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► **To cite this version:**

Mustapha Mokhtar-Kharroubi. On Kato classes and self-adjointness of many-body convolution type Hamiltonians. 2020. hal-02922387

HAL Id: hal-02922387

<https://hal.science/hal-02922387>

Preprint submitted on 26 Aug 2020

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On Kato classes and self-adjointness of many-body convolution type Hamiltonians

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Abstract

We give a functional analytic L^1 approach to L^2 form-bounds for many-body convolution type Hamiltonians and explore new aspects of Kato classes for convolution semigroups. These classes are explored in terms of L^1 weak compactness properties and also in terms of asymptotics of averages over suitable shells. In particular, various membership criteria are given.

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1 Introduction

This paper is an abridged and improved version of the recent preprint [22]. It is well-known that the mathematical theory of Schrödinger operators was born in 1951 with Kato's famous self-adjointness theorem [13] for atomic Hamiltonians in $L^2(\mathbb{R}^{3N})$

$$-\sum_{i=1}^N (2\mu_i)^{-1} \Delta_i + \sum_{i=1}^N V^i(x_i) + \sum_{i<j} V^{ij}(x_i - x_j) \quad (1)$$

on the domain $H^2(\mathbb{R}^{3N})$ provided that $V^i, V^{ij} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$; (Δ_i is the Laplacian with respect to the variable $x_i \in \mathbb{R}^3$ and the μ_i 's are positive constants). Indeed, in this case, the potential

$$\sum_{i=1}^N V^i(x_i) + \sum_{i<j} V^{ij}(x_i - x_j)$$

(as a multiplication operator) has zero relative operator bound with respect to $\sum_{i=1}^N (2\mu_i)^{-1} \Delta_i$ and the self-adjointness property follows from the Kato-Rellich theorem. Of course, the perturbed Hamiltonian can be defined in terms of quadratic forms (via the KLMN theorem) for much more general potentials, see e.g. [8]. The literature on the subject is considerable and cannot be summarized; to get an idea, see e.g. [27][28][26][7][23][11].

If we stay within the realm of L^1_{loc} potentials, a sufficient (but *not* necessary) condition is that the negative parts of the potentials belong to the *Kato class*. Besides its interest for self-adjointness (see the second famous Kato's paper [14]), this class of potentials turns out to play also a key role in the exploration of "*Schrödinger semigroups*" as shown in the classical paper by M. Aizenman and B. Simon [1] where the connections of the Kato class with Brownian motion, Harnack's inequality and L^p properties of Schrödinger semigroups are analyzed; we refer to the survey [29] and to [31][9][10] for more information. We note that this Kato class is attached to the Laplacian but other Kato classes are attached to different generators, see R. Carmona, W. Ch. Masters and B. Simon [4]. The present paper revisits these Kato classes in different new directions. Actually, our object here is twofold:

1. We give first general (abstract) form-bound estimates by means of L^1 functional analytic tools.
2. Secondly, we explore new aspects of Kato classes relative to general convolution semigroups. In particular, we explore them in terms of L^1 weak

compactness properties and also in terms of asymptotics of averages over suitable spherical shells.

This paper improves, in several directions, some previous results [18] and an unpublished work on multi-particle convolution Hamiltonians [19]. This is an abridged and improved version of a recent preprint [22]; (see Remark 13 below).

We deal with many-body Hamiltonians on $L^2(\mathbb{R}^{3N})$

$$\mathcal{H} := - \sum_{i=1}^N T_{(i)} + \sum_{i=1}^N V^i(x_i) + \sum_{i < j} V^{ij}(x_i - x_j) \quad (2)$$

where $T_{(i)}$ is a generator (with respect to the variable $x_i \in \mathbb{R}^3$) of a general symmetric convolution semigroup $(S_{(i)}(t))_{t \geq 0}$ depending a priori of the index i ($1 \leq i \leq N$) and $V^i, V^{ij} : \mathbb{R}^3 \rightarrow \mathbb{R}$ are measurable potentials. The introduction of this general class of Hamiltonians is motivated by both the non-relativistic Hamiltonian (1) and the quasi-relativistic one where

$$T_{(i)} = - \left(\sqrt{-c^2 h^2 \Delta_i + m_i^2 c^4} - m_i c^2 \right)$$

(see [15] Chapter 8) and also by the combination of the two (see [11] Example 2.6). The self-adjointness of \mathcal{H} is of course a prerequisite to build the unitary group $(e^{-it\mathcal{H}})_{t \in \mathbb{R}}$ which solves the Schrödinger equation

$$i \frac{df}{dt} = \mathcal{H}f, \quad f(0) = f_0 \in L^2(\mathbb{R}^{3N}).$$

Section 2 is devoted to some useful reminders; in particular, on symmetric convolution semigroups $(S^1(t))_{t \geq 0}$ on $L^1(\mathbb{R}^d)$

$$S^1(t) : f \in L^1(\mathbb{R}^d) \rightarrow \int_{\mathbb{R}^d} f(x-y) m_t(dy) \in L^1(\mathbb{R}^d)$$

where $\{m_t\}_{t \geq 0}$ are Borel sub-probability *measures* on \mathbb{R}^d (see Section 2 below for the details). Our main assumption is that the resolvent of the generator T^1 is a *kernel* operator

$$(\lambda - T^1)^{-1} f = \int_{\mathbb{R}^d} E_\lambda(x-y) f(y) dy$$

where

$$E_\lambda(\cdot) \in L^1(\mathbb{R}^d) \quad (3)$$

is the so-called λ -potential kernel; in particular, we do *not* need a priori that $\int_{\mathbb{R}^d} e^{-tF(\zeta)} d\zeta < +\infty$ ($t > 0$) where $F(\cdot)$ is the characteristic exponent (see Section 2 below). For the simplicity of notations (and depending on the context), E_λ will denote both the λ -potential kernel and the resolvent operator $(\lambda - T^1)^{-1}$

$$E_\lambda : L^1(\mathbb{R}^d) \ni f \rightarrow \int_{\mathbb{R}^d} E_\lambda(x-y)f(y)dy \in L^1(\mathbb{R}^d).$$

Our main result on form-perturbation theory is the following general statement:

Let V_-^i and V_-^{ij} (e.g. the negative parts of V^i and V^{ij}) be $T_{(i)}$ -bounded in $L^1(\mathbb{R}^3)$ and let

$$\delta_i := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-^i(\lambda - T_{(i)})^{-1}], \quad \delta_{ij} := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-^{ij}(\lambda - T_{(i)})^{-1}]$$

where r_σ refers to the spectral radius in $L^1(\mathbb{R}^3)$. Then the multiplication operator on $L^2(\mathbb{R}^{3N})$ by

$$-\mathcal{V}_-(x_1, \dots, x_N) := -\sum_{i=1}^N V_-^i(x_i) - \sum_{i < j} V_-^{ij}(x_i - x_j)$$

is form-bounded with respect to the positive free Hamiltonian

$$-\mathcal{T} := -\sum_{i=1}^N T_{(i)}$$

with relative form bound less than or equal to

$$\delta := \max_{1 \leq i \leq N} (\delta_i) + \max_{1 \leq i \leq N-1} (\widehat{\delta}_i)$$

where $\widehat{\delta}_i = \sum_{j=i+1}^N \delta_{ij}$ ($i \leq N-1$); (see Theorem 4). This result is derived from a preliminary L^1 -generation result, see the proof of Theorem 1. (We give also another hilbertian result on the *identification* of two self-adjoint operators, see Theorem 3.) We note that if $\delta < 1$ then the form-sum operators on $L^2(\mathbb{R}^{3N})$

$$(-\mathcal{T}) \oplus (-\mathcal{V}_-) \tag{4}$$

we obtain admit $C_c^\infty(\mathbb{R}^{3N})$ and $\mathcal{S}(\mathbb{R}^{3N})$ as form-cores; (see Remark 6). In particular, this is the case when

$$\lim_{\lambda \rightarrow +\infty} \left\| V_-^i(\lambda - T_{(i)})^{-1} \right\|_{\mathcal{L}(L^1(\mathbb{R}^d))} = \lim_{\lambda \rightarrow +\infty} \left\| V_-^{ij}(\lambda - T_{(i)})^{-1} \right\|_{\mathcal{L}(L^1(\mathbb{R}^d))} = 0$$

i.e. when V_-^i and V_-^{ij} are Kato class potentials relative to $T_{(i)}$ in $L^1(\mathbb{R}^3)$; (see below for more information). It is quite standard (see e.g. [8] Theorem 4.1, p. 24) to capture (2) as a form-sum of the positive potential

$$\sum_{i=1}^N V_+^i(x_i) + \sum_{i<j} V_+^{ij}(x_i - x_j)$$

and the lower bounded Hamiltonian (4); however, if $V_+^i, V_+^{ij} \in L_{loc}^1(\mathbb{R}^3)$, the question whether $C_c^\infty(\mathbb{R}^{3N})$ is a core of the new form deserves a separate study which is not considered here.

