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FLUID DYNAMIC LIMIT OF BOLTZMANN EQUATION FOR GRANULAR HARD-SPHERES IN A NEARLY ELASTIC REGIME

RICARDO J. ALONSO, BERTRAND LODS, AND ISABELLE TRISTANI

ABSTRACT. In this paper, we provide the first rigorous derivation of hydrodynamic equations from the Boltzmann equation for inelastic hard spheres with small inelasticity. The hydrodynamic system that we obtain is an incompressible Navier-Stokes-Fourier system with self-consistent forcing terms and, to our knowledge, it is thus the first hydrodynamic system that properly describes rapid granular flows consistent with the kinetic formulation. To this end, we write our Boltzmann equation in a non dimensional form using the dimensionless Knudsen number which is intended to be sent to 0. There are several difficulties in such derivation, the first one coming from the fact that the original Boltzmann equation is free-cooling and, thus, requires a self-similar change of variables to introduce an homogeneous steady state. Such a homogeneous state is not explicit and is heavy-tailed, which is a major obstacle to adapting energy estimates and spectral analysis. Additionally, a central challenge is to understand the relation between the restitution coefficient, which quantifies the energy loss at the microscopic level, and the Knudsen number. This is achieved by identifying the correct *nearly elastic regime* to capture nontrivial hydrodynamic behavior. We are, then, able to prove exponential stability uniformly with respect to the Knudsen number for solutions of the rescaled Boltzmann equation in a close to equilibrium regime. Finally, we prove that solutions to the Boltzmann equation converge in a specific weak sense towards a hydrodynamic limit which depends on time and space variables only through macroscopic quantities. Such macroscopic quantities are solutions to a suitable modification of the incompressible Navier-Stokes-Fourier system which appears to be new in this context.

Mathematics Subject Classification (2010): 76P05 Rarefied gas flows, Boltzmann equation [See also 82B40, 82C40, 82D05]; 76T25 Granular flows [See also 74C99, 74E20]; 47H20 Semigroups of nonlinear operators [See also 37L05, 47J35, 54H15, 58D07], 35Q35 PDEs in connection with fluid mechanics; 35Q30 Navier-Stokes equations [See also 76D05, 76D07, 76N10].

Keywords: Inelastic Boltzmann equation; Granular flows; Nearly elastic regime; Long-time asymptotic; Incompressible Navier-Stokes hydrodynamical limit; Knudsen number.

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1. INTRODUCTION

1.1. Multiscale descriptions of granular gases. Granular materials are ubiquitous in nature and understanding the behaviour of granular matter is a relevant challenge from both the physics and mathematics viewpoints. Various descriptions of granular matter have been proposed in the literature [28]. An especially relevant one consists in viewing granular systems as clusters of a large number of discrete macroscopic particles (with size exceeding $1\ \mu\text{m}$, significantly larger than the one of a typical particle described in classical kinetic theory) suffering dissipative interactions. One speaks then of *rapid granular flows* or *gaseous granular matter*. If the number of particles is large enough, it is then common to adopt a kinetic modelling based upon suitable modification of the Boltzmann equation. As usual in kinetic theory, it is then particularly relevant to deduce from this kinetic description the fluid behaviour of the system. This means, roughly speaking, that we look at the granular gas at a scale larger than the mesoscopic and aim to capture the hydrodynamical features of it through the evolution of macroscopic quantities like *density*, *bulk velocity* and *temperature* of the gas which satisfy suitable hydrodynamics equations.

One of the main objects of the present work is to make a first rigorous link between these two co-existing descriptions by deriving a suitable modification of incompressible Navier-Stokes equation from the Boltzmann equation for inelastic hard-spheres as the Knudsen number goes to zero.

Recall that the Knudsen number ε is proportional to the mean free path between collisions and in order to derive hydrodynamic equations from the Boltzmann equation, the usual strategy consists, roughly speaking, in performing a perturbation analysis in the limit $\varepsilon \rightarrow 0$ (meaning that the mean free path is negligible when compared to the typical physical scale length). We point out that these questions are perfectly understood in the elastic case (*molecular gases*) for which rigorous results on the hydrodynamic limits of the Boltzmann equation have been obtained, we refer to the next Section 1.6 for more details.

The picture in the context of granular gases is quite different. In fact, a satisfying hydrodynamic equation that properly describes rapid granular flow is still a controversial issue among the physics community. The continuous loss of kinetic energy makes granular gases an *open system* as far as thermodynamics is concerned. Moreover, no non-trivial steady states exist in granular gases without an external energy supply which makes granular gases a prototype of *non-equilibrium* systems. This is an important obstacle in the derivation of hydrodynamical equations from the kinetic description since it is expected that *equilibrium states* play the role of the typical *hydrodynamic solution* where time-space dependence of the single-particle distribution function $F(t, x, v)$ occurs only through suitable hydrodynamic fields like density $\varrho(t, x)$, bulk velocity $u(t, x)$, and temperature $\theta(t, x)$. An additional difficulty is related to the size of particles and scale separation. Recall that granular gases involve *macroscopic particles* whose size is much larger than the ones described by the usual Boltzmann equation with elastic interactions referred to as *molecular gases*. As the hydrodynamic description occurs on large time scales (compared to the mean free time) and on large spatial scales (compared to the mean free path) the mesoscopic –

continuum scale separation is problematic to justify in full generality for granular gases. We refer to [28, Section 3.1, p. 102] for more details on this point and observe here that the main concern is related to the time scale induced by the evolution of the temperature (see (1.9) hereafter). In particular, as observed in [28], this problem can only be answered with a fine spectral analysis of the linearized Boltzmann equation that ensures that the $d+2$ hydrodynamic modes associated to density, velocity and temperature decay more slowly than the remaining kinetic excitations at large times. This is the only way that the hydrodynamic excitations emerge as the dominant dynamics. All these physically grounded obstacles make the derivation of hydrodynamic equations from the Boltzmann equation associated to granular gases a reputedly challenging open problem. Quoting [13]:

“the context of the hydrodynamic equations remains uncertain. What are the relevant space and time scales? How much inelasticity can be described in this way?”

The present paper is, to the best of our knowledge, the first rigorous answer to these relevant problems, at least in dimension $d \geq 2$. We already mentioned that the key point in our analysis is to identify the correct regime which allows to answer these questions: *nearly elastic*. In this regime the energy dissipation rate in the systems happens in a controlled fashion since the inelasticity parameter is compensated accordingly to the number of collisions per time unit. This process mimics viscoelasticity as particle collisions become more elastic as the collision dissipation mechanism increases in the limit $\varepsilon \rightarrow 0$ (see Assumption 1.2 below). In this way, we are able to consider a re-scaling of the kinetic equation in which a peculiar intermediate asymptotic emerge and prevent the total cooling of the granular gas.

Other regimes can be considered depending on the rate that the kinetic energy is dissipated, for example, an interesting regime is the *mono-kinetic* which considers the extreme case of infinite energy dissipation rate. In this way, the limit is formally described by plugging in a Dirac mass solution in the kinetic equation yielding the *pressureless Euler system* (corresponding to sticky particles). Such a regime has been rigorously addressed in the one-dimensional framework in the interesting contribution [39]. It is an open question to extend such analysis to higher dimensions since the approach of [39] uses the so-called Bony functional which is a tool specifically tailored for $1D$ kinetic equations.

1.2. The Boltzmann equation for granular gases. We consider here the (freely cooling) Boltzmann equation which provides a statistical description of identical smooth hard spheres suffering binary and *inelastic collisions*:

$$\partial_t F(t, x, v) + v \cdot \nabla_x F(t, x, v) = \mathcal{Q}_\alpha(F, F) \quad (1.1)$$

supplemented with initial condition $F(0, x, v) = F_{\text{in}}(x, v)$, where $F(t, x, v)$ is the density of granular gases having position $x \in \mathbb{T}^d$ and velocity $v \in \mathbb{R}^d$ at time $t \geq 0$. We consider here for simplicity the case of *flat torus*

$$\mathbb{T}_\ell^d = \mathbb{R}^d / (2\pi \ell \mathbb{Z})^d \quad (1.2)$$

for some typical length-scale $\ell > 0$. This corresponds to periodic boundary conditions:

$$F(t, x + 2\pi \ell e_i, v) = F(t, x, v) \quad i = 1, \dots, d$$

where e_i is the i -th vector of the canonical basis of \mathbb{R}^d . The collision operator \mathcal{Q}_α is defined in weak form as

$$\int_{\mathbb{R}^d} \mathcal{Q}_\alpha(g, f)(v) \psi(v) dv = \frac{1}{2} \int_{\mathbb{R}^{2d}} f(v) g(v_*) |v - v_*| \mathcal{A}_\alpha[\psi](v, v_*) dv_* dv, \quad (1.3)$$

where

$$\mathcal{A}_\alpha[\psi](v, v_*) = \int_{\mathbb{S}^{d-1}} (\psi(v') + \psi(v'_*) - \psi(v) - \psi(v_*)) b(\sigma \cdot \bar{q}) d\sigma, \quad (1.4)$$

and the post-collisional velocities (v', v'_*) are given by

$$\begin{aligned} v' &= v + \frac{1+\alpha}{4} (|q|\sigma - q), & v'_* &= v_* - \frac{1+\alpha}{4} (|q|\sigma - q), \\ \text{where } q &= v - v_*, & \bar{q} &= q/|q|. \end{aligned} \quad (1.5)$$

Here, $d\sigma$ denotes the Lebesgue measure on \mathbb{S}^{d-1} and the angular part $b = b(\cos \theta)$ of the collision kernel appearing in (1.4) is a non-measurable mapping integrable over \mathbb{S}^{d-1} . There is no loss of generality assuming

$$\int_{\mathbb{S}^{d-1}} b(\sigma \cdot \bar{q}) d\sigma = 1, \quad \forall \bar{q} \in \mathbb{S}^{d-1}.$$

The fundamental distinction between the classical elastic Boltzmann equation and the associated to granular gases lies in the role of the parameter $\alpha \in (0, 1)$, the coefficient of restitution. This coefficient is given by the ratio between the magnitude of the normal component (along the line of separation between the centers of the two spheres at contact) of the relative velocity after and before the collision (see Appendix A.1 for the detailed microscopic velocities). The case $\alpha = 1$ corresponds to perfectly elastic collisions where kinetic energy is conserved. However, when $\alpha < 1$, part of the kinetic energy of the relative motion is lost since

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\frac{1-\alpha^2}{4} |q|^2 (1 - \sigma \cdot \bar{q}) \leq 0.$$

It is assumed in this work that α is independent of the relative velocity q (refer to [1], [6], and [7] for the viscoelastic restitution coefficient case). Notice that the microscopic description (1.5) preserves the momentum

$$v' + v'_* = v + v_*$$

and, taking $\psi = 1$ and then $\psi(v) = v$ in (1.3) yields to the following conservation of macroscopic density and bulk velocity

$$\frac{d}{dt} \mathbf{R}(t) = \frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{T}_\ell^d} F(t, x, v) dv dx = 0, \quad \frac{d}{dt} \mathbf{U}(t) = \frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{T}_\ell^d} v F(t, x, v) dv dx = 0.$$

Consequently, there is no loss of generality in assuming that

$$\mathbf{R}(t) = \mathbf{R}(0) = 1, \quad \mathbf{U}(t) = \mathbf{U}(0) = 0 \quad \forall t \geq 0.$$

As mentioned, the main contrast between elastic and inelastic gases is that in the latter the *granular temperature*

$$\mathbf{T}(t) := \frac{1}{|\mathbb{T}_\ell^d|} \int_{\mathbb{R}^d \times \mathbb{T}_\ell^d} |v|^2 F(t, x, v) dv dx$$

is constantly decreasing

$$\frac{d}{dt} \mathbf{T}(t) = -(1 - \alpha^2) \mathcal{D}_\alpha(F(t), F(t)) \leq 0,$$

where $\mathcal{D}_\alpha(g, g)$ denotes the normalised energy dissipation associated to \mathcal{Q}_α , see [51], given by

$$\mathcal{D}_\alpha(g, g) := \frac{\gamma_b}{4} \int_{\mathbb{T}_\ell^d} \frac{dx}{|\mathbb{T}_\ell^d|} \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, v) g(x, v_*) |v - v_*|^3 dv dv_* \quad (1.6)$$

with

$$\gamma_b := \int_{\mathbb{S}^{d-1}} \frac{1 - \sigma \cdot \bar{q}}{2} b(\sigma \cdot \bar{q}) d\sigma = |\mathbb{S}^{d-2}| \int_0^\pi b(\cos \theta) (\sin \theta)^{d-2} \sin^2 \left(\frac{\theta}{2} \right) d\theta.$$

In fact, it is possible to show that

$$\lim_{t \rightarrow \infty} \mathbf{T}(t) = 0$$

which expresses the *total cooling of granular gases*. Determining the exact dissipation rate of the granular temperature is an important question known as *Haff's law* [37].

1.3. Navier-Stokes scaling. To capture some hydrodynamic behaviour of the gas, we need to write the above equation in *nondimensional form* introducing the dimensionless Knudsen number

$$\varepsilon := \frac{\text{mean free path}}{\text{spatial length-scale}}$$

which is assumed to be small. We introduce then a rescaling of time and space to capture the hydrodynamic limit and introduce the particle density

$$F_\varepsilon(t, x, v) = F \left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, v \right), \quad t \geq 0. \quad (1.7)$$

In this case, we choose for simplicity $\ell = \varepsilon$ in (1.2) which ensures now that F_ε is defined on $\mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d$ with $\mathbb{T}^d = \mathbb{T}_1^d$. It is well-known that this scaling leads to the incompressible Navier-Stokes, however, other scalings are possible that yield different hydrodynamic models. Under such a scaling, the typical number of collisions per particle per time unit is ε^{-2} , more specifically, F_ε satisfies the rescaled Boltzmann equation

$$\varepsilon^2 \partial_t F_\varepsilon(t, x, v) + \varepsilon v \cdot \nabla_x F_\varepsilon(t, x, v) = \mathcal{Q}_\alpha(F_\varepsilon, F_\varepsilon), \quad (x, v) \in \mathbb{T}^d \times \mathbb{R}^d, \quad (1.8a)$$

supplemented with initial condition

$$F_\varepsilon(0, x, v) = F_{\text{in}}^\varepsilon(x, v) = F_{\text{in}}\left(\frac{x}{\varepsilon}, v\right). \quad (1.8b)$$

Conservation of mass and density is preserved under this scaling, consequently, we assume that

$$\mathbf{R}_\varepsilon(t) = \int_{\mathbb{R}^d \times \mathbb{T}^d} F_\varepsilon(t, x, v) dv dx = 1, \quad \mathbf{U}_\varepsilon(t) = \int_{\mathbb{R}^d \times \mathbb{T}^d} F_\varepsilon(t, x, v) v dv dx = 0, \quad \forall t \geq 0,$$

whereas the cooling of the granular gas is given by the equation

$$\frac{d}{dt} \mathbf{T}_\varepsilon(t) = -\frac{1-\alpha^2}{\varepsilon^2} \mathcal{D}_\alpha(F_\varepsilon(t), F_\varepsilon(t)), \quad (1.9)$$

where $\mathbf{T}_\varepsilon(t) = \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{R}^d \times \mathbb{T}^d} |v|^2 F_\varepsilon(t, x, v) dv dx$.

Remark 1.1. *From now on we will always assume that*

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} F_{\text{in}}^\varepsilon(x, v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv dx = \begin{pmatrix} 1 \\ 0 \\ E_{\text{in}} \end{pmatrix}$$

with $E_{\text{in}} > 0$ fixed and independent of ε . It is important to emphasize that, in all the sequel, all the threshold values on ε and the various constants involved are actually depending only on this initial choice.

1.4. Self-similar variable and homogeneous cooling state. Various forcing terms have been added to (1.8a) depending on the underlying physics. Forcing terms prevent the total cooling of the gas (heated bath, thermal bath, see [61] for details) since they act as an energy supply source to the system and induce the existence of a non-trivial steady state. These are, however, systems different from the free-cooling Boltzmann equation (1.8a) that we aim to investigate here.

To understand better this free-cooling scenario, it is still possible to introduce an intermediate asymptotic and a steady state to work with. This is done by performing a self-similar change of variables

$$F_\varepsilon(t, x, v) = V_\varepsilon(t)^d f_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v), \quad (1.10a)$$

with

$$\tau_\varepsilon(t) := \frac{1}{c_\varepsilon} \ln(1 + c_\varepsilon t), \quad V_\varepsilon(t) = (1 + c_\varepsilon t), \quad t \geq 0, \quad c_\varepsilon > 0. \quad (1.10b)$$

With the special choice

$$c_\varepsilon = \frac{1-\alpha}{\varepsilon^2}, \quad (1.10c)$$

we can prove that f_ε satisfies

$$\varepsilon^2 \partial_t f_\varepsilon(t, x, v) + \varepsilon v \cdot \nabla_x f_\varepsilon(t, x, v) + \kappa_\alpha \nabla_v \cdot (v f_\varepsilon(t, x, v)) = \mathcal{Q}_\alpha(f_\varepsilon, f_\varepsilon), \quad (1.11)$$

with initial condition

$$f_\varepsilon(0, x, v) = F_{\text{in}}^\varepsilon(x, v).$$

Here

$$\kappa_\alpha = 1 - \alpha > 0, \quad \forall \alpha \in (0, 1).$$

The underlying drift term $\kappa_\alpha \nabla_v \cdot (vf(t, x, v))$ acts as an energy supply which prevents the total cooling down of the gas. Indeed, it has been shown in a series of papers [50, 51, 52] that there exists a *spatially homogeneous* steady state G_α to (1.11) which is unique for $\alpha \in (\alpha_0, 1)$ in an explicit threshold value $\alpha_0 \in (0, 1)$. More specifically, for $\alpha \in (\alpha_0, 1)$, there exists a unique solution G_α to the spatially homogeneous steady equation

$$\kappa_\alpha \nabla_v \cdot (vG_\alpha(v)) = \mathcal{Q}_\alpha(G_\alpha, G_\alpha),$$

with

$$\int_{\mathbb{R}^d} G_\alpha(v) dv = 1, \quad \int_{\mathbb{R}^d} G_\alpha(v) v dv = 0.$$

Moreover,

$$\lim_{\alpha \rightarrow 1^-} \|G_\alpha - \mathcal{M}\|_{L^1_2} = 0, \quad (1.12)$$

where \mathcal{M} is the Maxwellian distribution

$$\mathcal{M}(v) = G_1(v) = (2\pi\vartheta_1)^{-d/2} \exp\left(-\frac{|v|^2}{2\vartheta_1}\right), \quad v \in \mathbb{R}^d, \quad (1.13)$$

for some explicit temperature $\vartheta_1 > 0$. The Maxwellian distribution $\mathcal{M}(v)$ is a steady solution for $\alpha = 1$ and its prescribed temperature ϑ_1 (which ensures (1.12) to hold) will play a role in the rest of the analysis. We refer to Appendix A for more details and explanation of the role of ϑ_1 .

Three main questions are addressed in this work regarding the solution to (1.11):

(Q1) First, we aim to prove the existence and uniqueness of solutions to (1.11) in a close to equilibrium setting, i.e. solutions which are defined *globally* in time and such that

$$\sup_{t \geq 0} \|f_\varepsilon(t) - G_\alpha\| \leq \delta \quad (1.14)$$

for some positive and explicit $\delta > 0$ in a suitable norm $\|\cdot\|$ of a functional space to be identified. The close to equilibrium setting is quite relevant for very small Knudsen numbers given the large number of collisions per unit time which keep the system thermodynamically relaxed.

(Q2) More importantly (though closely related), the scope here is to provide estimates on the constructed solutions f_ε which are *uniform with respect to ε* . This means that, in the previous point, $\delta > 0$ is independent of ε . In fact, we are able to prove *exponential time decay* for the difference $\|f_\varepsilon(t) - G_\alpha\|$.

(Q3) Finally, we aim to prove that, as $\varepsilon \rightarrow 0$, the solution $f_\varepsilon(t)$ converges towards some hydrodynamic solution which depends on (t, x) only through macroscopic quantities $(\rho(t, x), u(t, x), \theta(t, x))$ which are solutions to a suitable modification of the incompressible Navier-Stokes system.

The central underlying assumption in the previous program is the following relation between the restitution coefficient and the Knudsen number.

Assumption 1.2. *The restitution coefficient $\alpha(\cdot)$ is a continuously decreasing function of the Knudsen number ε satisfying the optimal scaling behaviour*

$$\alpha(\varepsilon) = 1 - \lambda_0 \varepsilon^2 + o(\varepsilon^2) \quad (1.15)$$

with $\lambda_0 \geq 0$.

Indeed, a careful spectral analysis of the linearized collision operator around G_α shows that unless one assumes $1 - \alpha$ comparable to ε^2 the eigenfunction associated to the energy dissipation would explode and prevent (1.14) to hold true. In fact, we require λ_0 to be relatively small with respect to the eigenvalues associated to other kinetic excitations. As mentioned before, in this regime the energy dissipation rate is controlled along time by mimicking a viscoelastic property in the granular gas which is at contrast to other regimes such as the mono-kinetic limit. In viscoelastic models, nearly elastic regimes emerges naturally, see [11, 6, 7] for details.

Because $\varepsilon \rightarrow 0$, Assumption 1.2 means that the limit produces a model of the cumulative effect of *nearly elastic* collisions in the *hydrodynamic regime*. Two situations are of interest in our analysis

Case 1: If $\lambda_0 = 0$ the cumulative effect of the inelasticity is too weak in the hydrodynamic scale and the expected model is the classical Navier-Stokes equations.

Case 2: If $0 < \lambda_0 < \infty$, the cumulative effect is visible in the hydrodynamic scale and we expect a different model to the Navier-Stokes equation accounting for that. As we mentioned, we require λ_0 to be relatively small compared to some explicit quantities completely determined by the mass and energy of the initial datum, say, $0 < \lambda_0 \ll 1$ with some explicit upper bounds on λ_0 .

1.5. Main results. The main results are both concern with the solutions to (1.11). The first one is the following Cauchy theorem regarding the existence and uniqueness of close-to-equilibrium solutions to (1.11). Notations for the functional spaces are introduced in Section 1.8.

Theorem 1.3. *Assume Assumption 1.2. Let*

$$m > 2d, \quad m - 1 \geq k \geq 1, \quad q \geq 4,$$

be fixed. There exist a triple $(\varepsilon^\dagger, \lambda^\dagger, \mathcal{K}_0^\dagger)$ depending only on the mass and energy of $F_{\text{in}}^\varepsilon$ and m, k, q such that, for $\varepsilon \in (0, \varepsilon^\dagger)$, $\lambda_\varepsilon \in (0, \lambda^\dagger)$, and $\mathcal{K}_0 \in (0, \mathcal{K}_0^\dagger)$, if

$$\|F_{\text{in}}^\varepsilon - G_{\alpha(\varepsilon)}\|_{\mathbb{W}_x^{m,1} \mathbb{W}_v^{k,1}(\langle v \rangle^q)} \leq \varepsilon \sqrt{\mathcal{K}_0}$$

then the inelastic Boltzmann equation (1.11) has a unique solution

$$f_\varepsilon \in \mathcal{C}([0, \infty); \mathbb{W}_x^{m,1} \mathbb{W}_v^{k,1}(\langle v \rangle^q))$$

satisfying for $t > 0$

$$\begin{aligned} \|f_\varepsilon(t) - G_{\alpha(\varepsilon)}\|_{\mathbb{W}_x^{m,1}\mathbb{W}_v^{k,1}(\langle v \rangle^q)} &\leq C\varepsilon\sqrt{\mathcal{K}_0} \exp(-\lambda_\varepsilon t), \\ \text{and } \int_0^t \|f_\varepsilon(\tau) - G_{\alpha(\varepsilon)}\|_{\mathbb{W}_x^{m,1}\mathbb{W}_v^{k,1}(\langle v \rangle^{q+1})} d\tau &\leq C\varepsilon\sqrt{\mathcal{K}_0} \min\left\{1+t, 1+\frac{1}{\lambda_\varepsilon}\right\}, \end{aligned}$$

for some positive constant $C > 0$ independent of ε and where $\lambda_\varepsilon \simeq \frac{1-\alpha(\varepsilon)}{\varepsilon^2}$ is the energy eigenvalue of the linearized operator (see Theorem 1.7 hereafter).

Theorem 1.3 completely answers queries (Q1) and (Q2) where the functional space is chosen to be a L^1 -based Sobolev space

$$\mathbb{W}_x^{m,1}\mathbb{W}_v^{k,1}(\langle v \rangle^q)$$

and the close-to-equilibrium solutions are shown to decay with a rate that can be made uniform with respect to the Knudsen number ε . Recall here that, since Assumption 1.2 is met, the homogeneous cooling state depends on ε and $G_{\alpha(\varepsilon)} \rightarrow \mathcal{M}$ as $\varepsilon \rightarrow 0$.

The estimates on the solution f_ε provided by Theorem 1.3 are enough to answer (Q3). This is done under some additional assumption on the initial datum. Namely

Theorem 1.4. *Under the Assumptions of Theorem 1.3, set*

$$f_\varepsilon(t, x, v) = G_{\alpha(\varepsilon)} + \varepsilon h_\varepsilon(t, x, v),$$

with $h_\varepsilon(0, x, v) = h_{\text{in}}^\varepsilon(x, v) = \varepsilon^{-1} (F_{\text{in}}^\varepsilon - G_{\alpha(\varepsilon)})$ such that

$$\lim_{\varepsilon \rightarrow 0} \|\pi_0 h_{\text{in}}^\varepsilon - h_0\|_{\mathbb{W}_x^{m,1}L_v^1} = 0,$$

where π_0 stands for the projection over the elastic linearized Boltzmann operator (see Section 6 for a precise definition)

$$h_0(x, v) = (\varrho_0(x) + u_0(x) \cdot v + \frac{1}{2}\theta_0(x)(|v|^2 - d\vartheta_1)) \mathcal{M}(v),$$

with \mathcal{M} being the Maxwellian distribution introduced in (1.13) and

$$(\varrho_0, u_0, \theta_0) \in \mathscr{W}_m,$$

where we set $\mathscr{W}_\ell := \mathbb{W}_x^{\ell,1}(\mathbb{T}^d) \times \left(\mathbb{W}_x^{\ell,1}(\mathbb{T}^d)\right)^d \times \mathbb{W}_x^{\ell,1}(\mathbb{T}^d)$ for any $\ell \in \mathbb{N}$.

Then, for any $T > 0$ and $\{h_\varepsilon\}_\varepsilon$ converges in some weak sense to a limit $\mathbf{h} = \mathbf{h}(t, x, v)$ which is such that

$$\mathbf{h}(t, x, v) = \left(\varrho(t, x) + u(t, x) \cdot v + \frac{1}{2}\theta(t, x)(|v|^2 - d\vartheta_1) \right) \mathcal{M}(v), \quad (1.16)$$

where \mathcal{M} is the Maxwellian distribution introduced in (1.13) and

$$(\varrho, u, \theta) \in \mathcal{C}([0, T]; \mathscr{W}_{m-2}) \cap L^1((0, T); \mathscr{W}_m),$$

is solution to the following incompressible Navier-Stokes-Fourier system with forcing

$$\begin{cases} \partial_t u - \frac{\nu}{\vartheta_1} \Delta_x u + \vartheta_1 u \cdot \nabla_x u + \nabla_x p = \lambda_0 u, \\ \partial_t \theta - \frac{\gamma}{\vartheta_1^2} \Delta_x \theta + \vartheta_1 u \cdot \nabla_x \theta = \frac{\lambda_0 \bar{c}}{2(d+2)} \sqrt{\vartheta_1} \theta, \\ \operatorname{div}_x u = 0, \quad \varrho + \vartheta_1 \theta = 0, \end{cases} \quad (1.17)$$

subject to initial conditions $(\varrho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}})$ given by

$$u_{\text{in}} =: u(0) = \mathcal{P}u_0, \quad \theta_{\text{in}} = \theta(0) = \frac{d}{d+2} \theta_0 - \frac{2}{(d+2)\vartheta_1} \varrho_0, \quad \varrho_{\text{in}} = \varrho(0) := -\vartheta_1 \theta_{\text{in}},$$

where $\mathcal{P}u_0$ is the Leray projection of u_0 on divergence-free vector fields. The viscosity $\nu > 0$ and heat conductivity $\gamma > 0$ are explicit and $\lambda_0 > 0$ is the parameter appearing in (1.15). The parameter $\bar{c} > 0$ is depending on the collision kernel $b(\cdot)$.

The precise notion of weak convergence in the above Theorem 1.4 is very peculiar and strongly related to the *a priori* estimates used for the proof of Theorem 1.3. The mode of convergence is detailed in Theorem 6.3, see also Section 6.2 for more details.

It is classical for incompressible Navier-Stokes equations, see [48, Section 1.8, Chapter I], that the pressure term p acts as a Lagrange multiplier due to the constraint $\operatorname{div}_x u = 0$ and it is recovered (up to a constant) from the knowledge of (ϱ, u, θ) .

We point out that the above incompressible Navier-Stokes-Fourier system (1.17) with the self-consistent forcing terms on the right-hand-side is a new system of hydrodynamic equations that, to our knowledge, has never been rigorously derived earlier to describe granular flows. We also notice that the last two identities in (1.17) give respectively the incompressibility condition and a strong Boussinesq relation (see the discussion in Section 6). It is important to point out that in the case $\lambda_0 = 0$, one recovers the classical incompressible Navier-Stokes-Fourier system derived from elastic Boltzmann equation, see [56]. This proves continuity with respect to the restitution coefficient α .

We finally mention that the above Theorem 1.4 together with the relations (1.10) provide also a quite precise description of the hydrodynamic behaviour of the original problem (1.8a) in physical variables. In this framework, the above mentioned Case 2 for which $\lambda_0 > 0$ enjoys some special features for which uniform-in-time error estimates can be obtained. Turning back to the original problem (1.8a) not only gives a precise answer to *Haff's law* (with an explicit cooling rate of the granular temperature $T_\varepsilon(t)$) but also describes the cooling rate of the *local temperature* $\int_{\mathbb{R}^d} F_\varepsilon(t, x, v) |v|^2 dv$. We refer to Section 6.6 and Appendix A.2 for a more detailed discussion.

1.6. Hydrodynamic limits in the elastic case. The derivation of hydrodynamic limits from the elastic Boltzmann equation is an important problem which received a lot of attention and its origin can be traced back at least to D. Hilbert exposition of its 6th problem at the 1900 International Congress of Mathematicians. We refer the reader to [56, 33] for an up-to-date description of the mathematically relevant results in the field. Roughly

speaking three main approaches are adopted for the rigorous derivation of hydrodynamic limits.

- A) Many of the early mathematical justifications of hydrodynamic limits of the Boltzmann equation are based on (truncated) asymptotic expansions of the solution around some hydrodynamic solution

$$F_\varepsilon(t, x, v) = F_0(t, x, v) \left(1 + \sum_n \varepsilon^n F_n(t, x, v) \right) \quad (1.18)$$

where, typically

$$F_0(t, x, v) = \frac{\varrho(t, x)}{(2\pi\theta(t, x))^{d/2}} \exp\left(-\frac{|v - u(t, x)|^2}{2\theta(t, x)}\right) \quad (1.19)$$

is a *local Maxwellian* associated to the macroscopic fields which is required to satisfy the limiting fluid dynamic equation. This approach (or a variant of it based upon Chapman-Enskog expansion) leads to the first rigorous justification of the compressible Euler limit up to the first singular time for the solution of the Euler system in [17] (see also [43] for more general initial data and a study of initial layers). In the same way, a justification of the incompressible Navier-Stokes limit has been obtained in [22]. This approach deals mainly with strong solutions for both the kinetic and fluid equations.

- B) Another important line of research concerns weak solutions and a whole program on this topic has been introduced in [8, 9]. The goal is to prove the convergence of the renormalized solutions to the Boltzmann equation (as obtained in [24]) towards weak solutions to the compressible Euler system or to the incompressible Navier-Stokes equations. This program has been continued exhaustively and the convergence have been obtained in important results (see [30, 31, 40, 44, 46, 47] to mention just a few). We remark that, in the notion of renormalized solutions for the classical Boltzmann equation, a crucial role is played by the entropy dissipation (*H*-theorem) which asserts that the entropy of solutions to the Boltzmann equation is non increasing

$$\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{T}^d} F_\varepsilon \log F_\varepsilon(t, x, v) dv dx \leq 0.$$

This *a priori* estimate is fully exploited in the construction of renormalized solutions to the classical Boltzmann equation and is also fundamental in some justification arguments for the Euler limit, see [57].

- C) A third line of research deals with strong solutions close to equilibrium and exploits a careful spectral analysis of the linearized Boltzmann equation. Strong solutions to the Boltzmann equation close to equilibrium have been obtained in a weighted L^2 -framework in the work [60] and the *local-in-time* convergence of these solutions towards solution to the compressible Euler equations have been derived in [54]. For the limiting incompressible Navier-Stokes solution, a similar result have been carried out in [10] for smooth global solutions in \mathbb{R}^3 with a small initial velocity field. The smallness assumption has been recently removed in [27] allowing to recover non global in

time solutions to the Navier-Stokes equation. These results as well as [14] exploit a very careful description of the spectrum of the linearized Boltzmann equation derived in [26]. We notice that they are framed in the space $L^2(\mathcal{M}^{-1})$ where the linearized Boltzmann operator is self-adjoint and coercive.

We mention in particular two papers whose approaches are the closest to the ones adopted here. The work [15] was the main inspiration to answer questions (Q1)-(Q2). Indeed, in [15], the first estimates on the elastic Boltzmann equation in Sobolev spaces with polynomial weight (based on L^1) are obtained *uniformly with respect to the Knudsen number* ε . Also, the work [41] deals with an energy method in $L^2(\mathcal{M}^{-1})$ spaces (see also [35, 36]) in order to prove the *strong convergence* of the solutions to the Boltzmann equation towards the incompressible Navier-Stokes equation without resorting to the work of [26]. We adopt a similar strategy to answer (Q3).

1.7. The challenge of hydrodynamic limits for granular gases. There are several reasons which make the derivation of hydrodynamic limits for granular gases a challenging question at the physical level. In regard of the mathematical aspects of the hydrodynamical limit, several hurdles stand on way when trying to adapt the aforementioned approaches:

- I) With respect to the strategy given in A), the main difficulty lies in the identification of the typical *hydrodynamic solution*. Such solution is such that the time-space dependence of the one-particle distribution function $F(t, x, v)$ occurs only through suitable hydrodynamic fields like density $\varrho(t, x)$, bulk velocity $u(t, x)$, and temperature $\theta(t, x)$. This is the role played by the Maxwellian F_0 in (1.19) whenever $\alpha = 1$ and one wonders if the homogeneous cooling state G_α plays this role here. This is indeed the case up to first order capturing the fat tails of inelastic distributions, yet surprisingly, a suitable Maxwellian plays the role of the hydrodynamic solution in the ε -order correction. This Gaussian behaviour emerges in the *hydrodynamic limit* because of the near elastic regime that we treat here.¹
- II) The direction promoted in B) appears for the moment out of reach in the context of granular gases. Renormalized solutions in the context of the inelastic Boltzmann equation (1.20) have not been obtained due to the lack of an H -Theorem for granular gases. It is unclear if the classical entropy (or a suitable modification of it) remains bounded in general for granular gases.
- III) Homogeneous cooling states G_α are not explicit, this is a technical difficulty when adapting the approach of [26] for the spectral analysis of the linearized inelastic Boltzmann equation in the spatial Fourier variable. Partial interesting results have been obtained in [55] (devoted to diffusively heated granular gases) but they do not give a complete asymptotic expansion of eigenvalues and eigenfunctions up to the order leading to the Navier-Stokes asymptotic. We mention that obtaining an

¹See the interesting discussion in [61], especially the Section 2.8 entitled “What Is the Trouble with Non-Gaussianity”

analogue of the work [26] for granular gases would allow, in particular, to quantify the convergence rate towards the limiting model as in the recent work [27].

- IV) A major obstacle to adapt energy estimates and spectral approach lies in the choice of functional spaces. While the linearized Boltzmann operator associated to elastic interactions is self-adjoint and coercive in the weighted L^2 -space $L^2(\mathcal{M}^{-1})$, there is no such “self-adjoint” space for the inelastic case. This yields technical difficulties in the study of the spectral analysis of the linearized operator². Moreover, the energy estimates of [35, 36, 40, 41] are essentially based upon the coercivity of the linearized operator. For granular gases, it seems that one needs to face the problem *directly* in a L^1 -setting. Points III) and IV) make the approach C) difficult to directly adapt.

1.8. Notations and definition. Let us introduce some useful notations for function spaces. For any nonnegative weight function $m : \mathbb{R}^d \rightarrow \mathbb{R}^+$, we define, for all $p > 1$ and $q > 0$ the space $L^p(m)$ through the norm

$$\|f\|_{L^p(m)} := \left(\int_{\mathbb{R}^d} |f(\xi)|^p m(\xi)^p d\xi \right)^{1/p},$$

We also define, for $p \geq 1$

$$\mathbb{W}^{k,p}(m) = \left\{ f \in L^p(m) ; \partial_\xi^\beta f \in L^p(m) \forall |\beta| \leq k \right\}$$

with the usual norm, i.e., for $k \in \mathbb{N}$:

$$\|f\|_{\mathbb{W}^{k,p}(m)}^p = \sum_{|\beta| \leq k} \|\partial_\xi^\beta f\|_{L^p(m)}^p.$$

For $m \equiv 1$, we simply denote the associated spaces by L^p and $\mathbb{W}^{k,p}$. Notice that all the weights we consider here will depend only on velocity, i.e. $m = m(v)$.

We consider in the sequel the general weight

$$\varpi_s(v) = (1 + |v|^2)^{\frac{s}{2}}, \quad v \in \mathbb{R}^d, \quad s \geq 0.$$

On the complex plane, for any $a \in \mathbb{R}$, we set

$$\mathbb{C}_a := \{z \in \mathbb{C} ; \operatorname{Re} z > -a\}, \quad \mathbb{C}_a^* := \mathbb{C}_a \setminus \{0\}$$

and, for any $r > 0$, we set

$$\mathbb{D}(r) = \{z \in \mathbb{C} ; |z| \leq r\}.$$

We also introduce the following notion of hypo-dissipativity in a general Banach space.

Definition 1.5. Let $(X, \|\cdot\|)$ be a given Banach space. A closed (unbounded) linear operator $A : \mathcal{D}(A) \subset X \rightarrow X$ is said to be hypo-dissipative on X if there exists a norm, denoted by $\|\!\| \cdot \|\!\|$, equivalent to the $\|\cdot\|$ -norm such that A is dissipative on the space $(X, \|\!\| \cdot \|\!\|)$, that is,

$$\|\!\|(\lambda - A)h\|\!\| \geq \lambda \|\!\|h\|\!\|, \quad \forall \lambda > 0, \quad h \in \mathcal{D}(A).$$

²Recall that the powerful enlargement techniques for the elastic Boltzmann equation are based on the knowledge of the spectral structure in the space $L^2(\mathcal{M}^{-1})$ (and Sobolev spaces built on it) which can be extended to the more natural L^1 -setting.

Remark 1.6. This is equivalent to the following, see [25, Proposition 3.23, p. 88]: if $\|\cdot\|_*$ denotes the norm on the dual space X^* , for all $h \in \mathcal{D}(A)$, there exists $\mathbf{u}_h \in X^*$ such that

$$[\mathbf{u}_h, h] = \|h\|^2 = \|\mathbf{u}_h\|_*^2 \quad \text{and} \quad \operatorname{Re}[\mathbf{u}_h, Ah] \leq 0,$$

where $[\cdot, \cdot]$ denotes the duality bracket between $(X^*, \|\cdot\|_*)$ and $(X, \|\cdot\|)$.

For two tensors $A = (A_{i,j}), B = (B_{i,j}) \in \mathcal{M}_d(\mathbb{R})$, we denote by $A : B$ the scalar $(A : B) = \sum_{i,j} A_{i,j} B_{i,j} \in \mathbb{R}$ as the trace of the matrix product AB whereas, for a vector function $w = w(x) \in \mathbb{R}^d$, the tensor $(\partial_{x_i} w_j)_{i,j}$ is denoted as $\nabla_x w$. We also write $(\operatorname{Div}_x A)^i = \sum_j \partial_{x_j} A_{i,j}(x)$.

1.9. Strategy of the proof. The strategy used to prove the main results Theorems 1.3 and 1.4 yields to several intermediate results of independent interest. The approach is perturbative in essence since we are dealing with close to equilibrium solutions to (1.11). This means that, in the study of (1.11), we introduce the fluctuation h_ε around the equilibrium G_α defined through

$$f_\varepsilon(t, x, v) = G_\alpha(v) + \varepsilon h_\varepsilon(t, x, v),$$

and h_ε satisfies

$$\begin{cases} \partial_t h_\varepsilon(t, x, v) + \frac{1}{\varepsilon} v \cdot \nabla_x h_\varepsilon(t, x, v) - \frac{1}{\varepsilon^2} \mathcal{L}_\alpha h_\varepsilon(t, x, v) = \frac{1}{\varepsilon} \mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon)(t, x, v), \\ h_\varepsilon(0, x, v) = h_\varepsilon^{\text{in}}(x, v), \end{cases} \quad (1.20)$$

where \mathcal{L}_α is the linearized collision operator (local in the x -variable) defined as

$$\mathcal{L}_\alpha h(x, v) = \mathbf{L}_\alpha(h)(x, v) - \kappa_\alpha \nabla_v \cdot (vh(x, v)),$$

with

$$\mathbf{L}_\alpha(h) = 2\tilde{\mathcal{Q}}_\alpha(G_\alpha, h),$$

where we set

$$\tilde{\mathcal{Q}}_\alpha(f, g) = \frac{1}{2} \{ \mathcal{Q}_\alpha(f, g) + \mathcal{Q}_\alpha(g, f) \}.$$

We also denote by \mathcal{L}_1 the linearized operator around $G_1 = \mathcal{M}$, that is,

$$\mathcal{L}_1(h) = \mathbf{L}_1(h) = \mathcal{Q}_1(\mathcal{M}, h) + \mathcal{Q}_1(h, \mathcal{M}).$$

The method of proof requires first a careful spectral analysis of the full linearized operator appearing in (1.20):

$$\mathcal{G}_{\alpha,\varepsilon} h := -\varepsilon^{-1} v \cdot \nabla_x h + \varepsilon^{-2} \mathcal{L}_\alpha h.$$

Such a spectral analysis has to be performed in a suitable L^1 -based Sobolev space and we borrow for this idea from [34] and extended to the case $\varepsilon \neq 1$ in [15].

A central point in the approach is that we treat $\mathcal{G}_{\alpha,\varepsilon}$ as a *perturbation*³ of the elastic linearized operator $\mathcal{G}_{1,\varepsilon}$. The spectrum of $\mathcal{G}_{1,\varepsilon}$ in $\mathbb{W}_x^{s,1}\mathbb{W}_v^{k,1}(\langle v \rangle^q)$ is well-understood [15], so, it is possible to deduce from this characterisation the spectrum of $\mathcal{G}_{\alpha,\varepsilon}$ using ideas from [59]. We *only* study the spectrum of $\mathcal{G}_{\alpha,\varepsilon}$ without requiring knowledge of the decay of the semigroup associated to $\mathcal{G}_{\alpha,\varepsilon}$. This simplifies the technicalities of the spectral analysis performed in Section 3 related to Dyson-Phillips iterates which leads to the spectral mapping theorem [34, 59]. Most notably, in this simplified approach one is able to identify the optimal scaling (1.15) of the restitution coefficient.

The scaling (1.15) is precisely the one which allows to preserve exactly $d+2$ eigenvalues in the neighbourhood of zero (recall that 0 is an eigenvalue of multiplicity $d+2$ in the elastic case). Recalling that, in any reasonable space, the elastic operator has a spectral gap of size $\mu_\star > 0$, i.e.

$$\mathfrak{S}(\mathcal{G}_{1,\varepsilon}) \cap \{z \in \mathbb{C} ; \operatorname{Re} z > -\mu_\star\} = \{0\}$$

where 0 is an eigenvalue of algebraic multiplicity $d+2$ associated to the eigenfunctions $\{\mathcal{M}, v_j \mathcal{M}, |v|^2 \mathcal{M}, j = 1, \dots, d\}$, one can prove the following theorem.

Theorem 1.7. *Set*

$$\mathbb{X} := \mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q), \quad \ell \in \mathbb{N}, s \geq 0, \quad \ell \geq s+1, \quad q > 2$$

and assume that Assumption 1.2 is met. There exists some explicit $\nu_\star > 0$ such that, if $\mu \in (\mu_\star - \nu_\star, \mu_\star)$, there is some explicit $\bar{\varepsilon} > 0$ depending only on $\mu_\star - \mu$ and such that, for all $\varepsilon \in (0, \bar{\varepsilon})$, the linearized operator

$$\mathcal{G}_{\alpha,\varepsilon} : \mathcal{D}(\mathcal{G}_{\alpha,\varepsilon}) \subset \mathbb{X} \rightarrow \mathbb{X}$$

has the spectral property:

$$\mathfrak{S}(\mathcal{G}_{\alpha,\varepsilon}) \cap \{z \in \mathbb{C} ; \operatorname{Re} z \geq -\mu\} = \{\lambda_1(\varepsilon), \dots, \lambda_{d+2}(\varepsilon)\}, \quad (1.21)$$

where $\lambda_1(\varepsilon), \dots, \lambda_{d+2}(\varepsilon)$ are eigenvalues of \mathcal{G}_ε (not necessarily distinct) with

$$|\lambda_j(\varepsilon)| \leq \mu_\star - \mu \quad \text{for } j = 1, \dots, d+2.$$

More precisely, it follows that

$$\begin{aligned} \mathfrak{S}(\mathcal{G}_{\alpha,\varepsilon}) \cap \{z \in \mathbb{C} ; \operatorname{Re} z \geq -\mu\} &= \mathfrak{S}(\varepsilon^{-2} \mathcal{L}_\alpha) \cap \{z \in \mathbb{C} ; \operatorname{Re} z \geq -\mu\} \\ &= \{\lambda_1(\varepsilon), \dots, \lambda_{d+2}(\varepsilon)\}, \end{aligned}$$

with

$$\lambda_1(\varepsilon) = 0, \quad \lambda_j(\varepsilon) = \varepsilon^{-2} \kappa_{\alpha(\varepsilon)}, \quad j = 2, \dots, d+1,$$

and

$$\lambda_{d+2}(\varepsilon) = -\lambda_\varepsilon = -\frac{1 - \alpha(\varepsilon)}{\varepsilon^2} + \mathcal{O}(\varepsilon^2), \quad \text{for } \varepsilon \simeq 0.$$

³This perturbation does not fall into the realm of the classical perturbation theory of the unbounded operator as described in [42]. Typically, the domain of $\mathcal{G}_{\alpha,\varepsilon}$ is much smaller than the one of $\mathcal{G}_{1,\varepsilon}$ (because of the drift term in velocity) and the relative bound between $\mathcal{G}_{1,\varepsilon}$ and $\mathcal{G}_{\alpha,\varepsilon}$ does not converges to zero in the elastic limit $\alpha \rightarrow 1$.

Remark 1.8. Notice that the eigenvalue $\lambda_{d+2}(\varepsilon) = -\lambda_\varepsilon$ can be seen as the energy eigenvalue in the sense that

$$\int_{\mathbb{R}^d} \mathcal{G}_{\alpha,\varepsilon} \varphi(v) |v|^2 dv = -\lambda_\varepsilon \int_{\mathbb{R}^d} \varphi(v) |v|^2 dv$$

for any smooth test-function φ .

To prove Theorem 1.7, it is necessary to strengthen several results of [52] and obtain sharp convergence rate in the elastic limit for the linearized operator. Typically, one needs to prove that, for suitable topology

$$\mathcal{L}_\alpha - \mathcal{L}_1 \simeq (1 - \alpha)$$

which gives an estimate of the type

$$(\mathcal{G}_{\alpha,\varepsilon} - \mathcal{G}_{1,\varepsilon}) \simeq \frac{1 - \alpha}{\varepsilon^2}.$$

This is done in Section 2.

