are scattered in the literature and full proof of the convergence of nonlinear terms is sometimes only sketched where most of the full detailed proofs are dealing with the more delicate case of renormalized solutions Golse & Saint-Raymond (2004, 2009); Levermore & Masmoudi (2010). In our framework, the terms involving the quadratic operator $\mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon)$ are treated as source terms which converge in distributions to zero whereas the drift term and the dissipation of energy function \mathcal{D}_α are the objects responsible for the terms in the right-side of the Navier-Stokes system (1.19). We also observe that the derivation of the strong Boussinesq relation is not as straightforward as in the elastic case. Actually, the classical

Boussinesq relation

$$\nabla (\varrho(t, x) + \vartheta_1 \theta(t, x)) = 0$$

is established as in the elastic case. In the elastic case, this relation implies the strong form of Boussinesq relation mainly because the two functions $\varrho(t, x)$ and $\theta(t, x)$ have zero spatial averages. This cannot be deduced directly in the granular context due to the dissipation of energy.

1.10. Organization of the paper. The paper is divided into 6 Sections and three Appendices. In the following Section 2, we collect several results regarding the collision operator \mathcal{L}_α and introduce the splitting of the operator in $\mathcal{L}_\alpha = \mathcal{A}_\alpha + \mathcal{B}_\alpha$ as well as the splitting of the full linearized operator $\mathcal{G}_{\alpha,\varepsilon}$. As mentioned, even if our final goal is to study the collision operator in spaces built on L_v^1 -spaces with polynomial weights, we shall also need to resort to estimates of \mathcal{L}_α in L_v^2 -spaces. Section 3 is devoted to the spectral analysis of $\mathcal{G}_{\alpha,\varepsilon}$ culminating with the proof of Theorem 1.8. In Section 4, we derive the fundamental *a priori* estimates on the close-to-equilibrium solutions to (1.22). It is the most technical part of the work and fully exploits the splitting of the operator $\mathcal{G}_{\alpha,\varepsilon}$ as explained earlier. Section 5 gives the proof of Theorem 1.3 whereas Section 6 gives the full proof of the hydrodynamic limit (Theorem 1.5). In Appendix A, we recall some facts about the granular Boltzmann equation and gives the full proof of a technical result of Section 2. Appendix B deals with some properties of the solution to (1.10a), i.e. dealing with the original variables, and provides some insights about the local version of Haff’s law. In Appendix C, we collect some well-known properties useful for the hydrodynamic limit as well as some technical proofs used in Section 6. Finally, Appendix D gives the proof of two technical results of Section 2.

Acknowledgements. RA gratefully acknowledges the support from O Conselho Nacional de Desenvolvimento Científico e Tecnológico, Bolsa de Produtividade em Pesquisa - CNPq (303325/2019-4). BL gratefully acknowledges the financial support from the Italian Ministry of Education, University and Research (MIUR), “Dipartimenti di Eccellenza” grant 2018-2022. Part of this research was performed while the second author was visiting the “Département de Mathématiques et Applications,” at École Normale Supérieure, Paris in February 2019. He wishes to express his gratitude for the financial support and warm hospitality offered by this Institution. IT thanks the ANR EFI: ANR-17-CE40-0030 and the ANR SALVE: ANR-19-CE40-0004 for their support. We thank Isabelle Gallagher for stimulating discussions and precious advices.

2. SUMMARY OF USEFUL RESULTS ABOUT THE COLLISION OPERATOR

2.1. The linearized operators \mathcal{L}_α and \mathcal{L}_1 . In all the sequel, we will use well-known estimates for the bilinear operator $\mathcal{Q}_\alpha(f, g)$ and $\mathcal{Q}_1(f, g)$ in several different functional spaces. We refer to Alonso et al. (2010); Alonso & Gamba (2011); Mischler & Mouhot (2009) for precise statements. A crucial role in our analysis will be played by the fact that, in some suitable sense, \mathcal{L}_α is close to the elastic linearized operator \mathcal{L}_1 for $\alpha \simeq 1$. Let us begin with the following crucial result which also justifies the optimal scaling (1.17) and optimise the rate of convergence previously derived in

(Mischler & Mouhot, 2009, Proposition 3.1 (iii)) for weights different to the ones considered here. The technical proof is postponed to Appendix A.

Lemma 2.1. *Let $a = \max\{d - 1, 2\}$ and $\kappa > \frac{d}{2}$. For $k \in \mathbb{N}$ and $q \geq 0$, there is a positive constant $c_{k,q} > 0$ such that for any $\alpha \in (0, 1]$,*

$$\begin{aligned} \|\mathcal{Q}_1(g, f) - \mathcal{Q}_\alpha(g, f)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} &\leq c_{k,q} \frac{1-\alpha}{\alpha^a} \left(\|f\|_{\mathbb{W}_v^{k,1}(\varpi_{q+2})} \|g\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+2})} \right. \\ &\quad \left. + \|g\|_{\mathbb{W}_v^{k,1}(\varpi_{q+2})} \|f\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+2})} \right) \end{aligned}$$

and a constant $c_{k,q}(\kappa) > 0$ such that, for any $\alpha \in (0, 1]$,

$$\begin{aligned} \|\mathcal{Q}_1(g, f) - \mathcal{Q}_\alpha(g, f)\|_{\mathbb{W}_v^{k,2}(\varpi_q)} &\leq c_{k,q} \frac{1-\alpha}{\alpha^a} \left(\|f\|_{\mathbb{W}_v^{k,2}(\varpi_{q+\kappa+2})} \|g\|_{\mathbb{W}_v^{k+1,2}(\varpi_{q+\kappa+2})} \right. \\ &\quad \left. + \|g\|_{\mathbb{W}_v^{k,2}(\varpi_{q+\kappa+2})} \|f\|_{\mathbb{W}_v^{k+1,2}(\varpi_{q+\kappa+2})} \right). \end{aligned}$$

By using the previous lemma and estimates on the difference between G_α and \mathcal{M} , one can get an estimate on the difference between \mathbf{L}_1 and \mathbf{L}_α with loss of regularity on the argument (see the first part of Proposition 2.4 below). In our analysis, we will also need an estimate on the difference between \mathbf{L}_1 and \mathbf{L}_α with no loss of regularity (i.e. an estimate in the *graph norm*), even if the rate is not anymore optimal. To this end, we here state a lemma which is in the spirit of (Mischler & Mouhot, 2009, Proposition 3.2) except from the fact that one of the argument is fixed to be the Maxwellian \mathcal{M} . Note that (Mischler & Mouhot, 2009, Proposition 3.2) gives an estimate on the difference between \mathcal{Q}_1 and \mathcal{Q}_α for general arguments but the proof heavily relies on the exponential weights considered. It turns out that we can not adapt easily the proof of (Mischler & Mouhot, 2009, Proposition 3.2) for polynomial weights. However, by using decay properties of \mathcal{M} , we are able, to get an estimate on $\mathcal{Q}_1(\mathcal{M}, \cdot) - \mathcal{Q}_\alpha(\mathcal{M}, \cdot)$ and its symmetric, which is enough for our purpose. The proof is also postponed to Appendix A.

Lemma 2.2. *Let $q \geq 0$ and $k \in \mathbb{N}$. There exist some explicit $\tilde{c}_q > 0$, $\tilde{c}_{k,q} > 0$, $p \in (0, 1)$ and $\alpha_1 \in (0, 1)$ such that*

$$\begin{aligned} \|\mathcal{Q}_1(\mathcal{M}, f) - \mathcal{Q}_\alpha(\mathcal{M}, f)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \\ + \|\mathcal{Q}_1(f, \mathcal{M}) - \mathcal{Q}_\alpha(f, \mathcal{M})\|_{\mathbb{W}_v^{k,1}(\varpi_q)} &\leq \tilde{c}_{k,q} (1-\alpha)^p \|f\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})}, \quad \alpha \in [\alpha_1, 1], \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{Q}_1(\mathcal{M}, f) - \mathcal{Q}_\alpha(\mathcal{M}, f)\|_{\mathbb{W}_v^{k,2}(\varpi_q)} \\ + \|\mathcal{Q}_1(f, \mathcal{M}) - \mathcal{Q}_\alpha(f, \mathcal{M})\|_{\mathbb{W}_v^{k,2}(\varpi_q)} &\leq \tilde{c}_{k,q} (1-\alpha)^p \|f\|_{\mathbb{W}_v^{k,2}(\varpi_{q+1})}, \quad \alpha \in [\alpha_1, 1]. \end{aligned}$$

Let us now investigate the rate of convergence of the equilibrium G_α towards \mathcal{M} . An optimal convergence rate in L^1 -spaces is given in (Mischler & Mouhot, 2009, Step 2, proof of Lemma 4.4): there is $C > 0$ and $\alpha_2 > 0$ such that

$$\|\mathcal{M} - G_\alpha\|_{L_v^1(\langle \cdot \rangle^m)} \leq C(1-\alpha), \quad \alpha \in [\alpha_2, 1]. \quad (2.1)$$

for $m(v) = \exp(a|v|)$, $a > 0$ small enough. We need to extend this optimal rate of convergence to the Sobolev spaces $\mathbb{W}_v^{k,j}(\varpi_q)$ for $j = 1, 2$ we are considering here. The proof of this technical result is postponed to Appendix A.

Lemma 2.3. *Let $k \in \mathbb{N}$, $q \geq 1$ be given. There exist some explicit $\alpha_3 \in (0, 1)$ and $C > 0$ such that*

$$\|\mathcal{M} - G_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_q)} + \|\mathcal{M} - G_\alpha\|_{\mathbb{W}_v^{k,2}(\varpi_q)} \leq C(1 - \alpha), \quad \alpha \in [\alpha_3, 1].$$

For $j = 1, 2$, on the underlying space $\mathbb{W}_v^{k,j}(\varpi_q)$, introduce the operator $T_\alpha : \mathcal{D}(T_\alpha) \subset \mathbb{W}_v^{k,j}(\varpi_q) \rightarrow \mathbb{W}_v^{k+1,j}(\varpi_{q+1})$ defined by $\mathcal{D}(T_\alpha) = \mathbb{W}_v^{k+1,j}(\varpi_{q+1})$ and

$$T_\alpha h(v) = -\kappa_\alpha \nabla_v \cdot (v h(v)), \quad h \in \mathcal{D}(T_\alpha).$$

One sees that T_α is one of the operators responsible for the discrepancy between the domain of \mathcal{L}_1 and \mathcal{L}_α . Because of this, we set

$$\mathcal{P}_\alpha : \mathcal{D}(\mathcal{P}_\alpha) \subset \mathbb{W}_v^{k,j}(\varpi_q) \rightarrow \mathbb{W}_v^{k,j}(\varpi_q)$$

as $\mathcal{P}_\alpha = \mathbf{L}_\alpha - \mathbf{L}_1$ with domain

$$\mathcal{D}(\mathcal{P}_\alpha) = \mathcal{D}(\mathcal{L}_1) = \mathbb{W}_v^{k,j}(\varpi_{q+1}).$$

One has then the following Proposition in L_v^1 and L_v^2 -based spaces:

Proposition 2.4. *Consider $k \in \mathbb{N}$ and $q \geq 0$. There exist some explicit constant $C_{k,q} > 0$ and $\alpha_\star \in (0, 1)$ such that for any $h \in \mathbb{W}_v^{k+1,1}(\varpi_{q+2})$*

$$\|\mathcal{P}_\alpha h\|_{\mathbb{W}_v^{k,1}(\varpi_q)} = \|\mathbf{L}_\alpha h - \mathbf{L}_1 h\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq C_{k,q}(1 - \alpha) \|h\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+2})}, \quad \alpha \in [\alpha_\star, 1], \quad (2.2)$$

As a consequence, for any $h \in \mathbb{W}_v^{k+1,1}(\varpi_{q+2})$

$$\|\mathcal{L}_\alpha h - \mathcal{L}_1 h\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq (C_{k,q}(1 - \alpha) + \kappa_\alpha) \|h\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+2})}, \quad \alpha \in [\alpha_\star, 1]. \quad (2.3)$$

In the same way, for any $\kappa > \frac{d}{2}$ and $q \geq 2$, there exist some explicit $C_{k,q}(\kappa) > 0$ such that for any $h \in \mathbb{W}_v^{k,2}(\varpi_{q+\kappa+2})$ it holds

$$\|\mathcal{P}_\alpha h\|_{\mathbb{W}_v^{k,2}(\varpi_q)} = \|\mathbf{L}_\alpha h - \mathbf{L}_1 h\|_{\mathbb{W}_v^{k,2}(\varpi_q)} \leq C_{k,q}(\kappa)(1 - \alpha) \|h\|_{\mathbb{W}_v^{k+1,2}(\varpi_{q+\kappa+2})}, \quad \alpha \in [\alpha_\star, 1], \quad (2.4)$$

As a consequence, for any $h \in \mathbb{W}_v^{k+1,2}(\varpi_{q+\kappa+2})$

$$\|\mathcal{L}_\alpha h - \mathcal{L}_1 h\|_{\mathbb{W}_v^{k,2}(\varpi_q)} \leq (C_{k,q}(\kappa)(1 - \alpha) + \kappa_\alpha) \|h\|_{\mathbb{W}_v^{k+1,2}(\varpi_{q+\kappa+2})}, \quad \alpha \in [\alpha_\star, 1].$$

Proof. Recall (see [Alonso et al. \(2010\)](#); [Alonso & Gamba \(2011\)](#)) that, for any $q \geq 0$, there exists some universal positive constant $C_q > 0$ such that for any $\alpha \in (0, 1]$,

$$\|\mathcal{Q}_\alpha^\pm(g, f)\|_{L_v^1(\varpi_q)} \leq C_q \|g\|_{L_v^1(\varpi_{q+1})} \|f\|_{L_v^1(\varpi_{q+1})}, \quad \forall f, g \in L_v^1(\varpi_{q+1}), \quad (2.5)$$

where we have denoted by \mathcal{Q}_α^+ (resp. \mathcal{Q}_α^-) the gain (resp. loss) part of the operator \mathcal{Q}_α . Then, we have

$$\begin{aligned} \mathbf{L}_\alpha h(v) - \mathbf{L}_1 h(v) &= \mathcal{Q}_\alpha(h, G_\alpha - \mathcal{M})(v) + \mathcal{Q}_\alpha(G_\alpha - \mathcal{M}, h)(v) \\ &\quad + [\mathcal{Q}_\alpha(h, \mathcal{M})(v) - \mathcal{Q}_1(h, \mathcal{M})(v)] + [\mathcal{Q}_\alpha(\mathcal{M}, h)(v) - \mathcal{Q}_1(\mathcal{M}, h)(v)]. \end{aligned} \quad (2.6)$$

One thus deduce from (2.5) and Lemma 2.1 that

$$\begin{aligned} \|\mathcal{P}_\alpha h\|_{L_v^1(\varpi_q)} &\leq 2C_q \|h\|_{L_v^1(\varpi_{q+1})} \|G_\alpha - \mathcal{M}\|_{L_v^1(\varpi_{q+1})} \\ &\quad + 2c_q \frac{1-\alpha}{\alpha^a} \|h\|_{\mathbb{W}_v^{1,1}(\varpi_{q+2})} \|\mathcal{M}\|_{\mathbb{W}_v^{1,1}(\varpi_{q+2})} \end{aligned}$$

where we recall that $a = \max\{d-1, 2\}$. Using now (2.1), one can conclude the proof of (2.2) for $k = 0$. In order to prove the result for higher-order derivatives, one argues using the fact that

$$\nabla_v \mathcal{Q}_\alpha(g, f) = \mathcal{Q}_\alpha(\nabla_v g, f) + \mathcal{Q}_\alpha(g, \nabla_v f). \quad (2.7)$$

Then, using (2.6) with the help of the estimate

$$\|T_\alpha h\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq \kappa_\alpha \|h\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+1})}$$

one deduces (2.3) from (2.2).

We prove the result in L_v^2 -based space in a similar way. From Theorem A.1, there exists $C_q > 0$ such that

$$\|\mathcal{Q}_\alpha^+(g, f)\|_{L_v^2(\varpi_q)} + \|\mathcal{Q}_\alpha^+(f, g)\|_{L_v^2(\varpi_q)} \leq C_q \|f\|_{L_v^1(\varpi_{q+1})} \|g\|_{L_v^2(\varpi_{q+1})}.$$

Indeed, since we have supposed that the condition (1.6) is satisfied for some $r > 2$, it in particular holds true for $r = 2$, which implies that we can apply Theorem A.1 for $r = 2$. On the other hand, it is immediate to check that for $\kappa > \frac{d}{2}$, there exists $C_q(\kappa) > 0$ such that

$$\|\mathcal{Q}_\alpha^-(g, f)\|_{L_v^2(\varpi_q)} \leq C_q(\kappa) \|f\|_{L_v^2(\varpi_{q+1})} \|g\|_{L_v^2(\varpi_{\kappa+1})}$$

where we used Cauchy Schwarz inequality. One deduces that

$$\begin{aligned} &\|\mathcal{Q}_\alpha(h, G_\alpha - \mathcal{M})\|_{L_v^2(\varpi_q)} + \|\mathcal{Q}_\alpha(G_\alpha - \mathcal{M}, h)\|_{L_v^2(\varpi_q)} \\ &\leq C_q(\kappa) \|h\|_{L_v^2(\varpi_{q+1})} \left(\|G_\alpha - \mathcal{M}\|_{L_v^1(\varpi_{q+1})} + \|G_\alpha - \mathcal{M}\|_{L_v^2(\varpi_{\kappa+1})} \right) \\ &\quad + C_q(\kappa) \|h\|_{L_v^2(\varpi_{\kappa+1})} \|G_\alpha - \mathcal{M}\|_{L_v^2(\varpi_{q+1})}. \end{aligned}$$

Then from Lemma 2.1, for any $\kappa > \frac{d}{2}$, we have:

$$\begin{aligned} &\|\mathcal{Q}_\alpha(h, \mathcal{M}) - \mathcal{Q}_1(h, \mathcal{M})\|_{L_v^2(\varpi_q)} + \|\mathcal{Q}_\alpha(\mathcal{M}, h) - \mathcal{Q}_1(\mathcal{M}, h)\|_{L_v^2(\varpi_q)} \\ &\leq c_q(\kappa) \frac{1-\alpha}{\alpha^a} \|\mathcal{M}\|_{\mathbb{W}_v^{1,2}(\varpi_{q+\kappa+2})} \|h\|_{\mathbb{W}_v^{1,2}(\varpi_{q+\kappa+2})} \end{aligned}$$

where $a = \max(d-1, 2)$. One can then conclude that (2.4) holds true for $k = 0$ thanks to Lemma 2.3 and the proof for $k > 0$ follows from (2.7). Finally, since

$$\|T_\alpha h\|_{\mathbb{W}_v^{k,2}(\varpi_q)} \leq \kappa_\alpha \|h\|_{\mathbb{W}_v^{k+1,2}(\varpi_{q+1})}$$

one also deduces the estimate for $\|\mathcal{L}_\alpha h - \mathcal{L}_1 h\|_{\mathbb{W}_v^{k,2}(\varpi_q)}$. \square

We will need also to derive an estimate for \mathcal{P}_α in its graph norm, at the price of loosing the sharp convergence rate $(1 - \alpha)$. The estimate is easy to deduce in L_v^1 -space but the extension to the L_v^2 case requires the use of Corollary A.2.

Lemma 2.5. *For any $k \in \mathbb{N}$ and $q \geq 0$, there exists $C_{k,q} > 0$ such that*

$$\|\mathcal{P}_\alpha h\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq C_{k,q} (1 - \alpha)^p \|h\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})}, \quad \alpha \in [\alpha_*, 1], \quad (2.8)$$

where p is defined in Lemma 2.2 (is independent of both k and q). Moreover, there exists $\bar{p} \in (0, 1)$ (independent of k, q) such that

$$\|\mathcal{P}_\alpha h\|_{\mathbb{W}_v^{k,2}(\varpi_q)} \leq C_{k,q} (1 - \alpha)^{\bar{p}} \|h\|_{\mathbb{W}_v^{k,2}(\varpi_{q+1})}, \quad \alpha \in [\alpha_*, 1]. \quad (2.9)$$

Proof. The proof of (2.8) is a direct consequence of (2.5) and Lemma 2.2 which give

$$\|\mathcal{P}_\alpha h\|_{L_v^1(\varpi_q)} \leq 2C_q \|h\|_{L_v^1(\varpi_{q+1})} \|G_\alpha - \mathcal{M}\|_{L_v^1(\varpi_{q+1})} + \tilde{c}_q (1 - \alpha)^p \|h\|_{L_v^1(\varpi_{q+1})}.$$

The proof of (2.9) is then deduced from (2.8) by Riesz-Thorin interpolation. Indeed, (2.8) asserts that

$$\mathcal{P}_\alpha \in \mathcal{B}(L_v^1(\varpi_{q+1}), L_v^1(\varpi_q)), \quad \|\mathcal{P}_\alpha\|_{\mathcal{B}(L_v^1(\varpi_{q+1}), L_v^1(\varpi_q))} \leq C_{0,q} (1 - \alpha)^p.$$

Now, using Corollary A.2, and recalling that b is such that (1.6) is satisfied for some $r > 2$, one proves easily that, for some $r > 2$,

$$\mathcal{P}_\alpha \in \mathcal{B}(L_v^r(\varpi_{q+1}), L_v^r(\varpi_q)) \quad \sup_{\alpha \in (0,1)} \|\mathcal{P}_\alpha\|_{\mathcal{B}(L_v^r(\varpi_{q+1}), L_v^r(\varpi_q))} = C_r(q) < \infty.$$

Then, from Riesz-Thorin interpolation Theorem (Grafakos, 2014, Theorem 1.3.4)

$$\mathcal{P}_\alpha \in \mathcal{B}(L_v^2(\varpi_{q+1}), L_v^2(\varpi_q)) \quad \text{with} \quad \|\mathcal{P}_\alpha\|_{\mathcal{B}(L_v^2(\varpi_{q+1}), L_v^2(\varpi_q))} \leq C (1 - \alpha)^{p\theta}$$

where $\theta = \frac{r-2}{2(r-1)}$ is such that $\frac{1}{2} = \theta + \frac{1-\theta}{r}$. This proves (2.9) for $k = 0$ setting $\bar{p} = p\theta$ and its extension to $k > 0$ follows from (2.7). \square

The above extends to functional spaces $\mathbb{W}_v^{k,1} L_x^2(\varpi_q)$ and $\mathbb{W}_v^{k,2} L_x^2(\varpi_q)$ in an easy way:

Corollary 2.6. *Consider $k \in \mathbb{N}$ and $q \geq 0$. There exists some explicit constant $\tilde{C}_{k,q} > 0$ such that*

$$\|\mathcal{P}_\alpha h\|_{\mathbb{W}_v^{k,1} L_x^2(\varpi_q)} \leq \tilde{C}_{k,q} (1 - \alpha)^{\frac{p}{2}} \|h\|_{\mathbb{W}_v^{k,1} L_x^2(\varpi_{q+1})}, \quad \alpha \in [\alpha_*, 1], \quad (2.10)$$

and similarly,

$$\|\mathcal{P}_\alpha h\|_{\mathbb{W}_v^{k,2} L_x^2(\varpi_q)} \leq \tilde{C}_{k,q} (1 - \alpha)^{\bar{p}} \|h\|_{\mathbb{W}_v^{k,2} L_x^2(\varpi_{q+1})}, \quad \alpha \in [\alpha_*, 1],$$

where $p > 0$ and \bar{p} are defined respectively in Lemmas 2.2 and 2.5.

Proof. On the one hand, the $\mathbb{W}_v^{k,1}L_x^1(\varpi_q)$ -norm of $\mathcal{P}_\alpha h$ is estimated using Fubini theorem and (2.8) in Lemma 2.5:

$$\|\mathcal{P}_\alpha h\|_{\mathbb{W}_v^{k,1}L_x^1(\varpi_q)} \leq C_{k,q}(1-\alpha)^p \|h\|_{\mathbb{W}_v^{k,1}L_x^1(\varpi_{q+1})}.$$

On the other hand, using (2.5) and the fact that Q_α^\pm are local in x , one can show that

$$\|\mathcal{P}_\alpha h\|_{\mathbb{W}_v^{k,1}L_x^\infty(\varpi_q)} \leq \tilde{C}_{k,q} \|h\|_{\mathbb{W}_v^{k,1}L_x^\infty(\varpi_{q+1})}.$$

We obtain (2.10) by interpolation. The proof for $L_v^2L_x^2$ -based spaces is simpler since it is deduced directly from Fubini Theorem. \square

2.2. Decomposition of \mathcal{L}_α . Let us now recall the following decomposition of \mathcal{L}_0 introduced in Gualdani et al. (2017); Tristani (2016). For any $\delta \in (0, 1)$, we consider the cutoff function $0 \leq \Theta_\delta = \Theta_\delta(\xi, \xi_*, \sigma) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1})$, assumed to be bounded by 1, which equals 1 on

$$J_\delta := \left\{ (\xi, \xi_*, \sigma) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \mid |\xi| \leq \delta^{-1}, 2\delta \leq |\xi - \xi_*| \leq \delta^{-1}, |\cos \theta| \leq 1 - 2\delta \right\},$$

and whose support is included in $J_{\delta/2}$ (where $\cos \theta = \langle \frac{\xi - \xi_*}{|\xi - \xi_*|}, \sigma \rangle$). We then set

$$\begin{aligned} \mathcal{L}_1^{S,\delta} h(\xi) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} [\mathcal{M}(\xi'_*)h(\xi') + \mathcal{M}(\xi')h(\xi'_*) - \mathcal{M}(\xi)h(\xi_*)] \\ &\quad \times |\xi - \xi_*| \Theta_\delta(\xi, \xi_*, \sigma) d\xi_* d\sigma, \\ \mathcal{L}_1^{R,\delta} h(\xi) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} [\mathcal{M}(\xi'_*)h(\xi') + \mathcal{M}(\xi')h(\xi'_*) - \mathcal{M}(\xi)h(\xi_*)] \\ &\quad \times |\xi - \xi_*| (1 - \Theta_\delta(\xi, \xi, \sigma)) d\xi_* d\sigma, \end{aligned}$$

so that $\mathcal{L}_1 h = \mathcal{L}_1^{S,\delta} h + \mathcal{L}_1^{R,\delta} h - h \Sigma_{\mathcal{M}}$ where $\Sigma_{\mathcal{M}}$ denotes the mapping

$$\Sigma_{\mathcal{M}}(\xi) = \int_{\mathbb{R}^d} \mathcal{M}(\xi_*) |\xi - \xi_*| d\xi_*, \quad \xi \in \mathbb{R}^d. \quad (2.11)$$

Recall that there exist $\sigma_0 > 0$ and $\sigma_1 > 0$ such that

$$\sigma_0 \varpi_1(\xi) \leq \Sigma_{\mathcal{M}}(\xi) \leq \sigma_1 \varpi_1(\xi), \quad \xi \in \mathbb{R}^d. \quad (2.12)$$

Introduce

$$\mathcal{A}^{(\delta)}(h) := \mathcal{L}_1^{S,\delta}(h) \quad \text{and} \quad \mathcal{B}_1^{(\delta)}(h) := \mathcal{L}_1^{R,\delta} - \Sigma_{\mathcal{M}}$$

so that $\mathcal{L}_1 = \mathcal{A}^{(\delta)} + \mathcal{B}_1^{(\delta)}$. Let us now recall the known hypo-dissipativity results for the elastic Boltzmann operator in $L_v^1L_x^2$ and $L_{v,x}^2$ -based Sobolev spaces, see (Gualdani et al., 2017, Lemmas 4.12, 4.14 & Lemma 4.16):

Lemma 2.7. *For any $k \in \mathbb{N}$ and $\delta > 0$, there are two positive constants $C_{k,\delta} > 0$ and $R_\delta > 0$ such that $\text{supp}(\mathcal{A}^{(\delta)} f) \subset B(0, R_\delta)$ and*

$$\|\mathcal{A}^{(\delta)} f\|_{\mathbb{W}_v^{k,2}(\mathbb{R}^d)} \leq C_{k,\delta} \|f\|_{L_v^1(\varpi_1)}, \quad \forall f \in L_v^1(\varpi_1). \quad (2.13)$$

Moreover, the following holds

(1) For any $q > 2$ and any $\delta \in (0, 1)$ it holds

$$\int_{\mathbb{R}^d} \|h(\cdot, v)\|_{L_x^2}^{-1} \left(\int_{\mathbb{T}^d} \left(\mathcal{B}_1^{(\delta)} h(x, v) \right) h(x, v) dx \right) \varpi_q dv \leq \left(\Lambda_q^{(1)}(\delta) - 1 \right) \|h\|_{L_v^1 L_x^2(\varpi_q \Sigma_{\mathcal{M}})} \quad (2.14)$$

where $\Lambda_q^{(1)} : (0, 1) \rightarrow \mathbb{R}^+$ is some explicit function such that $\lim_{\delta \rightarrow 0} \Lambda_q^{(1)}(\delta) = \frac{4}{q+2}$.

(2) For any $q > q^*$ and any $\delta \in (0, 1)$,

$$\int_{\mathbb{R}^d \times \mathbb{T}^d} \left(\mathcal{B}_1^{(\delta)} h(x, v) \right) h(x, v) dx \varpi_q^2(v) dv \leq \left(\sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) - \sqrt{\sigma_0} \right) \|h\|_{L_{v,x}^2(\varpi_q \sqrt{\Sigma_{\mathcal{M}}})} \quad (2.15)$$

where $\Lambda_q^{(2)} : (0, 1) \rightarrow \mathbb{R}^+$ is explicit and such that $\lim_{\delta \rightarrow 0} \Lambda_q^{(2)}(\delta) = \frac{8}{2q-3}$.

Remark 2.8. Notice that this lemma directly comes from [Gualdani et al. \(2017\)](#) but the constants involved in the final estimates are not the same as in Lemma 4.14 of [Gualdani et al. \(2017\)](#) where it seems that some multiplicative constants coming from (2.12) have been omitted in some computations of their proof.

This leads to the following decomposition of \mathcal{L}_α :

$$\mathcal{L}_\alpha = \mathcal{B}_\alpha^{(\delta)} + \mathcal{A}^{(\delta)}, \quad \text{where } \mathcal{B}_\alpha^{(\delta)} = \mathcal{B}_1^{(\delta)} + [\mathcal{L}_\alpha - \mathcal{L}_1]. \quad (2.16)$$

2.3. The complete linearized operator. The complete linearized operator is given by

$$\mathcal{G}_{\alpha,\varepsilon} h = \varepsilon^{-2} \mathcal{L}_\alpha(h) - \varepsilon^{-1} v \cdot \nabla_x h, \quad \forall \alpha \in (0, 1].$$

With previous decomposition, we have that

$$\mathcal{G}_{\alpha,\varepsilon} = \mathcal{A}_\varepsilon^{(\delta)} + \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$$

where

$$\mathcal{A}_\varepsilon^{(\delta)} = \varepsilon^{-2} \mathcal{A}^{(\delta)}, \quad \mathcal{B}_{\alpha,\varepsilon}^{(\delta)} = \varepsilon^{-2} \mathcal{B}_\alpha^{(\delta)} - \varepsilon^{-1} v \cdot \nabla_x.$$

Notice that

$$\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} - \mathcal{B}_{1,\varepsilon}^{(\delta)} = \mathcal{G}_{\alpha,\varepsilon} - \mathcal{G}_{1,\varepsilon} = \varepsilon^{-2} \mathcal{P}_\alpha + \varepsilon^{-2} T_\alpha.$$

We set

$$q_1 := 2, \quad q_2 = q^* := 4 \sqrt{\frac{\sigma_1}{\sigma_0}} + \frac{3}{2}, \quad (2.17)$$

where we recall that σ_0 and σ_1 are defined in (2.12). One has the following properties of $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ in $L_v^1 L_x^2$ and $L_{v,x}^2$ -based spaces.

Proposition 2.9. For $j = 1, 2$, for any $\ell \geq s \geq 0$ and $q > q_j$ there exist $\alpha_{j,\ell,s,q}^\dagger > 0$, $\delta_{j,\ell,s,q}^\dagger > 0$ and $\nu_{j,\ell,s,q} > 0$ such that for any $\varepsilon \in (0, 1]$,

$$\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2} \nu_{j,\ell,s,q} \text{ is hypo-dissipative in } \mathbb{W}_v^{s,j} \mathbb{W}_x^{\ell,2}(\varpi_q)$$

for any $\alpha \in (\alpha_{j,\ell,s,q}^\dagger, 1)$, and $\delta \in (0, \delta_{j,\ell,s,q}^\dagger)$.

Proof. We present here the proof in the space $\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)$, the proof for $\mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\varpi_q)$ follows the same lines and is given in Appendix A. Notice that derivatives with respect to the x -variable commute with the operator $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ and this allows to prove the result, without loss of generality, in the special case $\ell = s$. We divide the proof in several steps:

- We first consider the case $\ell = 0$. We write $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}(h) = \sum_{i=0}^3 C_i(h)$ with

$$\begin{aligned} C_0(h) &= \varepsilon^{-2} \mathcal{B}_1^{(\delta)} h, & C_1(h) &= -\varepsilon^{-1} v \cdot \nabla_x h, \\ C_2(h) &= \varepsilon^{-2} \mathcal{P}_\alpha h, & C_3(h) &= \varepsilon^{-2} T_\alpha h = -\varepsilon^{-2} \kappa_\alpha \nabla_v \cdot (v h(x, v)), \end{aligned}$$

and correspondingly and with obvious notations,

$$\int_{\mathbb{R}^d \times \mathbb{T}^d} \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}(h)(x, v) h(x, v) \varpi_q^2(v) dx dv =: \sum_{i=0}^3 I_i(h).$$

First, $I_1(h) = 0$ since

$$\int_{\mathbb{T}^d} (v \cdot \nabla_x h(x, v)) h(x, v) dx = \frac{1}{2} \int_{\mathbb{T}^d} v \cdot \nabla_x h^2(x, v) dx = 0.$$

Then, from (2.15), one has

$$I_0(h) \leq \varepsilon^{-2} \left(\sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) - \sqrt{\sigma_0} \right) \|h\|_{L_{v,x}^2(\varpi_q \sqrt{\Sigma_{\mathcal{M}}})}.$$

Recalling that $\lim_{\delta \rightarrow 0} \Lambda_q^{(2)}(\delta) = 8/(2q - 3)$, one can choose $\delta > 0$ small enough so that, for $q > q^*$,

$$\sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) < \sqrt{\sigma_0}.$$

Then, for $q > q^*$ and δ small enough, we have

$$I_0(h) \leq \varepsilon^{-2} \sqrt{\sigma_0} \left(\sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) - \sqrt{\sigma_0} \right) \|h\|_{L_{v,x}^2(\varpi_{q+1/2})}. \quad (2.18)$$

Moreover, it follows from Cauchy-Schwarz inequality and (2.9) that

$$\begin{aligned} I_2(h) &\leq \varepsilon^{-2} \|\mathcal{P}_\alpha h\|_{L_{v,x}^2(\varpi_{q-\frac{1}{2}})} \|h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})} \\ &\leq \varepsilon^{-2} \mathcal{C}_{0,q-\frac{1}{2}} (1 - \alpha)^{\bar{p}} \|h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})}^2 \end{aligned}$$

Finally, for I_3 , one can compute

$$\begin{aligned} &\int_{\mathbb{R}^d \times \mathbb{T}^d} \nabla_v \cdot (v h(x, v)) h(x, v) \varpi_q^2(v) dx dv \\ &= d \|h\|_{L_{v,x}^2(\varpi_q)}^2 + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{T}^d} v \cdot \nabla_v h^2(x, v) \varpi_q^2(v) dx dv \\ &= \frac{d}{2} \|h\|_{L_{v,x}^2(\varpi_q)}^2 - \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{T}^d} h^2(x, v) v \cdot \nabla_v \varpi_q^2(v) dx dv \end{aligned}$$

Since $v \cdot \nabla_v \varpi_q^2(v) = 2q\varpi_q^2(v) - 2q\varpi_{q-1}^2(v)$ we get

$$I_3(h) \leq q\kappa_\alpha \varepsilon^{-2} \|h\|_{L_{v,x}^2(\varpi_q)}^2 \leq q\kappa_\alpha \varepsilon^{-2} \|h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})}^2. \quad (2.19)$$

Gathering the previous estimates, one obtains

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{T}^d} \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}(h)(x,v) h(x,v) \varpi_q^2(v) \, dx dv \\ & \leq \varepsilon^{-2} \left(C_{0,q-\frac{1}{2}} (1-\alpha)^{\bar{p}} + \sqrt{\sigma_0} \left(\sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) - \sqrt{\sigma_0} \right) + q\kappa_\alpha \right) \|h\|_{L_{v,x}^2(\varpi_{q+1/2})}. \end{aligned}$$

Recalling that $\kappa_\alpha = 1 - \alpha$ while $\lim_{\delta \rightarrow 0} \sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) < \sqrt{\sigma_0}$ for $q > q^*$ we can pick $\delta_{2,0,0,q}^\dagger$ small enough and then $\alpha_{2,0,0,q}^\dagger \in (0, 1)$ close enough to 1 so that

$\nu_{2,0,0,q} :=$

$$- \inf \left\{ C_{0,q-\frac{1}{2}} (1-\alpha)^{\bar{p}} + \sqrt{\sigma_0} \left(\sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) - \sqrt{\sigma_0} \right) + q\kappa_\alpha; \alpha \in (\alpha_{2,0,0,q}^\dagger, 1), \delta \in (0, \delta_{2,0,0,q}^\dagger) \right\}$$

is positive and get, for any $\delta \in (0, \delta_{2,0,0,q}^\dagger)$ and $\alpha \in (\alpha_{2,0,0,q}^\dagger, 1)$,

$$\begin{aligned} \mathcal{I} & := \int_{\mathbb{R}^d \times \mathbb{T}^d} \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}(h)(x,v) h(x,v) \varpi_q^2(v) \, dx dv \leq -\varepsilon^{-2} \nu_{2,0,0,q} \|h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})}^2 \\ & \leq -\varepsilon^{-2} \nu_{2,0,0,q} \|h\|_{L_{v,x}^2(\varpi_q)}^2 \end{aligned} \quad (2.20)$$

which implies that $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2} \nu_{2,0,0,q}$ is dissipative in $L_{v,x}^2(\varpi_q)$.

Let now investigate the case $\ell = 1$. We consider the norm

$$\|h\|^2 = \|h\|_{L_{v,x}^2(\varpi_q)}^2 + \|\nabla_x h\|_{L_{v,x}^2(\varpi_q)}^2 + \eta \|\nabla_v h\|_{L_{v,x}^2(\varpi_q)}^2,$$

for some $\eta > 0$, the value of which shall be fixed later on. This norm is equivalent to the classical $\mathbb{W}_{v,x}^{1,2}(\varpi_q)$ -norm. We shall prove that for some $\nu_{2,1,1,q} > 0$, $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2} \nu_{2,1,1,q}$ is dissipative in $\mathbb{W}_{v,x}^{1,2}(\varpi_q)$ for the norm $\|\cdot\|$. Notice first that the x -derivative commutes with all the above terms $C_i(h)$, $i = 0, \dots, 3$, i.e.

$$\nabla_x \mathcal{B}_{\alpha,\varepsilon}^{(\delta)} h(x,v) = \mathcal{B}_{\alpha,\varepsilon}^{(\delta)} (\nabla_x h)(x,v)$$

so that, according to the previous step

$$\begin{aligned} \mathcal{J}_x & := \int_{\mathbb{R}^d \times \mathbb{T}^d} \nabla_x \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}(h)(x,v) \cdot \nabla_x h(x,v) \varpi_q^2(v) \, dx dv \\ & \leq -\varepsilon^{-2} \nu_{2,0,0,q} \|\nabla_x h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})}^2. \end{aligned} \quad (2.21)$$

Consider now the quantity

$$\mathcal{J}_v := \int_{\mathbb{R}^d \times \mathbb{T}^d} \nabla_v \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}(h)(x,v) \cdot \nabla_v h(x,v) \varpi_q^2(v) \, dx dv.$$

Using the notations above, one notices that $\nabla_v C_1(h) = -\varepsilon^{-1} \nabla_x h + C_1(\nabla_v h)$, so that

$$\begin{aligned} \nabla_v(\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} h(x,v)) &= \varepsilon^{-2} \nabla_v(\mathcal{B}_1^{(\delta)} h) - \varepsilon^{-1} \nabla_x h + C_1(\nabla_v h) \\ &\quad + \varepsilon^{-2} \nabla_v(\mathcal{P}_\alpha h) + \varepsilon^{-2} \nabla_v(T_\alpha h) \end{aligned} \quad (2.22)$$

Then, it follows from Corollary 2.6 that

$$\|\nabla_v(\mathcal{P}_\alpha h)\|_{L_{v,x}^2(\varpi_{q-\frac{1}{2}})} \leq \tilde{C}_{1,q-\frac{1}{2}} (1-\alpha)^{\bar{p}} \left(\|h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})} + \|\nabla_v h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})} \right). \quad (2.23)$$

Now,

$$\nabla_v \mathcal{B}_1^{(\delta)} h = \nabla_v [\mathcal{L}_1^{R,\delta} h - \Sigma_{\mathcal{M}} h] = \mathcal{L}_1^{R,\delta}(\nabla_v h) - \Sigma_{\mathcal{M}} \nabla_v h + \mathcal{R}(h),$$

where

$$\mathcal{R}(h) = \mathcal{Q}_1(h, \nabla_v \mathcal{M}) + \mathcal{Q}_1(\nabla_v \mathcal{M}, h) - (\nabla_v \mathcal{A}^{(\delta)})(h) - \mathcal{A}^{(\delta)}(\nabla_v h).$$

From the proof of (Gualdani et al., 2017, Lemma 4.14), we have that

$$\|\mathcal{R}(h)\|_{L_{v,x}^2(\varpi_{q-\frac{1}{2}})} \leq C_\delta \|h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})}.$$

while, according to (2.18), one has

$$\begin{aligned} \varepsilon^{-2} \int_{\mathbb{R}^d \times \mathbb{T}^d} \left[\mathcal{L}_1^{R,\delta}(\nabla_v h) - \Sigma_{\mathcal{M}} \nabla_v h \right] \cdot \nabla_v h \varpi_q^2 dx dv \\ = I_0(\nabla_v h) \leq \varepsilon^{-2} \sqrt{\sigma_0} \left(\sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) - \sqrt{\sigma_0} \right) \|\nabla_v h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})}^2. \end{aligned}$$

Therefore, the contribution of $\nabla_v \mathcal{B}_1^{(\delta)} h$ is

$$\begin{aligned} \varepsilon^{-2} \int_{\mathbb{R}^d \times \mathbb{T}^d} \nabla_v \mathcal{B}_1^{(\delta)} h \cdot \nabla_v h \varpi_q^2 dx dv \\ \leq \varepsilon^{-2} C_\delta \|h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})}^2 + \varepsilon^{-2} \sqrt{\sigma_0} \left(\sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) - \sqrt{\sigma_0} \right) \|\nabla_v h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})}^2 \end{aligned} \quad (2.24)$$

where $\sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) - \sqrt{\sigma_0} < 0$ for δ small enough and $q > q^*$. Finally, using the short-hand notation

$$\nabla_v \cdot (v \nabla_v h) = (\nabla_v \cdot (v \partial_{v_1} h), \dots, \nabla_v \cdot (v \partial_{v_d} h)),$$

we have

$$\nabla_v h \cdot \nabla_v (\nabla_v \cdot (v h)) = |\nabla_v h|^2 + [\nabla_v \cdot (v \nabla_v h)] \cdot \nabla_v h.$$

Doing similar computations as the ones leading to (2.19), we obtain:

$$\int_{\mathbb{R}^d \times \mathbb{T}^d} \nabla_v T_\alpha(h)(x,v) \cdot \nabla_v h(x,v) \varpi_q^2(v) dx dv \leq (q-1) \kappa_\alpha \|\nabla_v h\|_{L_{v,x}^2(\varpi_{q+\frac{1}{2}})}^2. \quad (2.25)$$

Coming back to (2.22), Cauchy-Schwarz inequality and estimates (2.23), (2.24) and (2.25) give that

$$\begin{aligned} \mathcal{J}_v &\leq \varepsilon^{-2} (C_\delta + \tilde{\mathcal{C}}_{1,q-1/2} (1-\alpha)^{\bar{p}}) \|h\|_{L^2_{v,x}(\varpi_{q+\frac{1}{2}})}^2 + \varepsilon^{-1} \|\nabla_x h\|_{L^2_{v,x}(\varpi_q)} \|\nabla_v h\|_{L^2_{v,x}(\varpi_q)} \\ &\quad + \varepsilon^{-2} \left(\tilde{\mathcal{C}}_{1,q-\frac{1}{2}} (1-\alpha)^{\bar{p}} + (q-1)\kappa_\alpha + \sqrt{\sigma_0} \left(\sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) - \sqrt{\sigma_0} \right) \right) \|\nabla_v h\|_{L^2_{v,x}(\varpi_{q+\frac{1}{2}})}^2, \end{aligned}$$

where we used that the contribution to \mathcal{J}_v of the term $C_1(\nabla_v h)$ vanishes. Hence, combining this estimate with (2.20) and (2.21) and using that $\varepsilon \leq 1$, one obtains, for any $\tau > 0$,

$$\begin{aligned} \mathcal{I} + \mathcal{J}_x + \eta \mathcal{J}_v &\leq \varepsilon^{-2} \left(\left[-\nu_{2,0,0,q} + \eta \left(C_\delta + \tilde{\mathcal{C}}_{1,q-\frac{1}{2}} (1-\alpha)^{\bar{p}} \right) \right] \|h\|_{L^2_{v,x}(\varpi_{q+\frac{1}{2}})}^2 \right. \\ &\quad \left. - \left(\nu_{2,0,0,q} - \frac{1}{4\tau} \eta \right) \|\nabla_x h\|_{L^2_{v,x}(\varpi_{q+\frac{1}{2}})}^2 \right. \\ &\quad \left. + \eta \left[\tilde{\mathcal{C}}_{1,q-\frac{1}{2}} (1-\alpha)^{\bar{p}} + (q-1)\kappa_\alpha + \sqrt{\sigma_0} \left(\sqrt{\sigma_1} \Lambda_q^{(2)}(\delta) - \sqrt{\sigma_0} \right) + \tau \right] \|\nabla_v h\|_{L^2(\varpi_{q+\frac{1}{2}})}^2 \right) \end{aligned}$$

where we used Young's inequality to estimate the mixed term

$$\|\nabla_x h\|_{L^2_{v,x}(\varpi_q)} \|\nabla_v h\|_{L^2_{v,x}(\varpi_q)} \leq \frac{1}{4\tau} \|\nabla_x h\|_{L^2_{v,x}(\varpi_q)}^2 + \tau \|\nabla_v h\|_{L^2_{v,x}(\varpi_q)}^2, \quad \tau > 0.$$

There exist $\alpha_{2,1,1,q}^\dagger > 0$ and $\delta_{2,1,1,q}^\dagger > 0$ and $\tau > 0$ small enough so that for any $\alpha \in (\alpha_{2,1,1,q}^\dagger, 1)$ and any $\delta \in (0, \delta_{2,1,1,q}^\dagger)$,

$$\left(\tilde{\mathcal{C}}_{1,q-\frac{1}{2}} (1-\alpha)^{\bar{p}} + (q-1)\kappa_\alpha + \sqrt{\sigma_0} \left(\Lambda_q^{(2)}(\delta) - \sqrt{\sigma_0} \right) \right) + \tau < 0.$$

One chooses then $\eta > 0$ small enough such that

$$\nu_{2,0,0,q} - \eta \max \left(\frac{1}{4\tau}, \sup_{\delta \in (0, \delta_{2,1,1,q}^\dagger)} C_\delta + \tilde{\mathcal{C}}_{1,q-\frac{1}{2}} (1-\alpha_{2,1,1,q}^\dagger)^{\bar{p}} \right) > 0,$$

we finally obtain that there exists $\nu_{2,1,1,q} > 0$ such that for $\alpha \in (\alpha_{2,1,1,q}^\dagger, 1)$ and $\delta \in (0, \delta_{2,1,1,q}^\dagger)$,

$$\begin{aligned} \mathcal{I} + \mathcal{J}_x + \eta \mathcal{J}_v &\leq -\varepsilon^{-2} \nu_{2,1,1,q} \left[\|h\|_{L^2_{x,v}(\varpi_{q+\frac{1}{2}})}^2 + \|\nabla_x h\|_{L^2_{v,x}(\varpi_{q+\frac{1}{2}})}^2 + \eta \|\nabla_v h\|_{L^2_{v,x}(\varpi_{q+\frac{1}{2}})}^2 \right] \\ &\leq -\varepsilon^{-2} \nu_{2,1,1,q} \|h\|^2. \end{aligned}$$

This proves that $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2} \nu_{2,1,1,q}$ is hypo-dissipative in $\mathbb{W}_{v,x}^{1,2}(\varpi_q)$. We prove the result for higher order derivatives in the same way considering now the norm

$$\|h\|^2 = \sum_{|\beta_1|+|\beta_2| \leq k} \eta^{|\beta_1|} \left\| \nabla_v^{|\beta_1|} \nabla_x^{|\beta_2|} h \right\|_{L^2_v L^2_x(\varpi_q)}^2$$

for some $\eta > 0$ to be chosen sufficiently small. \square

Remark 2.10. *It is important to notice that the equivalent norms constructed in the Proposition 2.9 are independent of ε . This means that the hypo-dissipativity of $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2}\nu_{j,\ell,s,q}$ on $\mathbb{W}_v^{s,j}\mathbb{W}_x^{\ell,2}(\varpi_q)$, $j = 1, 2$ can be re-written as*

$$\|(\lambda - \varepsilon^{-2}\nu_{j,\ell,s,q} - \mathcal{B}_{\alpha,\varepsilon}^{(\delta)})g\|_{\mathbb{W}_v^{s,j}\mathbb{W}_x^{\ell,2}(\varpi_q)} \geq C\lambda\|g\|_{\mathbb{W}_v^{s,j}\mathbb{W}_x^{\ell,2}(\varpi_q)}, \quad j = 1, 2$$

for any $\lambda > 0$, $g \in \mathcal{D}(\mathcal{B}_{\alpha,\varepsilon}^{(\delta)})$, and some constant $C > 0$ depending on j, ℓ, s, q but not on ε .

We now state some semigroup generation result.