Section 5 is devoted to the exploration in *various new directions* of the Kato classes relative to general convolution semigroups $(S^1(t))_{t \geq 0}$ with generators T^1 . We recall that a measurable potential

$$V : \mathbb{R}^d \rightarrow \mathbb{R}_+$$

belongs to the Kato class (relative to $(S^1(t))_{t \geq 0}$) if V is T^1 -bounded in $L^1(\mathbb{R}^d)$ and

$$\left\| V (\lambda - T^1)^{-1} \right\|_{\mathcal{L}(L^1(\mathbb{R}^d))} \rightarrow 0 \quad (\lambda \rightarrow +\infty). \quad (5)$$

As far as we know, the most general class of convolution semigroups whose Kato class is analyzed is the one such that

$$\int_{\mathbb{R}^d} e^{-tF(\zeta)} d\zeta < +\infty \quad (t > 0)$$

($F(\cdot)$ is the characteristic exponent) and $m_t(dy) = k_t(y)dy$ is such that $k_t(\cdot)$ is *radial and nonincreasing*, i.e.

$$k_t(y) = \widehat{k}_t(|y|) \text{ and } \rho \rightarrow \widehat{k}_t(\rho) \text{ nonincreasing.}$$

Indeed, in this case, a Kato class potential is characterized by

$$\lim_{\varepsilon \rightarrow 0} \sup_{y \in \mathbb{R}^d} \int_{\{|z| \leq \varepsilon\}} V(y+x) E_\lambda(x) dx = 0 \quad (\lambda > 0); \quad (6)$$

(see [4] Theorem III1).

We show here that (6) is *still a sufficient membership criterion* to the Kato class for the more general class of convolution semigroups admitting just a λ -potential kernel $E_\lambda(\cdot)$ bounded outside any neighborhood of the origin provided we restrict ourselves *a priori* to potentials V of the form

$$V \in \sum_{j=1}^{j=m} L^{p_j}(\mathbb{R}^d), \quad p_j \in [1, +\infty] \quad (7)$$

(see Theorem 9 for a more detailed statement). Note that (7) is a very general condition which is easily checkable in practice by decomposing V according to its different singularities.

For the sequel, we call Kato potentials those satisfying (6) (regardless of the occurrence of (5)) and *local* Kato potentials those satisfying

$$\lim_{\varepsilon \rightarrow 0} \sup_{|y| \leq C} \int_{\{|z| \leq \varepsilon\}} V(y+x)E_\lambda(x)dx = 0 \quad (\lambda > 0, C > 0). \quad (8)$$

We provide different (new) *characterizations* of local Kato potentials. Indeed, we show that (8) is equivalent to

$$VE_\lambda : L^1(\mathbb{R}^d) \rightarrow L^1_{loc}(\mathbb{R}^d) \text{ is weakly compact} \quad (9)$$

or to

$$VE_\lambda : L^1(\mathbb{R}^d) \rightarrow L^1_{loc}(\mathbb{R}^d) \text{ is compact.} \quad (10)$$

Furthermore, these properties are also equivalent to the following local equi-integrability property

$$\lim_{|\Omega| \rightarrow 0} \sup_{\Omega \subset B(0,R)} \sup_{|y| \leq C} \left(\int_{\Omega} V(x)E_\lambda(x-y)dx \right) = 0 \quad (C > 0) \quad (11)$$

where $|\Omega|$ is the Lebesgue measure of Ω ; (see Theorem 10). Note that (10) extends to more general operators a known characterization of the local Kato class of the Laplacian ([1] Theorem 4.18) while (9) and (11) are new even for the Laplacian.

We provide also *different* (sufficient) weak compactness criteria. Indeed, note that the T^1 -boundedness of V can be formulated as

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} V_y(z)E_\lambda(z)dz < +\infty$$

where

$$V_y : z \in \mathbb{R}^d \rightarrow V(y+z).$$

In particular

$$\sup_{y \in \mathbb{R}^d} \int_{|z| \leq 1} V_y(z)E_\lambda(z)dz < +\infty$$

i.e. $\{V_y; y \in \mathbb{R}^d\}$ is a bounded subset of $L^1(B(0,1); \mu(dz))$ where

$$\mu(dz) = E_\lambda(z)dz.$$

We show first that V is a local Kato potential once

$$\mathbb{R}^d \ni y \rightarrow V_y \in L^1(B(0, 1); \mu(dz)) \text{ is continuous;}$$

(see Theorem 14). This result extends a result in the same spirit for the Laplacian ([1] Theorem 4.15).

More generally, we show that V is a Kato (resp. a local Kato) potential provided that $\{V_y; y \in \mathbb{R}^d\}$ (resp. $\{V_y; |y| \leq C\}$, $C > 0$) is an equi-integrable subset of $L^1(B(0, 1); \mu(dz))$; (see Theorem 16). It follows that V is a Kato (resp. a local Kato) potential provided there exists $p > 1$ such that

$$\sup_{y \in \mathbb{R}^d} \int_{|z| \leq 1} V_y(z)^p E_\lambda(z) dz < +\infty \text{ (resp. } \sup_{|y| \leq C} \int_{|z| \leq 1} V_y(z)^p E_\lambda(z) dz < +\infty, C > 0);$$

(see Corollary 17).

We give other membership criteria to Kato classes in terms of asymptotics of averages over suitable spherical shells. When $E_\lambda(\cdot)$ is radially decreasing, i.e.

$$E_\lambda(z) = \widehat{E}_\lambda(|z|) \text{ and } \rho \rightarrow \widehat{E}_\lambda(\rho) \text{ is nonincreasing,}$$

we check that V is a Kato potential in terms of asymptotics ($k \rightarrow \infty$) of

$$\eta_k(y) := \int_{2^{-(k+1)} \leq |z| \leq 2^{-k}} V_y(z) dz, \quad (y \in \mathbb{R}^d, k \in \mathbb{N}).$$

Indeed, we show that V is a Kato potential *if and only if* the series

$$\sum_{j=1}^{\infty} \left(\widehat{E}_\lambda(2^{-j}) \right) \eta_j(y)$$

converges uniformly in $y \in \mathbb{R}^d$, in particular when

$$\limsup_{k \rightarrow \infty} (\widehat{\eta}_k)^{\frac{1}{k}} < \left(\limsup_{k \rightarrow \infty} \left(\widehat{E}_\lambda(2^{-k}) \right)^{\frac{1}{k}} \right)^{-1}$$

where

$$\widehat{\eta}_k := \sup_{y \in \mathbb{R}^d} \eta_k(y);$$

(see Theorem 20).

The parameter $\limsup_{k \rightarrow \infty} \left(\widehat{E}_\lambda(2^{-k}) \right)^{\frac{1}{k}}$ is also estimated

$$\limsup_{k \rightarrow \infty} \left(\widehat{E}_\lambda(2^{-k}) \right)^{\frac{1}{k}} \leq 2^{\frac{d}{s}}$$

where

$$s := \sup \left\{ p \geq 1; \int_{\{|x| \leq 1\}} (E_\lambda(x))^p dx < +\infty \right\};$$

(see Lemma 21). It follows that V is a Kato class potential provided that

$$\limsup_{j \rightarrow \infty} (\widehat{\eta}_j)^{\frac{1}{j}} < 2^{-\frac{d}{s}}; \quad (12)$$

(see Corollary 23). In the usual examples,

$$\widehat{E}_\lambda(\rho) \sim \frac{1}{\rho^{d-\alpha}} \quad (\rho \rightarrow 0) \quad (0 < \alpha \leq 2); \quad (13)$$

($\alpha = 2$ for the heat semigroup, $0 < \alpha < 2$ for the α -stable semigroup and $\alpha = 1$ for the relativistic semigroup) so V is a Kato class potential provided that

$$\limsup_{k \rightarrow \infty} (\widehat{\eta}_k)^{\frac{1}{k}} < 2^{-(d-\alpha)};$$

(see Remark 24).

It is well known ([1] Theorem 1.4 (iii)) that for the Laplacian, V is a Kato class potential provided that $V \in L_{loc,unif}^p$, i.e.

$$\sup_{y \in \mathbb{R}^d} \int_{\{|x| \leq 1\}} (V(y+x))^p dx < +\infty,$$

for some $p > \frac{d}{2}$. Of course, this result can also be formulated, with $p > \frac{d}{\alpha}$, for general generators satisfying (13). We can derive this last result from the membership criterion (12); (see Theorem 25). We can derive it also from the $\mu(dz)$ -equi-integrability criterion given in Theorem 16; (see Remark 26).

As far as we know, our results are new and appear here for the first time. Finally, we mention that a part of this work extends to higher-order elliptic systems where the loss of positivity is compensated by the existence of suitable kernel estimates [21].