After the spectral analysis is performed, in order to prove Theorem 1.3 several *a priori estimates* for the solutions to (1.11) are required. This is done in Section 4. The crucial point in the analysis lies in the splitting of (1.11) into a system of two equations mimicking a spectral enlargement method from a PDE perspective (see [53, Section 2.3] and [15] for pioneering ideas on such a splitting). More precisely, the splitting performed in Sections 4 and 5 amounts to look for a solution of (1.20) of the form

$$h_\varepsilon(t) = h_\varepsilon^0(t) + h_\varepsilon^1(t)$$

where $h_\varepsilon^1(t)$ is solution to the *linearized elastic equation* with a source term involving the reminder $h_\varepsilon^0(t)$, namely,

$$\partial_t h_\varepsilon^1(t) = \mathcal{G}_{1,\varepsilon} h_\varepsilon^1 + \varepsilon^{-1} \mathcal{Q}_1(h_\varepsilon^1, h_\varepsilon^1) + \mathcal{A}_\varepsilon h_\varepsilon^0 \quad (1.22)$$

having zero initial datum and where \mathcal{A}_ε is a regularizing operator (see Section 3 for a precise definition). In this way we seek $h_\varepsilon^1(t)$ in the Hilbert space

$$h_\varepsilon^1(t) \in \mathbb{W}_{x,v}^{m,2}(\mathcal{M}^{-1/2}) =: \mathcal{H}$$

with $m > 2d$ and prove bounds of the type

$$\sup_{t \geq 0} \left(\|h_\varepsilon^1(t)\|_{\mathcal{H}}^2 + \int_0^t e^{-\nu(t-\tau)} \|h_\varepsilon^1(\tau)\|_{\mathcal{H}_1}^2 d\tau \right) \leq C\mathcal{K}_0$$

where \mathcal{H}_1 is the domain of $\mathcal{G}_{1,\varepsilon}$ in \mathcal{H} , \mathcal{K}_0 depends only on the initial datum $h_{\text{in}}^\varepsilon$ and ν is of the order of the spectral gap of $\mathcal{G}_{1,\varepsilon}$ on \mathcal{H} . With such a splitting, it is possible to fully exploit the elastic problem and treat h_ε as a perturbation of this solution. This is the role of Section 4.

In Section 5, we prove Theorem 1.3 introducing a suitable iterative scheme based upon the coupling $(h_\varepsilon^0(t), h_\varepsilon^1(t))$. We show in practice that the coupled system of kinetic equations satisfied by h^0 and h^1 is well-posed. It is fair to say that the bounds for h_ε^0 and h_ε^1 given in sections 4 and 5 play the role of suitable energy estimates as the ones established

in the purely Hilbert setting [35, 36, 41]. In particular, these bounds are sufficient to deduce a very peculiar type of weak convergence of $h_\varepsilon(t)$ towards an element in the kernel of the linearized operator \mathcal{L}_1 , in particular, the limit of h_ε is necessarily of the form (1.16). The notion of weak convergence we use here fully exploits the splitting $h_\varepsilon = h_\varepsilon^0 + h_\varepsilon^1$ where we proved that h_ε^0 converges to 0 *strongly* in $L^1((0, T); \mathbb{W}_x^{m,1} L_v^1(\varpi_q))$ whereas h_ε^1 converges to \mathbf{h} *weakly* in $L^2((0, T); \mathbb{W}_x^{m,2} L_v^2(\mathcal{M}^{-\frac{1}{2}}))$.

Finally, in Section 6, the regularity of (ϱ, u, θ) obtained via a simple use of Ascoli-Arzelà Theorem and the identification of the limiting equations these macroscopic fields satisfy is presented. With the notion of weak convergence at hand presented above, the approach is simpler but reminiscent of the program established in [8, 9]. In particular, we can adapt some of the main ideas of [30] regarding the delicate convergence of nonlinear convection terms. Detailed computations are included to make the paper as much self-contained as possible also because, even in the classical “elastic” case, it is difficult to find a full proof of the convergence towards hydrodynamic limit for the weak solutions we consider here. For such solutions, details of proof are scattered in the literature and full proof of the convergence of nonlinear terms is sometimes only sketched where most of the full detailed proofs are dealing with the more delicate case of renormalized solutions [30, 31, 44]. In our framework, the terms involving the quadratic operator $\mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon)$ are treated as source terms which converge in distributions to zero whereas the drift term and the dissipation of energy function \mathcal{D}_α are the objects responsible for the terms in the right-side of the Navier-Stokes system (1.17). We also observe that the derivation of the strong Boussinesq relation is not as straightforward as in the elastic case. Actually, the classical Boussinesq relation

$$\nabla(\varrho(t, x) + \vartheta_1 \theta(t, x)) = 0$$

is established as in the elastic case. In the elastic case, this relation implies the strong form of Boussinesq mainly because the two functions $\varrho(t, x)$ and $\theta(t, x)$ have zero spatial averages. This cannot be deduced directly in the granular context due to the dissipation of energy.

1.10. Organization of the paper. The paper is divided into 6 Sections and three Appendices. In the following Section 2, we collect several results regarding the collision operator $\mathcal{L}_{\alpha(\varepsilon)}$ and introduce the splitting of the operator in $\mathcal{L}_{\alpha(\varepsilon)} = \mathcal{A}_\alpha + \mathcal{B}_\alpha$ as well as the splitting of the full linearized operator $\mathcal{G}_{\alpha,\varepsilon}$. Section 3 is devoted to the spectral analysis of $\mathcal{G}_{\alpha,\varepsilon}$ culminating with the proof of Theorem 1.7. In Section 4, we derive the fundamental *a priori* estimates on the close-to-equilibrium solutions to (1.20). It is the most technical part of the work and fully exploits the splitting of the operator $\mathcal{G}_{\alpha,\varepsilon}$ as explained earlier. Section 5 gives the proof of Theorem 1.3 whereas Section 6 gives the full proof of the hydrodynamic limit (Theorem 1.4). In Appendix A, we recall some facts about the granular Boltzmann equation and gives the full proof of a technical result of Section 2. In Appendix B, we collect some well-known properties useful for the hydrodynamic limit as well as some technical proofs used in Section 6. Finally, Appendix C gives the proof of two technical results of Section 2.

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2. SUMMARY OF USEFUL RESULTS ABOUT THE COLLISION OPERATOR

2.1. The linearized operators \mathcal{L}_α and \mathcal{L}_1 . In all the sequel, we will use well-known estimates for the bilinear operator $\mathcal{Q}_\alpha(f, g)$ and $\mathcal{Q}_1(f, g)$ in several different functional spaces. We refer to [3, 4, 52] for precise statements. A crucial role in our analysis will be played by the fact that, in some suitable sense, \mathcal{L}_α is close to the elastic linearized operator \mathcal{L}_1 for $\alpha \simeq 1$. Let us begin with the following crucial observation which also justifies the optimal scaling (1.15) and optimise the rate of convergence previously derived in [52, Proposition 3.1 (iii)] for weights different to the ones considered here. The technical proof is postponed to Appendix A since it based upon the so-called n -representation of the collision operator:

Lemma 2.1. *Assume $\alpha \in (0, 1]$. For all $q \geq 2$ there is a positive constant c_q such that*

$$\|\mathcal{Q}_1(g, f) - \mathcal{Q}_\alpha(g, f)\|_{L_v^1(\varpi_q)} \leq c_q \frac{1-\alpha}{\alpha^2} \|f\|_{\mathbb{W}_v^{1,1}(\varpi_{q+2})} \|g\|_{\mathbb{W}_v^{1,1}(\varpi_{q+2})}.$$

More generally, for $k \in \mathbb{N}$ and $q \geq 2$, there is a positive constant $c_{k,q} > 0$ such that

$$\|\mathcal{Q}_1(g, f) - \mathcal{Q}_\alpha(g, f)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq c_{k,q} \frac{1-\alpha}{\alpha^2} \|f\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+2})} \|g\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+2})}.$$

Let us now investigate the rate of convergence of the equilibrium G_α towards \mathcal{M} . An optimal convergence rate in L^1 -spaces is given in [52, Step 2, proof of Lemma 4.4]: there is $C > 0$ and $\alpha_* > 0$ such that

$$\|\mathcal{M} - G_\alpha\|_{L_v^1(\langle \cdot \rangle_{m-1})} \leq C(1-\alpha), \quad \alpha \in [\alpha_*, 1]. \quad (2.1)$$

for $m(v) = \exp(a|v|)$, $a > 0$ small enough. We need to extend this optimal rate of convergence to the Sobolev spaces $\mathbb{W}_v^{s,1}(\varpi_q)$ we are considering here.

Lemma 2.2. *Let $k \in \mathbb{N}$, $q \geq 2$ be given. There exist some explicit $\alpha_* \in (0, 1)$ and $C > 0$ such that*

$$\|\mathcal{M} - G_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} \leq C(1-\alpha), \quad k \in \mathbb{N}.$$

Proof. We slightly modify here a strategy adopted in [5] which consists in combining a *nonlinear* estimate for $\|G_\alpha - \mathcal{M}\|_{\mathbb{W}_v^{k,1}(\varpi_q)}$ together with non-quantitative convergence. We fix k, q and we divide the proof into three steps:

First step: non quantitative convergence. We prove that

$$\lim_{\alpha \rightarrow 1} \|\mathcal{M} - G_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} = 0. \quad (2.2)$$

We argue here as in [5, Theorem 4.1]. We sketch only the main steps. First, as already noticed in [52], there is $\alpha_0 > 0$ such that

$$\sup_{\alpha \in (\alpha_0, 1)} \|G_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} < \infty.$$

Then, there is a sequence $(\alpha_n)_n$ converging to 1 such that $(G_{\alpha_n})_n$ converges weakly, in $\mathbb{W}_v^{k,1}(\varpi_{q+1})$ to some limit \bar{G} (notice that, a priori, the limit function \bar{G} depends on the choice of k and q). Using the decay of (G_α) and compact embedding for Sobolev spaces, this convergence is actually strong, i.e. $\lim_n \|G_{\alpha_n} - \bar{G}\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} = 0$. According to (2.1), one necessarily has $\bar{G} = \mathcal{M}$ and one deduces easily that whole net $(G_\alpha)_\alpha$ is converging to \mathcal{M} . This proves (2.2).

Second step: nonlinear estimate. We first consider the Maxwellian \mathcal{M}_α with same mass, momentum and energy of G_α and we consider the linearized elastic collision operator around that Maxwellian

$$\mathbf{L}g = \mathcal{Q}_1(g, \mathcal{M}_\alpha) + \mathcal{Q}_1(\mathcal{M}_\alpha, g), \quad g \in \mathbb{W}_v^{k,1}(\varpi_{q+1}).$$

One simply notices that, since $\mathcal{Q}_1(\mathcal{M}_\alpha, \mathcal{M}_\alpha) = 0$,

$$\begin{aligned} \mathbf{L}(G_\alpha) &= \mathcal{Q}_1(G_\alpha - \mathcal{M}_\alpha, \mathcal{M}_\alpha - G_\alpha) + \mathcal{Q}_\alpha(G_\alpha, G_\alpha) + \left[\mathcal{Q}_1(G_\alpha, G_\alpha) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha) \right] \\ &= \mathcal{Q}_1(G_\alpha - \mathcal{M}_\alpha, \mathcal{M}_\alpha - G_\alpha) - (1 - \alpha) \nabla \cdot (v G_\alpha) + \left[\mathcal{Q}_1(G_\alpha, G_\alpha) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha) \right]. \end{aligned}$$

Therefore, using classical estimates for \mathcal{Q}_1 (see [3, 4])

$$\begin{aligned} \|\mathbf{L}(G_\alpha)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} &\leq \|\mathcal{Q}_1(G_\alpha - \mathcal{M}_\alpha, \mathcal{M}_\alpha - G_\alpha)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} + (1 - \alpha) \|G_\alpha\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+1})} \\ &\quad + \|\mathcal{Q}_1(G_\alpha, G_\alpha) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \\ &\leq C_1 \|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})}^2 + C(1 - \alpha) \|G_\alpha\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+1})} + C_1(1 - \alpha) \|G_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+2})} \end{aligned}$$

where we used Lemma 2.1 for estimating the difference $\mathcal{Q}_1(G_\alpha, G_\alpha) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha)$. Since $\sup_\alpha \|G_\alpha\|_{\mathbb{W}_v^{k+1,1}(\varpi_{q+2})} < \infty$, we obtain that there is a positive constant $C_2 > 0$ such that

$$\|\mathbf{L}(G_\alpha)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq C_2(1 - \alpha) + C_2 \|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})}^2.$$

We can write $\mathbf{L}(G_\alpha) = \mathbf{L}(G_\alpha - \mathcal{M}_\alpha)$ and, as $G_\alpha - \mathcal{M}_\alpha$ has zero mass, momentum and energy, there is a positive constant $c > 0$ (that can be taken independent of α) such that

$$\|\mathbf{L}(G_\alpha - \mathcal{M}_\alpha)\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \geq c \|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})}.$$

Recall that the constant $c > 0$ is actually the norm of the inverse of \mathbf{L} on the subspace of functions with zero mass, momentum and energy; recall that this inverse maps $\mathbb{W}_v^{k,1}(\varpi_q)$ into $\mathcal{D}(\mathbf{L}) = \mathbb{W}_v^{k,1}(\varpi_{q+1})$. Therefore, with $C_3 = C_2/c$

$$\|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} \leq C_3(1 - \alpha) + C_3\|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})}^2, \quad \alpha \in (\alpha_0, 1). \quad (2.3)$$

Third step: conclusion. Setting

$$\vartheta_\alpha := \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 \mathcal{M}_\alpha(v) dv = \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 G_\alpha(v) dv,$$

one sees easily from (2.1) that $|\vartheta_1 - \vartheta_\alpha| \leq C(1 - \alpha)$ and then, one can check without difficulty that there is some positive constant $C_{k,q} > 0$ such that

$$\|\mathcal{M}_\alpha - \mathcal{M}\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq C_{k,q}(1 - \alpha), \quad \forall \alpha \in [\alpha_*, 1]. \quad (2.4)$$

Thanks to (2.2), we can then find $\alpha_1 > \alpha_0$ such that

$$C_3\|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} \leq \frac{1}{2}$$

where $C_3 > 0$ is the positive constant in (2.3). Then, (2.3) reads simply as

$$\|G_\alpha - \mathcal{M}_\alpha\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} \leq 2C_3(1 - \alpha), \quad \alpha \in (\alpha_1, 1),$$

and, using (2.4), we end up with

$$\|G_\alpha - \mathcal{M}\|_{\mathbb{W}_v^{k,1}(\varpi_{q+1})} \leq C(1 - \alpha), \quad \alpha \in (\alpha_1, 1),$$

which gives also a quantitative lower bound on α_1 . \square

Remark 2.3. *The existence of the self-similar profile G_α has been obtained in [51] and the uniqueness in [52] for $1 - \alpha$ sufficiently small. The uniqueness of the profile has been proved in spaces with exponential weights, but from the a posteriori estimates provided therein, any self-similar profile G_α belongs to some L^1 -space with exponential weight. Therefore, self-similar profiles are unique in spaces with polynomial weights ϖ_q as well, provided $1 - \alpha$ is sufficiently small.*

We recall also the following spectral properties of \mathcal{L}_α as established in [52].

Proposition 2.4. *On the space*

$$L_v^1(\varpi_q) \cap \left\{ h \in L_v^1(\varpi_1); \int_{\mathbb{R}^d} h(v) dv = \int_{\mathbb{R}^d} h(v) v dv = 0 \right\}, \quad q \geq 2$$

the spectrum of \mathcal{L}_α is such that there exists $\bar{\mu} > 0$ such that

$$\mathfrak{S}(\mathcal{L}_\alpha) \cap \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > -\bar{\mu}\} = \{-\mu_\alpha\} \quad (2.5)$$

where μ_α is a simple eigenvalue of \mathcal{L}_α with

$$\mu_\alpha = (1 - \alpha) + O((1 - \alpha)^2) \quad \text{as } \alpha \rightarrow 1. \quad (2.6)$$

Moreover, denoting by ϕ_α the unique associated eigenfunction such that $\|\phi_\alpha\|_{L_v^1(\varpi_2)} = 1$ and $\phi_\alpha(0) < 0$, it holds

$$\lim_{\alpha \rightarrow 1} \phi_\alpha(v) = c_0 (|v|^2 - d\vartheta_1) \mathcal{M}. \quad (2.7)$$

Remark 2.5. We stress that the results of [52] have been established for exponential weights $m(v) = \exp(a|v|)$. An elementary adaption can be made for the weights considered here. An alternative route is to extend the spaces from $L^1(\exp(a|v|)dv)$ to $L^1(\varpi_q)$ using enlargement techniques, see [34].

On the underlying space $\mathbb{W}_v^{k,1}(\varpi_q)$, introduce the operator $T_\alpha : \mathcal{D}(T_\alpha) \subset \mathbb{W}_v^{k,1}(\varpi_q) \rightarrow \mathbb{W}_v^{k,1}(\varpi_q)$ defined by $\mathcal{D}(T_\alpha) = \mathbb{W}_v^{k+1,1}(\varpi_{q+1})$ and

$$T_\alpha h(v) = -\kappa_\alpha \operatorname{div}(v h(v)), \quad h \in \mathcal{D}(T_\alpha).$$

One sees that the operator T_α is the one responsible for the discrepancy between the domain of \mathcal{L}_1 and \mathcal{L}_α . Because of this, we set

$$\mathcal{P}_\alpha : \mathcal{D}(\mathcal{P}_\alpha) \subset \mathbb{W}_v^{k,1}(\varpi_q) \rightarrow \mathbb{W}_v^{k,1}(\varpi_q)$$

as $\mathcal{P}_\alpha = \mathbf{L}_\alpha - \mathbf{L}_1$ with domain

$$\mathcal{D}(\mathcal{P}_\alpha) = \mathcal{D}(\mathcal{L}_1) = \mathbb{W}_v^{k,1}(\varpi_{1+q}).$$

One has then the following Proposition.

Proposition 2.6. For any $k \in \mathbb{N}$, $q \geq 0$, there exists some explicit constant $C_{k,q} > 0$ such that for any $h \in \mathbb{W}_v^{k,1}(\varpi_{1+q})$

$$\|\mathcal{P}_\alpha h\|_{\mathbb{W}_v^{k,1}(\varpi_q)} = \|\mathbf{L}_\alpha h - \mathbf{L}_1 h\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq C_{k,q}(1 - \alpha) \|h\|_{\mathbb{W}_v^{k+1,1}(\varpi_{2+q})}. \quad (2.8)$$

As a consequence, for any $h \in \mathbb{W}_v^{k+1,1}(\varpi_{2+q})$

$$\|\mathcal{L}_\alpha h - \mathcal{L}_1 h\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq (C_{k,q}(1 - \alpha) + \kappa_\alpha) \|h\|_{\mathbb{W}_v^{k+1,1}(\varpi_{2+q})}. \quad (2.9)$$

Proof. Recall (see [3, 4]) that, for any $q \geq 0$, there is some universal positive constant $C_q > 0$ such that

$$\|\mathcal{Q}_\alpha(g, f)\|_{L_v^1(\varpi_q)} \leq C_q \|g\|_{L_v^1(\varpi_{q+1})} \|f\|_{L_v^1(\varpi_{q+1})}, \quad \forall f, g \in L_v^1(\varpi_{q+1}). \quad (2.10)$$

Then, since

$$\begin{aligned} \mathbf{L}_\alpha h(v) - \mathbf{L}_1 h(v) &= \mathcal{Q}_\alpha(h, G_\alpha - \mathcal{M})(v) + \mathcal{Q}_\alpha(G_\alpha - \mathcal{M}, h)(v) \\ &\quad + [\mathcal{Q}_\alpha(h, \mathcal{M})(v) - \mathcal{Q}_1(\mathcal{M}, h)(v)] + [\mathcal{Q}_\alpha(\mathcal{M}, h)(v) - \mathcal{Q}_1(h, \mathcal{M})(v)], \end{aligned} \quad (2.11)$$

one deduces from (2.10) and Lemma 2.1 that

$$\begin{aligned} \|\mathcal{P}_\alpha h\|_{L_v^1(\varpi_q)} &\leq 2C_q \|h\|_{L_v^1(\varpi_{1+q})} \|G_\alpha - \mathcal{M}\|_{L_v^1(\varpi_{1+q})} \\ &\quad + 2c_q \frac{1 - \alpha}{\alpha^2} \|h\|_{\mathbb{W}_v^{1,1}(\varpi_{2+q})} \|\mathcal{M}\|_{\mathbb{W}_v^{1,1}(\varpi_{2+q})}. \end{aligned}$$

Using now Lemma 2.1, this proves (2.8) for $k = 0$ with $C_{0,q} = 2C_q C + 2c_q \alpha_0^{-2} \|\mathcal{M}\|_{\mathbb{W}_v^{1,1}(\varpi_{2+q})}$.

In order to prove the result for higher-order derivatives, one argues using the fact that

$$\nabla_v \mathcal{Q}_\alpha(g, f) = \mathcal{Q}_\alpha(\nabla_v g, f) + \mathcal{Q}_\alpha(g, \nabla_v f).$$

Then, using (2.11) with the help of the estimate

$$\|T_\alpha h\|_{\mathbb{W}_v^{k,1}(\varpi_q)} \leq \kappa_\alpha \|h\|_{\mathbb{W}_v^{k+1,1}(\varpi_{1+q})}$$

one deduces (2.9) from (2.8). \square

2.2. Decomposition of \mathcal{L}_α . Let us now recall the following decomposition of \mathcal{L}_0 introduced in [34, 59]. For any $\delta \in (0, 1)$, we consider the cutoff function $0 \leq \Theta_\delta = \Theta_\delta(\xi, \xi_*, \sigma) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1})$, assumed to be bounded by 1, which equals 1 on

$$J_\delta := \left\{ (\xi, \xi_*, \sigma) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \mid |\xi| \leq \delta^{-1}, 2\delta \leq |\xi - \xi_*| \leq \delta^{-1}, |\cos \theta| \leq 1 - 2\delta \right\},$$

and whose support is included in $J_{\delta/2}$ (where $\cos \theta = \langle \frac{\xi - \xi_*}{|\xi - \xi_*|}, \sigma \rangle$). We then set

$$\begin{aligned} \mathcal{L}_1^{S,\delta} h(\xi) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} [\mathcal{M}(\xi'_*) h(\xi') + \mathcal{M}(\xi') h(\xi'_*) - \mathcal{M}(\xi) h(\xi_*)] \\ &\quad \times |\xi - \xi_*| \Theta_\delta(\xi, \xi_*, \sigma) d\xi_* d\sigma, \\ \mathcal{L}_1^{R,\delta} h(\xi) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} [\mathcal{M}(\xi'_*) h(\xi') + \mathcal{M}(\xi') h(\xi'_*) - \mathcal{M}(\xi) h(\xi_*)] \\ &\quad \times |\xi - \xi_*| (1 - \Theta_\delta(\xi, \xi_*, \sigma)) d\xi_* d\sigma, \end{aligned}$$

so that $\mathcal{L}_1 h = \mathcal{L}_1^{S,\delta} h + \mathcal{L}_1^{R,\delta} h - h \Sigma_{\mathcal{M}}$ where $\Sigma_{\mathcal{M}}$ denotes the mapping

$$\Sigma_{\mathcal{M}}(\xi) = \int_{\mathbb{R}^d} \mathcal{M}(\xi_*) |\xi - \xi_*| d\xi_*, \quad \xi \in \mathbb{R}^d. \quad (2.12)$$

Introduce

$$\mathcal{A}^{(\delta)}(h) := \mathcal{L}_1^{S,\delta}(h) \quad \text{and} \quad \mathcal{B}_1^{(\delta)}(h) := \mathcal{L}_1^{R,\delta} - \Sigma_{\mathcal{M}}$$

so that $\mathcal{L}_1 = \mathcal{A}^{(\delta)} + \mathcal{B}_1^{(\delta)}$. Let us recall [34, Lemmas 4.12, 4.14 & Lemma 4.16]:

Lemma 2.7. *For any $k \in \mathbb{N}$ and $\delta > 0$, there are two positive constants $C_{k,\delta} > 0$ and $R_\delta > 0$ such that $\text{supp}(\mathcal{A}^{(\delta)} f) \subset B(0, R_\delta)$ and*

$$\|\mathcal{A}^{(\delta)} f\|_{\mathbb{W}_v^{k,1}(\mathbb{R}^d)} \leq C_{k,\delta} \|f\|_{L_v^1(\varpi_1)}, \quad \forall f \in L^1(\mathbb{R}^d, \varpi_1(v) dv). \quad (2.13)$$

Moreover, for any $q > 2$ and any $\delta \in (0, 1)$ it holds

$$\int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} \langle \xi \rangle^q \text{sign}(h) \mathcal{B}_1^{(\delta)} h d\xi \leq (\Lambda_q(\delta) - 1) \|h\|_{L_x^1 L_\xi^1(\varpi_{q+1})} \quad (2.14)$$

where $\Lambda_q : (0, 1) \rightarrow \mathbb{R}^+$ is some explicit function such that $\lim_{\delta \rightarrow 0} \Lambda_q(\delta) = \frac{4}{q+2}$.

This leads to the following decomposition of \mathcal{L}_α :

$$\mathcal{L}_\alpha = \mathcal{B}_\alpha^{(\delta)} + \mathcal{A}^{(\delta)}, \quad \text{where} \quad \mathcal{B}_\alpha^{(\delta)} = \mathcal{B}_1^{(\delta)} + [\mathcal{L}_\alpha - \mathcal{L}_1].$$

2.3. The complete linearized operator. The complete linearized operator is given by

$$\mathcal{G}_{\alpha,\varepsilon}h = \varepsilon^{-2}\mathcal{L}_\alpha(h) - \varepsilon^{-1}v \cdot \nabla_x h, \quad \forall \alpha \in (0, 1].$$

With previous decomposition, we have that

$$\mathcal{G}_{\alpha,\varepsilon} = \mathcal{A}_\varepsilon^{(\delta)} + \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$$

where

$$\mathcal{A}_\varepsilon^{(\delta)} = \varepsilon^{-2}\mathcal{A}^{(\delta)}, \quad \mathcal{B}_{\alpha,\varepsilon}^{(\delta)} = \varepsilon^{-2}\mathcal{B}_\alpha^{(\delta)} - \varepsilon^{-1}v \cdot \nabla_x.$$

Notice that

$$\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} - \mathcal{B}_{1,\varepsilon}^{(\delta)} = \mathcal{G}_{\alpha,\varepsilon} - \mathcal{G}_{1,\varepsilon} = \varepsilon^{-2}\mathcal{P}_\alpha + \varepsilon^{-2}T_\alpha.$$

One has the following properties of $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ (see [59, Lemmas 2.7, 2.8, 2.9] for a similar result).

Proposition 2.8. *For any $k \geq s \geq 0$ and $q > 2$ there exist $\alpha_{\ell,s,q}^\dagger > 0$, $\delta_{\ell,s,q}^\dagger > 0$ and $\nu_{\ell,s,q} > 0$ such that*

$$\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2}\nu_{\ell,s,q} \text{ is hypo-dissipative in } \mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q), \quad \forall \alpha \in (\alpha_{\ell,s,q}^\dagger, 1), \quad \delta \in (0, \delta_{\ell,s,q}^\dagger).$$

Proof. Notice that derivatives with respect to the x -variable commute with the operator $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ and this allows to prove the result, without loss of generality, in the special case $\ell = s$. We divide the proof in several steps:

- We first consider the case $\ell = 0$. We write $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}(h) = \sum_{i=0}^3 C_i(h)$ with

$$\begin{aligned} C_0(h) &= \varepsilon^{-2}\mathcal{B}_1^{(\delta)}h, & C_1(h) &= -\varepsilon^{-1}v \cdot \nabla_x h, \\ C_2(h) &= \varepsilon^{-2}\mathcal{P}_\alpha h, & C_3(h) &= \varepsilon^{-2}T_\alpha h = -\varepsilon^{-2}\kappa_\alpha \nabla_v \cdot (v h(x, v)), \end{aligned}$$

and correspondingly and with obvious notations,

$$\int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}(h)(x, v) \text{sign}(h(v)) \varpi_q(v) dv =: \sum_{i=0}^3 I_i(h).$$

First,

$$\begin{aligned} I_1(h) &= \varepsilon^{-1} \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} \text{sign}(h(x, v)) v \cdot \nabla_x h(x, v) \varpi_q(v) dv \\ &= \varepsilon^{-1} \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} v \cdot \nabla_x |h(x, v)| \varpi_q(v) dv = 0 \end{aligned}$$

while, according to Eq. (2.14)

$$I_0(h) \leq \varepsilon^{-2} (\Lambda_q(\delta) - 1) \|h\|_{L_x^1 L_v^1(\varpi_{q+1})}.$$

Moreover, it follows from Proposition 2.6 that

$$I_2(h) \leq \varepsilon^{-2} \int_{\mathbb{T}^d} \|\mathcal{P}_\alpha h(x, \cdot)\|_{L_v^1(\varpi_q)} dx \leq \varepsilon^{-2} \mathbf{C}_{0,q} (1 - \alpha) \|h\|_{L_x^1 L_v^1(\varpi_{q+1})}.$$

Finally, since $h \nabla_v \text{sign} h = 0$, one has

$$\begin{aligned} I_3(h) &= -\varepsilon^{-2} \kappa_\alpha \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} \text{div}_v(v|h(x, v)|) \varpi_q(v) dv \\ &= \varepsilon^{-2} \kappa_\alpha \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} |h(x, v)| v \cdot \nabla_v \varpi_q(v) dv \end{aligned}$$

Since $v \cdot \nabla_v \varpi_q(v) = q \varpi_q(v) - q \varpi_{q-2}(v)$ we get

$$I_3(h) \leq q \kappa_\alpha \varepsilon^{-2} \|h\|_{L_x^1 L_v^1(\varpi_{q+1})}.$$

Gathering the previous estimates, one obtains

$$\begin{aligned} \mathcal{I} &:= \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} \mathcal{B}_{\alpha, \varepsilon}^{(\delta)}(h)(x, v) \text{sign}(h(x, v)) \varpi_q(v) dv \\ &\leq \varepsilon^{-2} (\mathbf{C}_{0, q}(1 - \alpha) + \Lambda_q(\delta) - 1 + q \kappa_\alpha) \|h\|_{L_x^1 L_v^1(\varpi_{q+1})}. \end{aligned} \quad (2.15)$$

Recalling that $\kappa_\alpha = 1 - \alpha$ while $\lim_{\delta \rightarrow 0} (\Lambda_q(\delta) - 1) = -\frac{q-2}{q+2} < 0$ we can pick $\delta_{0,0,q}^\dagger$ small enough and then $\alpha_{0,0,q}^\dagger \in (0, 1)$ close enough to 1 so that

$$\nu_{0,0,q} := -\inf \left\{ \mathbf{C}_{0, q}(1 - \alpha) + \Lambda_q(\delta) - 1 + q \kappa_\alpha; \alpha \in (\alpha_{0,0,q}^\dagger, 1), \delta \in (0, \delta_{0,0,q}^\dagger) \right\} > 0$$

and get the result.

Let us investigate the case $k = 1$ first. We consider the norm

$$\|h\| = \|h\|_{L_x^1 L_v^1(\varpi_q)} + \|\nabla_x h\|_{L_x^1 L_v^1(\varpi_q)} + \eta \|\nabla_v h\|_{L_x^1 L_v^1(\varpi_q)},$$

for some $\eta > 0$, the value of which shall be fixed later on. This norm is equivalent to the classical $\mathbb{W}_{x,v}^{1,1}(\varpi_q)$ -norm. We shall prove that for some $\nu_{1,1,q} > 0$, $\mathcal{B}_{\alpha, \varepsilon}^{(\delta)} + \varepsilon^{-2} \nu_{1,1,q}$ is dissipative in $W_{x,v}^{1,1}(\varpi_q)$ for the norm $\|\cdot\|$. Notice first that the x -derivative commutes with all the above terms $C_i(h)$, $i = 0, \dots, 3$, i.e.

$$\nabla_x \mathcal{B}_{\alpha, \varepsilon}^{(\delta)} h(x, v) = \mathcal{B}_{\alpha, \varepsilon}^{(\delta)} \nabla_x h(x, v)$$

so that, according to the previous step

$$\begin{aligned} \mathcal{J}_x &:= \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} \nabla_x \mathcal{B}_{\alpha, \varepsilon}^{(\delta)}(h)(x, v) \text{sign}(\nabla_x h(x, v)) \varpi_q(v) dv \\ &\leq -\varepsilon^{-2} \nu_{0,0,q} \|\nabla_x h\|_{L_x^1 L_v^1(\varpi_q)} \end{aligned} \quad (2.16)$$

where we used the short-hand notation

$$\text{sign}(\nabla_x h(x, v)) = (\text{sign}(\partial_{x_1} h(x, v)), \dots, \text{sign}(\partial_{x_d} h(x, v))).$$

Consider now the quantity

$$\mathcal{J}_v := \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} \nabla_v (\mathcal{B}_{\alpha, \varepsilon}^{(\delta)} h(x, v)) \cdot \text{sign}(\nabla_v h(x, v)) \varpi_q(v) dv.$$

Using the notations above, one notices that $\nabla_v C_1(h) = -\varepsilon^{-1} \nabla_x h + C_1(\nabla_v h)$, so that

$$\begin{aligned} \nabla_v(\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} h(x,v)) &= \varepsilon^{-2} \nabla_v(\mathcal{B}_1^{(\delta)} h) - \varepsilon^{-1} \nabla_x h + C_1(\nabla_v h) \\ &\quad + \varepsilon^{-2} \nabla_v(\mathcal{P}_\alpha h) + \varepsilon^{-2} \nabla_v(\mathcal{T}_\alpha h) \\ &= \varepsilon^{-2} \nabla_v[\mathcal{L}_1^{R,\delta} h - \Sigma_{\mathcal{M}} h] - \varepsilon^{-1} \nabla_x h + C_1(\nabla_v h) \\ &\quad + \varepsilon^{-2} \nabla_v(\mathcal{P}_\alpha h) + \varepsilon^{-2} \nabla_v(\mathcal{T}_\alpha h). \end{aligned} \quad (2.17)$$

Then, it follows from Proposition 2.6 that

$$\|\nabla_v(\mathcal{P}_\alpha h)\|_{L_v^1(\varpi_q)} \leq C_{1,q}(1-\alpha) \left(\|h\|_{L_v^1(\varpi_{1+q})} + \|\nabla_v h\|_{L_v^1(\varpi_{1+q})} \right). \quad (2.18)$$

Now,

$$\nabla_v[\mathcal{L}_1^{R,\delta} h - \Sigma_{\mathcal{M}} h] = \mathcal{L}_1^{R,\delta}(\nabla_v h) - \Sigma_{\mathcal{M}} \nabla_v h + \mathcal{R}(h),$$

where

$$\mathcal{R}(h) = \mathcal{Q}_1(h, \nabla_v \mathcal{M}) + \mathcal{Q}_1(\nabla_v \mathcal{M}, h) - (\nabla_v \mathcal{A}^{(\delta)})(h) - \mathcal{A}^{(\delta)}(\nabla_v h).$$

As in [59, p. 1942], an integration by parts leads to

$$\|(\nabla \mathcal{A}^{(\delta)})(h)\|_{L_v^1(\varpi_q)} + \|\mathcal{A}^{(\delta)}(\nabla_v h)\|_{L_v^1(\varpi_q)} \leq C_\delta \|h\|_{L_v^1(\varpi_q)},$$

for some constant $C_\delta > 0$. This and some classical estimates on $\mathcal{Q}_1(h, \nabla_v \mathcal{M}) + \mathcal{Q}_1(\nabla_v \mathcal{M}, h)$ yield

$$\|\mathcal{R}(h)\|_{L_v^1(\varpi_q)} \leq C_\delta \|h\|_{L_v^1(\varpi_{q+1})}.$$

Again, as in [59, Eq. (2.10)], one has

$$\|\mathcal{L}_1^{R,\delta}(\nabla_v h)\|_{L_v^1(\varpi_q)} \leq \tau(\delta) \|\nabla_v h\|_{L_v^1(\varpi_{q+1})},$$

where $\lim_{\delta \rightarrow 0} \tau(\delta) = 0$. Then, since $\Sigma_{\mathcal{M}}(\xi) \geq \sigma_0(\xi)$, one has that

$$\begin{aligned} - \int_{\mathbb{R}^d} \Sigma_{\mathcal{M}}(\xi) \nabla_\xi h(x, \xi) \cdot \text{sign}(\nabla_\xi h(x, \xi)) \varpi_q(\xi) \, d\xi &= - \int_{\mathbb{R}^d} \Sigma_{\mathcal{M}}(\xi) |\nabla_\xi h(\xi)| \varpi_q(\xi) \, d\xi \\ &\leq -\sigma_0 \|\nabla_v h\|_{L_v^1(\varpi_{q+1})}. \end{aligned}$$

Therefore,

$$\|\nabla_v[\mathcal{L}_1^{R,\delta}(h) - \Sigma_{\mathcal{M}} h]\|_{L_v^1(\varpi_q)} \leq C_\delta \|h\|_{L_v^1(\varpi_{q+1})} + (\tau(\delta) - \sigma_0) \|\nabla_v h\|_{L_v^1(\varpi_{q+1})} \quad (2.19)$$

where $\lim_{\delta \rightarrow 0^+} \tau(\delta) = 0$. Finally,

$$\begin{aligned} \int_{\mathbb{R}^d} \nabla_v(\mathcal{T}_\alpha h(x,v)) \cdot \text{sign}(\nabla_v h(v)) \varpi_q(v) \, dv &= -(d+1) \kappa_\alpha \int_{\mathbb{R}^d} |\nabla_v h(x,v)| \varpi_q(v) \, dv \\ &\quad + \kappa_\alpha \int_{\mathbb{R}^d} |\nabla_v h(x,v)| \nabla_v \cdot (v \varpi_q(v)) \, dv \leq C \kappa_\alpha \|\nabla_v h\|_{L_v^1(\varpi_{q+1})}. \end{aligned} \quad (2.20)$$

Combining (2.17) with the estimates (2.18), (2.19) and (2.20), one obtains that

$$\begin{aligned} \mathcal{J}_v &\leq \varepsilon^{-2}(C_\delta + \mathbf{C}_{1,q}(1 - \alpha)) \|h\|_{L_x^1 L_v^1(\varpi_{q+1})} \\ &\quad + \varepsilon^{-2}(\mathbf{C}_{1,q}(1 - \alpha) + C\kappa_\alpha + \tau(\delta) - \sigma_0) \|\nabla_v h\|_{L_x^1 L_v^1(\varpi_{q+1})}, \end{aligned}$$

where we used that the contribution to \mathcal{J}_v of the divergence term $-\varepsilon^{-1}\nabla_x h + C_1(\nabla_v h)$ vanishes. Hence, combining this estimate with (2.15) and (2.16) it follows that

$$\begin{aligned} \mathcal{I} + \mathcal{J}_x + \eta \mathcal{J}_v &\leq \varepsilon^{-2} \left([-\nu_{0,0,q} + \eta(C_\delta + \mathbf{C}_{1,q}(1 - \alpha))] \|h\|_{L_x^1 L_v^1(\varpi_{q+1})} \right. \\ &\quad \left. - \nu_{0,0,q} \|\nabla_x h\|_{L_x^1 L_v^1(\varpi_{q+1})} + \eta[\mathbf{C}_{1,q}(1 - \alpha) + C\kappa_\alpha + \tau(\delta) - \sigma_0] \|\nabla_v h\|_{L_x^1 L_v^1(\varpi_{q+1})} \right). \end{aligned}$$

Consequently, there exists $\alpha_{1,1,q}^\dagger > 0$ and $\delta_{1,1,q}^\dagger > 0$ so that

$$\mathbf{C}_{1,q}(1 - \alpha) + C\kappa_\alpha + \tau(\delta) - \sigma_0 < 0 \quad \forall \alpha \in (\alpha_{1,1,q}^\dagger, 1), \quad \delta \in (0, \delta_{1,1,q}^\dagger).$$

Choosing $\eta > 0$ small enough such that $\nu_{0,0,q} - \eta(C_\delta + \mathbf{C}_{1,q}(1 - \alpha)) > 0$, we finally obtain

$$\begin{aligned} \mathcal{I} + \mathcal{J}_x + \eta \mathcal{J}_v &\leq -\varepsilon^{-2} \nu_{1,1,q} \left[\|h\|_{L_x^1 L_v^1(\varpi_{q+1})} + \|\nabla_x h\|_{L_x^1 L_v^1(\varpi_{q+1})} + \eta \|\nabla_v h\|_{L_x^1 L_v^1(\varpi_{q+1})} \right] \\ &\leq -\varepsilon^{-2} \nu_{1,1,q} \|h\|, \end{aligned}$$

with $\nu_{1,1,q} := \min(\nu_{0,0,q} - \eta(C_\delta + \mathbf{C}_{1,q}(\alpha)), \sigma_0 - (\mathbf{C}_{1,q}(\alpha) + C\kappa_\alpha + \tau(\delta)))$. This proves that $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2}\nu_{1,1,q}$ is hypo-dissipative in $\mathbb{W}_{x,v}^{1,1}(\varpi_q)$. We prove the result for higher order derivatives in the same way considering now the norm

$$\|h\| = \sum_{|\beta_1| + |\beta_2| \leq k} \eta^{|\beta_1|} \left\| \nabla_v^{|\beta_1|} \nabla_x^{|\beta_2|} h \right\|_{L_x^1 L_v^1(\varpi_q)}$$

for some $\eta > 0$ to be chosen sufficiently small. \square

Remark 2.9. *It is important to notice that the equivalent norms constructed in the Proposition 2.8 are independent of ε . This means that the hypo-dissipativity of $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2}\nu_{\ell,s,q}$ on $\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)$ can be re-written as*

$$\|(\lambda - \varepsilon^{-2}\nu_{\ell,s,q} - \mathcal{B}_{\alpha,\varepsilon}^{(\delta)})g\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)} \geq C\lambda \|g\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)}$$

for any $\lambda > 0$, $g \in \mathcal{D}(\mathcal{B}_{\alpha,\varepsilon}^{(\delta)})$, and some constant $C > 0$ depending on ℓ, s, q but not on ε .

2.4. The elastic semigroup. The spectral analysis of $\mathcal{G}_{1,\varepsilon}$ and the generation of its associated semigroup has been performed in [15, Theorem 2.1]. We need a slightly more precise estimate on the decay of the semigroup independently of ε . Our main result concerning $\mathcal{G}_{1,\varepsilon}$ is the following whose proof is postponed to Appendix C:

Theorem 2.10. *There exists $\varepsilon_0 \in (0, 1)$ such that, for all $\ell, s \in \mathbb{N}$ with $\ell \geq s$ and $q > 2$ and any $\varepsilon \in (0, \varepsilon_0)$, the full transport operator $\mathcal{G}_{1,\varepsilon}$ generates a C_0 -semigroup $\{\mathcal{V}_{1,\varepsilon}(t) ; t \geq 0\}$ on $\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)$ and there exist $C_0 > 0$ and $\mu_\star > 0$ (both independent of ε) such that*

$$\begin{aligned} & \|\mathcal{V}_{1,\varepsilon}(t)[h - \mathbf{P}_0 h]\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)} \\ & \leq C_0 \exp(-\mu_\star t) \|h - \mathbf{P}_0 h\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)}, \quad \forall t \geq 0, \end{aligned} \quad (2.21)$$

holds true for any $h \in \mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)$, where \mathbf{P}_0 is the spectral projection onto $\text{Ker}(\mathcal{G}_{1,\varepsilon}) = \text{Ker}(\mathcal{L}_1)$ which is independent of ε and given by

$$\mathbf{P}_0 h = \sum_{i=1}^{d+2} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} h \Psi_i \, dx dv \right) \Psi_i \mathcal{M} \quad (2.22)$$

where $\Psi_1(v) = 1$, $\Psi_i(v) = \frac{1}{\sqrt{\vartheta_1}} v_{i-1}$ ($i = 2, \dots, d+1$) and $\Psi_{d+2}(v) = \frac{|v|^2 - d\vartheta_1}{\vartheta_1 \sqrt{2d}}$ ($v \in \mathbb{R}^d$).

Remark 2.11. *Theorem 2.10 is known to be true on the Hilbert space $H_{x,v}^\ell(\mathcal{M}^{-\frac{1}{2}})$, see [14, Theorem 2.1] whereas, in the present context, a similar result was obtained in [15, Theorem 2.1] with the important difference that the estimate (2.21) was shown only for $t > t_\star > 0$. This actually comes from the use of a general enlargement theorem from [34] which yields*

$$\|\mathcal{V}_{1,\varepsilon}(t)h - \mathbf{P}_0 h\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)} \leq C_0 \frac{t^N}{\varepsilon^{N(2+s)}} \exp(-\mu t) \|h - \mathbf{P}_0 h\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)}, \quad t \geq 0$$

for some $N \in \mathbb{N}$ and $s > 0$ and $\mu > \mu_\star$. It is important for the rest of our analysis to be able to remove this strong dependence on ε in the decay estimate of $\mathcal{V}_{1,\varepsilon}(t)(\mathbf{Id} - \mathbf{P}_0)$. This is done in Appendix C.

An important consequence of the Theorem 2.10 is the following proposition.

Proposition 2.12. *Let $\ell, s \in \mathbb{N}$ with $\ell \geq s$ and $q > 2$. There exists $C_1 > 0$ such that*

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q))} \leq C_1 \max\left(\frac{1}{|\lambda|}, \frac{1}{\text{Re}\lambda + \mu_\star}\right), \quad \forall \lambda \in \mathbb{C}_{\mu_\star}^\star, \quad \forall \varepsilon \in (0, \varepsilon_0),$$

where ε_0 and μ_\star have been defined in Theorem 2.10, C_1 being independent of ε .

Proof. On the space $\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)$, the spectrum of $\mathcal{G}_{1,\varepsilon}$ satisfies

$$\mathfrak{S}(\mathcal{G}_{1,\varepsilon}) \cap \{z \in \mathbb{C} ; \text{Re}z > -\mu_\star\} = \{0\}$$

and the above projection \mathbf{P}_0 is nothing but the spectral projection of $\mathfrak{S}(\mathcal{G}_{1,\varepsilon})$ associated to the zero eigenvalue given by

$$\mathbf{P}_0 = \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(z, \mathcal{G}_{1,\varepsilon}) dz, \quad \gamma_r := \{z \in \mathbb{C} ; |z| = r\}, \quad r < \mu_\star.$$

Notice also

$$\dim(\text{Range}(\mathbf{P}_0)) = \dim \text{Ker}(\mathcal{G}_{1,\varepsilon}) = d + 2,$$

which means that the algebraic multiplicity of the zero eigenvalue coincides with its geometrical multiplicity and, as such, 0 is a *simple* pole of the resolvent $\mathcal{R}(\cdot, \mathcal{G}_{1,\varepsilon})$ (see [42, III.5]). Denote by $\|\cdot\|$ the operator norm in $\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)$ and fix $\mu \in (0, \mu_*)$. Since $\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) = \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\mathbf{P}_0 + \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})(\mathbf{Id} - \mathbf{P}_0)$ and \mathbf{P}_0 commutes with $\mathcal{G}_{1,\varepsilon}$, we only need to estimate independently

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\mathbf{P}_0\| \quad \text{and} \quad \|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})[\mathbf{Id} - \mathbf{P}_0]\|$$

for any $\lambda \in \mathbb{C}_{\mu}^*$. Since the multiplicity of the pole 0 is one, one has $\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\mathbf{P}_0 = \frac{1}{\lambda}\mathbf{P}_0$ and

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\mathbf{P}_0\| \leq \frac{\|\mathbf{P}_0\|}{|\lambda|}, \quad \lambda \in \mathbb{C}_{\mu_*}^*.$$

On the other hand, since for any $\lambda \in \mathbb{C}_{\mu_*}$

$$\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})[\mathbf{Id} - \mathbf{P}_0] = \int_0^\infty e^{-\lambda t} \mathcal{V}_{1,\varepsilon}(t) [\mathbf{Id} - \mathbf{P}_0] dt,$$

one deduces from Theorem 2.10 that

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})[\mathbf{Id} - \mathbf{P}_0]\| \leq C_0 \int_0^\infty e^{-\operatorname{Re}\lambda t} e^{-\mu_* t} \|\mathbf{Id} - \mathbf{P}_0\| dt,$$

which gives that

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})[\mathbf{Id} - \mathbf{P}_0]\| \leq C_0 \|\mathbf{Id} - \mathbf{P}_0\| \frac{1}{\operatorname{Re}\lambda + \mu_*}, \quad \forall \lambda \in \mathbb{C}_{\mu_*}.$$

This gives the desired estimate with $C_1 = \|\mathbf{P}_0\| + C_0 \|\mathbf{Id} - \mathbf{P}_0\|$ independent of ε and μ . \square

We end this section with the following semigroup generation result.