Proposition 2.11. *For $j = 1, 2$, for any $\ell \geq s \geq 0$, $q > q_j$, $\alpha \in (\alpha_{j,\ell,s,q}^\dagger, 1)$, $\delta \in (0, \delta_{j,\ell,s,q}^\dagger)$ and $\varepsilon > 0$, the operator*

$$\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} : \mathcal{D}(\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}) \subset \mathbb{W}_v^{s,j}\mathbb{W}_x^{\ell,2}(\varpi_q) \longrightarrow \mathbb{W}_v^{s,j}\mathbb{W}_x^{\ell,2}(\varpi_q)$$

is the generator of a C_0 -semigroup $\{\mathcal{S}_{\alpha,\varepsilon}^{(\delta)}(t) ; t \geq 0\}$ in $\mathbb{W}_v^{s,j}\mathbb{W}_x^{\ell,2}(\varpi_q)$ and there exist $0 < \nu_* < \nu_{j,\ell,s,q}$ and $C_{j,\ell,s,q} > 0$ such that

$$\left\| \mathcal{S}_{\alpha,\varepsilon}^{(\delta)}(t) \right\|_{\mathcal{B}(\mathbb{W}_v^{s,j}\mathbb{W}_x^{\ell,2}(\varpi_q))} \leq C_{j,\ell,s,q} \exp(-\varepsilon^{-2}\nu_* t), \quad \forall t \geq 0. \quad (2.26)$$

As a consequence,

$$\mathcal{G}_{\alpha,\varepsilon} : \mathcal{D}(\mathcal{G}_{\alpha,\varepsilon}) \subset \mathbb{W}_v^{s,j}\mathbb{W}_x^{\ell,2}(\varpi_q) \longrightarrow \mathbb{W}_v^{s,j}\mathbb{W}_x^{\ell,2}(\varpi_q)$$

is the generator of a C_0 -semigroup $\{\mathcal{V}_{\alpha,\varepsilon}(t) ; t \geq 0\}$ in $\mathbb{W}_v^{s,j}\mathbb{W}_x^{\ell,2}(\varpi_q)$.

Proof. The fact that $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ is a generator of a C_0 -semigroup in $\mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\varpi_q)$ is proven in Appendix D. Since we already proved that $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2}\nu_{j,\ell,s,q}$ is hypo-dissipative, we deduce directly (2.26). Finally, because $\mathcal{A}_\varepsilon^{(\delta)}$ is a bounded operator in $\mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\varpi_q)$, we deduce from the bounded perturbation theorem that $\mathcal{G}_{\alpha,\varepsilon} = \mathcal{A}_\varepsilon^{(\delta)} + \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ generates a C_0 -semigroup in $\mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\varpi_q)$. \square

2.4. The elastic semigroup. The spectral analysis of $\mathcal{G}_{1,\varepsilon}$ and the generation of its associated semigroup has been performed in Theorem 2.1 of Briant et al. (2019). We need a slightly more precise estimate on the decay of the semigroup independently of ε . Our main result concerning $\mathcal{G}_{1,\varepsilon}$ is the following whose proof is postponed to Appendix D:

Theorem 2.12. *There exists $\varepsilon_0 \in (0, 1)$ such that, for all $\ell, s \in \mathbb{N}$ with $\ell \geq s$ and $q > q^*$ and any $\varepsilon \in (0, \varepsilon_0)$, the full transport operator $\mathcal{G}_{1,\varepsilon}$ generates a C_0 -semigroup $\{\mathcal{V}_{1,\varepsilon}(t) ; t \geq 0\}$ on $\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)$. Moreover, there exist $C_0 > 0$ and $\mu_* > 0$ (both independent of ε) such that,*

$$\begin{aligned} \left\| \mathcal{V}_{1,\varepsilon}(t) [h - \mathbf{P}_0 h] \right\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)} \\ \leq C_0 \exp(-\mu_* t) \|h - \mathbf{P}_0 h\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)}, \quad \forall t \geq 0, \end{aligned} \quad (2.27)$$

holds true for any $h \in \mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q)$, where \mathbf{P}_0 is the spectral projection onto $\text{Ker}(\mathcal{G}_{1,\varepsilon}) = \text{Ker}(\mathcal{L}_1)$ which is independent of ε and given by

$$\mathbf{P}_0 h = \sum_{i=1}^{d+2} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} h \Psi_i \, dx dv \right) \Psi_i \mathcal{M} \quad (2.28)$$

where $\Psi_1(v) = 1$, $\Psi_i(v) = \frac{1}{\sqrt{\vartheta_1}} v_{i-1}$ ($i = 2, \dots, d+1$) and $\Psi_{d+2}(v) = \frac{|v|^2 - d\vartheta_1}{\vartheta_1 \sqrt{2d}}$ ($v \in \mathbb{R}^d$).

Remark 2.13. Theorem 2.12 is known to be true on the Hilbert space $\mathbb{W}_{v,x}^{\ell,2}(\mathcal{M}^{-\frac{1}{2}})$, see (Briant, 2015, Theorems 2.1 and 2.4). Notice that Theorem 2.1 from Briant (2015) only provides an exponential decay of a norm of the solution which depends on ε . The introduction of a new norm which is equivalent to the usual $\mathbb{W}_{v,x}^{\ell,2}$ norm uniformly in ε then allows the author to recover in Theorem 2.4 a uniform in ε exponential decay of the solution to the whole nonlinear problem. One can of course proceed similarly to obtain a uniform in ε exponential decay of the semigroup. In the present context of polynomial weighted spaces, a similar result was obtained in (Briant et al., 2019, Theorem 2.1) with the important difference that the estimate (2.27) was shown only for $t > t_\star > 0$. This actually comes from the use of a general enlargement theorem from Gualdani et al. (2017) which yields

$$\|\mathcal{V}_{1,\varepsilon}(t)h - \mathbf{P}_0 h\|_{\mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q)} \leq C_0 \frac{t^N}{\varepsilon^{N(2+s)}} \exp(-\mu t) \|h - \mathbf{P}_0 h\|_{\mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q)}, \quad t \geq 0 \quad (2.29)$$

for some $N \in \mathbb{N}$ and $s > 0$ and $\mu < \mu_\star$. It is important for the rest of our analysis to be able to remove this strong dependence on ε in the decay estimate of $\mathcal{V}_{1,\varepsilon}(t)(\mathbf{Id} - \mathbf{P}_0)$. This is done in Appendix D. The key point is that, in our case, the enlargement argument is developed with $\mathbb{W}_{v,x}^{\ell,2}(\mathcal{M}^{-1/2})$ as the small space and $\mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q)$ as the larger one. One can remark that in this context, we do not need any gain of regularity in the space variable. The fact that the rate degenerates for small times in (2.29) actually comes from the use of an averaging lemma to gain regularity in x , which is no longer necessary in our framework. Note that Theorem 2.12 also holds in $\mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,2}(\varpi_q)$ for $q > 2$ for the same reasons.

Remark 2.14. Notice that, unfortunately, it seems that the above result is not true on the natural space $\mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,1}(\varpi_q)$: indeed, whereas (2.29) still holds in such a space, it seems that the “initial layer” dependence on ε is not removable in this case because \mathcal{A}_ε has no regularizing properties on the x -variable. This is the main reason why we need in our approach to deal with spaces built upon L_x^2 .

An important consequence of the Theorem 2.12 is the following proposition.

Proposition 2.15. Let $\ell, s \in \mathbb{N}$ with $\ell \geq s$ and $q > q^\star$. There exists $C_1 > 0$ such that,

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q))} \leq C_1 \max \left(\frac{1}{|\lambda|}, \frac{1}{\text{Re} \lambda + \mu_\star} \right), \quad \forall \lambda \in \mathbb{C}_{\mu_\star}^\star, \quad \forall \varepsilon \in (0, \varepsilon_0),$$

where ε_0 and μ_\star have been defined in Theorem 2.12, C_1 being independent of ε .

Proof. On the space $\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)$, the spectrum of $\mathcal{G}_{1,\varepsilon}$ satisfies

$$\mathfrak{S}(\mathcal{G}_{1,\varepsilon}) \cap \{z \in \mathbb{C} ; \operatorname{Re} z > -\mu_\star\} = \{0\}$$

and the above projection \mathbf{P}_0 is nothing but the spectral projection of $\mathfrak{S}(\mathcal{G}_{1,\varepsilon})$ associated to the zero eigenvalue given by

$$\mathbf{P}_0 = \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(z, \mathcal{G}_{1,\varepsilon}) dz, \quad \gamma_r := \{z \in \mathbb{C} ; |z| = r\}, \quad r < \mu_\star.$$

Notice also

$$\dim(\operatorname{Range}(\mathbf{P}_0)) = \dim \operatorname{Ker}(\mathcal{G}_{1,\varepsilon}) = d + 2,$$

which means that the algebraic multiplicity of the zero eigenvalue coincides with its geometrical multiplicity and, as such, 0 is a *simple* pole of the resolvent $\mathcal{R}(\cdot, \mathcal{G}_{1,\varepsilon})$ (see (Kato, 1980, III.5)). Denote by $\|\cdot\|$ the operator norm in $\mathcal{B}(\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q))$ and fix $\mu \in (0, \mu_\star)$. Since $\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) = \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\mathbf{P}_0 + \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})(\mathbf{Id} - \mathbf{P}_0)$ and \mathbf{P}_0 commutes with $\mathcal{G}_{1,\varepsilon}$, we only need to estimate independently

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\mathbf{P}_0\| \quad \text{and} \quad \|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})[\mathbf{Id} - \mathbf{P}_0]\|$$

for any $\lambda \in \mathbb{C}_\mu^\star$. Since the multiplicity of the pole 0 is one, one has $\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\mathbf{P}_0 = \frac{1}{\lambda}\mathbf{P}_0$ and

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\mathbf{P}_0\| \leq \frac{\|\mathbf{P}_0\|}{|\lambda|}, \quad \lambda \in \mathbb{C}_{\mu_\star}^\star.$$

On the other hand, since for any $\lambda \in \mathbb{C}_{\mu_\star}$

$$\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})[\mathbf{Id} - \mathbf{P}_0] = \int_0^\infty e^{-\lambda t} \mathcal{V}_{1,\varepsilon}(t) [\mathbf{Id} - \mathbf{P}_0] dt,$$

one deduces from Theorem 2.12 that

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})[\mathbf{Id} - \mathbf{P}_0]\| \leq C_0 \int_0^\infty e^{-\operatorname{Re}\lambda t} e^{-\mu_\star t} \|\mathbf{Id} - \mathbf{P}_0\| dt,$$

which gives that

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})[\mathbf{Id} - \mathbf{P}_0]\| \leq C_0 \|\mathbf{Id} - \mathbf{P}_0\| \frac{1}{\operatorname{Re}\lambda + \mu_\star}, \quad \forall \lambda \in \mathbb{C}_{\mu_\star}.$$

This gives the desired estimate with $C_1 = \|\mathbf{P}_0\| + C_0\|\mathbf{Id} - \mathbf{P}_0\|$ independent of ε and μ . \square

3. LINEAR THEORY IN THE WEAKLY INELASTIC REGIME

We start this part by giving some results on the spectrum of the homogeneous operator \mathcal{L}_α in the weakly inelastic regime.

Proposition 3.1. *For α close enough to 1, on the spaces*

$$L_v^1(\varpi_q) \cap \left\{ h \in L_v^1(\varpi_1) ; \int_{\mathbb{R}^d} h(v) dv = \int_{\mathbb{R}^d} h(v) v dv = 0 \right\}, \quad q > 2$$

and

$$L_v^2(\varpi_q) \cap \left\{ h \in L_v^1(\varpi_1) ; \int_{\mathbb{R}^d} h(v) dv = \int_{\mathbb{R}^d} h(v) v dv = 0 \right\}, \quad q > q^*,$$

the spectrum of \mathcal{L}_α is such that there exists $\bar{\mu} > 0$ such that

$$\mathfrak{S}(\mathcal{L}_\alpha) \cap \{ \lambda \in \mathbb{C} ; \operatorname{Re} \lambda > -\bar{\mu} \} = \{ -\mu_\alpha \} \quad (3.1)$$

where μ_α is a simple eigenvalue of \mathcal{L}_α with

$$\mu_\alpha = (1 - \alpha) + O((1 - \alpha)^2) \quad \text{as } \alpha \rightarrow 1. \quad (3.2)$$

Moreover, denoting by ϕ_α the unique associated eigenfunction such that $\|\phi_\alpha\|_{L_v^1(\varpi_2)} = 1$ and $\phi_\alpha(0) < 0$, it holds

$$\lim_{\alpha \rightarrow 1} \phi_\alpha(v) = c_0 (|v|^2 - d\vartheta_1) \mathcal{M}. \quad (3.3)$$

Remark 3.2. *In Mischler & Mouhot (2009), the authors obtained the exact same result but only in exponentially weighted L_v^1 -spaces. But it is an easy matter to enlarge and shrink the space in which this type of result holds thanks to the enlargement and shrinkage arguments developed in Mischler & Mouhot (2016) and thanks to the splitting of \mathcal{L}_α exhibited in (2.16). Indeed, one can check that the assumptions of (Mischler & Mouhot, 2016, Theorem 2.2) are satisfied thanks to Lemma 2.7 and Proposition 2.9.*

The final goal of this section is to prove Theorem 1.8 in both families of spaces considered there. Let $\kappa > \frac{d}{2}$ be fixed. For simplicity, in this section, we use the following notation:

$$\mathbb{Y} := \mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q), \quad \ell \in \mathbb{N}, s \in \mathbb{N}^*, \quad \ell \geq s + 1, \quad q > q^* + \kappa + 2.$$

We will actually prove that the Theorem 1.8 holds true first in the space \mathbb{Y} for any choice of the parameter $q > q^* + \kappa + 2$ and then will use a factorization argument to deduce it also holds in the spaces \mathcal{E} introduced in Theorem 1.8. We introduce then the two spaces

$$\mathbb{Y}_{-1} := \mathbb{W}_v^{s-1,2} \mathbb{W}_x^{\ell,2}(\varpi_{q-\kappa-2}), \quad \mathbb{Y}_1 := \mathbb{W}_v^{s+1,2} \mathbb{W}_x^{\ell,2}(\varpi_{q+\kappa+2})$$

and will also use the notation $\mathbb{Y}_0 := \mathbb{Y}$. We recall that, in the space \mathbb{Y} , the full linearized operator is given by

$$\mathcal{G}_{\alpha,\varepsilon} h = \varepsilon^{-2} \mathcal{L}_\alpha(h) - \varepsilon^{-1} v \cdot \nabla_x h, \quad \forall \alpha \in (0, 1],$$

with domain $\mathcal{D}(\mathcal{G}_{\alpha,\varepsilon}) = \mathbb{W}_v^{s+1,2} \mathbb{W}_x^{\ell+1,2}(\varpi_{q+1})$.

Clearly, any spatially homogeneous eigenfunction of \mathcal{L}_α associated to an eigenvalue $\lambda \in \mathbb{C}$ is an eigenfunction to $\mathcal{G}_{\alpha,\varepsilon}$ with associated eigenvalue $\varepsilon^{-2} \lambda$. In particular

$$\operatorname{Ker}(\mathcal{L}_\alpha) \subset \operatorname{Ker}(\mathcal{G}_{\alpha,\varepsilon}).$$

Notice that, in contrast to [Briant \(2015\)](#); [Briant et al. \(2019\)](#), it is not clear whether such kernels agree. We deduce in particular from [Proposition 3.1](#) that, on the space $L^2_{v,x}(\varpi_q)$,

$$-\varepsilon^{-2}\mu_\alpha \in \mathfrak{S}(\mathcal{G}_{\alpha,\varepsilon})$$

with associated eigenfunction ϕ_α , that is,

$$\mathcal{G}_{\alpha,\varepsilon}\phi_\alpha = -\varepsilon^{-2}\mu_\alpha\phi_\alpha.$$

For the eigenvalue $-\varepsilon^{-2}\mu_\alpha$ to stay sufficiently close to 0, we assume that $\alpha = \alpha(\varepsilon)$ satisfies [Assumption 1.2](#) and write

$$\mathcal{G}_\varepsilon = \mathcal{G}_{\alpha(\varepsilon),\varepsilon},$$

and keep the notation $\mathcal{G}_{1,\varepsilon}$ for the elastic operator. Similarly, for all the operators introduced in [Section 2.1](#) the double subscript (α, ε) will be replaced by ε except when $\alpha = 1$. More precisely, to fix notations, we have

$$\mathcal{G}_\varepsilon h = \varepsilon^{-2}\mathcal{L}_{\alpha(\varepsilon)}h - \varepsilon^{-1}v \cdot \nabla_x h,$$

with

$$\mathcal{L}_{\alpha(\varepsilon)}h = \mathbf{L}_{\alpha(\varepsilon)}h - \kappa_{\alpha(\varepsilon)}\nabla_v \cdot (vh),$$

and

$$\mathbf{L}_{\alpha(\varepsilon)}h = \mathcal{Q}_{\alpha(\varepsilon)}(h, G_{\alpha(\varepsilon)}) + \mathcal{Q}_{\alpha(\varepsilon)}(G_{\alpha(\varepsilon)}, h).$$

In the sequel, since j, ℓ, s, q are fixed, we set

$$\delta^\dagger := \min \{ \delta_{2,\ell,s-1,q-\kappa-2}^\dagger, \delta_{2,\ell,s,q}^\dagger, \delta_{2,\ell,s+1,q+\kappa+2}^\dagger \},$$

and

$$\alpha^\dagger := \max \{ \alpha_{2,\ell,s-1,q-\kappa-2}^\dagger, \alpha_{2,\ell,s,q}^\dagger, \alpha_{2,\ell,s+1,q+\kappa+2}^\dagger \},$$

so that, for $\delta \in (0, \delta^\dagger)$ and $\alpha \in (\alpha^\dagger, 1)$, the results of the previous section hold in all the spaces \mathbb{Y}_j . Moreover, we denote by $\varepsilon^\dagger > 0$ the *unique* solution to

$$\alpha(\varepsilon^\dagger) = \alpha^\dagger.$$

We consider $\delta \in (0, \delta^\dagger)$, $\varepsilon \in (0, \varepsilon^\dagger)$ (which implies $\alpha(\varepsilon) \in (\alpha^\dagger, 1)$), and write

$$\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon^{(\delta)}, \quad \mathcal{B}_\varepsilon = \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}.$$

Our scope here is to obtain a result similar to [Lemma 2.16](#) of [Tristani \(2016\)](#) regarding the invertibility of \mathcal{G}_ε . We actually drastically simplify the proof given there by exploiting the fact that the difference operator $\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon}$ does not involve any spatial derivatives. Precisely, one starts with the following estimate for this difference:

Lemma 3.3. *There exists some positive constant C_0 such that, for any $\varepsilon \in (0, \varepsilon^\dagger)$*

$$\max \left(\|\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon}\|_{\mathcal{B}(\mathbb{Y}_0, \mathbb{Y}_{-1})}, \|\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon}\|_{\mathcal{B}(\mathbb{Y}_1, \mathbb{Y}_0)} \right) \leq C_0 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2}. \quad (3.4)$$

Proof. Observe that, in the difference $\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon}$, the transport term $v \cdot \nabla_x$ vanishes so that

$$\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon} = \varepsilon^{-2} [\mathcal{L}_{\alpha(\varepsilon)} - \mathcal{L}_1].$$

Let us only prove the estimate in $\mathcal{B}(\mathbb{Y}_0, \mathbb{Y}_{-1})$, the other being the same (changing only the value of s and q). For a given $h \in \mathbb{Y}_0$, one has, since ∇_x^k commutes with both $\mathcal{L}_{\alpha(\varepsilon)}$ and \mathcal{L}_1 ,

$$\begin{aligned} \|(\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon})h\|_{\mathbb{Y}_{-1}}^2 &= \varepsilon^{-4} \sum_{\substack{0 \leq k \leq \ell, 0 \leq r \leq s-1, \\ r+k \leq \ell}} \int_{\mathbb{R}^d} \|\nabla_x^k \nabla_v^r [\mathcal{L}_{\alpha(\varepsilon)} - \mathcal{L}_1] h\|_{L_x^2}^2 \varpi_{q-\kappa-2}^2(v) dv \\ &= \varepsilon^{-4} \sum_{0 \leq k \leq \ell} \int_{\mathbb{T}^d} \left\| [\mathcal{L}_{\alpha(\varepsilon)} - \mathcal{L}_1] \nabla_x^k h \right\|_{\mathbb{W}_v^{s-1,2}(\varpi_{q-\kappa-2})}^2 dx \end{aligned}$$

According to (2.4), we deduce that there is $C(s, q) > 0$ such that

$$\begin{aligned} \|(\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon})h\|_{\mathbb{Y}_{-1}} &\leq C(s, q) \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \left(\sum_{0 \leq k \leq \ell} \int_{\mathbb{T}^d} \|\nabla_x^k h\|_{\mathbb{W}_v^{s,2}(\varpi_q)}^2 dx \right)^{\frac{1}{2}} \\ &= C(s, q) \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \|h\|_{\mathbb{Y}_0} \end{aligned}$$

which is the desired result. \square

Remark 3.4. The above result does seem to be true if one replace \mathbb{Y} with $\mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,2}(\varpi_q)$ here even if $\mathcal{L}_\alpha - \mathcal{L}_1$ satisfies nice estimates on $\mathbb{W}_v^{s,1}(\varpi_q)$. The use of Fubini Theorem is fundamental here to be able to get our rate.

Remark 3.5. Notice that the exact same argument together with Lemma 2.1 show that for $\ell, s \in \mathbb{N}$ and $q \geq 0$,

$$\begin{aligned} \|\mathcal{Q}_{\alpha(\varepsilon)}(g, h) - \mathcal{Q}_1(g, h)\|_{\mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,2}(\varpi_q)} \\ \leq C(1 - \alpha(\varepsilon)) \|g\|_{\mathbb{W}_v^{s+1,2} \mathbb{W}_x^{\ell,2}(\varpi_{q+2\kappa+2})} \|h\|_{\mathbb{W}_v^{s+1,2} \mathbb{W}_x^{\ell,2}(\varpi_{q+2\kappa+2})} \end{aligned}$$

with $\kappa > \frac{d}{2}$.

One can now adapt Lemma 2.16 of [Tristani \(2016\)](#):

Proposition 3.6. For all $\lambda \in \mathbb{C}_{\mu_\star}^\star$, let

$$\mathcal{J}_\varepsilon(\lambda) = (\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon}) \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon).$$

Then, $\mathcal{J}_\varepsilon(\lambda) \in \mathcal{B}(\mathbb{Y})$. Furthermore, for any $\mu \in (0, \mu_\star)$ and

$$\lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu) = \{z \in \mathbb{C} ; \operatorname{Re} z > -\mu, |z| > \mu_\star - \mu\},$$

it holds that

$$\|\mathcal{J}_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{Y})} \leq \frac{C}{\mu_\star - \mu} \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \quad (3.5)$$

for a universal constant $C > 0$.

In addition, there exists $\varepsilon^* \in (0, \varepsilon^\dagger)$ such that $\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda)$ and $\lambda - \mathcal{G}_\varepsilon$ are invertible in \mathbb{Y} with

$$\mathcal{R}(\lambda, \mathcal{G}_\varepsilon) = \Gamma_\varepsilon(\lambda)(\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu), \quad \varepsilon \in (0, \varepsilon^*), \quad (3.6)$$

where $\Gamma_\varepsilon(\lambda) = \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) + \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\mathcal{A}_\varepsilon\mathcal{R}(\lambda, \mathcal{B}_\varepsilon)$. Finally, there exists some constant $C > 0$ such that

$$\|\mathcal{R}(\lambda, \mathcal{G}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y})} \leq \frac{C}{\mu_\star - \mu}, \quad \forall \lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu), \quad \varepsilon \in (0, \varepsilon^*). \quad (3.7)$$

Proof. We adapt the method of [Tristani \(2016\)](#) but simplifies it in several aspects. For $\operatorname{Re}\lambda > -\mu_\star$, $\lambda \neq 0$, one knows from [Proposition 2.15](#) that $\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \in \mathcal{B}(\mathbb{Y}_1)$ since we assumed $\ell \geq s + 1$ and there is $C_1 > 0$ such that, for any $\varepsilon \in (0, \varepsilon^\dagger)$, it holds that

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y}_1)} \leq C_1 \max\left(\frac{1}{|\lambda|}, \frac{1}{\operatorname{Re}\lambda + \mu_\star}\right), \quad \lambda \in \mathbb{C}_{\mu_\star}^*.$$

Moreover, from [Proposition 2.9](#), there is $\nu > 0$ such that $\mathcal{B}_\varepsilon + \varepsilon^{-2}\nu$ is hypo-dissipative in \mathbb{Y} . In particular (see [Remark 2.10](#)) there exists $C_2 > 0$, independent of ε , such that

$$\|\mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y})} \leq \frac{C_2}{\operatorname{Re}\lambda + \varepsilon^{-2}\nu}, \quad \forall \operatorname{Re}\lambda > -\mu_\star.$$

Therefore, as soon as $\varepsilon^{-2}\nu > 2\mu_\star$, one gets

$$\|\mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y})} \leq \frac{C_2 \varepsilon^2}{\varepsilon^2 \operatorname{Re}\lambda + \nu} \leq C_3 \varepsilon^2, \quad \forall \operatorname{Re}\lambda > -\mu_\star, \quad (3.8)$$

with $C_3 = 2C_2/\nu$. A similar estimate holds true if \mathbb{Y} is replaced with $\mathbb{Y}_{\pm 1}$. Notice that the regularization properties of \mathcal{A}_ε in both velocity regularity and tail behaviour implies that there exists $C > 0$ (independent of ε) such that

$$\|\mathcal{A}_\varepsilon\|_{\mathcal{B}(\mathbb{Y}_i, \mathbb{Y}_j)} \leq C\varepsilon^{-2}, \quad i \leq j, \quad i, j \in \{-1, 0, 1\} \quad (3.9)$$

from which in particular,

$$\|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y}, \mathbb{Y}_1)} \leq C_4, \quad \forall \operatorname{Re}\lambda > -\mu_\star,$$

with $C_4 = C_3 C$. We deduce with this that, for any $\operatorname{Re}\lambda > -\mu_\star$, $\lambda \neq 0$, the operator $\mathcal{J}_\varepsilon(\lambda) \in \mathcal{B}(\mathbb{Y})$ is well-defined and, for any $r \in (0, \mu_\star)$

$$\begin{aligned} \|\mathcal{J}_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{Y})} &\leq \|\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon}\|_{\mathcal{B}(\mathbb{Y}_1, \mathbb{Y})} \|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y}_1)} \|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y}, \mathbb{Y}_1)} \\ &\leq C_5 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \max\left(\frac{1}{|\lambda|}, \frac{1}{\operatorname{Re}\lambda + \mu_\star}\right), \quad \lambda \in \mathbb{C}_{\mu_\star}^*, \end{aligned}$$

with $C_5 := C_0 C_1 C_4 > 0$ independent of ε . Then, for $\mu \in (0, \mu_\star)$ it holds

$$\|\mathcal{J}_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{Y})} \leq C_5 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \max\left(\frac{1}{|\lambda|}, \frac{1}{\mu_\star - \mu}\right), \quad \lambda \in \mathbb{C}_\mu^*, \quad (3.10)$$

which gives (3.5). With this, under Assumptions 1.2, one can choose ε^* small enough, depending on the difference $|\mu_\star - \mu|$, so that

$$\rho(\varepsilon) = \frac{C_5}{\mu_\star - \mu} \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} < 1, \quad \forall \varepsilon \in (0, \varepsilon^*). \quad (3.11)$$

Under such an assumption, one sees that, for all $\lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$, $\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda)$ is invertible in \mathbb{Y} with

$$(\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} = \sum_{p=0}^{\infty} [\mathcal{J}_\varepsilon(\lambda)]^p, \quad \forall \varepsilon \in (0, \varepsilon^*).$$

Let us fix then $\varepsilon \in (0, \varepsilon^*)$ and $\lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$. The range of $\Gamma_\varepsilon(\lambda)$ is clearly included in $\mathcal{D}(\mathcal{B}_\varepsilon) = \mathcal{D}(\mathcal{G}_{1,\varepsilon})$. Then, writing $\mathcal{G}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ we easily get that

$$(\lambda - \mathcal{G}_\varepsilon)\Gamma_\varepsilon(\lambda) = \mathbf{Id} - \mathcal{J}_\varepsilon(\lambda)$$

i.e. $\Gamma_\varepsilon(\lambda)(\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1}$ is a right-inverse of $(\lambda - \mathcal{G}_\varepsilon)$. To prove that $\lambda - \mathcal{G}_\varepsilon$ is invertible, it is therefore enough to prove that it is one-to-one. Consider the eigenvalue problem

$$\mathcal{G}_\varepsilon h = \lambda h, \quad h \in \mathcal{D}(\mathcal{G}_\varepsilon),$$

Writing this as $(\lambda - \mathcal{G}_{1,\varepsilon})h = \mathcal{G}_\varepsilon h - \mathcal{G}_{1,\varepsilon}h$, there is a positive constant $C_6 > 0$ independent of ε such that

$$\|h\|_{\mathbb{Y}} = \|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})(\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon})h\|_{\mathbb{Y}} \leq C_6 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \|h\|_{\mathbb{Y}_1} \quad (3.12)$$

where we used Proposition 2.15 to estimate $\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y})}$ on $\mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$ and (3.4) for the difference $(\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon})h$. Let us now estimate $\|h\|_{\mathbb{Y}_1}$. Since $\mathcal{G}_\varepsilon h = \lambda h$, one has $(\lambda - \mathcal{B}_\varepsilon)h = \mathcal{A}_\varepsilon h$ and $h = \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\mathcal{A}_\varepsilon h$, so that, thanks to (3.8),

$$\|h\|_{\mathbb{Y}_1} \leq \|\mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y}_1)} \|\mathcal{A}_\varepsilon h\|_{\mathbb{Y}_1} \leq C_3 \varepsilon^2 \|\mathcal{A}_\varepsilon h\|_{\mathbb{Y}_1} \leq \bar{C}_3 \|h\|_{\mathbb{Y}}$$

where we used (3.9). Combining this with the above estimate (3.12), we end up with

$$\|h\|_{\mathbb{Y}} \leq C_7 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \|h\|_{\mathbb{Y}}$$

with $C_7 := C_6 \bar{C}_3$ independent of ε . One sees that, up to reducing ε^* , one can assume that $C_7 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} < 1$ for $\varepsilon \in (0, \varepsilon^*)$ which implies that $h = 0$. This proves that $\lambda - \mathcal{G}_\varepsilon$ is one-to-one and its right-inverse is, actually, its inverse. Thus, for $\varepsilon \in (0, \varepsilon^*)$, $\mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$ belongs to the resolvent set of \mathcal{G}_ε and this shows (3.6). To estimate now $\|\mathcal{R}(\lambda, \mathcal{G}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y})}$ one simply notices that

$$\|(\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1}\|_{\mathcal{B}(\mathbb{Y})} \leq \sum_{p=0}^{\infty} \|\mathcal{J}_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{Y})}^p \leq \frac{1}{1 - \rho(\varepsilon)}, \quad \forall \lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu) \quad (3.13)$$

from which, as soon as $\lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$,

$$\|\mathcal{R}(\lambda, \mathcal{G}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y})} \leq \frac{1}{1 - \rho(\varepsilon)} \|\Gamma_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{Y})}.$$

One checks, using the previous computations, that for $\lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$,

$$\|\Gamma_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{Y})} \leq C_3\varepsilon^2 + C_3\|\mathcal{A}\|_{\mathcal{B}(\mathbb{Y})}\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y})} \quad (3.14)$$

and deduces (3.7). This achieves the proof. \square

Remark 3.7. *Of course, the above result is relevant mainly for $\frac{1}{2}\mu_\star < \mu < \mu_\star$ for which $\mathbb{D}(\mu_\star - \mu) \subset \mathbb{C}_\mu$, see Figure 1. Notice also that, in previous statement, the parameter ε^\star is depending only on the gap*

$$\chi := \mu_\star - \mu.$$

From (3.11) we consider ε for which

$$\lambda_0 := \lim_{\varepsilon \rightarrow 0^+} \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \ll \chi,$$

therefore, λ_0 is a fraction of χ .

A first obvious consequence of Proposition 3.6 is that, for any $\mu \in (0, \mu_\star)$, there is $\varepsilon^\star \in (0, \varepsilon^\dagger)$ depending only on $\chi = \mu_\star - \mu$ such that, on \mathbb{Y}

$$\mathfrak{S}(\mathcal{G}_\varepsilon) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > -\mu\} \subset \{z \in \mathbb{C}; |z| \leq \mu_\star - \mu\}, \quad \forall \varepsilon \in (0, \varepsilon^\star).$$

We denote by \mathbf{P}_ε the spectral projection associated to the set

$$\mathfrak{S}_\varepsilon := \mathfrak{S}(\mathcal{G}_\varepsilon) \cap \mathbb{C}_\mu = \mathfrak{S}(\mathcal{G}_\varepsilon) \cap \mathbb{D}(\mu_\star - \mu).$$

One can deduce then the following lemma whose proof is similar to (Tristani, 2016, Lemma 2.17).

Lemma 3.8. *For any $\mu \in (0, \mu_\star)$ there is some $\varepsilon_0^\star \in (0, \varepsilon^\star)$ depending only on $\mu_\star - \mu$ and such that*

$$\|\mathbf{P}_\varepsilon - \mathbf{P}_0\|_{\mathcal{B}(\mathbb{Y})} < 1, \quad \forall \varepsilon \in (0, \varepsilon_0^\star).$$

In particular,

$$\dim \operatorname{Range}(\mathbf{P}_\varepsilon) = \dim \operatorname{Range}(\mathbf{P}_0) = d + 2, \quad \forall \varepsilon \in (0, \varepsilon_0^\star). \quad (3.15)$$

Proof. Let $\frac{\mu_\star}{2} < \mu < \mu_\star$ and $0 < r < \chi := \mu_\star - \mu$. Recall that ε^\star depends only on χ . One has $\mathbb{D}(r) \subset \mathbb{C}_\mu^\star$. We set $\gamma_r := \{z \in \mathbb{C}; |z| = r\}$. Recall that

$$\mathbf{P}_\varepsilon := \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{G}_\varepsilon) d\lambda, \quad \mathbf{P}_0 := \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) d\lambda.$$

For $\lambda \in \gamma_r$, set

$$\mathcal{Z}_\varepsilon(\lambda) = \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)$$

so that $\Gamma_\varepsilon(\lambda) = \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) + \mathcal{Z}_\varepsilon(\lambda)$. Recall from (3.6) that, for $\lambda \in \gamma_r$,

$$\begin{aligned} \mathcal{R}(\lambda, \mathcal{G}_\varepsilon) &= \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} + \mathcal{Z}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} \\ &= \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) + \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) \mathcal{J}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} + \mathcal{Z}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} \end{aligned}$$

where we wrote $(\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} = \mathbf{Id} + \mathcal{J}_\varepsilon(\lambda)(\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1}$. In the same way, one sees that

$$\begin{aligned} \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) &= \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) + \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\mathcal{A}_\varepsilon\mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) \\ &= \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) + \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\mathcal{A}_\varepsilon [\mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) - \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)] + \mathcal{Z}_\varepsilon(\lambda). \end{aligned}$$

Since the mappings $\lambda \in \mathbb{D}(r) \mapsto \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)$ and $\lambda \in \mathbb{D}(r) \mapsto \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})$ are analytic, one has

$$\oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) d\lambda = \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) d\lambda = 0,$$

so that

$$\mathbf{P}_\varepsilon = \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) \mathcal{J}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} d\lambda + \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{Z}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} d\lambda,$$

whereas

$$\mathbf{P}_0 = \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon [\mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) - \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)] d\lambda + \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{Z}_\varepsilon(\lambda) d\lambda.$$

Consequently, one easily obtains that

$$\begin{aligned} \mathbf{P}_\varepsilon - \mathbf{P}_0 &= \frac{1}{2i\pi} \oint_{\gamma_r} \Gamma_\varepsilon(\lambda) \mathcal{J}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} d\lambda \\ &\quad + \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon [\mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})] d\lambda. \end{aligned}$$

Using (3.10), (3.13), and (3.7), one notices that there exists $C > 0$ independent of ε such that

$$\|\Gamma_\varepsilon(\lambda) \mathcal{J}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1}\|_{\mathcal{B}(\mathbb{Y})} \leq \frac{C}{r^2(1-\rho(\varepsilon))} \frac{1-\alpha(\varepsilon)}{\varepsilon^2}, \quad \forall \lambda \in \gamma_r,$$

where we used that $0 < r < \mu_* - \mu$ and noticed that $\|\Gamma_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{Y})} \leq C/r$ by virtue of (3.14). Moreover, from Proposition 2.15, it follows that

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon [\mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})]\|_{\mathcal{B}(\mathbb{Y})} \leq \frac{C_1}{r} \|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y})}$$

for any $\lambda \in \gamma_r$, from which

$$\|\mathbf{P}_\varepsilon - \mathbf{P}_0\|_{\mathcal{B}(\mathbb{Y})} \leq \frac{C_0}{r} \left(\frac{1}{r(1-\rho(\varepsilon))} \frac{1-\alpha(\varepsilon)}{\varepsilon^2} + \sup_{\lambda \in \gamma_r} \|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y})} \right)$$

for some positive constant $C_0 > 0$ independent of ε . We only need to estimate

$$\|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y})}$$

for $\lambda \in \gamma_r$. Observe that, for $\lambda \in \gamma_r$,

$$\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) = \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) [\mathcal{B}_\varepsilon - \mathcal{B}_{1,\varepsilon}] \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})$$

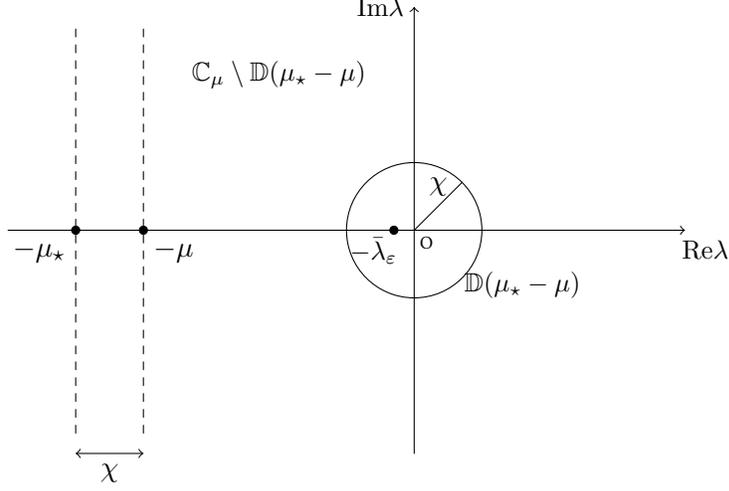


FIGURE 1. The set $\mathbb{C}_\mu \setminus \mathbb{D}(\mu_* - \mu)$ and the eigenvalue $-\bar{\lambda}_\varepsilon$.

and, with the notations of the proof of Proposition 3.6,

$$\begin{aligned} & \|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y})} \\ & \leq \|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y}_{-1}, \mathbb{Y}_0)} \|\mathcal{B}_\varepsilon - \mathcal{B}_{1,\varepsilon}\|_{\mathcal{B}(\mathbb{Y}_0, \mathbb{Y}_{-1})} \|\mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y}_0)}. \end{aligned}$$

Now, as in Proposition 3.6 (see (3.8)) there is a positive constant $C > 0$ independent of ε such that

$$\|\mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y}_{-1})} \leq C\varepsilon^2 \quad \|\mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y}_0)} \leq C, \quad \lambda \in \gamma_r,$$

where we used the hypo-dissipativity of $\mathcal{B}_\varepsilon + \varepsilon^{-2}\nu_0$ in \mathbb{Y}_{-1} thanks to Proposition 2.9. Now, using (3.9) with $i = -1, j = 0$, we deduce from (3.4) that

$$\|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y})} \leq C \frac{1 - \alpha(\varepsilon)}{\varepsilon^2}, \quad \forall \lambda \in \gamma_r.$$

Gathering the previous estimates, it follows that, for any $0 < r < \chi = \mu_* - \mu$,

$$\|\mathbf{P}_\varepsilon - \mathbf{P}_0\|_{\mathcal{B}(\mathbb{Y})} \leq \frac{C}{r} \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \left(\frac{1}{r(1 - \rho(\varepsilon))} + 1 \right) := \ell(\varepsilon) \quad (3.16)$$

and, thanks to Assumption 1.2, one can find ε_* depending only on χ such that $\ell(\varepsilon) < 1$ for any $\varepsilon \in (0, \varepsilon^*)$. In particular, we deduce (3.15) from (Kato, 1980, Paragraph I.4.6). \square

With Lemma 3.8 we can prove Theorem 1.8:

Proof of Theorem 1.8. We prove the result first in the space

$$\mathcal{E} = \mathbb{Y} = \mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q), \quad \ell \geq s + 1, \quad q > q^* + \kappa + 2$$

where we recall that $\kappa > \frac{d}{2}$. The structure of the spectrum of $\mathfrak{S}(\mathcal{G}_\varepsilon) \cap \mathbb{C}_\mu$ in the space \mathbb{Y} comes directly from Lemma 3.8 together with Proposition 3.6. To describe more precisely the spectrum, one first recalls that

$$\mathfrak{S}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \{z \in \mathbb{C} ; \operatorname{Re} z > -\mu\} \subset \mathfrak{S}(\mathcal{G}_\varepsilon) \cap \{z \in \mathbb{C} ; \operatorname{Re} z > -\mu\}.$$

Since, for ε small enough, the spectral projection $\Pi_{\mathcal{L}_{\alpha(\varepsilon)}}$ associated to $\mathfrak{S}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \mathbb{C}_\mu$ satisfies

$$\dim(\operatorname{Range}(\Pi_{\mathcal{L}_{\alpha(\varepsilon)}})) = \dim(\operatorname{Range}(\Pi_{\mathcal{L}_1})) = d + 2 = \dim(\operatorname{Range}(\mathbf{P}_\varepsilon)),$$

we get that

$$\mathfrak{S}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \mathbb{C}_\mu = \mathfrak{S}(\mathcal{G}_\varepsilon) \cap \mathbb{C}_\mu, \quad (3.17)$$

that is, the eigenvalues $\lambda_j(\varepsilon)$ are actually eigenvalues of $\mathcal{L}_{\alpha(\varepsilon)}$. In particular, one has that

$$\lambda_{d+2}(\varepsilon) = -\varepsilon^{-2} \mu_{\alpha(\varepsilon)} = -\frac{1 - \alpha(\varepsilon)}{\varepsilon^2} + \mathcal{O}\left(\left(\frac{1 - \alpha(\varepsilon)}{\varepsilon}\right)^2\right), \quad \text{for } \varepsilon \simeq 0,$$

according to (3.1) and (3.2). We set

$$\bar{\lambda}_\varepsilon := -\lambda_{d+2}(\varepsilon) > 0, \quad \lambda_\varepsilon \simeq -\varepsilon^{-2}(1 - \alpha(\varepsilon)).$$

For the other eigenvalues, one notices that

$$\int_{\mathbb{R}^d} \mathcal{L}_{\alpha(\varepsilon)} \varphi(v) dv = 0, \quad \forall \varphi \in \mathcal{D}(\mathcal{L}_{\alpha(\varepsilon)}) \subset \mathbb{Y}.$$

Of course, the spatial variable x plays no role here since $\mathcal{L}_{\alpha(\varepsilon)}$ is local in x . We begin with understanding the eigenfunctions in

$$\bar{Y} = L_v^2 L_x^2(\varpi_{q+\kappa}).$$

Recall that

$$\int_{\mathbb{R}^d} \mathcal{L}_{\alpha(\varepsilon)} \varphi(v) dv = 0, \quad \forall \varphi \in \mathcal{D}(\mathcal{L}_{\alpha(\varepsilon)}) \subset \bar{Y},$$

which implies that

$$\varpi_{q+\kappa}^{-1} \in \mathcal{D}(\mathcal{L}_{\alpha(\varepsilon)}^*) \quad \text{with} \quad \mathcal{L}_{\alpha(\varepsilon)}^*(\varpi_{q+\kappa}^{-1}) = 0,$$

that is, 0 is an eigenvalue of the adjoint $\mathcal{L}_{\alpha(\varepsilon)}^*$ in \bar{Y}^* and therefore an eigenvalue of $\mathcal{L}_{\alpha(\varepsilon)}$ in \bar{Y} .

With the same reasoning, since

$$\int_{\mathbb{R}^d} \mathcal{L}_{\alpha(\varepsilon)} \varphi(v) v_i dv = -\varepsilon^{-2} \kappa_{\alpha(\varepsilon)} \int_{\mathbb{R}^d} v_i \nabla \cdot (v \varphi(v)) dv = \varepsilon^{-2} \kappa_{\alpha(\varepsilon)} \int_{\mathbb{R}^d} v_i \varphi(v) dv$$

one sees that, for any $i = 1, \dots, d$, $m_i^*(v) := v_i \varpi_{q+\kappa}^{-1}(v) \in \mathcal{D}(\mathcal{L}_{\alpha(\varepsilon)}^*)$ satisfies

$$\mathcal{L}_{\alpha(\varepsilon)}^* m_i^* = \varepsilon^{-2} \kappa_{\alpha(\varepsilon)} m_i^*,$$

that is, $\varepsilon^{-2} \kappa_{\alpha(\varepsilon)}$ is an eigenvalue of $\mathcal{L}_{\alpha(\varepsilon)}^*$ of multiplicity d and, as such, an eigenvalue of $\mathcal{L}_{\alpha(\varepsilon)}$ with same multiplicity in the space \bar{Y} . With this, we found $d + 1$ eigenvalues of $\mathcal{L}_{\alpha(\varepsilon)}$ in the space \bar{Y} . To prove that these $d + 1$ eigenvalues are still eigenvalues of $\mathcal{L}_{\alpha(\varepsilon)}$ in the smaller

space \mathbb{Y} , we proceed as follows. Let \bar{g} be an eigenfunction of $\mathcal{L}_{\alpha(\varepsilon)}$ in \bar{Y} associated to the $\varepsilon^{-2}\kappa_{\alpha(\varepsilon)}$ eigenvalue, i.e

$$\mathcal{L}_{\alpha(\varepsilon)} \bar{g} = \varepsilon^{-2}\kappa_{\alpha(\varepsilon)} \bar{g}, \quad \bar{g} \in \mathcal{D}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \bar{Y}.$$

With the splitting $\mathcal{L}_{\alpha(\varepsilon)} = \mathcal{B}_{\alpha}^{\delta} + \mathcal{A}^{(\delta)}$, where we recall $\alpha = \alpha(\varepsilon)$ and δ is sufficiently small, one deduces from this that

$$\left(\varepsilon^{-2}\kappa_{\alpha(\varepsilon)} - \mathcal{B}_{\alpha}^{\delta} \right) \bar{g} = \mathcal{A}^{(\delta)} \bar{g}.$$

Using the fact that for $\varepsilon^{-2}\kappa_{\alpha(\varepsilon)} < \mu_{*} - \mu < \nu_{*}$ the operator $\varepsilon^{-2}\kappa_{\alpha(\varepsilon)} - \mathcal{B}_{\alpha}^{\delta}$ is invertible in both \mathbb{Y} and \bar{Y} thanks to Proposition 2.11 and

$$\bar{g} = \mathcal{R}(\varepsilon^{-2}\kappa_{\alpha(\varepsilon)}, \mathcal{B}_{\alpha}^{(\delta)}) \mathcal{A}^{(\delta)} \bar{g}.$$

Because \bar{g} is depending on the velocity only, using the regularizing effect of $\mathcal{A}^{(\delta)}$ and the hypo-dissipativity property of the operator $\varepsilon^{-2}\kappa_{\alpha(\varepsilon)} - \mathcal{B}_{\alpha}^{\delta}$ one concludes that $\mathcal{A}^{(\delta)} \bar{g} \in \mathbb{Y}$ and, by previous identity, so is \bar{g} . Therefore, any eigenfunction of $\mathcal{L}_{\alpha(\varepsilon)}$ associated to the eigenvalue $\varepsilon^{-2}\kappa_{\alpha(\varepsilon)}$ in \bar{Y} lies in \mathbb{Y} as well, consequently, it is an eigenvalue of $\mathcal{L}_{\alpha(\varepsilon)}$ in \mathbb{Y} . It has the same multiplicity d as in \bar{Y} since the reasoning is valid for *any* eigenfunction \bar{g} . In the same way, we prove that 0 is a simple eigenvalue of $\mathcal{L}_{\alpha(\varepsilon)}$ in \mathbb{Y} . We just found exactly $d + 1$ eigenvalues and exhausted $\mathfrak{S}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \mathbb{C}_{\mu} = \mathfrak{S}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \mathbb{D}(\mu_{*} - \mu)$ under the assumption that $\varepsilon^{-2}\kappa_{\alpha(\varepsilon)} < \mu_{*} - \mu$ which gives the desired result.

Now, let us prove the result in the space

$$\mathcal{E} := \mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,2}(\varpi_q), \quad q > 2, \quad \ell \geq s.$$

As mentioned earlier, we will resort to Theorem 2.1 of Gualdani et al. (2017). We observe that, since $\kappa > \frac{d}{2}$,

$$E := \mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell+1,2}(\varpi_{q+q^{*}+\kappa}) \hookrightarrow \mathcal{E}$$

with a continuous embedding, with of course E dense in \mathcal{E} . We observe that $q+q^{*}+\kappa > q^{*}+\kappa+2$ and $\ell + 1 \geq s + 1$. From the first part of the proof, the conclusion of Theorem 1.8 holds in E . It follows then from a simple application of Theorem 2.1 in Gualdani et al. (2017) that the part of the spectrum of $\mathcal{G}_{\varepsilon}$ lying in the half plane $\{z \in \mathbb{C} ; \operatorname{Re} z \geq -\mu\}$ coincides in both the spaces \mathcal{E} and E , i.e. Theorem 1.8 holds in \mathcal{E} . Notice that all assumptions of Theorem 2.1 in Gualdani et al. (2017) are met for the pair of Banach spaces (E, \mathcal{E}) in a straightforward way due to the splitting

$$\mathcal{G}_{\varepsilon} = \mathcal{A}_{\varepsilon} + \mathcal{B}_{\varepsilon}$$

and using Lemma 2.7, Propositions 2.9–2.11. In particular, due to the regularizing effect of $\mathcal{A}_{\varepsilon}$ in velocity, Hypothesis (H2) (iii) of (Gualdani et al., 2017, Theorem 2.1) is satisfied with $n = 1$. The extension to spaces of type $\mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q)$ with $q > q^{*}$ and $\ell \geq s$ is similar. \square

Remark 3.9. *It is an interesting open question to determine whether the conclusion of Theorem 1.8 remains true in spaces of the type $\mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,1}(\varpi_q)$. The obstacle here is of course the fact that the regularizing operator \mathcal{A}_ε acts in velocity only and do not induce any gain of integrability in the space variable.*

4. NONLINEAR ANALYSIS

We now apply the results obtained so far to the study of Eq. (1.22). In all this section we assume that

$$\mathcal{E} = \mathbb{W}_v^{k,1}\mathbb{W}_x^{m,2}(\varpi_q) \quad (4.1)$$

with

$$m > d, \quad m - 1 \geq k \geq 0, \quad q \geq 3, \quad (4.2)$$

and introduce also the Hilbert space on which \mathcal{L}_1 is symmetric

$$\mathcal{H} := \mathbb{W}_{v,x}^{m,2}(\mathcal{M}^{-1/2}).$$

We recall here that \mathcal{M} is the steady state of \mathcal{L}_1 whereas \mathcal{H} is a Hilbert space on which the elastic Boltzmann equation is well-understood Briant (2015).