2 Reminders on absorption semigroups

Before giving our results, we recall some facts on the theory of absorption semigroups [31]. (Other properties of absorption semigroups are given in [20] but we do not need them here.) Let $(S^p(t))_{t \geq 0}$ be a positive contraction C_0 -semigroup on some $L^p(\Omega; \mu)$ space with generator T^p and let $V : \Omega \rightarrow \mathbb{R}$ be measurable. We assume that $V = V_+ - V_-$ is decomposed as a difference of two nonnegative measurable functions V_{\pm} (not necessarily the positive and negative parts of V) such that $V_{\pm}(x) < +\infty$ μ -a.e. To define " $T^p - V_+$ " as a generator (for the time being we are *not* interested in " $T^p - V$ "), we approximate V_+ monotonically from below by $V_+ \wedge j$ ($j \in \mathbb{N}$); the corresponding sequence of semigroups converges strongly to a semigroup $(S_{V_+}^p(t))_{t \geq 0}$ which need not be strongly continuous at $t = 0$. We say that V_+ is admissible if $(S_{V_+}^p(t))_{t \geq 0}$ is strongly continuous and denote by $T_{V_+}^p$ its generator. In this case,

$$T_{V_+}^p \supset T^p - V_+$$

([31] Cor 2.7); this occurs e.g. if

$$D(T^p) \cap D(V_+) \text{ is dense in } L^p(\mu)$$

([31] Prop 2.9). Note that if $p = 1$, if $(S^1(t))_{t \geq 0}$ is mass preserving and if $D(T^1) \cap D(V_+)$ is a core for T^1 then $T_{V_+}^1 = T^1 - V_+$ ([31] Cor 4.3 and Prop 4.4). If $(S(t))_{t \geq 0}$ is a symmetric sub-Markov semigroup, i.e. acts in all $L^p(\mu)$ spaces as a positive contraction C_0 -semigroup $(S^p(t))_{t \geq 0}$ with generator T^p and $(S^2(t))_{t \geq 0}$ is self-adjoint then the admissibility of V_+ is p -independent, the dual of $(S_{V_+}^p(t))_{t \geq 0}$ is equal to $(S_{V_+}^q(t))_{t \geq 0}$ (q is the conjugate exponent) and $(S_{V_+}^2(t))_{t \geq 0}$ is self-adjoint ([31] Prop 3.2). Moreover,

$$S_{V_+}^t(t)|_{L^p \cap L^s} = S_{V_+}^s(t)|_{L^p \cap L^s}$$

([31] Prop 3.1). To avoid cumbersome notations, we write $(S_+^p(t))_{t \geq 0}$ instead of $(S_{V_+}^p(t))_{t \geq 0}$ and T_+^p instead of $T_{V_+}^p$. We consider now symmetric convolution semigroups

$$S^p(t) : f \in L^p(\mathbb{R}^d) \rightarrow \int_{\mathbb{R}^d} f(x-y) m_t(dy) \in L^p(\mathbb{R}^d)$$

where $\{m_t\}_{t \geq 0}$ are (symmetric with respect to the origin) Borel *sub*-probability measures on \mathbb{R}^d such that $m_0 = \delta_0$ (Dirac measure at zero), $m_t * m_s = m_{t+s}$

and $m_t \rightarrow m_0$ vaguely as $t \rightarrow 0_+$. Such convolution semigroups are related to Lévy processes and cover many examples of practical interest such as Gaussian semigroups, α -stable semigroups, relativistic Schrödinger semigroups etc. Note that $(S^p(t))_{t \geq 0}$ is a positive contraction C_0 -semigroup on $L^p(\mathbb{R}^d)$ ($1 \leq p < +\infty$) with generator

$$T^p : D(T^p) \subset L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d).$$

The sub-probability measures $\{m_t\}_{t \geq 0}$ are characterized by

$$\widehat{m}_t(\zeta) := (2\pi)^{-\frac{d}{2}} \int e^{-i\zeta \cdot x} m_t(dx) = (2\pi)^{-\frac{d}{2}} e^{-tF(\zeta)}, \quad \zeta \in \mathbb{R}^d$$

where F , the so-called characteristic exponent, is a continuous negative definite function (see [12] Definition 3.6.5, p. 122) and has the representation

$$F(\zeta) = c + \zeta \cdot C \zeta + \int_{\mathbb{R}^d \setminus \{0\}} [1 - \cos(x \cdot \zeta)] \mu(dx)$$

with $c \geq 0$, C is a real symmetric matrix such that $\zeta \cdot C \zeta \geq 0 \quad \forall \zeta \in \mathbb{R}^d$ and μ , the so-called Lévy measure, is a positive (symmetric with respect to the origin) Borel measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int \min(1, |x|^2) \mu(dx) < +\infty$. Note that $\{m_t\}_{t \geq 0}$ are probability measures if $F(0) = 0$, i.e. $c = 0$. We recall that $F \geq 0$ and $F(\zeta) \leq c_F(1 + |\zeta|^2)$. Note that

$$T^2 \varphi = -(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\zeta \cdot x} F(\zeta) \widehat{\varphi}(\zeta) d\zeta$$

with domain

$$D(T^2) = \left\{ \varphi \in L^2(\mathbb{R}^d); F \widehat{\varphi} \in L^2(\mathbb{R}^d) \right\}.$$

We refer e.g. to [11][12] for more information on convolution semigroups. Finally, we recall that $\mathcal{S}(\mathbb{R}^d)$ is a core of T^p ([6] Thm 2.1.15, p. 38). In particular, if $V_+ \in L^1_{loc}(\mathbb{R}^d)$ then $C_c^\infty(\mathbb{R}^d) \subset D(T_1) \cap D(V_+)$ so V_+ is admissible with respect to $\{S^1(t)\}_{t \geq 0}$ and therefore with respect to $\{S^p(t)\}_{t \geq 0}$ for all $p \geq 1$. Note finally that if $m_t(dy)$ is a probability measure then $\{S^1(t)\}_{t \geq 0}$ is mass preserving. It follows that if $\mathcal{S}(\mathbb{R}^d) \subset D(V_+)$ then $D(T_1) \cap D(V_+)$ is a core of T^1 and $T^1_{V_+} = T^1 - V_+$.

Note first that $(\lambda - T^1)^{-1}$ is a convolution operator

$$(\lambda - T^1)^{-1} f = \int_{\mathbb{R}^d} f(x - y) m^\lambda(dy)$$

where $m^\lambda = \int_0^{+\infty} e^{-\lambda t} m_t dt$ ($\lambda > 0$) is a vaguely convergent integral, i.e.

$$\int_{\mathbb{R}^d} f(x) m^\lambda(dx) := \int_0^{+\infty} e^{-\lambda t} \left[\int_{\mathbb{R}^d} f(x) m_t(dx) \right] dt, \quad f \in C_0(\mathbb{R}^d).$$

Our *general* assumption here is that m^λ is absolutely continuous with respect to Lebesgue measure, i.e.

$$m^\lambda(dx) = E_\lambda(x) dx, \quad E_\lambda(\cdot) \in L^1(\mathbb{R}^d) \quad (14)$$

so that

$$(\lambda - T^1)^{-1} f = \int_{\mathbb{R}^d} E_\lambda(x - y) f(y) dy, \quad f \in L^1(\mathbb{R}^d) \quad (15)$$

and $E_\lambda(\cdot)$ is called the λ -potential kernel according to the terminology in [4]. We recall that for the usual examples we have

$$\int_{\mathbb{R}^d} e^{-tF(\zeta)} d\zeta < +\infty \quad (t > 0) \quad (16)$$

so that the bounded measure m_t ($t > 0$) is absolutely continuous with respect to Lebesgue measure, i.e.

$$m_t(dx) = k_t(x) dx \quad (t > 0) \quad (17)$$

where $k_t \in L^1_+(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ is even and

$$E_\lambda(z) = \int_0^{+\infty} e^{-\lambda t} k_t(z) dt \quad (\lambda > 0).$$

Note however that we do not assume (16) in this paper.

3 Form-bounds for one-body Hamiltonians

As noted in the previous section, if

$$V_+ \in L^1_{loc}(\mathbb{R}^d) \quad (18)$$

then we can define an absorption convolution semigroup $\{S_+^p(t)\}_{t \geq 0}$ on $L^p(\mathbb{R}^d)$ with generator T_+^p . We assume that V_- is T_1 -bounded in $L^1(\mathbb{R}^d)$ i.e.

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} V_-(x) E_\lambda(x - y) dx < +\infty \quad (\lambda > 0) \quad (19)$$

where $E_\lambda(x-y)$ is the kernel of $(\lambda - T_1)^{-1}$. Note that (19) is λ -independent. Of course, (19) implies that for any $\delta > 0$

$$\sup_{y \in \mathbb{R}^d} \int_{\{|x-y| \leq \delta\}} V_-(x) E_\lambda(x-y) dx < +\infty \quad (\lambda > 0). \quad (20)$$

According to Desch's theorem [5] (see also [32] or [16] Chapter 8), if

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_1)^{-1}] < 1$$

then $A^1 := T_+^1 + V_-$ with domain $D(A^1) = D(T_+^1)$ generates a positive C_0 -semigroup $\{W^1(t)\}_{t \geq 0}$ on $L^1(\mathbb{R}^d)$. Moreover, $W^1(t)$ maps $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ into itself and, for any $p > 1$,

$$W^1(t) : L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$$

extends uniquely to a C_0 -semigroup $\{W^p(t)\}_{t \geq 0}$ on $L^p(\mathbb{R}^d)$; see [18]. If we denote by A^p its generator then

$$\left\{ f \in D(T_+^1) \cap L^p(\mathbb{R}^d); T_+^1 f + V_- f \in L^p(\mathbb{R}^d) \right\}$$

is a core of A^p and A^2 is self-adjoint; see [18] for the details. The following result is already given in ([18] Theorem 21) under the assumption that $V_- \in L_{Loc}^2$. We aim here at removing this L_{Loc}^2 assumption by using an approximation argument combined to suitable a priori spectral estimates.