Proposition 2.13. *For any $\ell \geq s \geq 0$, $q > 2$, $\alpha \in (\alpha_{\ell,s,q}^\dagger, 1)$, $\delta \in (0, \delta_{\ell,s,q}^\dagger)$ and $\varepsilon > 0$, the operator*

$$\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} : \mathcal{D}(\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}) \subset \mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q) \longrightarrow \mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)$$

is the generator of a C_0 -semigroup $\{\mathcal{S}_{\alpha,\varepsilon}^{(\delta)}(t) ; t \geq 0\}$ in $\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)$ and there exist $0 < \nu_ < \nu_{\ell,s,q}$ and $C_{\ell,s,q} > 0$ such that*

$$\left\| \mathcal{S}_{\alpha,\varepsilon}^{(\delta)}(t) \right\|_{\mathcal{B}(\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q))} \leq C_{\ell,s,q} \exp(-\varepsilon^{-2}\nu_* t), \quad \forall t \geq 0. \quad (2.23)$$

As a consequence,

$$\mathcal{G}_{\alpha,\varepsilon} : \mathcal{D}(\mathcal{G}_{\alpha,\varepsilon}) \subset \mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q) \longrightarrow \mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)$$

is the generator of a C_0 -semigroup $\{\mathcal{V}_{\alpha,\varepsilon}(t) ; t \geq 0\}$ in $\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)$.

Proof. The fact that $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ is a generator of a C_0 -semigroup in $\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)$ is proven in Appendix C. Since we already proved that $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2}\nu_{\ell,s,q}$ is hypo-dissipative, we deduce directly (2.23). Finally, because $\mathcal{A}_\varepsilon^{(\delta)}$ is a bounded operator in $\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)$, we deduce from the bounded perturbation theorem that $\mathcal{G}_{\alpha,\varepsilon} = \mathcal{A}_\varepsilon^{(\delta)} + \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ generates a C_0 -semigroup in $\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)$. \square

3. LINEAR THEORY IN THE WEAKLY INELASTIC REGIME

The final goal of this section is to prove Theorem 1.7, so, in the sequel we define

$$\begin{aligned} \mathbb{X} &:= \mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q), & \mathbb{Y} &:= \mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s+1,1}(\varpi_{q+2}) \\ & \ell, s \in \mathbb{N}, \quad \ell \geq s+1 \quad \text{and} \quad q > 2 \end{aligned}$$

so that $\mathbb{Y} \subset \mathbb{X}$.

We recall that, in the space \mathbb{X} , the full linearized operator is given by

$$\mathcal{G}_{\alpha,\varepsilon} h = \varepsilon^{-2} \mathcal{L}_\alpha(h) - \varepsilon^{-1} v \cdot \nabla_x h, \quad \forall \alpha \in (0, 1],$$

with domain $\mathcal{D}(\mathcal{G}_{\alpha,\varepsilon}) = \mathbb{W}_x^{\ell+1} \mathbb{W}_v^{s+1,1}(\varpi_{q+1})$.

Clearly, any spatially homogeneous eigenfunction of \mathcal{L}_α associated to an eigenvalue $\lambda \in \mathbb{C}$ is an eigenfunction to $\mathcal{G}_{\alpha,\varepsilon}$ with associated eigenvalue $\varepsilon^{-2}\lambda$. In particular

$$\text{Ker}(\mathcal{L}_\alpha) \subset \text{Ker}(\mathcal{G}_{\alpha,\varepsilon}).$$

Notice that, in contrast to [14, 15], it is not clear whether such spaces agree. We deduce in particular from Proposition 2.4 that, on the space $L_x^1 L_v^1(\varpi_q)$,

$$-\varepsilon^{-2} \mu_\alpha \in \mathfrak{S}(\mathcal{G}_{\alpha,\varepsilon})$$

with associated eigenfunction ϕ_α , that is,

$$\mathcal{G}_{\alpha,\varepsilon} \phi_\alpha = -\varepsilon^{-2} \mu_\alpha \phi_\alpha.$$

For the eigenvalue $-\varepsilon^{-2} \mu_\alpha$ to stay sufficiently close to 0, we assume that $\alpha = \alpha(\varepsilon)$ satisfies Assumption 1.2 and write

$$\mathcal{G}_\varepsilon = \mathcal{G}_{\alpha(\varepsilon),\varepsilon},$$

and keep the notation $\mathcal{G}_{1,\varepsilon}$ for the elastic operator. Similarly, for all the operators introduced in Section 2.1 the double subscript (α, ε) will be replaced by ε except when $\alpha = 1$. More precisely, to fix notations, we have

$$\mathcal{G}_\varepsilon h = \varepsilon^{-2} \mathcal{L}_{\alpha(\varepsilon)} h - \varepsilon^{-1} v \cdot \nabla_x h,$$

with

$$\mathcal{L}_{\alpha(\varepsilon)} h = \mathbf{L}_{\alpha(\varepsilon)} h - \kappa_{\alpha(\varepsilon)} \nabla_v \cdot (vh),$$

and

$$\mathbf{L}_{\alpha(\varepsilon)} h = \mathcal{Q}_{\alpha(\varepsilon)}(h, G_{\alpha(\varepsilon)}) + \mathcal{Q}_{\alpha(\varepsilon)}(G_{\alpha(\varepsilon)}, h).$$

In the sequel, since ℓ, s, q are fixed, we set

$$\delta^\dagger := \min \{ \delta_{\ell,s,q}^\dagger, \delta_{\ell,s+1,q+2}^\dagger \}, \quad \alpha^\dagger := \max \{ \alpha_{\ell,s,q}^\dagger, \alpha_{\ell,s+1,q+2}^\dagger \},$$

so that, for $\delta \in (0, \delta^\dagger)$ and $\alpha \in (\alpha^\dagger, 1)$, the results of the previous section hold in both the spaces \mathbb{X}, \mathbb{Y} . Moreover, we denote by $\varepsilon^\dagger > 0$ the *unique* solution to

$$\alpha(\varepsilon^\dagger) = \alpha^\dagger.$$

We consider $\delta \in (0, \delta^\dagger), \varepsilon \in (0, \varepsilon^\dagger)$ (which implies $\alpha(\varepsilon) \in (\alpha^\dagger, 1)$), and write

$$\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon^{(\delta)}, \quad \mathcal{B}_\varepsilon = \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}.$$

One has the following result which is similar to [59, Lemma 2.16]. We adapt and drastically simplify the proof given there by exploiting the fact that the difference operator $\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon}$ does not involve any spatial derivatives:

Proposition 3.1. *For all $\lambda \in \mathbb{C}_{\mu_\star}^\star$, let*

$$\mathcal{J}_\varepsilon(\lambda) = (\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon}) \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon).$$

Then, $\mathcal{J}_\varepsilon(\lambda) \in \mathcal{B}(\mathbb{X})$. Furthermore, for any $\mu \in (0, \mu_\star)$ and

$$\lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu) = \{z \in \mathbb{C} ; \operatorname{Re} z > -\mu, |z| > \mu_\star - \mu\},$$

it holds that

$$\|\mathcal{J}_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{X})} \leq \frac{C}{\mu_\star - \mu} \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \quad (3.1)$$

for a universal constant $C > 0$.

In addition, there exists $\varepsilon^\star \in (0, \varepsilon^\dagger)$ such that $\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda)$ and $\lambda - \mathcal{G}_\varepsilon$ are invertible in \mathbb{X} with

$$\mathcal{R}(\lambda, \mathcal{G}_\varepsilon) = \Gamma_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu), \quad \varepsilon \in (0, \varepsilon^\star), \quad (3.2)$$

where $\Gamma_\varepsilon(\lambda) = \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) + \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)$. Finally, there exists some constant $C > 0$ such that

$$\|\mathcal{R}(\lambda, \mathcal{G}_\varepsilon)\|_{\mathcal{B}(\mathbb{X})} \leq \frac{C}{\mu_\star - \mu}, \quad \forall \lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu), \quad \varepsilon \in (0, \varepsilon^\star). \quad (3.3)$$

Proof. We adapt the method of [59, Lemma 2.16] but simplifies it in several aspects. For $\operatorname{Re} \lambda > -\mu_\star$, $\lambda \neq 0$, one knows from Proposition 2.12 that $\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \in \mathcal{B}(\mathbb{Y})$ since we assumed $k \geq s + 1$ and there is $C_1 > 0$ such that, for any $\varepsilon \in (0, \varepsilon^\dagger)$, it holds that

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y})} \leq C_1 \max \left(\frac{1}{|\lambda|}, \frac{1}{\operatorname{Re} \lambda + \mu_\star} \right), \quad \lambda \in \mathbb{C}_{\mu_\star}^\star.$$

Moreover, from Proposition 2.8, there is $\nu > 0$ such that $\mathcal{B}_\varepsilon + \varepsilon^{-2}\nu$ is hypo-dissipative in both \mathbb{Y} and \mathbb{X} . In particular (see Remark 2.9) there exists $C_2 > 0$, independent of ε , such that

$$\|\mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{X})} \leq \frac{C_2}{\operatorname{Re} \lambda + \varepsilon^{-2}\nu}, \quad \forall \operatorname{Re} \lambda > -\mu_\star.$$

Therefore, as soon as $\varepsilon^{-2}\nu > 2\mu_\star$, one gets

$$\|\mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{X})} \leq \frac{C_2 \varepsilon^2}{\varepsilon^2 \operatorname{Re} \lambda + \nu} \leq C_3 \varepsilon^2, \quad \forall \operatorname{Re} \lambda > -\mu_\star, \quad (3.4)$$

with $C_3 = 2C_2/\nu$. A similar estimate holds true if \mathbb{X} is replaced with \mathbb{Y} . Notice that the regularization properties of \mathcal{A}_ε in both velocity regularity and tail behaviour implies that there exists $C > 0$ (independent of ε) such that $\|\mathcal{A}_\varepsilon\|_{\mathcal{B}(\mathbb{X}, \mathbb{Y})} \leq C\varepsilon^{-2}$ from which

$$\|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{X}, \mathbb{Y})} \leq C_4, \quad \forall \operatorname{Re} \lambda > -\mu_\star,$$

with $C_4 = C_3 C$. Finally, notice that in the difference $\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon} = \varepsilon^{-2} [\mathcal{L}_{\alpha(\varepsilon)} - \mathcal{L}_1]$, the transport term $v \cdot \nabla_x$ vanishes and, according to (2.9), $\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon} \in \mathcal{B}(\mathbb{Y}, \mathbb{X})$ for $\varepsilon \in (0, \varepsilon^\dagger)$ with

$$\|\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon}\|_{\mathcal{B}(\mathbb{Y}, \mathbb{X})} \leq C_0 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \quad (3.5)$$

for some positive constant C_0 independent of ε . We deduce with this that, for any $\operatorname{Re} \lambda > -\mu_\star$, $\lambda \neq 0$, the operator $\mathcal{J}_\varepsilon(\lambda) \in \mathcal{B}(\mathbb{X})$ is well-defined and, for any $r \in (0, \mu_\star)$

$$\begin{aligned} \|\mathcal{J}_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{X})} &\leq \|\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon}\|_{\mathcal{B}(\mathbb{Y}, \mathbb{X})} \|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{Y})} \|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{X}, \mathbb{Y})} \\ &\leq C_5 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \max\left(\frac{1}{|\lambda|}, \frac{1}{\operatorname{Re} \lambda + \mu_\star}\right), \quad \lambda \in \mathbb{C}_{\mu_\star}^\star, \end{aligned}$$

with $C_5 := C_0 C_1 C_4 > 0$ independent of ε . Then, for $\mu \in (0, \mu_\star)$ it holds

$$\|\mathcal{J}_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{X})} \leq C_5 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \max\left(\frac{1}{|\lambda|}, \frac{1}{\mu_\star - \mu}\right), \quad \lambda \in \mathbb{C}_\mu^\star, \quad (3.6)$$

which gives (3.1). With this, under Assumptions 1.2, one can choose ε^\star small enough, depending on the difference $|\mu_\star - \mu|$, so that

$$\rho(\varepsilon) = \frac{C_5}{\mu_\star - \mu} \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} < 1, \quad \forall \varepsilon \in (0, \varepsilon^\star). \quad (3.7)$$

Under such an assumption, one sees that, for all $\lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$, $\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda)$ is invertible in \mathbb{X} with

$$(\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} = \sum_{p=0}^{\infty} [\mathcal{J}_\varepsilon(\lambda)]^p, \quad \forall \varepsilon \in (0, \varepsilon^\star).$$

Let us fix then $\varepsilon \in (0, \varepsilon^\star)$ and $\lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$. The range of $\Gamma_\varepsilon(\lambda)$ is clearly included in $\mathcal{D}(\mathcal{B}_\varepsilon) = \mathcal{D}(\mathcal{G}_{1,\varepsilon})$. Then, writing $\mathcal{G}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ we easily get that

$$(\lambda - \mathcal{G}_\varepsilon)\Gamma_\varepsilon(\lambda) = \mathbf{Id} - \mathcal{J}_\varepsilon(\lambda)$$

i.e. $\Gamma_\varepsilon(\lambda)(\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1}$ is a right-inverse of $(\lambda - \mathcal{G}_\varepsilon)$. To prove that $\lambda - \mathcal{G}_\varepsilon$ is invertible, it is therefore enough to prove that it is one-to-one. Consider the eigenvalue problem

$$\mathcal{G}_\varepsilon h = \lambda h, \quad h \in \mathcal{D}(\mathcal{G}_\varepsilon),$$

Writing this as $(\lambda - \mathcal{G}_{1,\varepsilon})h = \mathcal{G}_\varepsilon h - \mathcal{G}_{1,\varepsilon}h$, there is a positive constant $C_6 > 0$ independent of ε such that

$$\|h\|_{\mathbb{X}} = \|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})(\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon})h\|_{\mathbb{X}} \leq C_6 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \|h\|_{\mathbb{Y}} \quad (3.8)$$

where we used Proposition 2.12 to estimate $\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{X})}$ on $\mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$ and (3.5) for the difference $(\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon})h$. Let us now estimate $\|h\|_{\mathbb{Y}}$. Since $\mathcal{G}_\varepsilon h = \lambda h$, one has $(\lambda - \mathcal{B}_\varepsilon)h = \mathcal{A}_\varepsilon h$ and $h = \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\mathcal{A}_\varepsilon h$, so that, thanks to (3.4),

$$\|h\|_{\mathbb{Y}} \leq \|\mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y})} \|\mathcal{A}_\varepsilon h\|_{\mathbb{Y}} \leq C_3 \varepsilon^2 \|\mathcal{A}_\varepsilon h\|_{\mathbb{Y}} \leq C_3 \|\mathcal{A}\|_{\mathcal{B}(\mathbb{X}, \mathbb{Y})} \|h\|_{\mathbb{X}}$$

where we recall that $\mathcal{A}_\varepsilon = \varepsilon^{-2}\mathcal{A} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$. Combining this with the above estimate (3.8), we end up with

$$\|h\|_{\mathbb{X}} \leq C_7 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \|h\|_{\mathbb{X}}$$

with $C_7 := C_6 C_3 \|\mathcal{A}\|_{\mathcal{B}(\mathbb{X}, \mathbb{Y})}$ independent of ε . One sees that, up to reducing ε^* , one can assume that $C_7 \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} < 1$ for $\varepsilon \in (0, \varepsilon^*)$ which implies that $h = 0$. This proves that $\lambda - \mathcal{G}_\varepsilon$ is one-to-one and its right-inverse is, actually, its inverse. Thus, for $\varepsilon \in (0, \varepsilon^*)$, $\mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$ belongs to the resolvent set of \mathcal{G}_ε and this shows (3.2). To estimate now $\|\mathcal{R}(\lambda, \mathcal{G}_\varepsilon)\|_{\mathcal{B}(\mathbb{X})}$ one simply notices that

$$\|(\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1}\|_{\mathcal{B}(\mathbb{X})} \leq \sum_{p=0}^{\infty} \|\mathcal{J}_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{X})}^p \leq \frac{1}{1 - \rho(\varepsilon)}, \quad \forall \lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu) \quad (3.9)$$

from which, as soon as $\lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$,

$$\|\mathcal{R}(\lambda, \mathcal{G}_\varepsilon)\|_{\mathcal{B}(\mathbb{X})} \leq \frac{1}{1 - \rho(\varepsilon)} \|\Gamma_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{X})}.$$

One checks, using the previous computations, that for $\lambda \in \mathbb{C}_\mu \setminus \mathbb{D}(\mu_\star - \mu)$,

$$\|\Gamma_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{X})} \leq C_3 \varepsilon^2 + C_3 \|\mathcal{A}\|_{\mathcal{B}(\mathbb{X})} \|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{X})} \quad (3.10)$$

and deduces (3.3). This achieves the proof. \square

Remark 3.2. *Of course, the above result is relevant mainly for $\frac{1}{2}\mu_\star < \mu < \mu_\star$ for which $\mathbb{D}(\mu_\star - \mu) \subset \mathbb{C}_\mu$, see Figure 1. Notice also that, in previous statement, the parameter ε^* is depending only on the gap*

$$\chi := \mu_\star - \mu.$$

From (3.7) we consider ε for which

$$\lambda_0 := \lim_{\varepsilon \rightarrow 0^+} \frac{1 - \alpha(\varepsilon)}{\varepsilon^2} \ll \chi,$$

therefore, λ_0 is a fraction of χ .

A first obvious consequence of Proposition 3.1 is that, for any $\mu \in (0, \mu_\star)$, there is $\varepsilon^* \in (0, \varepsilon^\dagger)$ depending only on $\chi = \mu_\star - \mu$ such that

$$\mathfrak{S}(\mathcal{G}_\varepsilon) \cap \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > -\mu\} \subset \{z \in \mathbb{C}; |z| \leq \mu_\star - \mu\}, \quad \forall \varepsilon \in (0, \varepsilon^*).$$

We denote by \mathbf{P}_ε the spectral projection associated to the set

$$\mathfrak{S}_\varepsilon := \mathfrak{S}(\mathcal{G}_\varepsilon) \cap \mathbb{C}_\mu = \mathfrak{S}(\mathcal{G}_\varepsilon) \cap \mathbb{D}(\mu_\star - \mu).$$

One can deduce then the following lemma whose proof is similar to [59, Lemma 2.17].

Lemma 3.3. *For any $\mu \in (0, \mu_\star)$ there is some $\varepsilon_0^* \in (0, \varepsilon^*)$ depending only on $\mu_\star - \mu$ and such that*

$$\|\mathbf{P}_\varepsilon - \mathbf{P}_0\|_{\mathcal{B}(\mathbb{X})} < 1, \quad \forall \varepsilon \in (0, \varepsilon_0^*).$$

In particular,

$$\dim \operatorname{Range}(\mathbf{P}_\varepsilon) = \dim \operatorname{Range}(\mathbf{P}_0) = d + 2, \quad \forall \varepsilon \in (0, \varepsilon_0^*). \quad (3.11)$$

Proof. Let $\frac{\mu_*}{2} < \mu < \mu_*$ and $0 < r < \chi := \mu_* - \mu$. Recall that ε^* depends only on χ . One has $\mathbb{D}(r) \subset \mathbb{C}_\mu^*$. We set $\gamma_r := \{z \in \mathbb{C} ; |z| = r\}$. Recall that

$$\mathbf{P}_\varepsilon := \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{G}_\varepsilon) d\lambda, \quad \mathbf{P}_0 := \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) d\lambda.$$

For $\lambda \in \gamma_r$, set

$$\mathcal{Z}_\varepsilon(\lambda) = \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)$$

so that $\Gamma_\varepsilon(\lambda) = \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) + \mathcal{Z}_\varepsilon(\lambda)$. Recall from (3.2) that, for $\lambda \in \gamma_r$,

$$\begin{aligned} \mathcal{R}(\lambda, \mathcal{G}_\varepsilon) &= \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} + \mathcal{Z}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} \\ &= \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) + \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) \mathcal{J}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} + \mathcal{Z}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} \end{aligned}$$

where we wrote $(\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} = \mathbf{Id} + \mathcal{J}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1}$. In the same way, one sees that

$$\begin{aligned} \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) &= \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) + \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) \\ &= \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) + \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon [\mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) - \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)] + \mathcal{Z}_\varepsilon(\lambda). \end{aligned}$$

Since the mappings $\lambda \in \mathbb{D}(r) \mapsto \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)$ and $\lambda \in \mathbb{D}(r) \mapsto \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})$ are analytic, one has

$$\oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) d\lambda = \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) d\lambda = 0,$$

so that

$$\mathbf{P}_\varepsilon = \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) \mathcal{J}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} d\lambda + \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{Z}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} d\lambda,$$

whereas

$$\mathbf{P}_0 = \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon [\mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) - \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)] d\lambda + \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{Z}_\varepsilon(\lambda) d\lambda.$$

Consequently, one easily obtains that

$$\begin{aligned} \mathbf{P}_\varepsilon - \mathbf{P}_0 &= \frac{1}{2i\pi} \oint_{\gamma_r} \Gamma_\varepsilon(\lambda) \mathcal{J}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1} d\lambda \\ &\quad + \frac{1}{2i\pi} \oint_{\gamma_r} \mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon [\mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})] d\lambda. \end{aligned}$$

Using (3.6), (3.9), and (3.3), one notices that there exists $C > 0$ independent of ε such that

$$\|\Gamma_\varepsilon(\lambda) \mathcal{J}_\varepsilon(\lambda) (\mathbf{Id} - \mathcal{J}_\varepsilon(\lambda))^{-1}\|_{\mathcal{B}(\mathbb{X})} \leq \frac{C}{r^2(1-\rho(\varepsilon))} \frac{1-\alpha(\varepsilon)}{\varepsilon^2}, \quad \forall \lambda \in \gamma_r,$$

where we used that $0 < r < \mu_* - \mu$ and noticed that $\|\Gamma_\varepsilon(\lambda)\|_{\mathcal{B}(\mathbb{X})} \leq C/r$ by virtue of (3.10). Moreover, from Proposition 2.12, it follows that

$$\|\mathcal{R}(\lambda, \mathcal{G}_{1,\varepsilon}) \mathcal{A}_\varepsilon [\mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})]\|_{\mathcal{B}(\mathbb{X})} \leq \frac{C_1}{r} \|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{X})}$$

for any $\lambda \in \gamma_r$, from which

$$\|\mathbf{P}_\varepsilon - \mathbf{P}_0\|_{\mathcal{B}(\mathbb{X})} \leq \frac{C_0}{r^2(1-\rho(\varepsilon))} \left(\frac{1-\alpha(\varepsilon)}{\varepsilon^2} + \sup_{\lambda \in \gamma_r} \|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{X})} \right)$$

for some positive constant $C_0 > 0$ independent of ε . We only need to estimate

$$\|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{X})}$$

for $\lambda \in \gamma_r$. Observe that, for $\lambda \in \gamma_r$,

$$\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon}) = \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) [\mathcal{B}_\varepsilon - \mathcal{B}_{1,\varepsilon}] \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})$$

and, with the notations of the proof of Proposition 3.1,

$$\begin{aligned} \|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon) - \mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{X})} \\ \leq \|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y}, \mathbb{X})} \|\mathcal{B}_\varepsilon - \mathcal{B}_{1,\varepsilon}\|_{\mathcal{B}(\mathbb{X}, \mathbb{Y})} \|\mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{X})} \end{aligned}$$

where, as in Proposition 3.1, there is a positive constant $C > 0$ independent of ε such that

$$\|\mathcal{A}_\varepsilon \mathcal{R}(\lambda, \mathcal{B}_\varepsilon)\|_{\mathcal{B}(\mathbb{Y}, \mathbb{X})} \leq C, \quad \|\mathcal{R}(\lambda, \mathcal{B}_{1,\varepsilon})\|_{\mathcal{B}(\mathbb{X})} \leq C, \quad \lambda \in \gamma_r,$$

whereas

$$\|\mathcal{B}_\varepsilon - \mathcal{B}_{1,\varepsilon}\|_{\mathcal{B}(\mathbb{X}, \mathbb{Y})} = \|\mathcal{G}_\varepsilon - \mathcal{G}_{1,\varepsilon}\|_{\mathcal{B}(\mathbb{X}, \mathbb{Y})} \leq C \frac{1-\alpha(\varepsilon)}{\varepsilon^2}.$$

Gathering the previous estimates, it follows that, for any $0 < r < \chi = \mu_* - \mu$,

$$\|\mathbf{P}_\varepsilon - \mathbf{P}_0\|_{\mathcal{B}(\mathbb{X})} \leq \frac{C}{r^2(1-\rho(\varepsilon))} \frac{1-\alpha(\varepsilon)}{\varepsilon^2} := \ell(\varepsilon) \quad (3.12)$$

and, thanks to Assumption 1.2, one can find ε_* depending only on χ such that $\ell(\varepsilon) < 1$ for any $\varepsilon \in (0, \varepsilon^*)$. In particular, we deduce (3.11) from [42, Paragraph I.4.6]. \square

With Lemma 3.3 we can prove Theorem 1.7

Proof of Theorem 1.7. The structure of the spectrum of $\mathfrak{S}(\mathcal{G}_\varepsilon) \cap \mathbb{C}_\mu$ in the space \mathbb{X} comes directly from Lemma 3.3 together with Proposition 3.1. To describe more precisely the spectrum, one first recalls that

$$\mathfrak{S}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \{z \in \mathbb{C}; \operatorname{Re} z > -\mu\} \subset \mathfrak{S}(\mathcal{G}_\varepsilon) \cap \{z \in \mathbb{C}; \operatorname{Re} z > -\mu\}.$$

Since, for ε small enough, the spectral projection $\Pi_{\mathcal{L}_{\alpha(\varepsilon)}}$ associated to $\mathfrak{S}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \mathbb{C}_\mu$ satisfies

$$\dim(\operatorname{Range}(\Pi_{\mathcal{L}_{\alpha(\varepsilon)}})) = \dim(\operatorname{Range}(\Pi_{\mathcal{L}_1})) = d+2 = \dim(\operatorname{Range}(\mathbf{P}_\varepsilon)),$$

we get that

$$\mathfrak{S}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \mathbb{C}_\mu = \mathfrak{S}(\mathcal{G}_\varepsilon) \cap \mathbb{C}_\mu, \quad (3.13)$$

that is, the eigenvalues $\lambda_j(\varepsilon)$ are actually eigenvalues of $\mathcal{L}_{\alpha(\varepsilon)}$. In particular, one has that

$$\lambda_{d+2}(\varepsilon) = -\varepsilon^{-2} \mu_{\alpha(\varepsilon)} = -\frac{1-\alpha(\varepsilon)}{\varepsilon^2} + \mathcal{O}\left(\left(\frac{1-\alpha(\varepsilon)}{\varepsilon}\right)^2\right), \quad \text{for } \varepsilon \simeq 0,$$

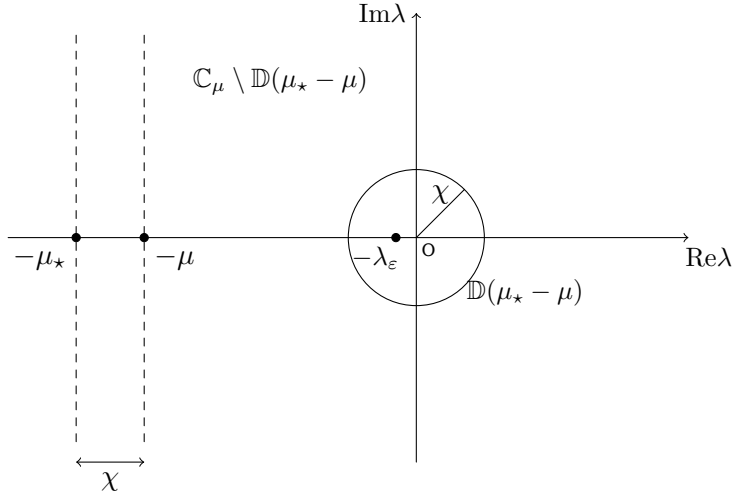


FIGURE 1. The set $\mathbb{C}_\mu \setminus \mathbb{D}(\mu_* - \mu)$ and the eigenvalue $-\lambda_\varepsilon$.

according to (2.5) and (2.6). We set

$$\lambda_\varepsilon := -\lambda_{d+2}(\varepsilon) > 0, \quad \lambda_\varepsilon \simeq -\varepsilon^{-2}(1 - \alpha(\varepsilon)).$$

For the other eigenvalues, one notices that

$$\int_{\mathbb{R}^d} \mathcal{L}_{\alpha(\varepsilon)} \varphi(v) dv = 0, \quad \forall \varphi \in \mathcal{D}(\mathcal{L}_{\alpha(\varepsilon)}) \subset \mathbb{Y},$$

where we recall that $\mathbb{Y} = \mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s+1,1}(\varpi_{q+2})$. Of course, the spatial variable x plays no role here since $\mathcal{L}_{\alpha(\varepsilon)}$ is local in x . We begin with understanding the eigenfunctions in

$$\mathbb{X}_0 = L_x^1 L_v^1(\varpi_q).$$

Recall that

$$\int_{\mathbb{R}^d} \mathcal{L}_{\alpha(\varepsilon)} \varphi(v) dv = 0, \quad \forall \varphi \in \mathcal{D}(\mathcal{L}_{\alpha(\varepsilon)}) \subset \mathbb{X}_0,$$

which implies that

$$\langle \mathcal{L}_{\alpha(\varepsilon)} \varphi, \varpi_q^{-1} \rangle_{\mathbb{X}_0, \mathbb{X}_0^*} = 0$$

where $\langle \cdot, \cdot \rangle_{\mathbb{X}_0, \mathbb{X}_0^*}$ denotes the duality bracket between \mathbb{X}_0 and its dual \mathbb{X}_0^* . This proves that

$$\varpi_q^{-1} \in \mathcal{D}(\mathcal{L}_{\alpha(\varepsilon)}^*) \quad \text{with} \quad \mathcal{L}_{\alpha(\varepsilon)}^*(\varpi_q^{-1}) = 0,$$

that is, 0 is an eigenvalue of $\mathcal{L}_{\alpha(\varepsilon)}^*$ in \mathbb{X}_0^* and therefore an eigenvalue of $\mathcal{L}_{\alpha(\varepsilon)}$ in \mathbb{X}_0 . With the same reasoning, since

$$\int_{\mathbb{R}^d} \mathcal{L}_{\alpha(\varepsilon)} \varphi(v) v_i dv = -\varepsilon^{-2} \kappa_{\alpha(\varepsilon)} \int_{\mathbb{R}^d} v_i \nabla \cdot (v \varphi(v)) dv = \varepsilon^{-2} \kappa_{\alpha(\varepsilon)} \int_{\mathbb{R}^d} v_i \varphi(v) dv$$

one sees that, for any $i = 1, \dots, d$, $m_i^*(v) := v_i \varpi_q^{-1}(v) \in \mathcal{D}(\mathcal{L}_{\alpha(\varepsilon)}^*)$ satisfies

$$\mathcal{L}_{\alpha(\varepsilon)}^* m_i^* = \varepsilon^{-2} \kappa_{\alpha(\varepsilon)} m_i^*,$$

that is, $\varepsilon^{-2} \kappa_{\alpha(\varepsilon)}$ is an eigenvalue of $\mathcal{L}_{\alpha(\varepsilon)}^*$ of multiplicity d and, as such, an eigenvalue of $\mathcal{L}_{\alpha(\varepsilon)}$ with same multiplicity in the space \mathbb{X}_0 . With this, we found $d + 1$ eigenvalues of $\mathcal{L}_{\alpha(\varepsilon)}$ in the space \mathbb{X}_0 . To prove that these $d + 1$ eigenvalues are still eigenvalues of $\mathcal{L}_{\alpha(\varepsilon)}$ in the smaller space \mathbb{X} , we proceed as follows. Let \bar{g} be an eigenfunction of $\mathcal{L}_{\alpha(\varepsilon)}$ in \mathbb{X}_0 associated to the $\varepsilon^{-2} \kappa_{\alpha(\varepsilon)}$ eigenvalue, i.e

$$\mathcal{L}_{\alpha(\varepsilon)} \bar{g} = \varepsilon^{-2} \kappa_{\alpha(\varepsilon)} \bar{g}, \quad \bar{g} \in \mathcal{D}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \mathbb{X}_0.$$

With the splitting $\mathcal{L}_{\alpha(\varepsilon)} = \mathcal{B}_\alpha^\delta + \mathcal{A}^{(\delta)}$, where we recall $\alpha = \alpha(\varepsilon)$ and δ is sufficiently small, one deduces from this that

$$\left(\varepsilon^{-2} \kappa_{\alpha(\varepsilon)} - \mathcal{B}_\alpha^\delta \right) \bar{g} = \mathcal{A}^{(\delta)} \bar{g}.$$

Using the fact that for $\varepsilon^{-2} \kappa_{\alpha(\varepsilon)} < \mu_* - \mu < \nu_*$ the operator $\varepsilon^{-2} \kappa_{\alpha(\varepsilon)} - \mathcal{B}_\alpha^\delta$ is invertible in both \mathbb{X} and \mathbb{X}_0 thanks to Proposition 2.13 and

$$\bar{g} = \mathcal{R}(\varepsilon^{-2} \kappa_{\alpha(\varepsilon)}, \mathcal{B}_\alpha^\delta) \mathcal{A}^{(\delta)} \bar{g}.$$

Because \bar{g} is depending on the velocity only, using the regularizing effect of $\mathcal{A}^{(\delta)}$ and the hypo-dissipativity property of the operator $\varepsilon^{-2} \kappa_{\alpha(\varepsilon)} - \mathcal{B}_\alpha^\delta$ one concludes that $\mathcal{A}^{(\delta)} \bar{g} \in \mathbb{X}$ and, by previous identity, so is \bar{g} . Therefore, any eigenfunction of $\mathcal{L}_{\alpha(\varepsilon)}$ associated to the eigenvalue $\varepsilon^{-2} \kappa_{\alpha(\varepsilon)}$ in \mathbb{X}_0 lies in \mathbb{X} as well, consequently, it is an eigenvalue of $\mathcal{L}_{\alpha(\varepsilon)}$ in \mathbb{X} . It has the same multiplicity d as in \mathbb{X}_0 since the reasoning is valid for any eigenfunction \bar{g} . In the same way, we prove that 0 is a simple eigenvalue of $\mathcal{L}_{\alpha(\varepsilon)}$ in \mathbb{X} . We just found exactly $d + 1$ eigenvalues and exhausted $\mathfrak{S}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \mathbb{C}_\mu = \mathfrak{S}(\mathcal{L}_{\alpha(\varepsilon)}) \cap \mathbb{D}(\mu_* - \mu)$ under the assumption that $\varepsilon^{-2} \kappa_{\alpha(\varepsilon)} < \mu_* - \mu$ which gives the desired result. \square

4. NONLINEAR ANALYSIS

We now apply the results obtained so far to the study of Eq. (1.20). In all this section we assume that

$$\mathcal{E} = \mathbb{W}_x^{m,1} \mathbb{W}_v^{k,1}(\varpi_q) \tag{4.1}$$

with

$$m > 2d, \quad m - 1 \geq k \geq 1, \quad q \geq 4, \tag{4.2}$$

and introduce also the Hilbert space on which \mathcal{L}_1 is symmetric

$$\mathcal{H} := \mathbb{W}_{x,v}^{m,2}(\mathcal{M}^{-1/2}).$$

We recall here that \mathcal{M} is the steady state of \mathcal{L}_1 whereas \mathcal{H} is a Hilbert space on which the elastic Boltzmann equation is well-understood [14].

We also denote

$$\begin{aligned} \mathcal{E}_1 &:= \mathbb{W}_x^{m,1} \mathbb{W}_v^{k,1}(\varpi_{q+1}), & \mathcal{E}_2 &:= \mathbb{W}_x^{m,1} \mathbb{W}_v^{k+1,1}(\varpi_{q+2}) \\ \mathcal{H}_1 &:= \mathbb{W}_{x,v}^{m,2} \left(\mathcal{M}^{-1/2} \langle \cdot \rangle^{1/2} \right) & \text{and} & \mathcal{E}_{-1} := \mathbb{W}_x^{m,1} \mathbb{W}_v^{k,1}(\varpi_{q-1}) \end{aligned} \quad (4.3)$$

where k, m, q satisfy (4.2).

The analysis of the elastic case in [15, 14] holds in $\mathbb{W}_{x,v}^{\beta,2}(\mathcal{M}^{-1/2})$ for $\beta > d$. We need, however, the \mathcal{H} -norm to control the \mathcal{E} -norm, which constrains $\beta \geq m$. At the same time, it is needed that $\mathcal{A}_\varepsilon \in \mathcal{B}(\mathcal{E}, \mathcal{H})$ and, because \mathcal{A}_ε has no regularisation effect on the spatial variable, we are forced to choose $\beta \leq m$. This explains the choice of $\beta = m$. Moreover, we need the constraint $m > 2d$ to carry out our nonlinear analysis, more precisely, we use that the embedding $\mathbb{W}_x^{m/2,1}(\mathbb{T}^d) \hookrightarrow L_x^\infty(\mathbb{T}^d)$ is continuous if $m/2 > d$ which provides us an algebra structure. Notice that the analysis of [14] is in particular valid under this condition since it only required $m > d$ to ensure that the embedding $\mathbb{W}_x^{m/2,2}(\mathbb{T}^d) \hookrightarrow L_x^\infty(\mathbb{T}^d)$ is continuous. Taking $q > 3$ would be enough to control the dissipation of kinetic energy $\int_{\mathbb{R}^d} \mathcal{Q}_\alpha(f, f) |v|^2 dv$ but we require $q \geq 4$ in order to be able to control some terms in the study of the hydrodynamic limit (see Subsection 6.3). Finally, the restriction $k \leq m - 1$ in (4.2) implies the continuous embedding $\mathcal{H} \hookrightarrow \mathcal{E}_2$ and the restriction $k \geq 1$ is due to the loss of one derivative in the estimate of $\mathcal{Q}_\alpha - \mathcal{Q}_1$ (see Lemma 2.1).

For $\mathbf{A}, \mathbf{B} > 0$, we will indicate in the sequel $\mathbf{A} \lesssim \mathbf{B}$ whenever there is a positive constant $C > 0$ depending on the mass and energy of the $h(0)$, but not on parameters like t, ε or Δ_0 , such that $\mathbf{A} \leq C \mathbf{B}$.

We adapt the approach of [15] and decompose the solution h_ε into

$$h_\varepsilon(t, x, v) = h^0(t, x, v) + h^1(t, x, v)$$

where $h^0 = h_\varepsilon^0 \in \mathcal{E}$ and $h^1 = h_\varepsilon^1 \in \mathcal{H}$ are the solutions to the following system of equations

$$\begin{cases} \partial_t h^0 &= \mathcal{B}_{\alpha(\varepsilon), \varepsilon} h^0 + \varepsilon^{-1} \mathcal{Q}_{\alpha(\varepsilon)}(h^0, h^0) + \varepsilon^{-1} \left[\mathcal{Q}_{\alpha(\varepsilon)}(h^0, h^1) + \mathcal{Q}_{\alpha(\varepsilon)}(h^1, h^0) \right] \\ &+ \left[\mathcal{G}_\varepsilon h^1 - \mathcal{G}_{1, \varepsilon} h^1 \right] + \varepsilon^{-1} \left[\mathcal{Q}_{\alpha(\varepsilon)}(h^1, h^1) - \mathcal{Q}_1(h^1, h^1) \right], \\ h^0(0, x, v) &= h_{\text{in}}^\varepsilon(x, v) \in \mathcal{E}. \end{cases} \quad (4.4)$$

and

$$\begin{cases} \partial_t h^1 &= \mathcal{G}_{1, \varepsilon} h^1 + \varepsilon^{-1} \mathcal{Q}_1(h^1, h^1) + \mathcal{A}_\varepsilon h^0, \\ h^1(0, x, v) &= 0. \end{cases} \quad (4.5)$$

In this section, we omit the dependence on ε for h^0 and h^1 . We recall that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} F_{\text{in}}^\varepsilon(x, v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv dx = 0 \implies \int_{\mathbb{T}^d \times \mathbb{R}^d} f_\varepsilon(t, x, v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv dx = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and, in particular, the fluctuation $h_\varepsilon(t, x, v)$ also satisfies

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} h_\varepsilon(t, x, v) \begin{pmatrix} 1 \\ v \end{pmatrix} dx dv = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.6)$$

4.1. Estimating h^0 . For the part of the solution $h^0(t)$ in \mathcal{E} we have the following estimate.

Proposition 4.1. *Assume that $h^0 \in \mathcal{E}$, $h^1 \in \mathcal{E}_1$ are such that*

$$\sup_{t \geq 0} (\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{E}_1}) \leq \Delta_0 < \infty.$$

Let $\nu_0 := \nu_{m,k,q}$ given in Proposition 2.8. Then, for $\mu_0 \in (0, \nu_0)$ there exists an explicit $\varepsilon_1 > 0$ (that can be chosen less than ε_0 defined in Theorem C.1) such that:

$$\begin{aligned} \|h^0(t)\|_{\mathcal{E}} &\lesssim \|h^0(0)\|_{\mathcal{E}} e^{-\frac{\mu_0}{\varepsilon^2}t} + \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2} ds \\ &+ \varepsilon \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds, \quad \forall \varepsilon \in (0, \varepsilon_1). \end{aligned} \quad (4.7)$$

As a consequence, for any $\varepsilon \in (0, \varepsilon_1)$,

$$\begin{aligned} \|h^0(t)\|_{\mathcal{E}}^2 &\lesssim \|h^0(0)\|_{\mathcal{E}}^2 e^{-\frac{2\mu_0}{\varepsilon^2}t} + \frac{1}{\mu_0} (\varepsilon \lambda_\varepsilon)^2 \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds \\ &+ \frac{1}{\mu_0} (\varepsilon^2 \lambda_\varepsilon)^2 \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds. \end{aligned} \quad (4.8)$$

Proof. In the subsequent proof, we denote by $\|\cdot\|_{\mathcal{E}_1}$ and $\|\cdot\|_{\mathcal{E}}$ the norms on \mathcal{E}_1 and \mathcal{E} that are equivalent to the standard ones (with multiplicative constants independent of ε) and that make $\varepsilon^{-2}\nu_0 + \mathcal{B}_{\alpha(\varepsilon), \varepsilon}$ dissipative. The conclusion with standard norms will simply follow by equivalence. We first observe that

$$\begin{aligned} \frac{d}{dt} \|h^0(t)\|_{\mathcal{E}} &\leq -\frac{\nu_0}{\varepsilon^2} \|h^0(t)\|_{\mathcal{E}_1} + \varepsilon^{-1} \left(\|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t), h^0(t))\|_{\mathcal{E}} + \|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t), h^1(t))\|_{\mathcal{E}} \right. \\ &+ \|\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t), h^0(t))\|_{\mathcal{E}} \left. + \|\mathcal{G}_\varepsilon h^1(t) - \mathcal{G}_{1,\varepsilon} h^1(t)\|_{\mathcal{E}} \right. \\ &\quad \left. + \varepsilon^{-1} \|\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t), h^1(t)) - \mathcal{Q}_1(h^1(t), h^1(t))\|_{\mathcal{E}} \right). \end{aligned}$$

Using classical estimates for $\mathcal{Q}_{\alpha(\varepsilon)}$ and \mathcal{Q}_1 , (see [3, 4]), together with Lemma 2.1, there exist $C > 0$ independent of ε such that

$$\begin{aligned} \|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t), h^0(t))\|_{\mathcal{E}} + \|\mathcal{Q}_{\alpha(\varepsilon)}(h^0(t), h^1(t))\|_{\mathcal{E}} \\ + \|\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t), h^0(t))\|_{\mathcal{E}} \leq C \left(\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{E}_1} \right) \|h^0(t)\|_{\mathcal{E}_1}, \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{G}_\varepsilon h^1(t) - \mathcal{G}_{1,\varepsilon} h^1(t)\|_{\mathcal{E}} + \varepsilon^{-1} \|\mathcal{Q}_{\alpha(\varepsilon)}(h^1(t), h^1(t)) - \mathcal{Q}_1(h^1(t), h^1(t))\|_{\mathcal{E}} \\ \leq C(1 - \alpha(\varepsilon)) \|h^1(t)\|_{\mathcal{E}_2} \left(\varepsilon^{-2} + \varepsilon^{-1} \|h^1(t)\|_{\mathcal{E}_2} \right). \end{aligned}$$

Notice that such estimate is exactly what motivated the definition of \mathcal{E}_2 . We conclude that

$$\begin{aligned} \frac{d}{dt} \|h^0(t)\|_{\mathcal{E}} &\leq -\varepsilon^{-2} \left(\nu_0 - \varepsilon C (\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{E}_1}) \right) \|h^0(t)\|_{\mathcal{E}_1} \\ &\quad + C(1 - \alpha(\varepsilon))\varepsilon^{-2} \|h^1(t)\|_{\mathcal{E}_2} + C(1 - \alpha(\varepsilon))\varepsilon^{-1} \|h^1(t)\|_{\mathcal{E}_2}^2. \end{aligned}$$

For any $\mu_0 \in (0, \nu_0)$, we pick $\varepsilon_1 \in (0, \varepsilon_0)$ as $\nu_0 - \varepsilon_1 C \Delta_0 \geq \mu_0$. Therefore,

$$\nu_0 - \varepsilon C (\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{E}_1}) \geq \mu_0, \quad \forall \varepsilon \in (0, \varepsilon_1).$$

Consequently, we obtain that

$$\begin{aligned} \frac{d}{dt} \|h^0(t)\|_{\mathcal{E}} &\leq -\frac{\mu_0}{\varepsilon^2} \|h^0(t)\|_{\mathcal{E}_1} + C(1 - \alpha(\varepsilon))\varepsilon^{-2} \|h^1(t)\|_{\mathcal{E}_2} \\ &\quad + C(1 - \alpha(\varepsilon))\varepsilon^{-1} \|h^1(t)\|_{\mathcal{E}_2}^2, \tag{4.9} \\ &\leq -\frac{\mu_0}{\varepsilon^2} \|h^0(t)\|_{\mathcal{E}_1} + C\lambda_\varepsilon \|h^1(t)\|_{\mathcal{E}_2} + C\varepsilon\lambda_\varepsilon \|h^1(t)\|_{\mathcal{E}_2}^2, \quad \forall t \geq 0, \end{aligned}$$

where we used that $\varepsilon^2\lambda_\varepsilon \simeq 1 - \alpha(\varepsilon)$ which gives (4.7) after integration. To prove (4.8), we use the fact that, for any nonnegative mapping $t \mapsto \zeta(t)$ and $\alpha > 0$

$$\begin{aligned} \left(\int_0^t e^{-\alpha(t-s)} \zeta(s) ds \right)^2 &\leq \int_0^t e^{-\alpha(t-s)} ds \int_0^t e^{-\alpha(t-s)} \zeta(s)^2 ds \\ &\leq \frac{1}{\alpha} \int_0^t e^{-\alpha(t-s)} \zeta(s)^2 ds, \quad \forall t \geq 0, \end{aligned} \tag{4.10}$$

which gives the result. \square

On the basis of Theorem 1.7, there exist Ψ_ε^j , with $j = 1, \dots, d+2$, linearly independent and such that

$$\mathcal{G}_\varepsilon \Psi_\varepsilon^j = \mathcal{L}_{\alpha(\varepsilon)} \Psi_\varepsilon^j = \lambda_j(\varepsilon) \Psi_\varepsilon^j$$

with moreover $\Psi_\varepsilon^{d+2} = \phi_{\alpha(\varepsilon)}$. We denote by Π_ε the spectral projection associated to $\lambda_{d+2}(\varepsilon) = -\lambda_\varepsilon$ and $\mathbb{P}_\varepsilon = \mathbf{P}_\varepsilon - \Pi_\varepsilon$, that is,

$$\text{Range}(\mathbb{P}_\varepsilon) = \text{Span}\{\Psi_\varepsilon^1, \dots, \Psi_\varepsilon^{d+1}\}, \quad \text{Range}(\Pi_\varepsilon) = \text{Span}(\Psi_\varepsilon^{d+2}).$$

In the same way, with the notations of Theorem C.1, we introduce

$$\mathbb{P}_0 h = \sum_{i=1}^{d+1} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} h \Psi_i dx dv \right) \Psi_i \mathcal{M}, \quad \Pi_0 h = \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} h \Psi_{d+2} dx dv \right) \Psi_{d+2} \mathcal{M}, \tag{4.11}$$

where recall that

$$\Psi_1 = 1, \quad \Psi_i = \frac{v_{i-1}}{\sqrt{\vartheta_1}} \text{ for } i = 2, \dots, d+1, \quad \text{and} \quad \Psi_{d+2} = \frac{1}{\vartheta_1 \sqrt{2d}} (|v|^2 - d\vartheta_1).$$

Of course, see (2.22), one has $\mathbb{P}_0 = \mathbf{P}_0 - \Pi_0$. Recall that the eigenfunctions Ψ_j are such that

$$\int_{\mathbb{R}^3} \Psi_i(v) \Psi_j(v) \mathcal{M}(v) dv = \delta_{i,j} \quad i, j = 1, \dots, d+2,$$

which in particular implies that, in the Hilbert space \mathcal{H}^4 , one has $\text{Id} - \mathbf{P}_0 = \mathbf{P}_0^\perp$.