We also denote

$$\begin{aligned} \mathcal{E}_1 &:= \mathbb{W}_v^{k,1}\mathbb{W}_x^{m,2}(\varpi_{q+1}), & \mathcal{E}_2 &:= \mathbb{W}_v^{k+1,2}\mathbb{W}_x^{m,2}(\varpi_{q+2\kappa+2}), & \kappa &> \frac{d}{2} \\ \mathcal{H}_1 &:= \mathbb{W}_{v,x}^{m,2}(\mathcal{M}^{-1/2}\langle \cdot \rangle^{1/2}) & \text{and} & & \mathcal{E}_{-1} &:= \mathbb{W}_v^{k,1}\mathbb{W}_x^{m,2}(\varpi_{q-1}) \end{aligned} \quad (4.3)$$

where k, m, q satisfy (4.2).

The analysis of the elastic case in Briant et al. (2019); Briant (2015) holds in $\mathbb{W}_{v,x}^{\beta,2}(\mathcal{M}^{-1/2})$ for $\beta > d$. We need, however, the \mathcal{H} -norm to control the \mathcal{E} -norm, which constrains $\beta \geq m$. At the same time, it is needed that $\mathcal{A}_\varepsilon \in \mathcal{B}(\mathcal{E}, \mathcal{H})$ and, because \mathcal{A}_ε has no regularisation effect on the spatial variable, we are forced to choose $\beta \leq m$. This explains the choice of $\beta = m$. Moreover, we need the constraint $m > d$ to carry out our nonlinear analysis, more precisely, we use that the embedding $\mathbb{W}_x^{m/2,2}(\mathbb{T}^d) \hookrightarrow L_x^\infty(\mathbb{T}^d)$ is continuous if $m > d$ which provides us an algebra structure. Notice that the analysis of Briant (2015) is also valid under this condition. Taking $q \geq 3$ allows us to control the dissipation of kinetic energy $\int_{\mathbb{R}^d} \mathcal{Q}_\alpha(f, f)|v|^2 dv$ and to apply the results of Section 3. Finally, the restriction $k \leq m - 1$ in (4.2) implies the continuous embedding $\mathcal{H} \hookrightarrow \mathcal{E}_2$.

For $\mathbf{A}, \mathbf{B} > 0$, we will indicate in the sequel $\mathbf{A} \lesssim \mathbf{B}$ whenever there is a positive constant $C > 0$ depending on the mass and energy of the $h(0)$, but not on parameters like t, ε or Δ_0 , such that $\mathbf{A} \leq C \mathbf{B}$.

We adapt the approach of Briant et al. (2019) and decompose the solution h_ε into

$$h_\varepsilon(t, x, v) = h^0(t, x, v) + h^1(t, x, v)$$

where $h^0 = h_\varepsilon^0 \in \mathcal{E}$ and $h^1 = h_\varepsilon^1 \in \mathcal{H}$ are the solutions to the following system of equations

$$\begin{cases} \partial_t h^0 &= \mathcal{B}_{\alpha(\varepsilon), \varepsilon} h^0 + \varepsilon^{-1} \mathcal{Q}_{\alpha(\varepsilon)}(h^0, h^0) + \varepsilon^{-1} \left[\mathcal{Q}_{\alpha(\varepsilon)}(h^0, h^1) + \mathcal{Q}_{\alpha(\varepsilon)}(h^1, h^0) \right] \\ &+ \left[\mathcal{G}_\varepsilon h^1 - \mathcal{G}_{1, \varepsilon} h^1 \right] + \varepsilon^{-1} \left[\mathcal{Q}_{\alpha(\varepsilon)}(h^1, h^1) - \mathcal{Q}_1(h^1, h^1) \right], \\ h^0(0, x, v) &= h_{\text{in}}^\varepsilon(x, v) \in \mathcal{E}. \end{cases} \quad (4.4)$$

and

$$\begin{cases} \partial_t h^1 &= \mathcal{G}_{1, \varepsilon} h^1 + \varepsilon^{-1} \mathcal{Q}_1(h^1, h^1) + \mathcal{A}_\varepsilon h^0, \\ h^1(0, x, v) &= 0. \end{cases} \quad (4.5)$$

In this section, we omit the dependence on ε for h^0 and h^1 . We recall that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} F_{\text{in}}^\varepsilon(x, v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv dx = 0 \implies \int_{\mathbb{T}^d \times \mathbb{R}^d} f_\varepsilon(t, x, v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv dx = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and, in particular, the fluctuation $h_\varepsilon(t, x, v)$ also satisfies

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} h_\varepsilon(t, x, v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.6)$$

Recalling the definition of \mathbf{P}_0 in Theorem 2.12, we define

$$\mathbb{P}_0 h = \sum_{i=1}^{d+1} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} h \Psi_i dv dx \right) \Psi_i \mathcal{M}, \quad \Pi_0 h = \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} h \Psi_{d+2} dv dx \right) \Psi_{d+2} \mathcal{M}, \quad (4.7)$$

where recall that

$$\Psi_1 = 1, \quad \Psi_i = \frac{v_{i-1}}{\sqrt{\vartheta_1}} \text{ for } i = 2, \dots, d+1, \quad \text{and} \quad \Psi_{d+2} = \frac{1}{\vartheta_1 \sqrt{2d}} (|v|^2 - d\vartheta_1).$$

Of course, see (2.28), one has $\mathbb{P}_0 = \mathbf{P}_0 - \Pi_0$. Recall that the eigenfunctions Ψ_j are such that

$$\int_{\mathbb{R}^3} \Psi_i(v) \Psi_j(v) \mathcal{M}(v) dv = \delta_{i,j} \quad i, j = 1, \dots, d+2,$$

which in particular implies that, in the Hilbert space \mathcal{H}^4 , one has $\mathbf{Id} - \mathbf{P}_0 = \mathbf{P}_0^\perp$. We begin with two basic observations. The first one is related to \mathbb{P}_0 :

Lemma 4.1. *For $i = 1, \dots, d+1$, it holds that*

$$\left| \int_{\mathbb{T}^d \times \mathbb{R}^d} h^1(t, x, v) \Psi_i(v) dv dx \right| \leq \max \left(1, \frac{1}{\sqrt{\vartheta_1}} \right) \|h^0(t)\|_{\mathcal{E}}.$$

As a consequence,

$$\|\mathbb{P}_0 h^1(t)\|_{\mathcal{E}} \leq C \|h^0(t)\|_{\mathcal{E}},$$

for some constant $C > 0$ depending only on \mathcal{M} .

⁴Recall here that, on the space $L_v^2(\mathcal{M}^{-\frac{1}{2}})$ the inner product is $\langle f, g \rangle = \int_{\mathbb{R}^d} f(v)g(v)\mathcal{M}^{-1}(v)dv$.

Proof. Note that total mass and momentum conservation leads to

$$0 = \int_{\mathbb{T}^d \times \mathbb{R}^d} h(t, x, v) \Psi_i(v) dv dx = \int_{\mathbb{T}^d \times \mathbb{R}^d} h^0(t, x, v) \Psi_i(v) dv dx + \int_{\mathbb{T}^d \times \mathbb{R}^d} h^1(t, x, v) \Psi_i(v) dv dx, \quad i = 1, \dots, d+1.$$

Thus, for any $i = 1, \dots, d+1$,

$$\left| \int_{\mathbb{T}^d \times \mathbb{R}^d} h^1(t, x, v) \Psi_i(v) dv dx \right| = \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} h^0(t, x, v) \Psi_i(v) dv dx \right| \leq \max \left(1, \frac{1}{\sqrt{\vartheta_1}} \right) \|h^0(t)\|_{\mathcal{E}}$$

thanks to Cauchy-Schwarz inequality and since $|\Psi_i(v)| \leq \max \left(1, \frac{1}{\sqrt{\vartheta_1}} \right) \varpi_q(v)$ for any $i = 1, \dots, d+1$. Regarding the estimate for the projection, it follows from the previous inequality and (4.7) by taking for example $C := \max_{i=1, \dots, d+1} \|\Psi_i \mathcal{M}\|_{\mathcal{E}}$. \square

A second observation regards the action of Π_0 on the linearized operator \mathcal{G}_ε :

Lemma 4.2. *For any solution $h = h(t, x, v)$ to (1.22), one has*

$$\Pi_0 [\mathcal{G}_\varepsilon h(t)] = -\bar{\lambda}_\varepsilon (1 + r_\varepsilon) \Pi_0 h(t) + s_\varepsilon(t) \phi_0, \quad t \geq 0 \quad (4.8)$$

where $r_\varepsilon \in \mathbb{R}$ (independent of t) and $s_\varepsilon(t) \in \mathbb{R}$ are such that

$$|r_\varepsilon| \leq C_b (1 - \alpha(\varepsilon)), \quad |s_\varepsilon(t)| \leq C_b \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \|(\mathbf{I} - \mathbf{P}_0) h(t)\|_{L^1_{x,v}(\varpi_3)} \quad (4.9)$$

for some positive constant C_b independent of ε and $t \geq 0$.

Proof. The proof is by direct inspection. One first notices that

$$\Pi_0 h = \frac{1}{2dc_0\vartheta_1^2} \left(\int_{\mathbb{R}^d \times \mathbb{T}^d} h (|v|^2 - d\vartheta_1) dv dx \right) \phi_0(v) =: \beta_0[h] \phi_0.$$

Notice that, since $\int_{\mathbb{R}^d} \mathcal{G}_\varepsilon h dv = 0$ and $\int_{\mathbb{T}^d} v \cdot \nabla_x h dx = 0$ one has

$$\begin{aligned} \Pi_0 [\mathcal{G}_\varepsilon h(t)] &= \frac{1}{2d\varepsilon^2 c_0 \vartheta_1^2} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} \mathcal{L}_{\alpha(\varepsilon)} h(t, x, v) |v|^2 dv dx \right) \phi_0 \\ &= \frac{1}{2d\varepsilon^2 c_0 \vartheta_1^2} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} (\mathbf{L}_{\alpha(\varepsilon)} h(t, x, v) - \kappa_{\alpha(\varepsilon)} \nabla_v \cdot (vh(t, x, v))) |v|^2 dv dx \right) \phi_0 \\ &= \frac{1}{2d\varepsilon^2 c_0 \vartheta_1^2} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} (\mathbf{L}_{\alpha(\varepsilon)} h(t, x, v) + 2\kappa_{\alpha(\varepsilon)} h(t, x, v)) |v|^2 dv dx \right) \phi_0 \end{aligned}$$

Now, as in (1.8), one can check that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} \mathbf{L}_{\alpha(\varepsilon)} h(t, x, v) |v|^2 dv dx = -\frac{1 - \alpha^2(\varepsilon)}{2} \gamma_b \int_{\mathbb{T}^d \times \mathbb{R}^{2d}} h(t, x, v) G_{\alpha(\varepsilon)}(v_*) |v - v_*|^3 dv_* dv dx$$

which, writing first $h = \Pi_0 h + (\mathbf{I} - \Pi_0) h = \beta_0[h]\phi_0 + (\mathbf{I} - \Pi_0) h$ and then $G_{\alpha(\varepsilon)} = \mathcal{M}_1 + (G_{\alpha(\varepsilon)} - \mathcal{M}_1)$ gives

$$\Pi_0 [\mathcal{G}_\varepsilon h(t)] = a_\varepsilon \beta_0[h]\phi_0 + s_\varepsilon(t)\phi_0 = a_\varepsilon \Pi_0 h + s_\varepsilon(t)\phi_0$$

where

$$s_\varepsilon(t) := -\frac{1 - \alpha^2(\varepsilon)}{4d\varepsilon^2 c_0 \vartheta_1^2} \gamma_b \left(\int_{\mathbb{T}^d \times \mathbb{R}^{2d}} [\mathbf{I} - \Pi_0] h(t, x, v) G_{\alpha(\varepsilon)}(v_*) |v - v_*|^3 dv_* dv dx \right)$$

and

$$a_\varepsilon := \frac{1}{\varepsilon^2} \left\{ 2\kappa_{\alpha(\varepsilon)} - \frac{1 - \alpha^2(\varepsilon)}{4dc_0\vartheta_1^2} \gamma_b \int_{\mathbb{R}^{2d}} \phi_0(v) \mathcal{M}_1(v_*) |v - v_*|^3 dv_* dv \right. \\ \left. - \frac{1 - \alpha^2(\varepsilon)}{4dc_0\vartheta_1^2} \gamma_b \int_{\mathbb{R}^{2d}} \phi_0(v) [G_{\alpha(\varepsilon)}(v_*) - \mathcal{M}_1(v_*)] |v - v_*|^3 dv_* dv \right\}.$$

One has (see (Mischler & Mouhot, 2009, Lemma 5.19, Eq. (5.10)))

$$\frac{\gamma_b}{4} \int_{\mathbb{R}^{2d}} \phi_0(v) \mathcal{M}_1(v_*) |v - v_*|^3 dv dv_* = \frac{3d}{2} c_0 \vartheta_1^2$$

which, recalling that $\kappa_\alpha = 1 - \alpha$, results easily in

$$a_\varepsilon = \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \left\{ 2 - \frac{3}{2}(1 + \alpha(\varepsilon)) \right. \\ \left. - \frac{1 + \alpha(\varepsilon)}{4dc_0\vartheta_1^2} \gamma_b \int_{\mathbb{R}^{2d}} \phi_0(v) [G_{\alpha(\varepsilon)}(v_*) - \mathcal{M}_1(v_*)] |v - v_*|^3 dv_* dv \right\}.$$

Writing simply $1 + \alpha(\varepsilon) = 2 - (1 - \alpha(\varepsilon))$ one sees that

$$a_\varepsilon = \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} (-1 + \tilde{r}_\varepsilon),$$

with

$$|\tilde{r}_\varepsilon| \leq \frac{3}{2} (1 - \alpha(\varepsilon)) + \frac{\gamma_b}{2dc_0\vartheta_1^2} \|\phi_0\|_{L_v^1(\varpi_3)} \|G_{\alpha(\varepsilon)} - \mathcal{M}_1\|_{L_v^1(\varpi_3)} \leq C(1 - \alpha(\varepsilon))$$

thanks to Lemma 2.3. The bound on $s_\varepsilon(t)$ is also obvious since, for solution h to (1.22), conservation of mass and momentum implies that $\Pi_0 h(t) = \mathbf{P}_0 h(t)$. Then, since

$$-\bar{\lambda}_\varepsilon = -\frac{1 - \alpha(\varepsilon)}{\varepsilon^2} + \mathcal{O}\left(\left(\frac{1 - \alpha(\varepsilon)}{\varepsilon^2}\right)^2\right)$$

we get the desired result. \square

Remark 4.3. If we denote by Π_ε the spectral projection associated to \mathcal{G}_ε and its eigenvalue $-\bar{\lambda}_\varepsilon$, it may appear at first sight preferable to rather deal with the projection Π_ε (since $\Pi_\varepsilon \mathcal{G}_\varepsilon h = -\bar{\lambda}_\varepsilon \Pi_\varepsilon h$)

but we face then two different problems: first, Π_ε is not fully explicit whereas Π_0 is; second, applying Π_ε to the equation satisfied by h

$$\partial_t h = \mathcal{G}_\varepsilon h + \varepsilon^{-1} \mathcal{Q}_{\alpha(\varepsilon)}(h, h),$$

nothing guarantees that $\Pi_\varepsilon [\varepsilon^{-1} \mathcal{Q}_{\alpha(\varepsilon)}(h, h)]$ remains of order 1 with respect to ε whereas we will see later on (see Lemma 4.5) that, due to the dissipation of kinetic energy, $\Pi_0 [\varepsilon^{-1} \mathcal{Q}_{\alpha(\varepsilon)}(h, h)]$ is actually of order ε .

In all the sequel, we will denote

$$\lambda_\varepsilon := \bar{\lambda}_\varepsilon (1 + r_\varepsilon), \quad \lambda_\varepsilon > 0.$$

Notice that λ_ε does not exactly corresponds to the eigenvalue $\bar{\lambda}_\varepsilon$ of \mathcal{G}_ε but we observe that

$$\lambda_\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} \bar{\lambda}_\varepsilon$$

with $\lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon = \lim_{\varepsilon \rightarrow 0} \bar{\lambda}_\varepsilon = \lambda_0$ where we recall λ_0 is defined in Assumption 1.2. In the rest of this Section, we estimate separately h^0 and h^1 .

4.1. **Estimating h^0 .** For the part of the solution $h^0(t)$ in \mathcal{E} we have the following estimate.

Proposition 4.4. *Assume that $h^0 \in \mathcal{E}$, $h^1 \in \mathcal{H}$ are such that*

$$\sup_{t \geq 0} (\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{H}}) \leq \Delta_0 < \infty.$$

Let $\nu_0 := \min\{\nu_{1,m,k,q}, \nu_{1,m,k,q+1}\}$ given in Proposition 2.9. Then, for $\mu_0 \in (0, \nu_0)$ there exists an explicit $\varepsilon_1 > 0$ (that can be chosen less than ε_0 defined in Theorem 2.12) such that:

$$\begin{aligned} \|h^0(t)\|_{\mathcal{E}} &\lesssim \|h^0(0)\|_{\mathcal{E}} e^{-\frac{\mu_0}{\varepsilon^2} t} + \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2} ds \\ &+ \varepsilon \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds, \quad \forall \varepsilon \in (0, \varepsilon_1). \end{aligned} \quad (4.10)$$

As a consequence, for any $\varepsilon \in (0, \varepsilon_1)$,

$$\begin{aligned} \|h^0(t)\|_{\mathcal{E}}^2 &\lesssim \|h^0(0)\|_{\mathcal{E}}^2 e^{-\frac{2\mu_0}{\varepsilon^2} t} + (\varepsilon \lambda_\varepsilon)^2 \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds \\ &+ (\varepsilon^2 \lambda_\varepsilon)^2 \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds. \end{aligned} \quad (4.11)$$

Proof. In the subsequent proof, we denote by $\|\cdot\|_{\mathcal{E}_1}$ and $\|\cdot\|_{\mathcal{E}}$ the norms on \mathcal{E}_1 and \mathcal{E} that are equivalent to the standard ones (with multiplicative constants independent of ε) and that make $\varepsilon^{-2}\nu_0 + \mathcal{B}_{\alpha(\varepsilon),\varepsilon}$ dissipative.⁵ The conclusion with standard norms will simply follows by

⁵ More precisely, we shall use here norms such that

$$\text{if } \partial_t g = \mathcal{B}_{\alpha(\varepsilon),\varepsilon} g \quad \text{then} \quad \frac{d}{dt} \|g(t)\|_{\mathcal{E}} \leq -\varepsilon^{-2}\nu_0 \|g(t)\|_{\mathcal{E}_1}$$

equivalence. We first observe that

$$\begin{aligned} \frac{d}{dt} \|h^0(t)\|_{\mathcal{E}} &\leq -\frac{\nu_0}{\varepsilon^2} \|h^0(t)\|_{\mathcal{E}_1} + \varepsilon^{-1} \left(\|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t), h^0(t))\|_{\mathcal{E}} + \|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t), h^1(t))\|_{\mathcal{E}} \right. \\ &\quad \left. + \|\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t), h^0(t))\|_{\mathcal{E}} \right) + \|\mathcal{G}_{\varepsilon} h^1(t) - \mathcal{G}_{1,\varepsilon} h^1(t)\|_{\mathcal{E}} \\ &\quad + \varepsilon^{-1} \|\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t), h^1(t)) - \mathcal{Q}_1(h^1(t), h^1(t))\|_{\mathcal{E}}. \end{aligned}$$

Using classical estimates for $\mathcal{Q}_{\alpha(\varepsilon)}$ and \mathcal{Q}_1 , (see [Alonso et al. \(2010\)](#); [Alonso & Gamba \(2011\)](#)), there exist $C > 0$ independent of ε such that

$$\begin{aligned} \|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t), h^0(t))\|_{\mathcal{E}} + \|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t), h^1(t))\|_{\mathcal{E}} \\ + \|\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t), h^0(t))\|_{\mathcal{E}} \leq C \left(\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{E}_1} \right) \|h^0(t)\|_{\mathcal{E}_1}, \end{aligned}$$

and, thanks to [Remark 3.5](#)

$$\begin{aligned} \|\mathcal{G}_{\varepsilon} h^1(t) - \mathcal{G}_{1,\varepsilon} h^1(t)\|_{\mathcal{E}} + \varepsilon^{-1} \|\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t), h^1(t)) - \mathcal{Q}_1(h^1(t), h^1(t))\|_{\mathcal{E}} \\ \leq C(1 - \alpha(\varepsilon)) \|h^1(t)\|_{\mathcal{E}_2} \left(\varepsilon^{-2} + \varepsilon^{-1} \|h^1(t)\|_{\mathcal{E}_2} \right). \end{aligned}$$

Notice that such estimate is exactly what motivated the definition of \mathcal{E}_2 . We conclude that

$$\begin{aligned} \frac{d}{dt} \|h^0(t)\|_{\mathcal{E}} &\leq -\varepsilon^{-2} \left(\nu_0 - \varepsilon C (\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{E}_1}) \right) \|h^0(t)\|_{\mathcal{E}_1} \\ &\quad + C(1 - \alpha(\varepsilon)) \varepsilon^{-2} \|h^1(t)\|_{\mathcal{E}_2} + C(1 - \alpha(\varepsilon)) \varepsilon^{-1} \|h^1(t)\|_{\mathcal{E}_2}^2. \end{aligned}$$

For any $\mu_0 \in (0, \nu_0)$, we pick $\varepsilon_1 \in (0, \varepsilon_0)$ as $\nu_0 - \varepsilon_1 C \Delta_0 \geq \mu_0$. Therefore,

$$\nu_0 - \varepsilon C (\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{E}_1}) \geq \mu_0, \quad \forall \varepsilon \in (0, \varepsilon_1).$$

Consequently, we obtain that

$$\begin{aligned} \frac{d}{dt} \|h^0(t)\|_{\mathcal{E}} &\leq -\frac{\mu_0}{\varepsilon^2} \|h^0(t)\|_{\mathcal{E}_1} + C(1 - \alpha(\varepsilon)) \varepsilon^{-2} \|h^1(t)\|_{\mathcal{E}_2} \\ &\quad + C(1 - \alpha(\varepsilon)) \varepsilon^{-1} \|h^1(t)\|_{\mathcal{E}_2}^2, \tag{4.12} \\ &\leq -\frac{\mu_0}{\varepsilon^2} \|h^0(t)\|_{\mathcal{E}_1} + C\lambda_{\varepsilon} \|h^1(t)\|_{\mathcal{E}_2} + C\varepsilon\lambda_{\varepsilon} \|h^1(t)\|_{\mathcal{E}_2}^2, \quad \forall t \geq 0, \end{aligned}$$

where we used that $\varepsilon^2 \lambda_{\varepsilon} \simeq \varepsilon^2 \bar{\lambda}_{\varepsilon} \simeq 1 - \alpha(\varepsilon)$ which gives [\(4.10\)](#) after integration. To prove [\(4.11\)](#), we use the fact that by Cauchy-Schwarz inequality, for any nonnegative mapping $t \mapsto \zeta(t)$ and $\beta > 0$, we have that for any $r \in (0, 1)$,

$$\begin{aligned} \left(\int_0^t e^{-\beta(t-s)} \zeta(s) ds \right)^2 &\leq \left(\int_0^t e^{-2r\beta(t-s)} ds \right) \left(\int_0^t e^{-2(1-r)\beta(t-s)} \zeta(s)^2 ds \right) \\ &\leq \frac{1}{2r\beta} \int_0^t e^{-2(1-r)\beta(t-s)} \zeta(s)^2 ds, \quad \forall t \geq 0. \end{aligned} \tag{4.13}$$

Notice that such norm exists from [Proposition 2.9](#).

This inequality applied with $r = \frac{1}{2}$ gives the result. \square

4.2. **Estimating $\mathbf{P}_0 h^1$.** One has the following fundamental estimate for $\mathbf{P}_0 h^1(t)$.

Lemma 4.5. *We have that*

$$\begin{aligned} \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}} &\lesssim \|\Pi_0 h(0)\|_{\mathcal{E}} e^{-\lambda_\varepsilon t} + \|h^0(t)\|_{\mathcal{E}} \\ &\quad + \varepsilon \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \left(\|h^1(s)\|_{\mathcal{E}}^2 + \|h^0(s)\|_{\mathcal{E}}^2 \right) ds \\ &\quad + \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \left(\|h^0(s)\|_{\mathcal{E}} + \|(\mathbf{I} - \mathbf{P}_0) h^1(s)\|_{\mathcal{E}} \right) ds \end{aligned} \quad (4.14)$$

for any $t \geq 0$.

Proof. The equation for h is given by

$$\partial_t h = \mathcal{G}_\varepsilon h + \varepsilon^{-1} \mathcal{Q}_{\alpha(\varepsilon)}(h, h).$$

Thus, applying the projection Π_0 and using (4.8)

$$\partial_t (\Pi_0 h) = -\bar{\lambda}_\varepsilon (1 + r_\varepsilon) \Pi_0 h + s_\varepsilon(t) \phi_0 + \varepsilon^{-1} \Pi_0 \mathcal{Q}_{\alpha(\varepsilon)}(h, h),$$

so that

$$\Pi_0 h(t) = \Pi_0 h(0) e^{-\lambda_\varepsilon t} + \int_0^t e^{-\lambda_\varepsilon(t-s)} \left(\varepsilon^{-1} \Pi_0 \mathcal{Q}_{\alpha(\varepsilon)}(h(s), h(s)) + s_\varepsilon(s) \phi_0 \right) ds, \quad (4.15)$$

where $\lambda_\varepsilon = \bar{\lambda}_\varepsilon (1 + r_\varepsilon)$. Notice that, according to (4.7), $\Pi_0 \mathcal{Q}_{\alpha(\varepsilon)}$ is explicit with

$$\|\Pi_0 \mathcal{Q}_{\alpha(\varepsilon)}(h(s), h(s))\|_{\mathcal{E}_{-1}} = (1 - \alpha^2(\varepsilon)) \left| \mathcal{D}_{\alpha(\varepsilon)}(h(s), h(s)) \right| \|\Psi_{d+2} \mathcal{M}\|_{\mathcal{E}_{-1}},$$

where $\mathcal{D}_\alpha(g, g)$ denotes the normalized energy dissipation associated to \mathcal{Q}_α , namely,

$$\begin{aligned} \mathcal{D}_\alpha(g, g) &= -\frac{1}{1 - \alpha^2} \int_{\mathbb{T}^d \times \mathbb{R}^d} \Psi_{d+2}(v) \mathcal{Q}_\alpha(g, g) dv dx \\ &= \gamma_b \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, v) g(x, v_*) |v - v_*|^3 dv_* dv \end{aligned}$$

for some nonnegative γ_b independent of α , see (1.8). Now, one clearly has

$$|\mathcal{D}_\alpha(h(s), h(s))| \leq C \int_{\mathbb{T}^d} \left[\int_{\mathbb{R}^d} \varpi_3(v) |h(s, x, v)| dv \right]^2 dx$$

and, using Minkowski's integral inequality, we deduce that

$$|\mathcal{D}_\alpha(h(s), h(s))| \leq C \left[\int_{\mathbb{R}^d} \varpi_3(v) \left(\int_{\mathbb{T}^d} |h(s, x, v)|^2 dx \right)^{\frac{1}{2}} dv \right]^2 = C \|h(s)\|_{L_v^1 L_x^2(\varpi_3)}^2.$$

Therefore,

$$\|\Pi_0 \mathcal{Q}_{\alpha(\varepsilon)}(h(s), h(s))\|_{\mathcal{E}_{-1}} \lesssim (1 - \alpha(\varepsilon)) \|h(s)\|_{\mathcal{E}}^2$$

because $\varpi_q(v) \geq \langle v \rangle^3$ for any $v \in \mathbb{R}^3$. Thus, applying the $\|\cdot\|_{\mathcal{E}_{-1}}$ -norm in (4.15), one obtains

$$\begin{aligned} \|\Pi_0 h(t)\|_{\mathcal{E}_{-1}} &\lesssim \|\Pi_0 h(0)\|_{\mathcal{E}} e^{-\lambda_\varepsilon t} + \frac{1-\alpha(\varepsilon)}{\varepsilon} \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h(s)\|_{\mathcal{E}}^2 ds \\ &\quad + \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \|(\mathbf{I} - \mathbf{P}_0) h(s)\|_{\mathcal{E}} ds \end{aligned} \quad (4.16)$$

where we used (4.9) to estimate $\|s_\varepsilon(s)\phi_0\|_{\mathcal{E}_{-1}}$. As already observed, according to (4.6), one has

$$\mathbf{P}_0 h(t) = \Pi_0 h(t)$$

for any $t \geq 0$. Since $\mathbf{P}_0 h^1(t) = \mathbf{P}_0 h(t) - \mathbf{P}_0 h^0(t)$, we can reformulate the above (4.16) in terms of the relevant functions h^1 and h^0 to obtain the desired estimate recalling that $\varepsilon \lambda_\varepsilon \simeq \frac{1-\alpha(\varepsilon)}{\varepsilon}$. \square

We make more precise our estimates of $\mathbf{P}_0 h^1(t)$ in the following

Proposition 4.6. *There exists an explicit $\varepsilon_2 \in (0, \varepsilon_1)$ such that for any $\varepsilon \in (0, \varepsilon_2)$ and $t \geq 0$, it holds that*

$$\begin{aligned} \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}} &\lesssim \left(\|\Pi_0 h(0)\|_{\mathcal{E}} + \|h^0(0)\|_{\mathcal{E}} + \varepsilon^3 \lambda_\varepsilon \|h^0(0)\|_{\mathcal{E}}^2 \right) e^{-\lambda_\varepsilon t} \\ &\quad + \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2} ds + \varepsilon \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds \\ &\quad + \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \|(\mathbf{I} - \mathbf{P}_0) h^1(s)\|_{\mathcal{E}} ds + \varepsilon^2 \lambda_\varepsilon^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}} ds \\ &\quad + \varepsilon \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}}^2 ds + \varepsilon^7 \lambda_\varepsilon^3 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds. \end{aligned}$$

Proof. We insert the bound for $\|h^0(t)\|_{\mathcal{E}}^i$ for $i = 1, 2$ in (4.10) and (4.11) in the estimate of Lemma 4.5. Assuming $\mu_0 \geq 2\varepsilon^2 \lambda_\varepsilon$ and recalling that $1 - \alpha(\varepsilon) \simeq \varepsilon^2 \lambda_\varepsilon$, we first deduce from (4.10)–(4.14) that

$$\begin{aligned} \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}} &\lesssim \left(\|\Pi_0 h(0)\|_{\mathcal{E}} + \|h^0(0)\|_{\mathcal{E}} \right) e^{-\lambda_\varepsilon t} + \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2} ds \\ &\quad + \varepsilon \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds + \varepsilon \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}}^2 ds \\ &\quad + \lambda_\varepsilon \|h^0(0)\|_{\mathcal{E}} \int_0^t e^{-\lambda_\varepsilon(t-s)} e^{-\frac{\mu_0}{\varepsilon^2}s} ds + \lambda_\varepsilon^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2} d\tau \\ &\quad + \varepsilon \lambda_\varepsilon^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^2 d\tau + \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \|(\mathbf{I} - \mathbf{P}_0) h^1(s)\|_{\mathcal{E}} ds \\ &\quad + \varepsilon \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^0(s)\|_{\mathcal{E}}^2 ds. \end{aligned} \quad (4.17)$$

Now, using (4.11) for the last integral, we obtain

$$\begin{aligned} \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^0(s)\|_{\mathcal{E}}^2 ds &\lesssim \|h^0(0)\|_{\mathcal{E}}^2 \int_0^t e^{-\frac{2\mu_0}{\varepsilon^2}s - \lambda_\varepsilon(t-s)} ds \\ &+ (\varepsilon \lambda_\varepsilon)^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^2 d\tau \\ &+ (\varepsilon^2 \lambda_\varepsilon)^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^4 d\tau. \end{aligned}$$

Using that, for any $\beta > \alpha > 0$ and nonnegative mapping $t \mapsto \zeta(t)$

$$\begin{aligned} \int_0^t e^{-\alpha(t-s)} ds \int_0^s e^{-\beta(s-\tau)} \zeta(\tau) d\tau &= e^{-\alpha t} \int_0^t e^{\beta\tau} \zeta(\tau) d\tau \int_\tau^t e^{-(\beta-\alpha)s} ds \\ &\leq \frac{1}{\beta - \alpha} \int_0^t e^{-\alpha(t-\tau)} \zeta(\tau) d\tau \end{aligned} \quad (4.18)$$

we have, for $\mu_0 \geq 2\varepsilon^2 \lambda_\varepsilon$, that

$$\int_0^t e^{-\lambda_\varepsilon(t-s)} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^k d\tau \leq \frac{2\varepsilon^2}{\mu_0} \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^k ds, \quad k = 1, 2, 4, \quad (4.19)$$

so that

$$\begin{aligned} \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^0(s)\|_{\mathcal{E}}^2 ds &\lesssim \varepsilon^2 \|h^0(0)\|_{\mathcal{E}}^2 e^{-\lambda_\varepsilon t} + (\varepsilon^2 \lambda_\varepsilon)^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds \\ &+ (\varepsilon^3 \lambda_\varepsilon)^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds. \end{aligned}$$

Using again the above (4.19) to estimate the sixth and seventh terms of (4.17) and keeping only the dominant terms, we get the desired estimate. \square

Remark 4.7. We will also need an estimate for $\|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^2$. Using (4.13), we obtain that for any $r \in (0, 1)$,

$$\begin{aligned} \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^2 &\lesssim \left(\|\Pi_0 h(0)\|_{\mathcal{E}}^2 + \|h^0(0)\|_{\mathcal{E}}^2 + (\varepsilon^3 \lambda_\varepsilon)^2 \|h^0(0)\|_{\mathcal{E}}^4 \right) e^{-2\lambda_\varepsilon t} \\ &+ (\varepsilon \lambda_\varepsilon)^2 \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds + (\varepsilon^2 \lambda_\varepsilon)^2 \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds \\ &+ \frac{\lambda_\varepsilon}{r} \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)} \|(\mathbf{I} - \mathbf{P}_0) h^1(s)\|_{\mathcal{E}}^2 ds + \frac{\varepsilon^4 \lambda_\varepsilon^3}{r} \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds \\ &+ \frac{\varepsilon^2 \lambda_\varepsilon}{r} \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds + \frac{\varepsilon^4 (\varepsilon^2 \lambda_\varepsilon)^5}{r} \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^8 ds. \end{aligned}$$

where the multiplicative constant does not depend on r .

4.3. **Estimating the complement** $(\mathbf{Id} - \mathbf{P}_0)h^1$. Let us focus on an estimate on $\mathbf{P}_0^\perp h^1(t)$ with $\mathbf{P}_0^\perp = \mathbf{Id} - \mathbf{P}_0$, the orthogonal projection onto $(\text{Ker}(\mathcal{G}_{1,\varepsilon}))^\perp$ in the Hilbert space $L_{v,x}^2(\mathcal{M}^{-1/2})$. The same notation for the operator $\mathcal{G}_{1,\varepsilon}$ in the spaces \mathcal{E} and \mathcal{H} is used. We begin with the following lemma where, we recall that $\Sigma_{\mathcal{M}}$ is defined in (2.11).

Lemma 4.8. *With the notations of Theorem 2.12, let $\varepsilon \in (0, \varepsilon_0)$, $\mu \in (0, \mu_\star)$ and assume that*

$$\sup_{t \geq 0} (\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{H}}) \leq \Delta_0$$

with $\Delta_0 \leq 1$ small enough so that

$$\nu := \frac{2\mu}{\sigma_0^2} - c_0 \Delta_0^2 > 0 \quad (4.20)$$

where $\sigma_0 := \inf_{\xi \in \mathbb{R}^d} \Sigma_{\mathcal{M}}(\xi) > 0$ and $c_0 > 0$ is a universal constant depending only on \mathcal{M} defined in (4.24). Set

$$\Psi(t) = h^1(t) - \mathbf{P}_0 h^1(t), \quad \forall t \geq 0.$$

Then, there exists $C_0 > 0$ independent of $\varepsilon > 0$ such that

$$\|\Psi(t)\|_{\mathcal{H}}^2 \leq C_0 \int_0^t e^{-\nu(t-s)} \|\mathbf{P}_0 h^1(s)\|_{\mathcal{E}_{-1}}^4 ds + \frac{C_0}{\varepsilon^2} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} \|h^0(s)\|_{\mathcal{E}} ds \quad (4.21)$$

for any $t \geq 0$. In particular,

$$\|\Psi(t)\|_{\mathcal{H}}^2 \leq C_0 \Delta_0^2 \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{C_0}{\varepsilon^2} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} \|h^0(s)\|_{\mathcal{E}} ds \quad (4.22)$$

for any $t \geq 0$.

Proof. We start by recalling that $h^1(0) = 0$ so that $\Psi(0) = 0$. One checks from (4.5) that

$$\partial_t \Psi = \mathcal{G}_{1,\varepsilon} \Psi + \mathbf{P}_0^\perp (\varepsilon^{-1} \mathcal{Q}_1(h^1, h^1) + \mathcal{A}_\varepsilon h^0) = \mathcal{G}_{1,\varepsilon} \Psi + \varepsilon^{-1} \mathcal{Q}_1(h^1, h^1) + \mathbf{P}_0^\perp \mathcal{A}_\varepsilon h^0,$$

where for the later we used that $\mathbf{P}_0 \mathcal{Q}_1(h^1, h^1) = 0$. Hereafter, we denote by $\|\cdot\|_{\mathcal{H}}$ a hypocoercive norm which is equivalent to the usual one independently of ε and which allows us to write nice energy estimates. It is worth mentioning that such a norm has been exhibited in (Briant, 2015, Theorem 2.4). Using (Briant et al., 2019, Theorem 4.7), one obtains as in (Briant et al., 2019, Eq. (4.8)) that, for any $\mu \in (0, \mu_\star)$ there is some positive constant $C > 0$ such that

$$\frac{d}{dt} \|\Psi(t)\|_{\mathcal{H}}^2 \leq -\frac{2\mu}{\sigma_0^2} \|\Psi(t)\|_{\mathcal{H}_1}^2 + C \|h^1(t)\|_{\mathcal{H}}^2 \|h^1(t)\|_{\mathcal{H}_1}^2 + \|\Psi(t)\|_{\mathcal{H}} \|\mathbf{P}_0^\perp \mathcal{A}_\varepsilon h^0(t)\|_{\mathcal{H}}. \quad (4.23)$$

Writing $h^1 = \mathbf{P}_0 h^1 + \Psi$, we obtain

$$\begin{aligned} \|h^1(t)\|_{\mathcal{H}}^2 \|h^1(t)\|_{\mathcal{H}_1}^2 &\leq 2 \|h^1(t)\|_{\mathcal{H}}^2 \left(\|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}_1}^2 + \|\Psi(t)\|_{\mathcal{H}_1}^2 \right) \\ &\leq 2 \Delta_0^2 \|\Psi(t)\|_{\mathcal{H}_1}^2 + 4 \|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}_1}^2 \left(\|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}}^2 + \|\Psi(t)\|_{\mathcal{H}}^2 \right). \end{aligned}$$

In particular, since there exists a positive constant $c > 0$ depending only on \mathcal{M} such that

$$\|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}}^2 \leq \|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}_1}^2 \leq c \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^2,$$

we deduce that

$$\|h^1(t)\|_{\mathcal{H}}^2 \|h^1(t)\|_{\mathcal{H}_1}^2 \leq c_0 \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^4 + c_0 \Delta_0^2 \|\Psi(t)\|_{\mathcal{H}_1}^2 \quad (4.24)$$

for some universal constant $c_0 > 0$ depending only on \mathcal{M} . Therefore, assuming that Δ_0 is small enough so that

$$\nu := \frac{2\mu}{\sigma_0^2} - c_0 \Delta_0^2 > 0$$

we deduce that

$$\frac{d}{dt} \|\Psi(t)\|_{\mathcal{H}}^2 \leq -\nu \|\Psi(t)\|_{\mathcal{H}_1}^2 + C \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^4 + \|\Psi(t)\|_{\mathcal{H}} \|\mathbf{P}_0^\perp \mathcal{A}_\varepsilon h^0(t)\|_{\mathcal{H}}, \quad \forall t \geq 0$$

for some $C > 0$ independent of t and ε . Moreover, we also have that

$$\|\mathbf{P}_0^\perp \mathcal{A}_\varepsilon h^0(t)\|_{\mathcal{H}} \lesssim \frac{1}{\varepsilon^2} \|h^0(t)\|_{\mathcal{E}}, \quad \|\Psi(t)\|_{\mathcal{H}} \lesssim \|h^1(t)\|_{\mathcal{H}}, \quad \forall t \geq 0,$$

from which we get the desired estimate (4.21) after integration of the previous differential inequality. We deduce then (4.22) from (4.21) we use that using the estimate

$$\|\cdot\|_{\mathcal{E}_{-1}} \lesssim \|\cdot\|_{\mathcal{H}}, \quad \text{and} \quad \|h^1\|_{\mathcal{H}}^{i+2} \leq \Delta_0^i \|h^1\|_{\mathcal{H}}^2, \quad \forall i \geq 0, \quad (4.25)$$

together with the fact that $\|\mathbf{P}_0 h^1\|_{\mathcal{H}} \leq \|h^1\|_{\mathcal{H}}$. \square

To complete the estimate of $\|\Psi(t)\|_{\mathcal{H}}^2$ we need to estimate the last integral in (4.22):

Lemma 4.9. *With the notation of Lemma 4.8, there is an explicit $\varepsilon_3 \in (0, \varepsilon_2)$ such that for any $\delta > 0$, $\varepsilon \in (0, \varepsilon_3)$, and $t \geq 0$*

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} \|h^0(s)\|_{\mathcal{E}} ds &\lesssim \frac{\delta}{\varepsilon^2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{1}{\delta} \|h^0(0)\|_{\mathcal{E}}^2 e^{-\nu t} \\ &\quad + \left(\delta + \frac{\lambda_\varepsilon^2}{\delta} \right) \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds. \end{aligned}$$

Proof. We use the estimate of $\|h^0(s)\|_{\mathcal{E}}$ provided in (4.10) which gives

$$\varepsilon^{-2} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} \|h^0(s)\|_{\mathcal{E}} ds \lesssim I_1(t) + I_2(t) + I_3(t)$$

with

$$\begin{aligned} I_1(t) &= \varepsilon^{-2} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} \|h^0(0)\|_{\mathcal{E}} e^{-\frac{\mu_0}{\varepsilon^2} s} ds, \\ I_2(t) &= \varepsilon^{-2} \lambda_\varepsilon \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2} d\tau, \end{aligned}$$

and

$$I_3(t) = \varepsilon^{-1} \lambda_\varepsilon \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} \, ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^2 \, d\tau.$$

Using Young's inequality, for any $\delta > 0$ it holds that

$$\|h^1(s)\|_{\mathcal{H}} \|h^0(0)\|_{\mathcal{E}} \leq \delta \|h^1(s)\|_{\mathcal{H}}^2 + \frac{1}{4\delta} \|h^0(0)\|_{\mathcal{E}}^2,$$

so that, since $\mu_0 - \varepsilon^2 \nu \geq \frac{\mu_0}{2}$,

$$\begin{aligned} I_1(t) &\leq \delta \varepsilon^{-2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2}s} \|h^1(s)\|_{\mathcal{H}}^2 \, ds + \frac{1}{4\delta \varepsilon^2} \|h^0(0)\|_{\mathcal{E}}^2 \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2}s} \, ds \\ &\leq \delta \varepsilon^{-2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2}s} \|h^1(s)\|_{\mathcal{H}}^2 \, ds + \frac{1}{2\delta \mu_0} \|h^0(0)\|_{\mathcal{E}}^2 e^{-\nu t}. \end{aligned}$$

Similarly, Young's inequality implies, for any $\delta > 0$, that

$$\begin{aligned} I_2(t) &\leq \delta \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 \, ds \\ &\quad + \frac{(\varepsilon^{-2} \lambda_\varepsilon)^2}{4\delta} \int_0^t e^{-\nu(t-s)} \, ds \left(\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2} \, d\tau \right)^2, \end{aligned}$$

and, using (4.13) with $r = \frac{1}{2}$ and (4.18) to estimate the square of the last integral, we get for $\mu_0 \geq 2\varepsilon^2 \nu$ that

$$I_2(t) \leq \delta \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 \, ds + \frac{1}{2\delta \mu_0^2} \lambda_\varepsilon^2 \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 \, ds.$$

In the same way, it follows that

$$I_3(t) \leq \delta \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 \, ds + \frac{1}{2\delta \mu_0^2} (\varepsilon \lambda_\varepsilon)^2 \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 \, ds.$$

Combining these estimates yields

$$\begin{aligned} \varepsilon^{-2} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} \|h^0(s)\|_{\mathcal{E}} \, ds &\lesssim \delta \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 \, ds + \frac{1}{\delta} \|h^0(0)\|_{\mathcal{E}}^2 e^{-\nu t} \\ &\quad + \frac{\delta}{\varepsilon^2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2}s} \|h^1(s)\|_{\mathcal{H}}^2 \, ds + \frac{\lambda_\varepsilon^2}{\delta} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 \, ds \\ &\quad + \frac{(\varepsilon \lambda_\varepsilon)^2}{\delta} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 \, ds. \quad (4.26) \end{aligned}$$

We conclude thanks to (4.25). \square

We deduce from the previous the following main estimate for $\|\Psi(t)\|_{\mathcal{H}}$

Proposition 4.10. *Under the Assumptions of Lemma 4.8, there exist $\varepsilon_4 \in (0, \varepsilon_3)$, $\lambda_4 > 0$, $c > 0$ a positive universal constant that depends on μ_0 and ν such that, for any $\delta > 0$, $t \geq 0$, $\varepsilon \in (0, \varepsilon_4)$, $\lambda_\varepsilon \in (0, \lambda_4)$,*

$$\begin{aligned} \frac{1}{c} \|\Psi(t)\|_{\mathcal{H}}^2 &\leq \frac{1}{\delta} \|h^0(0)\|_{\mathcal{E}}^2 e^{-\nu t} + \mathbf{C}_2 \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &\quad + \frac{\delta}{\varepsilon^2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds, \end{aligned} \quad (4.27)$$

with $\mathbf{C}_2 = \mathbf{C}_2(\delta, \varepsilon, \Delta_0) := \left[\delta + \Delta_0^2 + \frac{\lambda_\varepsilon^2}{\delta} \right]$. In particular, there is a positive constant c_1 depending only on μ_0 and ν such that for any $r \in (0, 1)$,

$$\begin{aligned} \frac{1}{c_1} \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)} \|\Psi(s)\|_{\mathcal{H}}^2 ds \\ \leq \frac{1}{\delta} \|h^0(0)\|_{\mathcal{E}}^2 e^{-2(1-r)\lambda_\varepsilon t} + \mathbf{C}_2 \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ + \frac{\delta}{\varepsilon^2} \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds \end{aligned} \quad (4.28)$$

holds for any $t \geq 0$, $\delta > 0$.

Proof. Inserting the estimate obtained in Lemma 4.9 into (4.22) we get directly (4.27). Using (4.18) twice, we deduce then easily (4.28) from (4.27) after integration by choosing λ_5 small enough such that $\nu \geq 4\lambda_\varepsilon$ and thus $\nu - 2(1-r)\lambda_\varepsilon \geq \nu - 2\lambda_\varepsilon \geq \frac{\nu}{2}$ for $\lambda_\varepsilon \in (0, \lambda_4)$. \square

We deduce from the above the following

Proposition 4.11. *Under the assumptions of Lemma 4.8, there exist $\varepsilon_5 \in (0, \varepsilon_4)$ and $c_2 > 0$ that depends on μ_0 and ν such that, for any $\delta \in (0, 1)$, $t \geq 0$, $\varepsilon \in (0, \varepsilon_5)$, $\lambda_\varepsilon \in (0, \lambda_4)$ and $r \in (0, 1)$,*

$$\begin{aligned} \frac{1}{c_2} \|h^1(t)\|_{\mathcal{H}}^2 &\leq \frac{1}{\delta r} \mathcal{K}_0 e^{-2(1-r)\lambda_\varepsilon t} + \left(\delta + \frac{\lambda_\varepsilon^2}{\delta} + \Delta_0^2 \right) \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &\quad + \frac{\lambda_\varepsilon}{r} \left(\delta + \frac{\lambda_\varepsilon^2}{\delta} + \Delta_0^2 \right) \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &\quad + \frac{\delta}{\varepsilon^2} \left(1 + \frac{\lambda_\varepsilon}{r} \right) \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds \end{aligned} \quad (4.29)$$

where

$$\mathcal{K}_0 := \|h^0(0)\|_{\mathcal{E}}^2 + \|\Pi_0 h(0)\|_{\mathcal{E}}^2 + \|h^0(0)\|_{\mathcal{E}}^4$$

depends only on $h(0)$.