Theorem 1 *Let $\delta := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_+^1)^{-1}] < 1$ and let (14)(18)(19) be satisfied. Then V_- is form-bounded with respect to $-T_+^2$ in $L^2(\mathbb{R}^d)$ with relative form-bound less than or equal to δ .*

Proof. Let

$$s(A^1) = \sup \{ \operatorname{Re} \lambda; \lambda \in \sigma(A^1) \}$$

be the spectral bound of A^1 . We recall that the type of a positive semigroup in L^p spaces coincides with the spectral bound of its generator, see e.g. [33]. We introduce now a nondecreasing sequence of bounded potentials $\{V_-^n\}_n$ converging pointwisely to V_- , e.g. we can choose $V_-^n = V_- \wedge n$. We note the uniform bound

$$r_\sigma [V_-^n(\lambda - T_+^1)^{-1}] \leq r_\sigma [V_-(\lambda - T_+^1)^{-1}] \quad \forall n.$$

Similarly,

$$A_n^1 = T_+^1 + V_-^n : D(T_+^1) \subset L^1(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d)$$

is a generator of a positive C_0 -semigroup $\{W_n^1(t)\}_{t \geq 0}$ in $L^1(\mathbb{R}^d)$. Moreover, $V_-^n \leq V_-$ implies $W_n^1(t) \leq W^1(t)$ and then $s(A_n^1) \leq s(A^1) \quad \forall n$. By a symmetry argument, $\{W_n^1(t)\}_{t \geq 0}$ interpolates on all $L^p(\mathbb{R}^d)$ providing C_0 -semigroups $\{W_n^p(t)\}_{t \geq 0}$ in $L^p(\mathbb{R}^d)$ with generator A_n^p where A_n^2 is self-adjoint in $L^2(\mathbb{R}^d)$. Actually, since V_-^n is bounded A_n^2 is nothing but

$$T_+^2 + V_-^n : D(T_+^2) \rightarrow L^2(\mathbb{R}^d).$$

On the other hand, since $s(A_n^1)$ is the type of $\{W_n^1(t)\}_{t \geq 0}$ then for every $\varepsilon > 0$ there exists C_ε such that

$$\|W_n^1(t)\|_{\mathcal{L}(L^1(\mathbb{R}^3))} \leq C_\varepsilon e^{(s(A_n^1) + \varepsilon)t}$$

$$\|(W_n^1(t))'\|_{\mathcal{L}(L^\infty(\mathbb{R}^3))} \leq C_\varepsilon e^{(s(A_n^1) + \varepsilon)t}$$

where $(W_n^1(t))'$ is the dual semigroup on $L^\infty(\mathbb{R}^d)$. Riesz-Thorin's interpolation theorem implies

$$\|W_n^p(t)\|_{\mathcal{L}(L^2(\mathbb{R}^3))} \leq C_\varepsilon e^{(s(A_n^1) + \varepsilon)t}$$

showing thus that $s(A_n^2) \leq s(A_n^1)$ since $\varepsilon > 0$ is arbitrary. Finally we get the uniform estimate

$$s(A_n^2) \leq s(A^1) \quad \forall n.$$

Since V_-^n is bounded then

$$(A_n^{2,c} \varphi, \varphi) \leq s(A^1) \|\varphi\|_{L^2(\mathbb{R}^d)}^2 \quad \forall \varphi \in D(T_+^2) \quad \forall n.$$

If we choose c arbitrarily such that $1 < c < \frac{1}{\delta}$ then

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [cV_-(\lambda - T_1)^{-1}] = c\delta < 1$$

and then, arguing as previously,

$$A_n^{2,c} := T_+^2 + cV_-^n : D(T_+^2) \rightarrow L^2(\mathbb{R}^d)$$

is self-adjoint and

$$(A_n^{2,c} \varphi, \varphi) \leq s(A_1^c) \|\varphi\|_{L^2(\mathbb{R}^d)}^2 \quad \forall \varphi \in D(T_+^2) \quad \forall n$$

where $s(A_1^c)$ denotes the spectral bound of

$$A_1^c = T_+^1 + cV : D(T_1) \subset L^1(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d).$$

Thus

$$(A_n^{2,c}\varphi, \varphi) = (T_+^2\varphi + cV_-^n\varphi, \varphi) = -\left\|\sqrt{-T_+^2}\varphi\right\|^2 + c\int V_-^n|\varphi|^2 dx$$

so that, by the density of $D(T_+^2)$ in $D(\sqrt{-T_+^2})$ (for the graph norm of $\sqrt{-T_+^2}$),

$$c\int V_-^n|\varphi|^2 dx \leq \left\|\sqrt{-T_+^2}\varphi\right\|^2 + s(A_1^c)\|\varphi\|_{L^2(\mathbb{R}^d)}^2 \quad \forall n, \forall \varphi \in D(\sqrt{-T_+^2}).$$

Letting $n \rightarrow \infty$ we get

$$\int V_-|\varphi|^2 dx \leq c^{-1}\left\|\sqrt{-T_+^2}\varphi\right\|^2 + c^{-1}s(A_1^c)\|\varphi\|_{L^2(\mathbb{R}^d)}^2 \quad \forall \varphi \in D(\sqrt{-T_+^2}).$$

This ends the proof since c^{-1} can be chosen as close to δ as we want. \blacksquare

Corollary 2 *If V_- is T_+^1 -bounded in $L^1(\mathbb{R}^d)$ then V_- is form-bounded with respect to $-T_+^2$ in $L^2(\mathbb{R}^d)$.*

Proof. The limit $\delta := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_+^1)^{-1}]$ always exists. Let $\varepsilon > 0$ and $V_-^{\delta+\varepsilon} = (\delta + \varepsilon)^{-1}V_-$. Then

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-^{\delta+\varepsilon}(\lambda - T_+^1)^{-1}] = (\delta + \varepsilon)^{-1}\delta < 1$$

so by theorem 1 $(\delta + \varepsilon)^{-1}V_-$ is form-bounded with respect to $-T_+^2$ in $L^2(\mathbb{R}^d)$ with relative form-bound less than or equal to $(\delta + \varepsilon)^{-1}\delta$. \blacksquare

According to Theorem 1, we can define

$$(-T_+^2) \oplus (-V_-)$$

(a form-sum operator) via the KLMN theorem (see e.g. [30] Theorem 6. 24, p. 150). A very natural conjecture is that this operator coincides with the self-adjoint operator $-A^2$ where A^2 was obtained previously by interpolation arguments from A^1 . Indeed, this is the case.

Theorem 3 *Let $\delta := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_+^1)^{-1}] < 1$ and let (14)(18)(19) be satisfied. Then $-A^2$ is equal to $(-T_+^2) \oplus (-V_-)$.*

Proof. The first observation is

$$(\lambda - T_+^1 - V_-^n)^{-1} \rightarrow (\lambda - T_+^1 - V_-)^{-1} \quad (n \rightarrow +\infty)$$

strongly in $L^1(\mathbb{R}^d)$. Indeed, let $\varepsilon > 0$ and let λ be large enough so that

$$r_\sigma [V_-(\lambda - T_+^1)^{-1}] < \delta + \varepsilon < 1.$$

Then $r_\sigma [V_-^n(\lambda - T_+^1)^{-1}] \leq r_\sigma [V_-(\lambda - T_+^1)^{-1}] \forall n$ and for all $\varphi \in L_+^1(\mathbb{R}^d)$

$$\begin{aligned} (\lambda - T_+^1 - V_-^n)^{-1}\varphi &= (\lambda - T_+^1)^{-1} \sum_{j=0}^{\infty} [V_-^n(\lambda - T_+^1)^{-1}]^j \varphi \\ &\rightarrow (\lambda - T_+^1)^{-1} \sum_{j=0}^{\infty} [V_-(\lambda - T_+^1)^{-1}]^j \varphi = (\lambda - T_+^1 - V_-)^{-1}\varphi \end{aligned}$$

by the monotone convergence theorem and we are done since $L_+^1(\mathbb{R}^d)$ is generating. It follows by Riesz-Thorin's interpolation theorem that

$$(\lambda - A_n^2)^{-1} \rightarrow (\lambda - A^2)^{-1} \text{ strongly in } L^2(\mathbb{R}^d). \quad (21)$$

On the other hand, since V_-^n is a bounded operator then $-A_n^2$ is also the form-sum $-A_n^2 = (-T_+^2) \oplus (-V_-^n)$. A key point is that the resolvent of the form-sum operator $(-T_+^2) \oplus (-V_-^n)$ is given by

$$(\lambda + (-T_+^2) \oplus (-V_-^n))^{-1} = (\lambda - T_+^2)^{-\frac{1}{2}} (I - C_n(\lambda))^{-1} (\lambda - T_+^2)^{-\frac{1}{2}}$$