Remark 4.2. Notice that, since $\Psi_\varepsilon^{d+2} = \phi_{\alpha(\varepsilon)}$ one deduces from (2.7) that

$$\lim_{\varepsilon \rightarrow 0} \|\Pi_\varepsilon - \Pi_0\|_{\mathcal{B}(\mathcal{E})} = 0.$$

The rate of convergence is actually explicit (see [52, Lemma 5.17]). For any $s \in \mathbb{N}$, $p \geq 0$, there is $C > 0$ such that

$$\|\Pi_\varepsilon - \Pi_0\|_{\mathcal{B}(\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_p))} \leq C(1 - \alpha(\varepsilon)), \quad \ell \geq 0.$$

Lemma 4.3. For $i = 1, \dots, d+1$, it holds that

$$\left| \int_{\mathbb{T}^d \times \mathbb{R}^d} h^1(t, x, v) \Psi_i(v) dv dx \right| \leq \max\left(1, \frac{1}{\sqrt{\vartheta_1}}\right) \|h^0(t)\|_{\mathcal{E}}.$$

As a consequence,

$$\|\mathbb{P}_0 h^1(t)\|_{\mathcal{E}} \leq C \|h^0(t)\|_{\mathcal{E}},$$

for some constant $C > 0$ depending only on \mathcal{M} .

Proof. Note that total mass and momentum conservation leads to

$$\begin{aligned} 0 &= \int_{\mathbb{T}^d \times \mathbb{R}^d} h(t, x, v) \Psi_i(v) dv dx = \int_{\mathbb{T}^d \times \mathbb{R}^d} h^0(t, x, v) \Psi_i(v) dv dx \\ &\quad + \int_{\mathbb{T}^d \times \mathbb{R}^d} h^1(t, x, v) \Psi_i(v) dv dx, \quad i = 1, \dots, d+1. \end{aligned}$$

Thus, for any $i = 1, \dots, d+1$,

$$\left| \int_{\mathbb{T}^d \times \mathbb{R}^d} h^1(t, x, v) \Psi_i(v) dv dx \right| = \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} h^0(t, x, v) \Psi_i(v) dv dx \right| \leq \max\left(1, \frac{1}{\sqrt{\vartheta_1}}\right) \|h^0(t)\|_{\mathcal{E}}$$

since $|\Psi_i(v)| \leq \max\left(1, \frac{1}{\sqrt{\vartheta_1}}\right) \varpi_q(v)$ for any $i = 1, \dots, d+1$. Regarding the estimate for the projection, it follows from the previous inequality and (4.11) by taking for example $C := \max_{i=1, \dots, d+1} \|\Psi_i \mathcal{M}\|_{\mathcal{E}}$. \square

4.2. Estimating $\mathbf{P}_0 h^1(t)$. One has the following fundamental estimate for $\mathbf{P}_0 h^1(t)$.

Lemma 4.4. We have that

$$\begin{aligned} \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}} &\lesssim \|\Pi_\varepsilon h(0)\|_{\mathcal{E}} e^{-\lambda_\varepsilon t} + (1 - \alpha(\varepsilon)) \|h^1(t)\|_{\mathcal{E}} + \|h^0(t)\|_{\mathcal{E}} \\ &\quad + \varepsilon^{-1} (1 - \alpha(\varepsilon)) \int_0^t e^{-\lambda_\varepsilon(t-s)} \left(\|h^1(s)\|_{\mathcal{E}}^2 + \|h^0(s)\|_{\mathcal{E}}^2 \right) ds \end{aligned}$$

for any $t \geq 0$.

⁴Recall here that, on the space $L_v^2(\mathcal{M}^{-\frac{1}{2}})$ the inner product is $\langle f, g \rangle = \int_{\mathbb{R}^d} f(v)g(v)\mathcal{M}^{-1}(v)dv$.

Proof. The equation for h is given by

$$\partial_t h = \mathcal{G}_\varepsilon h + \varepsilon^{-1} \mathcal{Q}_{\alpha(\varepsilon)}(h, h).$$

Thus,

$$\partial_t (\Pi_\varepsilon h) = \mathcal{G}_\varepsilon (\Pi_\varepsilon h) + \varepsilon^{-1} \Pi_\varepsilon \mathcal{Q}_{\alpha(\varepsilon)}(h, h).$$

Therefore,

$$\Pi_\varepsilon h(t) = \Pi_\varepsilon h(0) e^{-\lambda_\varepsilon t} + \varepsilon^{-1} \int_0^t e^{-\lambda_\varepsilon(t-s)} \Pi_\varepsilon \mathcal{Q}_{\alpha(\varepsilon)}(h(s), h(s)) ds. \quad (4.12)$$

Observe the following facts

$$\|\Pi_\varepsilon \mathcal{Q}_{\alpha(\varepsilon)}(h(s), h(s))\|_{\mathcal{E}_{-1}} \leq \|(\Pi_\varepsilon - \Pi_0) \mathcal{Q}_{\alpha(\varepsilon)}(h(s), h(s))\|_{\mathcal{E}_{-1}} + \|\Pi_0 \mathcal{Q}_{\alpha(\varepsilon)}(h(s), h(s))\|_{\mathcal{E}_{-1}},$$

where

$$\begin{aligned} \|(\Pi_\varepsilon - \Pi_0) \mathcal{Q}_{\alpha(\varepsilon)}(h(s), h(s))\|_{\mathcal{E}_{-1}} &\lesssim (1 - \alpha(\varepsilon)) \|\mathcal{Q}_{\alpha(\varepsilon)}(h(s), h(s))\|_{\mathcal{E}_{-1}} \\ &\lesssim (1 - \alpha(\varepsilon)) \|h(s)\|_{\mathcal{E}}^2. \end{aligned}$$

Notice that, according to (4.11), $\Pi_0 \mathcal{Q}_{\alpha(\varepsilon)}$ is explicit with

$$\|\Pi_0 \mathcal{Q}_{\alpha(\varepsilon)}(h(s), h(s))\|_{\mathcal{E}_{-1}} = (1 - \alpha^2(\varepsilon)) \left| \mathcal{D}_{\alpha(\varepsilon)}(h(s), h(s)) \right| \|\Psi_{d+2} \mathcal{M}\|_{\mathcal{E}_{-1}},$$

where $\mathcal{D}_\alpha(g, g)$ denotes the normalised energy dissipation associated to \mathcal{Q}_α , namely,

$$\begin{aligned} \mathcal{D}_\alpha(g, g) &= -\frac{1}{1 - \alpha^2} \int_{\mathbb{T}^d \times \mathbb{R}^d} \Psi_{d+2}(v) \mathcal{Q}_\alpha(g, g) dv dx \\ &= -\frac{1}{\vartheta_1 \sqrt{2d}} \frac{1}{1 - \alpha^2} \frac{1}{\vartheta_1 \sqrt{2d}} \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} \mathcal{Q}_\alpha(g, g) |v|^2 dv \\ &= \gamma_b \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, v) g(x, v_*) |v - v_*|^3 dv dv_* \end{aligned}$$

for some nonnegative γ_b independent of α , see [52]. Therefore,

$$\|\Pi_0 \mathcal{Q}_{\alpha(\varepsilon)}(h(s), h(s))\|_{\mathcal{E}_{-1}} \lesssim (1 - \alpha(\varepsilon)) \|h(s)\|_{\mathcal{E}}^2$$

because $\varpi_q(v) \geq \langle v \rangle^3$ for any $v \in \mathbb{R}^3$.

Thus, applying the $\|\cdot\|_{\mathcal{E}_{-1}}$ -norm in (4.12) one obtains

$$\|\Pi_\varepsilon h(t)\|_{\mathcal{E}_{-1}} \lesssim \|\Pi_\varepsilon h(0)\|_{\mathcal{E}} e^{-\lambda_\varepsilon t} + \varepsilon^{-1} (1 - \alpha(\varepsilon)) \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h(s)\|_{\mathcal{E}}^2 ds$$

where we recall that $\lambda_\varepsilon = -\lambda_{d+2}(\varepsilon) > 0$. Consequently, using again Remark 4.2, we conclude that

$$\begin{aligned} \|\Pi_0 h(t)\|_{\mathcal{E}_{-1}} &\lesssim \|\Pi_\varepsilon h(0)\|_{\mathcal{E}} e^{-\lambda_\varepsilon t} + (1 - \alpha(\varepsilon)) \|h(t)\|_{\mathcal{E}} \\ &\quad + \varepsilon^{-1} (1 - \alpha(\varepsilon)) \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h(s)\|_{\mathcal{E}}^2 ds. \end{aligned} \quad (4.13)$$

According to (4.6), one has $\mathbb{P}_0 h(t) = 0$ for any $t \geq 0$ from which we deduce that

$$\mathbf{P}_0 h(t) = \Pi_0 h(t), \quad \forall t \geq 0.$$

Since $\mathbf{P}_0 h^1(t) = \mathbf{P}_0 h(t) - \mathbf{P}_0 h^0(t)$, we can reformulate the above (4.13) in terms of the relevant functions h^1 and h^0 to obtain that

$$\begin{aligned} \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}} &\lesssim \|\Pi_\varepsilon h(0)\|_{\mathcal{E}} e^{-\lambda_\varepsilon t} + (1 - \alpha(\varepsilon)) (\|h^1(t)\|_{\mathcal{E}} + \|h^0(t)\|_{\mathcal{E}}) + \|h^0(t)\|_{\mathcal{E}} \\ &\quad + \varepsilon^{-1} (1 - \alpha(\varepsilon)) \int_0^t e^{-\lambda_\varepsilon(t-s)} (\|h^0(s)\|_{\mathcal{E}}^2 + \|h^1(s)\|_{\mathcal{E}}^2) ds, \end{aligned}$$

which is the desired lemma. \square

Remark 4.5. Since $\varepsilon \lambda_\varepsilon \simeq \frac{1-\alpha(\varepsilon)}{\varepsilon}$, we get $\frac{1-\alpha^2(\varepsilon)}{\varepsilon \lambda_\varepsilon} \simeq \varepsilon(1 + \alpha(\varepsilon)) \simeq 2\varepsilon$ as $\varepsilon \rightarrow 0$. Therefore,

$$\frac{(1 - \alpha^2(\varepsilon))}{\varepsilon \lambda_\varepsilon} \lesssim \varepsilon.$$

Proposition 4.6. *There exists an explicit $\varepsilon_2 \in (0, \varepsilon_1)$ such that for any $\varepsilon \in (0, \varepsilon_2)$ and $t \geq 0$, it holds that*

$$\begin{aligned} \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}} &\lesssim \left(\|\Pi_\varepsilon h(0)\|_{\mathcal{E}} + \|h^0(0)\|_{\mathcal{E}} + \frac{\varepsilon^3 \lambda_\varepsilon}{\mu_0} \|h^0(0)\|_{\mathcal{E}}^2 \right) e^{-\lambda_\varepsilon t} + \varepsilon^2 \lambda_\varepsilon \|h^1(t)\|_{\mathcal{E}} \\ &\quad + \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2} ds + \varepsilon \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds \\ &\quad + \varepsilon \lambda_\varepsilon (1 + \mu_0^{-2} \varepsilon^4 \lambda_\varepsilon^2) \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}}^2 ds \\ &\quad + \mu_0^{-2} \varepsilon^7 \lambda_\varepsilon^3 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds. \end{aligned}$$

Proof. We insert the bound for $\|h^0(t)\|_{\mathcal{E}}^i$ for $i = 1, 2$ in (4.7) and (4.8) in the estimate of Lemma 4.4. Assuming $\mu_0 \geq 2\varepsilon^2 \lambda_\varepsilon$ and recalling that $1 - \alpha(\varepsilon) \simeq \varepsilon^2 \lambda_\varepsilon$, we first deduce from (4.7) that

$$\begin{aligned} \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}} &\lesssim \left(\|\Pi_\varepsilon h(0)\|_{\mathcal{E}} + \|h^0(0)\|_{\mathcal{E}} \right) e^{-\lambda_\varepsilon t} + \varepsilon^2 \lambda_\varepsilon \|h^1(t)\|_{\mathcal{E}} \\ &\quad + \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2} ds + \varepsilon \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds \\ &\quad + \varepsilon \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}}^2 ds + \varepsilon \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^0(s)\|_{\mathcal{E}}^2 ds. \quad (4.14) \end{aligned}$$

Now, using (4.8) for the last integral, we obtain

$$\begin{aligned} \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^0(s)\|_{\mathcal{E}}^2 ds &\lesssim \|h^0(0)\|_{\mathcal{E}}^2 \int_0^t e^{-\frac{2\mu_0}{\varepsilon^2}s - \lambda_\varepsilon(t-s)} ds \\ &\quad + \mu_0^{-1}(\varepsilon \lambda_\varepsilon)^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^2 d\tau \\ &\quad + \mu_0^{-1}(\varepsilon^2 \lambda_\varepsilon)^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^4 ds. \end{aligned}$$

Using that, for any $\beta > \alpha > 0$ and nonnegative mapping $t \mapsto \zeta(t)$

$$\begin{aligned} \int_0^t e^{-\alpha(t-s)} ds \int_0^s e^{-\beta(s-\tau)} \zeta(\tau) d\tau &= e^{-\alpha t} \int_0^t e^{\beta\tau} \zeta(\tau) d\tau \int_\tau^t e^{-(\beta-\alpha)s} ds \\ &\leq \frac{1}{\beta - \alpha} \int_0^t e^{-\alpha(t-\tau)} \zeta(\tau) d\tau \end{aligned} \quad (4.15)$$

we have, for $\mu_0 \geq 2\varepsilon^2 \lambda_\varepsilon$, that

$$\int_0^t e^{-\lambda_\varepsilon(t-s)} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^{2i} d\tau \leq \frac{2\varepsilon^2}{\mu_0} \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^{2i} ds, \quad i = 1, 2,$$

so that

$$\begin{aligned} \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^0(s)\|_{\mathcal{E}}^2 ds &\lesssim \frac{\varepsilon^2}{\mu_0} \|h^0(0)\|_{\mathcal{E}}^2 e^{-\lambda_\varepsilon t} + \mu_0^{-2} \varepsilon^4 \lambda_\varepsilon^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds \\ &\quad + \mu_0^{-2} (\varepsilon^3 \lambda_\varepsilon)^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds. \end{aligned}$$

Inserting this in (4.14) gives the desired estimate. \square

Remark 4.7. We will also need an estimate for $\|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^2$. This is easy to do using Cauchy-Schwarz inequality as we did for the proof of (4.8). We obtain that

$$\begin{aligned} \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^2 &\lesssim \left(\|\Pi_\varepsilon h(0)\|_{\mathcal{E}}^2 + \|h^0(0)\|_{\mathcal{E}}^2 + \frac{(\varepsilon^3 \lambda_\varepsilon)^2}{\mu_0^2} \|h^0(0)\|_{\mathcal{E}}^4 \right) e^{-2\lambda_\varepsilon t} + \varepsilon^4 \lambda_\varepsilon^2 \|h^1(t)\|_{\mathcal{E}}^2 \\ &\quad + \frac{(\varepsilon \lambda_\varepsilon)^2}{\mu_0} \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds + \frac{(\varepsilon^2 \lambda_\varepsilon)^2}{\mu_0} \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds \\ &\quad + \varepsilon^2 \lambda_\varepsilon \left(1 + \mu_0^{-2} (\varepsilon^2 \lambda_\varepsilon)^2 \right)^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds \\ &\quad + \mu_0^{-4} \varepsilon^4 (\varepsilon^2 \lambda_\varepsilon)^5 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^8 ds. \end{aligned}$$

4.3. Estimating the complement $(\text{Id} - \mathbf{P}_0)h^1$. Let us focus on an estimate on $\mathbf{P}_0^\perp h^1(t)$ with $\mathbf{P}_0^\perp = \text{Id} - \mathbf{P}_0$, the orthogonal projection onto $(\text{Ker}(\mathcal{G}_{1,\varepsilon}))^\perp$ in the Hilbert space $L_{x,v}^2(\mathcal{M}^{-1/2})$. The same notation for the operator $\mathcal{G}_{1,\varepsilon}$ in the spaces \mathcal{E} and \mathcal{H} is used. We begin with the following lemma where, we recall, $\Sigma_{\mathcal{M}}(\xi)$ is defined in (2.12).

Proposition 4.8. *With the notations of Theorem C.1, let $\varepsilon \in (0, \varepsilon_0)$, $\mu \in (0, \mu_*)$ and assume that*

$$\sup_{t \geq 0} (\|h^0(t)\|_{\mathcal{E}} + \|h^1(t)\|_{\mathcal{H}}) \leq \Delta_0$$

with $\Delta_0 \leq 1$ small enough so that

$$\nu := \frac{2\mu}{\sigma_0^2} - c_0 \Delta_0^2 > 0 \quad (4.16)$$

where $\sigma_0 := \inf_{\xi \in \mathbb{R}^d} \Sigma_{\mathcal{M}}(\xi) > 0$ and $c_0 > 0$ is a universal constant depending only on \mathcal{M} defined in (4.19). Set

$$\Psi(t) = h^1(t) - \mathbf{P}_0 h^1(t), \quad \forall t \geq 0.$$

Then, there exists $C_0 > 0$ independent of $\varepsilon > 0$ such that

$$\begin{aligned} \|\Psi(t)\|_{\mathcal{H}}^2 &\leq \|\Psi(0)\|_{\mathcal{H}}^2 e^{-\nu t} + C_0 \int_0^t e^{-\nu(t-s)} \|\mathbf{P}_0 h^1(s)\|_{\mathcal{E}_{-1}}^4 ds \\ &\quad + \frac{C_0}{\varepsilon^2} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} \|h^0(s)\|_{\mathcal{E}} ds \end{aligned} \quad (4.17)$$

for any $t \geq 0$.

Proof. One checks from (4.5) that

$$\partial_t \Psi = \mathcal{G}_{1,\varepsilon} \Psi + \mathbf{P}_0^\perp (\varepsilon^{-1} \mathcal{Q}_1(h^1, h^1) + \mathcal{A}_\varepsilon h^0) = \mathcal{G}_{1,\varepsilon} \Psi + \varepsilon^{-1} \mathcal{Q}_1(h^1, h^1) + \mathbf{P}_0^\perp \mathcal{A}_\varepsilon h^0,$$

where for the later we used that $\mathbf{P}_0 \mathcal{Q}_1(h^1, h^1) = 0$. Using [15, Theorem 4.7], one obtains as in [15, Eq. (4.8)] that, for any $\mu \in (0, \mu_*)$ there is some positive constant $C > 0$ such that

$$\frac{d}{dt} \|\Psi(t)\|_{\mathcal{H}}^2 \leq -\frac{2\mu}{\sigma_0^2} \|\Psi(t)\|_{\mathcal{H}_1}^2 + C \|h^1(t)\|_{\mathcal{H}}^2 \|h^1(t)\|_{\mathcal{H}_1}^2 + \|\Psi(t)\|_{\mathcal{H}} \|\mathbf{P}_0^\perp \mathcal{A}_\varepsilon h^0(t)\|_{\mathcal{H}}. \quad (4.18)$$

Writing $h^1 = \mathbf{P}_0 h^1 + \Psi$, we obtain

$$\begin{aligned} \|h^1(t)\|_{\mathcal{H}}^2 \|h^1(t)\|_{\mathcal{H}_1}^2 &\leq 2 \|h^1(t)\|_{\mathcal{H}}^2 \left(\|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}_1}^2 + \|\Psi(t)\|_{\mathcal{H}_1}^2 \right) \\ &\leq 2\Delta_0^2 \|\Psi(t)\|_{\mathcal{H}_1}^2 + 4 \|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}_1}^2 (\|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}}^2 + \|\Psi(t)\|_{\mathcal{H}}^2). \end{aligned}$$

In particular, since there exists a positive constant $c > 0$ depending only on \mathcal{M} such that

$$\|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}_1}^2 \leq c \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^2,$$

we deduce that

$$\|h^1(t)\|_{\mathcal{H}}^2 \|h^1(t)\|_{\mathcal{H}_1}^2 \leq c_0 \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^4 + c_0 \Delta_0^2 \|\Psi(t)\|_{\mathcal{H}_1}^2 \quad (4.19)$$

for some universal constant $c_0 > 0$ depending only on \mathcal{M} . Therefore, assuming that Δ_0 is small enough so that

$$\nu := \frac{2\mu}{\sigma_0^2} - c_0 \Delta_0^2 > 0$$

we deduce that

$$\frac{d}{dt} \|\Psi(t)\|_{\mathcal{H}}^2 \leq -\nu \|\Psi(t)\|_{\mathcal{H}_1}^2 + C \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^4 + \|\Psi(t)\|_{\mathcal{H}} \|\mathbf{P}_0^\perp \mathcal{A}_\varepsilon h^0(t)\|_{\mathcal{H}}, \quad \forall t \geq 0$$

for some $C > 0$ independent of t and ε . Moreover, we also have that

$$\|\mathbf{P}_0^\perp \mathcal{A}_\varepsilon h^0(t)\|_{\mathcal{H}} \lesssim \frac{1}{\varepsilon^2} \|h^0(t)\|_{\mathcal{E}}, \quad \|\Psi(t)\|_{\mathcal{H}} \lesssim \|h^1(t)\|_{\mathcal{H}}, \quad \forall t \geq 0,$$

from which we get the desired estimate after integration of the previous differential inequality. \square

Lemma 4.9. *With the notation of Proposition 4.8, there is an explicit $\varepsilon_3 \in (0, \varepsilon_2)$ such that for for any $\delta > 0$, $\varepsilon \in (0, \varepsilon_3)$, and $t \geq 0$*

$$\begin{aligned} \varepsilon^{-2} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} \|h^0(s)\|_{\mathcal{E}} ds &\lesssim \delta \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{1}{\delta \mu_0} \|h^0(0)\|_{\mathcal{E}}^2 e^{-\nu t} \\ &+ \frac{\delta}{\varepsilon^2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{1}{\delta \mu_0^2} \lambda_\varepsilon^2 \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds \\ &+ \frac{1}{\delta \mu_0^2} (\varepsilon \lambda_\varepsilon)^2 \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds. \end{aligned}$$

Proof. We use the estimate of $\|h^0(s)\|_{\mathcal{E}}$ provided in (4.7) which gives

$$\varepsilon^{-2} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} \|h^0(s)\|_{\mathcal{E}} ds \lesssim I_1(t) + I_2(t) + I_3(t)$$

with

$$I_1(t) = \varepsilon^{-2} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} \|h^0(0)\|_{\mathcal{E}} e^{-\frac{\mu_0}{\varepsilon^2} s} ds,$$

$$I_2(t) = \varepsilon^{-2} \lambda_\varepsilon \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2} d\tau,$$

and

$$I_3(t) = \varepsilon^{-1} \lambda_\varepsilon \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^2 d\tau.$$

Using Young's inequality, for any $\delta > 0$ it holds that

$$\|h^1(s)\|_{\mathcal{H}} \|h^0(0)\|_{\mathcal{E}} \leq \delta \|h^1(s)\|_{\mathcal{H}}^2 + \frac{1}{4\delta} \|h^0(0)\|_{\mathcal{E}}^2,$$

so that, since $\mu_0 - \varepsilon^2 \nu \geq \frac{\mu_0}{2}$,

$$\begin{aligned} I_1(t) &\leq \delta \varepsilon^{-2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{1}{4\delta \varepsilon^2} \|h^0(0)\|_{\mathcal{E}}^2 \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} ds \\ &\leq \delta \varepsilon^{-2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{1}{2\delta \mu_0} \|h^0(0)\|_{\mathcal{E}}^2 e^{-\nu t}. \end{aligned}$$

Similarly, Young's inequality implies, for any $\delta > 0$, that

$$I_2(t) \leq \delta \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{(\varepsilon^{-2}\lambda_\varepsilon)^2}{4\delta} \int_0^t e^{-\nu(t-s)} ds \left(\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2} d\tau \right)^2,$$

and, using (4.10) and (4.15) to estimate the square of the last integral, we get for $\mu_0 \geq 2\varepsilon^2\nu$ that

$$I_2(t) \leq \delta \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{1}{2\delta\mu_0^2} \lambda_\varepsilon^2 \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^2 ds.$$

In the same way, it follows that

$$I_3(t) \leq \delta \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{1}{2\delta\mu_0^2} (\varepsilon\lambda_\varepsilon)^2 \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds.$$

Combining these estimates yields the result. \square

It remains to estimate the term involving $\|\mathbf{P}_0 h^1\|_{\mathcal{E}_{-1}}^4$ in (4.17). This is done in the following lemma.

Lemma 4.10. *Under the assumptions and notations of Proposition 4.8, there are explicit constants $\varepsilon_4 \in (0, \varepsilon_3)$ and $\lambda_4 > 0$ such that, for any $\delta > 0$, $\varepsilon \in (0, \varepsilon_4)$ and $\lambda_\varepsilon \in (0, \lambda_4)$,*

$$\begin{aligned} \int_0^t e^{-\nu(t-s)} \|\mathbf{P}_0 h^1(s)\|_{\mathcal{E}_{-1}}^4 ds &\lesssim \nu^{-1} \left(\|\Pi_\varepsilon h(0)\|_{\mathcal{E}}^4 + \|h^0(0)\|_{\mathcal{E}}^4 + \frac{(\varepsilon^3\lambda_\varepsilon)^4}{\mu_0^4} \|h^0(0)\|_{\mathcal{E}}^8 \right) e^{-4\lambda_\varepsilon t} \\ &\quad + (\varepsilon^2\lambda_\varepsilon)^4 \mathbf{C}_0(\varepsilon, \mu_0, \Delta_0) \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &\quad + \varepsilon^4 \lambda_\varepsilon \mathbf{C}_1(\varepsilon, \mu_0, \nu, \Delta_0) \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \end{aligned}$$

with $\mathbf{C}_0(\varepsilon, \mu_0, \Delta_0) := \Delta_0^2 + \frac{\Delta_0^2}{\mu_0^4} + \frac{\varepsilon^4}{\mu_0^4} \Delta_0^6$ and

$$\mathbf{C}_1(\varepsilon, \mu_0, \nu, \Delta_0) := \frac{(\varepsilon^3\lambda_\varepsilon)^8}{\nu\mu_0^8} \Delta_0^{14} + \nu^{-1} (1 + \mu_0^{-2}(\varepsilon^2\lambda_\varepsilon)^2)^4 \Delta_0^6.$$

Remark 4.11. *Since $\varepsilon\lambda_\varepsilon \ll 1$ and $\Delta_0 \leq 1$, it follows that for ε small enough*

$$\mathbf{C}_0(\varepsilon, \mu_0, \Delta_0) \lesssim (1 + \mu_0^{-4}) \Delta_0^2 \quad \text{and} \quad \mathbf{C}_1(\varepsilon, \mu_0, \nu, \Delta_0) \lesssim \nu^{-1} (1 + \mu_0^{-8}) \Delta_0^2.$$

Proof. We start with the estimate for $\|\mathbf{P}_0 h^1\|_{\mathcal{E}_{-1}}^2$ in Remark 4.7 which gives

$$\begin{aligned} \|\mathbf{P}_0 h^1(s)\|_{\mathcal{E}_{-1}}^4 &\lesssim \left(\|\Pi_\varepsilon h(0)\|_{\mathcal{E}}^4 + \|h^0(0)\|_{\mathcal{E}}^4 + \frac{(\varepsilon^3 \lambda_\varepsilon)^4}{\mu_0^4} \|h^0(0)\|_{\mathcal{E}}^8 \right) e^{-4\lambda_\varepsilon s} + \varepsilon^8 \lambda_\varepsilon^4 \|h^1(t)\|_{\mathcal{E}}^4 \\ &+ \frac{(\varepsilon \lambda_\varepsilon)^4}{\mu_0^2} \left(\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^2 d\tau \right)^2 + \frac{(\varepsilon^2 \lambda_\varepsilon)^4}{\mu_0^2} \left(\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^4 d\tau \right)^2 \\ &+ \varepsilon^4 \lambda_\varepsilon^2 \left(1 + \mu_0^{-2} (\varepsilon^2 \lambda_\varepsilon)^2 \right)^4 \left(\int_0^s e^{-\lambda_\varepsilon(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^4 d\tau \right)^2 \\ &+ \mu_0^{-8} \varepsilon^8 (\varepsilon^2 \lambda_\varepsilon)^{10} \left(\int_0^s e^{-\lambda_\varepsilon(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^8 d\tau \right)^2. \end{aligned}$$

A repeated use of (4.10) gives

$$\begin{aligned} \|\mathbf{P}_0 h^1(s)\|_{\mathcal{E}_{-1}}^4 &\lesssim \left(\|\Pi_\varepsilon h(0)\|_{\mathcal{E}}^4 + \|h^0(0)\|_{\mathcal{E}}^4 + \frac{(\varepsilon^3 \lambda_\varepsilon)^4}{\mu_0^4} \|h^0(0)\|_{\mathcal{E}}^8 \right) e^{-4\lambda_\varepsilon s} + \varepsilon^8 \lambda_\varepsilon^4 \|h^1(s)\|_{\mathcal{E}}^4 \\ &+ \frac{(\varepsilon \lambda_\varepsilon)^4}{\mu_0^3} \varepsilon^2 \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^4 d\tau + \frac{(\varepsilon^2 \lambda_\varepsilon)^4}{\mu_0^3} \varepsilon^2 \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^8 d\tau \\ &+ \varepsilon^4 \lambda_\varepsilon \left(1 + \mu_0^{-2} (\varepsilon^2 \lambda_\varepsilon)^2 \right)^4 \int_0^s e^{-\lambda_\varepsilon(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^8 d\tau \\ &+ \lambda_\varepsilon^{-1} \mu_0^{-8} \varepsilon^8 (\varepsilon^2 \lambda_\varepsilon)^{10} \int_0^s e^{-\lambda_\varepsilon(s-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^{16} d\tau. \end{aligned}$$

Multiplying this inequality with $e^{-\nu(t-s)}$, using (4.15) and integrating over $(0, t)$ yields

$$\begin{aligned} \int_0^t e^{-\nu(t-s)} \|\mathbf{P}_0 h^1(s)\|_{\mathcal{E}_{-1}}^4 ds &\lesssim \nu^{-1} \left(\|\Pi_\varepsilon h(0)\|_{\mathcal{E}}^4 + \|h^0(0)\|_{\mathcal{E}}^4 + \frac{(\varepsilon^3 \lambda_\varepsilon)^4}{\mu_0^4} \|h^0(0)\|_{\mathcal{E}}^8 \right) e^{-4\lambda_\varepsilon t} \\ &+ (\varepsilon^2 \lambda_\varepsilon)^4 \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{E}}^4 ds + \frac{(\varepsilon^2 \lambda_\varepsilon)^4}{\mu_0^4} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^4 ds \\ &+ \frac{(\varepsilon^3 \lambda_\varepsilon)^4}{\mu_0^4} \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{E}_2}^8 ds \\ &+ \nu^{-1} \varepsilon^4 \lambda_\varepsilon \left(1 + \mu_0^{-2} (\varepsilon^2 \lambda_\varepsilon)^2 \right)^4 \int_0^t e^{-\lambda_\varepsilon(t-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^8 d\tau \\ &+ \nu^{-1} \lambda_\varepsilon^{-1} \mu_0^{-8} \varepsilon^8 (\varepsilon^2 \lambda_\varepsilon)^{10} \int_0^t e^{-\lambda_\varepsilon(t-\tau)} \|h^1(\tau)\|_{\mathcal{E}_2}^{16} d\tau \end{aligned}$$

as soon as $\mu_0 \geq 2\varepsilon^2\nu$ and $\nu \geq 2\lambda_\varepsilon$. Using now the estimates

$$\|\cdot\|_{\mathcal{E}_2} \lesssim \|\cdot\|_{\mathcal{H}}, \quad \text{and} \quad \|h^1\|_{\mathcal{H}}^{i+2} \leq \Delta_0^i \|h^1\|_{\mathcal{H}}^2, \quad i \geq 0, \quad (4.20)$$

we obtain the result. \square

Proposition 4.12. *There exist $\varepsilon_5 \in (0, \varepsilon_4)$, $\lambda_5 \in (0, \lambda_4)$, C_1 is a positive universal constant and C_2 is a positive constant that depends on μ_0 and ν such that, for any $\delta > 0$, $t \geq 0$,*

$\varepsilon \in (0, \varepsilon_5)$ and $\lambda_\varepsilon \in (0, \lambda_5)$,

$$\begin{aligned} \|h^1(t)\|_{\mathcal{H}}^2 &\leq C_1 \mathcal{K}_0 e^{-2\lambda_\varepsilon t} + C_2 \left(\delta + \frac{\lambda_\varepsilon^2}{\delta \mu_0^2} \right) \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &+ C_2 \varepsilon^2 \lambda_\varepsilon \Delta_0^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds + C_2 \frac{\delta}{\varepsilon^2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds, \end{aligned} \quad (4.21)$$

where

$$\begin{aligned} \mathcal{K}_0 &:= \|h^1(0)\|_{\mathcal{H}}^2 + \frac{1}{\delta \mu_0} \|h^0(0)\|_{\mathcal{E}}^2 \\ &+ \frac{1}{\nu} \|\Pi_\varepsilon h(0)\|_{\mathcal{E}}^4 + \frac{1}{\nu} \|h^0(0)\|_{\mathcal{E}}^4 + \frac{1}{\nu \mu_0^4} \|h^0(0)\|_{\mathcal{E}}^8 \\ &+ \|\Pi_\varepsilon h(0)\|_{\mathcal{E}}^2 + \|h^0(0)\|_{\mathcal{E}}^2 + \frac{1}{\mu_0^2} \|h^0(0)\|_{\mathcal{E}}^4 \end{aligned}$$

depends only on $h(0)$, μ_0 and ν .

Proof. Combining the two previous Lemmas with Proposition 4.8, we first have the following estimate for $\|\Psi(t)\|_{\mathcal{H}}^2$. For simplicity, we set

$$b_\varepsilon := \|\Pi_\varepsilon h(0)\|_{\mathcal{E}}^2 + \|h^0(0)\|_{\mathcal{E}}^2 + \frac{(\varepsilon^3 \lambda_\varepsilon)^2}{\mu_0^2} \|h^0(0)\|_{\mathcal{E}}^4,$$

which depends only on $h(0)$ and μ_0 . Using again (4.20) to estimate $\|h^1\|_{\mathcal{E}_2}$ in Lemma 4.9, we get that there exists a positive constant $C > 0$ such that, for any $\delta > 0$,

$$\begin{aligned} \|\Psi(t)\|_{\mathcal{H}}^2 &\lesssim \left[\|\Psi(0)\|_{\mathcal{H}}^2 + \frac{1}{\delta \mu_0} \|h^0(0)\|_{\mathcal{E}}^2 \right] e^{-\nu t} + \nu^{-1} b_\varepsilon^2 e^{-4\lambda_\varepsilon t} \\ &+ \left(\delta + (\varepsilon^2 \lambda_\varepsilon)^4 C_0(\varepsilon, \mu_0, \Delta_0) + \frac{\lambda_\varepsilon^2}{\delta \mu_0^2} (1 + \varepsilon^2 \Delta_0^2) \right) \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &+ \varepsilon^4 \lambda_\varepsilon C_1(\varepsilon, \mu_0, \nu, \Delta_0) \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{\delta}{\varepsilon^2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds. \end{aligned}$$

Therefore, for ε sufficiently small, using Remark 4.11, since $(\varepsilon^2 \lambda_\varepsilon)^4 \ll (\varepsilon^2 \lambda_\varepsilon)^2 \ll \varepsilon^4 \lambda_\varepsilon \ll \varepsilon^2 \lambda_\varepsilon$ and $4\lambda_\varepsilon < \nu$, one obtains

$$\begin{aligned} \|\Psi(t)\|_{\mathcal{H}}^2 &\lesssim \left[\|\Psi(0)\|_{\mathcal{H}}^2 + \frac{1}{\delta \mu_0} \|h^0(0)\|_{\mathcal{E}}^2 + \nu^{-1} b_\varepsilon^2 \right] e^{-4\lambda_\varepsilon t} \\ &+ \left(\delta + (\varepsilon^2 \lambda_\varepsilon)^2 (1 + \mu_0^{-4}) \Delta_0^2 + \frac{\lambda_\varepsilon^2}{\delta \mu_0^2} \right) \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &+ \nu^{-1} \varepsilon^2 \lambda_\varepsilon (1 + \mu_0^{-8}) \Delta_0^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{\delta}{\varepsilon^2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds. \end{aligned}$$

Adding $\|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}}^2$ to both sides and, since

$$\|h^1(t)\|_{\mathcal{H}}^2 \leq \|\mathbf{P}_0 h^1(t)\|_{\mathcal{H}}^2 + \|\Psi(t)\|_{\mathcal{H}}^2 \lesssim \|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^2 + \|\Psi(t)\|_{\mathcal{H}}^2,$$

we obtain, using the estimate of $\|\mathbf{P}_0 h^1(t)\|_{\mathcal{E}_{-1}}^2$ in Remark 4.7, that

$$\begin{aligned} \|h^1(t)\|_{\mathcal{H}}^2 &\lesssim \left[\|h^1(0)\|_{\mathcal{H}}^2 + \frac{1}{\delta \mu_0} \|h^0(0)\|_{\mathcal{E}}^2 + \nu^{-1} b_\varepsilon^2 + b_\varepsilon \right] e^{-2\lambda_\varepsilon t} + \varepsilon^4 \lambda_\varepsilon^2 \|h^1(t)\|_{\mathcal{E}}^2 \\ &+ \left(\delta + (\varepsilon^2 \lambda_\varepsilon)^2 (1 + \mu_0^{-4}) \Delta_0^2 + \frac{\lambda_\varepsilon^2}{\delta \mu_0^2} + \frac{\varepsilon^2 \lambda_\varepsilon^2}{\mu_0} + \frac{(\varepsilon^2 \lambda_\varepsilon)^2}{\mu_0} \Delta_0^2 \right) \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &+ \varepsilon^2 \lambda_\varepsilon \Delta_0^2 \left(\nu^{-1} (1 + \mu_0^{-8}) + (1 + \mu_0^{-4}) + \mu_0^{-4} \varepsilon^4 (\varepsilon^2 \lambda_\varepsilon)^4 \Delta_0^4 \right) \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &\quad + \frac{\delta}{\varepsilon^2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds, \end{aligned}$$

where we used that $\nu \leq \frac{\mu_0}{\varepsilon^2}$. Therefore, there exist positive universal constants C_0 and C_1 and a positive constant C_2 that depends on μ_0 and ν , for ε sufficiently small, such that

$$\begin{aligned} (1 - C_0 \varepsilon^4 \lambda_\varepsilon^2) \|h^1(t)\|_{\mathcal{H}}^2 &\leq C_1 \left[\|h^1(0)\|_{\mathcal{H}}^2 + \frac{1}{\delta \mu_0} \|h^0(0)\|_{\mathcal{E}}^2 + \nu^{-1} b_\varepsilon^2 + b_\varepsilon \right] e^{-2\lambda_\varepsilon t} \\ &\quad + C_2 \left(\delta + \frac{\lambda_\varepsilon^2}{\delta \mu_0^2} \right) \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &\quad + C_2 \varepsilon^2 \lambda_\varepsilon \Delta_0^2 \int_0^t e^{-\lambda_\varepsilon(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds + C_2 \frac{\delta}{\varepsilon^2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds. \end{aligned}$$

Take ε small enough so that $1 - C_0 \varepsilon^4 \lambda_\varepsilon^2 \geq \frac{1}{2}$ to deduce the result. \square

Corollary 4.13. *There exist $\varepsilon_6 \in (0, \varepsilon_5)$, $\lambda_6 \in (0, \lambda_5)$, and $C > 0$ depending on ν, μ_0, Δ_0 such that*

$$\|h^1(t)\|_{\mathcal{H}}^2 \leq C \mathcal{K}_0 \exp(-\lambda_\varepsilon t), \quad \forall t \geq 0, \quad (4.22)$$

for any $\varepsilon \in (0, \varepsilon_6)$ and $\lambda_\varepsilon \in (0, \lambda_6)$. Here \mathcal{K}_0 is given in Proposition 4.12.

Proof. Set for simplicity

$$\mathbf{x}(t) := e^{\lambda_\varepsilon t} \|h^1(t)\|_{\mathcal{H}}^2, \quad t \geq 0.$$

Inequality (4.21) yields

$$\begin{aligned} \frac{1}{c_1} \mathbf{x}(t) &\leq \mathcal{K}_0 + \left(\delta + \frac{\lambda_\varepsilon^2}{\delta \mu_0^2} \right) \int_0^t e^{-(\nu - \lambda_\varepsilon)(t-s)} \mathbf{x}(s) ds \\ &\quad + \varepsilon^2 \lambda_\varepsilon \Delta_0^2 \int_0^t \mathbf{x}(s) ds + \frac{\delta}{\varepsilon^2} \int_0^t e^{-\frac{\mu_0}{\varepsilon^2} s} \mathbf{x}(s) ds. \end{aligned}$$

We use a Gronwall type argument to prove the result. For notational simplicity introduce

$$C_\delta(\lambda_\varepsilon) = c_1 \left(\delta + \frac{\lambda_\varepsilon^2}{\delta \mu_0^2} \right), \quad A_0 = c_1 \mathcal{K}_0, \quad \xi(t) = c_1 \varepsilon^2 \lambda_\varepsilon \Delta_0^2 + \frac{c_1 \delta}{\varepsilon^2} e^{-\frac{\mu_0}{\varepsilon^2} t}, \quad t \geq 0,$$

from which one obtains that

$$0 \leq \mathbf{x}(t) \leq A_0 + C_\delta(\lambda_\varepsilon) \int_0^t e^{-(\nu - \lambda_\varepsilon)(t-s)} \mathbf{x}(s) ds + \int_0^t \xi(s) \mathbf{x}(s) ds =: \Upsilon(t). \quad (4.23)$$

Thus,

$$\begin{aligned} \frac{d}{dt}\Upsilon(t) &= -(\nu - \lambda_\varepsilon)C_\delta(\lambda_\varepsilon) \int_0^t e^{-(\nu - \lambda_\varepsilon)(t-s)} \mathbf{x}(s) ds + (C_\delta(\lambda_\varepsilon) + \xi(t)) \mathbf{x}(t) \\ &= -(\nu - \lambda_\varepsilon) \left(\Upsilon(t) - A_0 - \int_0^t \xi(s) \mathbf{x}(s) ds \right) + (C_\delta(\lambda_\varepsilon) + \xi(t)) \mathbf{x}(t). \end{aligned}$$

Using (4.23), which reads $\mathbf{x}(t) \leq \Upsilon(t)$, we deduce that

$$\frac{d}{dt}\Upsilon(t) \leq (\nu - \lambda_\varepsilon)A_0 + [C_\delta(\lambda_\varepsilon) - (\nu - \lambda_\varepsilon) + \xi(t)]\Upsilon(t) + (\nu - \lambda_\varepsilon) \int_0^t \xi(s)\Upsilon(s) ds.$$

Clearly, it is possible to choose $\delta_\star > 0$ sufficiently small depending only on ν , and λ_\star sufficiently small depending only on ν, μ_0, Δ_0 , so that

$$C_\delta(\lambda_\varepsilon) - (\nu - \lambda_\varepsilon) + \xi(t) \leq -\frac{\nu}{2} + \frac{\delta c_1}{\varepsilon^2} e^{-\frac{\mu_0}{\varepsilon^2} t}, \quad \forall t \geq 0,$$

holds true for any $\delta \in (0, \delta_\star)$ and $\lambda_\varepsilon \in (0, \lambda_\star)$. Fixing δ and λ_ε in this range, we introduce

$$\mathbf{z}(t) = -\frac{\nu}{2}t + \frac{\delta c_1}{\varepsilon^2} \int_0^t e^{-\frac{\mu_0}{\varepsilon^2} s} ds$$

to deduce that

$$\begin{aligned} \frac{d}{dt} \left(e^{-\mathbf{z}(t)} \Upsilon(t) \right) &\leq (\nu - \lambda_\varepsilon) A_0 e^{-\mathbf{z}(t)} + (\nu - \lambda_\varepsilon) e^{-\mathbf{z}(t)} \int_0^t \xi(s) \Upsilon(s) ds \\ &\leq \nu A_0 e^{-\mathbf{z}(t)} + \nu e^{-\mathbf{z}(t)} \int_0^t \xi(s) \Upsilon(s) ds. \end{aligned}$$

Integration of this differential inequality yields (recalling that $\Upsilon(0) = A_0$)

$$\begin{aligned} \Upsilon(t) &\leq A_0 e^{\mathbf{z}(t)} + \nu A_0 \int_0^t e^{\mathbf{z}(t)-\mathbf{z}(s)} ds + \nu \int_0^t e^{\mathbf{z}(t)-\mathbf{z}(s)} ds \int_0^s \xi(\tau) \Upsilon(\tau) d\tau \\ &= A_0 e^{\mathbf{z}(t)} + \nu A_0 \int_0^t e^{\mathbf{z}(t)-\mathbf{z}(s)} ds + \nu \int_0^t \xi(\tau) \Upsilon(\tau) \left(\int_\tau^t e^{\mathbf{z}(t)-\mathbf{z}(s)} ds \right) d\tau. \end{aligned}$$

Notice that $\mathbf{z}(t) - \mathbf{z}(s) \leq \frac{\delta c_1}{\mu_0} - \frac{\nu}{2}(t-s)$, for $0 \leq s \leq t$, from which we conclude that

$$\int_\tau^t e^{\mathbf{z}(t)-\mathbf{z}(s)} ds \leq \frac{2}{\nu} e^{\frac{\delta c_1}{\mu_0}}, \quad 0 \leq \tau \leq t.$$

Consequently,

$$\Upsilon(t) \leq 3A_0 e^{\frac{\delta c_1}{\mu_0}} + 2e^{\frac{\delta c_1}{\mu_0}} \int_0^t \xi(s) \Upsilon(s) ds,$$

which, thanks to Gronwall lemma, implies that

$$\Upsilon(t) \leq 3A_0 e^{\frac{\delta c_1}{\mu_0}} \exp \left(2e^{\frac{\delta c_1}{\mu_0}} \int_0^t \xi(s) ds \right).$$

Noticing that

$$\int_0^t \xi(s) ds \leq c_1 \varepsilon^2 \lambda_\varepsilon \Delta_0^2 t + \frac{c_1 \delta}{\mu_0},$$

one can choose $\varepsilon_\star > 0$ sufficiently small so that

$$2e^{\frac{\delta c_1}{\mu_0}} c_1 \varepsilon^2 \Delta_0^2 \leq \frac{1}{2} \quad \text{for any } \varepsilon \in (0, \varepsilon_\star). \quad (4.24)$$

Consequently,

$$\Upsilon(t) \leq C A_0 e^{\frac{\lambda_\varepsilon}{2} t}$$

for some positive constant C depending only on ν, μ_0, Δ_0 . Recalling the definition of A_0 , such estimate combined with (4.23) gives that

$$\|h^1(t)\|_{\mathcal{H}}^2 \leq C \mathcal{K}_0 \exp\left(-\frac{\lambda_\varepsilon}{2} t\right), \quad \forall t \geq 0. \quad (4.25)$$

One can upgrade the relaxation rate up to λ_ε by bootstrapping. To see this, one suitably uses (4.25) instead of (4.20), namely

$$\|h^1(t)\|_{\mathcal{H}}^4 \leq C \mathcal{K}_0 \exp\left(-\frac{\lambda_\varepsilon}{2} t\right) \|h^1(t)\|_{\mathcal{H}}^2.$$

Then, (4.21) changes to

$$\begin{aligned} \frac{1}{c_1} \|h^1(t)\|_{\mathcal{H}}^2 &\leq \mathcal{K}_0 e^{-2\lambda_\varepsilon t} + \left(\delta + \frac{\lambda_\varepsilon^2}{\delta \mu_0^2}\right) \int_0^t e^{-\nu(t-s)} \|h^1(s)\|_{\mathcal{H}}^2 ds \\ &+ \varepsilon^2 \lambda_\varepsilon C \mathcal{K}_0 \int_0^t e^{-\lambda_\varepsilon(t-s)} e^{-\frac{\lambda_\varepsilon}{2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds + \frac{\delta}{\varepsilon^2} \int_0^t e^{-\nu(t-s) - \frac{\mu_0}{\varepsilon^2} s} \|h^1(s)\|_{\mathcal{H}}^2 ds. \end{aligned}$$

Consequently, one can redo the argument above with $\tilde{\xi}(t)$ instead of $\xi(t)$, where

$$\tilde{\xi}(t) = c_1 \varepsilon^2 \lambda_\varepsilon C \mathcal{K}_0 e^{-\frac{\lambda_\varepsilon}{2} t} + \frac{c_1 \delta}{\varepsilon^2} e^{-\frac{\mu_0}{\varepsilon^2} t}.$$

Since,

$$\int_0^t \tilde{\xi}(s) ds \leq 2c_1 \varepsilon^2 C \mathcal{K}_0 + \frac{c_1 \delta}{\mu_0} \leq 2 \frac{c_1 \delta}{\mu_0},$$

provided $\varepsilon \in (0, \varepsilon_\star)$ for sufficiently small $\varepsilon_\star := \varepsilon_\star(\nu, \mu_0, \Delta_0)$, one is led to (4.22). \square

Estimate (4.22) leads to the main result of this section.