Proof. The previous Proposition gives an estimate of $\|\Psi(t)\|_{\mathcal{H}}^2$ in terms of $\|h^1(t)\|_{\mathcal{H}}^2$. Adding $\|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}}^2$ to both sides of (4.27) and, since

$$\|h^1(t)\|_{\mathcal{H}}^2 \leq \|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}}^2 + \|\Psi(t)\|_{\mathcal{H}}^2 \lesssim \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^2 + \|\Psi(t)\|_{\mathcal{H}}^2,$$

we only need to estimate $\|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^2$ in terms of $\|h^1(t)\|_{\mathcal{H}}^2$. We invoke the estimate in Remark 4.7 which, using that $\frac{\mu_0}{\varepsilon^2} \geq 2(1-r)\lambda_\varepsilon$ and assuming ε small enough and keeping only the dominant terms reads simply

$$\begin{aligned} \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^2 &\lesssim \left(\|\Pi_0 h(0)\|_{\mathcal{E}}^2 + \|h^0(0)\|_{\mathcal{E}}^2 + \|h^0(0)\|_{\mathcal{E}}^4 \right) e^{-2\lambda_\varepsilon t} \\ &\quad + \frac{\varepsilon^2 \lambda_\varepsilon}{r} \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{\lambda_\varepsilon}{r} \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)} \|\Psi(s)\|_{\mathcal{E}}^2 ds \end{aligned} \quad (4.30)$$

where we used (4.25) repeatedly. Then, using now (4.28) to estimate the last integral and adding (4.30) to (4.27), there is $c_0 > 0$ depending only on μ_0, ν such that

$$\begin{aligned} \frac{1}{c_0} \|h^1(t)\|_{\mathcal{H}}^2 &\leq \frac{1}{\delta r} \mathcal{K}_0 e^{-2(1-r)\lambda_\varepsilon t} + \mathbf{C}_2 \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &\quad + \frac{\lambda_\varepsilon}{r} (\varepsilon^2 + \mathbf{C}_2) \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &\quad + \frac{\delta}{\varepsilon^2} \left(1 + \frac{\lambda_\varepsilon}{r} \right) \int_0^t e^{-2(1-r)\lambda_\varepsilon(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds \end{aligned}$$

where we used that $\lambda_\varepsilon \lesssim 1$ to obtain the first bound in the right-hand-side of the inequality and $\nu \geq 2(1-r)\lambda_\varepsilon$ for the last term. This gives the desired conclusion once we noticed that, for ε small enough (independent of δ), it holds $\lambda_\varepsilon (\varepsilon^2 + \mathbf{C}_2) \lesssim \lambda_\varepsilon \left(\delta + \frac{\lambda_\varepsilon^2}{\delta} + \Delta_0^2 \right)$. \square

We derive from this the following decay rate for $\|h^1(t)\|_{\mathcal{H}}$:

Corollary 4.12. *Let $r \in (0, 1)$. There exist $\varepsilon_6 \in (0, \varepsilon_5)$, $\lambda_6 \in (0, \lambda_4)$ and $C > 0$ depending on ν, μ_0, Δ_0 and r such that*

$$\|h^1(t)\|_{\mathcal{H}}^2 \leq C \mathcal{K}_0 \exp(-2(1-r)\bar{\lambda}_\varepsilon t), \quad \forall t \geq 0, \quad (4.31)$$

for any $\varepsilon \in (0, \varepsilon_6)$, $\lambda_\varepsilon \in (0, \lambda_6)$ and where $-\bar{\lambda}_\varepsilon$ denotes the eigenvalue of $\mathcal{G}_{\alpha(\varepsilon), \varepsilon}$ obtained in Theorem 1.8, Eq. (1.25) and \mathcal{K}_0 is given in Proposition 4.11.

Proof. Set for simplicity $r_0 := \frac{r}{2}$ and

$$\mathbf{x}(t) := e^{2(1-r_0)\lambda_\varepsilon t} \|h^1(t)\|_{\mathcal{H}}^2, \quad t \geq 0.$$

We choose $\lambda_6 > 0$ small enough so that $\lambda_\varepsilon \leq r_0$ for any $\lambda_\varepsilon \in (0, \lambda_6)$. Inequality (4.29) (applied with r_0 instead of r) then yields

$$\begin{aligned} \frac{1}{c_2} \mathbf{x}(t) &\leq \frac{1}{\delta r_0} \mathcal{K}_0 + \left(\delta + \frac{\lambda_\varepsilon^2}{\delta} + \Delta_0^2 \right) \int_0^t e^{-(\nu-2(1-r_0)\lambda_\varepsilon)(t-s)} \mathbf{x}(s) ds \\ &\quad + \frac{\lambda_\varepsilon}{r_0} \left(\delta + \frac{\lambda_\varepsilon^2}{\delta} + \Delta_0^2 \right) \int_0^t \mathbf{x}(s) ds + \frac{\delta}{\varepsilon^2} \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}s} \mathbf{x}(s) ds \end{aligned}$$

where c_2 is defined in the previous proposition. We use a Gronwall type argument to prove the result. For notational simplicity introduce

$$C_\delta(\lambda_\varepsilon, \Delta_0) = c_2 \left(\delta + \frac{\lambda_\varepsilon^2}{\delta} + \Delta_0^2 \right), \quad A_0 = \frac{c_2 \mathcal{K}_0}{\delta r_0}, \quad \xi_\delta(t) = \frac{\lambda_\varepsilon}{r_0} C_\delta(\lambda_\varepsilon, \Delta_0) + \frac{c_2 \delta}{\varepsilon^2} e^{-\frac{\mu_0}{\varepsilon^2}t}, \quad t \geq 0,$$

from which one obtains that

$$0 \leq \mathbf{x}(t) \leq A_0 + C_\delta(\lambda_\varepsilon, \Delta_0) \int_0^t e^{-(\nu-2(1-r_0)\lambda_\varepsilon)(t-s)} \mathbf{x}(s) ds + \int_0^t \xi_\delta(s) \mathbf{x}(s) ds =: \Upsilon(t). \quad (4.32)$$

Thus,

$$\begin{aligned} \frac{d}{dt} \Upsilon(t) &= -(\nu - 2(1 - r_0)\lambda_\varepsilon) C_\delta(\lambda_\varepsilon, \Delta_0) \int_0^t e^{-(\nu-2(1-r_0)\lambda_\varepsilon)(t-s)} \mathbf{x}(s) ds \\ &\quad + (C_\delta(\lambda_\varepsilon, \Delta_0) + \xi_\delta(t)) \mathbf{x}(t) \\ &= -(\nu - 2(1 - r_0)\lambda_\varepsilon) \left(\Upsilon(t) - A_0 - \int_0^t \xi_\delta(s) \mathbf{x}(s) ds \right) + (C_\delta(\lambda_\varepsilon, \Delta_0) + \xi_\delta(t)) \mathbf{x}(t). \end{aligned}$$

Using (4.32), which reads $\mathbf{x}(t) \leq \Upsilon(t)$, we deduce that

$$\begin{aligned} \frac{d}{dt} \Upsilon(t) &\leq (\nu - 2(1 - r_0)\lambda_\varepsilon) A_0 + [C_\delta(\lambda_\varepsilon, \Delta_0) - (\nu - 2(1 - r_0)\lambda_\varepsilon) + \xi_\delta(t)] \Upsilon(t) \\ &\quad + (\nu - 2(1 - r_0)\lambda_\varepsilon) \int_0^t \xi_\delta(s) \Upsilon(s) ds. \end{aligned}$$

Clearly, it is possible to choose $\delta \in (0, 1)$ and Δ_0 sufficiently small depending on μ_0 and ν and then λ_\star sufficiently small depending on μ_0, ν, δ and r such that

$$C_\delta(\lambda_\varepsilon, \Delta_0) - (\nu - 2(1 - r_0)\lambda_\varepsilon) + \xi_\delta(t) \leq -\frac{\nu}{2} + \frac{\delta c_2}{\varepsilon^2} e^{-\frac{\mu_0}{\varepsilon^2}t}, \quad \forall t \geq 0,$$

and

$$e^{\frac{\delta c_2}{\mu_0}} c_2 \left(\delta + \Delta_0^2 + \frac{\lambda_\varepsilon^2}{\delta} \right) \leq r_0^2 \quad (4.33)$$

hold true for any $\lambda_\varepsilon \in (0, \lambda_\star)$. Fixing λ_ε in this range, we introduce

$$\mathbf{z}(t) = -\frac{\nu}{2}t + \frac{\delta c_2}{\varepsilon^2} \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}s} ds$$

to deduce that

$$\begin{aligned} \frac{d}{dt} \left(e^{-z(t)} \Upsilon(t) \right) &\leq (\nu - 2(1 - r_0)\lambda_\varepsilon) A_0 e^{-z(t)} + (\nu - 2(1 - r_0)\lambda_\varepsilon) e^{-z(t)} \int_0^t \xi_\delta(s) \Upsilon(s) ds \\ &\leq \nu A_0 e^{-z(t)} + \nu e^{-z(t)} \int_0^t \xi_\delta(s) \Upsilon(s) ds \end{aligned}$$

where A_0 is defined as before. Integration of this differential inequality yields (recalling that $\Upsilon(0) = A_0$)

$$\begin{aligned} \Upsilon(t) &\leq A_0 e^{z(t)} + \nu A_0 \int_0^t e^{z(t)-z(s)} ds + \nu \int_0^t e^{z(t)-z(s)} ds \int_0^s \xi_\delta(\tau) \Upsilon(\tau) d\tau \\ &= A_0 e^{z(t)} + \nu A_0 \int_0^t e^{z(t)-z(s)} ds + \nu \int_0^t \xi_\delta(\tau) \Upsilon(\tau) \left(\int_\tau^t e^{z(t)-z(s)} ds \right) d\tau. \end{aligned}$$

Notice that $z(t) - z(s) \leq \frac{\delta c_2}{\mu_0} - \frac{\nu}{2}(t - s)$, for $0 \leq s \leq t$, from which we conclude that

$$\int_\tau^t e^{z(t)-z(s)} ds \leq \frac{2}{\nu} e^{\frac{\delta c_2}{\mu_0}}, \quad 0 \leq \tau \leq t.$$

Consequently,

$$\Upsilon(t) \leq 3A_0 e^{\frac{\delta c_2}{\mu_0}} + 2e^{\frac{\delta c_2}{\mu_0}} \int_0^t \xi_\delta(s) \Upsilon(s) ds,$$

which, thanks to Gronwall lemma, implies that

$$\Upsilon(t) \leq 3A_0 e^{\frac{\delta c_2}{\mu_0}} \exp \left(2e^{\frac{\delta c_2}{\mu_0}} \int_0^t \xi_\delta(s) ds \right).$$

Notice that

$$\int_0^t \xi_\delta(s) ds \leq c_2 \frac{\lambda_\varepsilon}{r_0} \left(\delta + \Delta_0^2 + \frac{\lambda_\varepsilon^2}{\delta} \right) t + \frac{c_2 \delta}{\mu_0}.$$

Moreover, (4.33) implies that

$$2e^{\frac{\delta c_2}{\mu_0}} c_2 \frac{\lambda_\varepsilon}{r_0} \left(\delta + \Delta_0^2 + \frac{\lambda_\varepsilon^2}{\delta} \right) \leq r \lambda_\varepsilon \quad \text{for any } \varepsilon \in (0, \varepsilon_*).$$

Consequently,

$$\Upsilon(t) \leq C A_0 e^{r \lambda_\varepsilon t}$$

for some positive constant C depending on ν , μ_0 , Δ_0 and r . Recalling the definition of A_0 , such estimate combined with (4.32) gives that

$$\|h^1(t)\|_{\mathcal{H}}^2 \leq C \mathcal{K}_0 \exp(-2(1-r)\lambda_\varepsilon t), \quad \forall t \geq 0. \quad (4.34)$$

This estimate yields the final result since it can be proven for any $r \in (0, 1)$ and $\lambda_\varepsilon \sim_{\varepsilon \rightarrow 0} \overline{\lambda}_\varepsilon$. \square

Estimate (4.31) leads to the main result of this section.

Theorem 4.13. *Let $r \in (0, 1)$. There exist $\varepsilon^\dagger \in (0, \varepsilon_6)$, $\lambda^\dagger \in (0, \lambda_6)$, and $C > 0$ depending on ν, μ_0, Δ_0, r such that*

$$\|h(t)\|_{\mathcal{E}}^2 \leq C \left(\|h(0)\|_{\mathcal{E}}^2 + \|h(0)\|_{\mathcal{E}}^4 \right) \exp(-2(1-r)\bar{\lambda}_\varepsilon t), \quad \forall t \geq 0,$$

where $\bar{\lambda}_\varepsilon$ is defined in (1.25), for any $\varepsilon \in (0, \varepsilon^\dagger)$ and $\lambda_\varepsilon \in (0, \lambda^\dagger)$.

Proof. Using estimate (4.31) in estimate (4.11), it follows that

$$\|h^0(t)\|_{\mathcal{E}}^2 \leq 2\|h^0(0)\|_{\mathcal{E}}^2 e^{-\frac{2\mu_0}{\varepsilon^2}t} + C\mathcal{K}_0\varepsilon^2(\varepsilon\lambda_\varepsilon)^2 e^{-2(1-r)\bar{\lambda}_\varepsilon t} \leq C\mathcal{K}_0 e^{-2(1-r)\bar{\lambda}_\varepsilon t}. \quad (4.35)$$

Consequently,

$$\|h(t)\|_{\mathcal{E}}^2 \leq 2(\|h^0(t)\|_{\mathcal{E}}^2 + \|h^1(t)\|_{\mathcal{E}}^2) \leq C(\|h^0(t)\|_{\mathcal{E}}^2 + \|h^1(t)\|_{\mathcal{H}}^2) \leq C\mathcal{K}_0 e^{-2(1-r)\bar{\lambda}_\varepsilon t}. \quad (4.36)$$

Recalling that $h^0(0) = h(0)$ in the definition of \mathcal{K}_0 , estimate (4.36) gives the result. \square

Remark 4.14. *In the following sections, for sake of simplicity, we will use this result for the special value $r = \frac{1}{2}$ and denote by ε^\dagger and λ^\dagger the threshold values associated to $r = \frac{1}{2}$.*

We also point out the gain of decay in h in the following corollary.

Corollary 4.15. *Under the same conditions of Theorem 4.13 it follows that*

$$\int_0^t \|h(\tau)\|_{\mathcal{E}_1} d\tau \leq C\sqrt{\mathcal{K}_0} \min\left\{1+t, 1+\frac{1}{\lambda_\varepsilon}\right\}, \quad \forall t > 0.$$

In particular, $\|h(\cdot)\|_{\mathcal{E}_1}$ is integrable and exists a.e. in $(0, T)$ for any $T > 0$.

Proof. After performing time integration of equation (4.12) in $[0, t]$ one finds that

$$\begin{aligned} \|h^0(t)\|_{\mathcal{E}} + \frac{\mu_0}{\varepsilon^2} \int_0^t \|h^0(\tau)\|_{\mathcal{E}_1} d\tau \\ \leq \|h^0(0)\|_{\mathcal{E}} + C \int_0^t \left(\lambda_\varepsilon \|h^1(\tau)\|_{\mathcal{E}_1} + \varepsilon \lambda_\varepsilon \|h^1(\tau)\|_{\mathcal{E}_1}^2 \right) d\tau \\ \leq C \left(\sqrt{\mathcal{K}_0} + \varepsilon \mathcal{K}_0 \right), \quad \forall t > 0, \end{aligned} \quad (4.37)$$

where we used estimate (4.31) in the latter inequality. Thus,

$$\begin{aligned} \int_0^t \|h(\tau)\|_{\mathcal{E}_1} d\tau &\leq \int_0^t \|h^0(\tau)\|_{\mathcal{E}_1} d\tau + \int_0^t \|h^1(\tau)\|_{\mathcal{E}_1} d\tau \\ &\leq C(\sqrt{\mathcal{K}_0} + \varepsilon \mathcal{K}_0) + C\sqrt{\mathcal{K}_0} \int_0^t e^{-(1-r)\bar{\lambda}_\varepsilon \tau} d\tau, \end{aligned}$$

which gives the result. \square

Remark 4.16. *Of course, for a fixed $\varepsilon > 0$, one can replace $\min\{1+t, 1+\frac{1}{\lambda_\varepsilon}\}$ with $1+\frac{1}{\lambda_\varepsilon}$ and the above estimate shows that $h(t) = h_\varepsilon(t) \in L^1([0, \infty), \mathcal{E}_1)$. However, in the case in which $\lim_{\varepsilon \rightarrow 0} \bar{\lambda}_\varepsilon = 0$ then the bound is not uniform with respect to ε . In practice, two situations occur according to the value of λ_0 in Assumption 1.2:*

a) *If $\lambda_0 > 0$, then the family $\{h_\varepsilon(t)\}_{\varepsilon \geq 0}$ is bounded in $L^1([0, \infty), \mathcal{E}_1)$,*

b) *If $\lambda_0 = 0$ then for any $T > 0$, the family $\{h_\varepsilon(t)\}_{\varepsilon \geq 0}$ is bounded in $L^1([0, T], \mathcal{E}_1)$.*

5. CAUCHY THEORY

In this section, we will use the functional spaces introduced at the beginning of Section 4. Based on the a priori estimates derived in the previous section, we show in this section the well-posedness of the system (4.4)-(4.5).

5.1. Iteration scheme. Let us follow the iteration scheme of (Tristani, 2016, Section 3) with suitable modifications. We are seeking to approximate the solution to the inelastic Boltzmann equation using the iteration scheme

$$\begin{cases} \partial_t h_{n+1}(t) &= \mathcal{G}_\varepsilon h_{n+1}(t) + \varepsilon^{-1} \mathcal{Q}_{\alpha(\varepsilon)}(h_n(t), h_n(t)), & n \geq 1 \\ \partial_t h_1(t) &= \mathcal{G}_\varepsilon h_1(t), \\ h_n(0) &= h(0) \in \mathcal{E}, & n \geq 1, \end{cases} \quad (5.1)$$

where the initial perturbation $h(0)$ has zero mass and momentum. This is done using the decomposition of previous section. More precisely, writing $h_n = h_n^0 + h_n^1$ we consider solutions with the coupled system

$$\begin{cases} \partial_t h_{n+1}^0 &= \mathcal{B}_{\alpha(\varepsilon), \varepsilon} h_{n+1}^0 + \varepsilon^{-1} \mathcal{Q}_{\alpha(\varepsilon)}(h_n^0, h_n^0) + \varepsilon^{-1} [\mathcal{Q}_{\alpha(\varepsilon)}(h_n^0, h_n^1) + \mathcal{Q}_{\alpha(\varepsilon)}(h_n^1, h_n^0)] \\ &+ [\mathcal{G}_\varepsilon h_{n+1}^1 - \mathcal{G}_{1, \varepsilon} h_{n+1}^1] + \varepsilon^{-1} [\mathcal{Q}_{\alpha(\varepsilon)}(h_n^1, h_n^1) - \mathcal{Q}_1(h_n^1, h_n^1)], \\ h_{n+1}^0(0) &= h^0(0) \in \mathcal{E}, \end{cases} \quad (5.2)$$

and

$$\begin{cases} \partial_t h_{n+1}^1 &= \mathcal{G}_{1, \varepsilon} h_{n+1}^1 + \varepsilon^{-1} \mathcal{Q}_1(h_n^1, h_n^1) + \mathcal{A}_\varepsilon h_{n+1}^0, \\ h_{n+1}^1(0) &= h^1(0) \in \mathcal{H}. \end{cases} \quad (5.3)$$

Motivated by the a priori estimates of Section 4, we introduce the following norms

$$\|g\|_0 := \sup_{t \geq 0} \left(\|g(t)\|_{\mathcal{E}} + \varepsilon^{-2} \int_0^t \|g(\tau)\|_{\mathcal{E}_1} d\tau \right), \quad g \in \mathcal{C}([0, \infty), \mathcal{E}),$$

and

$$\|g\|_1 := \sup_{t \geq 0} \left(\|g(t)\|_{\mathcal{H}}^2 + \int_0^t \|g(\tau)\|_{\mathcal{H}_1}^2 d\tau \right)^{\frac{1}{2}}, \quad g \in \mathcal{C}([0, \infty), \mathcal{H}),$$

where we recall that $\mathcal{E}, \mathcal{E}_1, \mathcal{H}$ and \mathcal{H}_1 are defined in (4.3). Notice that $(\mathcal{C}([0, \infty), \mathcal{E}); \|\cdot\|_0)$ and $(\mathcal{C}([0, \infty), \mathcal{H}); \|\cdot\|_1)$ are Banach spaces. In particular, the space

$$\mathbb{B} := \mathcal{C}([0, \infty), \mathcal{E}) \times \mathcal{C}([0, \infty), \mathcal{H})$$

endowed with the norm

$$\|(g, h)\| := \|g\|_0 + \|h\|_1 \quad \text{for } (g, h) \in \mathbb{B},$$

is a Banach space. Define then

$$\begin{aligned} \mathcal{X}_0 &= \left\{ h^0 \in \mathcal{C}([0, \infty); \mathcal{E}) \mid \|h^0\|_0 \leq C\sqrt{\mathcal{K}_0} \right\}, \\ \mathcal{X}_1 &= \left\{ h^1 \in \mathcal{C}([0, \infty); \mathcal{H}) \mid \|h^1\|_1 \leq C\sqrt{\mathcal{K}_0} \right\}, \end{aligned} \quad (5.4)$$

for some positive constant $C > 0$ which can be explicitly estimated from the subsequent computations. The system (5.2)-(5.3) is a simplified coupled version of the system (4.4)-(4.5) with all nonlinear terms as sources. Notice however that the coupling between h_{n+1}^0 and h_{n+1}^1 in the system makes it *nonlinear*. However, because \mathcal{G}_ε is the generator of a C_0 -semigroup in \mathcal{E} , equation (5.1) is well-posed and

$$h_{n+1}(t) = \mathcal{V}_\varepsilon(t)h(0) + \varepsilon^{-1} \int_0^t \mathcal{V}_\varepsilon(t-s) \mathcal{Q}_{\alpha(\varepsilon)}(h_n(s), h_n(s)) ds$$

where $\{\mathcal{V}_\varepsilon(t); t \geq 0\}$ is the C_0 -semigroup in \mathcal{E} generated by \mathcal{G}_ε (i.e., with the notations of Prop. 2.11, $\mathcal{V}_\varepsilon(t) = \mathcal{V}_{\alpha(\varepsilon), \varepsilon}(t), t \geq 0$). With this at hands, substitute in (5.2) the term h_{n+1}^1 by $h_{n+1} - h_{n+1}^0$ and look at $h_{n+1}(t)$ as an additional source term. In the same way for (5.3), the system (5.2)-(5.3) becomes linear (in terms of h_{n+1}^0 and h_{n+1}^1) and admits, for any $n \in \mathbb{N}$, a unique solution. One can use a slight modification of the ideas of Section 4 to check that the iteration scheme is stable, that is, the mapping

$$(h_n^0, h_n^1) \in \mathcal{X}_0 \times \mathcal{X}_1 \mapsto (h_{n+1}^0, h_{n+1}^1) \in \mathcal{X}_0 \times \mathcal{X}_1$$

is well defined. Indeed, existence of the scheme is guaranteed by the linear theory as the iteration scheme is based on the linear equation. Moreover, note that (5.1) preserves the conservation laws: mass conservation and vanishing momentum, which were essential for the *a priori* estimates related to $\mathbf{P}_0 h^1$. Thus, stability holds true under the conditions of the *a priori* estimates, that is, for $\varepsilon \in (0, \varepsilon^\dagger)$ (where ε^\dagger is defined in Theorem 4.13-Remark 4.14) and

$$\sup_{t \geq 0} (\|h_n^1(t)\|_{\mathcal{H}} + \|h_n^0(t)\|_{\mathcal{E}}) \leq C\sqrt{\mathcal{K}_0} \leq \Delta_0, \quad n \in \mathbb{N}.$$

This latter condition is possible by taking \mathcal{K}_0 smaller than a threshold depending only on the initial mass and energy E_{in} ⁶

$$\mathcal{K}_0 \leq (\Delta_0/C)^2 =: \mathcal{K}_0^\dagger.$$

⁶since all the threshold values appearing here are prescribed by the choice of the initial mass and energy, see Remark 1.1.

We leave the details to the reader and focus in the next subsections on the convergence of the scheme.

5.2. Estimating $\|h_{n+1}^0 - h_n^0\|_{\mathcal{E}}$ and $\|h_{n+1}^1 - h_n^1\|_{\mathcal{H}}$. To prove the convergence of the scheme, we define for $n \in \mathbb{N}$

$$d_{n+1}^0 = h_{n+1}^0 - h_n^0, \quad d_{n+1}^1 = h_{n+1}^1 - h_n^1.$$

Then, one deduces from (5.2) and (5.3)

$$\begin{cases} \partial_t d_{n+1}^0 &= \mathcal{B}_{\alpha(\varepsilon), \varepsilon} d_{n+1}^0 + \left[\mathcal{G}_{\varepsilon} d_{n+1}^1 - \mathcal{G}_{1, \varepsilon} d_{n+1}^1 \right] + \varepsilon^{-1} \mathcal{F}_n^0, \\ d_{n+1}^0(0) &= 0, \end{cases} \quad (5.5)$$

and

$$\begin{cases} \partial_t d_{n+1}^1 &= \mathcal{G}_{1, \varepsilon} d_{n+1}^1 + \mathcal{A}_{\varepsilon} d_{n+1}^0 + \varepsilon^{-1} \mathcal{F}_n^1, \\ d_{n+1}^1(0) &= 0. \end{cases} \quad (5.6)$$

The sources \mathcal{F}_n^i , for $i \in \{0, 1\}$, correspond to the bilinear terms and depend only on the previous iterations $\{h_n^i, h_{n-1}^i\}$, for $i \in \{0, 1\}$ and $n \geq 2$ (see (5.9) and (5.11) for the precise expression). We introduce

$$\begin{cases} \Psi_n^1(t) &= \|h_n^0(t)\|_{\mathcal{E}_1} + \|h_{n-1}^0(t)\|_{\mathcal{E}_1} \\ \Psi_n^\infty(t) &= \|h_n^0(t)\|_{\mathcal{E}} + \|h_{n-1}^0(t)\|_{\mathcal{E}} + \|h_n^1(t)\|_{\mathcal{H}} + \|h_{n-1}^1(t)\|_{\mathcal{H}}, \end{cases}$$

which satisfy

$$\sup_{t \geq 0} \left(\Psi_n^\infty(t) + \varepsilon^{-2} \int_0^t \Psi_n^1(\tau) d\tau \right) \leq C \sqrt{\mathcal{K}_0}, \quad n \geq 2, \quad (5.7)$$

for $h_n^0, h_{n-1}^0 \in \mathcal{X}_0$, and $h_n^1, h_{n-1}^1 \in \mathcal{X}_1$. Consequently, the following estimate for d_{n+1}^0 follows under suitable modifications of the arguments leading to Proposition 4.4 (keep in mind that $\|\cdot\|_{\mathcal{E}_2} \lesssim \|\cdot\|_{\mathcal{H}}$).

Lemma 5.1. *Let $\varepsilon \in (0, \varepsilon^\dagger)$ and $\mathcal{K}_0 \leq \mathcal{K}_0^\dagger$. Then, we have that*

$$\begin{aligned} \|d_{n+1}^0(t)\|_{\mathcal{E}} &\lesssim \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} ds \\ &\quad + \varepsilon^{-1} \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \Psi_n^1(s) \left(\|d_n^0(s)\|_{\mathcal{E}} + \|d_n^1(s)\|_{\mathcal{H}} \right) ds \\ &\quad + \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \Psi_n^\infty(s) \left(\varepsilon^{-1} \|d_n^0(s)\|_{\mathcal{E}_1} + \varepsilon \lambda_\varepsilon \|d_n^1(s)\|_{\mathcal{H}} \right) ds. \end{aligned} \quad (5.8)$$

Proof. Here again, as in the proof of Proposition 4.4, we denote by $\|\cdot\|_{\mathcal{E}_1}$ and $\|\cdot\|_{\mathcal{E}}$ the norms on \mathcal{E}_1 and \mathcal{E} that are equivalent to the standard ones (with multiplicative constants independent of ε) and that make $\varepsilon^{-2}\nu_0 + \mathcal{B}_{\alpha(\varepsilon),\varepsilon}$ *dissipative* so that

$$\begin{aligned} \frac{d}{dt} \|d_{n+1}^0(t)\|_{\mathcal{E}} &\leq -\frac{\nu_0}{\varepsilon^2} \|d_{n+1}^0(t)\|_{\mathcal{E}_1} + \varepsilon^{-1} \|\mathcal{F}_n^0(t)\|_{\mathcal{E}} + \|\mathcal{G}_\varepsilon d_{n+1}^1(t) - \mathcal{G}_{1,\varepsilon} d_{n+1}^1(t)\|_{\mathcal{E}} \\ &\leq -\frac{\nu_0}{\varepsilon^2} \|d_{n+1}^0(t)\|_{\mathcal{E}_1} + \varepsilon^{-1} \|\mathcal{F}_n^0(t)\|_{\mathcal{E}} + C\lambda_\varepsilon \|d_{n+1}^1(t)\|_{\mathcal{H}}. \end{aligned}$$

We need to estimate $\|\mathcal{F}_n^0(t)\|_{\mathcal{E}}$. One has,

$$\begin{aligned} \mathcal{F}_n^0 &= \mathcal{Q}_{\alpha(\varepsilon)}(d_n^0, h_n^0) + \mathcal{Q}_{\alpha(\varepsilon)}(h_{n-1}^0, d_n^0) + 2\tilde{\mathcal{Q}}_{\alpha(\varepsilon)}(d_n^0, h_n^1) + 2\tilde{\mathcal{Q}}_{\alpha(\varepsilon)}(h_{n-1}^0, d_n^1) \\ &\quad + (\mathcal{Q}_{\alpha(\varepsilon)}(d_n^1, h_n^1) - \mathcal{Q}_1(d_n^1, h_n^1)) + (\mathcal{Q}_{\alpha(\varepsilon)}(h_{n-1}^1, d_n^1) - \mathcal{Q}_1(h_{n-1}^1, d_n^1)). \end{aligned} \quad (5.9)$$

Therefore, since $1 - \alpha(\varepsilon) \lesssim \varepsilon^2 \lambda_\varepsilon$, using Remark 3.5 and the usual estimates for \mathcal{Q}_α and \mathcal{Q}_1 :

$$\begin{aligned} \|\mathcal{F}_n^0\|_{\mathcal{E}} &\lesssim \|d_n^0\|_{\mathcal{E}_1} (\|h_n^0\|_{\mathcal{E}} + \|h_{n-1}^0\|_{\mathcal{E}}) + \|d_n^0\|_{\mathcal{E}} (\|h_n^0\|_{\mathcal{E}_1} + \|h_{n-1}^0\|_{\mathcal{E}_1}) \\ &\quad + \|d_n^0\|_{\mathcal{E}} \|h_n^1\|_{\mathcal{E}_1} + \|d_n^0\|_{\mathcal{E}_1} \|h_n^1\|_{\mathcal{E}} + \|d_n^1\|_{\mathcal{E}_1} \|h_{n-1}^0\|_{\mathcal{E}} \\ &\quad + \|d_n^1\|_{\mathcal{E}} \|h_{n-1}^0\|_{\mathcal{E}_1} + \varepsilon^2 \lambda_\varepsilon \|d_n^1\|_{\mathcal{E}_2} (\|h_n^1\|_{\mathcal{E}_2} + \|h_{n-1}^1\|_{\mathcal{E}_2}). \end{aligned}$$

Using that $\|\cdot\|_{\mathcal{E}_2} \lesssim \|\cdot\|_{\mathcal{H}}$ we get

$$\|\mathcal{F}_n^0(t)\|_{\mathcal{E}} \lesssim \|d_n^0(t)\|_{\mathcal{E}_1} \Psi_n^\infty(t) + \Psi_n^1(t) (\|d_n^0(t)\|_{\mathcal{E}} + \|d_n^1(t)\|_{\mathcal{H}}) + \varepsilon^2 \lambda_\varepsilon \|d_n^1(t)\|_{\mathcal{E}_2} \Psi_n^\infty(t).$$

This leads to the desired estimate since $\mu_0 < \nu_0$ (see the proof of Proposition 4.4). \square

Regarding the projection $\mathbf{P}_0 d_{n+1}^1(t)$, since the difference $h_{n+1} - h_n = d_{n+1}^0 + d_{n+1}^1$ has zero mass and momentum, one can follow the line of proof of Lemma 4.5 to deduce that

$$\begin{aligned} \|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{E}_{-1}} &\lesssim \|d_{n+1}^0(t)\|_{\mathcal{E}} + \varepsilon \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \Psi_n^\infty(s) (\|d_n^0(s)\|_{\mathcal{E}} + \|d_n^1(s)\|_{\mathcal{H}}) ds \\ &\quad + \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} (\|d_{n+1}^0(s)\|_{\mathcal{E}} + \|\mathbf{P}_0^\perp d_{n+1}^1(s)\|_{\mathcal{E}}) ds. \end{aligned}$$

Consequently, plugging (5.8) in the second term in the right side and recalling that

$$\|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{H}} \lesssim \|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{E}_{-1}} \lesssim \|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{H}}$$

we obtain the following lemma.

Lemma 5.2. *For any $t \geq 0$, we have that*

$$\begin{aligned}
\|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{H}} &\lesssim \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \, ds \\
&+ \varepsilon^{-1} \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \Psi_n^1(s) \left(\|d_n^0(s)\|_{\mathcal{E}} + \|d_n^1(s)\|_{\mathcal{H}} \right) \, ds \\
&+ \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \Psi_n^\infty(s) \left(\varepsilon^{-1} \|d_n^0(s)\|_{\mathcal{E}_1} + \varepsilon \lambda_\varepsilon \|d_n^1(s)\|_{\mathcal{H}} \right) \, ds \\
&+ \varepsilon \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \Psi_n^\infty(s) \left(\|d_n^0(s)\|_{\mathcal{E}} + \|d_n^1(s)\|_{\mathcal{H}} \right) \, ds \\
&+ \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \left(\|d_{n+1}^0(s)\|_{\mathcal{E}} + \|\mathbf{P}_0^\perp d_{n+1}^1(s)\|_{\mathcal{H}} \right) \, ds.
\end{aligned}$$

Let us focus on estimating $\mathbf{P}_0^\perp d_{n+1}^1(t)$. To do so, we introduce the functions Φ_n^1 and Φ_n^∞ defined by

$$\Phi_n^1(t) = \|h_n^1(t)\|_{\mathcal{H}_1}^2 + \|h_{n-1}^1(t)\|_{\mathcal{H}_1}^2 \quad \text{and} \quad \Phi_n^\infty(t) = \|h_n^1(t)\|_{\mathcal{H}}^2 + \|h_{n-1}^1(t)\|_{\mathcal{H}}^2$$

which satisfy

$$\sup_{t \geq 0} \left(\Phi_n^\infty(t) + \int_0^t \Phi_n^1(\tau) \, d\tau \right) \leq CK_0, \quad n \geq 2. \quad (5.10)$$

One has the following lemma.

Lemma 5.3. *Let $\varepsilon \in (0, \varepsilon^\dagger)$ and $\mathcal{K}_0 \leq \mathcal{K}_0^\dagger$. Then,*

$$\begin{aligned}
\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 &\lesssim \int_0^t e^{-\nu(t-s)} \Phi_n^1(s) \|d_n^1(s)\|_{\mathcal{H}}^2 \, ds \\
&+ \int_0^t e^{-\nu(t-s)} \Phi_n^\infty(s) \|d_n^1(s)\|_{\mathcal{H}_1}^2 \, ds + \lambda_\varepsilon \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 \, ds \\
&+ \varepsilon^{-1} \left(\sup_{s \geq 0} \|d_{n+1}^1(s)\|_{\mathcal{H}} \right) \int_0^t e^{-\nu(t-\tau)} \Psi_n^1(\tau) \left(\|d_n^0(\tau)\|_{\mathcal{E}} + \|d_n^1(\tau)\|_{\mathcal{H}} \right) \, d\tau \\
&+ \left(\sup_{s \geq 0} \|d_{n+1}^1(s)\|_{\mathcal{H}} \right) \int_0^t e^{-\nu(t-\tau)} \Psi_n^\infty(\tau) \left(\varepsilon^{-1} \|d_n^0(\tau)\|_{\mathcal{E}} + \varepsilon \lambda_\varepsilon \|d_n^1(\tau)\|_{\mathcal{H}} \right) \, d\tau.
\end{aligned}$$

Proof. One deduces from (5.6) that $\mathbf{P}_0^\perp d_{n+1}^1(t)$ is such that

$$\partial_t \mathbf{P}_0^\perp d_{n+1}^1(t) = \mathcal{G}_{1,\varepsilon} \mathbf{P}_0^\perp d_{n+1}^1(t) + \mathbf{P}_0^\perp \mathcal{A}_\varepsilon d_{n+1}^0(t) + \varepsilon^{-1} \mathcal{F}_n^1$$

where

$$\mathcal{F}_n^1 = \mathcal{Q}_1(h_n^1, h_n^1) - \mathcal{Q}_1(h_{n-1}^1, h_{n-1}^1) = \mathcal{Q}_1(d_n^1, h_n^1) + \mathcal{Q}_1(h_{n-1}^1, d_n^1). \quad (5.11)$$

Following the argument leading to inequality (4.23) (see also (Briant et al., 2019, Lemma 4.6, Theorem 4.7)) one deduces that

$$\begin{aligned} \|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 &\lesssim \int_0^t e^{-\nu(t-s)} \Phi_n^1(s) \|d_n^1(s)\|_{\mathcal{H}}^2 ds + \int_0^t e^{-\nu(t-s)} \Phi_n^\infty(s) \|d_n^1(s)\|_{\mathcal{H}_1}^2 ds \\ &\quad + \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \|\mathcal{A}_\varepsilon d_{n+1}^0(s)\|_{\mathcal{H}} ds. \end{aligned} \quad (5.12)$$

The latter term in the right side of (5.12) can be estimated using (5.8) and recalling that $\|\mathcal{A}_\varepsilon d_{n+1}^0\|_{\mathcal{H}} \lesssim \varepsilon^{-2} \|d_{n+1}^0\|_{\mathcal{E}}$. Thus,

$$\int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \|\mathcal{A}_\varepsilon d_{n+1}^0(s)\|_{\mathcal{H}} ds \lesssim \sum_{i=1}^3 \mathcal{T}_i,$$

with

$$\begin{aligned} \mathcal{T}_1 &= \frac{\lambda_\varepsilon}{\varepsilon^2} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \left(\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|d_{n+1}^1(\tau)\|_{\mathcal{H}} d\tau \right) ds, \\ \mathcal{T}_2 &= \varepsilon^{-3} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \left[\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \Psi_n^1(\tau) \left(\|d_n^0(\tau)\|_{\mathcal{E}} + \|d_n^1(\tau)\|_{\mathcal{H}} \right) d\tau \right] ds \\ \mathcal{T}_3 &= \varepsilon^{-2} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \left[\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \Psi_n^\infty(\tau) \left(\varepsilon^{-1} \|d_n^0(\tau)\|_{\mathcal{E}_1} + \varepsilon \lambda_\varepsilon \|d_n^1(\tau)\|_{\mathcal{H}} \right) d\tau \right] ds. \end{aligned}$$

It is easy to check, using (4.18), that

$$\mathcal{T}_2 \leq \frac{2}{\mu_0 \varepsilon} \left(\sup_{s \geq 0} \|d_{n+1}^1(s)\|_{\mathcal{H}} \right) \int_0^t e^{-\nu(t-\tau)} \Psi_n^1(\tau) \left(\|d_n^0(\tau)\|_{\mathcal{E}} + \|d_n^1(\tau)\|_{\mathcal{H}} \right) d\tau$$

and

$$\mathcal{T}_3 \leq \frac{2}{\mu_0} \left(\sup_{s \geq 0} \|d_{n+1}^1(s)\|_{\mathcal{H}} \right) \int_0^t e^{-\nu(t-\tau)} \Psi_n^\infty(\tau) \left(\varepsilon^{-1} \|d_n^0(\tau)\|_{\mathcal{E}_1} + \varepsilon \lambda_\varepsilon \|d_n^1(\tau)\|_{\mathcal{H}} \right) d\tau.$$

The estimate for \mathcal{T}_1 is a bit more involved. Thanks to Cauchy-Schwarz inequality one first has

$$\mathcal{T}_1 \leq \frac{\lambda_\varepsilon}{\varepsilon^2} \left(\int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t e^{-\nu(t-s)} Y^2(s) ds \right)^{\frac{1}{2}}$$

where

$$Y(s) := \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|d_{n+1}^1(\tau)\|_{\mathcal{H}} d\tau, \quad s \in (0, t).$$

Thanks to (4.13) applied with $r = \frac{1}{2}$,

$$Y^2(s) \leq \frac{\varepsilon^2}{\mu_0} \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|d_{n+1}^1(\tau)\|_{\mathcal{H}}^2 d\tau$$

and, using now (4.18) for $\mu_0 > 2\varepsilon^2\nu$,

$$\begin{aligned} \int_0^t e^{-\nu(t-s)} Y^2(s) ds &\leq \frac{\varepsilon^2}{\mu_0} \int_0^t e^{-\nu(t-s)} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|d_{n+1}^1(\tau)\|_{\mathcal{H}}^2 d\tau \\ &\leq \frac{\varepsilon^2}{\mu_0 \left(\frac{\mu_0}{\varepsilon^2} - \nu\right)} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 ds \leq \frac{2\varepsilon^4}{\mu_0^2} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 ds. \end{aligned}$$

We deduce finally that

$$\mathcal{T}_1 \leq \frac{\sqrt{2}\lambda_\varepsilon}{\mu_0} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 ds$$

and this, together with the estimates for \mathcal{T}_2 and \mathcal{T}_3 , gives the result. \square

Introducing now the quantities

$$\begin{aligned} \Xi_n^0 &= \sup_{t \geq 0} \left(\|d_n^0(t)\|_{\mathcal{E}} + \varepsilon^{-2} \int_0^t \|d_n^0(\tau)\|_{\mathcal{E}_1} d\tau \right), \\ \Xi_n^1 &= \sup_{t \geq 0} \left(\|d_n^1(t)\|_{\mathcal{H}}^2 + \int_0^t \|d_n^1(\tau)\|_{\mathcal{H}_1}^2 d\tau \right)^{\frac{1}{2}}, \quad n \geq 2, \end{aligned}$$

we can gather the three previous lemmas and use (5.7) to obtain the following result.

Proposition 5.4. *For any $n \in \mathbb{N}$ and $t \geq 0$*

$$\|d_{n+1}^0(t)\|_{\mathcal{E}} \lesssim \varepsilon^2 \lambda_\varepsilon \Xi_{n+1}^1 + \varepsilon \sqrt{\mathcal{K}_0} (\Xi_n^0 + \Xi_n^1), \quad (5.13)$$

while

$$\|d_{n+1}^1(t)\|_{\mathcal{H}} \lesssim \sqrt{\lambda_\varepsilon + \varepsilon} \Xi_{n+1}^1 + \sqrt{\varepsilon \mathcal{K}_0} \Xi_n^0 + \sqrt{\mathcal{K}_0} \Xi_n^1, \quad (5.14)$$

as long as $\varepsilon \in (0, \varepsilon^\dagger)$, $\mathcal{K}_0 \leq \mathcal{K}_0^\dagger$.

Proof. First, we claim that

$$\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}} \lesssim \sqrt{\lambda_\varepsilon + \varepsilon} \Xi_{n+1}^1 + \sqrt{\varepsilon \mathcal{K}_0} \Xi_n^0 + \sqrt{\mathcal{K}_0} \Xi_n^1, \quad (5.15)$$

Indeed, from Lemma 5.3, we have that

$$\begin{aligned} \|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 &\lesssim \lambda_\varepsilon [\Xi_{n+1}^1]^2 + [\Xi_n^1]^2 \left(\int_0^t e^{-\nu(t-s)} \Phi_n^1(s) ds + \sup_{s \geq 0} \Phi_n^\infty(s) \right) \\ &\quad + \varepsilon^{-1} \Xi_{n+1}^1 (\Xi_n^0 + \Xi_n^1) \int_0^t e^{-\nu(t-\tau)} \Psi_n^1(\tau) d\tau \\ &\quad + \Xi_{n+1}^1 \left(\sup_{s \geq 0} \Psi_n^\infty(s) \right) (\varepsilon \Xi_n^0 + \varepsilon \lambda_\varepsilon \Xi_n^1). \end{aligned}$$

We can thus invoke (5.7) and (5.10) to deduce that

$$\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 \lesssim \mathcal{K}_0 [\Xi_n^1]^2 + \lambda_\varepsilon [\Xi_{n+1}^1]^2 + \varepsilon \sqrt{\mathcal{K}_0} (\Xi_n^0 + \Xi_n^1) \Xi_{n+1}^1$$

where we used that $\varepsilon\lambda_\varepsilon < \varepsilon$. From Young's inequality, we deduce that

$$\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 \lesssim (\lambda_\varepsilon + \varepsilon) [\Xi_{n+1}^1]^2 + \mathcal{K}_0 [\Xi_n^1]^2 + \varepsilon \mathcal{K}_0 [\Xi_n^0]^2,$$

which proves (5.15). In the same way, the estimate (5.13) is easily deduced from Lemma 5.1. To end the proof, it remains to prove that

$$\|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{H}} \lesssim \sqrt{\lambda_\varepsilon + \varepsilon} \Xi_{n+1}^1 + \sqrt{\varepsilon \mathcal{K}_0} \Xi_n^0 + \sqrt{\mathcal{K}_0} \Xi_n^1. \quad (5.16)$$

This inequality is a consequence of Lemma 5.2 combined with (5.7), (5.13) and (5.15). \square

Proposition 5.5. *For any $t \geq 0$, we have that*

$$\left(\frac{\mu_0}{\varepsilon^2} - \nu\right) \int_0^t e^{-\nu(t-s)} \|d_{n+1}^0(s)\|_{\mathcal{E}_1} ds \lesssim \lambda_\varepsilon \Xi_{n+1}^1 + \varepsilon \sqrt{\mathcal{K}_0} (\Xi_n^0 + \Xi_n^1), \quad (5.17)$$

and

$$\left(\int_0^t e^{-\nu(t-\tau)} \|d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 d\tau\right)^{\frac{1}{2}} \lesssim \sqrt{\lambda_\varepsilon + \varepsilon} \Xi_{n+1}^1 + \sqrt{\varepsilon \mathcal{K}_0} \Xi_n^0 + \sqrt{\mathcal{K}_0} \Xi_n^1. \quad (5.18)$$

Proof. To prove (5.17), we follow the argument that led to Lemma 5.1 and thus in the subsequent proof, we again denote by $\|\cdot\|_{\mathcal{E}_1}$ and $\|\cdot\|_{\mathcal{E}}$ the norms on \mathcal{E}_1 and \mathcal{E} that are equivalent to the standard ones independently of ε and that make $\varepsilon^{-2}\nu_0 + \mathcal{B}_{\alpha(\varepsilon),\varepsilon}$ *dissipative* so that we can write

$$\frac{d}{dt} \|d_{n+1}^0(t)\|_{\mathcal{E}} \leq -\frac{\nu_0}{\varepsilon^2} \|d_{n+1}^0(t)\|_{\mathcal{E}_1} + \varepsilon^{-1} \|\mathcal{F}_n^0(t)\|_{\mathcal{E}} + C\lambda_\varepsilon \|d_{n+1}^1(t)\|_{\mathcal{H}},$$

which implies that,

$$\begin{aligned} \frac{d}{dt} \|d_{n+1}^0(t)\|_{\mathcal{E}} + \nu \|d_{n+1}^0(t)\|_{\mathcal{E}_1} &\leq -\left(\frac{\mu_0}{\varepsilon^2} - \nu\right) \|d_{n+1}^0(t)\|_{\mathcal{E}_1} \\ &\quad + \varepsilon^{-1} \|\mathcal{F}_n^0(t)\|_{\mathcal{E}} + C\lambda_\varepsilon \|d_{n+1}^1(t)\|_{\mathcal{H}} \end{aligned}$$

where we used that $\mu_0 < \nu_0$. After integration over $[0, t]$, using that $d_{n+1}^0(0) = 0$, we get that

$$\begin{aligned} \|d_{n+1}^0(t)\|_{\mathcal{E}} &\leq -\left(\frac{\mu_0}{\varepsilon^2} - \nu\right) \int_0^t e^{-\nu(t-s)} \|d_{n+1}^0(s)\|_{\mathcal{E}_1} ds + \varepsilon^{-1} \int_0^t e^{-\nu(t-s)} \|\mathcal{F}_n^0(s)\|_{\mathcal{E}} ds \\ &\quad + C\lambda_\varepsilon \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} ds, \end{aligned}$$

and, recalling that \mathcal{F}_n^0 is given by (5.9), we estimate $\|\mathcal{F}_n^0(s)\|_{\mathcal{E}}$ as in Lemma 5.1 to obtain that

$$\begin{aligned} \left(\frac{\mu_0}{\varepsilon^2} - \nu\right) \int_0^t e^{-\nu(t-s)} \|d_{n+1}^0(s)\|_{\mathcal{E}_1} ds &\lesssim \lambda_\varepsilon \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} ds \\ &\quad + \varepsilon^{-1} \int_0^t e^{-\nu(t-s)} \Psi_n^1(s) \left(\|d_n^0(s)\|_{\mathcal{E}} + \|d_n^1(s)\|_{\mathcal{H}} \right) ds \\ &\quad + \int_0^t e^{-\nu(t-s)} \Psi_n^\infty(s) \left(\varepsilon^{-1} \|d_n^0(s)\|_{\mathcal{E}_1} + \varepsilon\lambda_\varepsilon \|d_n^1(s)\|_{\mathcal{H}} \right) ds. \end{aligned}$$

This yields (5.17). In the same way, we adapt the proof of Lemma 5.3 to get that

$$\begin{aligned} \left(\frac{2\mu}{\sigma_0^2} - \nu\right) \int_0^t e^{-\nu(t-\tau)} \|\mathbf{P}_0^\perp d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 d\tau &\lesssim \int_0^t e^{-\nu(t-s)} \Phi_n^1(s) \|d_n^1(s)\|_{\mathcal{H}}^2 ds \\ &+ \int_0^t e^{-\nu(t-s)} \Phi_n^\infty(s) \|d_n^1(s)\|_{\mathcal{H}_1}^2 ds + \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \|\mathcal{A}_\varepsilon d_{n+1}^0(s)\|_{\mathcal{H}} ds \end{aligned}$$

where $2\mu/\sigma_0^2 - \nu = c_0\Delta_0^2$ (see (4.20)). This estimate is similar to (5.12) and therefore we can resume both the proofs of Lemma 5.3 and Proposition 5.4 to obtain that

$$\Delta_0^2 \left(\int_0^t e^{-\nu(t-\tau)} \|\mathbf{P}_0^\perp d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 d\tau \right)^{\frac{1}{2}} \lesssim \sqrt{\lambda_\varepsilon + \varepsilon} \Xi_{n+1}^1 + \sqrt{\varepsilon \mathcal{K}_0} \Xi_n^0 + \sqrt{\mathcal{K}_0} \Xi_n^1.$$

To conclude, we recall that $\|\mathbf{P}_0 d_{n+1}^1\|_{\mathcal{H}_1} \lesssim \|\mathbf{P}_0 d_{n+1}^1\|_{\mathcal{H}}$ so that a simple integration of (5.16) gives that

$$\left(\int_0^t e^{-\nu(t-\tau)} \|\mathbf{P}_0 d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 d\tau \right)^{\frac{1}{2}} \lesssim \sqrt{\lambda_\varepsilon + \varepsilon} \Xi_{n+1}^1 + \sqrt{\varepsilon \mathcal{K}_0} \Xi_n^0 + \sqrt{\mathcal{K}_0} \Xi_n^1.$$

Adding these two estimates, one deduces (5.18). \square

5.3. Convergence of the iteration scheme. We are now in position to conclude our analysis by proving the convergence of the iteration scheme. In the sequel, we indicate with a same letter C a positive constant depending on μ_0 , ν and Δ_0 that may change from line to line. Suitably adding (5.13) and (5.17) and taking the supremum in time, one has that

$$\Xi_{n+1}^0 \lesssim \lambda_\varepsilon \Xi_{n+1}^1 + \varepsilon \sqrt{\mathcal{K}_0} (\Xi_n^0 + \Xi_n^1) \quad (5.19)$$

where we used that $\mu_0 \geq 2\varepsilon^2\nu$. Similarly, adding (5.14) and (5.18) and taking the supremum in time it holds that

$$\Xi_{n+1}^1 \lesssim \sqrt{\lambda_\varepsilon + \varepsilon} \Xi_{n+1}^1 + \sqrt{\varepsilon \mathcal{K}_0} \Xi_n^0 + \sqrt{\mathcal{K}_0} \Xi_n^1. \quad (5.20)$$

Let us define $\mathcal{E}_n = \Xi_n^0 + \Xi_n^1$, for $n \geq 2$. Adding the estimates (5.19) and (5.20), we conclude that there exists $C > 0$ such that $\mathcal{E}_{n+1} \leq C\sqrt{\lambda_\varepsilon + \varepsilon} \mathcal{E}_{n+1} + C\sqrt{\mathcal{K}_0} \mathcal{E}_n$. Thus, choosing ε sufficiently small such that $C\sqrt{\lambda_\varepsilon + \varepsilon} < \frac{1}{2}$, we get that $\mathcal{E}_{n+1} \leq C\sqrt{\mathcal{K}_0} \mathcal{E}_n$ from which

$$\mathcal{E}_{n+1} \leq (C\sqrt{\mathcal{K}_0})^{n-1} \mathcal{E}_2, \quad \forall n \geq 2.$$

Choosing $\mathcal{K}_0 \leq \mathcal{K}_0^\dagger < C^{-2}$ so that

$$\theta := C\sqrt{\mathcal{K}_0} < 1$$

we deduce that, in the Banach space $(\mathbb{B}, \|\cdot\|)$, one has for $m > n \geq 1$,

$$\| (h_m^0, h_m^1) - (h_n^0, h_n^1) \| \leq \sum_{i=n}^{m-1} \mathcal{E}_{i+1} \leq \mathcal{E}_2 \frac{\theta^{n-1}}{1-\theta}, \quad \theta := C\sqrt{\mathcal{K}_0}.$$

Whence the sequence $\{(h_n^0, h_n^1)\}_n \subset \mathcal{X}_0 \times \mathcal{X}_1 \subset \mathbb{B}$ is a Cauchy sequence and it converges in $(\mathbb{B}, \|\cdot\|)$ to a limit $(h^0, h^1) \in \mathcal{X}_0 \times \mathcal{X}_1$. Of course, such limit satisfies equations (4.4) and (4.5). Thus, $h = h^0 + h^1$ is a solution to the inelastic Boltzmann problem (1.22). Such solution is unique in the class of functions that we consider since, at essence, we proved that the problem is a contraction on $\mathcal{X}_0 \times \mathcal{X}_1$. Let us write the conclusion as the main theorem of the section.