(see [30] Theorem 6.25, p. 150) where $C_n(\lambda)$ is the positive bounded self-adjoint operator on $L^2(\mathbb{R}^d)$ defined by the positive bounded quadratic form

$$\varphi \in L^2(\mathbb{R}^3) \rightarrow \int_{\mathbb{R}^d} V_-^n \left| (\lambda - T_+^2)^{-\frac{1}{2}} \varphi \right|^2$$

with $\|C_n(\lambda)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq c^{-1}$ for λ large enough (see [30] Theorem 6.25, p. 150). Similarly, the resolvent of the form-sum operator $(-T_+^2) \oplus (-V_-)$ is given by

$$(\lambda + (-T_+^2) \oplus (-V_-))^{-1} = (\lambda - T_+^2)^{-\frac{1}{2}} (I - C(\lambda))^{-1} (\lambda - T_+^2)^{-\frac{1}{2}}$$

where $C(\lambda)$ is the positive bounded self-adjoint operator on $L^2(\mathbb{R}^d)$ defined by the positive bounded quadratic form

$$\varphi \in L^2(\mathbb{R}^3) \rightarrow \int_{\mathbb{R}^d} V_- \left| (\lambda - T_+^2)^{-\frac{1}{2}} \varphi \right|^2$$

and $\|C(\lambda)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq c^{-1} < 1$ for λ large enough (see [30] Theorem 6.25, p. 150). The monotonic convergence of the quadratic forms

$$(C_n(\lambda)\varphi, \varphi) = \int_{\mathbb{R}^d} V_-^n \left| (\lambda - T_+^2)^{-\frac{1}{2}} \varphi \right|^2 \rightarrow (C(\lambda)\varphi, \varphi) = \int_{\mathbb{R}^d} V_- \left| (\lambda - T_+^2)^{-\frac{1}{2}} \varphi \right|^2$$

implies the strong convergence $(I - C_n(\lambda))^{-1} \rightarrow (I - C(\lambda))^{-1}$ ($n \rightarrow +\infty$) (see e.g. [25] Theorem S. 14, p. 373) and finally the strong convergence

$$(\lambda - T_+^2)^{-\frac{1}{2}} (I - C_n(\lambda))^{-1} (\lambda - T_+^2)^{-\frac{1}{2}} \rightarrow (\lambda - T_+^2)^{-\frac{1}{2}} (I - C(\lambda))^{-1} (\lambda - T_+^2)^{-\frac{1}{2}}$$

($n \rightarrow +\infty$) which shows the equality

$$(\lambda - A_2)^{-1} = (\lambda + (-T_+^2) \oplus (-V_-))^{-1}$$

i.e. $-A_2 = (-T_+^2) \oplus (-V_-)$. ■

4 Form-bounds for many-body Hamiltonians

We show now how to define (form-sum) Hamiltonians of the form

$$\left(-\sum_{i=1}^N T_{(i)} \right) \oplus \left(-\sum_{i=1}^N V_-^i(x_i) - \sum_{i<j} V_-^{ij}(x_i - x_j) \right)$$

where, for each i ($1 \leq i \leq N$), $T_{(i)}$ acts on the variable $x_i \in \mathbb{R}^3$ only as a generator of a symmetric convolution semigroup depending a priori on the index i . Thus, we consider a family of symmetric convolution semigroups on $L^2(\mathbb{R}^3)$ indexed by an integer j ($1 \leq j \leq N$) ($N \geq 2$)

$$S_j^2(t) : f \in L^2(\mathbb{R}^3) \rightarrow \int_{\mathbb{R}^3} f(x-y) m_t^j(dy) \in L^2(\mathbb{R}^3).$$

Let T_j^2 be the generator of $\left(S_j^2(t) \right)_{t \geq 0}$ and let F_j be the corresponding characteristic exponent. (We denote by $\left(S_j^1(t) \right)_{t \geq 0}$ its realization $L^1(\mathbb{R}^3)$ and denote by T_j^1 its generator.)

On $L^2((\mathbb{R}^3)^N)$, we define

$$\left\{ \begin{array}{l} T_{(j)}\varphi = -(2\pi)^{-\frac{3N}{2}} \int_{\mathbb{R}^{3N}} e^{i\zeta \cdot x} F_j(\zeta_j) \widehat{\varphi}(\zeta) d\zeta \\ D(T_{(j)}) = \{ \varphi \in L^2((\mathbb{R}^3)^N); F_j(\zeta_j) \widehat{\varphi}(\zeta) \in L^2((\mathbb{R}^3)^N) \} \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \mathcal{T}\varphi = -(2\pi)^{-\frac{3N}{2}} \int_{\mathbb{R}^{3N}} e^{i\zeta \cdot x} \tilde{F}(\zeta) \widehat{\varphi}(\zeta) d\zeta \\ D(\mathcal{T}) = \left\{ \varphi \in L^2((\mathbb{R}^3)^N); \tilde{F}(\zeta) \widehat{\varphi}(\zeta) \in L^2((\mathbb{R}^3)^N) \right\} \end{array} \right\} \quad (22)$$

where

$$\tilde{F}(\zeta) = \sum_{j=1}^N F_j(\zeta_j)$$

(the ζ_j 's are the component of $\zeta \in (\mathbb{R}^3)^N$). Note that \tilde{F} is also a continuous negative definite function on $(\mathbb{R}^3)^N$ ([12] Lemma 3.6.7, p. 123). Let

$$V^i, V^{ij} : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad (i, j \leq N)$$

be measurable and $|V^i(z)| + |V^{ij}(z)| < +\infty$ a.e. Let

$$V^i = V_+^i - V_-^i, \quad V^{ij} = V_+^{ij} - V_-^{ij}$$

be decompositions into differences of nonnegative functions (which need not be the standard positive and negative parts) and let

$$\mathcal{V}_-(x_1, \dots, x_N) := \sum_{i=1}^N V_-^i(x_i) + \sum_{i < j} V_-^{ij}(x_i - x_j). \quad (23)$$

We are ready to state our main (abstract) form-perturbation result.

Theorem 4 *Let (14) be satisfied by all T_i^1 . We assume that V_-^i and V_-^{ij} are T_i^1 -bounded in $L^1(\mathbb{R}^3)$. Let*

$$\delta_i := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-^i(\lambda - T_i^1)^{-1}], \quad \delta_{ij} := \lim_{\lambda \rightarrow +\infty} r_\sigma [V_-^{ij}(\lambda - T_i^1)^{-1}]$$

(where r_σ refers to spectral radius of bounded operators on $L^1(\mathbb{R}^3)$) and let $\widehat{\delta}_i = \sum_{j=i+1}^N \delta_{ij}$ ($i \leq N-1$). If

$$\delta := \max_{1 \leq i \leq N} (\delta_i) + \max_{1 \leq i \leq N-1} (\widehat{\delta}_i) < 1$$

then the multiplication operator on $L^2(\mathbb{R}^{3N})$ by the potential $-\mathcal{V}_-$ is form-bounded with respect to the positive self-adjoint operator $-\mathcal{T}$ with relative form bound less than or equal to δ .

Proof. According to Theorem 1, V_-^i is form-bounded (with respect to T_i^2) with relative form-bound less than or equal to δ_i . Thus, for each $\varepsilon > 0$ there exists $c_\varepsilon^i > 0$ such that (for $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N$ fixed) and for all $f \in C_c^\infty(\mathbb{R}^{3N})$

$$\begin{aligned} & \int_{\mathbb{R}^3} V_-^i(x_i) |f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N)|^2 dx_i \\ & \leq (\delta_i + \varepsilon) \int_{\mathbb{R}^3} \left| \sqrt{-T_i^2} f \right|^2 dx_i + c_\varepsilon^i \int_{\mathbb{R}^3} |f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N)|^2 dx_i \end{aligned}$$

so that integrating with respect to the remaining variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N$

$$\begin{aligned} \int_{\mathbb{R}^{3N}} V_-^i |f|^2 dx & \leq (\delta_i + \varepsilon) \left\| \sqrt{-T_{(i)}} f \right\|_{L^2(\mathbb{R}^{3N})}^2 + c_\varepsilon^i \int_{\mathbb{R}^{3N}} |f|^2 dx \\ & = (\delta_i + \varepsilon) \int_{\mathbb{R}^{3N}} F_i(\zeta_i) \left| \widehat{f}(\zeta) \right|^2 d\zeta + c_\varepsilon^i \int_{\mathbb{R}^{3N}} |f|^2 dx \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^N \int_{\mathbb{R}^{3N}} V_-^i |f|^2 dx & \leq (\max_i \delta_i + \varepsilon) \int_{\mathbb{R}^{3N}} \widetilde{F}(\zeta) \left| \widehat{f}(\zeta) \right|^2 d\zeta + \left(\sum_{i=1}^N c_\varepsilon^i \right) \|f\|_{L^2(\mathbb{R}^{3N})}^2 \\ & = (\max_i \delta_i + \varepsilon) \int_{\mathbb{R}^{3N}} \left| \sqrt{-T} f \right|^2 dx + \left(\sum_{i=1}^N c_\varepsilon^i \right) \|f\|_{L^2(\mathbb{R}^{3N})}^2. \end{aligned}$$

Similarly, V_-^{ij} is form-bounded (with respect to T_i^2) with relative form-bound less than or equal to δ_{ij} . Thus, for each $\varepsilon > 0$ there exists $c_\varepsilon^{ij} > 0$ such that (for $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N$ fixed)

$$\begin{aligned} & \int_{\mathbb{R}^3} V_{ij}^-(z) |f(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_N)|^2 dz \\ & \leq (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^3} \left| \sqrt{-T_i^2} f \right|^2 dz + c_\varepsilon^{ij} \int_{\mathbb{R}^3} |f(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_N)|^2 dz \end{aligned}$$

whence

$$\begin{aligned} & \int_{\mathbb{R}^3} V_{ij}^-(x_i - x_j) |f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N)|^2 dx_i \tag{24} \\ & = \int_{\mathbb{R}^3} V_{ij}^-(z) |f(x_1, \dots, x_{i-1}, z + x_j, x_{i+1}, \dots, x_N)|^2 dz \\ & \leq (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^3} \left| \sqrt{-T_i^2} f_{x_j} \right|^2 dz + c_\varepsilon^{ij} \int_{\mathbb{R}^3} |f_{x_j}(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_N)|^2 dz \\ & = (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^3} \left| \sqrt{-T_i^2} f \right|^2 dz + c_\varepsilon^{ij} \int_{\mathbb{R}^3} |f(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_N)|^2 dz \end{aligned}$$