Theorem 4.14. *There exist $\varepsilon^\dagger \in (0, \varepsilon_6)$, $\lambda^\dagger \in (0, \lambda_6)$, and $C > 0$ depending on ν, μ_0, Δ_0 such that*

$$\|h(t)\|_{\mathcal{E}}^2 \leq C \left(\|h(0)\|_{\mathcal{E}}^2 + \|h(0)\|_{\mathcal{E}}^4 + \|h(0)\|_{\mathcal{E}}^8 \right) \exp(-\lambda_\varepsilon t), \quad \forall t \geq 0,$$

for any $\varepsilon \in (0, \varepsilon^\dagger)$ and $\lambda_\varepsilon \in (0, \lambda^\dagger)$.

Proof. Using estimate (4.22) in estimate (4.8), it follows that

$$\|h^0(t)\|_{\mathcal{E}}^2 \leq 2\|h^0(0)\|_{\mathcal{E}}^2 e^{-\frac{2\mu_0}{\varepsilon^2}t} + \frac{C\mathcal{K}_0}{\mu_0^2}\varepsilon^2(\varepsilon\lambda_\varepsilon)^2 e^{-\lambda_\varepsilon t} \leq C\mathcal{K}_0 e^{-\lambda_\varepsilon t}. \quad (4.26)$$

Consequently,

$$\|h(t)\|_{\mathcal{E}}^2 \leq 2(\|h^0(t)\|_{\mathcal{E}}^2 + \|h^1(t)\|_{\mathcal{H}}^2) \leq C(\|h^0(t)\|_{\mathcal{E}}^2 + \|h^1(t)\|_{\mathcal{H}}^2) \leq C\mathcal{K}_0 e^{-\lambda_\varepsilon t}. \quad (4.27)$$

Recalling that $h^0(0) = h(0)$ and $h^1(0) = 0$ in the definition of \mathcal{K}_0 , estimate (4.27) gives the result. \square

We also point out the gain of decay in h in the following corollary.

Corollary 4.15. *Under the same conditions of Theorem 4.14 it follows that*

$$\int_0^t \|h(\tau)\|_{\mathcal{E}_1} d\tau \leq C\sqrt{\mathcal{K}_0} \min\left\{1+t, 1+\frac{1}{\lambda_\varepsilon}\right\}, \quad \forall t > 0.$$

In particular, $\|h(\cdot)\|_{\mathcal{E}_1}$ is integrable and exists a.e. in $(0, T)$ for any $T > 0$.

Proof. After performing time integration of equation (4.9) in $[0, t]$ one finds that

$$\begin{aligned} \|h^0(t)\|_{\mathcal{E}} + \frac{\mu_0}{\varepsilon^2} \int_0^t \|h^0(\tau)\|_{\mathcal{E}_1} d\tau \\ \leq \|h^0(0)\|_{\mathcal{E}} + C \int_0^t \left(\lambda_\varepsilon \|h^1(\tau)\|_{\mathcal{E}_1} + \varepsilon \lambda_\varepsilon \|h^1(\tau)\|_{\mathcal{E}_1}^2 \right) d\tau \\ \leq C \left(\sqrt{\mathcal{K}_0} + \varepsilon \mathcal{K}_0 \right), \quad \forall t > 0, \end{aligned} \quad (4.28)$$

where we used estimate (4.22) in the latter inequality. Thus,

$$\begin{aligned} \int_0^t \|h(\tau)\|_{\mathcal{E}_1} d\tau &\leq \int_0^t \|h^0(\tau)\|_{\mathcal{E}_1} d\tau + \int_0^t \|h^1(\tau)\|_{\mathcal{E}_1} d\tau \\ &\leq C \left(\sqrt{\mathcal{K}_0} + \varepsilon \mathcal{K}_0 \right) + C\sqrt{\mathcal{K}_0} \int_0^t e^{-\frac{\lambda_\varepsilon}{2}\tau} d\tau, \end{aligned}$$

which gives the result. \square

Remark 4.16. *Of course, for a fixed $\varepsilon > 0$, one can replace $\min\{1+t, 1+\frac{1}{\lambda_\varepsilon}\}$ with $1+\frac{1}{\lambda_\varepsilon}$ and the above estimate shows that $h(t) = h_\varepsilon(t) \in L^1([0, \infty), \mathcal{E}_1)$. However, in the case in which $\lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon = 0$ then the bound is not uniform with respect to ε . In practice, two situations occur according to the value of λ_0 in Assumption 1.2:*

a) *If $\lambda_0 > 0$, then the family $\{h_\varepsilon(t)\}_{\varepsilon \geq 0}$ is bounded in $L^1([0, \infty), \mathcal{E}_1)$,*

b) *If $\lambda_0 = 0$ then for any $T > 0$, the family $\{h_\varepsilon(t)\}_{\varepsilon \geq 0}$ is bounded in $L^1([0, T], \mathcal{E}_1)$.*

5. CAUCHY THEORY

In this section, we will use the functional spaces introduced at the beginning of Section 4. Based on the a priori estimates derived in the previous section, we show in this section the well-posedness of the system (4.4)-(4.5).

5.1. Iteration scheme. Let us follow the iteration scheme of [59, Section 3] with suitable modifications. We are seeking to approximate the solution to the inelastic Boltzmann equation using the iteration scheme

$$\begin{cases} \partial_t h_{n+1}(t) &= \mathcal{G}_\varepsilon h_{n+1}(t) + \varepsilon^{-1} \mathcal{Q}_{\alpha(\varepsilon)}(h_n(t), h_n(t)), & n \geq 1 \\ \partial_t h_1(t) &= \mathcal{G}_\varepsilon h_1(t), \\ h_n(0) &= h(0) \in \mathcal{E}, & n \geq 1, \end{cases} \quad (5.1)$$

where the initial perturbation $h(0)$ has zero mass and momentum. This is done using the decomposition of previous section. More precisely, writing $h_n = h_n^0 + h_n^1$ we consider solutions with the coupled system

$$\begin{cases} \partial_t h_{n+1}^0 &= \mathcal{B}_{\alpha(\varepsilon), \varepsilon} h_{n+1}^0 + \varepsilon^{-1} \mathcal{Q}_{\alpha(\varepsilon)}(h_n^0, h_n^0) + \varepsilon^{-1} \left[\mathcal{Q}_{\alpha(\varepsilon)}(h_n^0, h_n^1) + \mathcal{Q}_{\alpha(\varepsilon)}(h_n^1, h_n^0) \right] \\ &+ \left[\mathcal{G}_\varepsilon h_{n+1}^1 - \mathcal{G}_{1, \varepsilon} h_{n+1}^1 \right] + \varepsilon^{-1} \left[\mathcal{Q}_{\alpha(\varepsilon)}(h_n^1, h_n^1) - \mathcal{Q}_1(h_n^1, h_n^1) \right], \\ h_{n+1}^0(0) &= h^0(0) \in \mathcal{E}, \end{cases} \quad (5.2)$$

and

$$\begin{cases} \partial_t h_{n+1}^1 &= \mathcal{G}_{1, \varepsilon} h_{n+1}^1 + \varepsilon^{-1} \mathcal{Q}_1(h_n^1, h_n^1) + \mathcal{A}_\varepsilon h_{n+1}^0, \\ h_{n+1}^1(0) &= h^1(0) \in \mathcal{H}. \end{cases} \quad (5.3)$$

Motivated by the *a priori* estimates of Section 4, we introduce the following norms

$$\| \| g \| \|_0 := \sup_{t \geq 0} \left(\| g(t) \|_{\mathcal{E}} + \varepsilon^{-2} \int_0^t e^{-\nu(t-\tau)} \| g(\tau) \|_{\mathcal{E}_1} d\tau \right), \quad g \in \mathcal{C}([0, \infty), \mathcal{E}),$$

and

$$\| \| g \| \|_1 := \sup_{t \geq 0} \left(\| g(t) \|_{\mathcal{H}}^2 + \int_0^t e^{-\nu(t-\tau)} \| g(\tau) \|_{\mathcal{H}_1}^2 d\tau \right)^{\frac{1}{2}}, \quad g \in \mathcal{C}([0, \infty), \mathcal{H}),$$

where $\nu \sim \mu/\sigma_0^2 > 0$ was computed in (4.16) and we recall that $\mathcal{E}_1, \mathcal{E}, \mathcal{H}$ and \mathcal{H}_1 are defined in (4.3). Notice that $(\mathcal{C}([0, \infty), \mathcal{E}); \| \cdot \|_0)$ and $(\mathcal{C}([0, \infty), \mathcal{H}); \| \cdot \|_1)$ are Banach spaces. In particular, the space

$$\mathbb{B} := \mathcal{C}([0, \infty), \mathcal{E}) \times \mathcal{C}([0, \infty), \mathcal{H})$$

endowed with the norm

$$\| \| (g, h) \| \| := \| \| g \| \|_0 + \| \| h \| \|_1 \quad \text{for } (g, h) \in \mathbb{B},$$

is a Banach space. Define then

$$\begin{aligned} \mathcal{X}_0 &= \left\{ h^0 \in \mathcal{C}([0, \infty); \mathcal{E}) \mid \| \| h^0 \| \|_0 \leq C\sqrt{\mathcal{K}_0} \right\}, \\ \mathcal{X}_1 &= \left\{ h^1 \in \mathcal{C}([0, \infty); \mathcal{H}) \mid \| \| h^1 \| \|_1 \leq C\sqrt{\mathcal{K}_0} \right\}, \end{aligned} \quad (5.4)$$

for some positive constant $C > 0$ which can be explicitly estimated from the subsequent computations. The system (5.2)-(5.3) is a simplified coupled version of the system (4.4)-(4.5) with all nonlinear terms as sources. Notice however that the coupling between h_{n+1}^0 and h_{n+1}^1 in the system makes it *nonlinear*. However, because \mathcal{G}_ε is the generator of a C_0 -semigroup in \mathcal{E} , equation (5.1) is well-posed and

$$h_{n+1}(t) = \mathcal{V}_\varepsilon(t)h(0) + \varepsilon^{-1} \int_0^t \mathcal{V}_\varepsilon(t-s) \mathcal{Q}_{\alpha(\varepsilon)}(h_n(s), h_n(s)) ds$$

where $\{\mathcal{V}_\varepsilon(t); t \geq 0\}$ is the C_0 -semigroup in \mathcal{E} generated by \mathcal{G}_ε (i.e., with the notations of Prop. 2.13, $\mathcal{V}_\varepsilon(t) = \mathcal{V}_{\alpha(\varepsilon), \varepsilon}(t)$, $t \geq 0$). With this at hands, substitute in (5.2) the term h_{n+1}^1 by $h_{n+1} - h_{n+1}^0$ and look at $h_{n+1}(t)$ as an additional source term. In the same way for (5.3), the system (5.2)-(5.3) becomes linear (in terms of h_{n+1}^0 and h_{n+1}^1) and admits, for any $n \in \mathbb{N}$, a unique solution. One can use a slight modification of the ideas of Section 4 to check that the iteration scheme is stable, that is, the mapping

$$(h_n^0, h_n^1) \in \mathcal{X}_0 \times \mathcal{X}_1 \mapsto (h_{n+1}^0, h_{n+1}^1) \in \mathcal{X}_0 \times \mathcal{X}_1$$

is well defined. Indeed, existence of the scheme is guaranteed by the linear theory as the iteration scheme is based on the linear equation. Moreover, note that (5.1) preserves the conservation laws: mass conservation and vanishing momentum, which were essential for the *a priori* estimates related to $\mathbf{P}_0 h^1$. Thus, stability holds true under the conditions of the *a priori* estimates, that is, for $\varepsilon \in (0, \varepsilon^\dagger)$ (where ε^\dagger is defined in Theorem 4.14) and

$$\sup_{t \geq 0} (\|h_n^1(t)\|_{\mathcal{H}} + \|h_n^0(t)\|_{\mathcal{E}}) \leq C\sqrt{\mathcal{K}_0} \leq \Delta_0, \quad n \in \mathbb{N}.$$

This latter condition is possible by taking \mathcal{K}_0 smaller than a threshold depending only on the initial mass and energy E_{in} ⁵

$$\mathcal{K}_0 \leq (\Delta_0/C)^2 =: \mathcal{K}_0^\dagger.$$

We leave the details to the reader and focus in the next subsections on the convergence of the scheme.

5.2. Estimating $\|h_{n+1}^0 - h_n^0\|_{\mathcal{E}}$ and $\|h_{n+1}^1 - h_n^1\|_{\mathcal{H}}$. To prove the convergence of the scheme, we define for $n \in \mathbb{N}$

$$d_{n+1}^0 = h_{n+1}^0 - h_n^0, \quad d_{n+1}^1 = h_{n+1}^1 - h_n^1.$$

⁵since all the threshold values appearing here are prescribed by the choice of the initial mass and energy, see Remark 1.1.

Then, one deduces from (5.2) and (5.3)

$$\begin{cases} \partial_t d_{n+1}^0 &= \mathcal{B}_{\alpha(\varepsilon), \varepsilon} d_{n+1}^0 + \left[\mathcal{G}_\varepsilon d_{n+1}^1 - \mathcal{G}_{1, \varepsilon} d_{n+1}^1 \right] + \varepsilon^{-1} \mathcal{F}_n^0, \\ d_{n+1}^0(0) &= 0, \end{cases} \quad (5.5)$$

and

$$\begin{cases} \partial_t d_{n+1}^1 &= \mathcal{G}_{1, \varepsilon} d_{n+1}^1 + \mathcal{A}_\varepsilon d_{n+1}^0 + \varepsilon^{-1} \mathcal{F}_n^1, \\ d_{n+1}^1(0) &= 0. \end{cases} \quad (5.6)$$

The sources \mathcal{F}_n^i , for $i \in \{0, 1\}$, correspond to the bilinear terms and depend only on the previous iterations $\{h_n^i, h_{n-1}^i\}$, for $i \in \{0, 1\}$ and $n \geq 2$ (see (5.9) and (5.11) for the precise expression). We introduce

$$\begin{cases} \Psi_n^1(t) &= \|h_n^0(t)\|_{\mathcal{E}_2} + \|h_{n-1}^0(t)\|_{\mathcal{E}_2} \\ \Psi_n^\infty(t) &= \|h_n^0(t)\|_{\mathcal{E}} + \|h_{n-1}^0(t)\|_{\mathcal{E}} + \|h_n^1(t)\|_{\mathcal{H}} + \|h_{n-1}^1(t)\|_{\mathcal{H}}, \end{cases}$$

which satisfy

$$\sup_{t \geq 0} \left(\Psi_n^\infty(t) + \varepsilon^{-2} \int_0^t e^{-\nu(t-\tau)} \Psi_n^1(\tau) d\tau \right) \leq C \sqrt{\mathcal{K}_0}, \quad n \geq 2, \quad (5.7)$$

for $h_n^0, h_{n-1}^0 \in \mathcal{X}_0$, and $h_n^1, h_{n-1}^1 \in \mathcal{X}_1$. Consequently, the following estimate for d_{n+1}^0 follows under suitable modifications of the arguments leading to Proposition 4.1 (keep in mind that $\|\cdot\|_{\mathcal{E}_2} \lesssim \|\cdot\|_{\mathcal{H}}$).

Lemma 5.1. *Let $\varepsilon \in (0, \varepsilon^\dagger)$ and $\mathcal{K}_0 \leq \mathcal{K}_0^\dagger$. Then, we have that*

$$\begin{aligned} \|d_{n+1}^0(t)\|_{\mathcal{E}} &\lesssim \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} ds \\ &\quad + \varepsilon^{-1} \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \Psi_n^1(s) \left(\|d_n^0(s)\|_{\mathcal{E}} + \|d_n^1(s)\|_{\mathcal{H}} \right) ds \\ &\quad + \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \Psi_n^\infty(s) \left(\varepsilon^{-1} \|d_n^0(s)\|_{\mathcal{E}_1} + \varepsilon \lambda_\varepsilon \|d_n^1(s)\|_{\mathcal{H}} \right) ds. \end{aligned} \quad (5.8)$$

Proof. Here again, as in the proof of Proposition 4.1, we denote by $\|\cdot\|_{\mathcal{E}_1}$ and $\|\cdot\|_{\mathcal{E}}$ the norms on \mathcal{E}_1 and \mathcal{E} that are equivalent to the standard ones (with multiplicative constants independent of ε) and that make $\varepsilon^{-2}\nu_0 + \mathcal{B}_{\alpha(\varepsilon), \varepsilon}$ dissipative so that

$$\begin{aligned} \frac{d}{dt} \|d_{n+1}^0(t)\|_{\mathcal{E}} &\leq -\frac{\nu_0}{\varepsilon^2} \|d_{n+1}^0(t)\|_{\mathcal{E}_1} + \varepsilon^{-1} \|\mathcal{F}_n^0(t)\|_{\mathcal{E}} + \|\mathcal{G}_\varepsilon d_{n+1}^1(t) - \mathcal{G}_{1, \varepsilon} d_{n+1}^1(t)\|_{\mathcal{E}} \\ &\leq -\frac{\nu_0}{\varepsilon^2} \|d_{n+1}^0(t)\|_{\mathcal{E}_1} + \varepsilon^{-1} \|\mathcal{F}_n^0(t)\|_{\mathcal{E}} + C \lambda_\varepsilon \|d_{n+1}^1(t)\|_{\mathcal{H}}. \end{aligned}$$

We need to estimate $\|\mathcal{F}_n^0(t)\|_{\mathcal{E}}$. One has,

$$\begin{aligned} \mathcal{F}_n^0 &= \mathcal{Q}_{\alpha(\varepsilon)}(d_n^0, h_n^0) + \mathcal{Q}_{\alpha(\varepsilon)}(h_{n-1}^0, d_n^0) + 2\tilde{\mathcal{Q}}_{\alpha(\varepsilon)}(d_n^0, h_n^1) + 2\tilde{\mathcal{Q}}_{\alpha(\varepsilon)}(h_{n-1}^0, d_n^1) \\ &\quad + \left(\mathcal{Q}_{\alpha(\varepsilon)}(d_n^1, h_n^1) - \mathcal{Q}_1(d_n^1, h_n^1) \right) + \left(\mathcal{Q}_{\alpha(\varepsilon)}(h_{n-1}^1, d_n^1) - \mathcal{Q}_1(h_{n-1}^1, d_n^1) \right). \end{aligned} \quad (5.9)$$

Therefore, since $1 - \alpha(\varepsilon) \lesssim \varepsilon^2 \lambda_\varepsilon$, using Lemma 2.1 and the usual estimates for \mathcal{Q}_α and \mathcal{Q}_1 :

$$\begin{aligned} \|\mathcal{F}_n^0\|_\varepsilon &\lesssim \|d_n^0\|_{\varepsilon_1} (\|h_n^0\|_\varepsilon + \|h_{n-1}^0\|_\varepsilon) + \|d_n^0\|_\varepsilon (\|h_n^0\|_{\varepsilon_1} + \|h_{n-1}^0\|_{\varepsilon_1}) \\ &\quad + \|d_n^0\|_\varepsilon \|h_n^1\|_{\varepsilon_1} + \|d_n^0\|_{\varepsilon_1} \|h_n^1\|_\varepsilon + \|d_n^1\|_{\varepsilon_1} \|h_{n-1}^0\|_\varepsilon \\ &\quad + \|d_n^1\|_\varepsilon \|h_{n-1}^0\|_{\varepsilon_1} + \varepsilon^2 \lambda_\varepsilon \|d_n^1\|_{\varepsilon_2} (\|h_n^1\|_{\varepsilon_2} + \|h_{n-1}^1\|_{\varepsilon_2}). \end{aligned}$$

Using that $\|\cdot\|_{\varepsilon_2} \lesssim \|\cdot\|_{\mathcal{H}}$ we get

$$\|\mathcal{F}_n^0(t)\|_\varepsilon \lesssim \|d_n^0(t)\|_{\varepsilon_1} \Psi_n^\infty(t) + \Psi_n^1(t) (\|d_n^0(t)\|_\varepsilon + \|d_n^1(t)\|_{\mathcal{H}}) + \varepsilon^2 \lambda_\varepsilon \|d_n^1(t)\|_{\varepsilon_2} \Psi_n^\infty(t).$$

This leads to the desired estimate since $\mu_0 < \nu_0$ (see the proof of Proposition 4.1). \square

Regarding the projection $\mathbf{P}_0 d_{n+1}^1(t)$, since the difference $h_{n+1} - h_n = d_{n+1}^0 + d_{n+1}^1$ has zero mass and momentum, one can follow the line of proof of Lemma 4.4 to deduce that

$$\begin{aligned} \|\mathbf{P}_0 d_{n+1}^1(t)\|_{\varepsilon_{-1}} &\lesssim (1 - \alpha(\varepsilon)) \|d_{n+1}^1(t)\|_\varepsilon + \|d_{n+1}^0(t)\|_\varepsilon \\ &\quad + \varepsilon \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \Psi_n^\infty(s) (\|d_n^0(s)\|_\varepsilon + \|d_n^1(s)\|_\varepsilon) ds. \end{aligned}$$

Consequently, plugging (5.8) in the second term in the right side and recalling that

$$\|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{H}} \lesssim \|\mathbf{P}_0 d_{n+1}^1(t)\|_{\varepsilon_{-1}} \lesssim \|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{H}}$$

we obtain the following lemma.

Lemma 5.2. *For any $t \geq 0$, we have that*

$$\begin{aligned} \|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{H}} &\lesssim \lambda_\varepsilon \varepsilon^2 \|d_{n+1}^1(t)\|_{\mathcal{H}} + \lambda_\varepsilon \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} ds \\ &\quad + \varepsilon^{-1} \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \Psi_n^1(s) (\|d_n^0(s)\|_\varepsilon + \|d_n^1(s)\|_{\mathcal{H}}) ds \\ &\quad + \int_0^t e^{-\frac{\mu_0}{\varepsilon^2}(t-s)} \Psi_n^\infty(s) (\varepsilon^{-1} \|d_n^0(s)\|_{\varepsilon_1} + \varepsilon \lambda_\varepsilon \|d_n^1(s)\|_{\mathcal{H}}) ds \\ &\quad + \varepsilon \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-s)} \Psi_n^\infty(s) (\|d_n^0(s)\|_\varepsilon + \|d_n^1(s)\|_{\mathcal{H}}) ds. \end{aligned}$$

Let us focus on estimating $\mathbf{P}_0^\perp d_{n+1}^1(t)$. To do so, we introduce the functions Φ_n^1 and Φ_n^∞ defined by

$$\Phi_n^1(t) = \|h_n^1(t)\|_{\mathcal{H}_1}^2 + \|h_{n+1}^1(t)\|_{\mathcal{H}_1}^2 \quad \text{and} \quad \Phi_n^\infty = \|h_n^1(t)\|_{\mathcal{H}}^2 + \|h_{n+1}^1(t)\|_{\mathcal{H}}^2$$

which satisfy

$$\sup_{t \geq 0} \left(\Phi_n^\infty(t) + \int_0^t e^{-\nu(t-\tau)} \Phi_n^1(\tau) d\tau \right) \leq C \mathcal{K}_0, \quad n \geq 2. \quad (5.10)$$

One has the following lemma.

Lemma 5.3. *Let $\varepsilon \in (0, \varepsilon^\dagger)$ and $\mathcal{K}_0 \leq \mathcal{K}_0^\dagger$. Then,*

$$\begin{aligned} \|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 &\lesssim \int_0^t e^{-\nu(t-s)} \Phi_n^1(s) \|d_n^1(s)\|_{\mathcal{H}}^2 ds \\ &\quad + \int_0^t e^{-\nu(t-s)} \Phi_n^\infty(s) \|d_n^1(s)\|_{\mathcal{H}_1}^2 ds + \frac{\lambda_\varepsilon}{\mu_0} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 ds \\ &\quad + \frac{1}{\mu_0 \varepsilon} \left(\sup_{t \geq 0} \|d_{n+1}^1(t)\|_{\mathcal{H}} \right) \int_0^t e^{-\nu(t-\tau)} \Psi_n^1(\tau) \left(\|d_n^0(\tau)\|_{\mathcal{E}} + \|d_n^1(\tau)\|_{\mathcal{H}} \right) d\tau \\ &\quad + \frac{1}{\mu_0} \left(\sup_{t \geq 0} \|d_{n+1}^1(t)\|_{\mathcal{H}} \right) \int_0^t e^{-\nu(t-\tau)} \Psi_n^\infty(\tau) \left(\varepsilon^{-1} \|d_n^0(\tau)\|_{\mathcal{E}} + \varepsilon \lambda_\varepsilon \|d_n^1(\tau)\|_{\mathcal{H}} \right) d\tau. \end{aligned}$$

Proof. One deduces from (5.6) that $\mathbf{P}_0^\perp d_{n+1}^1(t)$ is such that

$$\partial_t \mathbf{P}_0^\perp d_{n+1}^1(t) = \mathcal{G}_{1,\varepsilon} \mathbf{P}_0^\perp d_{n+1}^1(t) + \mathbf{P}_0^\perp \mathcal{A}_\varepsilon d_{n+1}^0(t) + \mathcal{F}_n^1$$

where

$$\mathcal{F}_n^1 = \mathcal{Q}_1(h_n^1, h_n^1) - \mathcal{Q}_1(h_{n-1}^1, h_{n-1}^1) = \mathcal{Q}_1(d_n^1, h_n^1) + \mathcal{Q}_1(h_{n-1}^1, d_n^1). \quad (5.11)$$

Following the argument leading to inequality (4.18) (see also [15, Lemma 4.6, Theorem 4.7]) one deduces that

$$\begin{aligned} \|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 &\lesssim \int_0^t e^{-\nu(t-s)} \Phi_n^1(s) \|d_n^1(s)\|_{\mathcal{H}}^2 ds + \int_0^t e^{-\nu(t-s)} \Phi_n^\infty(s) \|d_n^1(s)\|_{\mathcal{H}_1}^2 ds \\ &\quad + \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \|\mathcal{A}_\varepsilon d_{n+1}^0(s)\|_{\mathcal{H}} ds. \end{aligned} \quad (5.12)$$

The latter term in the right side of (5.12) can be estimated using (5.8) and recalling that $\|\mathcal{A}_\varepsilon d_{n+1}^0\|_{\mathcal{H}} \lesssim \varepsilon^{-2} \|d_{n+1}^0\|_{\mathcal{E}}$. Thus,

$$\int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \|\mathcal{A}_\varepsilon d_{n+1}^0(s)\|_{\mathcal{H}} ds \lesssim \sum_{i=1}^3 \mathcal{T}_i,$$

with

$$\begin{aligned} \mathcal{T}_1 &= \frac{\lambda_\varepsilon}{\varepsilon^2} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \left(\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|d_{n+1}^1(\tau)\|_{\mathcal{H}} d\tau \right) ds, \\ \mathcal{T}_2 &= \varepsilon^{-3} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \left[\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \Psi_n^1(\tau) \left(\|d_n^0(\tau)\|_{\mathcal{E}} + \|d_n^1(\tau)\|_{\mathcal{H}} \right) d\tau \right] ds \\ \mathcal{T}_3 &= \varepsilon^{-2} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \left[\int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \Psi_n^\infty(\tau) \left(\varepsilon^{-1} \|d_n^0(\tau)\|_{\mathcal{E}} + \varepsilon \lambda_\varepsilon \|d_n^1(\tau)\|_{\mathcal{H}} \right) d\tau \right] ds. \end{aligned}$$

It is easy to check, using (4.15), that

$$\mathcal{T}_2 \leq \frac{2}{\mu_0 \varepsilon} \left(\sup_{t \geq 0} \|d_{n+1}^1(t)\|_{\mathcal{H}} \right) \int_0^t e^{-\nu(t-\tau)} \Psi_n^1(\tau) \left(\|d_n^0(\tau)\|_{\mathcal{E}} + \|d_n^1(\tau)\|_{\mathcal{H}} \right) d\tau$$

and

$$\mathcal{T}_3 \leq \frac{2}{\mu_0} \left(\sup_{t \geq 0} \|d_{n+1}^1(t)\|_{\mathcal{H}} \right) \int_0^t e^{-\nu(t-\tau)} \Psi_n^\infty(\tau) \left(\varepsilon^{-1} \|d_n^0(\tau)\|_{\mathcal{E}_1} + \varepsilon \lambda_\varepsilon \|d_n^1(\tau)\|_{\mathcal{H}} \right) d\tau.$$

The estimate for \mathcal{T}_1 is a bit more involved. Thanks to Cauchy-Schwarz inequality one first has

$$\mathcal{T}_1 \leq \frac{\lambda_\varepsilon}{\varepsilon^2} \left(\int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t e^{-\nu(t-s)} Y^2(s) ds \right)^{\frac{1}{2}}$$

where

$$Y(s) := \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|d_{n+1}^1(\tau)\|_{\mathcal{H}} d\tau, \quad s \in (0, t).$$

Thanks to (4.10),

$$Y^2(s) \leq \frac{\varepsilon^2}{\mu_0} \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|d_{n+1}^1(\tau)\|_{\mathcal{H}}^2 d\tau$$

and, using now (4.15) for $\mu_0 > 2\varepsilon^2\nu$,

$$\begin{aligned} \int_0^t e^{-\nu(t-s)} Y^2(s) ds &\leq \frac{\varepsilon^2}{\mu_0} \int_0^t e^{-\nu(t-s)} ds \int_0^s e^{-\frac{\mu_0}{\varepsilon^2}(s-\tau)} \|d_{n+1}^1(\tau)\|_{\mathcal{H}}^2 d\tau \\ &\leq \frac{\varepsilon^2}{\mu_0 \left(\frac{\mu_0}{\varepsilon^2} - \nu \right)} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 ds \leq \frac{2\varepsilon^4}{\mu_0^2} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 ds. \end{aligned}$$

We deduce finally that

$$\mathcal{T}_1 \leq \frac{\sqrt{2}\lambda_\varepsilon}{\mu_0} \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}}^2 ds$$

and this, together with the estimates for \mathcal{T}_2 and \mathcal{T}_3 , gives the result. \square

Introducing now the quantities

$$\begin{aligned} \Xi_n^0 &= \sup_{t \geq 0} \left(\|d_n^0(t)\|_{\mathcal{E}} + \varepsilon^{-2} \int_0^t e^{-\nu(t-\tau)} \|d_n^0(\tau)\|_{\mathcal{E}_1} d\tau \right), \\ \Xi_n^1 &= \sup_{t \geq 0} \left(\|d_n^1(t)\|_{\mathcal{H}}^2 + \int_0^t e^{-\nu(t-\tau)} \|d_n^1(\tau)\|_{\mathcal{H}_1}^2 d\tau \right)^{\frac{1}{2}}, \quad n \geq 2, \end{aligned}$$

we can gather the three previous lemmas and use (5.7) to obtain the following result.

Proposition 5.4. *For any $n \in \mathbb{N}$ and $t \geq 0$*

$$\|d_{n+1}^0(t)\|_{\mathcal{E}} \lesssim \frac{\varepsilon^2 \lambda_\varepsilon}{\mu_0} \Xi_{n+1}^1 + \sqrt{\mathcal{K}_0} \varepsilon \left(\Xi_n^0 + \Xi_n^1 (1 + \mu_0^{-1}) \right). \quad (5.13)$$

while

$$\|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{H}} \lesssim \varepsilon^2 \lambda_\varepsilon (1 + \mu_0^{-1}) \Xi_{n+1}^1 + \sqrt{\mathcal{K}_0} \varepsilon \left(\Xi_n^0 + \Xi_n^1 (1 + \mu_0^{-1}) \right), \quad (5.14)$$

and

$$\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}} \lesssim \sqrt{\frac{\lambda_\varepsilon}{\nu\mu_0} + \varepsilon} \mathfrak{E}_{n+1}^1 + \frac{1}{\mu_0} \sqrt{\varepsilon\mathcal{K}_0} \mathfrak{E}_n^0 + \left(\frac{\sqrt{(1+\nu^{-1})\varepsilon}}{\mu_0} + 1\right) \sqrt{(1+\nu^{-1})\mathcal{K}_0} \mathfrak{E}_n^1. \quad (5.15)$$

In particular,

$$\begin{aligned} \|d_{n+1}^1(t)\|_{\mathcal{H}} &\lesssim \left((1+\mu_0^{-1})\sqrt{\varepsilon} + \sqrt{\frac{\lambda_\varepsilon}{\nu\mu_0}}\right) \mathfrak{E}_{n+1}^1 + (1+\mu_0^{-1})\sqrt{\varepsilon\mathcal{K}_0} \mathfrak{E}_n^0 \\ &\quad + \sqrt{1+\nu^{-1}} \left(\sqrt{1+\nu^{-1}}(1+\mu_0^{-1})\sqrt{\varepsilon} + 1\right) \sqrt{\mathcal{K}_0} \mathfrak{E}_n^1, \end{aligned} \quad (5.16)$$

as long as $\varepsilon \in (0, \varepsilon^\dagger)$, $\mathcal{K}_0 \leq \mathcal{K}_0^\dagger$.

Proof. We give a detailed proof of inequality (5.15). Notice directly from Lemma 5.3 that

$$\begin{aligned} \|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 &\lesssim \frac{\lambda_\varepsilon}{\mu_0} [\mathfrak{E}_{n+1}^1]^2 \int_0^t e^{-\nu(t-s)} ds \\ &\quad + [\mathfrak{E}_n^1]^2 \left(\int_0^t e^{-\nu(t-s)} \Phi_n^1(s) ds + \sup_{t \geq 0} \Phi_n^\infty(t) \int_0^t e^{-\nu(t-s)} ds \right) \\ &\quad + \frac{1}{\mu_0 \varepsilon} \mathfrak{E}_{n+1}^1 (\mathfrak{E}_n^0 + \mathfrak{E}_n^1) \int_0^t e^{-\nu(t-\tau)} \Psi_n^1(\tau) d\tau \\ &\quad + \frac{1}{\mu_0} \mathfrak{E}_{n+1}^1 (\sup_{t \geq 0} \Psi_n^\infty(t)) (\varepsilon \mathfrak{E}_n^0 + \nu^{-1} \varepsilon \lambda_\varepsilon \mathfrak{E}_n^1) \end{aligned}$$

since $\int_0^t e^{-\nu(t-s)} ds \leq \nu^{-1}$. We can thus invoke (5.7) and (5.10) to deduce that

$$\|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 \lesssim (1+\nu^{-1})\mathcal{K}_0 [\mathfrak{E}_n^1]^2 + \frac{\lambda_\varepsilon}{\nu\mu_0} [\mathfrak{E}_{n+1}^1]^2 + \frac{\varepsilon}{\mu_0} \sqrt{\mathcal{K}_0} (\mathfrak{E}_n^0 + \mathfrak{E}_n^1 (1+\nu^{-1})) \mathfrak{E}_{n+1}^1$$

where we used that $\varepsilon \lambda_\varepsilon < \varepsilon$. From Young's inequality, we deduce that

$$\begin{aligned} \|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2 &\lesssim \left(\frac{\lambda_\varepsilon}{\nu\mu_0} + \varepsilon\right) [\mathfrak{E}_{n+1}^1]^2 \\ &\quad + (1+\nu^{-1})\mathcal{K}_0 \left(1 + \frac{\varepsilon}{\mu_0^2} (1+\nu^{-1})\right) [\mathfrak{E}_n^1]^2 + \frac{\varepsilon}{\mu_0^2} \mathcal{K}_0 [\mathfrak{E}_n^0]^2. \end{aligned}$$

This proves (5.15). The other inequalities are easier to prove along the same lines, with (5.13) easily deduced from Lemma 5.1 and (5.14) deduced from Lemma 5.2 together with (5.7). The proof of (5.16) comes from the fact that $\|d_{n+1}^1(t)\|_{\mathcal{H}}^2 = \|\mathbf{P}_0 d_{n+1}^1(t)\|_{\mathcal{H}}^2 + \|\mathbf{P}_0^\perp d_{n+1}^1(t)\|_{\mathcal{H}}^2$. \square

Proposition 5.5. *For any $t \geq 0$, we have that*

$$\left(\frac{\mu_0}{\varepsilon^2} - \nu\right) \int_0^t e^{-\nu(t-s)} \|d_{n+1}^0(s)\|_{\mathcal{E}_1} ds \lesssim \nu^{-1} \lambda_\varepsilon \mathfrak{E}_{n+1}^1 + \varepsilon \sqrt{\mathcal{K}_0} (\mathfrak{E}_n^0 + (1+\nu^{-1})\mathfrak{E}_n^1) \quad (5.17)$$

$$\begin{aligned} \left(\int_0^t e^{-\nu(t-\tau)} \|\mathbf{P}_0 d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 d\tau \right)^{\frac{1}{2}} &\lesssim \nu^{-1/2} (1 + \mu_0^{-1}) \varepsilon^2 \lambda_\varepsilon \Xi_{n+1}^1 \\ &+ \nu^{-1/2} \sqrt{\mathcal{K}_0} \varepsilon (\Xi_n^0 + \Xi_n^1 (1 + \mu_0^{-1})). \end{aligned} \quad (5.18)$$

whereas

$$\begin{aligned} \Delta_0^2 \left(\int_0^t e^{-\nu(t-\tau)} \|\mathbf{P}_0^\perp d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 d\tau \right)^{\frac{1}{2}} &\lesssim \sqrt{\frac{\lambda_\varepsilon}{\nu\mu_0} + \varepsilon} \Xi_{n+1}^1 + \frac{1}{\mu_0} \sqrt{\varepsilon \mathcal{K}_0} \Xi_n^0 \\ &+ \left(\frac{\sqrt{(1+\nu^{-1})\varepsilon}}{\mu_0} + 1 \right) \sqrt{(1+\nu^{-1})\mathcal{K}_0} \Xi_n^1. \end{aligned} \quad (5.19)$$

In particular

$$\begin{aligned} \Delta_0^2 \left(\int_0^t e^{-\nu(t-\tau)} \|d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 d\tau \right)^{\frac{1}{2}} &\lesssim \left((1 + \nu^{-1/2})(1 + \mu_0^{-1})\sqrt{\varepsilon} + \sqrt{\frac{\lambda_\varepsilon}{\nu\mu_0}} \right) \Xi_{n+1}^1 \\ &+ (\nu^{-1/2} + \mu_0^{-1}) \sqrt{\varepsilon \mathcal{K}_0} \Xi_n^0 + \sqrt{1 + \nu^{-1}} \left(\sqrt{1 + \nu^{-1}} (1 + \mu_0^{-1}) \sqrt{\varepsilon} + 1 + \mu_0^{-1} \right) \sqrt{\mathcal{K}_0} \Xi_n^1. \end{aligned} \quad (5.20)$$

Proof. To prove (5.17), we follow the argument that led to Lemma 5.1 and thus in the subsequent proof, we again denote by $\|\cdot\|_{\mathcal{E}_1}$ and $\|\cdot\|_{\mathcal{E}}$ the norms on \mathcal{E}_1 and \mathcal{E} that are equivalent to the standard ones independently of ε and that make $\varepsilon^{-2}\nu_0 + \mathcal{B}_{\alpha(\varepsilon),\varepsilon}$ dissipative so that we can write

$$\frac{d}{dt} \|d_{n+1}^0(t)\|_{\mathcal{E}} \leq -\frac{\nu_0}{\varepsilon^2} \|d_{n+1}^0(t)\|_{\mathcal{E}_1} + \varepsilon^{-1} \|\mathcal{F}_n^0(t)\|_{\mathcal{E}} + C\lambda_\varepsilon \|d_{n+1}^1(t)\|_{\mathcal{H}},$$

which implies that,

$$\begin{aligned} \frac{d}{dt} \|d_{n+1}^0(t)\|_{\mathcal{E}} + \nu \|d_{n+1}^0(t)\|_{\mathcal{E}_1} &\leq -\left(\frac{\mu_0}{\varepsilon^2} - \nu\right) \|d_{n+1}^0(t)\|_{\mathcal{E}_1} \\ &+ \varepsilon^{-1} \|\mathcal{F}_n^0(t)\|_{\mathcal{E}} + C\lambda_\varepsilon \|d_{n+1}^1(t)\|_{\mathcal{H}} \end{aligned}$$

where we used that $\mu_0 < \nu_0$. After integration over $[0, t]$, using that $d_{n+1}^0(0) = 0$, we get that

$$\begin{aligned} \|d_{n+1}^0(t)\|_{\mathcal{E}} &\leq -\left(\frac{\mu_0}{\varepsilon^2} - \nu\right) \int_0^t e^{-\nu(t-s)} \|d_{n+1}^0(s)\|_{\mathcal{E}_1} ds + \varepsilon^{-1} \int_0^t e^{-\nu(t-s)} \|\mathcal{F}_n^1(s)\|_{\mathcal{E}} ds \\ &+ C\lambda_\varepsilon \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} ds, \end{aligned}$$

and, recalling that \mathcal{F}_n^0 is given by (5.9), we estimate $\|\mathcal{F}_n^0(s)\|_{\mathcal{E}}$ as in Lemma 5.1 to obtain that

$$\begin{aligned} \left(\frac{\mu_0}{\varepsilon^2} - \nu\right) \int_0^t e^{-\nu(t-s)} \|d_{n+1}^0(s)\|_{\mathcal{E}_1} ds &\lesssim \lambda_\varepsilon \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} ds \\ &+ \varepsilon^{-1} \int_0^t e^{-\nu(t-s)} \Psi_n^1(s) \left(\|d_n^0(s)\|_{\mathcal{E}} + \|d_n^1(s)\|_{\mathcal{H}} \right) ds \\ &+ \int_0^t e^{-\nu(t-s)} \Psi_n^\infty(s) \left(\varepsilon^{-1} \|d_n^0(s)\|_{\mathcal{E}_1} + \varepsilon \lambda_\varepsilon \|d_n^1(s)\|_{\mathcal{H}} \right) ds. \end{aligned}$$

This yields (5.17). In the same way, we adapt the proof of Lemma 5.3 to get that

$$\begin{aligned} \left(\frac{2\mu}{\sigma_0^2} - \nu\right) \int_0^t e^{-\nu(t-\tau)} \|\mathbf{P}_0^\perp d_{n+1}^1(\tau)\|_{\mathcal{H}_1}^2 d\tau &\lesssim \int_0^t e^{-\nu(t-s)} \Phi_n^1(s) \|d_n^1(s)\|_{\mathcal{H}}^2 ds \\ &+ \int_0^t e^{-\nu(t-s)} \Phi_n^\infty(s) \|d_n^1(s)\|_{\mathcal{H}_1}^2 ds + \int_0^t e^{-\nu(t-s)} \|d_{n+1}^1(s)\|_{\mathcal{H}} \|\mathcal{A}_\varepsilon d_{n+1}^0(s)\|_{\mathcal{H}} ds \end{aligned}$$

where $2\mu/\sigma_0^2 - \nu = c_0\Delta_0^2$ (see (4.16)). This estimate is similar to (5.12) and therefore we can resume both the proofs of Lemma 5.3 and Proposition 5.4 to conclude to (5.19). Recalling that $\|\mathbf{P}_0 d_{n+1}^1\|_{\mathcal{H}_1} \lesssim \|\mathbf{P}_0 d_{n+1}^1\|_{\mathcal{H}}$, a simple integration of (5.14) gives (5.18). \square

5.3. Convergence of the iteration scheme. In this Subsection, we do not mention anymore the dependences of positive multiplicative constants in terms of μ_0 , ν and Δ_0 so that they may depend on these parameters. We are in position to conclude our analysis and prove the convergence of the iteration scheme. Suitably adding (5.13) and (5.17) and taking the supremum in time, one has that

$$\Xi_{n+1}^0 \lesssim \lambda_\varepsilon \Xi_{n+1}^1 + \varepsilon \sqrt{\mathcal{K}_0} (\Xi_n^0 + \Xi_n^1) \quad (5.21)$$

where we used that $\mu_0 \geq 2\varepsilon^2\nu$. Similarly, adding (5.16) and (5.20) and taking the supremum in time it holds that

$$\Xi_{n+1}^1 \lesssim \sqrt{\lambda_\varepsilon + \varepsilon} \Xi_{n+1}^1 + \sqrt{\varepsilon\mathcal{K}_0} \Xi_n^0 + \sqrt{\mathcal{K}_0} \Xi_n^1. \quad (5.22)$$

Let us define $\mathcal{E}_n = \Xi_n^0 + \Xi_n^1$, for $n \geq 2$. Adding the estimates (5.21) and (5.22), we conclude that there exists $C > 0$ such that $\mathcal{E}_{n+1} \leq C(\lambda_\varepsilon + \sqrt{\lambda_\varepsilon + \varepsilon}) \mathcal{E}_{n+1} + C\sqrt{\mathcal{K}_0} \mathcal{E}_n$. Thus, choosing ε sufficiently small such that $C(\lambda_\varepsilon + \sqrt{\lambda_\varepsilon + \varepsilon}) < \frac{1}{2}$, we get that $\mathcal{E}_{n+1} \leq C\sqrt{\mathcal{K}_0} \mathcal{E}_n$ from which

$$\mathcal{E}_{n+1} \leq (C\sqrt{\mathcal{K}_0})^{n-1} \mathcal{E}_2, \quad \forall n \geq 2.$$

Whence, in the Banach space $(\mathbb{B}, \|\cdot\|)$, one has for $m > n \geq 1$,

$$\|(h_m^0, h_m^1) - (h_n^0, h_n^1)\| \leq \sum_{i=n}^{m-1} \mathcal{E}_{i+1} \leq \mathcal{E}_2 \frac{\theta^{n-1}}{1-\theta}, \quad \theta := C\sqrt{\mathcal{K}_0}.$$

Therefore, choosing $\mathcal{K}_0 \leq \mathcal{K}_0^\dagger < C^{-2}$ so that $\theta < 1$, we conclude that the sequence $\{(h_n^0, h_n^1)\}_n \subset \mathcal{X}_0 \times \mathcal{X}_1 \subset \mathbb{B}$ is a Cauchy sequence. Hence, it converges in $(\mathbb{B}, \|\cdot\|)$ to a

limit $(h^0, h^1) \in \mathcal{X}_0 \times \mathcal{X}_1$. Of course, such limit satisfies equations (4.4) and (4.5). Thus, $h = h^0 + h^1$ is a solution to the inelastic Boltzmann problem (1.20). Such solution is unique in the class of functions that we consider since, at essence, we proved that the problem is a contraction on $\mathcal{X}_0 \times \mathcal{X}_1$. Let us write the conclusion as the main theorem of the section.

Theorem 5.6. *Fix a nonnegative initial data $F_{\text{in}}^\varepsilon = G_\alpha + \varepsilon h_{\text{in}}^\varepsilon \in \mathcal{E}$ and assume that the initial perturbation $h_{\text{in}}^\varepsilon$ has zero total mass and momentum*

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} h_{\text{in}}^\varepsilon(t, x, v) dv dx = \int_{\mathbb{T}^d \times \mathbb{R}^d} h_{\text{in}}^\varepsilon(t, x, v) v dv dx = 0.$$

Then, there exist positive threshold values $(\varepsilon^\dagger, \lambda^\dagger, \mathcal{K}_0^\dagger)$ fully determined by the initial mass and energy E_{in} (see Remark 1.1) such that if

$$\|h_{\text{in}}^\varepsilon\|_{\mathcal{E}} \leq \sqrt{\mathcal{K}_0^\dagger},$$

and $\varepsilon \in (0, \varepsilon^\dagger)$, $\lambda_\varepsilon \in (0, \lambda^\dagger)$, the inelastic Boltzmann equation (1.20) has a unique solution $h_\varepsilon \in \mathcal{C}([0, \infty); \mathcal{E})$ satisfying for $t > 0$

$$\|h_\varepsilon(t)\|_{\mathcal{E}} \leq C \|h_{\text{in}}^\varepsilon\|_{\mathcal{E}} \exp(-\lambda_\varepsilon t) \quad \text{and} \quad \int_0^t \|h_\varepsilon(\tau)\|_{\mathcal{E}_1} d\tau \leq C \|h_{\text{in}}^\varepsilon\|_{\mathcal{E}} \min\left\{1 + t, 1 + \frac{1}{\lambda_\varepsilon}\right\}.$$

6. HYDRODYNAMIC LIMIT

In this last section, we will once again specify that h , h^0 and h^1 depend on ε by noting $h = h_\varepsilon$, $h^0 = h_\varepsilon^0$, $h^1 = h_\varepsilon^1$. On the other hand, to lighten notations, we will write α for $\alpha(\varepsilon)$ but recall that $\alpha = \alpha(\varepsilon)$ satisfies Assumption 1.2. Finally, we will consider m and q satisfying (4.2) as well as the spaces \mathcal{E} and \mathcal{E}_1 defined in (4.1)-(4.3).