Theorem 5.6. *Fix a nonnegative initial data $F_{\text{in}}^\varepsilon = G_\alpha + \varepsilon h_{\text{in}}^\varepsilon \in \mathcal{E}$ and assume that the initial perturbation $h_{\text{in}}^\varepsilon$ has zero total mass and momentum*

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} h_{\text{in}}^\varepsilon(t, x, v) dv dx = \int_{\mathbb{T}^d \times \mathbb{R}^d} h_{\text{in}}^\varepsilon(t, x, v) v dv dx = 0.$$

Then, there exist positive threshold values $(\varepsilon^\dagger, \lambda^\dagger, \mathcal{K}_0^\dagger)$ fully determined by the initial mass and energy E_{in} (see Remark 1.1) such that if

$$\|h_{\text{in}}^\varepsilon\|_{\mathcal{E}} \leq \sqrt{\mathcal{K}_0^\dagger},$$

and $\varepsilon \in (0, \varepsilon^\dagger)$, $\lambda_\varepsilon \in (0, \lambda^\dagger)$, the inelastic Boltzmann equation (1.22) has a unique solution $h_\varepsilon \in \mathcal{C}([0, \infty); \mathcal{E})$ satisfying for $t > 0$

$$\|h_\varepsilon(t)\|_{\mathcal{E}} \leq C \|h_{\text{in}}^\varepsilon\|_{\mathcal{E}} \exp\left(-\frac{\bar{\lambda}_\varepsilon}{2} t\right) \quad \text{and} \quad \int_0^t \|h_\varepsilon(\tau)\|_{\mathcal{E}_1} d\tau \leq C \|h_{\text{in}}^\varepsilon\|_{\mathcal{E}} \min\left\{1 + t, 1 + \frac{1}{\lambda_\varepsilon}\right\}.$$

6. HYDRODYNAMIC LIMIT

In this last section, we will once again specify that h , h^0 and h^1 depend on ε by noting $h = h_\varepsilon$, $h^0 = h_\varepsilon^0$, $h^1 = h_\varepsilon^1$. On the other hand, to lighten notations, we will write α for $\alpha(\varepsilon)$ but recall that $\alpha = \alpha(\varepsilon)$ satisfies Assumption 1.2. Finally, we will consider m , k and q satisfying

$$m > d, \quad m - 1 \geq k \geq 1, \quad q \geq 5.$$

as well as the corresponding spaces \mathcal{E} and \mathcal{E}_1 defined in (4.1)-(4.3).

6.1. Compactness and convergence. We start this section recalling the expression for the spectral projection π_0 onto the kernel $\text{Ker}(\mathbf{L}_1)$ of the linearized collision operator \mathbf{L}_1 seen as an operator acting in velocity only on the space $L_v^2(\mathcal{M}^{-\frac{1}{2}})$. We recall that, with the notations of Theorem 2.12,

$$\pi_0(g) := \sum_{i=1}^{d+2} \left(\int_{\mathbb{R}^d} g \Psi_i dv \right) \Psi_i \mathcal{M}, \quad (6.1)$$

where $\Psi_1(v) = 1$, $\Psi_i(v) = \frac{1}{\sqrt{\vartheta_1}} v_{i-1}$ ($i = 2, \dots, d+1$) and $\Psi_{d+2}(v) = \frac{|v|^2 - d\vartheta_1}{\vartheta_1 \sqrt{2d}}$. Note that the difference with respect to the spectral projection \mathbf{P}_0 for the operator $\mathcal{G}_{1,\varepsilon}$ in (2.28) is that no spatial integration is performed.

Consider now $h_\varepsilon = h_\varepsilon^0 + h_\varepsilon^1$ the solution constructed in Section 5. One can prove the following estimate for time-averages of $(\mathbf{Id} - \pi_0)h_\varepsilon(\tau)$ in spaces which do not involve derivatives in the v -variable:

Proposition 6.1. *For any $0 \leq \beta \leq m - 1$ there exists $C > 0$ independent of ε such that*

$$\int_{t_1}^{t_2} \|(\mathbf{Id} - \pi_0)h_\varepsilon(\tau)\|_{L_v^1 \mathbb{W}_x^{\beta,2}(\varpi_q)} d\tau \leq C \varepsilon \sqrt{\mathcal{K}_0} \max \{ \sqrt{t_2 - t_1}, t_2 - t_1 \} \quad (6.2)$$

holds true for any $0 \leq t_1 \leq t_2$.

Proof. For a given $0 \leq \beta \leq m - 1$, we introduce the hierarchy of Hilbert spaces

$$\tilde{H}_s = L_v^2 \mathbb{W}_x^{\beta,2}(\mathcal{M}^{-\frac{1}{2}} \langle v \rangle^{\frac{s}{2}}), \quad s \in \mathbb{R},$$

setting simply $\tilde{H} := \tilde{H}_0$. Recall that $-\mathbf{L}_1$ is (better than) coercive on $(\mathbf{Id} - \pi_0)\tilde{H}$ (see Briant (2015) for instance) and denote by $\tilde{\mu}_1$ the coercivity constant, namely

$$-\langle \mathbf{L}_1(\mathbf{Id} - \pi_0)g, (\mathbf{Id} - \pi_0)g \rangle_{\tilde{H}} \geq \tilde{\mu}_1 \|(\mathbf{Id} - \pi_0)g\|_{\tilde{H}_1}^2, \quad g \in \tilde{H}_1.$$

In the space \tilde{H} , we can compute the inner product between $\partial_t h_\varepsilon^1$ and $(\mathbf{Id} - \pi_0)h_\varepsilon^1$ where we recall that h_ε^1 solves

$$\partial_t h_\varepsilon^1 = \mathcal{G}_{1,\varepsilon} h_\varepsilon^1 + \varepsilon^{-1} \mathcal{Q}_1(h_\varepsilon^1, h_\varepsilon^1) + \mathcal{A}_\varepsilon h_\varepsilon^0.$$

We obtain, thanks to Cauchy-Schwarz inequality, that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}}^2 + \frac{\tilde{\mu}_1}{\varepsilon^2} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}_1}^2 \\ & \leq \langle \varepsilon^{-1} (\mathcal{Q}_1(h_\varepsilon^1, h_\varepsilon^1) - (\mathbf{Id} - \pi_0)(v \cdot \nabla_x h_\varepsilon^1)) + (\mathbf{Id} - \pi_0)(\mathcal{A}_\varepsilon h_\varepsilon^0), (\mathbf{Id} - \pi_0)h_\varepsilon^1 \rangle_{\tilde{H}} \\ & \leq \varepsilon^{-1} \left(\|\mathcal{Q}_1(h_\varepsilon^1, h_\varepsilon^1)\|_{\tilde{H}_{-1}} + \|v \cdot \nabla_x h_\varepsilon^1\|_{\tilde{H}_{-1}} \right) \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}_1} \\ & \quad + \|\mathcal{A}_\varepsilon h_\varepsilon^0\|_{\tilde{H}} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}}. \end{aligned}$$

We deduce easily then with a simple use of Young's inequality on the right-hand-side of this inequality that there is $C > 0$ independent of ε such that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}}^2 + \frac{\tilde{\mu}_1}{2\varepsilon^2} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}_1}^2 \\ & \leq C \left(\|\mathcal{Q}_1(h_\varepsilon^1, h_\varepsilon^1)\|_{\tilde{H}_{-1}}^2 + \|v \cdot \nabla_x h_\varepsilon^1\|_{\tilde{H}_{-1}}^2 + \varepsilon^2 \|\mathcal{A}_\varepsilon h_\varepsilon^0\|_{\tilde{H}}^2 \right). \end{aligned}$$

Thus, for some different $C > 0$, one has

$$\begin{aligned} & \frac{d}{dt} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1(t)\|_{\tilde{H}}^2 + \frac{\tilde{\mu}_1}{\varepsilon^2} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1(t)\|_{\tilde{H}_1}^2 \\ & \leq C \left(\|h_\varepsilon^1(t)\|_{\mathcal{H}}^4 + \|h_\varepsilon^1(t)\|_{\mathcal{H}}^2 + \|h_\varepsilon^0(t)\|_{\mathcal{E}}^2 \right) \leq CK_0 \end{aligned}$$

where the last estimate comes from the results obtained in Sections 4 and 5, $\mathcal{K}_0 \lesssim 1$ and also $\|\cdot\|_{\tilde{H}} \lesssim \|\cdot\|_{\mathcal{H}}$. We integrate this inequality over (t_1, t_2) to get

$$\begin{aligned} \frac{\tilde{\mu}_1}{\varepsilon^2} \int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(t)\|_{\tilde{H}_1}^2 dt &\leq \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(t_1)\|_{\tilde{H}}^2 + C\mathcal{K}_0(t_2 - t_1) \\ &\leq C\mathcal{K}_0 \max(1, t_2 - t_1), \end{aligned} \quad (6.3)$$

where we used (4.34). Introduce now the space $\tilde{E} = L_v^1 \mathbb{W}_x^{\beta, 2}(\boldsymbol{\varpi}_q)$. Noticing that

$$\|\cdot\|_{\tilde{E}} \lesssim \|\cdot\|_{\mathcal{E}} \quad \text{and} \quad \|\cdot\|_{\tilde{E}} \lesssim \|\cdot\|_{\tilde{H}}$$

and writing that $h_\varepsilon(\tau) = h_\varepsilon^1(\tau) + h_\varepsilon^0(\tau)$, one has

$$\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon(\tau)\|_{\tilde{E}} d\tau \leq \int_{t_1}^{t_2} \left(\|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(\tau)\|_{\tilde{E}} + \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^0(\tau)\|_{\tilde{E}} \right) d\tau. \quad (6.4)$$

Using Cauchy-Schwarz inequality

$$\begin{aligned} \int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon(\tau)\|_{\tilde{E}} d\tau &\lesssim \sqrt{t_2 - t_1} \left(\left(\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(\tau)\|_{\tilde{H}}^2 d\tau \right)^{\frac{1}{2}} + \left(\int_{t_1}^{t_2} \|h_\varepsilon^0(\tau)\|_{\mathcal{E}}^2 d\tau \right)^{\frac{1}{2}} \right). \end{aligned}$$

From (6.3), the first integral involving h_ε^1 is such that

$$\left(\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(\tau)\|_{\tilde{H}}^2 d\tau \right)^{\frac{1}{2}} \leq C\varepsilon\sqrt{\mathcal{K}_0} \max\{1, \sqrt{t_2 - t_1}\}$$

whereas, to estimate the integral involving h_ε^0 we use that $h_\varepsilon^0 \in \mathcal{X}_0$ as defined in Section 5 to get

$$\begin{aligned} \int_{t_1}^{t_2} \|h_\varepsilon^0(\tau)\|_{\mathcal{E}}^2 d\tau &\leq \sup_{t_1 \leq \tau \leq t_2} \|h_\varepsilon^0(\tau)\|_{\mathcal{E}} \int_{t_1}^{t_2} \|h_\varepsilon^0(\tau)\|_{\mathcal{E}_1} d\tau \\ &\leq \varepsilon^2 \|h_\varepsilon^0\|_0^2 \leq C\varepsilon^2 \mathcal{K}_0. \end{aligned}$$

This proves the result. \square

Remark 6.2. Notice that, if we are not interested in introducing a modulus of continuity in time for the above integral, we can directly deduce from (6.3) and (6.4) that

$$\begin{aligned} \int_0^T \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon(t)\|_{L_v^1 \mathbb{W}_x^{\beta, 2}(\boldsymbol{\varpi}_{q+1})} dt &\lesssim \int_0^T \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^0(t)\|_{\mathcal{E}_1} dt + \int_0^T \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(t)\|_{\tilde{H}_1} dt \\ &\lesssim \varepsilon \|h_\varepsilon^0\|_0 + \sqrt{T} \left(\int_0^T \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(t)\|_{\tilde{H}_1}^2 dt \right)^{\frac{1}{2}} \end{aligned}$$

which results in

$$\int_0^T \|(\mathbf{Id} - \pi_0) h_\varepsilon(t)\|_{L_v^1 \mathbb{W}_x^{\beta,2}(\varpi_{q+1})} dt \lesssim \varepsilon \sqrt{\mathcal{K}_0} (1 + \sqrt{T}) \quad (6.5)$$

for any $0 \leq \beta \leq m - 1$.

We deduce the following convergence result:

Theorem 6.3 (Weak convergence). *Fix $T > 0$, and let*

$$\{h_\varepsilon\}_\varepsilon \subset L^1((0, T); L_v^1 \mathbb{W}_x^{m,2}(\varpi_q))$$

be a sequence of solutions to the inelastic Boltzmann equation (1.22). Then, with the splitting $h_\varepsilon = h_\varepsilon^0 + h_\varepsilon^1$, up to extraction of a subsequence, one has

$$\begin{cases} \{h_\varepsilon^0\}_\varepsilon \text{ converges to 0 strongly in } L^1((0, T); \mathcal{E}_1) \\ \{h_\varepsilon^1\}_\varepsilon \text{ which converges to } \mathbf{h} \text{ weakly in } L^2\left((0, T); L_v^2 \mathbb{W}_x^{m,2}(\mathcal{M}^{-\frac{1}{2}})\right) \end{cases} \quad (6.6)$$

where $\mathbf{h} = \pi_0(\mathbf{h})$. In particular, there exist

$$\begin{aligned} \varrho &\in L^2\left((0, T); \mathbb{W}_x^{m,2}(\mathbb{T}^d)\right), & \theta &\in L^2\left((0, T); \mathbb{W}_x^{m,2}(\mathbb{T}^d)\right), \\ u &\in L^2\left((0, T); \left(\mathbb{W}_x^{m,2}(\mathbb{T}^d)\right)^d\right) \end{aligned}$$

such that

$$\mathbf{h}(t, x, v) = \left(\varrho(t, x) + u(t, x) \cdot v + \frac{1}{2} \theta(t, x) (|v|^2 - d\vartheta_1) \right) \mathcal{M}(v) \quad (6.7)$$

where \mathcal{M} is the Maxwellian distribution introduced in (1.15).

Proof. Let $T > 0$ be fixed. We use the notations of Proposition 6.1. The estimates obtained in Section 5, using the splitting $h_\varepsilon = h_\varepsilon^0(t) + h_\varepsilon^1(t)$ imply the following properties of the sequences of time-dependent vector-valued mappings $\{h_\varepsilon^1\}_\varepsilon$, $\{h_\varepsilon^0\}_\varepsilon$ and $\{h_\varepsilon\}_\varepsilon$:

$$\{h_\varepsilon^1\} \subset (L^1 \cap L^\infty)((0, T); \mathcal{H}) \quad \text{is bounded} \quad (6.8)$$

$$\int_0^T \|h_\varepsilon^0(t)\|_{\mathcal{E}_1} dt \lesssim \varepsilon^2 \quad (6.9)$$

From (6.8) and since $\|\cdot\|_{L_v^2 \mathbb{W}_x^{m,2}(\mathcal{M}^{-\frac{1}{2}})} \lesssim \|\cdot\|_{\mathcal{H}}$, we deduce that

$$\{h_\varepsilon^1\} \text{ is bounded in } L^2\left((0, T); L_v^2 \mathbb{W}_x^{m,2}(\mathcal{M}^{-\frac{1}{2}})\right)$$

and therefore, admits a subsequence, say $\{h_{\varepsilon'}^1\}_{\varepsilon'}$, which converges weakly to some \mathbf{h} in the space $L^2((0, T); L_v^2 \mathbb{W}_x^{m,2}(\mathcal{M}^{-\frac{1}{2}}))$. This, combined with (6.9) gives (6.6). From (6.3) we also have, for that subsequence,

$$\lim_{\varepsilon' \rightarrow 0} \int_0^T \|(\mathbf{Id} - \pi_0) h_{\varepsilon'}^1(t)\|_{L_v^2 \mathbb{W}_x^{m-1,2}(\mathcal{M}^{-\frac{1}{2}})}^2 dt = 0$$

so that $(\mathbf{Id} - \pi_0) \mathbf{h} = 0$. This gives the result. \square

Remark 6.4. *As observed in the previous proof, the convergence (6.6) can be made even more precise since we also have*

$$\{(\mathbf{Id} - \pi_0) h_{\varepsilon}^1\} \text{ converges strongly to 0 in } L^2\left((0, T); L_v^2 \mathbb{W}_x^{m-1,2}(\mathcal{M}^{-\frac{1}{2}})\right).$$

This means somehow that the only part of h_{ε} which prevents the strong convergence towards \mathbf{h} is $\{\pi_0 h_{\varepsilon}^1\}_{\varepsilon}$.

Because of Theorem 6.3 and for simplicity sake, from here on, we will write that our sequences converge even if it is true up to an extraction.

The above mode of convergence implies the following convergence of velocity averages of h_{ε} . For any function $f = f(t, x, v)$ we denote the velocity average by

$$\langle f \rangle = \int_{\mathbb{R}^d} f(t, x, v) dv$$

recalling of course that this is a function depending on (t, x) . We have then the following:

Lemma 6.5. *Let $\{h_{\varepsilon}\}$ be converging to \mathbf{h} in the sense of Theorem 6.3. Then, for any function $\psi = \psi(v)$ such that*

$$|\psi(v)| \lesssim \varpi_q(v)$$

one has

$$\langle \psi h_{\varepsilon} \rangle \longrightarrow \langle \psi \mathbf{h} \rangle \quad \text{in } \mathcal{D}'_{t,x} \quad (6.10)$$

whereas

$$\langle \psi \mathcal{Q}_1^r(h_{\varepsilon}, h_{\varepsilon}) \rangle \longrightarrow 0 \quad \text{in } \mathcal{D}'_{t,x} \quad (6.11)$$

where we set $\mathcal{Q}_1^r(h_{\varepsilon}, h_{\varepsilon}) = \mathcal{Q}_1(h_{\varepsilon}, h_{\varepsilon}) - \mathcal{Q}_1(\pi_0 h_{\varepsilon}, \pi_0 h_{\varepsilon})$.

Proof. Let ψ be such that $|\psi(v)| \lesssim \varpi_q(v)$ and let $\varphi = \varphi(t, x) \in \mathcal{C}_c^\infty((0, T) \times \mathbb{T}^d)$ be given. One computes

$$I_{\varepsilon} := \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \left(\langle \psi h_{\varepsilon} \rangle - \langle \psi \mathbf{h} \rangle \right) dx = I_{\varepsilon}^0 + I_{\varepsilon}^1$$

where

$$I_{\varepsilon}^0 = \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \langle \psi h_{\varepsilon}^0 \rangle dx, \quad I_{\varepsilon}^1 = \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \left(\langle \psi h_{\varepsilon}^1 \rangle - \langle \psi \mathbf{h} \rangle \right) dx.$$

Because $|I_\varepsilon^0| \lesssim \|\varphi\|_{L_{t,x}^\infty} \int_0^T \|h_\varepsilon^0(t)\|_{L_{v,x}^1(\varpi_q)} dt \lesssim \|\varphi\|_{L_{t,x}^\infty} \int_0^T \|h_\varepsilon^0(t)\|_{L_v^1 L_x^2(\varpi_q)} dt$, we deduce from (6.6) that $\lim_{\varepsilon \rightarrow 0} I_\varepsilon^0 = 0$. In the same way, one has

$$I_\varepsilon^1 = \int_0^T dt \int_{\mathbb{T}^d \times \mathbb{R}^d} (\psi(v) \mathcal{M}(v) \varphi(t, x)) (h_\varepsilon^1(t, x, v) - \mathbf{h}(t, x, v)) dx \mathcal{M}^{-1}(v) dv$$

and, since the mapping

$$(t, x, v) \mapsto \psi(v) \mathcal{M}(v) \varphi(t, x) \quad \text{belongs to } L^2((0, T); L_v^2 \mathbb{W}_x^{m,2}(\mathcal{M}^{-\frac{1}{2}})), \quad (6.12)$$

we deduce from (6.6) that $\lim_{\varepsilon \rightarrow 0} I_\varepsilon^1 = 0$. This proves (6.10). To prove (6.11), one sets

$$J_\varepsilon := \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \langle \psi \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) \rangle dx.$$

One writes $J_\varepsilon = J_\varepsilon^1 + J_\varepsilon^2$ where

$$\begin{aligned} J_\varepsilon^1 &= \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \langle \psi \mathcal{Q}_1((\mathbf{Id} - \pi_0)h_\varepsilon, (\mathbf{Id} - \pi_0)h_\varepsilon) \rangle dx \\ J_\varepsilon^2 &= 2 \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \langle \psi \tilde{\mathcal{Q}}_1((\mathbf{Id} - \pi_0)h_\varepsilon, \pi_0 h_\varepsilon) \rangle dx \end{aligned}$$

where we recall that $\tilde{\mathcal{Q}}_1$ is defined in (1.23). One has

$$\begin{aligned} |J_\varepsilon^1| &\lesssim \|\varphi\|_{L_{t,x}^\infty} \int_0^T \|\mathcal{Q}_1((\mathbf{Id} - \pi_0)h_\varepsilon, (\mathbf{Id} - \pi_0)h_\varepsilon)\|_{L_{v,x}^1(\varpi_q)} dt \\ &\lesssim \|\varphi\|_{L_{t,x}^\infty} \int_0^T \|\mathcal{Q}_1((\mathbf{Id} - \pi_0)h_\varepsilon, (\mathbf{Id} - \pi_0)h_\varepsilon)\|_{L_v^1 L_x^2(\varpi_q)} dt. \end{aligned}$$

Noticing that

$$\|\mathcal{Q}_1((\mathbf{Id} - \pi_0)h_\varepsilon, (\mathbf{Id} - \pi_0)h_\varepsilon)\|_{L_v^1 L_x^2(\varpi_q)} \lesssim \|h_\varepsilon\|_{\mathcal{E}} \|(\mathbf{Id} - \pi_0)h_\varepsilon\|_{L_v^1 \mathbb{W}_x^{m-1,2}(\varpi_{q+1})}.$$

we deduce from (6.5) and the fact that $\sup_{t \in (0, T)} \|h_\varepsilon(t)\|_{\mathcal{E}} < \infty$ that

$$\lim_{\varepsilon \rightarrow 0} |J_\varepsilon^1| = 0.$$

We prove exactly in the same way that

$$\lim_{\varepsilon \rightarrow 0} |J_\varepsilon^2| = 0.$$

This proves the result. \square

Regarding the characterisation (6.7) of the limit $\mathbf{h}(t)$, note that

$$\varrho(t, x) = \int_{\mathbb{R}^d} \mathbf{h}(t, x, v) dv, \quad u(t, x) = \frac{1}{\vartheta_1} \int_{\mathbb{R}^d} v \mathbf{h}(t, x, v) dv,$$

and

$$\varrho(t, x) + \vartheta_1 \theta(t, x) = \frac{1}{d\vartheta_1} \int_{\mathbb{R}^d} |v|^2 \mathbf{h}(t, x, v) dv.$$

Corollary 6.6. *With the notations of Theorem 6.3, for any $T > 0$, the limit $\mathbf{h}(t, x, v)$ given by (6.7) satisfies the incompressibility condition*

$$\operatorname{div}_x u(t, x) = 0, \quad t \in (0, T), \quad (6.13)$$

and Boussinesq relation

$$\nabla_x (\varrho + \vartheta_1 \theta) = 0. \quad (6.14)$$

As a consequence, introducing

$$E(t) = \int_{\mathbb{T}^d} \theta(t, x) dx, \quad t \in (0, T),$$

one has strengthened Boussinesq relation

$$\varrho(t, x) + \vartheta_1 (\theta(t, x) - E(t)) = 0, \quad \text{for a.e } (t, x) \in (0, T) \times \mathbb{T}^d. \quad (6.15)$$

Proof. Set

$$\varrho_\varepsilon(t, x) = \int_{\mathbb{R}^d} h_\varepsilon(t, x, v) dv, \quad \mathbf{u}_\varepsilon(t, x) = \frac{1}{\vartheta_1} \int_{\mathbb{R}^d} v h_\varepsilon(t, x, v) dv,$$

and, multiplying (1.22) with 1 and v and integrating in velocity, we get

$$\varepsilon \partial_t \varrho_\varepsilon + \vartheta_1 \operatorname{div}_x (\mathbf{u}_\varepsilon) = 0, \quad (6.16)$$

$$\varepsilon \partial_t \mathbf{u}_\varepsilon + \operatorname{Div}_x (\mathbf{J}_\varepsilon) = \frac{\kappa_\alpha}{\varepsilon} \mathbf{u}_\varepsilon, \quad (6.17)$$

where $\mathbf{J}_\varepsilon(t, x)$ denotes the tensor

$$\mathbf{J}_\varepsilon(t, x) := \frac{1}{\vartheta_1} \int_{\mathbb{R}^d} v \otimes v h_\varepsilon(t, x, v) dv,$$

since both \mathbf{L}_α and \mathbf{Q}_α conserve mass and momentum. The proof of (6.13) is straightforward since $\varepsilon \partial_t \varrho_\varepsilon \rightarrow 0$ and $\operatorname{div}_x (\mathbf{u}_\varepsilon) \rightarrow \operatorname{div}_x u$ in the distribution sense. Let us give the detail for the sake of completeness. Multiplying (6.16) with a function $\varphi \in \mathcal{C}_c^\infty((0, T) \times \mathbb{T}^d)$ and integrating over $(0, T) \times \mathbb{T}^d$ we get that

$$- \int_0^T dt \int_{\mathbb{T}^d} \nabla_x \varphi(t, x) \cdot \mathbf{u}_\varepsilon(t, x) dx = \varepsilon \int_0^T dt \int_{\mathbb{T}^d} \varrho_\varepsilon(t, x) \partial_t \varphi(t, x) dx,$$

which, taking the limit $\varepsilon \rightarrow 0$ and because $\varrho_\varepsilon \rightarrow \varrho$ and $\mathbf{u}_\varepsilon \rightarrow u$ in $\mathcal{D}'_{t,x}$, yields

$$\int_0^T dt \int_{\mathbb{T}^d} \nabla_x \varphi(t, x) \cdot u(t, x) dx = 0, \quad \forall \varphi \in \mathcal{C}_c^\infty((0, T) \times \mathbb{T}^d).$$

Since $u(t, x) \in L^2((0, T); (\mathbb{W}_x^{m,2}(\mathbb{T}^d)^d))$, the incompressibility condition (6.13) holds true. In the same way, for any $i = 1, \dots, d$ and $\varphi \in \mathcal{C}_c^\infty((0, T) \times \mathbb{T}^d)$, noticing that

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon \int_0^T \mathbf{u}_\varepsilon^i \partial_t \varphi(t, x) dx = \lim_{\varepsilon \rightarrow 0^+} \frac{\kappa_\alpha}{\varepsilon} \int_0^T dt \int_{\mathbb{T}^d} \mathbf{u}_\varepsilon^i(t, x) \varphi(t, x) dx = 0,$$

because $\kappa_\alpha = 1 - \alpha \leq C\varepsilon^2$ we get that

$$0 = \lim_{\varepsilon \rightarrow 0^+} \sum_{j=1}^d \int_0^T dt \int_{\mathbb{T}^d} \mathbf{J}_\varepsilon^{i,j}(t, x) \partial_{x_j} \varphi(t, x) dx = \sum_{j=1}^d \int_0^T dt \int_{\mathbb{T}^d} \mathbf{J}_0^{i,j}(t, x) \partial_{x_j} \varphi(t, x) dx,$$

where

$$\mathbf{J}_0^{i,j}(t, x) = \frac{1}{\vartheta_1} \int_{\mathbb{R}^d} v_i v_j \mathbf{h}(t, x, v) dv = (\varrho(t, x) + \vartheta_1 \theta(t, x)) \delta_{ij}, \quad i, j = 1, \dots, d.$$

Therefore, for any $i = 1, \dots, d$,

$$\int_0^T dt \int_{\mathbb{T}^d} (\varrho(t, x) + \vartheta_1 \theta(t, x)) \partial_{x_i} \varphi(t, x) dx = 0, \quad \forall \varphi \in \mathcal{C}_c^\infty((0, T) \times \mathbb{T}^d).$$

As before, this gives the Boussinesq relation (6.14). To show that Boussinesq relation can be strengthened, one notices that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{T}^d} \varrho_\varepsilon(t, x) dx = \int_{\mathbb{T}^d} \varrho(t, x) dx \quad \text{in } \mathcal{D}'_t$$

from which we deduce, from the conservation of mass for (6.19), that

$$\int_{\mathbb{T}^d} \varrho(t, x) dx = 0, \quad \text{for a.e. } t > 0.$$

With the definition of $E(t)$, this implies that

$$\int_{\mathbb{T}^d} (\varrho(t, x) + \vartheta_1 (\theta(t, x) - E(t))) dx = 0, \quad \text{for a.e. } t > 0,$$

and, this combined with (6.14) yields the strengthened form (6.15). \square

Remark 6.7. Using Boussinesq relation together with (6.7), one checks without major difficulty that

$$v \cdot \nabla_x \mathbf{h} = \mathcal{M}(v \otimes v) : \nabla_x u + \frac{1}{2} \mathcal{M}(|v|^2 - (d+2)\vartheta_1) v \cdot \nabla_x \theta. \quad (6.18)$$

Then, using the incompressibility condition (6.13) it holds that

$$\int_{\mathbb{R}^d} \Psi_j(v) v \cdot \nabla_x \mathbf{h} dv = 0, \quad \forall j = 1, \dots, d+2,$$

that is, $\pi_0(v \cdot \nabla_x \mathbf{h}) = 0$. In particular, $v \cdot \nabla_x \mathbf{h} \in \text{Range}(\mathbf{Id} - \pi_0) \subset \text{Range}(\mathbf{L}_1)$ (see (Kato, 1980, Eq. (6.34), p. 180)).

6.2. Identification of the limit. We aim here to fully characterise the limit $\mathbf{h}(t, x, v)$ obtained in Theorem 6.3. To do so, we identify the limit equation satisfied by the macroscopic quantities (ϱ, u, θ) in (6.7) following the path of Bardos et al. (1993); Golse & Saint-Raymond (2004) and exploiting the fact that the mode of convergence in Theorem 6.3 is stronger than the one of Bardos et al. (1993); Golse & Saint-Raymond (2004). The regime of weak inelasticity is central in the analysis.

We denote by $\{h_\varepsilon\}$ any subsequence which converges to \mathbf{h} in the above Theorem 6.3. We will see in the sequel, under some strong convergence assumption on the initial datum that all subsequences will share the same limit and, as such, the whole sequence will be convergent.

Recall (1.22)

$$\varepsilon \partial_t h_\varepsilon + v \cdot \nabla_x h_\varepsilon + \varepsilon^{-1} \kappa_\alpha \nabla_v \cdot (v h_\varepsilon) = \varepsilon^{-1} \mathbf{L}_\alpha h_\varepsilon + \mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon), \quad (6.19)$$

under the scaling hypothesis that $\alpha = 1 - \lambda_0 \varepsilon^2 + o(\varepsilon^2)$, $\lambda_0 \geq 0$ (see Assumption 1.2). Multiplying (6.19) respectively with 1, v , $\frac{1}{2}|v|^2$, we observe that the quantities

$$\langle h_\varepsilon \rangle, \quad \langle v h_\varepsilon \rangle, \quad \langle \tfrac{1}{2}|v|^2 \rangle, \quad \langle \tfrac{1}{2}|v|^2 v h_\varepsilon \rangle, \quad \text{and} \quad \langle v \otimes v h_\varepsilon \rangle,$$

are important. As in the classical elastic case, we write

$$\langle v \otimes v h_\varepsilon \rangle = \langle \mathbf{A} h_\varepsilon \rangle + p_\varepsilon \mathbf{Id}, \quad p_\varepsilon = \langle \tfrac{1}{d}|v|^2 h_\varepsilon \rangle,$$

where we introduce the traceless tensor

$$\mathbf{A} = \mathbf{A}(v) = v \otimes v - \frac{1}{d}|v|^2 \mathbf{Id}.$$

Properties of this tensor are established in Appendix C. In a more precise way, one obtains, after integrating (6.19) against 1, v_i , $\frac{1}{2}|v|^2$,

$$\partial_t \langle h_\varepsilon \rangle + \frac{1}{\varepsilon} \operatorname{div}_x \langle v h_\varepsilon \rangle = 0, \quad (6.20a)$$

$$\partial_t \langle v h_\varepsilon \rangle + \frac{1}{\varepsilon} \operatorname{Div}_x \langle \mathbf{A} h_\varepsilon \rangle + \frac{1}{\varepsilon} \nabla_x p_\varepsilon = \frac{\kappa_\alpha}{\varepsilon^2} \langle v h_\varepsilon \rangle, \quad (6.20b)$$

$$\partial_t \langle \tfrac{1}{2}|v|^2 h_\varepsilon \rangle + \frac{1}{\varepsilon} \operatorname{div}_x \langle \tfrac{1}{2}|v|^2 v h_\varepsilon \rangle = \frac{1}{\varepsilon^3} \mathcal{I}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2\kappa_\alpha}{\varepsilon^2} \langle \tfrac{1}{2}|v|^2 h_\varepsilon \rangle, \quad (6.20c)$$

where

$$\mathcal{I}_\alpha(f, f) = \int_{\mathbb{R}^d} [\mathcal{Q}_\alpha(f, f) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha)] |v|^2 dv.$$

Notice that, using (6.7) as well as Corollary 6.6,

$$\begin{aligned} \operatorname{div}_x \langle v h_\varepsilon \rangle &\longrightarrow \vartheta_1 \operatorname{div}_x u = 0, & \left\langle \frac{1}{2} |v|^2 h_\varepsilon \right\rangle &\longrightarrow \frac{d\vartheta_1}{2} (\varrho + \vartheta_1 \theta), \\ \nabla_x p_\varepsilon &\longrightarrow \frac{1}{d} \nabla_x \langle |v|^2 \mathbf{h} \rangle = \vartheta_1 \nabla_x (\varrho + \vartheta_1 \theta) = 0, \\ & & \langle \mathbf{A} h_\varepsilon \rangle &\longrightarrow \langle \mathbf{A} \mathbf{h} \rangle = 0, \\ \left\langle \frac{1}{2} |v|^2 v_j h_\varepsilon \right\rangle &\longrightarrow \left\langle \frac{1}{2} |v|^2 v_j \mathbf{h} \right\rangle = \frac{1}{2} u_j \langle |v|^2 v_j^2 \mathcal{M} \rangle = \frac{d+2}{2} \vartheta_1^2 u_j, & j = 1, \dots, d, \end{aligned}$$

where all the limits hold in $\mathcal{D}'_{t,x}$ and where $\langle \mathbf{A} \mathbf{h} \rangle = 0$ since $\mathbf{h} \in \operatorname{Ker}(\mathbf{L}_1)$ and $\mathbf{A} \in \operatorname{Range}(\mathbf{I} - \boldsymbol{\pi}_0)$. Moreover, under the above scaling

$$\frac{\kappa_\alpha}{\varepsilon^2} \langle v h_\varepsilon \rangle \longrightarrow \vartheta_1 \lambda_0 u, \quad \text{in } \mathcal{D}'_{t,x},$$

since $\lambda_0 = \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-2} \kappa_\alpha$. The limit of $\varepsilon^{-3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon)$ is handled in the following lemma.

Lemma 6.8. *It holds that*

$$\frac{1}{\varepsilon^3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) \longrightarrow \mathcal{J}_0 \quad \text{in } \mathcal{D}'_{t,x},$$

where

$$\mathcal{J}_0(t, x) = -\lambda_0 \bar{c} \vartheta_1^{\frac{3}{2}} \left(\varrho(t, x) + \frac{3}{4} \vartheta_1 \theta(t, x) \right)$$

for some positive constant \bar{c} depending only on the angular kernel $b(\cdot)$ and d . In particular,

$$\mathcal{J}_0 = -\lambda_0 \bar{c} \vartheta_1^{\frac{5}{2}} \left(E(t) - \frac{1}{4} \theta(t, x) \right).$$

Proof. We recall, see (1.8), that

$$\int_{\mathbb{R}^d} |v|^2 \mathcal{Q}_\alpha(g, f) dv = -(1 - \alpha^2) \frac{\gamma_b}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(v) g(v_*) |v - v_*|^3 dv dv_*$$

where $\gamma_b = \frac{1}{2} \int_{\mathbb{S}^{d-1}} (1 - \hat{u} \cdot \sigma) b(\hat{u} \cdot \sigma) d\sigma$. Thus, for $f_\varepsilon = G_\alpha + \varepsilon h_\varepsilon$ we obtain

$$\begin{aligned} \frac{1}{\varepsilon^3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) &= -\frac{\gamma_b}{4} \frac{1 - \alpha^2}{\varepsilon^2} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} [h_\varepsilon(v) G_\alpha(v_*) + h_\varepsilon(v_*) G_\alpha(v)] |v - v_*|^3 dv dv_* \right. \\ &\quad \left. + \varepsilon \int_{\mathbb{R}^d \times \mathbb{R}^d} h_\varepsilon(v) h_\varepsilon(v_*) |v - v_*|^3 dv dv_* \right). \quad (6.21) \end{aligned}$$

Recall that $\lim_{\varepsilon \rightarrow 0^+} \frac{1 - \alpha^2}{\varepsilon^2} = \lambda_0$. It is clear from Minkowski's integral inequality that the $\mathbb{W}_x^{m,2}(\mathbb{T}^d)$ norm of the last term in the right-side is controlled by $\|h_\varepsilon\|_{\mathcal{E}}^2$. Theorem 5.6 implies that the last term in (6.21) is converging to 0 in $L^1((0, T); \mathbb{W}_x^{m,2}(\mathbb{T}^d))$. One handles the first term in the

right-side using Theorem 6.3 and the fact that $G_\alpha \rightarrow \mathcal{M}$ strongly. Details are left to the reader. We then easily obtain the convergence of $\varepsilon^{-3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon)$ towards

$$\mathcal{J}_0 := -\lambda_0 \gamma_b \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{h}(t, x, v) \mathcal{M}(v_*) |v - v_*|^3 dv dv_*.$$

The expression of \mathcal{J}_0 is then obtained by direct inspection from (6.7) with

$$\bar{c} = \gamma_b a, \quad a = \frac{2\sqrt{2}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}|v|^2\right) |v|^3 dv,$$

where

$$\begin{aligned} \int_{\mathbb{R}^{2d}} \mathcal{M}(v) \mathcal{M}(v_*) |v - v_*|^3 dv dv_* &= \vartheta_1^{\frac{3}{2}} a, \\ \int_{\mathbb{R}^{2d}} \mathcal{M}(v) \mathcal{M}(v_*) |v|^2 |v - v_*|^3 dv dv_* &= \frac{2d+3}{2} \vartheta_1^{\frac{5}{2}} a. \end{aligned}$$

We refer to (Mischler & Mouhot, 2009, Lemma A.1) for these identities. The second part of the lemma follows from the strengthened Boussinesq relation (6.15). \square

6.3. About the equations of motion and temperature. We give here some preliminary result aiming at deriving the equations satisfied by the bulk velocity $u(t, x)$ and $\theta(t, x)$. As in Bardos et al. (1993); Golse & Saint-Raymond (2004), in order to investigate the limiting behaviour of the system (6.20) as $\varepsilon \rightarrow 0^+$, we need to investigate the limit in the distributional sense of

$$\varepsilon^{-1} \text{Div}_x \langle \mathbf{A} h_\varepsilon \rangle = -\varepsilon^{-1} \text{Div}_x \langle \phi \mathbf{L}_1 h_\varepsilon \rangle \quad (6.22)$$

and

$$\varepsilon^{-1} \text{div}_x \langle \mathbf{b} h_\varepsilon \rangle = -\varepsilon^{-1} \text{div}_x \langle \psi \mathbf{L}_1 h_\varepsilon \rangle \quad (6.23)$$

where ϕ and ψ are defined in Lemma C.1 and where we used that \mathbf{L}_1 is selfadjoint in $L_v^2(\mathcal{M}^{-1/2})$.

Since the limiting vector-field u is divergence-free, it turns out enough to investigate only the limit of $\mathcal{P} \text{Div}_x \langle \varepsilon^{-1} \mathbf{A} h_\varepsilon \rangle$ where we recall that \mathcal{P} is the Leray projection on divergence-free vector fields⁷. We begin with a strong compactness result

Lemma 6.9. Introduce

$$\begin{aligned} u_\varepsilon(t, x) &= \exp\left(-t \frac{\kappa_\alpha}{\varepsilon^2}\right) \mathcal{P} u_\varepsilon(t, x) \\ \text{and } \vartheta_\varepsilon(t, x) &= \left\langle \frac{1}{2} (|v|^2 - (d+2)\vartheta_1) h_\varepsilon \right\rangle, \quad t \in (0, T), \quad x \in \mathbb{T}^d. \end{aligned}$$

⁷Recall that, for a vector field \mathbf{u} , $\mathcal{P}\mathbf{u} = \mathbf{u} - \nabla \Delta^{-1}(\nabla \cdot \mathbf{u})$. On the torus, it can be defined via Fourier expansion, if $\mathbf{u} = \sum_{k \in \mathbb{Z}^d} \mathbf{a}_k e^{ik \cdot x}$, $\mathbf{a}_k \in \mathbb{C}^d$, then $\mathcal{P}\mathbf{u} = \sum_{k \in \mathbb{Z}^d} \left(\mathbf{I}_d - \frac{k \otimes k}{|k|^2} \right) \mathbf{a}_k e^{ik \cdot x}$.

Then, $\{\partial_t u_\varepsilon\}_\varepsilon$ and $\{\partial_t \vartheta_\varepsilon\}_\varepsilon$ are bounded in $L^1\left((0, T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d)\right)$. Consequently, up to the extraction of a subsequence,

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \|\mathcal{P}u_\varepsilon(t) - u(t)\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)} dt = 0 \quad (6.24)$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^T \|\vartheta_\varepsilon(t, \cdot) - \vartheta_0(t, \cdot)\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)} dt = 0 \quad (6.25)$$

where

$$\vartheta_0(t, x) = \left\langle \frac{1}{2}(|v|^2 - (d+2)\vartheta_1)\mathbf{h} \right\rangle = \frac{d\vartheta_1}{2}(\varrho(t, x) + \vartheta_1\theta(t, x)) - \frac{d+2}{2}\vartheta_1\varrho(t, x).$$

In other words, $\{\mathcal{P}u_\varepsilon\}_\varepsilon$ converges strongly to u in $L^1\left((0, T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d)\right)$ and $\{\vartheta_\varepsilon\}_\varepsilon$ converges strongly to ϑ_0 in $L^1\left((0, T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d)\right)$.

Proof. We begin with the proof of (6.24). We apply the Leray projection \mathcal{P} to (6.20b) to eliminate the pressure gradient term. Then, we have that

$$\partial_t u_\varepsilon = -\exp\left(-t\frac{\kappa_\alpha}{\varepsilon^2}\right) \mathcal{P}\left(\vartheta_1^{-1} \operatorname{Div}_x \left\langle \frac{1}{\varepsilon} \mathbf{A} h_\varepsilon \right\rangle\right).$$

Notice that, since $\{h_\varepsilon\}_\varepsilon$ is bounded in $L^1((0, T); \mathcal{E})$ by Minkowski's integral inequality, one has that

$$\{u_\varepsilon\}_\varepsilon \text{ is bounded in } L^1\left((0, T); \mathbb{W}_x^{m,2}(\mathbb{T}^d)\right).$$

Moreover, since $\mathbf{A} h_\varepsilon = \mathbf{A}(\mathbf{Id} - \pi_0) h_\varepsilon$ we deduce from Proposition 6.1 and Minkowski's integral inequality that

$$\sup_\varepsilon \int_0^T \left\| \mathcal{P}\left(\operatorname{Div}_x \left\langle \frac{1}{\varepsilon} \mathbf{A} h_\varepsilon \right\rangle\right) \right\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)} dt < \infty.$$

In particular

$$\{\partial_t u_\varepsilon\}_\varepsilon \text{ is bounded in } L^1\left((0, T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d)\right).$$

Applying (Simon, 1987, Corollary 4) with $X = \mathbb{W}_x^{m,2}(\mathbb{T}^d)$ and $B = Y = \mathbb{W}_x^{m-2,2}(\mathbb{T}^d)$ (so that the embedding of X into B is compact by Rellich-Kondrachov Theorem (Taylor, 1996, Proposition 3.4, p. 330)), we deduce that $\{u_\varepsilon\}_\varepsilon$ is relatively compact in $L^1\left((0, T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d)\right)$. The result of strong convergence follows easily since we already now that $\mathcal{P}u_\varepsilon$ converges to u in $\mathcal{D}'_{t,x}$ (see Lemma 6.5 and recall $u = \mathcal{P}u$ since u is divergence-free).

The proof of (6.25) is similar. We begin with observing that, multiplying (6.20a) with $-\frac{d+2}{2}\vartheta_1$ and add it to (6.20c) we obtain the evolution of $\vartheta_\varepsilon(t, x)$

$$\partial_t \vartheta_\varepsilon + \frac{1}{\varepsilon} \operatorname{div}_x \left\langle \mathbf{b} h_\varepsilon \right\rangle = \frac{1}{\varepsilon^3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2\kappa_\alpha}{\varepsilon^2} \left\langle \frac{1}{2}|v|^2 h_\varepsilon \right\rangle. \quad (6.26)$$

Notice that $\{\vartheta_\varepsilon\}_\varepsilon$ is bounded in $L^1\left((0, T); \mathbb{W}_x^{m,2}(\mathbb{T}^d)\right)$ while, because $\mathbf{b} h_\varepsilon = \mathbf{b} (\mathbf{Id} - \boldsymbol{\pi}_0) h_\varepsilon$ we deduce from Proposition 6.1 by Minkowski's integral inequality that

$$\sup_\varepsilon \int_0^T \left\| \left(\operatorname{div}_x \left\langle \frac{1}{\varepsilon} \mathbf{b} h_\varepsilon \right\rangle \right) \right\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)} dt < \infty.$$

It is easy to see that the right-hand side of (6.26) is also bounded in $L^1\left((0, T), \mathbb{W}_x^{m,2}(\mathbb{T}^d)\right)$ so that $\{\partial_t \vartheta_\varepsilon\}_\varepsilon$ is bounded in $L^1\left((0, T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d)\right)$. Using again (Simon, 1987, Corollary 4) together with Rellich-Kondrachov Theorem, we deduce as before that $\{\vartheta_\varepsilon\}_\varepsilon$ is relatively compact in $L^1\left((0, T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d)\right)$. Since we already know that ϑ_ε converges in the distributional sense to ϑ_0 (see Lemma 6.5), we get the result of strong convergence. \square

Remark 6.10. *We will see later that the convergence of $\{\mathcal{P}\mathbf{u}_\varepsilon\}_\varepsilon$ and $\{\vartheta_\varepsilon\}_\varepsilon$ can actually be strenghten for well-prepared initial datum (see Proposition 6.18).*

A first consequence of the above Lemma is the following which regards (6.22)

Lemma 6.11. *In the distributional sense,*

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{P}\operatorname{Div}_x \left(\left\langle \varepsilon^{-1} \mathbf{A} h_\varepsilon \right\rangle - \left\langle \phi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \right\rangle \right) = -\boldsymbol{\nu} \Delta_x u \quad (6.27)$$

where $\boldsymbol{\nu}$ is defined in Lemma C.1.