(where $f_{x_j} : z \rightarrow f(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_N)$ is the *translation* by x_j) since the quadratic form is *invariant* by translation. By integrating (24) with respect to the remaining variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N$ we get

$$\begin{aligned} \int_{\mathbb{R}^{3N}} V_{ij}^- |f|^2 dx &\leq (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^{3N}} \left| \sqrt{-T_{(i)}} f \right|^2 dx + c_\varepsilon^{ij} \|f\|_{L^2(\mathbb{R}^{3N})}^2 \\ &= (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^{3N}} F_i(\zeta_i) \left| \widehat{f}(\zeta) \right|^2 d\zeta + c_\varepsilon^{ij} \int_{\mathbb{R}^{3N}} |f|^2 dx \end{aligned}$$

and

$$\begin{aligned} &\sum_{i < j} \int_{\mathbb{R}^{3N}} V_{ij}^- |f|^2 dx \\ &\leq \sup_i \sum_{j=i+1}^N (\delta_{ij} + \varepsilon) \int_{\mathbb{R}^{3N}} \widetilde{F}(\zeta) \left| \widehat{f}(\zeta) \right|^2 d\zeta + \left(\sum_{i < j} c_\varepsilon^{ij} \right) \int_{\mathbb{R}^{3N}} |f|^2 dx \\ &= \sup_i \sum_{j=i+1}^N (\delta_{ij} + \varepsilon) \left\| \sqrt{-T} f \right\|_{L^2(\mathbb{R}^{3N})}^2 + \left(\sum_{i < j} c_\varepsilon^{ij} \right) \|f\|_{L^2(\mathbb{R}^{3N})}^2. \end{aligned}$$

Finally

$$\begin{aligned} &\sum_{i=1}^N \int_{\mathbb{R}^{3N}} V_i^- |f|^2 dx + \sum_{i < j} \int_{\mathbb{R}^{3N}} V_{ij}^- |f|^2 dx \\ &\leq \left[\left(\max_i (\delta_i) + \varepsilon \right) + \sup_i \sum_{j=i+1}^N (\delta_{ij} + \varepsilon) \right] \left\| \sqrt{-T} f \right\|_{L^2(\mathbb{R}^{3N})}^2 \\ &\quad + \left[\sum_{i=1}^N c_\varepsilon^i + \sum_{i < j} c_\varepsilon^{ij} \right] \|f\|_{L^2(\mathbb{R}^{3N})}^2 \end{aligned}$$

which ends the proof since $\varepsilon > 0$ is arbitrary. ■

Remark 5 By using Corollary 2, one sees that once the V_i^- 's and the V_{ij}^- 's are T_i^1 -bounded in $L^1(\mathbb{R}^3)$ then $-\delta^{-1}\mathcal{V}_-$ is form-bounded with respect to the positive self-adjoint operator $-\mathcal{T}$ with relative form bound < 1 where δ is given by $\max_{1 \leq i \leq N} (\delta_i) + \max_{1 \leq i < j \leq N-1} (\widehat{\delta}_i)$.

Remark 6 Note that $C_c^\infty(\mathbb{R}^{3N})$ and $\mathcal{S}(\mathbb{R}^{3N})$ are cores of $-\mathcal{T}$ (see e.g. [18] Theorem 2 (iii)). Since $D(-\mathcal{T})$ is a also core of $\sqrt{-\mathcal{T}}$ then $C_c^\infty(\mathbb{R}^{3N})$ and $\mathcal{S}(\mathbb{R}^{3N})$ are cores of $\sqrt{-\mathcal{T}}$, i.e. $C_c^\infty(\mathbb{R}^{3N})$ and $\mathcal{S}(\mathbb{R}^{3N})$ are form-cores of $-\mathcal{T}$. It follows from ([25] Theorem X.17, p. 167) that $(-\mathcal{T}) \oplus (-\mathcal{V}_-)$ admits $C_c^\infty(\mathbb{R}^{3N})$ and $\mathcal{S}(\mathbb{R}^{3N})$ as form-cores.

5 The Kato classes revisited

In this section, we explore various new aspects of Kato classes. Let $\{S^1(t)\}_{t \geq 0}$ be a convolution semigroup on $L^1(\mathbb{R}^d)$ with generator T^1 satisfying (14) and let

$$V : \mathbb{R}^d \rightarrow \mathbb{R}_+$$

be T^1 -bounded in $L^1(\mathbb{R}^d)$. According to (20)

$$\sup_{y \in \mathbb{R}^d} \int_{\{|z| \leq \varepsilon\}} V(y+x) E_\lambda(x) dx < +\infty \quad (\varepsilon \geq 0).$$

Definition 7 *We say that V is a Kato class potential relative to T^1 if*

$$\left\| V (\lambda - T^1)^{-1} \right\|_{\mathcal{L}(L^1(\mathbb{R}^d))} \rightarrow 0 \quad (\lambda \rightarrow +\infty).$$

5.1 On membership to Kato classes

We start with a known characterization of the Kato class for a *suitable class* of convolution semigroups.

Theorem 8 ([4] *Thm III1*). *Let (16) be satisfied and let $k_t(x)$ (given in (17)) be radial and non-increasing in $|x|$. Then V is a Kato class potential relative to T^1 if and only if*

$$\lim_{\varepsilon \rightarrow 0} \sup_{y \in \mathbb{R}^d} \int_{\{|z| \leq \varepsilon\}} V(y+x) E_\lambda(x) dx = 0. \quad (25)$$

We show now that (25) is a *sufficient criterion* of membership to the Kato class for *more general* convolution semigroups provided we a priori restrict slightly the class of potentials.

Theorem 9 *Let (14) be satisfied.*

(i) *Let $E_\lambda(\cdot)$ be bounded outside any neighborhood of the origin and let $V \in \sum_{j=1}^{j=m} L^{p_j}(\mathbb{R}^d)$ where $p_j \in (1, +\infty]$. Then V is a Kato class potential relative to T^1 provided that (25) is satisfied for some $\lambda > 0$.*

(ii) *Let $\lim_{\lambda \rightarrow +\infty} \sup_{|x| \geq \varepsilon} E_\lambda(x) = 0$ ($\varepsilon > 0$) and $V \in \sum_{j=1}^{j=m} L^{p_j}(\mathbb{R}^d)$ where $p_j \in [1, +\infty]$. Then V is a Kato class potential relative to T^1 provided that (25) is satisfied for some $\lambda > 0$.*

(iii) *If (16) is satisfied and if $k_t(x)$ (given in (17)) is radial and non-increasing in $|x|$ then $\lim_{\lambda \rightarrow +\infty} \sup_{|x| \geq \varepsilon} E_\lambda(x) = 0$ ($\varepsilon > 0$).*

Proof. Note first that

$$\left\| V(\lambda - T^1)^{-1} \right\|_{\mathcal{L}(L^1(\mathbb{R}^d))} = \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} V(x) E_\lambda(x-y) dx = \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} V(y+x) E_\lambda(x) dx.$$

Since

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} V(y+x) E_\lambda(x) dx \leq \sup_{y \in \mathbb{R}^d} \int_{|x| \leq \varepsilon} V(y+x) E_\lambda(x) dx + \sup_{y \in \mathbb{R}^d} \int_{|x| > \varepsilon} V(y+x) E_\lambda(x) dx \quad \forall \varepsilon > 0$$

then

$$\begin{aligned} \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} V(y+x) E_\lambda(x) dx &\leq \sup_{y \in \mathbb{R}^d} \int_{|x| \leq \varepsilon} V(y+x) E_\mu(x) dx \\ + \sup_{y \in \mathbb{R}^d} \int_{|x| > \varepsilon} V(y+x) E_\lambda(x) dx &\quad \forall \varepsilon > 0, \quad \forall \lambda \geq \mu. \end{aligned}$$

If

$$\sup_{y \in \mathbb{R}^d} \int_{|x| > \varepsilon} V(y+x) E_\lambda(x) dx \rightarrow 0 \quad (\lambda \rightarrow +\infty) \quad \forall \varepsilon > 0 \quad (26)$$

then

$$\lim_{\lambda \rightarrow +\infty} \left\| V(\lambda - T^1)^{-1} \right\|_{\mathcal{L}(L^1(\mathbb{R}^d))} \leq \sup_{y \in \mathbb{R}^d} \int_{|x| \leq \varepsilon} V(y+x) E_\mu(x) dx \quad \forall \varepsilon > 0$$

and consequently

$$\lim_{\lambda \rightarrow +\infty} \left\| V(\lambda - T^1)^{-1} \right\|_{\mathcal{L}(L^1(\mathbb{R}^d))} \leq \lim_{\varepsilon \rightarrow 0} \sup_{y \in \mathbb{R}^d} \int_{|x| \leq \varepsilon} V(y+x) E_\mu(x) dx = 0.$$

It suffices to check (26). It follows easily from (15) that

$$\|E_\lambda\|_{L^1(\mathbb{R}^d)} = \left\| (\lambda - T^1)^{-1} \right\|_{\mathcal{L}(L^1(\mathbb{R}^d))} \leq \frac{1}{\lambda} \rightarrow 0 \quad (\lambda \rightarrow +\infty).$$