6.1. Compactness and convergence. We start this section recalling the expression for the spectral projection π_0 onto the kernel $\text{Ker}(\mathbf{L}_1)$ of the linearized collision operator \mathbf{L}_1 seen as an operator acting in velocity only on the space $L_v^2(\mathcal{M}^{-\frac{1}{2}})$. We recall that, with the notations of Theorem 2.10,

$$\pi_0(g) := \sum_{i=1}^{d+2} \left(\int_{\mathbb{R}^d} g \Psi_i dv \right) \Psi_i \mathcal{M}, \quad (6.1)$$

where $\Psi_1(v) = 1$, $\Psi_i(v) = \frac{1}{\sqrt{\vartheta_1}} v_{i-1}$ ($i = 2, \dots, d+1$) and $\Psi_{d+2}(v) = \frac{|v|^2 - d\vartheta_1}{\vartheta_1 \sqrt{2d}}$. Note that the difference with respect to the spectral projection \mathbf{P}_0 for the operator $\mathcal{G}_{1,\varepsilon}$ in (2.22) is that no spatial integration is performed.

Consider now $h_\varepsilon = h_\varepsilon^0 + h_\varepsilon^1$ the solution constructed in Section 5. One can prove the following estimate for time-averages of $(\text{Id} - \pi_0)h_\varepsilon(\tau)$ in spaces which do not involve derivatives in the v -variable:

Proposition 6.1. *For any $0 \leq \beta \leq m - 1$ there exists $C > 0$ independent of ε such that*

$$\int_{t_1}^{t_2} \|(\mathbf{Id} - \pi_0)h_\varepsilon(\tau)\|_{\mathbb{W}_x^{\beta,1}L_v^1(\varpi_q)} d\tau \leq C \varepsilon \sqrt{\mathcal{K}_0} \sqrt{t_2 - t_1} \exp\left(\frac{\nu}{2}(t_2 - t_1)\right) \quad (6.2)$$

holds true for any $0 \leq t_1 \leq t_2$.

Proof. For a given $0 \leq \beta \leq m - 1$, we introduce the hierarchy of Hilbert spaces

$$\tilde{H}_s = \mathbb{W}_x^{\beta,2}L_v^2(\mathcal{M}^{-\frac{1}{2}}\langle v \rangle^{\frac{s}{2}}), \quad s \in \mathbb{R},$$

setting simply $\tilde{H} := \tilde{H}_0$. Recall that $-\mathbf{L}_1$ is (better than) coercive on $(\mathbf{Id} - \pi_0)\tilde{H}$ (see [14] for instance) and denote by $\tilde{\mu}_1$ the coercivity constant, namely

$$-\langle \mathbf{L}_1(\mathbf{Id} - \pi_0)g, (\mathbf{Id} - \pi_0)g \rangle_{\tilde{H}} \geq \tilde{\mu}_1 \|(\mathbf{Id} - \pi_0)g\|_{\tilde{H}_1}^2, \quad g \in \tilde{H}_1.$$

In the space \tilde{H} , we can compute the inner product between $\partial_t h_\varepsilon^1$ and $(\mathbf{Id} - \pi_0)h_\varepsilon^1$ where we recall that h_ε^1 solves

$$\partial_t h_\varepsilon^1 = \mathcal{G}_{1,\varepsilon} h_\varepsilon^1 + \varepsilon^{-1} \mathcal{Q}_1(h_\varepsilon^1, h_\varepsilon^1) + \mathcal{A}_\varepsilon h_\varepsilon^0.$$

We obtain, thanks to Cauchy-Schwarz inequality, that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}}^2 + \frac{\tilde{\mu}_1}{\varepsilon^2} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}_1}^2 \\ & \leq \langle \varepsilon^{-1}(\mathcal{Q}_1(h_\varepsilon^1, h_\varepsilon^1) - (\mathbf{Id} - \pi_0)(v \cdot \nabla_x h_\varepsilon^1)) + (\mathbf{Id} - \pi_0)(\mathcal{A}_\varepsilon h_\varepsilon^0), (\mathbf{Id} - \pi_0)h_\varepsilon^1 \rangle_{\tilde{H}} \\ & \leq \varepsilon^{-1} \left(\|\mathcal{Q}_1(h_\varepsilon^1, h_\varepsilon^1)\|_{\tilde{H}_{-1}} + \|v \cdot \nabla_x h_\varepsilon^1\|_{\tilde{H}_{-1}} \right) \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}_1} \\ & \quad + \|\mathcal{A}_\varepsilon h_\varepsilon^0\|_{\tilde{H}} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}}. \end{aligned}$$

We deduce easily then with a simple use of Young's inequality on the right-hand-side of this inequality that there is $C > 0$ independent of ε such that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}}^2 + \frac{\tilde{\mu}_1}{2\varepsilon^2} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1\|_{\tilde{H}_1}^2 \\ & \leq C \left(\|\mathcal{Q}_1(h_\varepsilon^1, h_\varepsilon^1)\|_{\tilde{H}_{-1}}^2 + \|v \cdot \nabla_x h_\varepsilon^1\|_{\tilde{H}_{-1}}^2 + \varepsilon^2 \|\mathcal{A}_\varepsilon h_\varepsilon^0\|_{\tilde{H}}^2 \right). \end{aligned}$$

Thus, for some different $C > 0$, one has

$$\begin{aligned} & \frac{d}{dt} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1(t)\|_{\tilde{H}}^2 + \frac{\tilde{\mu}_1}{\varepsilon^2} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1(t)\|_{\tilde{H}_1}^2 \\ & \leq C \left(\|h_\varepsilon^1(t)\|_{\mathcal{H}}^4 + \|h_\varepsilon^1(t)\|_{\mathcal{H}}^2 + \|h_\varepsilon^0(t)\|_{\mathcal{E}}^2 \right) \leq C\mathcal{K}_0 \end{aligned}$$

where the last estimate comes from the results obtained in Sections 4 and 5, $\mathcal{K}_0 \lesssim 1$ and $\|\cdot\|_{\tilde{H}} \lesssim \|\cdot\|_{\mathcal{H}}$. We integrate this inequality over (t_1, t_2) to get

$$\begin{aligned} & \frac{\tilde{\mu}_1}{\varepsilon^2} \int_{t_1}^{t_2} \|(\mathbf{Id} - \pi_0)h_\varepsilon^1(t)\|_{\tilde{H}_1}^2 dt \leq \|(\mathbf{Id} - \pi_0)h_\varepsilon^1(t_1)\|_{\tilde{H}}^2 + C\mathcal{K}_0(t_2 - t_1) \\ & \leq C\mathcal{K}_0 \max(1, t_2 - t_1), \end{aligned} \quad (6.3)$$

where we used (4.25). Introduce now the space $\tilde{E} = \mathbb{W}_x^{\beta,1} L_v^1(\varpi_q)$. Noticing that

$$\|\cdot\|_{\tilde{E}} \lesssim \|\cdot\|_{\mathcal{E}} \quad \text{and} \quad \|\cdot\|_{\tilde{E}} \lesssim \|\cdot\|_{\tilde{H}}$$

and writing that $h_\varepsilon(\tau) = h_\varepsilon^1(\tau) + h_\varepsilon^0(\tau)$, one has

$$\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon(\tau)\|_{\tilde{E}} d\tau \leq \int_{t_1}^{t_2} \left(\|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(\tau)\|_{\tilde{E}} + \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^0(\tau)\|_{\tilde{E}} \right) d\tau. \quad (6.4)$$

Using Cauchy-Schwarz inequality

$$\begin{aligned} \int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon(\tau)\|_{\tilde{E}} d\tau \\ \lesssim \sqrt{t_2 - t_1} \left(\left(\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(\tau)\|_{\tilde{H}}^2 d\tau \right)^{\frac{1}{2}} + \left(\int_{t_1}^{t_2} \|h_\varepsilon^0(\tau)\|_{\mathcal{E}}^2 d\tau \right)^{\frac{1}{2}} \right). \end{aligned}$$

From (6.3), the first integral involving h_ε^1 is such that

$$\left(\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(\tau)\|_{\tilde{H}}^2 d\tau \right)^{\frac{1}{2}} \leq C\varepsilon\sqrt{\mathcal{K}_0} \max\{1, \sqrt{t_2 - t_1}\}$$

whereas, to estimate the integral involving h_ε^0 we use that $h_\varepsilon^0 \in \mathcal{X}_0$ as defined in Section 5 to get

$$\begin{aligned} \int_{t_1}^{t_2} \|h_\varepsilon^0(\tau)\|_{\mathcal{E}}^2 d\tau &\leq \sup_{t_1 \leq \tau \leq t_2} \|h_\varepsilon^0(\tau)\|_{\mathcal{E}} \int_{t_1}^{t_2} \|h_\varepsilon^0(\tau)\|_{\mathcal{E}_1} d\tau \\ &\leq e^{\nu(t_2 - t_1)} \varepsilon^2 \|h_\varepsilon^0\|_0^2 \leq C\varepsilon^2 e^{\nu(t_2 - t_1)} \mathcal{K}_0. \end{aligned}$$

This proves the result. \square

Remark 6.2. Notice that, if we are not interested in introducing a modulus of continuity in time for the above integral, we can directly deduce from (6.3) and (6.4) that

$$\begin{aligned} \int_0^T \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon(t)\|_{\mathbb{W}_x^{\beta,1} L_v^1(\varpi_{q+1})} dt \\ \lesssim \int_0^T \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^0(t)\|_{\mathcal{E}_1} dt + \int_0^T \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(t)\|_{\tilde{H}_1} dt \\ \lesssim \varepsilon \|h_\varepsilon^0\|_0 + \sqrt{T} \left(\int_0^T \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon^1(t)\|_{\tilde{H}_1}^2 dt \right)^{\frac{1}{2}} \end{aligned}$$

which results in

$$\int_0^T \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon(t)\|_{\mathbb{W}_x^{\beta,1} L_v^1(\varpi_{q+1})} dt \lesssim \varepsilon \quad (6.5)$$

for any $0 \leq \beta \leq m - 1$.

We deduce the following convergence result:

Theorem 6.3 (Weak convergence). *Fix $T > 0$, and let*

$$\{h_\varepsilon\}_\varepsilon \subset L^1((0, T); \mathbb{W}_x^{m,1} L_v^1(\varpi_q))$$

be a sequence of solutions to the inelastic Boltzmann equation (1.20). Then, with the splitting $h_\varepsilon = h_\varepsilon^0 + h_\varepsilon^1$, up to extraction of a subsequence, one has

$$\begin{cases} \{h_\varepsilon^0\}_\varepsilon \text{ converges to 0 strongly in } L^1((0, T); \mathcal{E}_1) \\ \{h_\varepsilon^1\}_\varepsilon \text{ which converges to } \mathbf{h} \text{ weakly in } L^2\left((0, T); \mathbb{W}_x^{m,2} L_v^2(\mathcal{M}^{-\frac{1}{2}})\right) \end{cases} \quad (6.6)$$

where $\mathbf{h} = \pi_0(\mathbf{h})$. In particular, there exist

$$\begin{aligned} \varrho &\in L^2\left((0, T); \mathbb{W}_x^{m,2}(\mathbb{T}^d)\right), & \theta &\in L^2\left((0, T); \mathbb{W}_x^{m,2}(\mathbb{T}^d)\right), \\ u &\in L^2\left((0, T); \left(\mathbb{W}_x^{m,2}(\mathbb{T}^d)\right)^d\right) \end{aligned}$$

such that

$$\mathbf{h}(t, x, v) = \left(\varrho(t, x) + u(t, x) \cdot v + \frac{1}{2} \theta(t, x) (|v|^2 - d\vartheta_1) \right) \mathcal{M}(v) \quad (6.7)$$

where \mathcal{M} is the Maxwellian distribution introduced in (1.13).

Proof. Let $T > 0$ be fixed. We use the notations of Proposition 6.1. The estimates obtained in Section 5, using the splitting $h_\varepsilon = h_\varepsilon^0(t) + h_\varepsilon^1(t)$ imply the following properties of the sequences of time-dependent vector-valued mappings $\{h_\varepsilon^1\}_\varepsilon$, $\{h_\varepsilon^0\}_\varepsilon$ and $\{h_\varepsilon\}_\varepsilon$:

$$\{h_\varepsilon^1\} \subset (L^1 \cap L^\infty)((0, T); \mathcal{H}) \quad \text{is bounded} \quad (6.8)$$

$$\int_0^T \|h_\varepsilon^0(t)\|_{\mathcal{E}_1} dt \lesssim \varepsilon^2 \quad (6.9)$$

From (6.8) and since $\|\cdot\|_{\mathbb{W}_x^{m,2} L_v^2(\mathcal{M}^{-\frac{1}{2}})} \lesssim \|\cdot\|_{\mathcal{H}}$, we deduce that

$$\{h_\varepsilon^1\} \text{ is bounded in } L^2\left((0, T); \mathbb{W}_x^{m,2} L_v^2(\mathcal{M}^{-\frac{1}{2}})\right)$$

and therefore, admits a subsequence, say $\{h_{\varepsilon'}^1\}_{\varepsilon'}$ which converges weakly to some \mathbf{h} in $L^2((0, T); \mathbb{W}_x^{m,2} L_v^2(\mathcal{M}^{-\frac{1}{2}}))$. This, combined with (6.9) gives (6.6). From (6.3) we also have, for that subsequence,

$$\lim_{\varepsilon' \rightarrow 0} \int_0^T \|(\mathbf{Id} - \pi_0) h_{\varepsilon'}^1(t)\|_{\mathbb{W}_x^{m-1,2} L_v^2(\mathcal{M}^{-\frac{1}{2}})}^2 dt = 0$$

so that $(\mathbf{Id} - \pi_0) \mathbf{h} = 0$. This gives the result. \square

Remark 6.4. *As observed in the previous proof, the convergence (6.6) can be made even more precise since we also have*

$$\{(\mathbf{Id} - \pi_0) h_\varepsilon^1\} \text{ converges strongly to 0 in } L^2\left((0, T); \mathbb{W}_x^{m-1,2} L_v^2(\mathcal{M}^{-\frac{1}{2}})\right).$$

This means somehow that the only part of h_ε which prevents the strong convergence towards \mathbf{h} is $\{\pi_0 h_\varepsilon^1\}_\varepsilon$.

Because of Theorem 6.3 and for simplicity sake, from here on, we will write that our sequences converge even if it is true up to an extraction.

The above mode of convergence implies the following convergence of velocity averages of h_ε . For any function $f = f(t, x, v)$ we denote the velocity average by

$$\langle f \rangle = \int_{\mathbb{R}^d} f(t, x, v) dv$$

recalling of course that this is a function depending on (t, x) . We have then the following:

Lemma 6.5. *Let $\{h_\varepsilon\}$ be converging to \mathbf{h} in the sense of Theorem 6.3. Then, for any function $\psi = \psi(v)$ such that*

$$|\psi(v)| \lesssim \varpi_q(v)$$

one has

$$\langle \psi h_\varepsilon \rangle \longrightarrow \langle \psi \mathbf{h} \rangle \quad \text{in } \mathcal{D}'_{t,x} \quad (6.10)$$

whereas

$$\langle \psi \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) \rangle \longrightarrow 0 \quad \text{in } \mathcal{D}'_{t,x} \quad (6.11)$$

where we set $\mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) = \mathcal{Q}_1(h_\varepsilon, h_\varepsilon) - \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon)$.

Proof. Let ψ be such that $|\psi(v)| \lesssim \varpi_q(v)$ and let $\varphi = \varphi(t, x) \in \mathcal{C}_c^\infty((0, T) \times \mathbb{T}^d)$ be given. One computes

$$I_\varepsilon := \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \left(\langle \psi h_\varepsilon \rangle - \langle \psi \mathbf{h} \rangle \right) dx = I_\varepsilon^0 + I_\varepsilon^1$$

where

$$I_\varepsilon^0 = \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \langle \psi h_\varepsilon^0 \rangle dx, \quad I_\varepsilon^1 = \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \left(\langle \psi h_\varepsilon^1 \rangle - \langle \psi \mathbf{h} \rangle \right) dx.$$

Because $|I_\varepsilon^0| \lesssim \|\varphi\|_{L_{t,x}^\infty} \int_0^T \|h_\varepsilon^0(t)\|_{L_x^1 L_v^1(\varpi_q)} dt$, we deduce from (6.6) that $\lim_{\varepsilon \rightarrow 0} I_\varepsilon^0 = 0$. In the same way, one has

$$I_\varepsilon^1 = \int_0^T dt \int_{\mathbb{T}^d \times \mathbb{R}^d} (\psi(v) \mathcal{M}(v) \varphi(t, x)) (h_\varepsilon^1(t, x, v) - \mathbf{h}(t, x, v)) dx \mathcal{M}^{-1}(v) dv$$

and, since the mapping

$$(t, x, v) \longmapsto \psi(v) \mathcal{M}(v) \varphi(t, x) \quad \text{belongs to } L^2((0, T); \mathbb{W}_x^{m,2} L_v^2(\mathcal{M}^{-\frac{1}{2}})), \quad (6.12)$$

we deduce from (6.6) that $\lim_{\varepsilon \rightarrow 0} I_\varepsilon^1 = 0$. This proves (6.10). To prove (6.11), one sets

$$J_\varepsilon := \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \langle \psi \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) \rangle dx.$$

One writes $J_\varepsilon = J_\varepsilon^1 + J_\varepsilon^2$ where

$$J_\varepsilon^1 = \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \left\langle \psi \mathcal{Q}_1((\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon, (\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon) \right\rangle dx$$

$$J_\varepsilon^2 = 2 \int_0^T dt \int_{\mathbb{T}^d} \varphi(t, x) \left\langle \psi \tilde{\mathcal{Q}}_1((\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \right\rangle dx$$

One has

$$|J_\varepsilon^1| \lesssim \|\varphi\|_{L_{t,x}^\infty} \int_0^T \|\mathcal{Q}_1((\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon, (\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon)\|_{L_{x,v}^1(\boldsymbol{\varpi}_q)} dt.$$

Noticing that

$$\|\mathcal{Q}_1((\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon, (\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon)\|_{L_{x,v}^1(\boldsymbol{\varpi}_q)} \lesssim \|h_\varepsilon\|_\varepsilon \|(\mathbf{Id} - \boldsymbol{\pi}_0)h_\varepsilon\|_{\mathbb{W}_x^{m-1,1} L_v^1(\boldsymbol{\varpi}_{q+1})}.$$

we deduce from (6.5) and the fact that $\sup_{t \in (0, T)} \|h_\varepsilon(t)\|_\varepsilon < \infty$ that

$$\lim_{\varepsilon \rightarrow 0} |J_\varepsilon^1| = 0.$$

We prove exactly in the same way that

$$\lim_{\varepsilon \rightarrow 0} |J_\varepsilon^2| = 0.$$

This proves the result. □

Regarding the characterisation (6.7) of the limit $\mathbf{h}(t)$, note that

$$\varrho(t, x) = \int_{\mathbb{R}^d} \mathbf{h}(t, x, v) dv, \quad u(t, x) = \frac{1}{\vartheta_1} \int_{\mathbb{R}^d} v \mathbf{h}(t, x, v) dv,$$

and

$$\varrho(t, x) + \vartheta_1 \theta(t, x) = \frac{1}{d\vartheta_1} \int_{\mathbb{R}^d} |v|^2 \mathbf{h}(t, x, v) dv.$$

Corollary 6.6. *With the notations of Theorem 6.3, for any $T > 0$, the limit $\mathbf{h}(t, x, v)$ given by (6.7) satisfies the incompressibility condition*

$$\operatorname{div}_x u(t, x) = 0, \quad t \in (0, T), \quad (6.13)$$

and Boussinesq relation

$$\nabla_x (\varrho + \vartheta_1 \theta) = 0. \quad (6.14)$$

As a consequence, introducing

$$E(t) = \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} \theta(t, x) dx, \quad t \in (0, T),$$

one has strengthened Boussinesq relation

$$\varrho(t, x) + \vartheta_1 (\theta(t, x) - E(t)) = 0, \quad \text{for a.e } (t, x) \in (0, T) \times \mathbb{T}^d. \quad (6.15)$$

Proof. Set

$$\varrho_\varepsilon(t, x) = \int_{\mathbb{R}^d} h_\varepsilon(t, x, v) dv, \quad \mathbf{u}_\varepsilon(t, x) = \frac{1}{\vartheta_1} \int_{\mathbb{R}^d} v h_\varepsilon(t, x, v) dv,$$

and, multiplying (1.20) with 1 and v and integrating in velocity, we get

$$\varepsilon \partial_t \varrho_\varepsilon + \vartheta_1 \operatorname{div}_x (\mathbf{u}_\varepsilon) = 0, \quad (6.16)$$

$$\varepsilon \partial_t \mathbf{u}_\varepsilon + \operatorname{Div}_x (\mathbf{J}_\varepsilon) = \frac{\kappa_\alpha}{\varepsilon} \mathbf{u}_\varepsilon, \quad (6.17)$$

where $\mathbf{J}_\varepsilon(t, x)$ denotes the tensor

$$\mathbf{J}_\varepsilon(t, x) := \frac{1}{\vartheta_1} \int_{\mathbb{R}^d} v \otimes v h_\varepsilon(t, x, v) dv,$$

since both \mathbf{L}_α and \mathcal{Q}_α conserve mass and momentum. The proof of (6.13) is straightforward since $\varepsilon \partial_t \varrho_\varepsilon \rightarrow 0$ and $\operatorname{div}_x (\mathbf{u}_\varepsilon) \rightarrow \operatorname{div}_x u$ in the distribution sense. Let us give the detail for the sake of completeness. Multiplying (6.16) with a function $\varphi \in C_c^\infty((0, T) \times \mathbb{T}^d)$ and integrating over $(0, T) \times \mathbb{T}^d$ we get that

$$- \int_0^T dt \int_{\mathbb{T}^d} \nabla_x \varphi(t, x) \cdot \mathbf{u}_\varepsilon(t, x) dx = \varepsilon \int_0^T dt \int_{\mathbb{T}^d} \varrho_\varepsilon(t, x) \partial_t \varphi(t, x) dx,$$

which, taking the limit $\varepsilon \rightarrow 0$ and because $\varrho_\varepsilon \rightarrow \varrho$ and $\mathbf{u}_\varepsilon \rightarrow u$ in $\mathcal{D}'_{t,x}$, yields

$$\int_0^T dt \int_{\mathbb{T}^d} \nabla_x \varphi(t, x) \cdot u(t, x) dx = 0, \quad \forall \varphi \in C_c^\infty((0, T) \times \mathbb{T}^d).$$

Since $u(t, x) \in L^2((0, T); \mathbb{W}_x^{m,2}(\mathbb{T}^d))$, the incompressibility condition (6.13) holds true. In the same way, for any $i = 1, \dots, d$ and $\varphi \in C_c^\infty((0, T) \times \mathbb{T}^d)$, noticing that

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon \int_0^T \mathbf{u}_\varepsilon^i \partial_t \varphi(t, x) dx = \lim_{\varepsilon \rightarrow 0^+} \frac{\kappa_\alpha}{\varepsilon} \int_0^T dt \int_{\mathbb{T}^d} \mathbf{u}_\varepsilon^i(t, x) \varphi(t, x) dx = 0,$$

because $\kappa_\alpha = 1 - \alpha \leq C\varepsilon^2$ we get that

$$0 = \lim_{\varepsilon \rightarrow 0^+} \sum_{j=1}^d \int_0^T dt \int_{\mathbb{T}^d} \mathbf{J}_\varepsilon^{i,j}(t, x) \partial_{x_j} \varphi(t, x) dx = \sum_{j=1}^d \int_0^T dt \int_{\mathbb{T}^d} \mathbf{J}_0^{i,j}(t, x) \partial_{x_j} \varphi(t, x) dx,$$

where

$$\mathbf{J}_0^{i,j}(t, x) = \frac{1}{\vartheta_1} \int_{\mathbb{R}^d} v_i v_j \mathbf{h}(t, x, v) dv = (\varrho(t, x) + \vartheta_1 \theta(t, x)) \delta_{ij}, \quad i, j = 1, \dots, d.$$

Therefore, for any $i = 1, \dots, d$,

$$\int_0^T dt \int_{\mathbb{T}^d} (\varrho(t, x) + \vartheta_1 \theta(t, x)) \partial_{x_i} \varphi(t, x) dx = 0, \quad \forall \varphi \in C_c^\infty((0, T) \times \mathbb{T}^d).$$

As before, this gives the Boussinesq relation (6.14). To show that Boussinesq relation can be strengthened, one notices that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{T}^d} \varrho_\varepsilon(t, x) dx = \int_{\mathbb{T}^d} \varrho(t, x) dx \quad \text{in } \mathcal{D}'_t$$

from which we deduce, from the conservation of mass for (6.19), that

$$\int_{\mathbb{T}^d} \varrho(t, x) dx = 0, \quad \text{for a.e. } t > 0.$$

With the definition of $E(t)$, this implies that

$$\int_{\mathbb{T}^d} (\varrho(t, x) + \vartheta_1 (\theta(t, x) - E(t))) dx = 0, \quad \text{for a.e. } t > 0,$$

and, this combined with (6.14) yields the strengthened form (6.15). \square

Remark 6.7. *Using Boussinesq relation together with (6.7), one checks without major difficulty that*

$$v \cdot \nabla_x \mathbf{h} = \mathcal{M}(v \otimes v) : \nabla_x u + \frac{1}{2} \mathcal{M} (|v|^2 - (d+2)\vartheta_1) v \cdot \nabla_x \theta. \quad (6.18)$$

Then, using the incompressibility condition (6.13) it holds that

$$\int_{\mathbb{R}^d} \Psi_j(v) v \cdot \nabla_x \mathbf{h} dv = 0, \quad \forall j = 1, \dots, d+2,$$

that is, $\pi_0(v \cdot \nabla_x \mathbf{h}) = 0$. In particular, $v \cdot \nabla_x \mathbf{h} \in \text{Range}(\mathbf{Id} - \pi_0) \subset \text{Range}(\mathbf{L}_1)$ (see [42, Eq. (6.34), p. 180]).

6.2. Identification of the limit. We aim here to fully characterise the limit $\mathbf{h}(t, x, v)$ obtained in Theorem 6.3. To do so, we identify the limit equation satisfied by the macroscopic quantities (ϱ, u, θ) in (6.7) following the path of [9, 30] and exploiting the fact that the mode of convergence in Theorem 6.3 is stronger than the one of [9, 30]. The regime of weak inelasticity is central in the analysis.

We denote by $\{h_\varepsilon\}$ any subsequence which converges to \mathbf{h} in the above Theorem 6.3. We will see in the sequel, under some strong convergence assumption on the initial datum, all subsequences will share the same limit and, as such, the whole sequence will be convergent.

Recall (1.20)

$$\varepsilon \partial_t h_\varepsilon + v \cdot \nabla_x h_\varepsilon + \varepsilon^{-1} \kappa_\alpha \nabla_v \cdot (v h_\varepsilon) = \varepsilon^{-1} \mathbf{L}_\alpha h_\varepsilon + \mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon), \quad (6.19)$$

under the scaling hypothesis that $\alpha = 1 - \lambda_0 \varepsilon^2 + o(\varepsilon^2)$, $\lambda_0 \geq 0$. Multiplying (6.19) respectively with 1, v , $\frac{1}{2}|v|^2$, we observe that the quantities

$$\langle h_\varepsilon \rangle, \quad \langle v h_\varepsilon \rangle, \quad \langle \frac{1}{2}|v|^2 \rangle, \quad \langle \frac{1}{2}|v|^2 v h_\varepsilon \rangle, \quad \text{and} \quad \langle v \otimes v h_\varepsilon \rangle,$$

are important. As in the classical case, we write

$$\langle v \otimes v h_\varepsilon \rangle = \langle \mathbf{A} h_\varepsilon \rangle + p_\varepsilon \mathbf{Id}, \quad p_\varepsilon = \left\langle \frac{1}{d} |v|^2 h_\varepsilon \right\rangle,$$

where we introduce the traceless tensor

$$\mathbf{A} = \mathbf{A}(v) = v \otimes v - \frac{1}{d}|v|^2 \mathbf{Id}.$$

Properties of this tensor are established in Appendix B. In a more precise way, one obtains, after integrating (6.19) against $1, v_i, \frac{1}{2}|v|^2$,

$$\partial_t \langle h_\varepsilon \rangle + \frac{1}{\varepsilon} \operatorname{div}_x \langle v h_\varepsilon \rangle = 0, \quad (6.20a)$$

$$\partial_t \langle v h_\varepsilon \rangle + \frac{1}{\varepsilon} \operatorname{Div}_x \langle \mathbf{A} h_\varepsilon \rangle + \frac{1}{\varepsilon} \nabla_x p_\varepsilon = \frac{\kappa_\alpha}{\varepsilon^2} \langle v h_\varepsilon \rangle, \quad (6.20b)$$

$$\partial_t \langle \frac{1}{2}|v|^2 h_\varepsilon \rangle + \frac{1}{\varepsilon} \operatorname{div}_x \langle \frac{1}{2}|v|^2 v h_\varepsilon \rangle = \frac{1}{\varepsilon^3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2\kappa_\alpha}{\varepsilon^2} \langle \frac{1}{2}|v|^2 h_\varepsilon \rangle, \quad (6.20c)$$

where

$$\mathcal{J}_\alpha(f, f) = \int_{\mathbb{R}^d} [\mathcal{Q}_\alpha(f, f) - \mathcal{Q}_\alpha(G_\alpha, G_\alpha)] |v|^2 dv.$$

Notice that, using (6.7) as well as Corollary 6.6,

$$\begin{aligned} \operatorname{div}_x \langle v h_\varepsilon \rangle &\longrightarrow \vartheta_1 \operatorname{div}_x u = 0, & \langle \frac{1}{2}|v|^2 h_\varepsilon \rangle &\longrightarrow \frac{d\vartheta_1}{2} (\varrho + \vartheta_1 \theta), \\ \nabla_x p_\varepsilon &\longrightarrow \frac{1}{d} \nabla_x \langle |v|^2 \mathbf{h} \rangle = \vartheta_1 \nabla_x (\varrho + \vartheta_1 \theta) = 0, \\ \langle \mathbf{A} h_\varepsilon \rangle &\longrightarrow \langle \mathbf{A} \mathbf{h} \rangle = 0, \\ \langle \frac{1}{2}|v|^2 v_j h_\varepsilon \rangle &\longrightarrow \langle \frac{1}{2}|v|^2 v_j \mathbf{h} \rangle = \frac{1}{2} u_j \langle |v|^2 v_j^2 \mathcal{M} \rangle = \frac{d+2}{2} \vartheta_1^2 u_j, \quad j = 1, \dots, d, \end{aligned}$$

where all the limits hold in $\mathcal{D}'_{t,x}$ and where $\langle \mathbf{A} \mathbf{h} \rangle = 0$ since $\mathbf{h} \in \operatorname{Ker}(\mathbf{L}_1)$ and $\mathbf{A} \in \operatorname{Range}(\mathbf{I} - \pi_0)$. Moreover, under the above scaling

$$\frac{\kappa_\alpha}{\varepsilon^2} \langle v h_\varepsilon \rangle \longrightarrow \vartheta_1 \lambda_0 u, \quad \text{in } \mathcal{D}'_{t,x},$$

since $\lambda_0 = \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-2} \kappa_\alpha$. The limit of $\varepsilon^{-3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon)$ is handled in the following lemma.

Lemma 6.8. *It holds that*

$$\frac{1}{\varepsilon^3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) \longrightarrow \mathcal{J}_0 \quad \text{in } \mathcal{D}'_{t,x},$$

where

$$\mathcal{J}_0(t, x) = -\lambda_0 \bar{c} \vartheta_1^{\frac{3}{2}} \left(\varrho(t, x) + \frac{3}{4} \vartheta_1 \theta(t, x) \right)$$

for some positive constant \bar{c} depending only on the angular kernel $b(\cdot)$ and d . In particular,

$$\mathcal{J}_0 = -\lambda_0 \bar{c} \vartheta_1^{\frac{5}{2}} \left(E(t) - \frac{1}{4} \theta(t, x) \right).$$

Proof. We recall, see (1.6), that

$$\int_{\mathbb{R}^d} |v|^2 \mathcal{Q}_\alpha(f, f) dv = -(1 - \alpha^2) \frac{\gamma_b}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(v) f(v_*) |v - v_*|^3 dv dv_*$$

where $\gamma_b := \frac{1}{2} \int_{\mathbb{S}^{d-1}} (1 - \bar{q} \cdot \sigma) b(\bar{q} \cdot \sigma) d\sigma$ ($\bar{q} \in \mathbb{S}^{d-1}$). Thus, for $f_\varepsilon = G_\alpha + \varepsilon h_\varepsilon$ we obtain

$$\begin{aligned} \frac{1}{\varepsilon^3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) &= -\frac{\gamma_b}{4} \frac{1 - \alpha^2}{\varepsilon^2} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} [h_\varepsilon(v) G_\alpha(v_*) + h_\varepsilon(v_*) G_\alpha(v)] |v - v_*|^3 dv dv_* \right. \\ &\quad \left. + \varepsilon \int_{\mathbb{R}^d \times \mathbb{R}^d} h_\varepsilon(v) h_\varepsilon(v_*) |v - v_*|^3 dv dv_* \right). \end{aligned} \quad (6.21)$$

Recall that $\lim_{\varepsilon \rightarrow 0^+} \frac{1 - \alpha^2}{\varepsilon^2} = \lambda_0$. It is clear that the $\mathbb{W}_x^{m,1}(\mathbb{T}^d)$ norm of the last term in the right-side is controlled by $\|h_\varepsilon\|_\varepsilon^2$. Theorem 5.6 implies that the last term in (6.21) is converging to 0 in $L^1((0, T); \mathbb{W}_x^{m,1}(\mathbb{T}^d))$. One handles the first term in the right-side using Theorem 6.3 and the fact that $G_\alpha \rightarrow \mathcal{M}$ strongly. Details are left to the reader. We then easily obtain the convergence of $\varepsilon^{-3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon)$ towards

$$\mathcal{J}_0 := -\lambda_0 \gamma_b \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{h}(t, x, v) \mathcal{M}(v_*) |v - v_*|^3 dv dv_*.$$

The expression of \mathcal{J}_0 is then obtained by direct inspection from (6.7) with

$$\bar{c} = \gamma_b a, \quad a = \frac{2\sqrt{2}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}|v|^2\right) |v|^3 dv,$$

where

$$\begin{aligned} \int_{\mathbb{R}^{2d}} \mathcal{M}(v) \mathcal{M}(v_*) |v - v_*|^3 dv dv_* &= \vartheta_1^{\frac{3}{2}} a, \\ \int_{\mathbb{R}^{2d}} \mathcal{M}(v) \mathcal{M}(v_*) |v|^2 |v - v_*|^3 dv dv_* &= \frac{2d+3}{2} \vartheta_1^{\frac{5}{2}} a. \end{aligned}$$

We refer to [52, Lemma A.1] for these identities. The second part of the lemma follows from the strengthened Boussinesq relation (6.15). \square

6.3. About the equations of motion and temperature. We give here some preliminary result aiming at deriving the equations satisfied by the bulk velocity $u(t, x)$ and $\theta(t, x)$. As in [9, 30], in order to investigate the limiting behaviour of the system (6.20) as $\varepsilon \rightarrow 0^+$, we need to investigate the limit in the distributional sense of

$$\varepsilon^{-1} \text{Div}_x \langle \mathbf{A} h_\varepsilon \rangle = -\varepsilon^{-1} \text{Div}_x \langle \phi \mathbf{L}_1 h_\varepsilon \rangle \quad (6.22)$$

and

$$\varepsilon^{-1} \text{div}_x \langle \mathbf{b} h_\varepsilon \rangle = -\varepsilon^{-1} \text{div}_x \langle \psi \mathbf{L}_1 h_\varepsilon \rangle \quad (6.23)$$

where ϕ and ψ are defined in Lemma B.1 and where we used that \mathbf{L}_1 is selfadjoint in $L_v^2(\mathcal{M}^{-1/2})$.

Since the limiting vector-field u is divergence-free, it turns out enough to investigate only the limit of $\mathcal{P}\text{Div}_x \langle \varepsilon^{-1} \mathbf{A} h_\varepsilon \rangle$ where we recall that \mathcal{P} is the Leray projection on divergence-free vector fields⁶. We begin with a strong compactness result

Lemma 6.9. *Introduce*

$$u_\varepsilon(t, x) = \exp\left(-t \frac{\kappa_\alpha}{\varepsilon^2}\right) \mathcal{P} \mathbf{u}_\varepsilon(t, x)$$

$$\text{and } \vartheta_\varepsilon(t, x) = \left\langle \frac{1}{2}(|v|^2 - (d+2)\vartheta_1) h_\varepsilon \right\rangle, \quad t \in (0, T), \quad x \in \mathbb{T}^d.$$

Then, $\{\partial_t u_\varepsilon\}_\varepsilon$ and $\{\partial_t \vartheta_\varepsilon\}_\varepsilon$ are bounded in $L^1\left((0, T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)\right)$. Consequently, up to the extraction of a subsequence,

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \|\mathcal{P} \mathbf{u}_\varepsilon(t) - u(t)\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)} dt = 0 \quad (6.24)$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^T \|\vartheta_\varepsilon(t, \cdot) - \vartheta_0(t, \cdot)\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)} dt = 0 \quad (6.25)$$

where

$$\vartheta_0(t, x) = \left\langle \frac{1}{2}(|v|^2 - (d+2)\vartheta_1) \mathbf{h} \right\rangle = \frac{d\vartheta_1}{2} (\varrho(t, x) + \vartheta_1 \theta(t, x)) - \frac{d+2}{2} \vartheta_1 \varrho(t, x).$$

In other words, $\{\mathcal{P} \mathbf{u}_\varepsilon\}_\varepsilon$ converges strongly to u in $L^1\left((0, T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)\right)$ and $\{\vartheta_\varepsilon\}_\varepsilon$ converges strongly to ϑ_0 in $L^1\left((0, T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)\right)$.

Proof. We begin with the proof of (6.24). We apply the Leray projection \mathcal{P} to (6.20b) to eliminate the pressure gradient term. Then, we have that

$$\partial_t u_\varepsilon = -\exp\left(-t \frac{\kappa_\alpha}{\varepsilon^2}\right) \mathcal{P} \left(\vartheta_1^{-1} \text{Div}_x \left\langle \frac{1}{\varepsilon} \mathbf{A} h_\varepsilon \right\rangle \right).$$

Notice that, since $\{h_\varepsilon\}_\varepsilon$ is bounded in $L^1((0, T); \mathcal{E})$, one has that

$$\{u_\varepsilon\}_\varepsilon \text{ is bounded in } L^1\left((0, T); \mathbb{W}_x^{m,1}(\mathbb{T}^d)\right).$$

Moreover, since $\mathbf{A} h_\varepsilon = \mathbf{A} (\mathbf{Id} - \pi_0) h_\varepsilon$ we deduce from Proposition 6.1 that

$$\sup_\varepsilon \int_0^T \left\| \mathcal{P} \left(\text{Div}_x \left\langle \frac{1}{\varepsilon} \mathbf{A} h_\varepsilon \right\rangle \right) \right\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)} dt < \infty.$$

In particular

$$\{\partial_t u_\varepsilon\}_\varepsilon \text{ is bounded in } L^1\left((0, T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)\right).$$

⁶Recall that, for a vector field \mathbf{u} , $\mathcal{P} \mathbf{u} = \mathbf{u} - \nabla \Delta^{-1} (\nabla \cdot \mathbf{u})$. On the torus, it can be defined via Fourier expansion, if $\mathbf{u} = \sum_{k \in \mathbb{Z}^d} \mathbf{a}_k e^{ik \cdot x}$, $\mathbf{a}_k \in \mathbb{C}^d$, then $\mathcal{P} \mathbf{u} = \sum_{k \in \mathbb{Z}^d} \left(\mathbf{I}_d - \frac{k \otimes k}{|k|^2} \right) \mathbf{a}_k e^{ik \cdot x}$.

Applying [58, Corollary 4] with $X = \mathbb{W}_x^{m,1}(\mathbb{T}^d)$ and $B = Y = \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)$ (so that the embedding of X into B is compact by Rellich-Kondrachov Theorem [38, Theorem 2.9, p. 37]), we deduce that $\{u_\varepsilon\}_\varepsilon$ is relatively compact in $L^1\left((0, T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)\right)$. The result of strong convergence follows easily since we already now that $\mathcal{P}u_\varepsilon$ converges to u in $\mathcal{D}'_{t,x}$ (see Lemma 6.5 and recall $u = \mathcal{P}u$ since u is divergence-free).

The proof of (6.25) is similar. We begin with observing that, multiplying (6.20a) with $-\frac{d+2}{2}\vartheta_1$ and add it to (6.20c) we obtain the evolution of $\vartheta_\varepsilon(t, x)$

$$\partial_t \vartheta_\varepsilon + \frac{1}{\varepsilon} \operatorname{div}_x \langle \mathbf{b} h_\varepsilon \rangle = \frac{1}{\varepsilon^3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2\kappa_\alpha}{\varepsilon^2} \left\langle \frac{1}{2} |v|^2 h_\varepsilon \right\rangle. \quad (6.26)$$

Notice that $\{\vartheta_\varepsilon\}_\varepsilon$ is bounded in $L^1\left((0, T); \mathbb{W}_x^{m,1}(\mathbb{T}^d)\right)$ while, because $\mathbf{b} h_\varepsilon = \mathbf{b} (\mathbf{Id} - \pi_0) h_\varepsilon$ we deduce from Proposition 6.1 that

$$\sup_\varepsilon \int_0^T \left\| \left(\operatorname{div}_x \left\langle \frac{1}{\varepsilon} \mathbf{b} h_\varepsilon \right\rangle \right) \right\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)} dt < \infty.$$

It is easy to see that the right-hand side of (6.26) is also bounded in $L^1\left((0, T), \mathbb{W}_x^{m,1}(\mathbb{T}^d)\right)$ so that $\{\partial_t \vartheta_\varepsilon\}_\varepsilon$ is bounded in $L^1\left((0, T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)\right)$. Using again [58, Corollary 4] together with Rellich-Kondrachov Theorem, we deduce as before that $\{\vartheta_\varepsilon\}_\varepsilon$ is relatively compact in $L^1\left((0, T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)\right)$. Since we already know that ϑ_ε converges in the distributional sense to ϑ_0 (see Lemma 6.5), we get the result of strong convergence. \square

Remark 6.10. We will see later that the convergence of $\{\mathcal{P}u_\varepsilon\}_\varepsilon$ and $\{\vartheta_\varepsilon\}_\varepsilon$ can actually be strenghten for well-prepared initial datum (see Proposition 6.18).

A first consequence of the above Lemma is the following which regards (6.22)

Lemma 6.11. *In the distributional sense,*

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{P} \operatorname{Div}_x \left(\left\langle \varepsilon^{-1} \mathbf{A} h_\varepsilon \right\rangle - \left\langle \phi \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \right\rangle \right) = -\nu \Delta_x u \quad (6.27)$$

where ν is defined in Lemma B.1.

Proof. When compared to the elastic case, $\mathbf{L}_1 h_\varepsilon$ does not appear in (6.19). We add it, as well as the quadratic elastic Boltzmann operator, by force and rewrite the latter as

$$\begin{aligned} \varepsilon \partial_t h_\varepsilon + v \cdot \nabla_x h_\varepsilon - \varepsilon^{-1} \mathbf{L}_1 h_\varepsilon &= \mathcal{Q}_1(h_\varepsilon, h_\varepsilon) - \varepsilon^{-1} \kappa_\alpha \nabla_v \cdot (v h_\varepsilon) + \\ &\quad \varepsilon^{-1} (\mathbf{L}_\alpha h_\varepsilon - \mathbf{L}_1 h_\varepsilon) + \mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon) - \mathcal{Q}_1(h_\varepsilon, h_\varepsilon). \end{aligned} \quad (6.28)$$

We interpret the last three terms as a source term

$$\mathbf{S}_\varepsilon := \varepsilon^{-1} (\mathbf{L}_\alpha h_\varepsilon - \mathbf{L}_1 h_\varepsilon) + \mathcal{Q}_\alpha(h_\varepsilon, h_\varepsilon) - \mathcal{Q}_1(h_\varepsilon, h_\varepsilon) - \varepsilon^{-1} \kappa_\alpha \nabla_v \cdot (v h_\varepsilon). \quad (6.29)$$

Then, multiplying (6.28) by ϕ and integrating over \mathbb{R}^d , we get using (6.22) that, for any $i, j = 1, \dots, d$,

$$\begin{aligned} \varepsilon \partial_t \langle \phi^{i,j} h_\varepsilon \rangle + \operatorname{div}_x \langle v \phi^{i,j} h_\varepsilon \rangle - \varepsilon^{-1} \langle \phi^{i,j} \mathbf{L}_1 h_\varepsilon \rangle \\ = \langle \phi^{i,j} \mathcal{Q}_1(h_\varepsilon, h_\varepsilon) \rangle + \langle \phi^{i,j} \mathcal{S}_\varepsilon \rangle. \end{aligned} \quad (6.30)$$

One writes

$$\mathcal{Q}_1(h_\varepsilon, h_\varepsilon) = \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) + \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon)$$

so that (6.30) becomes

$$\begin{aligned} \varepsilon \partial_t \langle \phi^{i,j} h_\varepsilon \rangle + \operatorname{div}_x \langle v \phi^{i,j} h_\varepsilon \rangle - \varepsilon^{-1} \langle \phi^{i,j} \mathbf{L}_1 h_\varepsilon \rangle \\ = \langle \phi^{i,j} \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle + \langle \phi^{i,j} \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) \rangle + \langle \phi^{i,j} \mathcal{S}_\varepsilon \rangle. \end{aligned}$$

According to Lemma 6.5 we have that

$$\begin{aligned} \varepsilon \partial_t \langle \phi^{i,j} h_\varepsilon \rangle &\longrightarrow 0, & \operatorname{div}_x \langle v \phi^{i,j} h_\varepsilon \rangle &\longrightarrow \operatorname{div}_x \langle v \phi^{i,j} \mathbf{h} \rangle, \\ \langle \phi^{i,j} \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) \rangle &\longrightarrow 0, & \langle \phi^{i,j} \mathcal{S}_\varepsilon \rangle &\longrightarrow 0, \end{aligned}$$

where the limits are all meant in the distributional sense and where the last limit is deduced from the strong convergence of \mathcal{S}_ε to 0 in $L^1((0, T); L_x^1 L_v^1(\varpi_q))$ (see Lemma B.6).

From Lemma B.4 in Appendix B, one has

$$\langle v_\ell \phi^{i,j} \mathbf{h} \rangle = \begin{cases} \nu u_j & \text{if } i \neq j, \ell = i, \\ \nu u_i & \text{if } i \neq j, \ell = j, \\ -\frac{2}{d} \nu u_\ell + 2\nu u_i \delta_{i\ell} & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Therefore, using the incompressibility condition,

$$\operatorname{div}_x \langle v \phi^{i,j} \mathbf{h} \rangle = \nu (\partial_{x_j} u_i + \partial_{x_i} u_j).$$

We deduce that

$$\lim_{\varepsilon \rightarrow 0^+} \left(\varepsilon^{-1} \langle \phi^{i,j} \mathbf{L}_1 h_\varepsilon \rangle + \langle \phi^{i,j} \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle \right) = \nu (\partial_{x_j} u_i + \partial_{x_i} u_j),$$

in the distributional sense. Applying the Div_x operator one deduces that, in $\mathcal{D}'_{t,x}$

$$\lim_{\varepsilon \rightarrow 0^+} \operatorname{Div}_x^i \left(\varepsilon^{-1} \langle \phi \mathbf{L}_1 h_\varepsilon \rangle + \langle \phi \mathcal{Q}_1(\pi_0 h_\varepsilon, \pi_0 h_\varepsilon) \rangle \right) = \nu \Delta_x u_i,$$

where we use the incompressibility condition to deduce that $\operatorname{Div}_x^i (\partial_{x_j} u_i + \partial_{x_i} u_j) = \Delta_x u_i$. This proves the result. \square

In the same spirit, we have the following which now regards (6.23).