Proof. When compared to the elastic case, $\mathbf{L}_1 h_\varepsilon$ does not appear in (6.19). We add it, as well as the quadratic elastic Boltzmann operator, by force and rewrite the latter as

$$\begin{aligned} \varepsilon \partial_t h_\varepsilon + v \cdot \nabla_x h_\varepsilon - \varepsilon^{-1} \mathbf{L}_1 h_\varepsilon &= \mathcal{Q}_1(h_\varepsilon, h_\varepsilon) - \varepsilon^{-1} \kappa_\alpha \nabla_v \cdot (v h_\varepsilon) + \\ &\quad \varepsilon^{-1} (\mathbf{L}_\alpha h_\varepsilon - \mathbf{L}_1 h_\varepsilon) + \mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon) - \mathcal{Q}_1(h_\varepsilon, h_\varepsilon). \end{aligned} \quad (6.28)$$

We interpret the last three terms as a source term

$$\mathbf{S}_\varepsilon := \varepsilon^{-1} (\mathbf{L}_\alpha h_\varepsilon - \mathbf{L}_1 h_\varepsilon) + \mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon) - \mathcal{Q}_1(h_\varepsilon, h_\varepsilon) - \varepsilon^{-1} \kappa_\alpha \nabla_v \cdot (v h_\varepsilon). \quad (6.29)$$

Then, multiplying (6.28) by ϕ and integrating over \mathbb{R}^d , we get using (6.22) that, for any $i, j = 1, \dots, d$,

$$\begin{aligned} \varepsilon \partial_t \left\langle \phi^{i,j} h_\varepsilon \right\rangle + \operatorname{div}_x \left\langle v \phi^{i,j} h_\varepsilon \right\rangle - \varepsilon^{-1} \left\langle \phi^{i,j} \mathbf{L}_1 h_\varepsilon \right\rangle \\ = \left\langle \phi^{i,j} \mathcal{Q}_1(h_\varepsilon, h_\varepsilon) \right\rangle + \left\langle \phi^{i,j} \mathbf{S}_\varepsilon \right\rangle. \end{aligned} \quad (6.30)$$

One writes

$$\mathcal{Q}_1(h_\varepsilon, h_\varepsilon) = \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) + \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon)$$

so that (6.30) becomes

$$\begin{aligned} \varepsilon \partial_t \langle \phi^{i,j} h_\varepsilon \rangle + \operatorname{div}_x \langle v \phi^{i,j} h_\varepsilon \rangle - \varepsilon^{-1} \langle \phi^{i,j} \mathbf{L}_1 h_\varepsilon \rangle \\ = \langle \phi^{i,j} \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle + \langle \phi^{i,j} \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) \rangle + \langle \phi^{i,j} \mathbf{S}_\varepsilon \rangle. \end{aligned}$$

According to Lemma 6.5 we have that

$$\begin{aligned} \varepsilon \partial_t \langle \phi^{i,j} h_\varepsilon \rangle &\longrightarrow 0, & \operatorname{div}_x \langle v \phi^{i,j} h_\varepsilon \rangle &\longrightarrow \operatorname{div}_x \langle v \phi^{i,j} \mathbf{h} \rangle, \\ \langle \phi^{i,j} \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) \rangle &\longrightarrow 0, & \langle \phi^{i,j} \mathbf{S}_\varepsilon \rangle &\longrightarrow 0, \end{aligned}$$

where the limits are all meant in the distributional sense and where the last limit is deduced from the strong convergence of \mathbf{S}_ε to 0 in $L^1((0, T); L^1_v L^2_x(\boldsymbol{\varpi}_{q-1}))$ (see Lemma C.6).

From Lemma C.4 in Appendix C, one has

$$\langle v_\ell \phi^{i,j} \mathbf{h} \rangle = \begin{cases} \boldsymbol{\nu} u_j & \text{if } i \neq j, \ell = i, \\ \boldsymbol{\nu} u_i & \text{if } i \neq j, \ell = j, \\ -\frac{2}{d} \boldsymbol{\nu} u_\ell + 2 \boldsymbol{\nu} u_i \delta_{i\ell} & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Therefore, using the incompressibility condition,

$$\operatorname{div}_x \langle v \phi^{i,j} \mathbf{h} \rangle = \boldsymbol{\nu} (\partial_{x_j} u_i + \partial_{x_i} u_j).$$

We deduce that

$$\lim_{\varepsilon \rightarrow 0^+} \left(\varepsilon^{-1} \langle \phi^{i,j} \mathbf{L}_1 h_\varepsilon \rangle + \langle \phi^{i,j} \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle \right) = \boldsymbol{\nu} (\partial_{x_j} u_i + \partial_{x_i} u_j),$$

in the distributional sense. Applying the Div_x operator one deduces that, in $\mathcal{D}'_{t,x}$

$$\lim_{\varepsilon \rightarrow 0^+} \operatorname{Div}_x^i \left(\varepsilon^{-1} \langle \phi \mathbf{L}_1 h_\varepsilon \rangle + \langle \phi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle \right) = \boldsymbol{\nu} \Delta_x u_i,$$

where we use the incompressibility condition to deduce that $\operatorname{Div}_x^i (\partial_{x_j} u_i + \partial_{x_i} u_j) = \Delta_x u_i$. This proves the result. \square

In the same spirit, we have the following which now regards (6.23).

Lemma 6.12. *In the distributional sense,*

$$\lim_{\varepsilon \rightarrow 0^+} \left(\varepsilon^{-1} \operatorname{div}_x \langle \mathbf{b} h_\varepsilon \rangle + \operatorname{div}_x \langle \psi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle \right) = -\frac{d+2}{2} \gamma \Delta_x \theta. \quad (6.31)$$

Proof. We recall that

$$\frac{1}{\varepsilon} \operatorname{div}_x \langle \mathbf{b} h_\varepsilon \rangle = -\frac{1}{\varepsilon} \operatorname{div}_x \langle \mathbf{L}_1(h_\varepsilon) \psi \rangle.$$

Multiply (6.28) with ψ_i (recall that ψ is defined by (C.1)). As previously, it holds that

$$\varepsilon \partial_t \langle \psi_i h_\varepsilon \rangle + \operatorname{div}_x \langle v \psi_i h_\varepsilon \rangle - \varepsilon^{-1} \langle \psi_i \mathbf{L}_1 h_\varepsilon \rangle = \langle \psi_i \mathcal{Q}_1(h_\varepsilon, h_\varepsilon) \rangle + \langle \psi_i \mathbf{S}_\varepsilon \rangle,$$

and

$$\varepsilon \partial_t \langle \psi_i h_\varepsilon \rangle \longrightarrow 0, \quad \langle \psi_i \mathbf{S}_\varepsilon \rangle \longrightarrow 0,$$

in the distributional sense. Splitting again $\mathcal{Q}_1(h_\varepsilon, h_\varepsilon) = \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) + \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon)$, one has $\langle \psi_i \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) \rangle$ converges to 0 in $\mathcal{D}'_{t,x}$ so that, in the distributional sense, it follows that

$$\lim_{\varepsilon \rightarrow 0^+} \left(\varepsilon^{-1} \langle \psi_i \mathbf{L}_1 h_\varepsilon \rangle + \langle \psi_i \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle \right) = \operatorname{div}_x \langle v \psi_i \mathbf{h} \rangle = \frac{d+2}{2} \gamma \partial_{x_i} \theta$$

thanks to Lemma C.5 in Appendix C. This gives the result. \square

6.4. Convergence of the nonlinear terms. To determine the distributional limit of (6.22) and (6.23), we “only” need now to explicit the limit of

$$\mathcal{P}\operatorname{Div}_x \langle \phi \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle \quad \text{and} \quad \operatorname{div}_x \langle \psi \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle$$

respectively. Writing

$$\pi_0 h_\varepsilon = \left(\varrho_\varepsilon(t, x) + \mathbf{u}_\varepsilon(t, x) \cdot v + \frac{1}{2} \theta_\varepsilon(t, x) (|v|^2 - d\vartheta_1) \right) \mathcal{M}(v)$$

we first observe that, according to Lemma C.3 and Lemma C.5 in Appendix C,

$$\langle \phi \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle = \vartheta_1^2 \left[\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \frac{2}{d} |\mathbf{u}_\varepsilon|^2 \mathbf{Id} \right]$$

and

$$\langle \psi \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle = \frac{d+2}{2} \vartheta_1^3 (\theta_\varepsilon \mathbf{u}_\varepsilon).$$

Therefore,

$$\mathcal{P}\operatorname{Div}_x \langle \phi \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle = \vartheta_1^2 \mathcal{P}\operatorname{Div}_x (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)$$

since $\operatorname{Div}_x (|\mathbf{u}_\varepsilon|^2 \mathbf{Id})$ is a gradient term and

$$\operatorname{div}_x \langle \psi \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle = \frac{d+2}{2} \vartheta_1^3 \operatorname{div}_x (\theta_\varepsilon \mathbf{u}_\varepsilon).$$

One has the following whose proof is adapted from (Golse & Saint-Raymond, 2004, Corollary 5.7).

Lemma 6.13. *In the distributional sense (in $\mathcal{D}'_{t,x}$), one has*

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{P}\operatorname{Div}_x \langle \phi \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle = \vartheta_1^2 \mathcal{P}\operatorname{Div}_x (u \otimes u)$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \operatorname{div}_x \langle \psi \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle = \frac{d+2}{2} \vartheta_1^3 u \cdot \nabla_x \theta.$$

In particular

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{P}\text{Div}_x \left\langle \varepsilon^{-1} \mathbf{A} h_\varepsilon \right\rangle = -\nu \Delta_x u + \vartheta_1^2 \mathcal{P}\text{Div}_x (u \otimes u) \quad \text{in } \mathcal{D}'_{t,x} \quad (6.32)$$

while

$$\lim_{\varepsilon \rightarrow 0^+} \text{div}_x \left\langle \varepsilon^{-1} \mathbf{b} h_\varepsilon \right\rangle = -\frac{d+2}{2} (\gamma \Delta_x \theta - \vartheta_1^3 u \cdot \nabla_x \theta) \quad \text{in } \mathcal{D}'_{t,x}. \quad (6.33)$$

Proof. We write $\mathbf{u}_\varepsilon = \mathcal{P}\mathbf{u}_\varepsilon + (\mathbf{Id} - \mathcal{P})\mathbf{u}_\varepsilon$. Due to the strong convergence of $\mathcal{P}\mathbf{u}_\varepsilon$ towards u in $L^1 \left((0, T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d) \right)$ (see Lemma 6.9) and the weak convergence of \mathbf{u}_ε (see Lemma 6.5), we see that

$$\mathcal{P}\text{Div}_x (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - (\mathbf{Id} - \mathcal{P})\mathbf{u}_\varepsilon \otimes (\mathbf{Id} - \mathcal{P})\mathbf{u}_\varepsilon) \longrightarrow \mathcal{P}\text{Div}_x (u \otimes u) \quad \text{in } \mathcal{D}'_{t,x}.$$

So, to prove the first part of the Lemma, we only need to prove that

$$\mathcal{P}\text{Div}_x ((\mathbf{Id} - \mathcal{P})\mathbf{u}_\varepsilon \otimes (\mathbf{Id} - \mathcal{P})\mathbf{u}_\varepsilon) \longrightarrow 0 \quad (6.34)$$

in $\mathcal{D}'_{t,x}$. Moreover, as in (Golse & Saint-Raymond, 2004, Corollary 5.7), we set

$$\beta_\varepsilon := \frac{1}{d\vartheta_1} \left\langle |v|^2 h_\varepsilon \right\rangle = \varrho_\varepsilon + \vartheta_1 \theta_\varepsilon$$

which is such that $\theta_\varepsilon = \frac{2}{(d+2)\vartheta_1} \left(\beta_\varepsilon + \frac{1}{\vartheta_1} \vartheta_\varepsilon \right)$ and

$$\begin{aligned} \text{div}_x (\theta_\varepsilon \mathbf{u}_\varepsilon) &= \frac{2}{(d+2)\vartheta_1} \left(\text{div}_x \left(\beta_\varepsilon \mathbf{u}_\varepsilon + \frac{1}{\vartheta_1} \mathbf{u}_\varepsilon \vartheta_\varepsilon \right) \right) \\ &= \frac{2}{(d+2)\vartheta_1} \text{div}_x (\beta_\varepsilon (\mathbf{Id} - \mathcal{P}) \mathbf{u}_\varepsilon) + \frac{2}{(d+2)\vartheta_1} \left[\text{div}_x \left(\beta_\varepsilon \mathcal{P}\mathbf{u}_\varepsilon + \frac{1}{\vartheta_1} \mathbf{u}_\varepsilon \vartheta_\varepsilon \right) \right]. \end{aligned}$$

Therefore, using the strong convergence of ϑ_ε towards ϑ_0 in $L^1 \left((0, T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d) \right)$ given by Lemma 6.9 together with the weak convergence of \mathbf{u}_ε to u from Lemma 6.5, we get

$$\frac{2}{(d+2)\vartheta_1^2} \text{div}_x (\mathbf{u}_\varepsilon \vartheta_\varepsilon) \longrightarrow \frac{2}{(d+2)\vartheta_1^2} \text{div}_x (u \vartheta_0) \quad \text{in } \mathcal{D}'_{t,x}$$

whereas the strong convergence of $\mathcal{P}\mathbf{u}_\varepsilon$ to u with the weak convergence of β_ε towards $\varrho + \vartheta_1 \theta$ we get

$$\text{div}_x (\beta_\varepsilon \mathcal{P}\mathbf{u}_\varepsilon) \longrightarrow \text{div}_x (u (\varrho + \vartheta_1 \theta)) = 0 \quad \text{in } \mathcal{D}'_{t,x}$$

where we used both the incompressibility condition (6.13) together with Boussinesq relation (6.14).

Notice that, thanks to (6.13), it holds

$$\frac{2}{(d+2)\vartheta_1^2} \text{div}_x (u \vartheta_0) = \frac{2}{(d+2)\vartheta_1^2} u \cdot \nabla_x \vartheta_0 = u \cdot \nabla_x \theta$$

where we used the expression of ϑ_0 together with Boussinesq relation (6.14). This shows that

$$\text{div}_x (\theta_\varepsilon \mathbf{u}_\varepsilon) - \frac{2}{(d+2)\vartheta_1} \text{div}_x (\beta_\varepsilon (\mathbf{Id} - \mathcal{P}) \mathbf{u}_\varepsilon) \longrightarrow u \cdot \nabla_x \theta \quad \text{in } \mathcal{D}'_{t,x}$$

and, to get the second part of the result, we need to prove that

$$\operatorname{div}_x (\beta_\varepsilon (\mathbf{Id} - \mathcal{P}) \mathbf{u}_\varepsilon) \longrightarrow 0 \quad \text{in } \mathcal{D}'_{t,x}. \quad (6.35)$$

Let us now focus on the proof of (6.34) and (6.35). One observes that, Equation (6.20b) reads

$$\varepsilon \partial_t \mathbf{u}_\varepsilon + \nabla_x \beta_\varepsilon = \frac{\kappa_\alpha}{\varepsilon} \mathbf{u}_\varepsilon - \vartheta_1^{-1} \operatorname{Div}_x \langle \mathbf{A} h_\varepsilon \rangle \quad (6.36)$$

whereas (6.20c) can be reformulated as

$$\varepsilon \partial_t \beta_\varepsilon + \operatorname{div}_x \left\langle \frac{1}{d\vartheta_1} |v|^2 v h_\varepsilon \right\rangle = \frac{2}{d\vartheta_1 \varepsilon^2} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2\kappa_\alpha}{\varepsilon} \beta_\varepsilon \quad (6.37)$$

where we check easily that

$$\begin{aligned} \operatorname{div}_x \left\langle \frac{1}{d\vartheta_1} |v|^2 v h_\varepsilon \right\rangle &= \frac{2}{d\vartheta_1} \operatorname{div}_x \langle \mathbf{b} h_\varepsilon \rangle + \frac{d+2}{d} \vartheta_1 \operatorname{div}_x \mathbf{u}_\varepsilon \\ &= \frac{2}{d\vartheta_1} \operatorname{div}_x \langle \mathbf{b} h_\varepsilon \rangle + \frac{d+2}{d} \vartheta_1 \operatorname{div}_x (\mathbf{Id} - \mathcal{P}) \mathbf{u}_\varepsilon. \end{aligned}$$

Recall that from Theorem 5.6, $h_\varepsilon \in L^\infty((0, T); \mathcal{E})$ so that by Minkowski's integral inequality, $\beta_\varepsilon \in L^\infty((0, T); \mathbb{W}_x^{m,2}(\mathbb{T}^d))$ and using (Majda & Bertozzi, 2002, Proposition 1.6, p. 33)), we can write

$$(\mathbf{Id} - \mathcal{P}) \mathbf{u}_\varepsilon = \nabla_x \mathbf{U}_\varepsilon$$

with $\mathbf{U}_\varepsilon \in L^\infty\left((0, T); \left(\mathbb{W}_x^{m-1,2}(\mathbb{T}^d)\right)^d\right)$. After applying $(\mathbf{Id} - \mathcal{P})$ to (6.36) and reformulating (6.37), we obtain that \mathbf{U}_ε and β_ε satisfy

$$\begin{cases} \varepsilon \partial_t \nabla_x \mathbf{U}_\varepsilon + \nabla_x \beta_\varepsilon = \mathbf{F}_\varepsilon \\ \varepsilon \partial_t \beta_\varepsilon + \frac{d+2}{d} \vartheta_1 \Delta_x \mathbf{U}_\varepsilon = \mathbf{G}_\varepsilon \end{cases} \quad (6.38)$$

with

$$\begin{aligned} \mathbf{F}_\varepsilon &:= \frac{\kappa_\alpha}{\varepsilon} \nabla_x \mathbf{U}_\varepsilon - \vartheta_1^{-1} (\mathbf{Id} - \mathcal{P}) \operatorname{Div}_x \langle \mathbf{A} h_\varepsilon \rangle \\ \mathbf{G}_\varepsilon &:= -\frac{2}{d\vartheta_1} \operatorname{div}_x \langle \mathbf{b} h_\varepsilon \rangle + \frac{2}{d\vartheta_1 \varepsilon^2} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2\kappa_\alpha}{\varepsilon} \beta_\varepsilon. \end{aligned}$$

It is easy to see that

$$\|\mathbf{F}_\varepsilon\|_{L^1((0,T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d))} \lesssim \varepsilon, \quad \text{and} \quad \|\mathbf{G}_\varepsilon\|_{L^1((0,T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d))} \lesssim \varepsilon.$$

so that both \mathbf{F}_ε and \mathbf{G}_ε converge *strongly* to 0 in $L^1((0, T); L_x^2(\mathbb{T}^d))$ and

$$\mathbf{U}_\varepsilon \in L^\infty((0, T); (\mathbb{W}_x^{1,2}(\mathbb{T}^d))^d), \quad \beta_\varepsilon \in L^\infty((0, T); L_x^2(\mathbb{T}^d)).$$

Then, according to the *compensated compactness* argument of Lions & Masmoudi (1999) recalled in Proposition C.7 in Appendix C, we deduce that (6.34) and (6.35) hold true and this achieves

the proof. The proofs of (6.32) and (6.33) follow then from an application of Lemmas 6.11 and 6.12. \square

Coming back to the system of equations (6.20) and with the preliminary results of Section 6.2, we get the following where we wrote $\mathcal{P}\text{Div}_x(u \otimes u) = \text{Div}_x(u \otimes u) + \vartheta_1^{-1}\nabla_x p$, see (Majda & Bertozzi, 2002, Proposition 1.6).

Proposition 6.14. *The limit velocity $u(t, x)$ in (6.7) satisfies*

$$\partial_t u - \frac{\nu}{\vartheta_1} \Delta_x u + \vartheta_1 \text{Div}_x(u \otimes u) + \nabla_x p = \lambda_0 u \quad (6.39)$$

while the limit temperature $\theta(t, x)$ in (6.7) satisfies

$$\partial_t \theta - \frac{\gamma}{\vartheta_1^2} \Delta_x \theta + \vartheta_1 u \cdot \nabla_x \theta = \frac{2}{(d+2)\vartheta_1^2} \mathcal{J}_0 + \frac{2d\lambda_0}{d+2} E(t) + \frac{2}{d+2} \frac{d}{dt} E(t),$$

where

$$E(t) = \int_{\mathbb{T}^d} \theta(t, x) dx, \quad t \geq 0.$$

Notice that, due to (6.13), $\text{Div}_x(u \otimes u) = (u \cdot \nabla_x) u$ and (6.39) is nothing but a *damped* Navier-Stokes equation associated to a divergence-free source term given by $\lambda_0 u$.

Proof. The proof of (6.39) is a straightforward consequence of the previous limit. To obtain investigate the evolution of θ , we recall that ϑ_ε satisfies (6.26). We notice that

$$\frac{1}{\varepsilon^3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2\kappa_\alpha}{\varepsilon^2} \left\langle \frac{1}{2} |v|^2 h_\varepsilon \right\rangle \longrightarrow \mathcal{J}_0 + d\vartheta_1 \lambda_0 (\varrho + \vartheta_1 \theta),$$

whereas

$$\vartheta_\varepsilon \longrightarrow \left\langle \frac{1}{2} (|v|^2 - (d+2)\vartheta_1) \mathbf{h} \right\rangle = \frac{d\vartheta_1}{2} (\varrho + \vartheta_1 \theta) - \frac{d+2}{2} \vartheta_1 \varrho,$$

where the convergence is meant in $\mathcal{D}'_{t,x}$. We deduce from (6.33), performing the distributional limit of (6.26), that

$$\begin{aligned} \frac{d\vartheta_1}{2} \partial_t (\varrho + \vartheta_1 \theta) - \frac{d+2}{2} \vartheta_1 \partial_t \varrho - \frac{d+2}{2} \gamma \Delta_x \theta + \frac{d+2}{2} \vartheta_1^3 u \cdot \nabla_x \theta \\ = \mathcal{J}_0 + d\vartheta_1 \lambda_0 (\varrho + \vartheta_1 \theta). \end{aligned} \quad (6.40)$$

Using the strengthened Boussinesq relation (6.15), we see that

$$\partial_t (\varrho + \vartheta_1 \theta) = \vartheta_1 \frac{d}{dt} E(t), \quad \text{and} \quad \partial_t \varrho = -\vartheta_1 \left(\partial_t \theta - \frac{d}{dt} E(t) \right),$$

and get the result. \square

Proposition 6.15. *For any $t \geq 0$, it follows that*

$$E(t) = \int_{\mathbb{T}^d} \theta(t, x) dx = 0,$$

consequently, the limiting temperature $\theta(t, x)$ in (6.7) satisfies

$$\partial_t \theta - \frac{\gamma}{\vartheta_1^2} \Delta_x \theta + \vartheta_1 u \cdot \nabla_x \theta = \frac{\lambda_0 \bar{c}}{2(d+2)} \sqrt{\vartheta_1} \theta. \quad (6.41)$$

Moreover, the strong Boussinesq relation

$$\varrho(t, x) + \vartheta_1 \theta(t, x) = 0, \quad x \in \mathbb{T}^d, \quad (6.42)$$

holds true.

Proof. To capture the evolution of the temperature $E(t)$, we average equation (6.40) over \mathbb{T}^d and using the incompressibility condition (6.13) we deduce get that

$$\frac{d}{dt} E(t) = \frac{2}{d\vartheta_1^2} \int_{\mathbb{T}^d} \mathcal{J}_0(t, x) dx + 2\lambda_0 E(t). \quad (6.43)$$

And, from Lemma 6.8, it holds that

$$\frac{d}{dt} E(t) = \bar{c}_0 E(t), \quad \bar{c}_0 := 2\lambda_0 - \frac{3}{2d} \lambda_0 \bar{c} \sqrt{\vartheta_1},$$

so that,

$$E(t) = E(0) \exp(\bar{c}_0 t), \quad t \geq 0.$$

Now, return to the original equation (1.22) and recall that the solution $f_\varepsilon(t, x, v)$ is given by

$$f_\varepsilon(t) = G_\alpha + \varepsilon h_\varepsilon(t) = \mathcal{M} + (G_\alpha - \mathcal{M}) + \varepsilon h_\varepsilon(t), \quad t \geq 0,$$

where \mathcal{M} has the same global mass, momentum and energy as the initial datum $f_\varepsilon(0) = F_{\text{in}}^\varepsilon$, independent of $\varepsilon > 0$. For any test-function $\phi = \phi(x, v)$ we get that

$$\varepsilon^{-1} \int_{\mathbb{T}^d} \langle (f_\varepsilon(t) - \mathcal{M}) \phi \rangle dx - \varepsilon^{-1} \int_{\mathbb{T}^d} \langle (G_\alpha - \mathcal{M}) \phi \rangle dx = \int_{\mathbb{T}^d} \langle h_\varepsilon(t) \phi \rangle dx, \quad t \geq 0.$$

Using this equality for $\phi(x, v) = \frac{1}{2}|v|^2$ and $t = 0$ one is led to

$$-\varepsilon^{-1} \int_{\mathbb{T}^d} \langle \frac{1}{2}(G_\alpha - \mathcal{M}) |v|^2 \rangle dx = \int_{\mathbb{T}^d} \langle \frac{1}{2} h_\varepsilon(0) |v|^2 \rangle dx.$$

We recall that $\|G_\alpha - \mathcal{M}\|_{L_v^1(\varpi_2)} \leq C(1 - \alpha) \lesssim \varepsilon^2$, consequently

$$E(0) = \int_{\mathbb{T}^d} \theta(0, x) dx = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{T}^d} \langle h_\varepsilon(0) \frac{1}{2} |v|^2 \rangle dx = 0.$$

Therefore, $E(t) \equiv 0$ for all $t > 0$. This observation and (6.41) lead us to the equation for the energy and Boussinesq relation (6.42). \square

6.5. About the initial conditions. Before going into the proof of Theorem 1.5 and handle the problem of initial datum of our limit system, we begin by proving that our limits u and θ in (6.7) are actually continuous on $(0, T)$.

Lemma 6.16. *Consider the sequences $\{u_\varepsilon\}_\varepsilon$ and $\{\vartheta_\varepsilon\}_\varepsilon$ defined in Lemma 6.9. The time-dependent mappings*

$$t \in [0, T] \longmapsto \|\vartheta_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)} \quad \text{and} \quad t \in [0, T] \longmapsto \|u_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)}$$

are Hölder continuous uniformly in ε . As a consequence, the limiting mass ϱ , velocity u and temperature θ in (6.7) are continuous on $(0, T)$.

Proof. Recall that we set

$$\vartheta_\varepsilon(t, x) = \left\langle \frac{1}{2}(|v|^2 - (d+2)\vartheta_1)h_\varepsilon \right\rangle.$$

For any test-function $\varphi = \varphi(x) \in C_c^\infty(\mathbb{T}^d)$ and multi-index β with $|\beta| \leq m-2$, multiplying (6.26) with $\partial_x^\beta \varphi$ and integrating in time and space, one deduces that for any $0 \leq t_1 \leq t_2$,

$$\begin{aligned} \int_{\mathbb{T}^d} \left[\partial_x^\beta \vartheta_\varepsilon(t_2, x) - \partial_x^\beta \vartheta_\varepsilon(t_1, x) \right] \varphi(x) dx &= \int_{t_1}^{t_2} dt \int_{\mathbb{T}^d} \operatorname{div}_x \left\langle \varepsilon^{-1} \mathbf{b} \partial_x^\beta h_\varepsilon(t) \right\rangle \varphi(x) dx \\ &+ \int_{t_1}^{t_2} dt \int_{\mathbb{T}^d} \varepsilon^{-3} \partial_x^\beta \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) \varphi(x) dx \\ &+ \frac{2\kappa_\alpha}{\varepsilon^2} \int_{t_1}^{t_2} dt \int_{\mathbb{T}^d} \left\langle \frac{1}{2} |v|^2 \partial_x^\beta h_\varepsilon \right\rangle \varphi(x) dx. \end{aligned} \quad (6.44)$$

Notice that

$$\left| \int_{\mathbb{T}^d} \left\langle \frac{1}{2} |v|^2 \partial_x^\beta h_\varepsilon \right\rangle \varphi(x) dx \right| \leq \|\varphi\|_{L_x^2} \left\| \left\langle \frac{1}{2} |v|^2 \partial_x^\beta h_\varepsilon \right\rangle \right\|_{L_x^2} \leq \|\varphi\|_{L_x^2} \left\| \frac{1}{2} |v|^2 \partial_x^\beta h_\varepsilon \right\|_{L_v^1 L_x^2}$$

thanks to Minkowski's integral inequality. Clearly, since $\varepsilon^{-2} \kappa_\alpha \rightarrow \lambda_0$, there is $C > 0$ such that

$$\begin{aligned} \frac{2\kappa_\alpha}{\varepsilon^2} \left| \int_{t_1}^{t_2} dt \int_{\mathbb{T}^d} \left\langle \frac{1}{2} |v|^2 \partial_x^\beta h_\varepsilon \right\rangle \varphi(x) dx \right| &\leq C \|\varphi\|_{L^2} \int_{t_1}^{t_2} \|h_\varepsilon(t)\|_{L_v^1 \mathbb{W}_x^{m-1,2}(\mathfrak{a}_2)} dt \\ &\leq C \sqrt{\mathcal{K}_0} (t_2 - t_1) \end{aligned}$$

from the general estimate in Theorem 5.6. In the same way, since

$$\partial_x^\beta \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) = \mathcal{J}_\alpha(\partial_x^\beta f_\varepsilon, f_\varepsilon) + \mathcal{J}_\alpha(f_\varepsilon, \partial_x^\beta f_\varepsilon),$$

with $f_\varepsilon = G_\alpha + \varepsilon h_\varepsilon$, one deduces again from Theorem 5.6 that

$$\begin{aligned} \left| \int_{t_1}^{t_2} dt \int_{\mathbb{T}^d} \varepsilon^{-3} \partial_x^\beta \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) \varphi(x) dx \right| \\ \leq C \|\varphi\|_{L_x^2} \int_{t_1}^{t_2} \|h_\varepsilon(t)\|_{L_v^1 \mathbb{W}_x^{m-1,2}(\mathfrak{a}_3)} \left(1 + \|h_\varepsilon(t)\|_{L_v^1 \mathbb{W}_x^{m-1,2}(\mathfrak{a}_3)} \right) dt \leq C \sqrt{\mathcal{K}_0} (t_2 - t_1). \end{aligned}$$

Moreover, noticing that $\langle \mathbf{b}h_\varepsilon(t) \rangle = \langle \mathbf{b}(\mathbf{Id} - \pi_0)h_\varepsilon(t) \rangle$ for any $t \geq 0$, one deduces easily from Proposition 6.1 that

$$\left| \int_{t_1}^{t_2} \varepsilon^{-1} \operatorname{div}_x \langle \mathbf{b}h_\varepsilon(t) \rangle dt \right| \leq C \sqrt{t_2 - t_1}$$

for any $0 \leq t_2 - t_1 \leq 1$. Since ∂_x^β commutes with π_0 we deduce easily that there is $C > 0$ independent of ε such that for any $0 \leq \beta \leq m - 2$,

$$\left| \int_{t_1}^{t_2} \varepsilon^{-1} \operatorname{div}_x \langle \mathbf{b} \partial_x^\beta h_\varepsilon(t) \rangle dt \right| \leq C \sqrt{t_2 - t_1} \quad (6.45)$$

for any $0 \leq t_2 - t_1 \leq 1$. We conclude with (6.44) that

$$\left| \int_{\mathbb{T}^d} \left[\partial_x^\beta \vartheta_\varepsilon(t_2, x) - \partial_x^\beta \vartheta_\varepsilon(t_1, x) \right] \varphi(x) dx \right| \leq C \|\varphi\|_{L_x^2} \sqrt{\mathcal{K}_0} \sqrt{t_2 - t_1}$$

for some positive constant independent of ε and $0 \leq t_2 - t_1 \leq 1$. Since $\mathcal{C}_c^\infty(\mathbb{T}^d)$ is dense in $L^2(\mathbb{T}^d)$, the previous estimate is true for any $\varphi \in L^2(\mathbb{T}^d)$ and, taking the supremum over all $\varphi \in L^2(\mathbb{T}^d)$, we deduce that

$$\left\| \partial_x^\beta \vartheta_\varepsilon(t_2) - \partial_x^\beta \vartheta_\varepsilon(t_1) \right\|_{L_x^2} \leq C \sqrt{\mathcal{K}_0} \sqrt{t_2 - t_1} \quad (6.46)$$

and, the time-dependent mappings $t \in [0, T] \mapsto \|\vartheta_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)}$ are thus Hölder continuous uniformly in ε . Recall also that $\vartheta_\varepsilon(t)$ converges in $L^1((0, T); \mathbb{W}_x^{m-2,2}(\mathbb{T}^d))$ towards $\vartheta_0(t) = \frac{d+1}{2}(\varrho + \vartheta_1\theta) - \frac{d+2}{2}\vartheta_1\varrho$ from Lemma 6.9. As a consequence, there exists a subsequence $(\vartheta_{\varepsilon'})_{\varepsilon'}$ such that $\|\vartheta_{\varepsilon'}(t) - \vartheta_0(t)\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)}$ converges towards 0 for almost every $t \in [0, T]$. Using then the uniform in ε Hölder continuity obtained above, we can deduce that ϑ_0 is Hölder continuous on $(0, T)$. Recalling that $E(0) = 0$ according to Proposition 6.15, the strong Boussinesq relation (6.42) holds true and

$$\vartheta_0(t) = \frac{d+2}{2} \vartheta_1^2 \theta(t) = -\frac{d+2}{2} \vartheta_1 \varrho(t),$$

which gives the regularity of both ϱ and θ .

We recall that, setting

$$u_\varepsilon(t, x) = \frac{1}{\vartheta_1} \exp\left(-t \frac{\kappa_\alpha}{\varepsilon^2}\right) \mathcal{P} \langle v h_\varepsilon \rangle,$$

we have that

$$\partial_t u_\varepsilon + \exp\left(-t \frac{\kappa_\alpha}{\varepsilon^2}\right) \mathcal{P} \left(\operatorname{Div}_x \left\langle \frac{1}{\vartheta_1 \varepsilon} \mathbf{A} h_\varepsilon \right\rangle \right) = 0.$$

we multiply this identity by $\partial_x^\beta \varphi$ and integrate in both time and space to get

$$\begin{aligned} & \int_{\mathbb{T}^d} \left[\partial_x^\beta u_\varepsilon(t_2, x) - \partial_x^\beta u_\varepsilon(t_1, x) \right] \varphi(x) dx \\ &= \int_{t_1}^{t_2} \exp\left(-\frac{\kappa_\alpha}{\varepsilon^2} t\right) dt \int_{\mathbb{T}^d} \mathcal{P} \left(\operatorname{Div}_x \left\langle \frac{1}{\partial_1 \varepsilon} \mathbf{A} \partial_x^\beta h_\varepsilon \right\rangle \right) \varphi(x) dx. \end{aligned}$$

Arguing as in the proof of (6.45), we see that there is $C > 0$ independent of ε such that for any $0 \leq |\beta| \leq m - 2$,

$$\left| \int_{t_1}^{t_2} \varepsilon^{-1} \operatorname{Div}_x \left\langle \mathbf{A} \partial_x^\beta h_\varepsilon(t) \right\rangle dt \right| \leq C \sqrt{t_2 - t_1}$$

for any $0 \leq t_2 - t_1 \leq 1$. This gives easily

$$\left| \int_{\mathbb{T}^d} \left[\partial_x^\beta u_\varepsilon(t_2, x) - \partial_x^\beta u_\varepsilon(t_1, x) \right] \varphi(x) dx \right| \leq C \|\varphi\|_{L_x^2} \sqrt{t_2 - t_1} \sqrt{\mathcal{K}_0},$$

from which, as before, the time-dependent mapping $t \in [0, T] \mapsto \|u_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)}$ is Hölder continuous uniformly in ε . We deduce the result of regularity on u as previously done for ϱ and θ noticing that the limit of u_ε is $\exp(-t\lambda_0)\mathcal{P}u = \exp(-t\lambda_0)u$. \square

Recall that, in Theorem 6.3, the convergence of h_ε to the solution $\mathbf{h}(t, x)$ given by (6.7) is known to hold only for a subsequence and, in particular, different subsequences could converge towards different initial datum and therefore (ϱ, u, θ) could be different solutions to the Navier-Stokes system. We aim here to *prescribe* the initial datum by ensuring the convergence of the initial datum $h_{\text{in}}^\varepsilon$ towards a *single* possible limit.

Recall that the initial datum for (6.28) is denoted by $h_{\text{in}}^\varepsilon$. We write $h_{\text{in}}^\varepsilon = \pi_0 h_{\text{in}}^\varepsilon + (\mathbf{Id} - \pi_0) h_{\text{in}}^\varepsilon$ and introduce the following assumption.

Assumption 6.17. *Assume that there exists*

$$(\varrho_0, u_0, \theta_0) \in \mathbb{W}_x^{m,2}(\mathbb{T}^d) \times \left(\mathbb{W}_x^{m,2}(\mathbb{T}^d) \right)^d \times \mathbb{W}_x^{m,2}(\mathbb{T}^d),$$

such that

$$\lim_{\varepsilon \rightarrow 0} \|\pi_0 h_{\text{in}}^\varepsilon - h_0\|_{L_v^1 \mathbb{W}_x^{m,2}(\varpi_q)} = 0,$$

where

$$h_0(x, v) = \left(\varrho_0(x) + u_0(x) \cdot v + \frac{1}{2} \theta_0(x) (|v|^2 - d\vartheta_1) \right) \mathcal{M}(v).$$

Under this assumption we can prescribe the initial value of the solution (ϱ, u, θ) and strengthen the convergence.

Proposition 6.18. *We define the initial data for (ϱ, u, θ) as*

$$u_{\text{in}} = u(0) := \mathcal{P}u_0, \quad \theta_{\text{in}} = \theta(0) = \frac{d}{d+2}\theta_0 - \frac{2}{(d+2)\vartheta_1}\varrho_0, \\ \varrho_{\text{in}} = \varrho(0) := -\vartheta_1\theta_{\text{in}}, \quad (6.47)$$

where we recall that $\mathcal{P}u_0$ is the Leray projection on divergence-free vector fields. Then, as a consequence, for any $T > 0$, one has that

$$\vartheta_\varepsilon(t) = \left\langle \frac{1}{2}(|v|^2 - (d+2)\vartheta_1)h_\varepsilon \right\rangle \longrightarrow \frac{d+2}{2}\vartheta_1^2\theta, \quad \text{in } \mathcal{C}\left([0, T], \mathbb{W}_x^{m-2,2}(\mathbb{T}^d)\right),$$

and

$$\frac{1}{\vartheta_1}\mathcal{P}\left\langle v h_\varepsilon(t) \right\rangle \longrightarrow u, \quad \text{in } \mathcal{C}\left([0, T], \left(\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)\right)^d\right).$$

Proof. According to Lemma 6.16, we already have that the family of time-dependent mappings

$$\left\{ t \in [0, T] \longmapsto \|\vartheta_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)} \right\}_\varepsilon \quad (6.48)$$

is equicontinuous. At time $t = 0$ according to Assumption 6.17,

$$\vartheta_\varepsilon(0, x) = \left\langle \frac{1}{2}(|v|^2 - (d+2)\vartheta_1)h_{\text{in}}^\varepsilon \right\rangle \longrightarrow \vartheta_1 \left[\frac{d\vartheta_1}{2}\theta_0(x) - \varrho_0(x) \right],$$

and, by definition of $\varrho(0, x), \theta(0, x)$, we get that

$$\lim_{\varepsilon \rightarrow 0^+} \|\vartheta_\varepsilon(0, \cdot) - \vartheta_0(0, \cdot)\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)} = 0.$$

In particular, the family $\{\|\vartheta_\varepsilon(0)\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)}\}_\varepsilon$ is bounded and, since the family (6.48) is uniformly in ε Hölder continuous, for any $t \in [0, T]$, the family $\{\|\vartheta_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,2}(\mathbb{T}^d)}\}_\varepsilon$ is also bounded. Since it is also equicontinuous, Arzelà-Ascoli Theorem implies that the convergence holds in $\mathcal{C}([0, T]; \mathbb{W}_x^{m-2,2}(\mathbb{T}^d))$ and

$$\vartheta_0 \in \mathcal{C}([0, T]; \mathbb{W}_x^{m-2,2}(\mathbb{T}^d)).$$

As in the proof of Lemma 6.16, it implies the continuity on $[0, T]$ of both ϱ and θ .

We proceed in a similar way for the regularity of u . □

All the previous convergence results lead us to the fact the the limit

$$\mathbf{h}(t, x, v) = \left(\varrho(t, x) + u(t, x) \cdot v + \frac{1}{2}\theta(t, x)(|v|^2 - d\vartheta_1) \right) \mathcal{M}(v)$$

is such that

$$(\varrho, u, \theta) \in \mathcal{C}([0, T]; \mathscr{W}_{m-2}) \cap L^2((0, T); \mathscr{W}_m),$$

solve the following *incompressible Navier-Stokes-Fourier system* where the right-hand-side acts as a self-consistent forcing term

$$\begin{cases} \partial_t u - \frac{\nu}{\vartheta_1} \Delta_x u + \vartheta_1 u \cdot \nabla_x u + \nabla_x p = \lambda_0 u, \\ \partial_t \theta - \frac{\gamma}{\vartheta_1^2} \Delta_x \theta + \vartheta_1 u \cdot \nabla_x \theta = \frac{\lambda_0 \bar{c}}{2(d+2)} \sqrt{\vartheta_1} \theta, \\ \operatorname{div}_x u = 0, \quad \varrho + \vartheta_1 \theta = 0, \end{cases} \quad (6.49)$$

subject to the initial datum $(\varrho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}})$. This proves Theorem 1.5 in full.

6.6. About the original problem in the physical variables. The above considerations allow us to get a quite precise description of the asymptotic behaviour for the original physical problem (1.10a). Indeed, recalling the relations (1.12) together with Theorem 1.5 one has

$$\begin{aligned} F_\varepsilon(t, x, v) &= V_\varepsilon(t)^d f_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v) \\ &= V_\varepsilon(t)^d \left(G_{\alpha(\varepsilon)}(V_\varepsilon(t)v) + \varepsilon h_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v) \right) \\ &= V_\varepsilon(t)^d \left(G_{\alpha(\varepsilon)}(V_\varepsilon(t)v) + \varepsilon \mathbf{h}(\tau_\varepsilon(t), x, V_\varepsilon(t)v) \right) + \varepsilon e_\varepsilon(t, x, v), \end{aligned}$$

where the error term e_ε is given by

$$e_\varepsilon(t, x, v) = V_\varepsilon(t)^d \left(h_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v) - \mathbf{h}(\tau_\varepsilon(t), x, V_\varepsilon(t)v) \right).$$

Under Assumption 1.2, a relevant phenomenon occurs when considering the purely dissipative case $\lambda_0 > 0$. In such a case, the term $e_\varepsilon(t, x, v)$ becomes a *uniform in time* error term. The reason is that, when $\lambda_0 > 0$, the scaling $V_\varepsilon(t)$ increases up to infinity. More precisely,

$$V_\varepsilon(t) \approx (1 + \lambda_0 t), \quad \varepsilon \ll 1.$$

Indeed, Lemma B.1 guarantees that for any $a \in (0, 1/2)$, up to an extraction of a subsequence if necessary,

$$|\langle e_\varepsilon(t), |v|^\kappa \varphi \rangle| \leq C_\varphi \sqrt{\mathcal{K}_0} V_\varepsilon(t)^{-\kappa-a}, \quad \varphi \in \mathcal{C}_{v,b}^1 L_x^\infty, \quad 0 \leq \kappa \leq q-1, \quad (6.50)$$

where we denoted by $\mathcal{C}_{v,b}^1$ the set of \mathcal{C}^1 functions in v that are bounded as well as their first order derivatives. Consequently,

$$\begin{aligned} F_\varepsilon(t, x, v) &= V_\varepsilon(t)^d \left(G_{\alpha(\varepsilon)}(V_\varepsilon(t)v) + \varepsilon \left(\varrho(\tau_\varepsilon(t), x) + u(\tau_\varepsilon(t), x) \cdot (V_\varepsilon(t)v) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \theta(\tau_\varepsilon(t), x) (|V_\varepsilon(t)v|^2 - d\vartheta_1) \right) \mathcal{M}(V_\varepsilon(t)v) \right) + \mathcal{O}\left(V_\varepsilon(t)^{-\kappa-a} \right), \end{aligned} \quad (6.51)$$

in the *weak sense* described in (6.50). In particular, if $\varphi = 1$ and $\kappa = 2$, one finds from (6.51) an explicit expression for *Haff's law* is obtained. That is, the optimal cooling rate of the temperature

is described by

$$\begin{aligned}
 \mathbf{T}_\varepsilon(t) &= \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d \times \mathbb{R}^d} F_\varepsilon(t, x, v) |v|^2 dv dx \\
 &= \frac{1}{V_\varepsilon(t)^2} \left(\int_{\mathbb{R}^d} G_\alpha(v) |v|^2 dv + \frac{\varepsilon}{2|\mathbb{T}^d|} \int_{\mathbb{R}^d} (|v|^2 - d\vartheta_1) |v|^2 \mathcal{M}(v) dv \int_{\mathbb{T}^d} \theta(\tau_\varepsilon(t), x) dx \right. \\
 &\quad \left. + \frac{\varepsilon}{|\mathbb{T}^d|} \int_{\mathbb{R}^d} |v|^2 \mathcal{M}(v) dv \int_{\mathbb{T}^d} \varrho(\tau_\varepsilon(t), x) dx \right) + \mathcal{O}(V_\varepsilon(t)^{-2-a}) \\
 &\approx \frac{d\vartheta_1}{V_\varepsilon(t)^2} \left(1 + \frac{\varepsilon}{|\mathbb{T}^d|} \left(d\vartheta_1 \int_{\mathbb{T}^d} \theta(\tau_\varepsilon(t), x) dx + 2 \int_{\mathbb{T}^d} \varrho(\tau_\varepsilon(t), x) dx \right) \right), \quad t \gg \frac{1}{\lambda_0}.
 \end{aligned}$$

Recalling that the fluctuation h_ε is such that the average mass and temperature both vanish at all times, we deduce the precised Haff's law

$$\mathbf{T}_\varepsilon(t) \approx \frac{d\vartheta_1}{V_\varepsilon(t)^2}, \quad t \gg \frac{1}{\lambda_0}.$$

In the Appendix B we complement this discussion and, in particular, show that the Haff's law holds uniformly *locally* in space due to the boundedness of the solutions that we treat here. This is not expected in a general context.

APPENDIX A. ABOUT GRANULAR GASES IN THE SPATIAL HOMOGENEOUS SETTING

A.1. The collision operator. We collect several results about the Boltzmann collision operator \mathcal{Q}_α for granular gases. We shall consider a collision operator with more general collision kernel than the hard-spheres case considered in the paper, more precisely, a collision kernel $B(u, \sigma)$ of the form

$$B(u, \sigma) = \Phi(|u|) b(\hat{u} \cdot \sigma). \tag{A.1}$$

The kinetic potential $\Phi(\cdot)$ is a suitable nonnegative function in \mathbb{R}^d and the *angular kernel* $b(\cdot)$ is assumed in $L^1(-1, 1)$. The associated collision operator $\mathcal{Q}_{B,\alpha}$ is defined through the weak formulation

$$\int_{\mathbb{R}^d} \mathcal{Q}_{B,\alpha}(f, f)(v) \psi(v) dv = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(v) f(v_*) \mathcal{A}_{B,\alpha}[\psi](v, v_*) dv_* dv \tag{A.2}$$

for any test function $\psi = \psi(v)$ where

$$\mathcal{A}_{B,\alpha}[\psi](v, v_*) = \int_{\mathbb{S}^{d-1}} \left(\psi(v') + \psi(v'_*) - \psi(v) - \psi(v_*) \right) B(u, \sigma) d\sigma$$

where the post-collisional velocities (v', v'_*) are given by

$$\begin{aligned}
 v' &= v + \frac{1+\alpha}{4} (|u|\sigma - u), & v'_* &= v_* - \frac{1+\alpha}{4} (|u|\sigma - u), \\
 \text{where } u &= v - v_*, & \hat{u} &= \frac{u}{|u|}.
 \end{aligned} \tag{A.3}$$

This allows to split $\mathcal{Q}_{B,\alpha}$ into positive and negative parts

$$\mathcal{Q}_{B,\alpha} = \mathcal{Q}_{B,\alpha}^+ - \mathcal{Q}_{B,\alpha}^-$$

where \mathcal{Q}_{α}^+ and \mathcal{Q}_{α}^- are given, in strong form, by

$$\mathcal{Q}_{B,\alpha}^+(g, f)(v) = \frac{1}{\alpha^2} \int_{\mathbb{R}^d \times \mathbb{S}^d} \frac{|'u|}{|u|} B(u, \sigma \cdot \hat{u}) f('v) g('v_*) d\sigma dv_* \quad (\text{A.4})$$

and

$$\mathcal{Q}_{B,\alpha}^-(g, f)(v) = f(v) \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} g(v_*) B(u, \sigma \cdot \hat{u}) dv_* d\sigma.$$

where, for $\sigma \in \mathbb{S}^{d-1}$, $'v$ and $'v_*$ denote the pre-collisional velocities

$$\begin{cases} 'v &= \frac{v + v_*}{2} - \frac{1 - \alpha}{4\alpha} u + \frac{1 + \alpha}{4\alpha} |u| \sigma \\ 'v_* &= \frac{v + v_*}{2} + \frac{1 - \alpha}{4\alpha} u - \frac{1 + \alpha}{4\alpha} |u| \sigma, \quad u = v - v_* \end{cases}$$

A particularly relevant model is the one of hard-spheres corresponding to $\Phi(|u|) = |u|$ which is the model investigated in the core of the paper and, in that case, we simply denote the collision operator $\mathcal{Q}_{B,\alpha}$ by \mathcal{Q}_{α} .