Since $1_{\{|x| > \varepsilon\}} E_\lambda \in L^\infty(\mathbb{R}^d)$ then, by an interpolation argument

$$1_{\{|x| > \varepsilon\}} E_\lambda \in L^p(\mathbb{R}^d) \quad \text{and} \quad \|1_{\{|x| > \varepsilon\}} E_\lambda\|_{L^p(\mathbb{R}^d)} \rightarrow 0 \quad (\lambda \rightarrow +\infty) \quad \forall p \in [1, +\infty).$$

If $V \in L^{p_j}(\mathbb{R}^d)$ for some $p_j \in (1, +\infty]$ then

$$\begin{aligned} \int_{|x| > \varepsilon} V(y+x) E_\lambda(x) dx &\leq \left(\int_{|x| > \varepsilon} V(y+x)^{p_j} dx \right)^{\frac{1}{p_j}} \|1_{\{|x| > \varepsilon\}} E_\lambda\|_{L^{p_j^*}(\mathbb{R}^d)} \\ &\leq \left(\int_{\mathbb{R}^d} V(y+x)^{p_j} dx \right)^{\frac{1}{p_j}} \|1_{\{|x| > \varepsilon\}} E_\lambda\|_{L^{p_j^*}(\mathbb{R}^d)} \\ &= \left(\int_{\mathbb{R}^d} V(x)^{p_j} dx \right)^{\frac{1}{p_j}} \|1_{\{|x| > \varepsilon\}} E_\lambda\|_{L^{p_j^*}(\mathbb{R}^d)} \end{aligned}$$

and

$$\sup_{y \in \mathbb{R}^d} \int_{|x| > \varepsilon} V(y+x) E_\lambda(x) dx \leq \|V\|_{L^{p_j}(\mathbb{R}^d)} \|1_{\{|x| > \varepsilon\}} E_\lambda\|_{L^{p_j^*}(\mathbb{R}^d)} \rightarrow 0 \quad (\lambda \rightarrow +\infty).$$

The case $V \in \sum_{j=1}^{j=m} L^{p_j}(\mathbb{R}^d)$ follows by linearity. This proves (i).

If $V \in L^1(\mathbb{R}^d)$ then, arguing similarly, the assumption

$$\|1_{\{|x| > \varepsilon\}} E_\lambda\|_{L^\infty(\mathbb{R}^d)} \rightarrow 0 \quad (\lambda \rightarrow +\infty)$$

ends the proof of (ii).

Consider (iii). Since $k_t(z) = \widehat{k}_t(|z|)$ with $\widehat{k}_t(\rho)$ nonincreasing in ρ then

$$E_\lambda(z) = \int_0^{+\infty} e^{-\lambda t} k_t(z) dt$$

shows that

$$E_\lambda(z) = \widehat{E}_\lambda(|z|) := \int_0^{+\infty} e^{-\lambda t} \widehat{k}_t(|z|) dt$$

is nonincreasing in $|z|$. It follows that if $\widehat{E}_\lambda(|z|) = +\infty$ for some $|z| = \rho > 0$ then $E_\lambda(z) = +\infty$ for all $|z| \leq \rho$ which is not possible since $E_\lambda \in L^1(\mathbb{R}^d)$. Hence $E_\lambda(z) < +\infty$ for all $z \neq 0$. Moreover

$$E_\lambda(z) \leq \int_0^{+\infty} e^{-\lambda t} \widehat{k}_t(\varepsilon) dt \quad (|z| \geq \varepsilon)$$

and the dominated convergence shows $\int_0^{+\infty} e^{-\lambda t} \widehat{k}_t(\varepsilon) dt \rightarrow 0 \quad (\lambda \rightarrow +\infty)$. ■

5.2 Kato class vs weak compactness

We call (25) the Kato property and define the local Kato property by

$$\lim_{\varepsilon \rightarrow 0} \sup_{|y| \leq C} \int_{\{|z| \leq \varepsilon\}} V(y+x) E_\lambda(x) dx = 0 \quad (C > 0). \quad (27)$$

In this subsection, we show that the (local) Kato property can be formulated in terms of (local) weak compactness properties.

5.2.1 Weak compactness vs compactness

We introduce the local weak compactness property

$$\lim_{\substack{|\Omega| \rightarrow 0 \\ \Omega \subset B(0,R)}} \sup_{|y| \leq C} \left(\int_{\Omega} V(x) E_{\lambda}(x-y) dx \right) = 0 \quad (C > 0) \quad (28)$$

where $|\Omega|$ denotes the Lebesgue measure of Ω .

Theorem 10 *Let $E_{\lambda}(\cdot)$ be bounded outside any neighborhood of the origin and tends to zero at infinity. Then the following assertions are equivalent.*

- (i) V is a local Kato potential.
- (ii) (28) is satisfied.
- (iii) $VE_{\lambda} : L^1(\mathbb{R}^d) \rightarrow L^1_{loc}(\mathbb{R}^d)$ is weakly compact.
- (iv) $VE_{\lambda} : L^1(\mathbb{R}^d) \rightarrow L^1_{loc}(\mathbb{R}^d)$ is compact.

Proof. Note that (iii) (resp. (iv)) means that

$$\widehat{V}E_{\lambda} : L^1(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d)$$

is weakly compact (resp. compact) where \widehat{V} is the truncation de V (by zero) outside any ball $B(0, R) \subset \mathbb{R}^d$.

We observe that the weak compactness of $\widehat{V}E_{\lambda}$ amounts to

$$\lim_{\substack{|\Omega| \rightarrow 0 \\ \Omega \subset B(0,R)}} \sup_{\|\varphi\| \leq 1} \int_{\Omega} dx V(x) \left| \int_{\mathbb{R}^d} E_{\lambda}(x-y) \varphi(y) dy \right| = 0.$$

Without loss of generality, we can assume that $\varphi \geq 0$. Since

$$\int_{\Omega} dx V(x) \int_{\mathbb{R}^d} E_{\lambda}(x-y) \varphi(y) dy = \int_{\mathbb{R}^d} \left(\int_{\Omega} V(x) E_{\lambda}(x-y) dx \right) \varphi(y) dy$$

then

$$\sup_{\|\varphi\| \leq 1} \int_{\Omega} dx V(x) \int_{\mathbb{R}^d} E_{\lambda}(x-y) \varphi(y) dy = \sup_{y \in \mathbb{R}^d} \left(\int_{\Omega} V(x) E_{\lambda}(x-y) dx \right).$$

Hence (iii) amounts to

$$\lim_{\substack{|\Omega| \rightarrow 0 \\ \Omega \subset B(0,R)}} \sup_{y \in \mathbb{R}^d} \left(\int_{\Omega} V(x) E_{\lambda}(x-y) dx \right) = 0$$

for any ball $B(0, R) \subset \mathbb{R}^d$. Thus (iii) implies (ii). Let us show that (ii) implies (iii).

We note for $x \in B(0, R)$ and $|y| > R$ we have $|x - y| \geq |y| - R$ and then

$$\sup_{x \in \Omega} E_\lambda(x - y) \rightarrow 0 \quad (|y| \rightarrow \infty).$$

Thus, so for $|y| > R$

$$\begin{aligned} \int_{\Omega} |V(x)| E_\lambda(x - y) dx &\leq \sup_{x \in \Omega} E_\lambda(x - y) \int_{\Omega} |V(x)| dx \\ &\leq \sup_{x \in \Omega} E_\lambda(x - y) \int_{B(0, R)} |V(x)| dx \rightarrow 0 \quad (|y| \rightarrow \infty). \end{aligned}$$

This shows that it suffices to take C large enough and to use (ii).

Let us show that (iii) implies (iv). We observe that since $E_\lambda(\cdot) \in L^1(\mathbb{R}^d)$ then

$$E_\lambda : L^1(\mathbb{R}^d) \rightarrow L^1_{loc}(\mathbb{R}^d)$$

is compact by Kolmogorov's criterion (see e.g. [3] Corollary 4.28, p. 114).

Let

$$\widehat{V}_n := \widehat{V} \wedge n.$$

Then

$$\widehat{V}_n E_\lambda : L^1(\mathbb{R}^d) \rightarrow L^1(B(0, R)) \text{ is compact}$$

as a composition of a compact operator and a bounded one. It suffices to show that

$$\widehat{V}_n E_\lambda \rightarrow \widehat{V} E_\lambda \text{ in operator norm } (n \rightarrow \infty).$$

It is easy to see that

$$\begin{aligned} \left\| \widehat{V} E_\lambda - \widehat{V}_n E_\lambda \right\| &= \left\| (\widehat{V} - \widehat{V}_n) E_\lambda \right\| \\ &= \sup_{y \in \mathbb{R}^d} \int_{B(0, R)} (\widehat{V}(x) - \widehat{V}_n(x)) E_\lambda(x - y) dx \\ &= \sup_{y \in \mathbb{R}^d} \int_{B(0, R) \cap \{V(x) \geq n\}} V(x) E_\lambda(x - y) dx. \end{aligned}$$