Lemma 6.12. *In the distributional sense,*

$$\lim_{\varepsilon \rightarrow 0^+} \left(\varepsilon^{-1} \operatorname{div}_x \langle \mathbf{b} h_\varepsilon \rangle + \operatorname{div}_x \langle \psi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle \right) = -\frac{d+2}{2} \gamma \Delta_x \theta. \quad (6.31)$$

Proof. We recall that

$$\frac{1}{\varepsilon} \operatorname{div}_x \langle \mathbf{b} h_\varepsilon \rangle = -\frac{1}{\varepsilon} \operatorname{div}_x \langle \mathbf{L}_1(h_\varepsilon) \psi \rangle.$$

Multiply (6.28) with ψ_i (recall that ψ is defined by (B.1)). As previously, it holds that

$$\varepsilon \partial_t \langle \psi_i h_\varepsilon \rangle + \operatorname{div}_x \langle v \psi_i h_\varepsilon \rangle - \varepsilon^{-1} \langle \psi_i \mathbf{L}_1 h_\varepsilon \rangle = \langle \psi_i \mathcal{Q}_1(h_\varepsilon, h_\varepsilon) \rangle + \langle \psi_i \mathbf{S}_\varepsilon \rangle,$$

and

$$\varepsilon \partial_t \langle \psi_i h_\varepsilon \rangle \longrightarrow 0, \quad \langle \psi_i \mathbf{S}_\varepsilon \rangle \longrightarrow 0,$$

in the distributional sense. Splitting again $\mathcal{Q}_1(h_\varepsilon, h_\varepsilon) = \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) + \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon)$, one has $\langle \psi_i \mathcal{Q}_1^r(h_\varepsilon, h_\varepsilon) \rangle$ converges to 0 in $\mathcal{D}'_{t,x}$ so that, in the distributional sense, it follows that

$$\lim_{\varepsilon \rightarrow 0^+} \left(\varepsilon^{-1} \langle \psi_i \mathbf{L}_1 h_\varepsilon \rangle + \langle \psi_i \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle \right) = \operatorname{div}_x \langle v \psi_i \mathbf{h} \rangle = \frac{d+2}{2} \gamma \partial_{x_i} \theta$$

thanks to Lemma B.5 in Appendix B. This gives the result. \square

6.4. Convergence of the nonlinear terms. To determine the distributional limit of (6.22) and (6.23), we “only” need now to explicit the limit of

$$\mathcal{P}\operatorname{Div}_x \langle \phi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle \quad \text{and} \quad \operatorname{div}_x \langle \psi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle$$

respectively. Writing

$$\boldsymbol{\pi}_0 h_\varepsilon = \left(\varrho_\varepsilon(t, x) + \mathbf{u}_\varepsilon(t, x) \cdot v + \frac{1}{2} \theta_\varepsilon(t, x) (|v|^2 - d\vartheta_1) \right) \mathcal{M}(v)$$

we first observe that, according to Lemma B.3 and Lemma B.5 in Appendix B,

$$\langle \phi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle = \vartheta_1^2 \left[\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \frac{2}{d} |\mathbf{u}_\varepsilon|^2 \mathbf{Id} \right]$$

and

$$\langle \psi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle = \frac{d+2}{2} \vartheta_1^3 (\theta_\varepsilon \mathbf{u}_\varepsilon).$$

Therefore,

$$\mathcal{P}\operatorname{Div}_x \langle \phi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle = \vartheta_1^2 \mathcal{P}\operatorname{Div}_x (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)$$

since $\operatorname{Div}_x (|\mathbf{u}_\varepsilon|^2 \mathbf{Id})$ is a gradient term and

$$\operatorname{div}_x \langle \psi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \rangle = \frac{d+2}{2} \vartheta_1^3 \operatorname{div}_x (\theta_\varepsilon \mathbf{u}_\varepsilon).$$

One has the following whose proof is adapted from [30, Corollary 5.7].

Lemma 6.13. *In the distributional sense (in $\mathcal{D}'_{t,x}$), one has*

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{P}\text{Div}_x \left\langle \phi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \right\rangle = \vartheta_1^2 \mathcal{P}\text{Div}_x(u \otimes u)$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \text{div}_x \left\langle \psi \mathcal{Q}_1(\boldsymbol{\pi}_0 h_\varepsilon, \boldsymbol{\pi}_0 h_\varepsilon) \right\rangle = \frac{d+2}{2} \vartheta_1^3 u \cdot \nabla_x \theta.$$

In particular

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{P}\text{Div}_x \left\langle \varepsilon^{-1} \mathbf{A} h_\varepsilon \right\rangle = -\boldsymbol{\nu} \Delta_x u + \vartheta_1^2 \mathcal{P}\text{Div}_x(u \otimes u) \quad \text{in } \mathcal{D}'_{t,x} \quad (6.32)$$

while

$$\lim_{\varepsilon \rightarrow 0^+} \text{div}_x \left\langle \varepsilon^{-1} \mathbf{b} h_\varepsilon \right\rangle = -\frac{d+2}{2} (\gamma \Delta_x \theta - \vartheta_1^3 u \cdot \nabla_x \theta) \quad \text{in } \mathcal{D}'_{t,x}. \quad (6.33)$$

Proof. We write $\mathbf{u}_\varepsilon = \mathcal{P}\mathbf{u}_\varepsilon + (\mathbf{Id} - \mathcal{P})\mathbf{u}_\varepsilon$. Due to the strong convergence of $\mathcal{P}\mathbf{u}_\varepsilon$ towards u in $L^1\left((0, T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)\right)$ (see Lemma 6.9) and the weak convergence of \mathbf{u}_ε (see Lemma 6.5), we see that

$$\mathcal{P}\text{Div}_x(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - (\mathbf{Id} - \mathcal{P})\mathbf{u}_\varepsilon \otimes (\mathbf{Id} - \mathcal{P})\mathbf{u}_\varepsilon) \longrightarrow \mathcal{P}\text{Div}_x(u \otimes u) \quad \text{in } \mathcal{D}'_{t,x}.$$

So, to prove the first part of the Lemma, we only need to prove that

$$\mathcal{P}\text{Div}_x((\mathbf{Id} - \mathcal{P})\mathbf{u}_\varepsilon \otimes (\mathbf{Id} - \mathcal{P})\mathbf{u}_\varepsilon) \longrightarrow 0 \quad (6.34)$$

in $\mathcal{D}'_{t,x}$. Moreover, as in [30, Corollary 5.7], we set

$$\boldsymbol{\beta}_\varepsilon := \frac{1}{d\vartheta_1} \left\langle |v|^2 h_\varepsilon \right\rangle = \varrho_\varepsilon + \vartheta_1 \vartheta_\varepsilon$$

which is such that $\theta_\varepsilon = \frac{2}{(d+2)\vartheta_1} \left(\boldsymbol{\beta}_\varepsilon + \frac{1}{\vartheta_1} \vartheta_\varepsilon \right)$ and

$$\begin{aligned} \text{div}_x(\theta_\varepsilon \mathbf{u}_\varepsilon) &= \frac{2}{(d+2)\vartheta_1} \left(\text{div}_x \left(\boldsymbol{\beta}_\varepsilon \mathbf{u}_\varepsilon + \frac{1}{\vartheta_1} \mathbf{u}_\varepsilon \vartheta_\varepsilon \right) \right) \\ &= \frac{2}{(d+2)\vartheta_1} \text{div}_x(\boldsymbol{\beta}_\varepsilon (\mathbf{Id} - \mathcal{P})\mathbf{u}_\varepsilon) + \frac{2}{(d+2)\vartheta_1} \left[\text{div}_x \left(\boldsymbol{\beta}_\varepsilon \mathcal{P}\mathbf{u}_\varepsilon + \frac{1}{\vartheta_1} \mathbf{u}_\varepsilon \vartheta_\varepsilon \right) \right] \end{aligned}$$

Therefore, using the strong convergence of ϑ_ε towards ϑ_0 in $L^1\left((0, T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)\right)$ given by Lemma 6.9 together with the weak convergence of \mathbf{u}_ε to u from Lemma 6.5, we get

$$\frac{2}{(d+2)\vartheta_1^2} \text{div}_x(\mathbf{u}_\varepsilon \vartheta_\varepsilon) \longrightarrow \frac{2}{(d+2)\vartheta_1^2} \text{div}_x(u \vartheta_0) \quad \text{in } \mathcal{D}'_{t,x}$$

whereas the strong convergence of $\mathcal{P}\mathbf{u}_\varepsilon$ to u with the weak convergence of $\boldsymbol{\beta}_\varepsilon$ towards $\varrho + \vartheta_1 \theta$ we get

$$\text{div}_x(\boldsymbol{\beta}_\varepsilon \mathcal{P}\mathbf{u}_\varepsilon) \longrightarrow \text{div}_x(u(\varrho + \vartheta_1 \theta)) = 0 \quad \text{in } \mathcal{D}'_{t,x}$$

where we used both the incompressibility condition (6.13) together with Boussinesq relation (6.14). Notice that, thanks to (6.13), it holds

$$\frac{2}{(d+2)\vartheta_1^2} \operatorname{div}_x(u\vartheta_0) = \frac{2}{(d+2)\vartheta_1^2} u \cdot \nabla_x \vartheta_0 = u \cdot \nabla_x \theta$$

where we used the expression of ϑ_0 together with Boussinesq relation (6.14). This shows that

$$\operatorname{div}_x(\theta_\varepsilon \mathbf{u}_\varepsilon) - \frac{2}{(d+2)\vartheta_1} \operatorname{div}_x(\beta_\varepsilon (\mathbf{Id} - \mathcal{P}) \mathbf{u}_\varepsilon) \longrightarrow u \cdot \nabla_x \theta \quad \text{in } \mathcal{D}'_{t,x}$$

and, to get the second part of the result, we need to prove that

$$\operatorname{div}_x(\beta_\varepsilon (\mathbf{Id} - \mathcal{P}) \mathbf{u}_\varepsilon) \longrightarrow 0 \quad \text{in } \mathcal{D}'_{t,x}. \quad (6.35)$$

Let us now focus on the proof of (6.34) and (6.35). One observes that, Equation (6.20b) reads

$$\varepsilon \partial_t \mathbf{u}_\varepsilon + \nabla_x \beta_\varepsilon = \frac{\kappa_\alpha}{\varepsilon} \mathbf{u}_\varepsilon - \vartheta_1^{-1} \operatorname{Div}_x \langle \mathbf{A} h_\varepsilon \rangle \quad (6.36)$$

whereas (6.20c) can be reformulated as

$$\varepsilon \partial_t \beta_\varepsilon + \operatorname{div}_x \left\langle \frac{1}{d\vartheta_1} |v|^2 v h_\varepsilon \right\rangle = \frac{2}{d\vartheta_1 \varepsilon^2} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2\kappa_\alpha}{\varepsilon} \beta_\varepsilon \quad (6.37)$$

where we check easily that

$$\begin{aligned} \operatorname{div}_x \left\langle \frac{1}{d\vartheta_1} |v|^2 v h_\varepsilon \right\rangle &= \frac{2}{d\vartheta_1} \operatorname{div}_x \langle \mathbf{b} h_\varepsilon \rangle + \frac{d+2}{d} \vartheta_1 \operatorname{div}_x \mathbf{u}_\varepsilon \\ &= \frac{2}{d\vartheta_1} \operatorname{div}_x \langle \mathbf{b} h_\varepsilon \rangle + \frac{d+2}{d} \vartheta_1 \operatorname{div}_x (\mathbf{Id} - \mathcal{P}) \mathbf{u}_\varepsilon. \end{aligned}$$

Recall that from Theorem 5.6, $h_\varepsilon \in L^\infty((0, T); \mathcal{E})$ so that $\beta_\varepsilon \in L^\infty((0, T); \mathbb{W}_x^{m,1}(\mathbb{T}^d))$ and using [48, Proposition 1.6, p. 33]), we can write

$$(\mathbf{Id} - \mathcal{P}) \mathbf{u}_\varepsilon = \nabla_x \mathbf{U}_\varepsilon$$

with $\mathbf{U}_\varepsilon \in L^\infty\left((0, T); \left(\mathbb{W}_x^{m-1,1}(\mathbb{T}^d)\right)^d\right)$. After applying $(\mathbf{Id} - \mathcal{P})$ to (6.36) and reformulating (6.37), we obtain that \mathbf{U}_ε and β_ε satisfy

$$\begin{cases} \varepsilon \partial_t \nabla_x \mathbf{U}_\varepsilon + \nabla_x \beta_\varepsilon = \mathbf{F}_\varepsilon \\ \varepsilon \partial_t \beta_\varepsilon + \frac{d+2}{d} \vartheta_1 \Delta_x \mathbf{U}_\varepsilon = \mathbf{G}_\varepsilon \end{cases} \quad (6.38)$$

with

$$\begin{aligned} \mathbf{F}_\varepsilon &:= \frac{\kappa_\alpha}{\varepsilon} \nabla_x \mathbf{U}_\varepsilon - \vartheta_1^{-1} (\mathbf{Id} - \mathcal{P}) \operatorname{Div}_x \langle \mathbf{A} h_\varepsilon \rangle \\ \mathbf{G}_\varepsilon &:= -\frac{2}{d\vartheta_1} \operatorname{div}_x \langle \mathbf{b} h_\varepsilon \rangle + \frac{2}{d\vartheta_1 \varepsilon^2} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2\kappa_\alpha}{\varepsilon} \beta_\varepsilon. \end{aligned}$$

It is easy to see that

$$\|\mathbf{F}_\varepsilon\|_{L^1((0,T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d))} \lesssim \varepsilon, \quad \text{and} \quad \|\mathbf{G}_\varepsilon\|_{L^1((0,T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d))} \lesssim \varepsilon.$$

Since the embeddings $\mathbb{W}_x^{m,1}(\mathbb{T}^d) \hookrightarrow \mathbb{W}_x^{m-2,1}(\mathbb{T}^d) \hookrightarrow L_x^2(\mathbb{T}^d)$ are continuous (recall $m > 2d$), we see that both \mathbf{F}_ε and \mathbf{G}_ε converge *strongly* to 0 in $L^1((0, T); L_x^2(\mathbb{T}^d))$ and $\mathbf{U}_\varepsilon \in L^\infty((0, T); (\mathbb{W}_x^{1,2}(\mathbb{T}^d))^d)$, $\beta_\varepsilon \in L^\infty((0, T); L_x^2(\mathbb{T}^d))$. Then, according to the *compensated compactness* argument of [45] recalled in Proposition B.7 in Appendix B, we deduce that (6.34) and (6.35) hold true and this achieves the proof. The proofs of (6.32) and (6.33) follow then from an application of Lemmas 6.11 and 6.12. \square

Coming back to the system of equations (6.20) and with the preliminary results of Section 6.2, we get the following where we wrote $\mathcal{P}\text{Div}_x(u \otimes u) = \text{Div}_x(u \otimes u) + \vartheta_1^{-1} \nabla_x p$, see [48, Proposition 1.6].

Proposition 6.14. *The limit velocity $u(t, x)$ in (6.7) satisfies*

$$\partial_t u - \frac{\nu}{\vartheta_1} \Delta_x u + \vartheta_1 \text{Div}_x(u \otimes u) + \nabla_x p = \lambda_0 u \quad (6.39)$$

while the limit temperature $\theta(t, x)$ in (6.7) satisfies

$$\partial_t \theta - \frac{\gamma}{\vartheta_1^2} \Delta_x \theta + \vartheta_1 u \cdot \nabla_x \theta = \frac{2}{(d+2)\vartheta_1^2} \mathcal{J}_0 + \frac{2d\lambda_0}{d+2} E(t) + \frac{2}{d+2} \frac{d}{dt} E(t),$$

where

$$E(t) = \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} \theta(t, x) dx, \quad t \geq 0.$$

Notice that, due to (6.13), $\text{Div}_x(u \otimes u) = (u \cdot \nabla_x) u$ and (6.39) is nothing but a *damped* Navier-Stokes equation associated to a divergence-free source term given by $\lambda_0 u$.

Proof. The proof of (6.39) is a straightforward consequence of the previous limit. To obtain investigate the evolution of θ , we recall that ϑ_ε satisfies (6.26). We notice that

$$\frac{1}{\varepsilon^3} \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) + \frac{2\kappa_\alpha}{\varepsilon^2} \left\langle \frac{1}{2} |v|^2 h_\varepsilon \right\rangle \longrightarrow \mathcal{J}_0 + d\vartheta_1 \lambda_0 (\varrho + \vartheta_1 \theta),$$

whereas

$$\vartheta_\varepsilon \longrightarrow \left\langle \frac{1}{2} (|v|^2 - (d+2)\vartheta_1) \mathbf{h} \right\rangle = \frac{d\vartheta_1}{2} (\varrho + \vartheta_1 \theta) - \frac{d+2}{2} \vartheta_1 \varrho,$$

where the convergence is meant in $\mathcal{D}'_{t,x}$. We deduce from (6.33), performing the distributional limit of (6.26), that

$$\begin{aligned} \frac{d\vartheta_1}{2} \partial_t (\varrho + \vartheta_1 \theta) - \frac{d+2}{2} \vartheta_1 \partial_t \varrho - \frac{d+2}{2} \gamma \Delta_x \theta + \frac{d+2}{2} \vartheta_1^3 u \cdot \nabla_x \theta \\ = \mathcal{J}_0 + d\vartheta_1 \lambda_0 (\varrho + \vartheta_1 \theta). \end{aligned} \quad (6.40)$$

Using the strengthened Boussinesq relation (6.15), we see that

$$\partial_t (\varrho + \vartheta_1 \theta) = \vartheta_1 \frac{d}{dt} E(t), \quad \text{and} \quad \partial_t \varrho = -\vartheta_1 \left(\partial_t \theta - \frac{d}{dt} E(t) \right),$$

and get the result. \square

Proposition 6.15. *For any $t \geq 0$, it follows that*

$$E(t) = \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} \theta(t, x) dx = 0,$$

consequently, the limiting temperature $\theta(t, x)$ in (6.7) satisfies

$$\partial_t \theta - \frac{\gamma}{\vartheta_1^2} \Delta_x \theta + \vartheta_1 u \cdot \nabla_x \theta = \frac{\lambda_0 \bar{c}}{2(d+2)} \sqrt{\vartheta_1} \theta. \quad (6.41)$$

Moreover, the strong Boussinesq relation

$$\varrho(t, x) + \vartheta_1 \theta(t, x) = 0, \quad x \in \mathbb{T}^d, \quad (6.42)$$

holds true.

Proof. To capture the evolution of the temperature $E(t)$, we average equation (6.40) over \mathbb{T}^d and using the incompressibility condition (6.13) we deduce get that

$$\frac{d}{dt} E(t) = \frac{2}{d\vartheta_1^2} \int_{\mathbb{T}^d} \mathcal{J}_0(t, x) \frac{dx}{|\mathbb{T}^d|} + 2\lambda_0 E(t). \quad (6.43)$$

And, from Lemma 6.8, it holds that

$$\frac{d}{dt} E(t) = \bar{c}_0 E(t), \quad \bar{c}_0 := 2\lambda_0 - \frac{3}{2d} \lambda_0 \bar{c} \sqrt{\vartheta_1},$$

so that,

$$E(t) = E(0) \exp(\bar{c}_0 t), \quad t \geq 0.$$

Now, return to the original equation (1.20) and recall that the solution $f_\varepsilon(t, x, v)$ is given by

$$f_\varepsilon(t) = G_\alpha + \varepsilon h_\varepsilon(t) = \mathcal{M} + (G_\alpha - \mathcal{M}) + \varepsilon h_\varepsilon(t), \quad t \geq 0,$$

where \mathcal{M} has the same global mass, momentum and energy as the initial datum $f_\varepsilon(0) = F_{\text{in}}^\varepsilon$, independent of $\varepsilon > 0$. For any test-function $\phi = \phi(x, v)$ we get that

$$\varepsilon^{-1} \int_{\mathbb{T}^d} \langle (f_\varepsilon(t) - \mathcal{M}) \phi \rangle dx - \varepsilon^{-1} \int_{\mathbb{T}^d} \langle (G_\alpha - \mathcal{M}) \phi \rangle dx = \int_{\mathbb{T}^d} \langle h_\varepsilon(t) \phi \rangle dx, \quad t \geq 0.$$

Using this equality for $\phi(x, v) = \frac{1}{2}|v|^2$ and $t = 0$ one is led to

$$-\varepsilon^{-1} \int_{\mathbb{T}^d} \langle \frac{1}{2}(G_\alpha - \mathcal{M}) |v|^2 \rangle dx = \int_{\mathbb{T}^d} \langle \frac{1}{2} h_\varepsilon(0) |v|^2 \rangle dx.$$

We recall that $\|G_\alpha - \mathcal{M}\|_{L_v^1(\varpi_2)} \leq C(1 - \alpha) \sim C\varepsilon^2$, consequently

$$E(0) = \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} \theta(0, x) dx = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} \langle h_\varepsilon(0) \frac{1}{2} |v|^2 \rangle dx = 0.$$

Therefore, $E(t) \equiv 0$ for all $t > 0$. This observation and (6.41) lead us to the equation for the energy and Boussinesq relation (6.42). \square

6.5. About the initial conditions. Before going into the proof of Theorem 1.4 and handle the problem of initial datum of our limit system, we begin by proving that our limits u and θ in (6.7) are actually continuous on $(0, T)$.

Lemma 6.16. *Consider the sequences $\{u_\varepsilon\}_\varepsilon$ and $\{\vartheta_\varepsilon\}_\varepsilon$ defined in Lemma 6.9. The time-depending mappings*

$$t \in [0, T] \longmapsto \|\vartheta_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)} \quad \text{and} \quad t \in [0, T] \longmapsto \|u_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)}$$

are Hölder continuous uniformly in ε . As a consequence, the limiting mass ϱ , velocity u and temperature θ in (6.7) are continuous on $(0, T)$.

Proof. Recall that we set

$$\vartheta_\varepsilon(t, x) = \left\langle \frac{1}{2}(|v|^2 - (d+2)\vartheta_1)h_\varepsilon \right\rangle.$$

For any test-function $\varphi = \varphi(x) \in C_c^\infty(\mathbb{T}^d)$ and multi-index β with $|\beta| \leq m-2$, multiplying (6.26) with $\partial_x^\beta \varphi$ and integrating in time and space, one deduces that for any $0 \leq t_1 \leq t_2$,

$$\begin{aligned} \int_{\mathbb{T}^d} \left[\partial_x^\beta \vartheta_\varepsilon(t_2, x) - \partial_x^\beta \vartheta_\varepsilon(t_1, x) \right] \varphi(x) dx &= \int_{t_1}^{t_2} dt \int_{\mathbb{T}^d} \operatorname{div}_x \left\langle \varepsilon^{-1} \mathbf{b} \partial_x^\beta h_\varepsilon(t) \right\rangle \varphi(x) dx \\ &+ \int_{t_1}^{t_2} dt \int_{\mathbb{T}^d} \varepsilon^{-3} \partial_x^\beta \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) \varphi(x) dx \\ &+ \frac{2\kappa_\alpha}{\varepsilon^2} \int_{t_1}^{t_2} dt \int_{\mathbb{T}^d} \left\langle \frac{1}{2} |v|^2 \partial_x^\beta h_\varepsilon \right\rangle \varphi(x) dx. \end{aligned} \quad (6.44)$$

Clearly, since $\varepsilon^{-2}\kappa_\alpha \rightarrow \lambda_0$, there is $C > 0$ such that

$$\begin{aligned} \frac{2\kappa_\alpha}{\varepsilon^2} \left| \int_{t_1}^{t_2} dt \int_{\mathbb{T}^d} \left\langle \frac{1}{2} |v|^2 \partial_x^\beta h_\varepsilon \right\rangle \varphi(x) dx \right| &\leq C \|\varphi\|_\infty \int_{t_1}^{t_2} \|h_\varepsilon(t)\|_{\mathbb{W}_x^{m-1,1} L_v^1(\varpi_2)} dt \\ &\leq C \sqrt{\mathcal{K}_0} (t_2 - t_1) \end{aligned}$$

from the general estimate in Theorem 5.6. In the same way, since

$$\partial_x^\beta \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) = \mathcal{J}_\alpha(\partial_x^\beta f_\varepsilon, f_\varepsilon) + \mathcal{J}_\alpha(f_\varepsilon, \partial_x^\beta f_\varepsilon),$$

with $f_\varepsilon = G_\alpha + \varepsilon h_\varepsilon$, one deduces again from Theorem 5.6 that

$$\begin{aligned} \left| \int_{t_1}^{t_2} dt \int_{\mathbb{T}^d} \varepsilon^{-3} \partial_x^\beta \mathcal{J}_\alpha(f_\varepsilon, f_\varepsilon) \varphi(x) dx \right| \\ \leq C \|\varphi\|_\infty \int_{t_1}^{t_2} \|h_\varepsilon(t)\|_{\mathbb{W}_x^{m-1,1} L_v^1(\varpi_3)} \left(1 + \|h_\varepsilon(t)\|_{\mathbb{W}_x^{m-1,1} L_v^1(\varpi_3)} \right) dt \leq C \sqrt{\mathcal{K}_0} (t_2 - t_1). \end{aligned}$$

Moreover, noticing that $\langle \mathbf{b}h_\varepsilon(t) \rangle = \langle \mathbf{b}(\mathbf{Id} - \pi_0)h_\varepsilon(t) \rangle$ for any $t \geq 0$, one deduces easily from Proposition 6.1 that

$$\left| \int_{t_1}^{t_2} \varepsilon^{-1} \operatorname{div}_x \langle \mathbf{b}h_\varepsilon(t) \rangle dt \right| \leq C \sqrt{t_2 - t_1}$$

for any $0 \leq t_2 - t_1 \leq 1$. Since ∂_x^β commutes with π_0 we deduce easily that there is $C > 0$ independent of ε such that for any $0 \leq \beta \leq m - 2$,

$$\left| \int_{t_1}^{t_2} \varepsilon^{-1} \operatorname{div}_x \langle \mathbf{b} \partial_x^\beta h_\varepsilon(t) \rangle dt \right| \leq C \sqrt{t_2 - t_1} \quad (6.45)$$

for any $0 \leq t_2 - t_1 \leq 1$. We conclude with (6.44) that

$$\left| \int_{\mathbb{T}^d} \left[\partial_x^\beta \vartheta_\varepsilon(t_2, x) - \partial_x^\beta \vartheta_\varepsilon(t_1, x) \right] \varphi(x) dx \right| \leq C \|\varphi\|_\infty \sqrt{\mathcal{K}_0} \sqrt{t_2 - t_1}$$

for some positive constant independent of ε and $0 \leq t_2 - t_1 \leq 1$. Taking the supremum over all $\varphi \in L^\infty(\mathbb{T}^d)$, we deduce that

$$\left\| \partial_x^\beta \vartheta_\varepsilon(t_2) - \partial_x^\beta \vartheta_\varepsilon(t_1) \right\|_{L_x^1(\mathbb{T}^d)} \leq C \sqrt{\mathcal{K}_0} \sqrt{t_2 - t_1} \quad (6.46)$$

and, the time-depending mappings $t \in [0, T] \mapsto \|\vartheta_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)}$ are thus Hölder continuous uniformly in ε . Recall also that $\vartheta_\varepsilon(t)$ converges in $L^1((0, T); \mathbb{W}_x^{m-2,1}(\mathbb{T}^d))$ towards $\vartheta_0(t) = \frac{d+1}{2}(\varrho + \vartheta_1\theta) - \frac{d+2}{2}\vartheta_1\varrho$ from Lemma 6.9. As a consequence, there exists a subsequence $(\vartheta_{\varepsilon'})_{\varepsilon'}$ such that $\|\vartheta_{\varepsilon'}(t) - \vartheta_0(t)\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)}$ converges towards 0 for almost every $t \in [0, T]$. Using then the uniform in ε Hölder continuity obtained above, we can deduce that ϑ_0 is Hölder continuous on $(0, T)$. Recalling that $E(0) = 0$ according to Proposition 6.15, the strong Boussinesq relation (6.42) holds true and

$$\vartheta_0(t) = \frac{d+2}{2}\vartheta_1^2\theta(t) = -\frac{d+2}{2}\vartheta_1\varrho(t),$$

which gives the regularity of both ϱ and θ .

We recall that, setting

$$u_\varepsilon(t, x) = \frac{1}{\vartheta_1} \exp\left(-t \frac{\kappa_\alpha}{\varepsilon^2}\right) \mathcal{P} \langle v h_\varepsilon \rangle,$$

we have that

$$\partial_t u_\varepsilon + \exp\left(-t \frac{\kappa_\alpha}{\varepsilon^2}\right) \mathcal{P} \left(\operatorname{Div}_x \left\langle \frac{1}{\vartheta_1 \varepsilon} \mathbf{A} h_\varepsilon \right\rangle \right) = 0.$$

we multiply this identity by $\partial_x^\beta \varphi$ and integrate in both time and space to get

$$\begin{aligned} & \int_{\mathbb{T}^d} \left[\partial_x^\beta u_\varepsilon(t_2, x) - \partial_x^\beta u_\varepsilon(t_1, x) \right] \varphi(x) dx \\ &= \int_{t_1}^{t_2} \exp\left(-\frac{\kappa_\alpha}{\varepsilon^2} t\right) dt \int_{\mathbb{T}^d} \mathcal{P} \left(\operatorname{Div}_x \left\langle \frac{1}{\vartheta_1 \varepsilon} \mathbf{A} \partial_x^\beta h_\varepsilon \right\rangle \right) \varphi(x) dx. \end{aligned}$$

Arguing as in the proof of (6.45), we see that there is $C > 0$ independent of ε such that for any $0 \leq \beta \leq m - 2$,

$$\left| \int_{t_1}^{t_2} \varepsilon^{-1} \operatorname{Div}_x \left\langle \mathbf{A} \partial_x^\beta h_\varepsilon(t) \right\rangle dt \right| \leq C \sqrt{t_2 - t_1}$$

for any $0 \leq t_2 - t_1 \leq 1$. This gives easily

$$\left| \int_{\mathbb{T}^d} \left[\partial_x^\beta u_\varepsilon(t_2, x) - \partial_x^\beta u_\varepsilon(t_1, x) \right] \varphi(x) dx \right| \leq C \|\varphi\|_\infty \sqrt{t_2 - t_1} \sqrt{\mathcal{K}_0},$$

from which, as before, the time-depending mappings $t \in [0, T] \mapsto \|u_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)}$ are Hölder continuous uniformly in ε . We deduce the result of regularity on u as previously done for ϱ and θ noticing that the limit of u_ε is $\exp(-t\lambda_0)\mathcal{P}u = \exp(-t\lambda_0)u$. \square

Recall that, in Theorem 6.3, the convergence of h_ε to the solution $\mathbf{h}(t, x)$ given by (6.7) is known to hold only for a subsequence and, in particular, different subsequences could converge towards different initial datum and therefore (ϱ, u, θ) could be different solutions to the Navier-Stokes system. We aim here to *prescribe* the initial datum by ensuring the convergence of the initial datum $h_{\text{in}}^\varepsilon$ towards a *single* possible limit.

Recall that the initial datum for (6.28) is denoted by $h_{\text{in}}^\varepsilon$. We write $h_{\text{in}}^\varepsilon = \pi_0 h_{\text{in}}^\varepsilon + (\mathbf{Id} - \pi_0)h_{\text{in}}^\varepsilon$ and introduce the following assumption.

Assumption 6.17. *Assume that there exists*

$$(\varrho_0, u_0, \theta_0) \in \mathbb{W}_x^{m,1}(\mathbb{T}^d) \times \left(\mathbb{W}_x^{m,1}(\mathbb{T}^d) \right)^d \times \mathbb{W}_x^{m,1}(\mathbb{T}^d),$$

such that

$$\lim_{\varepsilon \rightarrow 0} \|\pi_0 h_{\text{in}}^\varepsilon - h_0\|_{\mathbb{W}_x^{m,1} L_v^1(\varpi_q)} = 0,$$

where

$$h_0(x, v) = (\varrho_0(x) + u_0(x) \cdot v + \frac{1}{2}\theta_0(x)(|v|^2 - d\vartheta_1)) \mathcal{M}(v).$$

Under this assumption we can prescribe the initial value of the solution (ϱ, u, θ) and strengthen the convergence.

Proposition 6.18. *We define the initial data for (ϱ, u, θ) as*

$$u_{\text{in}} = u(0) := \mathcal{P}u_0, \quad \theta_{\text{in}} = \theta(0) = \frac{d}{d+2}\theta_0 - \frac{2}{(d+2)\vartheta_1}\varrho_0, \\ \varrho_{\text{in}} = \varrho(0) := -\vartheta_1\theta_{\text{in}}, \quad (6.47)$$

where we recall that $\mathcal{P}u_0$ is the Leray projection on divergence-free vector fields. Then, as a consequence, for any $T > 0$, one has that

$$\vartheta_\varepsilon(t) = \left\langle \frac{1}{2}(|v|^2 - (d+2)\vartheta_1)h_\varepsilon \right\rangle \longrightarrow \frac{d+2}{2}\vartheta_1^2\theta, \quad \text{in } \mathcal{C}\left([0, T], \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)\right),$$

and

$$\frac{1}{\vartheta_1}\mathcal{P}\left\langle v h_\varepsilon(t) \right\rangle \longrightarrow u, \quad \text{in } \mathcal{C}\left([0, T], \left(\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)\right)^d\right).$$

Proof. According to Lemma 6.16, we already have that the family of time-dependent mappings

$$\left\{ t \in [0, T] \mapsto \|\vartheta_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)} \right\}_\varepsilon \quad (6.48)$$

is equicontinuous. At time $t = 0$ according to Assumption 6.17,

$$\vartheta_\varepsilon(0, x) = \left\langle \frac{1}{2}(|v|^2 - (d+2)\vartheta_1)h_{\text{in}}^\varepsilon \right\rangle \longrightarrow \vartheta_1 \left[\frac{d\vartheta_1}{2}\theta_0(x) - \varrho_0(x) \right],$$

and, by definition of $\varrho(0, x), \theta(0, x)$, we get that

$$\lim_{\varepsilon \rightarrow 0^+} \|\vartheta_\varepsilon(0, \cdot) - \vartheta_0(0, \cdot)\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)} = 0.$$

In particular, the family $\{\|\vartheta_\varepsilon(0)\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)}\}_\varepsilon$ is bounded and, since the family (6.48) is uniformly in ε Hölder continuous, for any $t \in [0, T]$, the family $\{\|\vartheta_\varepsilon(t)\|_{\mathbb{W}_x^{m-2,1}(\mathbb{T}^d)}\}_\varepsilon$ is also bounded. Since it is also equicontinuous, Arzelà-Ascoli Theorem implies that the convergence holds in $\mathcal{C}([0, T]; \mathbb{W}_x^{m-2,1}(\mathbb{T}^d))$ and

$$\vartheta_0 \in \mathcal{C}([0, T]; \mathbb{W}_x^{m-2,1}(\mathbb{T}^d)).$$

As in the proof of Lemma 6.16, it implies the continuity on $[0, T]$ of both ϱ and θ .

We proceed in a similar way for the regularity of u . □

All the previous convergence results lead us to the fact the the limit

$$\mathbf{h}(t, x, v) = \left(\varrho(t, x) + u(t, x) \cdot v + \frac{1}{2}\theta(t, x)(|v|^2 - d\vartheta_1) \right) \mathcal{M}(v)$$

is such that

$$(\varrho, u, \theta) \in \mathcal{C}([0, T]; \mathscr{W}_{m-2}) \cap L^2((0, T); \mathscr{W}_m),$$

solve the following *incompressible Navier-Stokes-Fourier system* where the right-hand-side acts as a self-consistent forcing term

$$\begin{cases} \partial_t u - \frac{\nu}{\vartheta_1} \Delta_x u + \vartheta_1 u \cdot \nabla_x u + \nabla_x p = \lambda_0 u, \\ \partial_t \theta - \frac{\gamma}{\vartheta_1^2} \Delta_x \theta + \vartheta_1 u \cdot \nabla_x \theta = \frac{\lambda_0 \bar{c}}{2(d+2)} \sqrt{\vartheta_1} \theta, \\ \operatorname{div}_x u = 0, \quad \varrho + \vartheta_1 \theta = 0, \end{cases} \quad (6.49)$$

subject to the initial datum $(\varrho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}})$. This proves Theorem 1.4 in full.

6.6. About the original problem in the physical variables. The above considerations allow us to get a quite precise description of the asymptotic behaviour for the original

physical problem (1.8a). Indeed, recalling the relations (1.10) together with Theorem 1.4 one has

$$\begin{aligned} F_\varepsilon(t, x, v) &= V_\varepsilon(t)^d f_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v) \\ &= V_\varepsilon(t)^d \left(G_{\alpha(\varepsilon)}(V_\varepsilon(t)v) + \varepsilon h_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v) \right) \\ &= V_\varepsilon(t)^d \left(G_{\alpha(\varepsilon)}(V_\varepsilon(t)v) + \varepsilon \mathbf{h}(\tau_\varepsilon(t), x, V_\varepsilon(t)v) \right) + \varepsilon e_\varepsilon(t, x, v), \end{aligned}$$

where the error term e_ε is given by

$$e_\varepsilon(t, x, v) = V_\varepsilon(t)^d \left(h_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v) - \mathbf{h}(\tau_\varepsilon(t), x, V_\varepsilon(t)v) \right).$$

Under Assumption 1.2, a relevant phenomenon occurs when considering the purely dissipative case $\lambda_0 > 0$. In such a case, the term $e_\varepsilon(t, x, v)$ becomes an *uniform in time* error term. The reason is that, when $\lambda_0 > 0$, the scaling $V_\varepsilon(t)$ increases up to infinity. More precisely,

$$V_\varepsilon(t) \approx (1 + \lambda_0 t), \quad \varepsilon \ll 1.$$

Indeed, Lemma A.1 guarantees that for any $a \in (0, 1/2)$, up to an extraction of a subsequence if necessary,

$$|\langle e_\varepsilon(t), |v|^\kappa \varphi \rangle| \leq C_\varphi \sqrt{\mathcal{K}_0} V_\varepsilon(t)^{-\kappa-a}, \quad \varphi \in L_x^\infty \mathcal{C}_{v,b}^1, \quad 0 \leq \kappa \leq q-1, \quad (6.50)$$

where we denoted by $\mathcal{C}_{v,b}^1$ the set of \mathcal{C}^1 functions in v that are bounded as well as their first order derivatives. Consequently,

$$\begin{aligned} F_\varepsilon(t, x, v) &= V_\varepsilon(t)^d \left(G_{\alpha(\varepsilon)}(V_\varepsilon(t)v) + \varepsilon \left(\varrho(\tau_\varepsilon(t), x) + u(\tau_\varepsilon(t), x) \cdot (V_\varepsilon(t)v) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \theta(\tau_\varepsilon(t), x) (|V_\varepsilon(t)v|^2 - d\vartheta_1) \right) \mathcal{M}(V_\varepsilon(t)v) \right) + \mathcal{O}\left(V_\varepsilon(t)^{-\kappa-a}\right), \end{aligned} \quad (6.51)$$

in the *weak sense* described in (6.50). In particular, if $\varphi = 1$ and $\kappa = 2$, one finds from (6.51) an explicit expression for *Haff's law* is obtained. That is, the optimal cooling rate of the temperature is described by

$$\begin{aligned} \mathbf{T}_\varepsilon(t) &= \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d \times \mathbb{R}^d} F_\varepsilon(t, x, v) |v|^2 dv dx \\ &= \frac{1}{V_\varepsilon(t)^2} \left(\int_{\mathbb{R}^d} G_\alpha(v) |v|^2 dv + \frac{\varepsilon}{2|\mathbb{T}^d|} \int_{\mathbb{R}^d} (|v|^2 - d\vartheta_1) |v|^2 \mathcal{M}(v) dv \int_{\mathbb{T}^d} \theta(\tau_\varepsilon(t), x) dx \right. \\ &\quad \left. + \frac{\varepsilon}{|\mathbb{T}^d|} \int_{\mathbb{R}^d} |v|^2 \mathcal{M}(v) dv \int_{\mathbb{T}^d} \varrho(\tau_\varepsilon(t), x) dx \right) + \mathcal{O}\left(V_\varepsilon(t)^{-2-a}\right) \\ &\approx \frac{d\vartheta_1}{V_\varepsilon(t)^2} \left(1 + \frac{\varepsilon}{|\mathbb{T}^d|} \left(d\vartheta_1 \int_{\mathbb{T}^d} \theta(\tau_\varepsilon(t), x) dx + 2 \int_{\mathbb{T}^d} \varrho(\tau_\varepsilon(t), x) dx \right) \right), \quad t \gg \frac{1}{\lambda_0}. \end{aligned}$$

Recalling that the fluctuation h_ε is such that the average mass and temperature both vanish at all times, we deduce the precised Haff's law

$$\mathbf{T}_\varepsilon(t) \approx \frac{d\vartheta_1}{V_\varepsilon(t)^2}, \quad t \gg \frac{1}{\lambda_0}.$$

In the Appendix A.2 we complement this discussion and, in particular, show that the Haff's law holds uniformly *locally* in space due to the boundedness of the solutions that we treat here. This is not expected in a general context.

APPENDIX A. ABOUT GRANULAR GASES IN THE SPATIAL HOMOGENEOUS SETTING

We collect several result about the Boltzmann collision operator \mathcal{Q}_α for granular gases. We recall the definition given in the weak form

$$\int_{\mathbb{R}^d} \mathcal{Q}_\alpha(g, f)(v) \psi(v) dv = \frac{1}{2} \int_{\mathbb{R}^{2d}} f(v) g(v_*) |v - v_*| \mathcal{A}_\alpha[\psi](v, v_*) dv_* dv,$$

where

$$\mathcal{A}_\alpha[\psi](v, v_*) = \int_{\mathbb{S}^{d-1}} (\psi(v') + \psi(v'_*) - \psi(v) - \psi(v_*)) b(\sigma \cdot \bar{q}) d\sigma,$$

and the post-collisional velocities (v', v'_*) are given by

$$\begin{aligned} v' &= v + \frac{1+\alpha}{4} (|q|\sigma - q), & v'_* &= v_* - \frac{1+\alpha}{4} (|q|\sigma - q), \\ \text{where } q &= v - v_*, & \bar{q} &= q/|q|. \end{aligned} \quad (\text{A.1})$$

A.1. Alternative representation of the velocities. As well-known, the above collision operator is a well-accepted model that describes collisions in a system composed by a large number of granular particles which are assumed to be hard-spheres with equal mass (that we take to be $m = 1$) and that undertake inelastic collisions. The collision mechanism and the role of the *coefficient of normal restitution* is easier to understand in an alternative representation of the post-collisional velocities. More precisely, if v and v_* denote the velocities of two particles before collision, their respective velocities v' and v'_* after collision are such that

$$(u' \cdot n) = -\alpha(u \cdot n). \quad (\text{A.2})$$

The unitary vector $n \in \mathbb{S}^{d-1}$ determines the impact direction, that is, n stands for the unit vector that points from the v -particle center to the v_* -particle center at the moment of impact. Here above

$$u = v - v_*, \quad u' = v' - v'_*, \quad (\text{A.3})$$

denote respectively the relative velocity before and after collision. The velocities after collision v' and v'_* are given, in virtue of (A.2) and the conservation of momentum, by

$$v' = v - \frac{1+\alpha}{2} (u \cdot n)n, \quad v'_* = v_* + \frac{1+\alpha}{2} (u \cdot n)n. \quad (\text{A.4})$$

In particular, the energy relation and the collision mechanism can be written as

$$|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2 - \frac{1-\alpha^2}{2} (u \cdot n)^2, \quad u \cdot n = -\alpha(u' \cdot n). \quad (\text{A.5})$$

Pre-collisional velocities (v', v'_*) (resulting in (v, v_*) after collision) can be therefore introduced through the relation

$$v = v' - \frac{1+\alpha}{2} (u' \cdot n)n, \quad v_* = v'_* + \frac{1+\alpha}{2} (u' \cdot n)n, \quad u = v - v_* \quad (\text{A.6})$$

This representation is of course equivalent to the one given in (A.1) (so-called σ -representation) by setting, for a given pair of velocities (v, v_*) ,

$$\sigma = \hat{u} - 2(\hat{u} \cdot n)n \in \mathbb{S}^{d-1}.$$

Such a description provides an alternative parametrization of the unit sphere \mathbb{S}^{d-1} in which the unit vector σ points in the post-collisional relative velocity direction in the case of elastic collisions. In this case, the impact velocity reads

$$|u \cdot n| = |u| |\hat{u} \cdot n| = |u| \sqrt{\frac{1 - \hat{u} \cdot \sigma}{2}}.$$

In the n -representation, we can explicit the strong form of the collision operator \mathcal{Q}_α . Namely, for a given pair of distributions $f = f(v)$ and $g = g(v)$ and a given *collision kernel* the Boltzmann collision operator is defined as the difference of two nonnegative operators (gain and loss operators respectively)

$$\mathcal{Q}_\alpha(g, f) = \mathcal{Q}_\alpha^+(g, f) - \mathcal{Q}_\alpha^-(g, f),$$

with

$$\begin{aligned} \mathcal{Q}_\alpha^+(g, f)(v) &= \frac{1}{\alpha^2} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u \cdot n| b_0(\hat{u} \cdot n) f(v') g(v'_*) dv_* dn, \\ \mathcal{Q}_\alpha^-(g, f)(v) &= f(v) \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u \cdot n| b_0(\hat{u} \cdot n) g(v_*) dv_* dn. \end{aligned} \tag{A.7}$$

where the new angular collision kernel $b_0(\cdot)$ is related to the original one $b(\cdot)$ through the relation

$$b_0(\hat{u} \cdot n) = 2^{d-1} |\hat{u} \cdot n|^{d-2} b(\hat{u} \cdot \sigma).$$

i.e.

$$b_0(x) = 2^{d-1} |x|^{d-2} b(1 - 2x^2), \quad x \in [-1, 1].$$

Using this representation we prove Lemma 2.1 in Section 2.1.