A.2. Alternative representation of the velocities. As well-known, the above collision operator is a well-accepted model that describes collisions in a system composed by a large number of granular particles which are assumed to be hard-spheres with equal mass (that we take to be $m = 1$) and that undertake inelastic collisions. The collision mechanism and the role of the *coefficient of normal restitution* is easier to understand in an alternative representation of the post-collisional velocities. More precisely, if v and v_* denote the velocities of two particles before collision, their respective velocities v' and v'_* after collision are such that

$$(u' \cdot n) = -\alpha(u \cdot n). \quad (\text{A.5})$$

The unitary vector $n \in \mathbb{S}^{d-1}$ determines the impact direction, that is, n stands for the unit vector that points from the v -particle center to the v_* -particle center at the moment of impact. Here above

$$u = v - v_*, \quad u' = v' - v'_*, \quad (\text{A.6})$$

denote respectively the relative velocity before and after collision. The velocities after collision v' and v'_* are given, in virtue of (A.5) and the conservation of momentum, by

$$v' = v - \frac{1 + \alpha}{2} (u \cdot n) n, \quad v'_* = v_* + \frac{1 + \alpha}{2} (u \cdot n) n. \quad (\text{A.7})$$

In particular, the energy relation and the collision mechanism can be written as

$$|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2 - \frac{1 - \alpha^2}{2} ('u \cdot n)^2, \quad u \cdot n = -\alpha('u \cdot n). \quad (\text{A.8})$$

Pre-collisional velocities $({}'v, {}'v_*)$ (resulting in (v, v_*) after collision) can be therefore introduced through the relation

$$v = {}'v - \frac{1+\alpha}{2}({}'u \cdot n)n, \quad v_* = {}'v_* + \frac{1+\alpha}{2}({}'u \cdot n)n, \quad {}'u = {}'v - {}'v_* \quad (\text{A.9})$$

This representation is of course equivalent to the one given in (A.3) (so-called σ -representation) by setting, for a given pair of velocities (v, v_*) ,

$$\sigma = \widehat{u} - 2(\widehat{u} \cdot n)n \in \mathbb{S}^{d-1}.$$

Such a description provides an alternative parametrization of the unit sphere \mathbb{S}^{d-1} in which the unit vector σ points in the post-collisional relative velocity direction in the case of elastic collisions. In this case, the impact velocity reads

$$|u \cdot n| = |u| |\widehat{u} \cdot n| = |u| \sqrt{\frac{1 - \widehat{u} \cdot \sigma}{2}}.$$

In the n -representation, we can also explicit the strong form of the collision operator \mathcal{Q}_α . Namely, for a given pair of distributions $f = f(v)$ and $g = g(v)$ and a given *collision kernel* the Boltzmann collision operator is defined as the difference of two nonnegative operators (gain and loss operators respectively)

$$\mathcal{Q}_\alpha(g, f) = \mathcal{Q}_\alpha^+(g, f) - \mathcal{Q}_\alpha^-(g, f),$$

with

$$\begin{aligned} \mathcal{Q}_\alpha^+(g, f)(v) &= \frac{1}{\alpha^2} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u \cdot n| b_0(\widehat{u} \cdot n) f({}'v) g({}'v_*) dndv_*, \\ \mathcal{Q}_\alpha^-(g, f)(v) &= f(v) \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u \cdot n| b_0(\widehat{u} \cdot n) g(v_*) dndv_*. \end{aligned} \quad (\text{A.10})$$

where the new angular collision kernel $b_0(\cdot)$ is related to the original one $b(\cdot)$ through the relation

$$b_0(\widehat{u} \cdot n) = 2^{d-1} |\widehat{u} \cdot n|^{d-2} b(\widehat{u} \cdot \sigma).$$

i.e.

$$b_0(x) = 2^{d-1} |x|^{d-2} b(1 - 2x^2), \quad x \in [-1, 1].$$

A.3. Estimates on the collision operator. Using the above representation, we prove Lemma 2.1 in Section 2.1.

Proof of Lemma 2.1. Note that

$$\mathcal{Q}_1(g, f) - \mathcal{Q}_\alpha(g, f) = \mathcal{I}_1(g, f) + \mathcal{I}_2(g, f), \quad (\text{A.11})$$

where

$$\mathcal{I}_1(g, f) = -\frac{1-\alpha^2}{\alpha^2} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} g({}'v_{*,\alpha}) f({}'v_\alpha) |u \cdot n| b_0(\widehat{u} \cdot n) dndv_*,$$

and

$$\mathcal{I}_2(g, f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (g(v_{*,\alpha})f(v_\alpha) - g(v_{*,1})f(v_1)) |u \cdot n| b_0(\hat{u} \cdot n) dndv_*.$$

Here we adopt the notation v_1 and v_α for the pre-collisional velocities associate to elastic ($\alpha = 1$) and inelastic ($0 < \alpha < 1$) interactions respectively in the n -representation (see (A.9)).

We begin with proving the estimates in the L_v^1 setting. By classical means, there is a positive $c_q > 0$ such that

$$\|\mathcal{I}_1(g, f)\|_{L_v^1(\varpi_q)} \leq \frac{1-\alpha}{\alpha^2} c_q \|b_0\|_{L^1(\mathbb{S}^{d-1})} \|g\|_{L_v^1(\varpi_{q+1})} \|f\|_{L_v^1(\varpi_{q+1})}.$$

We estimate then the difference $g(v_{*,\alpha})f(v_\alpha) - g(v_{*,1})f(v_1)$ by writing

$$g(v_{*,\alpha})f(v_\alpha) - g(v_{*,1})f(v_1) = g(v_{*,1})(f(v_\alpha) - f(v_1)) + (g(v_{*,\alpha}) - g(v_{*,1}))f(v_\alpha),$$

and splitting $\mathcal{I}_2(g, f)$ accordingly into $\mathcal{I}_2(g, f) = \mathcal{I}_2^1(g, f) + \mathcal{I}_2^2(g, f)$.

For the term $\mathcal{I}_2^1(g, f)$, we first notice that, according to (A.9),

$$v_\alpha = v - \frac{1+\alpha}{2\alpha}(u \cdot n)n, \quad v_{*,\alpha} = v - u + \frac{1+\alpha}{2\alpha}(u \cdot n)n.$$

Therefore, using the change of variable $w = \frac{1+\alpha}{2\alpha}u$ one has $v_\alpha \rightarrow v_1$ and

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} g(v_{*,1}) f(v_\alpha) |u \cdot n| b_0(\hat{u} \cdot n) dndv_* \\ = \left(\frac{2\alpha}{1+\alpha} \right)^{d+1} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} g(\tilde{v}_*) f(v_1) |u \cdot n| b_0(\hat{u} \cdot n) dndv_*, \end{aligned}$$

where

$$\tilde{v}_* = v - \frac{2\alpha}{1+\alpha}u + (u \cdot n)n = v_{*,1} + \frac{1-\alpha}{1+\alpha}u.$$

Consequently,

$$\begin{aligned} \mathcal{I}_2^1(g, f) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left(\left(\frac{2\alpha}{1+\alpha} \right)^{d+1} g(\tilde{v}_*) - g(v_{*,1}) \right) f(v_1) |u \cdot n| b_0(\hat{u} \cdot n) dndv_* \\ &= \left(\left(\frac{2\alpha}{1+\alpha} \right)^{d+1} - 1 \right) \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} g(\tilde{v}_*) f(v_1) |u \cdot n| b_0(\hat{u} \cdot n) dndv_* \\ &\quad + \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (g(\tilde{v}_*) - g(v_{*,1})) f(v_1) |u \cdot n| b_0(\hat{u} \cdot n) dndv_* =: \mathcal{I}_2^{1,1}(g, f) + \mathcal{I}_2^{1,2}(g, f). \end{aligned}$$

For the first term, thanks to the mean-value theorem, one notices that

$$\left| \left(\frac{2\alpha}{1+\alpha} \right)^{d+1} - 1 \right| \leq d \frac{1-\alpha}{1+\alpha}, \quad \forall \alpha \in (0, 1)$$

so that using the pre-post collisional change of variable $(v_*, v) \rightarrow (\tilde{v}_*, 'v_1)$ (with Jacobian computed as $\frac{1+\alpha}{2} \left(\frac{1+\alpha}{2\alpha}\right)^{d-1}$) one concludes that

$$\|\mathcal{I}_2^{1,1}(g, f)\|_{L_v^1(\varpi_q)} \leq (1-\alpha) \left(\frac{1+\alpha}{\alpha}\right)^{d-1} c_q \|b_0\|_{L^1(\mathbb{S}^{d-1})} \|g\|_{L_v^1(\varpi_{q+1})} \|f\|_{L_v^1(\varpi_{q+1})}$$

for some positive c_q depending only on q and the dimension d . For the second term one uses Taylor formula:

$$g(\tilde{v}_*) - g('v_{*,1}) = (\tilde{v}_* - 'v_{*,1}) \cdot \int_0^1 \nabla g(\tilde{v}_{*,t}) dt = \frac{1-\alpha}{1+\alpha} u \cdot \int_0^1 \nabla g(\tilde{v}_{*,t}) dt,$$

where

$$\tilde{v}_{*,t} = \tilde{v}_* t + 'v_{*,1} (1-t) = 'v_{*,1} - \frac{1-\alpha}{1+\alpha} u t, \quad \forall t \in (0, 1).$$

Consequently,

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} |\mathcal{I}_2^{1,2}(g, f)(v)| \langle v \rangle^q dv \\ & \leq \frac{1-\alpha}{1+\alpha} \int_0^1 dt \int_{\mathbb{S}^{d-1}} b_0(\hat{u} \cdot n) |\hat{u} \cdot n|^2 dn \int_{\mathbb{R}^{2d}} |f('v_1)| |\nabla g(\tilde{v}_{*,t})| |u|^2 \langle v \rangle^q dv_* dv. \end{aligned}$$

We can apply the pre-post collisional change of variable in the inner integral $(v_*, v) \rightarrow (\tilde{v}_{*,t}, 'v_1)$ with Jacobian

$$\frac{1+\alpha}{1+\alpha+(1-\alpha)t} \left(\frac{1+\alpha}{1+\alpha-(1-\alpha)t} \right)^{d-1} \leq \left(\frac{1+\alpha}{2\alpha} \right)^{d-1},$$

to obtain that

$$\|\mathcal{I}_2^{1,2}(g, f)\|_{L_v^1(\varpi_q)} \leq \left(\frac{1+\alpha}{2\alpha} \right)^{d-1} \frac{1-\alpha}{1+\alpha} c_q \|b_0\|_{L^1(\mathbb{S}^{d-1})} \|\nabla g\|_{L_v^1(\varpi_{q+2})} \|f\|_{L_v^1(\varpi_{q+2})}$$

for some positive constant c_q . Regarding the term $\mathcal{I}_2^2(g, f)$, one invokes Taylor formula again

$$g('v_{*,\alpha}) - g('v_{*,1}) = ('v_{*,\alpha} - 'v_{*,1}) \cdot \int_0^1 \nabla g('v_{*,t}) dt,$$

where we recall that, according to (A.9),

$$'v_{*,\alpha} - 'v_{*,1} = \frac{1-\alpha}{2\alpha} (u \cdot n) n,$$

and, for $0 < t < 1$,

$$'v_{*,t} = 'v_{*,\alpha} t + 'v_{*,1} (1-t) = 'v_{*,\alpha} - (1-t) \frac{1-\alpha}{2\alpha} (u \cdot n) n.$$

Therefore,

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} |\mathcal{I}_2^2(g, f)(v)| \langle v \rangle^q dv \\ & \leq \frac{1-\alpha}{2\alpha} \int_0^1 dt \int_{\mathbb{S}^{d-1}} b_0(\widehat{u} \cdot n) |\widehat{u} \cdot n|^2 dn \int_{\mathbb{R}^{2d}} |f(\tilde{u}_\alpha + v_{*,t})| |\nabla g(v_{*,t})| |u|^2 \langle v \rangle^q dudv \end{aligned}$$

with

$${}' \tilde{u}_\alpha := {}' v_\alpha - {}' v_{*,t} = u - 2(u \cdot n)n - (1+t) \frac{1-\alpha}{2\alpha} (u \cdot n)n.$$

Apply, for fixed n , the pre-post collisional change of variables $(v_*, u) \rightarrow ({}' v_{*,t}, {}' \tilde{u}_\alpha)$ (with Jacobian $\tilde{J}_\alpha(t) = 1 + (1+t) \frac{1-\alpha}{2\alpha} \geq 1$). Noticing that

$$|u| \leq {}' u_\alpha \leq \frac{2}{1+t} |{}' \tilde{u}_\alpha|, \quad |v| \leq {}' v_\alpha + {}' v_{*,\alpha} \leq 2|{}' v_{*,t}| + 4|{}' \tilde{u}_\alpha|,$$

it follows that

$$\begin{aligned} \int_{\mathbb{R}^d} |\mathcal{I}_2^2(g, f)(v)| \langle v \rangle^q dv & \leq \frac{1-\alpha}{2\alpha} c_q \int_0^1 \frac{1}{\tilde{J}_\alpha(t)} dt \int_{\mathbb{S}^{d-1}} b_0(\widehat{u} \cdot n) |\widehat{u} \cdot n|^2 dn \\ & \int_{\mathbb{R}^{2d}} |f(v+u) \langle v+u \rangle^{q+2}| |\nabla g(v) \langle v \rangle^{q+2}| dudv \\ & \leq \frac{1-\alpha}{2\alpha} c_q \|b_0\|_{L^1(\mathbb{S}^{d-1})} \|f\|_{L_v^1(\varpi_{q+2})} \|\nabla g\|_{L_v^1(\varpi_{q+2})}. \end{aligned}$$

Gathering previous estimates proves the first assertion of the Lemma. The proof for higher norms $\mathbb{W}_v^{k,1}(\varpi_q)$ is simply obtained differentiating and applying the previous estimates for each suitable difference.

Let us now see how to derive the estimates in the L_v^2 setting. One still starts with the representation (A.11). To estimate $\mathcal{I}_1(g, f)$ in $L_v^2(\varpi_q)$, we first use Cauchy-Schwarz inequality to get

$$\begin{aligned} & \|\mathcal{I}_1(g, f)\|_{L_v^2(\varpi_q)}^2 \\ & \leq C_\kappa \frac{(1-\alpha^2)^2}{\alpha^4} \int_{\mathbb{R}^{2d} \times \mathbb{S}^{d-1}} |g(v_{*,\alpha})|^2 |f(v_\alpha)|^2 |u \cdot n|^2 \langle u \rangle^{2\kappa} b_0(\widehat{u} \cdot n) \varpi_{2q}(v) dn dv_*, \end{aligned}$$

where

$$C_\kappa := \sup_{v \in \mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} b_0(\widehat{u} \cdot n) \langle u \rangle^{-2\kappa} dv_* dn = \|b_0\|_{L^1(\mathbb{S}^{d-1})} \|\langle \cdot \rangle^{-2\kappa}\|_{L^1(\mathbb{R}^d)} < \infty$$

for $\kappa > \frac{d}{2}$. We proceed then exactly as previously to obtain:

$$\|\mathcal{I}_1(g, f)\|_{L_v^2(\varpi_q)} \leq c_q \frac{1-\alpha}{\alpha^2} \|b_0\|_{L^1(\mathbb{S}^{d-1})} \|g\|_{L_v^2(\varpi_{q+\kappa+1})} \|f\|_{L_v^2(\varpi_{q+\kappa+1})}.$$

In a similar way, using Cauchy-Schwarz inequality, one can prove that

$$\|\mathcal{I}_2^1(g, f)\|_{L_v^2(\varpi_q)} \leq c_q \frac{1-\alpha}{2\alpha} \|b_0\|_{L^1(\mathbb{S}^{d-1})} \|g\|_{L_v^2(\varpi_{q+\kappa+2})} \|f\|_{\mathbb{W}_v^{1,2}(\varpi_{q+\kappa+2})}$$

and

$$\|\mathcal{I}_2^2(g, f)\|_{L_v^2(\varpi_q)} \leq c_q \frac{1-\alpha}{2\alpha} \|b_0\|_{L^1(\mathbb{S}^{d-1})} \|g\|_{\mathbb{W}_v^{1,2}(\varpi_{q+\kappa+2})} \|f\|_{L_v^2(\varpi_{q+\kappa+2})}.$$

which gives the result for $k = 0$. The proof for higher norms is obtained as before, differentiating and applying the above estimate to each suitable difference. \square

Let us now give the proof of Lemma 2.2 which is based on the σ -representation. Note that it is reminiscent of the proof of (Mischler & Mouhot, 2009, Proposition 3.2) and (Alonso & Lods, 2013, Theorem 3.11) but that we face some additional difficulties due to the polynomial weight.

Proof of Lemma 2.2. We only prove the first estimate on $\mathcal{Q}_1(\cdot, \mathcal{M}) - \mathcal{Q}_\alpha(\cdot, \mathcal{M})$, the other one can be treated exactly in the same way. Notice first that

$$\mathcal{Q}_1(f, \mathcal{M}) - \mathcal{Q}_\alpha(f, \mathcal{M}) = \mathcal{Q}_1^+(f, \mathcal{M}) - \mathcal{Q}_\alpha^+(f, \mathcal{M}).$$

As in the proofs of (Mischler & Mouhot, 2009, Proposition 3.2) and (Alonso & Lods, 2013, Theorem 3.11), we set $w := v + v_*$ and $\widehat{w} := w/|w|$ and define $\chi \in [0, \pi/2]$ through $|\cos \chi| := |\widehat{w} \cdot \sigma|$. Let $\delta \in (0, 1)$ and $R > 1$ be fixed and let $\eta_\delta \in \mathbb{W}^{1,\infty}(-1, 1)$ such that $\eta_\delta(s) = \eta_\delta(-s)$ for any $s \in (0, 1)$ and

$$\eta_\delta(s) = \begin{cases} 1 & \text{if } s \in (-1 + 2\delta, 1 - 2\delta) \\ 0 & \text{if } s \notin (-1 + \delta, 1 - \delta) \end{cases}$$

with moreover

$$0 \leq \eta_\delta(s) \leq 1 \quad \text{and} \quad |\eta'_\delta(s)| \leq \frac{3}{\delta} \quad \forall s \in (-1, 1).$$

Let us define also $\Theta_R(r) = \Theta(r/R)$ with $\Theta(x) = 1$ on $[0, 1]$, $\Theta(x) = 1 - x$ for $x \in [1, 2]$ and $\Theta(x) = 0$ on $[2, \infty)$. We define the set

$$A(\delta) := \{\sigma \in \mathbb{S}^{d-1}; \sin^2 \chi \geq \delta\}$$

we split \mathcal{Q}_α^+ into

$$\mathcal{Q}_\alpha^+ = \mathcal{Q}_{B_r, \alpha}^+ + \mathcal{Q}_{B_l, \alpha}^+ + \mathcal{Q}_{B_a, \alpha}^+$$

where the collision kernels $B_i(u, \widehat{u} \cdot \sigma)$, $i = l, r, a$, are defined by

$$B_r(u, \widehat{u} \cdot \sigma) = \eta_\delta(\widehat{u} \cdot \sigma) \Theta_R(u) |u| b(\widehat{u} \cdot \sigma), \quad B_l(u, \widehat{u} \cdot \sigma) := \mathbf{1}_{A(\delta)}(\sigma) (1 - \Theta_R(|u|)) |u| b(\widehat{u} \cdot \sigma)$$

and

$$B_a(u, \widehat{u} \cdot \sigma) = |u| b(\widehat{u} \cdot \sigma) ((1 - \eta_\delta(\widehat{u} \cdot \sigma)) \Theta_R(|u|) + (1 - \Theta_R(|u|)) \mathbf{1}_{A^c(\delta)}).$$

This splitting corresponds to a splitting for *small angles* (corresponding to the kernel B_a), *large velocities* (corresponding to B_l) and the reminder term (corresponding to B_r). The treatment of small angles and of the truncated operator is similar to the one of (Mischler & Mouhot, 2009,

Proposition 3.2) and we only recall the results obtained therein: there exists a constant $C_q > 0$ such that for any $\alpha \in (0, 1]$, $\delta \in (0, 1)$ and $R \in (1, \infty)$,

$$\|\mathcal{Q}_{B_a, \alpha}^+(f, \mathcal{M})\|_{L_v^1(\varpi_q)} \leq C_q \delta \|f\|_{L_v^1(\varpi_{q+1})}$$

and

$$\|\mathcal{Q}_{B_r, 1}^+(f, \mathcal{M}) - \mathcal{Q}_{B_r, \alpha}^+(f, \mathcal{M})\|_{L_v^1(\varpi_q)} \leq C_q (1 - \alpha) \left(\frac{R^2}{\delta} + \frac{R}{\delta^3} \right) \|f\|_{L_v^1(\varpi_q)}.$$

Let us now handle the case of large relative velocities. To this end, we will use the strong formulation of the σ -representation of our collision operator given in (A.4):

$$\begin{aligned} \mathcal{Q}_{B_l, \alpha}^+(f, \mathcal{M})(v) &= \frac{1}{\alpha^2} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \frac{|'u|}{|u|} B_l(u, \widehat{u} \cdot \sigma), \mathcal{M}(v) f(v_*) dv_* d\sigma \\ &= \frac{1}{\alpha^2} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |'u| b(\widehat{u} \cdot \sigma) \mathbf{1}_{A(\delta)}(\sigma) (1 - \Theta_R(|u|)) \mathcal{M}(v) f(v_*) dv_* d\sigma. \end{aligned}$$

We first observe that

$$|'u| b(\widehat{u} \cdot \sigma) \mathbf{1}_{A(\delta)}(\sigma) (1 - \Theta_R(|u|)) \leq \frac{|u|^2}{\alpha R} b(\widehat{u} \cdot \sigma) \mathbf{1}_{A(\delta)}(\sigma) \mathbf{1}_{\{R \leq |u| \leq 2R\}}. \quad (\text{A.12})$$

We then remark that

$$v' = v - \frac{1 + \alpha}{4\alpha} (u - |u|\sigma) = \frac{w}{2} - \frac{1}{2} \left[\frac{1 - \alpha}{2\alpha} u - \frac{1 + \alpha}{2\alpha} |u|\sigma \right].$$

Using now that, when $\sin^2 \chi \geq \delta$, then $|\widehat{w} \cdot \sigma| \leq \sqrt{1 - \delta} \leq 1 - \frac{\delta}{2}$, we have

$$\begin{aligned} |v'|^2 &\geq \frac{1}{4} (|w|^2 + |u|^2) - \frac{1 + \alpha}{4\alpha} |w||u| \left(1 - \frac{\delta}{2} \right) - \frac{1 - \alpha}{4\alpha} (|v|^2 - |v_*|^2) \\ &\geq |v|^2 \left(\frac{1}{2} - \frac{1 + \alpha}{4\alpha} \left(1 - \frac{\delta}{2} \right) - \frac{1 - \alpha}{4\alpha} \right) + |v_*|^2 \left(\frac{1}{2} - \frac{1 + \alpha}{4\alpha} \left(1 - \frac{\delta}{2} \right) + \frac{1 - \alpha}{4\alpha} \right) \\ &\geq \delta \frac{1 + \alpha}{8\alpha} |v_*|^2 + \left(\delta \frac{1 + \alpha}{8\alpha} - \frac{1 - \alpha}{2\alpha} \right) |v|^2. \end{aligned}$$

Then, if $\delta \geq 8 \frac{1 - \alpha}{1 + \alpha}$, we get:

$$|v_*|^2 \geq \delta \frac{1 + \alpha}{16\alpha} (|v|^2 + |v_*|^2) \geq \frac{\delta}{16} (|v|^2 + |v_*|^2).$$

As a consequence,

$$\mathcal{M}(v) \langle v_* \rangle^2 \varpi_{q+2}(v) \lesssim e^{-\delta \frac{|v|^2}{64\vartheta_1}} \varpi_{q+2}(v) e^{-\delta \frac{|v_*|^2}{64\vartheta_1}} \langle v_* \rangle^2 \mathcal{M}^{\frac{1}{2}}(v) \lesssim \frac{1}{\delta^{\frac{q+4}{2}}} \mathcal{M}^{\frac{1}{2}}(v). \quad (\text{A.13})$$

Using this, we can now estimate the norm of $\mathcal{Q}_{B_l, \alpha}^+(f, \mathcal{M})$ if $\delta \geq 8 \frac{1-\alpha}{1+\alpha}$ thanks to (A.12) to get

$$\begin{aligned} \|\mathcal{Q}_{B_l, \alpha}^+(f, \mathcal{M})\|_{L_v^1(\varpi_q)} &\lesssim \frac{1}{R} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |u|^2 b(\widehat{u} \cdot \sigma) \mathbf{1}_{A(\delta)}(\sigma) \mathcal{M}(v) |f(v_*)| \varpi_q(v) d\sigma dv_* dv \\ &\lesssim \frac{1}{R} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} \mathcal{M}(v) |f(v_*)| \langle v_* \rangle^2 \varpi_{q+2}(v) b(\widehat{u} \cdot \sigma) d\sigma dv_* dv \\ &\lesssim \frac{1}{R \delta^{\frac{q+4}{2}}} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} \mathcal{M}^{\frac{1}{2}}(v) |f(v_*)| b(\widehat{u} \cdot \sigma) d\sigma dv_* dv \\ &\lesssim \frac{1}{R \delta^{\frac{q+4}{2}}} \|f\|_{L_v^1} \end{aligned}$$

since $b \in L^1(\mathbb{S}^{d-1})$ and $\mathcal{M} \in L^\infty(\mathbb{R}^d)$. Gathering the previous estimates, we obtain if $\delta \geq 8 \frac{1-\alpha}{1+\alpha}$:

$$\|\mathcal{Q}_\alpha(f, \mathcal{M}) - \mathcal{Q}_1(f, \mathcal{M})\|_{L_v^1(\varpi_q)} \lesssim \left(\delta + (1-\alpha) \left(\frac{R^2}{\delta} + \frac{R}{\delta^3} \right) + \frac{1}{R \delta^{\frac{q+4}{2}}} \right) \|f\|_{L_v^1(\varpi_{q+1})}.$$

Picking now $\delta = (1-\alpha)^p$ for some $p \in (0, 1)$ (so that the condition $\delta \geq 8 \frac{1-\alpha}{1+\alpha}$ will be satisfied for α close enough to 1) and $R = (1-\alpha)^{-p - \frac{q+4}{2}p} = (1-\alpha)^{-p \frac{q+6}{2}}$, we then obtain for α close enough to 1 that

$$\begin{aligned} \|\mathcal{Q}_\alpha(f, \mathcal{M}) - \mathcal{Q}_1(f, \mathcal{M})\|_{L_v^1(\varpi_q)} \\ \lesssim \left((1-\alpha)^p + (1-\alpha)^{1-p(q+7)} + (1-\alpha)^{1-p \frac{q+12}{2}} \right) \|f\|_{L_v^1(\varpi_{q+1})}, \end{aligned}$$

and, with $p = \frac{1}{q+8}$, $1 - p(q+7) = p$ and

$$\begin{aligned} \|\mathcal{Q}_\alpha(f, \mathcal{M}) - \mathcal{Q}_1(f, \mathcal{M})\|_{L_v^1(\varpi_q)} &\lesssim \left((1-\alpha)^p + (1-\alpha)^{\frac{q+4}{2}p} \right) \|f\|_{L_v^1(\varpi_{q+1})} \\ &\lesssim (1-\alpha)^p \|f\|_{L_v^1(\varpi_{q+1})}. \end{aligned}$$

For higher norms, it is enough to differentiate and apply previous estimates for each suitable difference. \square

We now give the proof of Lemma 2.3 in Section 2.1 which is based on Lemma 2.1.

Proof of Lemma 2.3. We start with the case of $\mathbb{W}_v^{k,1}(\varpi_q)$. To this end, we slightly modify here a strategy adopted in Alonso & Lods (2013) which consists in combining a *nonlinear* estimate for $\|G_\alpha - \mathcal{M}\|_{\mathbb{W}_v^{k,1}(\varpi_q)}$ together with non-quantitative convergence. We fix k, q and we divide the proof into three steps:

First step: non quantitative convergence. We prove that

$$\lim_{\alpha \rightarrow 1} \|\mathcal{M} - G_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_q)} = 0. \quad (\text{A.14})$$

We argue here as in (Alonso & Lods, 2013, Theorem 4.1). We sketch only the main steps. First, as already noticed in Mischler & Mouhot (2009), there is $\alpha_0 > 0$ such that

$$\sup_{\alpha \in (\alpha_0, 1)} \|G_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_q)} < \infty.$$

Then, there is a sequence $(\alpha_n)_n$ converging to 1 such that $(G_{\alpha_n})_n$ converges weakly, in $\mathbb{W}_v^{k,1}(\varpi_{q+1})$ to some limit \bar{G} (notice that, a priori, the limit function \bar{G} depends on the choice of k and q). Using the decay of (G_α) and compact embedding for Sobolev spaces, this convergence is actually strong, i.e. $\lim_n \|G_{\alpha_n} - \bar{G}\|_{\mathbb{W}_v^{k,1}(\varpi_q)} = 0$. According to (2.1), one necessarily has $\bar{G} = \mathcal{M}$ and one deduces easily that whole net $(G_\alpha)_\alpha$ is converging to \mathcal{M} . This proves (A.14).

Second step: nonlinear estimate. We first consider the Maxwellian \mathcal{M}_α with same mass, momentum and energy of G_α and we consider the linearized elastic collision operator around that Maxwellian

$$\mathbf{L}g = \mathcal{Q}_1(g, \mathcal{M}_\alpha) + \mathcal{Q}_1(\mathcal{M}_\alpha, g), \quad g \in \mathbb{W}_v^{k,1}(\varpi_q).$$

One simply notices that, since $\mathcal{Q}_1(\mathcal{M}_\alpha, \mathcal{M}_\alpha) = 0$,

$$\begin{aligned} \mathbf{L}(G_\alpha) &= \mathcal{Q}_1(G_\alpha - \mathcal{M}_\alpha, \mathcal{M}_\alpha - G_\alpha) + \mathcal{Q}_\alpha(G_\alpha, G_\alpha) + \left[\mathcal{Q}_1(G_\alpha, G_\alpha) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha) \right] \\ &= \mathcal{Q}_1(G_\alpha - \mathcal{M}_\alpha, \mathcal{M}_\alpha - G_\alpha) - (1 - \alpha)\nabla_v \cdot (vG_\alpha) + \left[\mathcal{Q}_1(G_\alpha, G_\alpha) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha) \right]. \end{aligned}$$

Therefore, using classical estimates for \mathcal{Q}_1 (see Alonso et al. (2010); Alonso & Gamba (2011))

$$\begin{aligned} \|\mathbf{L}(G_\alpha)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} &\leq \|\mathcal{Q}_1(G_\alpha - \mathcal{M}_\alpha, \mathcal{M}_\alpha - G_\alpha)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} + (1 - \alpha)\|G_\alpha\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+1})} \\ &\quad + \|\mathcal{Q}_1(G_\alpha, G_\alpha) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \\ &\leq C_1 \|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})}^2 + C(1 - \alpha)\|G_\alpha\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+1})} + C_1(1 - \alpha)\|G_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+2})} \end{aligned}$$

where we used Lemma 2.1 for estimating the difference $\mathcal{Q}_1(G_\alpha, G_\alpha) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha)$. Since $\sup_\alpha \|G_\alpha\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+2})} < \infty$, we obtain that there is a positive constant $C_2 > 0$ such that

$$\|\mathbf{L}(G_\alpha)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq C_2(1 - \alpha) + C_2\|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})}^2.$$

We can write $\mathbf{L}(G_\alpha) = \mathbf{L}(G_\alpha - \mathcal{M}_\alpha)$ and, as $G_\alpha - \mathcal{M}_\alpha$ has zero mass, momentum and energy, there is a positive constant $c > 0$ (that can be taken independent of α) such that

$$\|\mathbf{L}(G_\alpha - \mathcal{M}_\alpha)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \geq c\|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})}.$$

Recall that the constant $c > 0$ is actually the norm of the inverse of \mathbf{L} on the subspace of functions with zero mass, momentum and energy; recall that this inverse maps $\mathbb{W}_v^{k,1}(\varpi_q)$ into $\mathcal{D}(\mathbf{L}) = \mathbb{W}_v^{k,1}(\varpi_{q+1})$. Therefore, with $C_3 = C_2/c$

$$\|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} \leq C_3(1 - \alpha) + C_3\|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})}^2, \quad \alpha \in (\alpha_0, 1). \quad (\text{A.15})$$

Third step: conclusion. Setting

$$\vartheta_\alpha := \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 \mathcal{M}_\alpha(v) dv = \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 G_\alpha(v) dv,$$

one sees easily from (2.1) that $|\vartheta_1 - \vartheta_\alpha| \leq C(1 - \alpha)$ and then, one can check without difficulty that there is some positive constant $C_{k,q} > 0$ such that

$$\|\mathcal{M}_\alpha - \mathcal{M}\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq C_{k,q}(1 - \alpha), \quad \alpha \in [\alpha_2, 1]. \quad (\text{A.16})$$

Thanks to (A.14), we can then find $\alpha_3 > \alpha_0$ such that

$$C_3 \|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} \leq \frac{1}{2}$$

where $C_3 > 0$ is the positive constant in (A.15). Then, (A.15) reads simply as

$$\|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} \leq 2C_3(1 - \alpha), \quad \alpha \in [\alpha_3, 1],$$

and, using (A.16), we end up with

$$\|G_\alpha - \mathcal{M}\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} \leq C(1 - \alpha), \quad \alpha \in [\alpha_3, 1],$$

which gives also a quantitative lower bound on α_3 .

The estimate on the $\mathbb{W}_v^{k,2}$ -norm actually comes from the $\mathbb{W}_v^{k,1}$ -one and some Sobolev embedding. Indeed for $k' > d$, one can write that

$$\begin{aligned} \|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,2}(\varpi_q)} &\leq C_k \|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_q)}^{1/2} \|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,\infty}(\varpi_q)}^{1/2} \\ &\leq C_k \|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_q)}^{1/2} \|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k+k',1}(\varpi_q)}^{1/2} \end{aligned}$$

which allows us to conclude thanks to the first part of the proof. \square

Finally, we state (Alonso et al. , 2010, Theorem 1) followed by an immediate corollary:

Theorem A.1 (Theorem 1, Alonso et al. (2010)). *Consider $q \geq 1$ and $r \in [1, \infty]$ such that condition (1.6) is satisfied. For any $f \in L_v^r(\varpi_q)$ and $g \in L_v^1(\varpi_q)$,*

$$\|\mathcal{Q}_\alpha^+(f, g)\|_{L_v^r(\varpi_{q-1})} \leq C_r(b) \|f\|_{L_v^r(\varpi_q)} \|g\|_{L_v^1(\varpi_q)},$$

and

$$\|\mathcal{Q}_\alpha^+(g, f)\|_{L_v^r(\varpi_{q-1})} \leq \tilde{C}_r(b) \|f\|_{L_v^r(\varpi_q)} \|g\|_{L_v^1(\varpi_q)}$$

with

$$C_r(b) = 2^{\frac{q+1}{2} + \frac{1}{r} + \frac{d}{r'}} |\mathbb{S}^{d-2}| \int_{-1}^1 \left(\frac{1-s}{2} \right)^{-\frac{d}{2r'}} b(s) (1-s^2)^{\frac{d-3}{2}} ds$$

and

$$\tilde{C}_r(b) = 2^{\frac{q+1}{2} + \frac{1}{r}} |\mathbb{S}^{d-2}| \int_{-1}^1 \left(\frac{1+s}{2} + \frac{(1-\alpha)^2}{4} \frac{1+s}{2} \right)^{-\frac{d}{2r'}} b(s) (1-s^2)^{\frac{d-3}{2}} ds$$

where $\frac{1}{r} + \frac{1}{r'} = 1$.

Corollary A.2. *Consider $q \geq 1$ and $r \in [1, \infty]$ such that condition (1.6) is satisfied. Then there exists $C(r, q) > 0$ such that*

$$\sup_{\alpha \in (0,1]} \left(\|\mathcal{Q}_\alpha^+(f, \mathcal{M})\|_{L_v^r(\varpi_{q-1})} + \|\mathcal{Q}_\alpha^+(\mathcal{M}, f)\|_{L_v^r(\varpi_{q-1})} \right) \leq C(r, q) \|f\|_{L_v^r(\varpi_q)}$$

and

$$\sup_{\alpha \in (0,1]} \left(\|\mathcal{Q}_\alpha^+(f, G_\alpha - \mathcal{M})\|_{L_v^r(\varpi_{q-1})} + \|\mathcal{Q}_\alpha^+(G_\alpha - \mathcal{M}, f)\|_{L_v^r(\varpi_{q-1})} \right) \leq C(r, q) \|f\|_{L_v^r(\varpi_q)}.$$

A.4. Dissipativity properties. We finally give the remaining part of the proof of Proposition 2.9.

Proof of Proposition 2.9. We need to prove the result in the space $\mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,2}(\varpi_q)$. The proof follows the same lines as the one presented earlier and in particular, we write again $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}(h) = \sum_{i=0}^3 C_i(h)$. As in the $\mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q)$ case, there is no loss of generality in assuming $\ell = s$.

• Assume first $\ell = 0$. With obvious notations,

$$\int_{\mathbb{R}^d} \|h(\cdot, v)\|_{L_x^2}^{-1} \left(\int_{\mathbb{T}^d} \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}(h)(x, v) h(x, v) dx \right) \varpi_q(v) dv =: \sum_{i=0}^3 I_i(h).$$

First, $I_1(h) = 0$ since

$$\int_{\mathbb{T}^d} (v \cdot \nabla_x h(x, v)) h(x, v) dx = \frac{1}{2} \int_{\mathbb{T}^d} v \cdot \nabla_x h^2(x, v) dx = 0.$$

According to (2.14), by taking δ small enough so that $\Lambda_q^{(1)}(\delta) < 1$ (which is possible since $q > 2$), we have

$$I_0(h) \leq \varepsilon^{-2} \sigma_0 (\Lambda_q(\delta) - 1) \|h\|_{L_v^1 L_x^2(\varpi_{q+1})}.$$

Moreover, it follows from Cauchy-Schwarz inequality and Corollary 2.6 that

$$I_2(h) \leq \varepsilon^{-2} \int_{\mathbb{R}^d} \|\mathcal{P}_\alpha h(\cdot, v)\|_{L_x^2} \varpi_q(v) dv \leq \varepsilon^{-2} \tilde{C}_{0,q} (1 - \alpha)^{\frac{q}{2}} \|h\|_{L_v^1 L_x^2(\varpi_{q+1})}.$$

Finally, for I_3 , one can compute

$$\begin{aligned} & \int_{\mathbb{R}^d} \|h(\cdot, v)\|_{L_x^2}^{-1} \int_{\mathbb{T}^d} \nabla_v \cdot (v h(x, v)) h(x, v) dx \varpi_q(v) dv \\ &= d \int_{\mathbb{R}^d} \|h(\cdot, v)\|_{L_x^2} \varpi_q(v) dv + \frac{1}{2} \int_{\mathbb{R}^d} \|h(\cdot, v)\|_{L_x^2}^{-1} \int_{\mathbb{T}^d} v \cdot \nabla_v h^2(x, v) dx \varpi_q(v) dv \\ &= d \int_{\mathbb{R}^d} \|h(\cdot, v)\|_{L_x^2} \varpi_q(v) dv + \frac{1}{2} \int_{\mathbb{R}^d} \|h(\cdot, v)\|_{L_x^2}^{-1} v \cdot \nabla_v \|h(\cdot, v)\|_{L_x^2}^2 \varpi_q(v) dv \\ &= - \int_{\mathbb{R}^d} \|h(\cdot, v)\|_{L_x^2} v \cdot \nabla_v \varpi_q(v) dv. \end{aligned}$$

Since $v \cdot \nabla_v \varpi_q(v) = q \varpi_q(v) - q \varpi_{q-2}(v)$ we get

$$I_3(h) \leq q \kappa_\alpha \varepsilon^{-2} \|h\|_{L_v^1 L_x^2(\varpi_{q+1})}. \quad (\text{A.17})$$

Gathering the previous estimates, one obtains

$$\begin{aligned} \mathcal{I} &:= \int_{\mathbb{R}^d} \|h(\cdot, v)\|_{L_x^2}^{-1} \left(\int_{\mathbb{T}^d} \mathcal{B}_{\alpha, \varepsilon}^{(\delta)}(h)(x, v) h(x, v) dx \right) \varpi_q(v) dv \\ &\leq \varepsilon^{-2} \left(\tilde{\mathcal{C}}_{0,q} (1 - \alpha)^{\frac{p}{2}} + \sigma_0(\Lambda_q(\delta) - 1) + q\kappa_\alpha \right) \|h\|_{L_v^1 L_x^2(\varpi_{q+1})}. \end{aligned} \quad (\text{A.18})$$

Recalling that $\kappa_\alpha = 1 - \alpha$ while $\lim_{\delta \rightarrow 0} (\Lambda_q(\delta) - 1) = -\frac{q-2}{q+2} < 0$ we can pick $\delta_{1,0,0,q}^\dagger$ small enough and then $\alpha_{1,0,0,q}^\dagger \in (0, 1)$ close enough to 1 so that

$$\nu_{1,0,0,q} := -\inf \left\{ \tilde{\mathcal{C}}_{0,q} (1 - \alpha)^{\frac{p}{2}} + \sigma_0(\Lambda_q(\delta) - 1) + q\kappa_\alpha; \alpha \in (\alpha_{1,0,0,q}^\dagger, 1), \delta \in (0, \delta_{1,0,0,q}^\dagger) \right\} > 0$$

and get the result.

Let us now consider the case $\ell = 1$ and introduce the norm

$$\| \| h \| \| = \| h \|_{L_v^1 L_x^2(\varpi_q)} + \| \nabla_x h \|_{L_v^1 L_x^2(\varpi_q)} + \eta \| \nabla_v h \|_{L_v^1 L_x^2(\varpi_q)},$$

for some $\eta > 0$, the value of which shall be fixed later on. This norm is equivalent to the classical $\mathbb{W}_v^{1,1} \mathbb{W}_x^{1,2}(\varpi_q)$ -norm. We shall prove that for some $\nu_{1,1,1,q} > 0$, $\mathcal{B}_{\alpha, \varepsilon}^{(\delta)} + \varepsilon^{-2} \nu_{1,1,1,q}$ is dissipative in $\mathbb{W}_v^{1,1} \mathbb{W}_x^{1,2}(\varpi_q)$ for the norm $\| \| \cdot \| \|$. Notice first that the x -derivative commutes with all the above terms $C_i(h)$, $i = 0, \dots, 3$, i.e.

$$\nabla_x \mathcal{B}_{\alpha, \varepsilon}^{(\delta)} h(x, v) = \mathcal{B}_{\alpha, \varepsilon}^{(\delta)} \nabla_x h(x, v)$$

so that, according to the previous step

$$\begin{aligned} \mathcal{J}_x &:= \int_{\mathbb{R}^d} \| \nabla_x h(\cdot, v) \|_{L_x^2}^{-1} \left(\int_{\mathbb{T}^d} \nabla_x \mathcal{B}_{\alpha, \varepsilon}^{(\delta)}(h)(x, v) \cdot \nabla_x h(x, v) dx \right) \varpi_q(v) dv \\ &\leq -\varepsilon^{-2} \nu_{1,0,0,q} \| \nabla_x h \|_{L_v^1 L_x^2(\varpi_q)}. \end{aligned} \quad (\text{A.19})$$

Consider now the quantity

$$\mathcal{J}_v := \int_{\mathbb{R}^d} \| \nabla_v h(\cdot, v) \|_{L_x^2}^{-1} \left(\int_{\mathbb{T}^d} \nabla_v \mathcal{B}_{\alpha, \varepsilon}^{(\delta)}(h)(x, v) \cdot \nabla_v h(x, v) dx \right) \varpi_q(v) dv.$$

Using the notations above, one notices that $\nabla_v C_1(h) = -\varepsilon^{-1} \nabla_x h + C_1(\nabla_v h)$, so that

$$\begin{aligned} \nabla_v (\mathcal{B}_{\alpha, \varepsilon}^{(\delta)} h(x, v)) &= \varepsilon^{-2} \nabla_v (\mathcal{B}_1^{(\delta)} h) - \varepsilon^{-1} \nabla_x h + C_1(\nabla_v h) \\ &\quad + \varepsilon^{-2} \nabla_v (\mathcal{P}_\alpha h) + \varepsilon^{-2} \nabla_v (T_\alpha h) \\ &= \varepsilon^{-2} \nabla_v [\mathcal{L}_1^{R, \delta} h - \Sigma_{\mathcal{M}} h] - \varepsilon^{-1} \nabla_x h + C_1(\nabla_v h) \\ &\quad + \varepsilon^{-2} \nabla_v (\mathcal{P}_\alpha h) + \varepsilon^{-2} \nabla_v (T_\alpha h). \end{aligned} \quad (\text{A.20})$$

Then, it follows from Corollary 2.6 that

$$\| \nabla_v (\mathcal{P}_\alpha h) \|_{L_v^1 L_x^2(\varpi_q)} \leq \tilde{\mathcal{C}}_{1,q} (1 - \alpha)^{\frac{p}{2}} \left(\| h \|_{L_v^1 L_x^2(\varpi_{q+1})} + \| \nabla_v h \|_{L_v^1 L_x^2(\varpi_{q+1})} \right). \quad (\text{A.21})$$

Now,

$$\nabla_v[\mathcal{L}_1^{R,\delta}h - \Sigma_{\mathcal{M}}h] = \mathcal{L}_1^{R,\delta}(\nabla_v h) - \Sigma_{\mathcal{M}}\nabla_v h + \mathcal{R}(h),$$

where

$$\mathcal{R}(h) = \mathcal{Q}_1(h, \nabla_v \mathcal{M}) + \mathcal{Q}_1(\nabla_v \mathcal{M}, h) - (\nabla_v \mathcal{A}^{(\delta)})(h) - \mathcal{A}^{(\delta)}(\nabla_v h).$$

From the proof of (Gualdani et al., 2017, Lemma 4.14), we have that

$$\|\mathcal{R}(h)\|_{L_v^1 L_x^2(\varpi_q)} \leq C_\delta \|h\|_{L_v^1 L_x^2(\varpi_{q+1})}.$$

From (Gualdani et al., 2017, Lemma 4.12), we also have that

$$\|\mathcal{L}_1^{R,\delta}(\nabla_v h)\|_{L_v^1 L_x^2(\varpi_q)} \leq \Lambda_q^{(1)}(\delta) \|\nabla_v h\|_{L_v^1 L_x^2(\varpi_q \Sigma_{\mathcal{M}})},$$

where $\Lambda_q^{(1)}(\delta)$ was introduced in Lemma 2.7. Then, one has that

$$-\int_{\mathbb{R}^d} \|\nabla_v h(\cdot, v)\|_{L_x^2}^{-1} \left(\int_{\mathbb{T}^d} \Sigma_{\mathcal{M}}(v) |\nabla_v h(x, v)|^2 dx \right) \varpi_q(v) dv \leq -\|\nabla_v h\|_{L_v^1 L_x^2(\varpi_q \Sigma_{\mathcal{M}})}.$$

Therefore, recalling that δ is such that $\Lambda_q^{(1)}(\delta) < 1$ and that $\Sigma_{\mathcal{M}}(v) \geq \sigma_0(v)$, we get

$$\|\nabla_v[\mathcal{L}_1^{R,\delta}(h) - \Sigma_{\mathcal{M}}h]\|_{L_v^1 L_x^2(\varpi_q)} \leq C_\delta \|h\|_{L_v^1 L_x^2(\varpi_{q+1})} + \sigma_0 \left(\Lambda_q^{(1)}(\delta) - 1 \right) \|\nabla_v h\|_{L_v^1 L_x^2(\varpi_{q+1})}. \quad (\text{A.22})$$

Finally, using the short-hand notation

$$\nabla_v \cdot (v \nabla_v h) = (\nabla_v \cdot (v \partial_{v_1} h), \dots, \nabla_v \cdot (v \partial_{v_d} h)),$$

we have

$$\nabla_v h \cdot \nabla_v (\nabla_v \cdot (v h)) = |\nabla_v h|^2 + [\nabla_v \cdot (v \nabla_v h)] \cdot \nabla_v h.$$

Doing similar computations as the ones leading to (A.17), we obtain:

$$\begin{aligned} \int_{\mathbb{R}^d} \|\nabla_v h(\cdot, v)\|_{L_x^2}^{-1} \left(\int_{\mathbb{T}^d} \nabla_v T_\alpha(h)(x, v) \cdot \nabla_v h(x, v) dx \right) \varpi_q(v) dv \\ \leq (q-1) \kappa_\alpha \|\nabla_v h\|_{L_v^1 L_x^2(\varpi_{q+1})}. \end{aligned} \quad (\text{A.23})$$

Coming back to (A.20), Cauchy-Schwarz inequality and estimates (A.21), (A.22) and (A.23) give that

$$\begin{aligned} \mathcal{J}_v \leq \varepsilon^{-2} (C_\delta + \tilde{\mathcal{C}}_{1,q} (1-\alpha)^{\frac{p}{2}}) \|h\|_{L_v^1 L_x^2(\varpi_{q+1})} + \varepsilon^{-1} \|\nabla_x h\|_{L_v^1 L_x^2(\varpi_q)} \\ + \varepsilon^{-2} (\tilde{\mathcal{C}}_{1,q} (1-\alpha)^{\frac{p}{2}} + C \kappa_\alpha + \sigma_0 (\Lambda_q^{(1)}(\delta) - 1)) \|\nabla_v h\|_{L_v^1 L_x^2(\varpi_{q+1})}, \end{aligned}$$

where we used that the contribution to \mathcal{J}_v of the term $C_1(\nabla_v h)$ vanishes. Hence, combining this estimate with (A.18) and (A.19) and using that $\varepsilon \leq 1$, it follows that

$$\begin{aligned} \mathcal{I} + \mathcal{J}_x + \eta \mathcal{J}_v \leq \varepsilon^{-2} \left(\left[-\nu_{1,0,0,q} + \eta \left(C_\delta + \tilde{\mathcal{C}}_{1,q} (1-\alpha)^{\frac{p}{2}} \right) \right] \|h\|_{L_v^1 L_x^2(\varpi_{q+1})} \right. \\ \left. - (\nu_{1,0,0,q} - \eta) \|\nabla_x h\|_{L_v^1 L_x^2(\varpi_{q+1})} + \eta \left[\tilde{\mathcal{C}}_{1,q} (1-\alpha)^{\frac{p}{2}} + C \kappa_\alpha + \sigma_0 (\Lambda_q^{(1)}(\delta) - 1) \right] \|\nabla_v h\|_{L_v^1 L_x^2(\varpi_{q+1})} \right). \end{aligned}$$

Consequently, there exists $\alpha_{1,1,q}^\dagger > 0$ and $\delta_{1,1,q}^\dagger > 0$ so that

$$\mathbf{C}_{1,q}(1 - \alpha) + C\kappa_\alpha + \sigma_0(\Lambda_q^{(1)}(\delta) - 1) < 0 \quad \forall \alpha \in (\alpha_{1,1,q}^\dagger, 1), \quad \delta \in (0, \delta_{1,1,q}^\dagger).$$

Choosing $\eta > 0$ small enough such that

$$\nu_{1,0,0,q} - \eta \max \left(1, \sup_{\delta \in (0, \delta_{1,1,q}^\dagger)} C_\delta + \tilde{\mathbf{C}}_{1,q}(1 - \alpha_{1,1,q}^\dagger)^{\frac{p}{2}} \right) > 0,$$

we finally obtain that there exists $\nu_{1,1,1,q} > 0$ such that for $\alpha \in (\alpha_{1,1,q}^\dagger, 1)$ and $\delta \in (0, \delta_{1,1,q}^\dagger)$,

$$\begin{aligned} \mathcal{I} + \mathcal{J}_x + \eta \mathcal{J}_v &\leq -\varepsilon^{-2} \nu_{1,1,1,q} \left[\|h\|_{L_v^1 L_x^2(\varpi_{q+1})} + \|\nabla_x h\|_{L_v^1 L_x^2(\varpi_{q+1})} + \eta \|\nabla_v h\|_{L_v^1 L_x^2(\varpi_{q+1})} \right] \\ &\leq -\varepsilon^{-2} \nu_{1,1,1,q} \|h\|. \end{aligned}$$

This proves that $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2} \nu_{1,1,1,q}$ is hypo-dissipative in $\mathbb{W}_v^{1,1} \mathbb{W}_x^{1,2}(\varpi_q)$. We prove the result for higher order derivatives in the same way considering now the norm

$$\|h\| = \sum_{|\beta_1| + |\beta_2| \leq k} \eta^{|\beta_1|} \left\| \nabla_v^{|\beta_1|} \nabla_x^{|\beta_2|} h \right\|_{L_v^1 L_x^2(\varpi_q)}$$

for some $\eta > 0$ to be chosen sufficiently small. \square

APPENDIX B. ABOUT THE ORIGINAL PROBLEM IN PHYSICAL VARIABLES

Let $F_\varepsilon(t, x, v)$ be the solution of the Boltzmann equation (1.10a) with associated Knudsen number ε . Recall that the time-scale functions $\tau_\varepsilon(t), V_\varepsilon(t)$ that relate the problem in original (physical) variables to its self-similar counterpart

$$F_\varepsilon(t, x, v) = V_\varepsilon(t)^d f_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v)$$

are given by

$$\tau_\varepsilon(t) := \frac{1}{c_\varepsilon} \ln(1 + c_\varepsilon t), \quad V_\varepsilon(t) = 1 + c_\varepsilon t, \quad t \geq 0,$$

where $c_\varepsilon = \frac{1 - \alpha(\varepsilon)}{\varepsilon^2}$. It follows that the explicit equation for f_ε is given by

$$\partial_\tau f_\varepsilon + \varepsilon^{-1} w \cdot \nabla_x f_\varepsilon = \varepsilon^{-2} \mathcal{Q}(f_\varepsilon, f_\varepsilon) - \varepsilon^{-2} (1 - \alpha) \nabla_w (w f_\varepsilon), \quad w = V_\varepsilon(t)v$$

as observed in (1.22). Set $f_\varepsilon(\tau, x, w) = G_{\alpha(\varepsilon)}(w) + \varepsilon h_\varepsilon(\tau, x, w)$ and denote $\mathbf{h}(\tau, x, w)$ the weak- \star limit in the space $L^\infty((0, \infty); \mathcal{E})$ of the (sub-)sequence $\{h_\varepsilon\}$. Define

$$e_\varepsilon(t, x, v) = V_\varepsilon(t)^d (h_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v) - \mathbf{h}(\tau_\varepsilon(t), x, V_\varepsilon(t)v)).$$

The following error estimate holds.

Lemma B.1. *Under Assumption 1.2 and in the regime $\varepsilon \ll 1$, $\lambda_0 > 0$, the following estimation holds for any $a \in (0, 1/2)$, up to possibly extracting a subsequence,*

$$|\langle e_\varepsilon(t), |v|^\kappa \varphi \rangle| \leq C_\varphi \sqrt{\mathcal{K}_0} V_\varepsilon(t)^{-\kappa-a}, \quad \varphi \in \mathcal{C}_{v,b}^1 L_x^\infty, \quad 0 \leq \kappa \leq q-1,$$

where we denoted by $\mathcal{C}_{v,b}^1$ the set of \mathcal{C}^1 functions in v that are bounded as well as their derivatives and where for any $t \geq 0$,

$$V_\varepsilon(t) \approx V_0(t) = (1 + \lambda_0 t), \quad \text{as } \varepsilon \approx 0.$$

Proof. After a change of variables it follows that, for any test-function φ ,

$$\begin{aligned} & \langle e_\varepsilon(t), |v|^\kappa \varphi \rangle \\ &= V_\varepsilon(t)^{-\kappa} \int_{\mathbb{T}^d \times \mathbb{R}^d} \left(h_\varepsilon(\tau_\varepsilon(t), x, v) - \mathbf{h}(\tau_\varepsilon(t), x, v) \right) |v|^\kappa \left(\varphi(x, V_\varepsilon(t)^{-1}v) - \varphi(x, 0) \right) dv dx \\ & \quad + V_\varepsilon(t)^{-\kappa} \int_{\mathbb{T}^d \times \mathbb{R}^d} \left(h_\varepsilon(\tau_\varepsilon(t), x, v) - \mathbf{h}(\tau_\varepsilon(t), x, v) \right) |v|^\kappa \varphi(x, 0) dv dx \\ &= \mathcal{I}_1(t) + \mathcal{I}_2(t). \end{aligned}$$

Note that, up to a subsequence, \mathbf{h} is the weak- \star limit of $\{h_\varepsilon\}_\varepsilon$ in $L^\infty((0, \infty); \mathcal{E})$. Thus, for any $t > 0$, $\|\mathbf{h}\|_{L^\infty((t, \infty); \mathcal{E})} \leq \liminf_{\varepsilon \searrow 0} \|h_\varepsilon\|_{L^\infty((t, \infty); \mathcal{E})}$. Consequently, thanks to Theorem 4.13, it holds that

$$\|\mathbf{h}\|_{L^\infty((t, \infty); \mathcal{E})} \leq C \sqrt{\mathcal{K}_0} e^{-\frac{\lambda_\varepsilon}{2}t}, \quad t > 0. \quad (\text{B.1})$$

As a consequence, recalling that $\lambda_\varepsilon \simeq c_\varepsilon$, it follows that for any $a \in (0, 1/2)$

$$\begin{aligned} & \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} \left(h_\varepsilon(\tau, x, v) - \mathbf{h}(\tau, x, v) \right) |v|^\kappa dv dx \right| \\ & \leq \|h_\varepsilon(\tau_\varepsilon(t)) - \mathbf{h}(\tau_\varepsilon(t))\|_{L_{v,x}^1(\mathfrak{w}_q)} \leq C \sqrt{\mathcal{K}_0} e^{-\frac{\lambda_\varepsilon}{2}\tau_\varepsilon(t)} \leq C V_\varepsilon(t)^{-a} \sqrt{\mathcal{K}_0}, \end{aligned}$$

where we used once again that $\|\cdot\|_{L_v^1(\mathfrak{w}_q)} \lesssim \|\cdot\|_{L_v^2(\mathfrak{w}_{q+\kappa})}$ for $\kappa > \frac{d}{2}$. Now, in regard of $\mathcal{I}_1(t)$, note that

$$|\varphi(x, V_\varepsilon(t)^{-1}v) - \varphi(x, 0)| \leq V_\varepsilon(t)^{-1} |v| \sup_x \sup_v |\partial_v \varphi(x, v)| = C_\varphi V_\varepsilon(t)^{-1} |v|,$$

so that the following holds:

$$|\mathcal{I}_1(t)| \leq C_\varphi V_\varepsilon(t)^{-\kappa-1} \|h_\varepsilon(\tau_\varepsilon(t)) - \mathbf{h}(\tau_\varepsilon(t))\|_{L_{v,x}^1(\mathfrak{w}_q)} \leq C_\varphi V_\varepsilon(t)^{-\kappa-1-a} \sqrt{\mathcal{K}_0}.$$

Similarly,

$$|\mathcal{I}_2(t)| \leq C_\varphi V_\varepsilon(t)^{-\kappa-a} \sqrt{\mathcal{K}_0},$$

which proves the desired estimate. \square

The above computations also allow to provide a *local version of Haff's Law*. Namely, note that

$$\begin{aligned} & \int_{\mathbb{R}^d} f_\varepsilon(\tau_\varepsilon(t), x, w) |w|^\kappa dw \\ &= \int_{\mathbb{R}^d} G_\alpha(w) |w|^\kappa dw + \varepsilon \int_{\mathbb{R}^d} h_\varepsilon(\tau_\varepsilon(t), x, w) |w|^\kappa dw, \quad 0 \leq \kappa \leq q. \end{aligned}$$

Thanks to Sobolev embedding it holds that

$$\left| \sup_{x \in \mathbb{T}^d} \int_{\mathbb{R}^d} h_\varepsilon(\tau_\varepsilon(t), x, w) |w|^\kappa dv \right| \leq C_\kappa \|h_\varepsilon(\tau_\varepsilon(t))\|_\varepsilon \leq C_\kappa \sqrt{K_0}.$$

Therefore, for sufficiently small $\varepsilon > 0$ there exists two positive constants C_κ and c_κ such that

$$c_\kappa \leq \int_{\mathbb{R}^d} f_\varepsilon(\tau_\varepsilon(t), x, w) |w|^\kappa dw \leq C_\kappa, \quad 0 \leq \kappa \leq q, \quad t \geq 0,$$

which leads, for the physical problem, to

$$V_\varepsilon(t)^{-\kappa} c_\kappa \leq \int_{\mathbb{R}^d} F_\varepsilon(t, x, v) |v|^\kappa dv \leq V_\varepsilon(t)^{-\kappa} C_\kappa, \quad 0 \leq \kappa \leq q, \quad t \geq 0.$$

In particular, this estimate renders a local version of Haff's law

$$\int_{\mathbb{R}^d} F_\varepsilon(t, x, v) |v|^2 dv \sim (1 + c_\varepsilon t)^{-2}, \quad \forall t \geq 0, \quad x \in \mathbb{T}^d.$$

APPENDIX C. TOOLS FOR THE HYDRODYNAMIC LIMIT

We collect several tools that are used in Section 6.2 to derive the modified incompressible Navier-Stokes system. Various known computations regarding the elastic Boltzmann operator are needed. As in the classical case, we introduce the traceless tensor

$$\mathbf{A}(v) = v \otimes v - \frac{1}{d} |v|^2 \mathbf{Id}.$$

Notice that that (6.18) can be rewritten thanks to (6.13) as

$$v \cdot \nabla_x \mathbf{h} = \mathbf{A}(v) \mathcal{M}(v) : \nabla_x u + \mathbf{b}(v) \mathcal{M}(v) \cdot \nabla_x \theta,$$

with

$$\mathbf{b}(v) = \frac{1}{2} (|v|^2 - (d+2)v_1) v \in \mathbb{R}^d.$$

Lemma C.1. *One has that $\mathbf{A}, \mathbf{b} \in \text{Range}(\mathbf{I} - \pi_0)$ and there exists two radial functions $\chi_i = \chi_i(|v|)$, $i = 1, 2$, such that*

$$\phi(v) = \chi_1(|v|) \mathbf{A}(v) \in \mathcal{M}_d(\mathbb{R}), \quad \text{and} \quad \psi(v) = \chi_2(|v|) \mathbf{b}(v) \in \mathbb{R}^d,$$

satisfy

$$\mathbf{L}_1(\phi \mathcal{M}) = -\mathbf{A} \mathcal{M}, \quad \mathbf{L}_1(\psi \mathcal{M}) = -\mathbf{b} \mathcal{M}. \quad (\text{C.1})$$

Moreover,

$$\begin{aligned} \left\langle \phi^{i,j} \mathbf{L}_1(\phi^{k,\ell} \mathcal{M}) \right\rangle &= -\nu \left(\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk} - \frac{2}{d} \delta_{ij} \delta_{kl} \right) \\ \left\langle \psi_i \mathbf{L}_1(\psi_j \mathcal{M}) \right\rangle &= -\frac{d+2}{2} \gamma \delta_{ij}, \quad i, j, k, \ell \in \{1, \dots, d\}, \end{aligned} \quad (\text{C.2})$$

with

$$\nu := -\frac{1}{(d-1)(d+2)} \left\langle \phi : \mathbf{L}_1(\phi \mathcal{M}) \right\rangle \geq 0, \quad \gamma := -\frac{2}{d(d+2)} \left\langle \psi \cdot \mathbf{L}_1(\psi \mathcal{M}) \right\rangle \geq 0.$$

Finally,

$$\phi^{i,j}(v) \lesssim \varpi_3(v), \quad \psi_i(v) \lesssim \varpi_4(v), \quad i, j \in \{1, \dots, d\}.$$

Proof. The tensor \mathbf{A} and the vector \mathbf{b} satisfy

$$\left\langle \mathbf{A}^{k,\ell} \Psi_i \mathcal{M} \right\rangle = 0, \quad \left\langle \mathbf{b} \Psi_i \mathcal{M} \right\rangle = 0, \quad \forall i = 1, \dots, d+2, \quad k, \ell \in \{1, \dots, d\}, \quad (\text{C.3})$$

from which we get that $\mathbf{A}, \mathbf{b} \in \text{Range}(\mathbf{I} - \pi_0)$. We refer to [Desvillettes & Golse \(1994\)](#) and [Bardos et al. \(1993\)](#) for the proof of the second part of the Lemma, just mind that the linearized Boltzmann operator considered in such references is defined as $Lg = -\mathcal{M}^{-1} \mathbf{L}_1(\mathcal{M}g)$. We refer to [\(Bardos et al. , 1993, Lemma 4.4\)](#) for the proof of [\(C.2\)](#). We refer to [\(Golse & Saint-Raymond , 2005, Proposition 6.5\)](#) for the last estimates on $\phi^{i,j}$ and ψ . \square

Remark C.2. Notice that if $\zeta = \zeta(|v|)$ is radially symmetric, then

$$\left\langle \zeta \mathbf{A}^{i,j} \mathcal{M} \right\rangle = \left\langle \zeta \mathbf{L}_1(\phi \mathcal{M}) \right\rangle = 0, \quad \forall i, j = 1, \dots, d.$$

Lemma C.3. For \mathbf{h} given by [\(6.7\)](#), it holds that

$$\left\langle \phi \mathcal{Q}_1(\mathbf{h}, \mathbf{h}) \right\rangle = \vartheta_1^2 \left(u \otimes u - \frac{2}{d} |u|^2 \mathbf{Id} \right),$$

for any $i, j = 1, \dots, d$.

Proof. As observed in [\(Cercignani et al. , 1970, Eq. \(60\)\)](#), if $g\mathcal{M} \in \text{Ker}(\mathbf{L}_1)$ then $\mathcal{Q}_1(g\mathcal{M}, g\mathcal{M}) = -\frac{1}{2} \mathbf{L}_1(g^2 \mathcal{M})$. Therefore, with $g = \varrho + u \cdot v + \frac{1}{2}(|v|^2 - 2\vartheta_1)$,

$$\mathcal{Q}_1(\mathbf{h}, \mathbf{h}) = -\frac{1}{2} \mathbf{L}_1((u \cdot v)^2 \mathcal{M}) - \frac{1}{8} \theta^2 \mathbf{L}_1(|v|^4 \mathcal{M}) + \theta u \cdot \mathbf{L}_1(\frac{1}{2}|v|^2 v \mathcal{M}). \quad (\text{C.4})$$

One checks that

$$\left\langle \phi^{i,j} \mathbf{L}_1(|v|^4 \mathcal{M}) \right\rangle = 0,$$

whereas $\mathbf{L}_1(\frac{1}{2}|v|^2 v \mathcal{M}) = \mathbf{L}_1(\mathbf{b} \mathcal{M})$, from which

$$\left\langle \phi^{i,j} \mathbf{L}_1(\frac{1}{2}|v|^2 v \mathcal{M}) \right\rangle = \left\langle \mathbf{b} \mathbf{L}_1(\phi^{i,j} \mathcal{M}) \right\rangle = -\left\langle \mathbf{b} \mathbf{A}^{i,j} \mathcal{M} \right\rangle = 0,$$

since $\mathbf{b} \mathbf{A}^{i,j}$ is an even function. Therefore, we obtain that

$$\left\langle \phi^{i,j} \mathcal{Q}_1(\mathbf{h}, \mathbf{h}) \right\rangle = -\frac{1}{2} \sum_{k,\ell} u_k u_\ell \left\langle \phi^{i,j} \mathbf{L}_1(v_k v_\ell \mathcal{M}) \right\rangle = \frac{1}{2} \sum_{k,\ell} u_k u_\ell \left\langle v_k v_\ell \mathbf{A}^{i,j} \mathcal{M} \right\rangle. \quad (\text{C.5})$$

As for (C.2), one checks that if $i \neq j$

$$\sum_{k,\ell} u_k u_\ell \left\langle v_k v_\ell \mathbf{A}^{i,j} \mathcal{M} \right\rangle = \sum_{\{k,\ell\}=\{i,j\}} u_k u_\ell \left\langle v_i^2 v_j^2 \mathcal{M} \right\rangle = 2u_i u_j \left\langle v_i^2 v_j^2 \mathcal{M} \right\rangle,$$

whereas, for $i = j$,

$$\sum_{k,\ell} u_k u_\ell \left\langle v_k v_\ell \mathbf{A}^{i,i} \mathcal{M} \right\rangle = \sum_{k=1}^d u_k^2 \left(\left\langle v_i^2 v_k^2 \mathcal{M} \right\rangle - \frac{1}{d} \left\langle v_k^2 |v|^2 \mathcal{M} \right\rangle \right).$$

Notice that $a := \left\langle v_i^2 v_j^2 \mathcal{M} \right\rangle$ is independent of i, j , thus, it is not difficult to check that

$$(d-1)a = \frac{1}{d} \int_{\mathbb{R}^d} |v|^4 \mathcal{M} dv - \int_{\mathbb{R}^d} v_1^4 \mathcal{M}(v) dv = (d-1)\vartheta_1^2,$$

that is, $a = \vartheta_1^2$. In the same way, for any $k \in \{1, \dots, d\}$

$$\left\langle v_k^2 |v|^2 \mathcal{M} \right\rangle = \frac{1}{d} \left\langle |v|^4 \mathcal{M} \right\rangle = (d+2)\vartheta_1^2,$$

whereas

$$\left\langle v_k^2 v_i^2 \mathcal{M} \right\rangle = \begin{cases} a = \vartheta_1^2 & \text{if } k \neq i, \\ \left\langle v_i^4 \mathcal{M} \right\rangle = 3\vartheta_1^2 & \text{if } k = i, \end{cases}$$

so that,

$$\sum_{k,\ell} u_k u_\ell \left\langle v_k v_\ell \mathbf{A}^{i,i} \mathcal{M} \right\rangle = \vartheta_1^2 \sum_{k \neq i} u_k^2 + 3\vartheta_1^2 u_i^2 - \frac{d+2}{d} |u|^2 \vartheta_1^2 = 2\vartheta_1^2 u_i^2 - \frac{2}{d} \vartheta_1^2 |u|^2.$$

Gathering these last computations, we get

$$\left\langle \phi^{i,j} \mathcal{Q}_1((u \cdot v) \mathcal{M}, (u \cdot v) \mathcal{M}) \right\rangle = \vartheta_1^2 \left(u_i u_j - \frac{2}{d} |u|^2 \delta_{i,j} \right),$$

which, combined with (C.5) gives the result. \square

Lemma C.4. *Let \mathbf{h} be given by (6.7). For any $i, j = 1, \dots, d$ it holds that*

$$\left\langle v_\ell \phi^{i,j} \mathbf{h} \right\rangle = \begin{cases} \nu u_j & \text{if } i \neq j, \ell = i, \\ \nu u_i & \text{if } i \neq j, \ell = j, \\ -\frac{2}{d} \nu u_\ell + 2\nu u_i \delta_{i\ell} & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Proof. Using the fact that χ_1 is radial, similar computations to that of Lemma C.3 imply that for $\ell \in \{1, \dots, d\}$,

$$\begin{aligned} \langle v_\ell \phi^{i,j} \mathbf{h} \rangle &= \sum_{k=1}^d u_k \langle v_\ell v_k \phi^{i,j} \mathcal{M} \rangle = \sum_{k=1}^d u_k \langle v_\ell v_k \phi^{i,j} \mathcal{M} \rangle \\ &= \sum_{k=1}^d u_k \left(\langle \mathbf{A}^{k,\ell} \phi^{i,j} \mathcal{M} \rangle + \frac{1}{d} \langle |v|^2 \phi^{i,j} \mathcal{M} \rangle \delta_{k\ell} \right) \\ &= - \sum_{k=1}^d u_k \langle \phi^{i,j} \mathbf{L}_1(\phi^{k,\ell} \mathcal{M}) \rangle, \end{aligned}$$

where we used that $\mathbf{L}_1(\phi \mathcal{M}) = -\mathbf{A} \mathcal{M}$ and $\langle |v|^2 \phi^{i,j} \mathcal{M} \rangle = 0$. This gives the result thanks to (C.2). \square

Lemma C.5. *Let \mathbf{h} be given by (6.7). For any $i = 1, \dots, d$ it holds that*

$$\langle \psi_i \mathcal{Q}_1(\mathbf{h}, \mathbf{h}) \rangle = \frac{d+2}{2} \vartheta_1^3 (\theta u_i),$$

and, if ϱ and θ satisfies Boussinesq relation (6.14), then

$$\operatorname{div}_x \langle \psi_i \mathbf{h} v \rangle = \gamma \frac{d+2}{2} \partial_{x_i} \theta.$$

Proof. On the one hand, using (C.4) it holds that

$$\langle \psi_i \mathcal{Q}_1(\mathbf{h}, \mathbf{h}) \rangle = \theta u \cdot \langle \psi_i \mathbf{L}_1(\frac{1}{2}|v|^2 v \mathcal{M}) \rangle = \theta u \cdot \langle \psi_i \mathbf{L}_1(\mathbf{b} \mathcal{M}) \rangle,$$

since, ψ_i being odd, one has $\langle \psi_i \mathbf{L}_1((u \cdot v)^2 \mathcal{M}) \rangle = \langle \psi_i \mathbf{L}_1(|v|^4 \mathcal{M}) \rangle = 0$. Now,

$$\langle \psi_i \mathbf{L}_1(\mathbf{b} \mathcal{M}) \rangle = \langle \mathbf{b} \mathbf{L}_1(\psi_i \mathcal{M}) \rangle = - \langle \mathbf{b} \mathcal{M} \mathbf{b}_i \rangle,$$

and a direct computations show that

$$\langle \mathbf{b}_j \mathbf{b}_i \mathcal{M} \rangle = -\frac{1}{4d} \langle (|v|^2 - (d+2)\vartheta_1)^2 |v|^2 \mathcal{M} \rangle \delta_{ij} = -\frac{d+2}{2} \vartheta_1^3 \delta_{ij},$$

which gives the expression for $\langle \psi_i \mathcal{Q}_1(\mathbf{h}, \mathbf{h}) \rangle$. On the other hand, using symmetry properties, one checks that

$$\langle \psi_i \mathbf{h} v_\ell \rangle = \varrho \langle \psi_i v_i \mathcal{M} \rangle \delta_{i\ell} + \frac{1}{2} \theta \langle \psi_i (|v|^2 - d\vartheta_1) v_i \mathcal{M} \rangle \delta_{i\ell},$$

from which

$$\operatorname{div}_x \langle \psi_i \mathbf{h} v \rangle = \langle \psi_i v_i \mathcal{M} \rangle \partial_{x_i} \varrho + \frac{1}{2} \langle \psi_i (|v|^2 - d\vartheta_1) v_i \mathcal{M} \rangle \partial_{x_i} \theta.$$

Writing $\frac{1}{2}\langle \psi_i(|v|^2 - d\vartheta_1)v_i \mathcal{M} \rangle = \langle \psi_i \mathbf{b}_i \mathcal{M} \rangle + \vartheta_1 \langle \psi_i v_i \mathcal{M} \rangle$ and using Boussinesq relation (6.14), one gets that

$$\operatorname{div}_x \langle \psi_i \mathbf{h} v \rangle = \langle \psi_i \mathbf{b}_i \mathcal{M} \rangle \partial_{x_i} \theta = \gamma \frac{d+2}{2} \partial_{x_i} \theta,$$

where the identity $\langle \psi_i \mathbf{b}_i \mathcal{M} \rangle = -\langle \psi_i \mathbf{L}_1(\psi_i \mathcal{M}) \rangle$ was used together with (C.2). \square

In Lemma 6.11, we study the convergence of some term involving the source term \mathcal{S}_ε defined in (6.29). To do that, we use the next Lemma which provides a strong convergence to 0 of this source term.

Lemma C.6. *Let \mathcal{S}_ε defined in (6.29). We have that*

$$\|\mathcal{S}_\varepsilon\|_{L^1((0,T);L^1_v L^2_x(\varpi_{q-1}))} \lesssim \varepsilon.$$

Proof. We decompose \mathcal{S}_ε into three parts using the splitting $h_\varepsilon = h_\varepsilon^0 + h_\varepsilon^1$; namely, $\mathcal{S}_\varepsilon = \mathcal{S}_\varepsilon^0 + \mathcal{S}_\varepsilon^1 + \mathcal{S}_\varepsilon^2$ with

$$\mathcal{S}_\varepsilon^j := \varepsilon^{-1} (\mathbf{L}_\alpha h_\varepsilon^j - \mathbf{L}_1 h_\varepsilon^j) + \mathcal{Q}_\alpha(h_\varepsilon^j, h_\varepsilon^j) - \mathcal{Q}_1(h_\varepsilon^j, h_\varepsilon^j) - \varepsilon^{-1} \kappa_\alpha \nabla_v \cdot (v h_\varepsilon^j), \quad j = 0, 1,$$

and

$$\mathcal{S}_\varepsilon^2 := 2\tilde{\mathcal{Q}}_\alpha(h_\varepsilon^0, h_\varepsilon^1) - 2\tilde{\mathcal{Q}}_1(h_\varepsilon^0, h_\varepsilon^1).$$

The terms $\mathcal{S}_\varepsilon^0$ and $\mathcal{S}_\varepsilon^2$ are treated using the estimate on h_ε^0 and h_ε^1 stated in (6.8)-(6.9). Indeed, using standard estimates on \mathcal{Q}_α and \mathcal{Q}_1 and Proposition 2.4, we have:

$$\begin{aligned} \|\mathcal{S}_\varepsilon^0 + \mathcal{S}_\varepsilon^2\|_{L^1((0,T);L^1_v L^2_x(\varpi_{q-1}))} &\lesssim \varepsilon \|h_\varepsilon^0\|_{L^1((0,T);\mathcal{E}_1)} \\ &+ \|h_\varepsilon^0\|_{L^1((0,T);\mathcal{E}_1)} (\|h_\varepsilon^0\|_{L^\infty((0,T);\mathcal{E})} + \|h_\varepsilon^1\|_{L^\infty((0,T);\mathcal{H})}) + \varepsilon \|h_\varepsilon^0\|_{L^1((0,T);\mathcal{E}_1)} \lesssim \varepsilon^2. \end{aligned}$$

Using now (6.8), Proposition 2.4 and Remark 3.5, we have:

$$\begin{aligned} \|\mathcal{S}_\varepsilon^1\|_{L^1((0,T);L^1_v L^2_x(\varpi_{q-1}))} &\lesssim \varepsilon \|h_\varepsilon^1\|_{L^1((0,T);\mathcal{H})} \\ &+ \varepsilon^2 \|h_\varepsilon^1\|_{L^1((0,T);\mathcal{H})} \|h_\varepsilon^1\|_{L^\infty((0,T);\mathcal{H})} + \varepsilon \|h_\varepsilon^1\|_{L^1((0,T);\mathcal{H})} \lesssim \varepsilon, \end{aligned}$$

which yields the result. \square

To handle the convergence of nonlinear terms, we will need to resort to the following compensated compactness result extracted from Lions & Masmoudi (1999) (see also (Golse & Saint-Raymond, 2004, Lemma 13.1, Appendix D). The original result in Lions & Masmoudi (1999) is proven in the whole space but is easily adapted to the case of the torus.

Proposition C.7. *Let $c \neq 0$ and $T > 0$. Consider two families $\{\phi_\varepsilon\}_\varepsilon$ and $\{\psi_\varepsilon\}$ bounded in $L^\infty((0, T); L^2_x(\mathbb{T}^d))$ and in $L^\infty((0, T); \mathbb{W}_x^{1,2}(\mathbb{T}^d))$ respectively, such that*

$$\begin{cases} \partial_t \nabla_x \psi_\varepsilon + \frac{c^2}{\varepsilon} \nabla_x \phi_\varepsilon = \frac{1}{\varepsilon} F_\varepsilon \\ \partial_t \phi_\varepsilon + \frac{1}{\varepsilon} \Delta_x \psi_\varepsilon = \frac{1}{\varepsilon} G_\varepsilon \end{cases}$$

where F_ε and G_ε converge strongly in $L^1((0, T); L_x^2(\mathbb{T}^d))$. Then,

$$\mathcal{P}\text{Div}_x (\nabla_x \psi_\varepsilon \otimes \nabla_x \psi_\varepsilon) \longrightarrow 0, \quad \text{div}_x (\phi_\varepsilon \nabla_x \psi_\varepsilon) \longrightarrow 0$$

in the sense of distributions on $(0, T) \times \mathbb{T}^d$.

APPENDIX D. PROOF OF THEOREM 2.12 AND PROPOSITION 2.11

Theorem D.1 (See Theorem 2.1, Briant et al. (2019)). *Let $j = 1, 2$. There exists $\varepsilon_0 \in (0, 1)$ such that, for all $\ell, s \in \mathbb{N}$ with $\ell \geq s$ and $q > q_j$ (where q_j is defined in (2.17)) and any $\varepsilon \in (0, \varepsilon_0)$, the full transport operator $\mathcal{G}_{1,\varepsilon}$ generates a C_0 -semigroup $\{\mathcal{V}_{1,\varepsilon}(t); t \geq 0\}$ on $\mathbb{W}_v^{s,j} \mathbb{W}_x^{\ell,2}(\varpi_{q_j})$ such that, for all $t_\star > 0$ there exist $C_0(t_\star), \mu_\star > 0$ satisfying*

$$\begin{aligned} & \|\mathcal{V}_{1,\varepsilon}(t)h - \mathbf{P}_0 h\|_{\mathbb{W}_v^{s,j} \mathbb{W}_x^{\ell,2}(\varpi_q)} \\ & \leq C_0(t_\star) \exp(-\mu_\star t) \|h - \mathbf{P}_0 h\|_{\mathbb{W}_v^{s,j} \mathbb{W}_x^{\ell,2}(\varpi_q)}, \quad \forall t > t_\star, j = 1, 2 \end{aligned} \quad (\text{D.1})$$

holds true for any $h_0 \in \mathbb{W}_v^{s,j} \mathbb{W}_x^{\ell,2}(\varpi_q)$, $j = 1, 2$ where \mathbf{P}_0 is the spectral projection onto $\text{Ker}(\mathcal{G}_{1,\varepsilon}) = \text{Ker}(\mathcal{L}_1)$ which is independent of ε and given by (2.28).

Note that a similar estimate also holds on the space $\mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,1}(\varpi_q)$. The difference between Theorems 2.12 and D.1 lies in the fact that, in Theorem 2.12, we allow $t_\star = 0$ in the decay estimate (D.1). The ‘‘initial layer’’ dependence on $t_\star > 0$ in (D.1) is inherent to the method of the enlargement semigroup theory of Gualdani et al. (2017) (see Remarks 2.13 and 2.14).

Theorem D.1 ensures that $\mathcal{G}_{1,\varepsilon}$ is the generator of a C_0 -semigroup $\{\mathcal{V}_{1,\varepsilon}(t); t \geq 0\}$ on \mathcal{E} as soon as $q > \frac{9}{2}$ and $\varepsilon \in (0, \varepsilon_0)$. We focus on extending (D.1) to $t_\star := 0$.

Proof of Theorem 2.12. We adopt the decomposition of the nonlinear part of Briant et al. (2019) that we used in Section 4. Namely, for some fixed $h \in \mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q)$ we set

$$f_{\text{in}} := h - \mathbf{P}_0 h,$$

and write $f(t) = \mathcal{V}_{1,\varepsilon}(t)f_{\text{in}}$ as $f(t) = f^0(t) + f^1(t)$ with $f^0 \in \mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q)$ solution to

$$\partial_t f^0(t) = \mathcal{B}_{1,\varepsilon} f^0, \quad f^0(0) = f_{\text{in}}, \quad (\text{D.2})$$

whereas $f^1 \in \mathcal{H} := \mathbb{W}_{v,x}^{\ell,2}(\mathcal{M}^{-\frac{1}{2}})$, is solution to

$$\partial_t f^1(t) = \mathcal{G}_{1,\varepsilon} f^1(t) + \mathcal{A}_\varepsilon f^0(t), \quad f^1(0) = 0. \quad (\text{D.3})$$

As before, the same notations for the operators $\mathcal{G}_{1,\varepsilon}, \mathcal{V}_{1,\varepsilon}(t)$ acting on various different spaces is used. The definition should be clear from the context. Of course,

$$f^0(t) = \mathcal{S}_{1,\varepsilon}(t)f_{\text{in}},$$

and

$$\|f^0(t)\|_{\mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q)} \leq C_0 \exp(-\varepsilon^{-2} \nu_0 t) \|f_{\text{in}}\|_{\mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q)}, \quad (\text{D.4})$$

since $\mathcal{B}_{1,\varepsilon}$ is $\varepsilon^{-2}\nu_0$ hypo-dissipative (ν_0 depends on ℓ, m). The constant C_0 is independent of ε . Let us investigate $\|f^1(t)\|_{\mathcal{H}}$. Notice that, since $\mathbf{P}_0 f = 0$, $\mathbf{P}_0 f^1 = -\mathbf{P}_0 f^0$ (recall that the projection is the same in \mathcal{H} and $\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)$ and independent of ε), the estimate for $\mathbf{P}_0 f^1$ is thus straightforward

$$\|\mathbf{P}_0 f^1(t)\|_{\mathcal{H}} \leq C_1 \exp(-\varepsilon^{-2}\nu_0 t) \|f_{\text{in}}\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)}, \quad (\text{D.5})$$

where the constant C_1 differs from C_0 just because the norm of the eigenfunctions are different in \mathcal{H} and $\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)$. We now focus on

$$\psi(t) = \mathbf{P}_0^\perp f^1(t) = (\mathbf{Id} - \mathbf{P}_0) f^1(t).$$

One has

$$\partial_t \psi(t) = \mathcal{G}_{1,\varepsilon} \psi(t) + \mathbf{P}_0^\perp \mathcal{A}_\varepsilon f^0(t),$$

and, arguing as in (Briant, 2015, Section 7.2) (see also (Briant et al., 2019, Theorem 4.7 and Remark 4.8)), one has

$$\frac{1}{2} \|\psi(t)\|_{\mathcal{H}}^2 \leq \frac{1}{2} \|\psi(0)\|_{\mathcal{H}}^2 e^{-\mu_\star t} + \int_0^t e^{-\mu_\star(t-s)} \|\psi(s)\|_{\mathcal{H}} \|\mathbf{P}_0^\perp \mathcal{A}_\varepsilon f^0(s)\|_{\mathcal{H}} ds$$

with $\mu_\star > 0$ independent of ε which is the size of the spectral gap of $\mathcal{G}_{1,\varepsilon}$. Recalling that $\psi(0) = 0$ and $\|\mathcal{A}_\varepsilon\|_{\mathcal{B}(\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q), \mathcal{H})} \leq C_A \varepsilon^{-2}$, we get that

$$\|\psi(t)\|_{\mathcal{H}}^2 \leq \frac{2C_A}{\varepsilon^2} \int_0^t e^{-\mu_\star(t-s)} \|\psi(s)\|_{\mathcal{H}} \|f^0(s)\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)} ds.$$

We use (D.4) to deduce that

$$\|\psi(t)\|_{\mathcal{H}}^2 \leq \frac{2C_0 C_A}{\varepsilon^2} \int_0^t e^{-\mu_\star(t-s)} e^{-\frac{\nu_0}{\varepsilon^2} s} \|\psi(s)\|_{\mathcal{H}} \|f_{\text{in}}\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)} ds.$$

Then, Young's inequality leads to

$$\begin{aligned} \|\psi(t)\|_{\mathcal{H}}^2 &\leq \frac{C_0 C_A e^{-\mu_\star t}}{\varepsilon^2} \int_0^t e^{-(\frac{\nu_0}{\varepsilon^2} - \mu_\star)s} \|\psi(s)\|_{\mathcal{H}}^2 ds \\ &\quad + \frac{C_0 C_A e^{-\mu_\star t}}{\varepsilon^2} \|f_{\text{in}}\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)}^2 \int_0^t e^{-(\frac{\nu_0}{\varepsilon^2} - \mu_\star)s} ds. \end{aligned}$$

If $\varepsilon^{-2}\nu_0 > 2\mu_\star$ we get after integration that

$$\|\psi(t)\|_{\mathcal{H}}^2 \leq \frac{2C_0 C_A e^{-\mu_\star t}}{\nu_0} \|f_{\text{in}}\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)}^2 + \frac{C_0 C_A e^{-\mu_\star t}}{\varepsilon^2} \int_0^t e^{-(\frac{\nu_0}{\varepsilon^2} - \mu_\star)s} \|\psi(s)\|_{\mathcal{H}}^2 ds.$$

With $x(t) = e^{\mu_\star t} \|\psi(t)\|_{\mathcal{H}}^2$ it follows that

$$x(t) \leq \frac{2C_0 C_A}{\nu_0} \|f_{\text{in}}\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)}^2 + \frac{C_0 C_A}{\varepsilon^2} \int_0^t e^{-\frac{\nu_0}{\varepsilon^2} s} x(s) ds,$$

and Gronwall's lemma gives

$$x(t) \leq \frac{2C_0C_A}{\nu_0} \|f_{\text{in}}\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)}^2 \exp\left(\frac{C_0C_A}{\nu_0}\right) =: C_2 \|f_{\text{in}}\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)}^2,$$

with $C_2 > 0$ independent of ε . Therefore

$$\|\psi(t)\|_{\mathcal{H}}^2 \leq C_2 \|f_{\text{in}}\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)}^2 \exp(-\mu_\star t).$$

This combined with (D.5) gives that

$$\|f^1(t)\|_{\mathcal{H}}^2 \leq (C_2 + C_1) \|f_{\text{in}}\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)}^2 \exp(-\mu_\star t).$$

Overall, the estimates for f^0 and f^1 lead to

$$\|f(t)\|_{\mathcal{E}} \leq C_3 \|f_{\text{in}}\|_{\mathbb{W}_v^{s,2}\mathbb{W}_x^{\ell,2}(\varpi_q)} \exp\left(-\frac{\mu_\star}{2}t\right), \quad \forall t \geq 0,$$

with C_3 independent of ε and given by $C_0 + \sqrt{C_1 + C_2}$ as long as $\nu_0\varepsilon^{-2} > \mu_\star$. \square

Proof of Proposition 2.11. We aim to prove here that, on the Banach space $\mathbb{W}_v^{s,j}\mathbb{W}_x^{\ell,2}(\varpi_q)$, $j = 1, 2$, the operator

$$\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} = \mathcal{B}_{1,\varepsilon}^{(\delta)} + \varepsilon^{-2}\mathcal{P}_\alpha + \varepsilon^{-2}T_\alpha$$

with domain $\mathcal{D}(\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}) = \mathbb{W}_v^{s+1,j}\mathbb{W}_x^{\ell+1,2}(\varpi_{q+1})$ generates a C_0 -semigroup. Since we will resort to an approach introduced in [Alonso et al. \(2020\)](#) and some computations made earlier in [Canizo & Lods \(2016\)](#), it will be more convenient to prove that $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ is the generator of a C_0 -semigroup on the space

$$\mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\overline{m}_{a,\beta})$$

where $\overline{m}_{a,\beta}$ is the exponential weight

$$\overline{m}_{a,\beta}(v) = \exp\left(a\langle v \rangle^\beta\right), \quad v \in \mathbb{R}^d, \quad a > 0, \beta \in (0, 1)$$

and then use some enlargement result ([Gualdani et al., 2017](#), Theorem 2.13). It is easy to adapt the proof of Proposition 2.9 and find an equivalent norm on $\mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\overline{m}_{a,\beta})$ for which $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2}\nu_{\ell,s}$ is dissipative (see [Tristani \(2016\)](#) for useful computations in spaces with exponential weights whenever $\varepsilon = 1$). According to Lumer-Phillips Theorem, see ([Engel & Nagel, 1999](#), Proposition 3.14 & Theorem 3.15), in order to show that $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ generates a C_0 -semigroup it suffices to prove that there exists $\lambda > 0$ large enough such that

$$\text{Range}(\lambda - \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}) = \mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\overline{m}_{a,\beta}). \quad (\text{D.6})$$

Clearly, one can replace without loss of generality $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ with $\varepsilon^2\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$. Denote for simplicity

$$X = \mathbb{W}_v^{s,1}\mathbb{W}_x^{\ell,2}(\mu_a), \quad B_\alpha := \varepsilon^2\mathcal{B}_{\alpha,\varepsilon}^{(\delta)},$$

omitting the dependence with respect to ε and δ . It follows that

$$B_\alpha = \mathcal{L}_1^{R,\delta} - \Sigma_{\mathcal{M}} - \varepsilon v \cdot \nabla_x + \mathcal{P}_\alpha + T_\alpha.$$

Introduce the following operator

$$\mathcal{T}_\alpha h := -\varepsilon v \cdot \nabla_x h + T_\alpha h - \Sigma_{\mathcal{M}} h = -\varepsilon v \cdot \nabla_x h - \kappa_\alpha \operatorname{div}_v(vh) - \Sigma_{\mathcal{M}} h$$

with domain $\mathcal{D}(\mathcal{T}_\alpha) = \mathbb{W}_v^{s+1,1} \mathbb{W}_x^{\ell+1,2}(\langle \cdot \rangle \overline{m}_{a,\beta})$. It is not difficult to check that \mathcal{T}_α generates a C_0 -semigroup in X given by

$$e^{t\mathcal{T}_\alpha} g(x, v) = \exp\left(-\int_0^t d\kappa_\alpha + \Sigma_{\mathcal{M}}(ve^{\kappa_\alpha(s-t)}) ds\right) g\left(x - \frac{\varepsilon}{\kappa_\alpha} (1 - e^{-\kappa_\alpha t}) v, ve^{-\kappa_\alpha t}\right).$$

In particular,

$$\lim_{\lambda \rightarrow \infty} \|\mathcal{R}(\lambda, \mathcal{T}_\alpha)\|_{\mathcal{B}(X)} = 0. \quad (\text{D.7})$$

Moreover, one has the following gain of integrability for the resolvent of \mathcal{T}_α : there is $\alpha_1 \in (0, 1)$ such that, for $\alpha \in (\alpha_1, 1)$ there is $c > 0$ and $\lambda(\alpha) > 0$

$$\|\mathcal{R}(\lambda, \mathcal{T}_\alpha)\|_{\mathcal{B}(\mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,2}(\overline{m}_{a,\beta}), \mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,2}(\langle \cdot \rangle \overline{m}_{a,\beta}))} \leq \frac{1}{\sigma - c\kappa_\alpha}, \quad \forall \lambda > \lambda(\alpha), \quad (\text{D.8})$$

where σ is an explicit positive constant depending only on $\Sigma_{\mathcal{M}}$. The proof of such a property is an easy adaptation of Lemma C.14 in [Alonso et al. \(2020\)](#) whenever $k = s = 0$ and extends to $k \geq s \geq 0$ following techniques from [Mouhot & Neumann \(2006\)](#), we leave the details to the reader. One also has the following result, see the proof of [\(Canizo & Lods, 2016, Lemma B.1 & Proposition B.2\)](#): there exists $\tau(\delta) > 0$ such that $\lim_{\delta \rightarrow 0} \tau(\delta) = 0$ and

$$\left\| \mathcal{L}_{1,+}^{R,\delta} \right\|_{\mathcal{B}(\mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,2}(\langle \cdot \rangle \overline{m}_{a,\beta}), \mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,2}(\overline{m}_{a,\beta}))} \leq \tau(\delta), \quad (\text{D.9})$$

while $\mathcal{L}_{1,-}^{R,\delta} \in \mathcal{B}(X)$. With these two properties, introduce the sum $\mathcal{C}_\alpha := \mathcal{L}_{1,+}^{R,\delta} + \mathcal{T}_\alpha$ with domain $\mathcal{D}(\mathcal{C}_\alpha) = \mathcal{D}(\mathcal{T}_\alpha)$. We have directly from the previous two properties [\(D.8\)](#) and [\(D.9\)](#)

$$\left\| \mathcal{L}_{1,+}^{R,\delta} \mathcal{R}(\lambda, \mathcal{T}_\alpha) \right\|_{\mathcal{B}(X)} \leq \frac{\tau(\delta)}{\sigma - c\kappa_\alpha}, \quad \forall \lambda > \lambda(\alpha),$$

from which, choosing $\delta > 0$ sufficiently small such that $\frac{\tau(\delta)}{\sigma - c\kappa_\alpha} < 1$, we obtain that $(\mathbf{Id} - \mathcal{L}_{1,+}^{R,\delta} \mathcal{R}(\lambda, \mathcal{T}_\alpha))$ is invertible. We deduce that

$$\begin{aligned} \mathcal{R}(\lambda, \mathcal{C}_\alpha) &= \mathcal{R}(\lambda, \mathcal{T}_\alpha) \left(\mathbf{Id} - \mathcal{L}_{1,+}^{R,\delta} \mathcal{R}(\lambda, \mathcal{C}_\alpha) \right)^{-1} \\ &= \mathcal{R}(\lambda, \mathcal{T}_\alpha) \sum_{n=0}^{\infty} \left[\mathcal{L}_{1,+}^{R,\delta} \mathcal{R}(\lambda, \mathcal{T}_\alpha) \right]^n, \quad \forall \lambda > \lambda(\alpha), \end{aligned}$$

simply observing that $(\lambda - \mathcal{C}_\alpha) = (\mathbf{Id} - \mathcal{L}_{1,+}^{R,\delta} \mathcal{R}(\lambda, \mathcal{C}_\alpha))(\lambda - \mathcal{T}_\alpha)$. In particular,

$$\|\mathcal{R}(\lambda, \mathcal{C}_\alpha)\|_{\mathcal{B}(X)} \leq \frac{1}{\sigma - c\kappa_\alpha - \tau(\delta)}, \quad \forall \lambda > \lambda(\alpha),$$

with

$$\lim_{\lambda \rightarrow \infty} \|\mathcal{R}(\lambda, \mathcal{C}_\alpha)\|_{\mathcal{B}(X)} = 0$$

by virtue of (D.7). Set then

$$\mathcal{C}_\alpha^1 := \mathcal{C}_\alpha + \mathcal{P}_\alpha.$$

With the estimate of \mathcal{P}_α

$$\|\mathcal{P}_\alpha \mathcal{R}(\lambda, \mathcal{C}_\alpha)\|_{\mathcal{B}(X)} \leq C \frac{1 - \alpha}{\sigma - c\kappa_\alpha - \tau(\delta)}$$

and, choosing α sufficiently close to 1, the operator $\mathbf{Id} + \mathcal{P}_\alpha \mathcal{R}(\lambda, \mathcal{C}_\alpha)$ is invertible and so is $\lambda - \mathcal{C}_\alpha^1$. Finally, since

$$B_\alpha = \mathcal{C}_\alpha^1 - \mathcal{L}_{1,-}^{R,\delta},$$

one can chose $\lambda > 0$ sufficiently large so that

$$\|\mathcal{L}_{1,-}^{R,\delta} \mathcal{R}(\lambda, \mathcal{C}_\alpha^1)\|_{\mathcal{B}(X)} \leq \|\mathcal{L}_{1,-}^{R,\delta}\|_{\mathcal{B}(X)} \|\mathcal{R}(\lambda, \mathcal{C}_\alpha^1)\|_{\mathcal{B}(X)} < 1$$

and obtain that $\lambda - B_\alpha$ is invertible. In particular, (D.6) holds true and this proves the result on the space X . As said before, we deduce that $\mathcal{B}_{\alpha,\varepsilon}$ still generates a C_0 -semigroup on the larger spaces $\mathbb{W}_v^{s,1} \mathbb{W}_x^{\ell,2}(\varpi_q)$ and $\mathbb{W}_v^{s,2} \mathbb{W}_x^{\ell,2}(\varpi_q)$ under the assumptions thanks to Theorem 2.13 and Remark 2.14 (i) of [Gualdani et al. \(2017\)](#). We leave the technical details to the reader. \square

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