Proof of Lemma 2.1. Note that

$$\mathcal{Q}_1(g, f) - \mathcal{Q}_\alpha(g, f) = \mathcal{I}_1(g, f) + \mathcal{I}_2(g, f),$$

where

$$\mathcal{I}_1(g, f) = -\frac{1 - \alpha^2}{\alpha^2} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} g(v_{*,\alpha}) f(v_\alpha) |u \cdot n| b_0(\hat{u} \cdot n) dn dv_*,$$

and

$$\mathcal{I}_2(g, f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (g(v_{*,\alpha}) f(v_\alpha) - g(v_{*,1}) f(v_1)) |u \cdot n| b_0(\hat{u} \cdot n) dn dv_*.$$

Here we adopt the notation $'v_1$ and $'v_\alpha$ for the pre-collisional velocities associate to elastic ($\alpha = 1$) and inelastic ($0 < \alpha < 1$) interactions respectively in the n -representation (see (A.6) in Appendix A.1). Therefore, by classical means, there is a positive $c_q > 0$ such that

$$\|\mathcal{I}_1(g, f)\|_{L_v^1(\varpi_q)} \leq \frac{1 - \alpha}{\alpha^2} c_q \|b_0\|_{L^1(\mathbb{S}^{d-1})} \|g\|_{L_v^1(\varpi_{q+1})} \|f\|_{L_v^1(\varpi_{q+1})}.$$

We estimate then the difference $g(b_{*,\alpha})f(b_\alpha) - g(b_*)f(b)$ thanks to Taylor formula

$$\begin{aligned} g(b_{*,\alpha})f(b_\alpha) - g(b_*)f(b) &= g(b_*)(f(b_\alpha) - f(b)) + (g(b_{*,\alpha}) - g(b_*))f(b_\alpha) \\ &= g(b_*)(b_\alpha - b) \cdot \int_0^1 \nabla f(b_t) dt + f(b_\alpha) (b_{*,\alpha} - b_*) \cdot \int_0^1 \nabla g(b_{*,t}) dt \end{aligned}$$

where we recall that, according to (A.6) in Appendix A.1

$$b_\alpha - b = -\frac{1-\alpha}{2\alpha}(u \cdot n)n, \quad b_{*,\alpha} - b_* = \frac{1-\alpha}{2\alpha}(u \cdot n)n,$$

and, for $0 < t < 1$,

$$\begin{aligned} b_t &= b_\alpha t + b(1-t) = b - t \frac{1-\alpha}{2\alpha}(u \cdot n)n, \\ b_{*,t} &= b_{*,\alpha} t + b_*(1-t) = b_{*,\alpha} - (1-t) \frac{1-\alpha}{2\alpha}(u \cdot n)n. \end{aligned}$$

We split $\mathcal{I}_2(g, f)$ accordingly into $\mathcal{I}_2(g, f) = \mathcal{I}_2^1(g, f) + \mathcal{I}_2^2(g, f)$. For the term $\mathcal{I}_2^1(g, f)$, we first notice that

$$\begin{aligned} &\int_{\mathbb{R}^d} |\mathcal{I}_2^1(g, f)(v)| \langle v \rangle^q dv \\ &\leq \frac{1-\alpha}{2\alpha} \int_0^1 dt \int_{\mathbb{S}^{d-1}} b_0(\hat{u} \cdot n) |\hat{u} \cdot n|^2 dn \int_{\mathbb{R}^{2d}} |g(b_t - \tilde{u})| |\nabla f(b_t)| |u|^2 \langle v \rangle^q du dv. \end{aligned}$$

We set

$$\tilde{u} := b_t - b_* = u - 2(u \cdot n)n - t \frac{1-\alpha}{2\alpha}(u \cdot n)n$$

and apply, for fixed n , the change of variables $(v, u) \rightarrow (b_t, \tilde{u})$ (with Jacobian $J_\alpha(t) = 1 + t \frac{1-\alpha}{2\alpha} \geq 1$). Together with the fact that

$$|u| \leq |\tilde{u}|, \quad |v| \leq |b_t| + |b_*| \leq 2|b_t| + 2|b_*|,$$

we obtain that

$$\begin{aligned} \int_{\mathbb{R}^d} |\mathcal{I}_2^1(g, f)(v)| \langle v \rangle^q dv &\leq \frac{1-\alpha}{2\alpha} c_q \int_0^1 \frac{1}{J_\alpha(t)} dt \int_{\mathbb{S}^{d-1}} b_0(\hat{u} \cdot n) |\hat{u} \cdot n|^2 dn \\ &\quad \int_{\mathbb{R}^{2d}} |g(v - u) \langle v - u \rangle^{q+2}| |\nabla f(v) \langle v \rangle^{q+2}| du dv \\ &\leq \frac{1-\alpha}{2\alpha} c_q \|b_0\|_{L^1(\mathbb{S}^{d-1})} \|g\|_{L_v^1(\mathfrak{W}_{q+2})} \|f\|_{\mathbb{W}_v^{1,1}(\mathfrak{W}_{q+2})}. \end{aligned}$$

For the term $\mathcal{I}_2^2(g, f)$ we begin with observing that

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} |\mathcal{I}_2^2(g, f)(v)| \langle v \rangle^q dv \\ &\leq \frac{1-\alpha}{2\alpha} \int_0^1 dt \int_{\mathbb{S}^{d-1}} b_0(\hat{u} \cdot n) |\hat{u} \cdot n|^2 dn \int_{\mathbb{R}^{2d}} |f(b_\alpha + v_{*,t})| |\nabla g(b_{*,t})| |u|^2 \langle v \rangle^q du dv \end{aligned}$$

and set

$$'\tilde{u}_\alpha := 'v_\alpha - 'v_{*,t} = u - 2(u \cdot n)n - (1+t)\frac{1-\alpha}{2\alpha}(u \cdot n)n$$

and apply, for fixed n , the change of variables $(v_*, u) \rightarrow ('v_{*,t}, '\tilde{u}_\alpha)$ (with Jacobian $\tilde{J}_\alpha(t) = 1 + (1+t)\frac{1-\alpha}{2\alpha} \geq 1$). Noticing that

$$|u| \leq 'u_\alpha \leq \frac{2}{1+t}'\tilde{u}_\alpha, \quad |v| \leq |'v_\alpha| + |'v_{*,\alpha}| \leq 2|'v_{*,t}| + 4|'\tilde{u}_\alpha|,$$

it follows that

$$\begin{aligned} \int_{\mathbb{R}^d} |\mathcal{I}_2^2(g, f)(v)| \langle v \rangle^q dv &\leq \frac{1-\alpha}{2\alpha} c_q \int_0^1 \frac{1}{\tilde{J}_\alpha(t)} dt \int_{\mathbb{S}^{d-1}} b_0(\hat{u} \cdot n) |\hat{u} \cdot n|^2 dn \\ &\quad \int_{\mathbb{R}^{2d}} |f(v+u) \langle v+u \rangle^{q+2}| |\nabla g(v) \langle v \rangle^{q+2}| dudv \\ &\leq \frac{1-\alpha}{2\alpha} c_q \|f\|_{L_v^1(\varpi_{q+2})} \|g\|_{\mathbb{W}_v^{1,1}(\varpi_{q+2})}. \end{aligned}$$

Gathering previous estimates proves the first assertion of the Lemma. For higher norms simply differentiate and apply previous estimates for each suitable difference. \square

A.2. About the original problem in physical variables. Let $F_\varepsilon(t, x, v)$ be the solution of the Boltzmann equation (1.8a) with associated Knudsen number ε . Recall that the time-scale functions $\tau_\varepsilon(t), V_\varepsilon(t)$ that relate the problem in original (physical) variables to its self-similar counterpart

$$F_\varepsilon(t, x, v) = V_\varepsilon(t)^d f_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v)$$

are given by

$$\tau_\varepsilon(t) := \frac{1}{c_\varepsilon} \ln(1 + c_\varepsilon t), \quad V_\varepsilon(t) = 1 + c_\varepsilon t, \quad t \geq 0,$$

where $c_\varepsilon = \frac{1-\alpha(\varepsilon)}{\varepsilon^2}$. It follows that the explicit equation for f_ε is given by

$$\partial_\tau f_\varepsilon + \varepsilon^{-1} w \cdot \nabla_x f_\varepsilon = \varepsilon^{-2} \mathcal{Q}(f_\varepsilon, f_\varepsilon) - \varepsilon^{-2} (1-\alpha) \nabla_w (w f_\varepsilon), \quad w = V_\varepsilon(t)v$$

as observed in (1.20). Set $f_\varepsilon(\tau, x, w) = G_{\alpha(\varepsilon)}(w) + \varepsilon h_\varepsilon(\tau, x, w)$ and denote $\mathbf{h}(\tau, x, w)$ the weak- \star limit in the space $L^\infty((0, \infty); \mathcal{E})$ of the (sub-)sequence $\{h_\varepsilon\}$. Define

$$e_\varepsilon(t, x, v) = V_\varepsilon(t)^d (h_\varepsilon(\tau_\varepsilon(t), x, V_\varepsilon(t)v) - \mathbf{h}(\tau_\varepsilon(t), x, V_\varepsilon(t)v)).$$

The following error estimate holds.

Lemma A.1. *Under Assumption 1.2 and in the regime $\varepsilon \ll 1$, $\lambda_0 > 0$, the following estimation holds for any $a \in (0, 1/2)$, up to possibly extracting a subsequence,*

$$|\langle e_\varepsilon(t), |v|^\kappa \varphi \rangle| \leq C_\varphi \sqrt{\mathcal{K}_0} V_\varepsilon(t)^{-\kappa-a}, \quad \varphi \in L_x^\infty \mathcal{C}_{v,b}^1, \quad 0 \leq \kappa \leq q-1,$$

where we denoted by $\mathcal{C}_{v,b}^1$ the set of \mathcal{C}^1 functions in v that are bounded as well as their derivatives and where for any $t \geq 0$,

$$V_\varepsilon(t) \approx V_0(t) = (1 + \lambda_0 t), \quad \text{as } \varepsilon \approx 0.$$

Proof. After a change of variables it follows that, for any test-function φ ,

$$\begin{aligned} & \langle e_\varepsilon(t), |v|^\kappa \varphi \rangle \\ &= V_\varepsilon(t)^{-\kappa} \int_{\mathbb{T}^d \times \mathbb{R}^d} \left(h_\varepsilon(\tau_\varepsilon(t), x, v) - \mathbf{h}(\tau_\varepsilon(t), x, v) \right) |v|^\kappa \left(\varphi(x, V_\varepsilon(t)^{-1}v) - \varphi(x, 0) \right) dv dx \\ & \quad + V_\varepsilon(t)^{-\kappa} \int_{\mathbb{T}^d \times \mathbb{R}^d} \left(h_\varepsilon(\tau_\varepsilon(t), x, v) - \mathbf{h}(\tau_\varepsilon(t), x, v) \right) |v|^\kappa \varphi(x, 0) dv dx \\ &= \mathcal{I}_1(t) + \mathcal{I}_2(t). \end{aligned}$$

Note that, up to a subsequence, \mathbf{h} is the weak- \star limit of $\{h_\varepsilon\}_\varepsilon$ in $L^\infty((0, \infty); \mathcal{E})$. Thus, for any $t > 0$, $\|\mathbf{h}\|_{L^\infty((t, \infty); \mathcal{E})} \leq \liminf_{\varepsilon \searrow 0} \|h_\varepsilon\|_{L^\infty((t, \infty); \mathcal{E})}$. Consequently, thanks to Theorem 4.14, it holds that

$$\|\mathbf{h}\|_{L^\infty((t, \infty); \mathcal{E})} \leq C \sqrt{K_0} e^{-\frac{\lambda_\varepsilon}{2}t}, \quad t > 0. \quad (\text{A.8})$$

As a consequence, recalling that $\lambda_\varepsilon \simeq c_\varepsilon$, it follows that for any $a \in (0, 1/2)$

$$\begin{aligned} & \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} \left(h_\varepsilon(\tau, x, v) - \mathbf{h}(\tau, x, v) \right) |v|^\kappa dv dx \right| \\ & \leq \|h_\varepsilon(\tau_\varepsilon(t)) - \mathbf{h}(\tau_\varepsilon(t))\|_{L^1_{x,v}(\varpi_q)} \leq C \sqrt{K_0} e^{-\frac{\lambda_\varepsilon}{2}\tau_\varepsilon(t)} \leq C V_\varepsilon(t)^{-a} \sqrt{K_0}. \end{aligned}$$

Now, in regard of $\mathcal{I}_1(t)$, note that

$$|\varphi(x, V_\varepsilon(t)^{-1}v) - \varphi(x, 0)| \leq V_\varepsilon(t)^{-1} |v| \sup_x \sup_v |\partial_v \varphi(x, v)| = C_\varphi V_\varepsilon(t)^{-1} |v|,$$

so that the following holds:

$$|\mathcal{I}_1(t)| \leq C_\varphi V_\varepsilon(t)^{-\kappa-1} \|h_\varepsilon(\tau_\varepsilon(t)) - \mathbf{h}(\tau_\varepsilon(t))\|_{L^1_{x,v}(\varpi_q)} \leq C_\varphi V_\varepsilon(t)^{-\kappa-1-a} \sqrt{K_0}.$$

Similarly,

$$|\mathcal{I}_2(t)| \leq C_\varphi V_\varepsilon(t)^{-\kappa-a} \sqrt{K_0},$$

which proves the desired estimate. \square

The above computations also allow to provide a **local version of Haff's Law**. Namely, note that

$$\begin{aligned} & \int_{\mathbb{R}^d} f_\varepsilon(\tau_\varepsilon(t), x, w) |w|^\kappa dw \\ &= \int_{\mathbb{R}^d} G_\alpha(w) |w|^\kappa dw + \varepsilon \int_{\mathbb{R}^d} h_\varepsilon(\tau_\varepsilon(t), x, w) |w|^\kappa dw, \quad 0 \leq \kappa \leq q. \end{aligned}$$

Thanks to Sobolev embedding it holds that

$$\left| \sup_{x \in \mathbb{T}^d} \int_{\mathbb{R}^d} h_\varepsilon(\tau_\varepsilon(t), x, w) |w|^\kappa dw \right| \leq C_\kappa \|h_\varepsilon(\tau_\varepsilon(t))\|_{\mathcal{E}} \leq C_\kappa \sqrt{K_0}.$$

Therefore, for sufficiently small $\varepsilon > 0$ there exists two positive constants C_κ and c_κ such that

$$c_\kappa \leq \int_{\mathbb{R}^d} f_\varepsilon(\tau_\varepsilon(t), x, w) |w|^\kappa dw \leq C_\kappa, \quad 0 \leq \kappa \leq q, \quad t \geq 0,$$

which leads, for the physical problem, to

$$V_\varepsilon(t)^{-\kappa} c_\kappa \leq \int_{\mathbb{R}^d} F_\varepsilon(t, x, v) |v|^\kappa dv \leq V_\varepsilon(t)^{-\kappa} C_\kappa, \quad 0 \leq \kappa \leq q, \quad t \geq 0.$$

In particular, this estimate renders a local version of Haff's law

$$\int_{\mathbb{R}^d} F_\varepsilon(t, x, v) |v|^2 dv \sim (1 + c_\varepsilon t)^{-2}, \quad \forall t \geq 0, \quad x \in \mathbb{T}^d.$$

APPENDIX B. TOOLS FOR THE HYDRODYNAMIC LIMIT

We collect several tools that are used in Section 6.2 to derive the modified incompressible Navier-Stokes system. Various known computations regarding the elastic Boltzmann operator are needed. As in the classical case, we introduce the traceless tensor

$$\mathbf{A}(v) = v \otimes v - \frac{1}{d} |v|^2 \mathbf{Id}.$$

Notice that that (6.18) can be rewritten thanks to (6.13) as

$$v \cdot \nabla_x \mathbf{h} = \mathbf{A}(v) \mathcal{M}(v) : \nabla_x u + \mathbf{b}(v) \mathcal{M}(v) \cdot \nabla_x \theta,$$

with

$$\mathbf{b}(v) = \frac{1}{2} (|v|^2 - (d+2)\vartheta_1) v \in \mathbb{R}^d.$$

Lemma B.1. *One has that $\mathbf{A}, \mathbf{b} \in \text{Range}(\mathbf{I} - \pi_0)$ and there exists two radial functions $\chi_i = \chi_i(|v|)$, $i = 1, 2$, such that*

$$\phi(v) = \chi_1(|v|) \mathbf{A}(v) \in \mathcal{M}_d(\mathbb{R}), \quad \text{and} \quad \psi(v) = \chi_2(|v|) \mathbf{b}(v) \in \mathbb{R}^d,$$

satisfy

$$\mathbf{L}_1(\phi \mathcal{M}) = -\mathbf{A} \mathcal{M}, \quad \mathbf{L}_1(\psi \mathcal{M}) = -\mathbf{b} \mathcal{M}. \quad (\text{B.1})$$

Moreover,

$$\begin{aligned} \left\langle \phi^{i,j} \mathbf{L}_1(\phi^{k,\ell} \mathcal{M}) \right\rangle &= -\nu \left(\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk} - \frac{2}{d} \delta_{ij} \delta_{kl} \right) \\ \left\langle \psi_i \mathbf{L}_1(\psi_j \mathcal{M}) \right\rangle &= -\frac{d+2}{2} \gamma \delta_{ij}, \quad i, j, k, \ell \in \{1, \dots, d\}, \end{aligned} \quad (\text{B.2})$$

with

$$\nu := -\frac{1}{(d-1)(d+2)} \left\langle \phi : \mathbf{L}_1(\phi \mathcal{M}) \right\rangle \geq 0, \quad \gamma := -\frac{2}{d(d+2)} \left\langle \psi \cdot \mathbf{L}_1(\psi \mathcal{M}) \right\rangle \geq 0.$$

Finally,

$$\phi^{i,j}(v) \lesssim \varpi_3(v), \quad \psi_i(v) \lesssim \varpi_4(v), \quad i, j \in \{1, \dots, d\}.$$

Proof. The tensor \mathbf{A} and the vector \mathbf{b} satisfy

$$\left\langle \mathbf{A}^{k,\ell} \Psi_i \mathcal{M} \right\rangle = 0, \quad \left\langle \mathbf{b} \Psi_i \mathcal{M} \right\rangle = 0, \quad \forall i = 1, \dots, d+2, \quad k, \ell \in \{1, \dots, d\}, \quad (\text{B.3})$$

from which we get that $\mathbf{A}, \mathbf{b} \in \text{Range}(\mathbf{I} - \pi_0)$. We refer to [23] and [9] for the proof of the second part of the Lemma, just mind that the linearized Boltzmann operator considered in such references is defined as $Lg = -\mathcal{M}^{-1} \mathbf{L}_1(\mathcal{M}g)$. We refer to [9, Lemma 4.4] for the proof of (B.2). We refer to [32, Proposition 6.5] for the last estimates on $\phi^{i,j}$ and ψ . \square

Remark B.2. Notice that if $\zeta = \zeta(|v|)$ is radially symmetric, then

$$\left\langle \zeta \mathbf{A}^{i,j} \mathcal{M} \right\rangle = \left\langle \zeta \mathbf{L}_1(\phi \mathcal{M}) \right\rangle = 0, \quad \forall i, j = 1, \dots, d.$$

Lemma B.3. For \mathbf{h} given by (6.7), it holds that

$$\left\langle \phi \mathcal{Q}_1(\mathbf{h}, \mathbf{h}) \right\rangle = \vartheta_1^2 \left(u \otimes u - \frac{2}{d} |u|^2 \mathbf{Id} \right),$$

for any $i, j = 1, \dots, d$.

Proof. As observed in [21, Eq. (60)], if $g\mathcal{M} \in \text{Ker}(\mathbf{L}_1)$ then $\mathcal{Q}_1(g\mathcal{M}, g\mathcal{M}) = -\frac{1}{2} \mathbf{L}_1(g^2 \mathcal{M})$. Therefore, with $g = \varrho + u \cdot v + \frac{1}{2}(|v|^2 - 2\vartheta_1)$,

$$\mathcal{Q}_1(\mathbf{h}, \mathbf{h}) = -\frac{1}{2} \mathbf{L}_1((u \cdot v)^2 \mathcal{M}) - \frac{1}{8} \theta^2 \mathbf{L}_1(|v|^4 \mathcal{M}) + \theta u \cdot \mathbf{L}_1(\frac{1}{2}|v|^2 v \mathcal{M}). \quad (\text{B.4})$$

One checks that

$$\left\langle \phi^{i,j} \mathbf{L}_1(|v|^4 \mathcal{M}) \right\rangle = 0,$$

whereas $\mathbf{L}_1(\frac{1}{2}|v|^2 v \mathcal{M}) = \mathbf{L}_1(\mathbf{b} \mathcal{M})$, from which

$$\left\langle \phi^{i,j} \mathbf{L}_1(\frac{1}{2}|v|^2 v \mathcal{M}) \right\rangle = \left\langle \mathbf{b} \mathbf{L}_1(\phi^{i,j} \mathcal{M}) \right\rangle = -\left\langle \mathbf{b} \mathbf{A}^{i,j} \mathcal{M} \right\rangle = 0,$$

since $\mathbf{b} \mathbf{A}^{i,j}$ is an even function. Therefore, we obtain that

$$\left\langle \phi^{i,j} \mathcal{Q}_1(\mathbf{h}, \mathbf{h}) \right\rangle = -\frac{1}{2} \sum_{k,\ell} u_k u_\ell \left\langle \phi^{i,j} \mathbf{L}_1(v_k v_\ell \mathcal{M}) \right\rangle = \frac{1}{2} \sum_{k,\ell} u_k u_\ell \left\langle v_k v_\ell \mathbf{A}^{i,j} \mathcal{M} \right\rangle. \quad (\text{B.5})$$

As for (B.2), one checks that if $i \neq j$

$$\sum_{k,\ell} u_k u_\ell \left\langle v_k v_\ell \mathbf{A}^{i,j} \mathcal{M} \right\rangle = \sum_{\{k,\ell\}=\{i,j\}} u_k u_\ell \left\langle v_i^2 v_j^2 \mathcal{M} \right\rangle = 2u_i u_j \left\langle v_i^2 v_j^2 \mathcal{M} \right\rangle,$$

whereas, for $i = j$,

$$\sum_{k,\ell} u_k u_\ell \left\langle v_k v_\ell \mathbf{A}^{i,i} \mathcal{M} \right\rangle = \sum_{k=1}^d u_k^2 \left(\left\langle v_i^2 v_k^2 \mathcal{M} \right\rangle - \frac{1}{d} \left\langle v_k^2 |v|^2 \mathcal{M} \right\rangle \right).$$

Notice that $a := \left\langle v_i^2 v_j^2 \mathcal{M} \right\rangle$ is independent of i, j , thus, it is not difficult to check that

$$(d-1)a = \frac{1}{d} \int_{\mathbb{R}^d} |v|^4 \mathcal{M} dv - \int_{\mathbb{R}^d} v_1^4 \mathcal{M}(v) dv = (d-1)\vartheta_1^2,$$

that is, $a = \vartheta_1^2$. In the same way, for any $k \in \{1, \dots, d\}$

$$\langle v_k^2 |v|^2 \mathcal{M} \rangle = \frac{1}{d} \langle |v|^4 \mathcal{M} \rangle = (d+2) \vartheta_1^2,$$

whereas

$$\langle v_k^2 v_i^2 \mathcal{M} \rangle = \begin{cases} a = \vartheta_1^2 & \text{if } k \neq i, \\ \langle v_i^4 \mathcal{M} \rangle = 3\vartheta_1^2 & \text{if } k = i, \end{cases}$$

so that,

$$\sum_{k,\ell} u_k u_\ell \langle v_k v_\ell \mathbf{A}^{i,i} \mathcal{M} \rangle = \vartheta_1^2 \sum_{k \neq i} u_k^2 + 3\vartheta_1^2 u_i^2 - \frac{d+2}{d} |u|^2 \vartheta_1^2 = 2\vartheta_1^2 u_i^2 - \frac{2}{d} \vartheta_1^2 |u|^2.$$

Gathering these last computations, we get

$$\langle \phi^{i,j} \mathcal{Q}_1((u \cdot v) \mathcal{M}, (u \cdot v) \mathcal{M}) \rangle = \vartheta_1^2 \left(u_i u_j - \frac{2}{d} |u|^2 \delta_{i,j} \right),$$

which, combined with (B.5) gives the result. \square

Lemma B.4. *Let \mathbf{h} be given by (6.7). For any $i, j = 1, \dots, d$ it holds that*

$$\langle v_\ell \phi^{i,j} \mathbf{h} \rangle = \begin{cases} \nu u_j & \text{if } i \neq j, \ell = i, \\ \nu u_i & \text{if } i \neq j, \ell = j, \\ -\frac{2}{d} \nu u_\ell + 2\nu u_i \delta_{i\ell} & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Proof. Using the fact that χ_1 is radial, similar computations to that of Lemma B.3 imply that for $\ell \in \{1, \dots, d\}$,

$$\begin{aligned} \langle v_\ell \phi^{i,j} \mathbf{h} \rangle &= \sum_{k=1}^d u_k \langle v_\ell v_k \phi^{i,j} \mathcal{M} \rangle = \sum_{k=1}^d u_k \langle v_\ell v_k \phi^{i,j} \mathcal{M} \rangle \\ &= \sum_{k=1}^d u_k \left(\langle \mathbf{A}^{k,\ell} \phi^{i,j} \mathcal{M} \rangle + \frac{1}{d} \langle |v|^2 \phi^{i,j} \mathcal{M} \rangle \delta_{k\ell} \right) \\ &= - \sum_{k=1}^d u_k \langle \phi^{i,j} \mathbf{L}_1(\phi^{k,\ell} \mathcal{M}) \rangle, \end{aligned}$$

where we used that $\mathbf{L}_1(\phi \mathcal{M}) = -\mathbf{A} \mathcal{M}$ and $\langle |v|^2 \phi^{i,j} \mathcal{M} \rangle = 0$. This gives the result thanks to (B.2). \square

Lemma B.5. *Let \mathbf{h} be given by (6.7). For any $i = 1, \dots, d$ it holds that*

$$\langle \psi_i \mathcal{Q}_1(\mathbf{h}, \mathbf{h}) \rangle = \frac{d+2}{2} \vartheta_1^3 (\theta u_i),$$

and, if ϱ and θ satisfies Boussinesq relation (6.14), then

$$\operatorname{div}_x \langle \psi_i \mathbf{h} v \rangle = \gamma \frac{d+2}{2} \partial_{x_i} \theta.$$

Proof. On the one hand, using (B.4) it holds that

$$\langle \psi_i \mathcal{Q}_1(\mathbf{h}, \mathbf{h}) \rangle = \theta u \cdot \langle \psi_i \mathbf{L}_1(\frac{1}{2}|v|^2 v \mathcal{M}) \rangle = \theta u \cdot \langle \psi_i \mathbf{L}_1(\mathbf{b} \mathcal{M}) \rangle,$$

since, ψ_i being odd, one has $\langle \psi_i \mathbf{L}_1((u \cdot v)^2 \mathcal{M}) \rangle = \langle \psi_i \mathbf{L}_1(|v|^4 \mathcal{M}) \rangle = 0$. Now,

$$\langle \psi_i \mathbf{L}_1(\mathbf{b} \mathcal{M}) \rangle = \langle \mathbf{b} \mathbf{L}_1(\psi_i \mathcal{M}) \rangle = -\langle \mathbf{b} \mathcal{M} \mathbf{b}_i \rangle,$$

and a direct computations show that

$$\langle \mathbf{b}_j \mathbf{b}_i \mathcal{M} \rangle = -\frac{1}{4d} \langle (|v|^2 - (d+2)\vartheta_1)^2 |v|^2 \mathcal{M} \rangle \delta_{ij} = -\frac{d+2}{2} \vartheta_1^3 \delta_{ij},$$

which gives the expression for $\langle \psi_i \mathcal{Q}_1(\mathbf{h}, \mathbf{h}) \rangle$. On the other hand, using symmetry properties, one checks that

$$\langle \psi_i \mathbf{h} v_\ell \rangle = \varrho \langle \psi_i v_i \mathcal{M} \rangle \delta_{i\ell} + \frac{1}{2} \theta \langle \psi_i (|v|^2 - d\vartheta_1) v_i \mathcal{M} \rangle \delta_{i\ell},$$

from which

$$\operatorname{div}_x \langle \psi_i \mathbf{h} v \rangle = \langle \psi_i v_i \mathcal{M} \rangle \partial_{x_i} \varrho + \frac{1}{2} \langle \psi_i (|v|^2 - d\vartheta_1) v_i \mathcal{M} \rangle \partial_{x_i} \theta.$$

Writing $\frac{1}{2} \langle \psi_i (|v|^2 - d\vartheta_1) v_i \mathcal{M} \rangle = \langle \psi_i \mathbf{b}_i \mathcal{M} \rangle + \vartheta_1 \langle \psi_i v_i \mathcal{M} \rangle$ and using Boussinesq relation (6.14), one gets that

$$\operatorname{div}_x \langle \psi_i \mathbf{h} v \rangle = \langle \psi_i \mathbf{b}_i \mathcal{M} \rangle \partial_{x_i} \theta = \gamma \frac{d+2}{2} \partial_{x_i} \theta,$$

where the identity $\langle \psi_i \mathbf{b}_i \mathcal{M} \rangle = -\langle \psi_i \mathbf{L}_1(\psi_i \mathcal{M}) \rangle$ was used together with (B.2). \square

In Lemma 6.11, we study the convergence of some term involving the source term \mathbf{S}_ε defined in (6.29). To do that, we use the next Lemma which provides a strong convergence to 0 of this source term.

Lemma B.6. *Let \mathbf{S}_ε defined in (6.29). We have that*

$$\|\mathbf{S}_\varepsilon\|_{L^1((0,T); L_x^1 L_v^1(\varpi_q))} \lesssim \varepsilon.$$

Proof. We decompose \mathbf{S}_ε into three parts using the splitting $h_\varepsilon = h_\varepsilon^0 + h_\varepsilon^1$: $\mathbf{S}_\varepsilon = \mathbf{S}_\varepsilon^0 + \mathbf{S}_\varepsilon^1 + \mathbf{S}_\varepsilon^2$ with

$$\mathbf{S}_\varepsilon^j := \varepsilon^{-1} (\mathbf{L}_\alpha h_\varepsilon^j - \mathbf{L}_1 h_\varepsilon^j) + \mathcal{Q}_\alpha(h_\varepsilon^j, h_\varepsilon^j) - \mathcal{Q}_1(h_\varepsilon^j, h_\varepsilon^j) - \varepsilon^{-1} \kappa_\alpha \nabla_v \cdot (v h_\varepsilon^j), \quad j = 0, 1,$$

and

$$\mathbf{S}_\varepsilon^2 := 2\tilde{\mathcal{Q}}_\alpha(h_\varepsilon^0, h_\varepsilon^1) - 2\tilde{\mathcal{Q}}_1(h_\varepsilon^0, h_\varepsilon^1).$$

The terms \mathbf{S}_ε^0 and \mathbf{S}_ε^2 are treated using the estimate on h_ε^0 and h_ε^1 stated in (6.8)-(6.9). Indeed, using standard estimates on \mathcal{Q}_α and \mathcal{Q}_1 , we have:

$$\begin{aligned} \|\mathbf{S}_\varepsilon^0 + \mathbf{S}_\varepsilon^2\|_{L^1((0,T);L_x^1L_v^1(\varpi_q))} &\lesssim \|h_\varepsilon^0\|_{L^1((0,T);\mathcal{E}_1)} \\ &+ \|h_\varepsilon^0\|_{L^1((0,T);\mathcal{E}_1)}(\|h_\varepsilon^0\|_{L^\infty((0,T);\mathcal{E})} + \|h_\varepsilon^1\|_{L^\infty((0,T);\mathcal{H})}) + \varepsilon \|h_\varepsilon^0\|_{L^1((0,T);\mathcal{E}_1)} \lesssim \varepsilon^2. \end{aligned}$$

Using now (6.8) and Lemma 2.1, we have:

$$\begin{aligned} \|\mathbf{S}_\varepsilon^1\|_{L^1((0,T);L_x^1L_v^1(\varpi_q))} &\lesssim \varepsilon \|h_\varepsilon^1\|_{L^1((0,T);\mathcal{H})} \\ &+ \varepsilon^2 \|h_\varepsilon^1\|_{L^1((0,T);\mathcal{H})} \|h_\varepsilon^1\|_{L^\infty((0,T);\mathcal{H})} + \varepsilon \|h_\varepsilon^1\|_{L^1((0,T);\mathcal{H})} \lesssim \varepsilon \end{aligned}$$

which yields the result. \square

To handle the convergence of nonlinear terms, we will need to resort to the following compensated compactness result extracted from [45] (see also [30, Lemma 13.1, Appendix D]). The original result in [45] is proven in the whole space but is easily adapted to the case of the torus.

Proposition B.7. *Let $c \neq 0$ and $T > 0$. Consider two families $\{\phi_\varepsilon\}_\varepsilon$ and $\{\psi_\varepsilon\}$ bounded in $L^\infty((0, T); L_x^2(\mathbb{T}^d))$ and in $L^\infty((0, T); \mathbb{W}_x^{1,2}(\mathbb{T}^d))$ respectively, such that*

$$\begin{cases} \partial_t \nabla_x \psi_\varepsilon + \frac{c^2}{\varepsilon} \nabla_x \phi_\varepsilon = \frac{1}{\varepsilon} F_\varepsilon \\ \partial_t \phi_\varepsilon + \frac{1}{\varepsilon} \Delta_x \psi_\varepsilon = \frac{1}{\varepsilon} G_\varepsilon \end{cases}$$

where F_ε and G_ε converge strongly in $L^1((0, T); L_x^2(\mathbb{T}^d))$. Then,

$$\mathcal{P} \operatorname{Div}_x (\nabla_x \psi_\varepsilon \otimes \nabla_x \psi_\varepsilon) \longrightarrow 0, \quad \operatorname{div}_x (\phi_\varepsilon \nabla_x \psi_\varepsilon) \longrightarrow 0$$

in the sense of distributions on $(0, T) \times \mathbb{T}^d$.

APPENDIX C. PROOF OF THEOREM 2.10 AND PROPOSITION 2.13

Theorem C.1 (See Theorem 2.1, [15]). *There exists $\varepsilon_0 \in (0, 1)$ such that, for all $\ell, s \in \mathbb{N}$ with $\ell \geq s$ and $q > 2$ and any $\varepsilon \in (0, \varepsilon_0)$, the full transport operator $\mathcal{G}_{1,\varepsilon}$ generates a C_0 -semigroup $\{\mathcal{V}_{1,\varepsilon}(t); t \geq 0\}$ on $\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)$ such that, for all $t_\star > 0$ there exist $C_0(t_\star)$, $\mu_\star > 0$ such that*

$$\begin{aligned} \|\mathcal{V}_{1,\varepsilon}(t)h - \mathbf{P}_0 h\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)} \\ \leq C_0(t_\star) \exp(-\mu_\star t) \|h - \mathbf{P}_0 h\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)}, \quad \forall t > t_\star, \quad (\text{C.1}) \end{aligned}$$

holds true for any $h_0 \in \mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)$, where \mathbf{P}_0 is the spectral projection onto $\operatorname{Ker}(\mathcal{G}_{1,\varepsilon}) = \operatorname{Ker}(\mathcal{L}_1)$ which is independent of ε and given by (2.22).

The difference between Theorems 2.10 and C.1 lies in the fact that, in Theorem 2.10, we allow $t_\star = 0$ in the decay estimate (C.1). The “initial layer” dependence on $t_\star > 0$ in (C.1) is inherent to the method of the enlargement semigroup theory of [34].

Theorem C.1 ensures that $\mathcal{G}_{1,\varepsilon}$ is the generator of a C_0 -semigroup $\{\mathcal{V}_{1,\varepsilon}(t) ; t \geq 0\}$ on \mathcal{E} as soon as $q > 2$ and $\varepsilon \in (0, \varepsilon_0)$. We focus on extending (C.1) to $t_\star := 0$.

Proof of Theorem 2.10. We adopt the decomposition of the nonlinear part of [15] that we used in Section 4. Namely, for some fixed $h \in \mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)$ we set

$$f_{\text{in}} := h - \mathbf{P}_0 h,$$

and write $f(t) = \mathcal{V}_{1,\varepsilon}(t)f_{\text{in}}$ as $f(t) = f^0(t) + f^1(t)$ with $f^0 \in \mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)$ solution to

$$\partial_t f^0(t) = \mathcal{B}_{1,\varepsilon} f^0, \quad f^0(0) = f_{\text{in}}, \quad (\text{C.2})$$

whereas $f^1 \in \mathcal{H} := H_{x,v}^\ell(\mathcal{M}^{-\frac{1}{2}})$, is solution to

$$\partial_t f^1(t) = \mathcal{G}_{1,\varepsilon} f^1(t) + \mathcal{A}_\varepsilon f^0(t), \quad f^1(0) = 0. \quad (\text{C.3})$$

As before, the same notations for the operators $\mathcal{G}_{1,\varepsilon}$, $\mathcal{V}_{1,\varepsilon}(t)$ acting on various different spaces is used. The definition should be clear from the context. Of course,

$$f^0(t) = \mathcal{S}_{1,\varepsilon}(t)f_{\text{in}},$$

and

$$\|f^0(t)\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)} \leq C_0 \exp(-\varepsilon^{-2} \nu_0 t) \|f_{\text{in}}\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)}, \quad (\text{C.4})$$

since $\mathcal{B}_{1,\varepsilon}$ is $\varepsilon^{-2} \nu_0$ hypo-dissipative (ν_0 depends on ℓ, m). The constant C_0 is independent of ε . Let us investigate $\|f^1(t)\|_{\mathcal{H}}$. Notice that, since $\mathbf{P}_0 f = 0$, $\mathbf{P}_0 f^1 = -\mathbf{P}_0 f^0$ (recall that the projection is the same in \mathcal{H} and $\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)$ and independent of ε), the estimate for $\mathbf{P}_0 f^1$ is straightforward

$$\|\mathbf{P}_0 f^1(t)\|_{\mathcal{H}} \leq C_1 \exp(-\varepsilon^{-2} \nu_0 t) \|f_{\text{in}}\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)}, \quad (\text{C.5})$$

where the constant C_1 differs from C_0 just because the norm of the eigenfunctions are different in \mathcal{H} and $\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)$. We focus on

$$\psi(t) = \mathbf{P}_0^\perp f^1(t) = (\mathbf{Id} - \mathbf{P}_0) f^1(t).$$

One has

$$\partial_t \psi(t) = \mathcal{G}_{1,\varepsilon} \psi(t) + \mathbf{P}_0^\perp \mathcal{A}_\varepsilon f^0(t),$$

and, arguing as in [14, Section 7.2] (see also [15, Theorem 4.7 and Remark 4.8]) one has

$$\frac{1}{2} \|\psi(t)\|_{\mathcal{H}}^2 \leq \frac{1}{2} \|\psi(0)\|_{\mathcal{H}}^2 e^{-\mu_\star t} + \int_0^t e^{-\mu_\star(t-s)} \|\psi(s)\|_{\mathcal{H}} \|\mathbf{P}_0^\perp \mathcal{A}_\varepsilon f^0(s)\|_{\mathcal{H}} ds$$

with $\mu_\star > 0$ independent of ε which is the size of the spectral gap of $\mathcal{G}_{1,\varepsilon}$. Recalling that $\psi(0) = 0$ and $\|\mathcal{A}_\varepsilon\|_{\mathcal{B}(\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q), \mathcal{H})} \leq C_A \varepsilon^{-2}$, we get that

$$\|\psi(t)\|_{\mathcal{H}}^2 \leq \frac{2C_A}{\varepsilon^2} \int_0^t e^{-\mu_\star(t-s)} \|\psi(s)\|_{\mathcal{H}} \|f^0(s)\|_{\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q)} ds.$$

We use (C.4) to deduce that

$$\|\psi(t)\|_{\mathcal{H}}^2 \leq \frac{2C_0C_A}{\varepsilon^2} \int_0^t e^{-\mu_\star(t-s)} e^{-\frac{\nu_0}{\varepsilon^2}s} \|\psi(s)\|_{\mathcal{H}} \|f_{\text{in}}\|_{\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)} ds.$$

Then, Young's inequality leads to

$$\begin{aligned} \|\psi(t)\|_{\mathcal{H}}^2 &\leq \frac{C_0C_A e^{-\mu_\star t}}{\varepsilon^2} \int_0^t e^{-(\frac{\nu_0}{\varepsilon^2}-\mu_\star)s} \|\psi(s)\|_{\mathcal{H}}^2 ds \\ &\quad + \frac{C_0C_A e^{-\mu_\star t}}{\varepsilon^2} \|f_{\text{in}}\|_{\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)}^2 \int_0^t e^{-(\frac{\nu_0}{\varepsilon^2}-\mu_\star)s} ds. \end{aligned}$$

If $\varepsilon^{-2}\nu_0 > 2\mu_\star$ we get after integration that

$$\|\psi(t)\|_{\mathcal{H}}^2 \leq \frac{2C_0C_A e^{-\mu_\star t}}{\nu_0} \|f_{\text{in}}\|_{\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)}^2 + \frac{C_0C_A e^{-\mu_\star t}}{\varepsilon^2} \int_0^t e^{-(\frac{\nu_0}{\varepsilon^2}-\mu_\star)s} \|\psi(s)\|_{\mathcal{H}}^2 ds.$$

With $x(t) = e^{\mu_\star t} \|\psi(t)\|_{\mathcal{H}}^2$ it follows that

$$x(t) \leq \frac{2C_0C_A}{\nu_0} \|f_{\text{in}}\|_{\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)}^2 + \frac{C_0C_A}{\varepsilon^2} \int_0^t e^{-\frac{\nu_0}{\varepsilon^2}s} x(s) ds,$$

and Gronwall's lemma gives

$$x(t) \leq \frac{2C_0C_A}{\nu_0} \|f_{\text{in}}\|_{\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)}^2 \exp\left(\frac{C_0C_A}{\nu_0}\right) = C_2 \|f_{\text{in}}\|_{\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)}^2,$$

with $C_2 > 0$ independent of ε . Therefore

$$\|\psi(t)\|_{\mathcal{H}}^2 \leq C_2 \|f_{\text{in}}\|_{\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)}^2 \exp(-\mu_\star t).$$

This combined with (C.5) gives that

$$\|f^1(t)\|_{\mathcal{H}}^2 \leq (C_2 + C_1) \|f_{\text{in}}\|_{\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)}^2 \exp(-\mu_\star t).$$

Overall, the estimates for f^0 and f^1 lead to

$$\|f(t)\|_{\mathcal{E}} \leq C_3 \|f_{\text{in}}\|_{\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)} \exp\left(-\frac{\mu_\star}{2}t\right), \quad \forall t \geq 0,$$

with C_3 independent of ε and given by $C_0 + \sqrt{C_1 + C_2}$ as long as $\nu_0\varepsilon^{-2} > \mu_\star$. \square

Proof of Proposition 2.13. Notations are those of Proposition 2.13. We recall here that, on the Banach space $\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)$,

$$\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} = \mathcal{B}_{1,\varepsilon}^{(\delta)} + \varepsilon^{-2}\mathcal{P}_\alpha + \varepsilon^{-2}\mathcal{T}_\alpha$$

with domain $\mathcal{D}(\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}) = \mathbb{W}_x^{\ell+1,1}\mathbb{W}_v^{s+1,1}(\varpi_{q+1})$. Recall that we can find an equivalent norm on $\mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q)$ for which $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)} + \varepsilon^{-2}\nu_{\ell,s}$ is dissipative. According to Lumer-Phillips Theorem, see [25, Proposition 3.14 & Theorem 3.15], in order to show that $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ generates a C_0 -semigroup it suffices to prove that there exists $\lambda > 0$ large enough such that

$$\text{Range}(\lambda - \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}) = \mathbb{W}_x^{\ell,1}\mathbb{W}_v^{s,1}(\varpi_q). \quad (\text{C.6})$$

Clearly, one can replace without loss of generality $\mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$ with $\varepsilon^2 \mathcal{B}_{\alpha,\varepsilon}^{(\delta)}$. Denote for simplicity

$$X = \mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q), \quad B_\alpha := \varepsilon^2 \mathcal{B}_{\alpha,\varepsilon}^{(\delta)},$$

omitting the dependence with respect to ε and δ . It follows that

$$B_\alpha = \mathcal{L}_1^{R,\delta} - \Sigma_{\mathcal{M}} - \varepsilon v \cdot \nabla_x + \mathcal{P}_\alpha + T_\alpha.$$

Introduce the following operator

$$\mathcal{T}_\alpha h := -\varepsilon v \cdot \nabla_x h + T_\alpha h - \Sigma_{\mathcal{M}} h = -\varepsilon v \cdot \nabla_x h - \kappa_\alpha \operatorname{div}_v(vh) - \Sigma_{\mathcal{M}} h$$

with domain $\mathcal{D}(\mathcal{T}_\alpha) = \mathbb{W}_x^{\ell+1,1} \mathbb{W}_v^{s+1,1}(\varpi_{q+1})$. It is not difficult to check that \mathcal{T}_α generates a C_0 -semigroup in X given by

$$e^{t\mathcal{T}_\alpha} g(x, v) = \exp\left(-\int_0^t d\kappa_\alpha + \Sigma_{\mathcal{M}}(ve^{\kappa_\alpha(s-t)}) ds\right) g\left(x - \frac{\varepsilon}{\kappa_\alpha} (1 - e^{-\kappa_\alpha t}) v, ve^{-\kappa_\alpha t}\right).$$

In particular,

$$\lim_{\lambda \rightarrow \infty} \|\mathcal{R}(\lambda, \mathcal{T}_\alpha)\|_{\mathcal{B}(X)} = 0. \quad (\text{C.7})$$

Moreover, one has the following gain of integrability for the resolvent of \mathcal{T}_α : there is $\alpha_1 \in (0, 1)$ such that, for $\alpha \in (\alpha_1, 1)$ there is $c > 0$ and $\lambda(\alpha) > 0$

$$\|\mathcal{R}(\lambda, \mathcal{T}_\alpha)\|_{\mathcal{B}(\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q), \mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_{q+1}))} \leq \frac{1}{\sigma - c\kappa_\alpha}, \quad \forall \lambda > \lambda(\alpha), \quad (\text{C.8})$$

where σ is an explicit positive constant depending only on $\Sigma_{\mathcal{M}}$. The proof of such a property is an easy adaptation of [2, Lemma C.14] whenever $k = s = 0$ and extends to $k \geq s \geq 0$ following techniques from [49], we leave the details to the reader. One also has the following result, see the proof of [18, Lemma B.1 & Proposition B.2]: there exists $\tau(\delta) > 0$ such that $\lim_{\delta \rightarrow 0} \tau(\delta) = 0$ and

$$\left\| \mathcal{L}_{1,+}^{R,\delta} \right\|_{\mathcal{B}(\mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_{q+1}), \mathbb{W}_x^{\ell,1} \mathbb{W}_v^{s,1}(\varpi_q))} \leq \tau(\delta), \quad (\text{C.9})$$

while $\mathcal{L}_{1,-}^{R,\delta} \in \mathcal{B}(X)$. With these two properties, introduce the sum $\mathcal{C}_\alpha := \mathcal{L}_{1,+}^{R,\delta} + \mathcal{T}_\alpha$ with domain $\mathcal{D}(\mathcal{C}_\alpha) = \mathcal{D}(\mathcal{T}_\alpha)$. We have directly from the previous two properties (C.8) and (C.9)

$$\left\| \mathcal{L}_{1,+}^{R,\delta} \mathcal{R}(\lambda, \mathcal{T}_\alpha) \right\|_{\mathcal{B}(X)} \leq \frac{\tau(\delta)}{\sigma - c\kappa_\alpha}, \quad \forall \lambda > \lambda(\alpha),$$

from which, choosing $\delta > 0$ sufficiently small such that $\frac{\tau(\delta)}{\sigma - c\kappa_\alpha} < 1$, we obtain that $(\mathbf{Id} - \mathcal{L}_{1,+}^{R,\delta} \mathcal{R}(\lambda, \mathcal{T}_\alpha))$ is invertible. We deduce that

$$\begin{aligned} \mathcal{R}(\lambda, \mathcal{C}_\alpha) &= \mathcal{R}(\lambda, \mathcal{T}_\alpha) \left(\mathbf{Id} - \mathcal{L}_{1,+}^{R,\delta} \mathcal{R}(\lambda, \mathcal{C}_\alpha) \right)^{-1} \\ &= \mathcal{R}(\lambda, \mathcal{T}_\alpha) \sum_{n=0}^{\infty} \left[\mathcal{L}_{1,+}^{R,\delta} \mathcal{R}(\lambda, \mathcal{T}_\alpha) \right]^n, \quad \forall \lambda > \lambda(\alpha), \end{aligned}$$

simply observing that $(\lambda - \mathcal{C}_\alpha) = (\mathbf{Id} - \mathcal{L}_{1,+}^{R,\delta} \mathcal{R}(\lambda, \mathcal{C}_\alpha))(\lambda - \mathcal{T}_\alpha)$. In particular,

$$\|\mathcal{R}(\lambda, \mathcal{C}_\alpha)\|_{\mathcal{B}(X)} \leq \frac{1}{\sigma - c\kappa_\alpha - \tau(\delta)}, \quad \forall \lambda > \lambda(\alpha),$$

with

$$\lim_{\lambda \rightarrow \infty} \|\mathcal{R}(\lambda, \mathcal{C}_\alpha)\|_{\mathcal{B}(X)} = 0$$

by virtue of (C.7). Set then

$$\mathcal{C}_\alpha^1 := \mathcal{C}_\alpha + \mathcal{P}_\alpha.$$

With the estimate of \mathcal{P}_α

$$\|\mathcal{P}_\alpha \mathcal{R}(\lambda, \mathcal{C}_\alpha)\|_{\mathcal{B}(X)} \leq C \frac{1 - \alpha}{\sigma - c\kappa_\alpha - \tau(\delta)}$$

and, choosing α sufficiently close to 1, the operator $\mathbf{Id} + \mathcal{P}_\alpha \mathcal{R}(\lambda, \mathcal{C}_\alpha)$ is invertible and so is $\lambda - \mathcal{C}_\alpha^1$. Finally, since

$$B_\alpha = \mathcal{C}_\alpha^1 - \mathcal{L}_{1,-}^{R,\delta},$$

one can chose $\lambda > 0$ sufficiently large so that

$$\|\mathcal{L}_{1,-}^{R,\delta} \mathcal{R}(\lambda, \mathcal{C}_\alpha^1)\|_{\mathcal{B}(X)} \leq \|\mathcal{L}_{1,-}^{R,\delta}\|_{\mathcal{B}(X)} \|\mathcal{R}(\lambda, \mathcal{C}_\alpha^1)\|_{\mathcal{B}(X)} < 1$$

and obtain that $\lambda - B_\alpha$ is invertible. In particular, (C.6) holds true and this proves the result. \square

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