Note that $V(x) < +\infty$ a.e. so

$$|B(0, R) \cap \{V(x) \geq n\}| \rightarrow 0 \quad (n \rightarrow \infty).$$

Since $VE_\lambda : L^1(\mathbb{R}^d) \rightarrow L^1(B(0, R))$ is weakly compact then

$$\lim_{\substack{|\Omega| \rightarrow 0 \\ \Omega \subset B(0, R)}} \sup_{y \in \mathbb{R}^d} \left(\int_{\Omega} V(x) E_\lambda(x - y) dx \right) = 0$$

in particular

$$\sup_{y \in \mathbb{R}^d} \int_{B(0, R) \cap \{|V(x) \geq n\}} V(x) E_\lambda(x - y) dx \rightarrow 0 \quad (n \rightarrow \infty)$$

so $\|\widehat{V}E_\lambda \rightarrow \widehat{V}_n E_\lambda\| \rightarrow 0$ ($n \rightarrow \infty$). It remains to show the equivalence of (i) and (ii). Assume (i). We have to show (ii), i.e.

$$\lim_{\substack{|\Omega| \rightarrow 0 \\ \Omega \subset B(0, R)}} \sup_{|y| \leq C} \left(\int_{\Omega} V(x) E_\lambda(x - y) dx \right) = 0$$

for any $C > R$. On the other hand,

$$\begin{aligned} \int_{\Omega} V(x) E_\lambda(x - y) dx &= \int_{\Omega \cap \{|x-y| \leq \varepsilon\}} V(x) E_\lambda(x - y) dx \\ &\quad + \int_{\Omega \cap \{|x-y| > \varepsilon\}} V(x) E_\lambda(x - y) dx \\ &= \int_{\{\Omega - y\} \cap \{|z| \leq \varepsilon\}} V(y + z) E_\lambda(z) dz \\ &\quad + \int_{\Omega \cap \{|x-y| > \varepsilon\}} V(x) E_\lambda(x - y) dx \\ &\leq \int_{\{|z| \leq \varepsilon\}} |V(y + z)| E_\lambda(z) dz + \sup_{|z| > \varepsilon} E_\lambda(z) \int_{\Omega} |V(x)| dx. \end{aligned}$$

Since $|y| \leq C$ then, by (i)

$$\sup_{|y| \leq C} \int_{\{|z| \leq \varepsilon\}} |V(y + z)| E_\lambda(z) dz \rightarrow 0 \quad (\varepsilon \rightarrow 0)$$

and for a fixed ε small enough

$$\sup_{|z| > \varepsilon} E_\lambda(z) \int_{\Omega} |V(x)| dx \rightarrow 0 \quad (|\Omega| \rightarrow 0).$$

Thus (i) implies (ii). Conversely,

$$\begin{aligned}
\int_{|z|\leq\delta} |V|(y+z)E_\lambda(z)dz &= \int_{|x-y|\leq\delta} |V|(x)E_\lambda(x-y)dx \\
&= \int_{B(y;\delta)} |V|(x)E_\lambda(x-y)dx \\
&= \int_{\Omega_y} |V|(x)E_\lambda(x-y)dx
\end{aligned}$$

where $\Omega_y := B(y; \delta)$ is such that

$$\Omega_y \subset B(0, C+1) \text{ if } \delta < 1.$$

Since (ii) gives

$$\lim_{|\Omega|\rightarrow 0} \sup_{\Omega \subset B(0, C+1)} \sup_{y \in \mathbb{R}^d} \int_{\Omega} |V|(x)E_\lambda(x-y)dx = 0$$

then

$$\sup_{|y|\leq C} \int_{\Omega_y} |V|(x)E_\lambda(x-y)dx \rightarrow 0 \quad (\delta \rightarrow 0)$$

so (ii) implies (i). ■

Remark 11 *The characterization of the local Kato class in terms of a weak compactness property is new even for the Laplacian.*

Remark 12 *The equivalence (i) \Leftrightarrow (iv) is known for the Laplacian; ([1] Theorem 4.18).*

Remark 13 *In the preprint [22], the author has introduced a class of potentials (a priori larger than the Kato class) which satisfy the local weak compactness property (28) and which are "Kato at infinity" in the sense*

$$\lim_{\varepsilon \rightarrow 0} \lim_{|y| \rightarrow \infty} \sup_{\{|z|\leq\varepsilon\}} \int V(y+x)E_\lambda(x)dx = 0.$$

Actually, because of the equivalence (iii) \Leftrightarrow (iv) in Theorem 10 above, one can check that the two classes coincide ! Thus the present paper replaces [22].

5.2.2 Weak compactness continued

We give now different (sufficient) weak compactness criteria. We note that the T^1 -boundedness of V , i.e. (19), can be formulated as

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} V_y(z) E_\lambda(z) dz < +\infty$$

where

$$V_y : z \in \mathbb{R}^d \rightarrow V(y + z).$$

In particular

$$\sup_{y \in \mathbb{R}^d} \int_{|z| \leq 1} V_y(z) E_\lambda(z) dz < +\infty$$

i.e. $\{V_y; y \in \mathbb{R}^d\}$ is a bounded subset of

$$L^1(B(0, 1); \mu(dz))$$

where $\mu(dz) = E_\lambda(z) dz$.

We start with a particular case:

Theorem 14 *If*

$$\mathbb{R}^d \ni y \rightarrow V_y \in L^1(B(0, 1); \mu(dz)) \text{ is continuous}$$

then V is a local Kato potential.

Proof. By assumption $\{V_y; |y| \leq C\}$ is a compact subset of $L^1(B(0, 1); \mu(dz))$. It follows that this set is equi-integrable, in particular

$$\lim_{\varepsilon \rightarrow 0} \sup_{|y| \leq C} \int_{|z| \leq \varepsilon} V_y(z) \mu(dz) = 0$$

since $\int_{|z| \leq \varepsilon} \mu(dz) \rightarrow 0$ ($\varepsilon \rightarrow 0$). ■

Remark 15 *This result extends to more general operators a result in the same spirit for the Laplacian (see [1] Theorem 4.15).*

Actually, more generally, *both* local and global Kato property can be formulated in terms equi-integrability with respect to $\mu(dz)$.

Theorem 16 *A potential V is Kato (resp. local Kato) potential if*

$$\left\{ V_y; y \in \mathbb{R}^d \right\}, \text{ (resp. } \{V_y; |y| \leq C\} \text{ (} C > 0 \text{))}$$

is equi-integrable subset of $L^1(B(0, 1); \mu(dz))$.

Proof. We have

$$\begin{aligned}
\int_{\{|z|\leq\varepsilon\}} V(y+z)E_\lambda(z)dz &= \int_{\{|z|\leq\varepsilon\}} V_y(z)E_\lambda(z)dz \\
&= \int_{\{|z|\leq\varepsilon\}\cap\{V_y(z)\geq j\}} V_y(z)E_\lambda(z)dz \\
&\quad + \int_{\{|z|\leq\varepsilon\}\cap\{V_y(z)< j\}} V_y(z)E_\lambda(z)dz \\
&\leq \int_{\{|z|\leq 1\}\cap\{V_y(z)\geq j\}} V_y(z)E_\lambda(z)dz \\
&\quad + j \int_{\{|z|\leq\varepsilon\}} E_\lambda(z)dz.
\end{aligned}$$

Let $\{V_y; y \in \mathbb{R}^d\}$ be an equi-integrable subset of $L^1(B(0,1); \mu(dz))$. From

$$\begin{aligned}
&\sup_{y \in \mathbb{R}^d} \int_{\{|z|\leq\varepsilon\}} V(y+z)E_\lambda(z)dz \\
&\leq \sup_{y \in \mathbb{R}^d} \int_{\{|z|\leq 1\}\cap\{V_y(z)\geq j\}} V_y(z)E_\lambda(z)dz + j \int_{\{|z|\leq\varepsilon\}} E_\lambda(z)dz
\end{aligned}$$

and the criterion of equi-integrability (see e.g. [2] Theorem 4.7.20, p. 287), we fix j large enough so that $\sup_{y \in \mathbb{R}^d} \int_{\{|z|\leq 1\}\cap\{V_y(z)\geq j\}} V_y(z)E_\lambda(z)dz$ is as small as we want and then let $\varepsilon \rightarrow 0$. to show that V is Kato potential. We argue similarly to show that V is local Kato potential. ■

We can derive easily a practical criterion relying on the equi-integrability criterion given by Theorem 16.

Corollary 17 *A potential V is Kato (resp. local Kato) potential if there exists $p > 1$ such that*

$$\sup_{y \in \mathbb{R}^d} \int_{|z|\leq 1} V_y(z)^p E_\lambda(z)dz < +\infty \text{ (resp. } \sup_{|y|\leq C} \int_{|z|\leq 1} V_y(z)^p E_\lambda(z)dz < +\infty, C > 0).$$

Proof. Let the $\mu(dz)$ -measure of $\Omega \subset B(0,1)$ tends to zero, i.e.

$$\int_{\{|x|\leq 1\}\cap\Omega} E_\lambda(x)dx \rightarrow 0.$$

Then the estimate

$$\begin{aligned}
\int_{\{|x|\leq 1\}\cap\Omega} V_y(x)\mu(dx) &= \int_{\{|x|\leq 1\}\cap\Omega} V_y(x)E_\lambda(x)dx \\
&\leq \left(\int_{\{|x|\leq 1\}} V_y(x)^p E_\lambda(x)dx \right)^{\frac{1}{p}} \left(\int_{\{|x|\leq 1\}\cap\Omega} E_\lambda(x)dx \right)^{\frac{1}{p'}} \rightarrow 0
\end{aligned}$$

(p^* is the conjugate exponent of p) ends the proof. ■

Remark 18 See Remark 26 below for a concrete application of Corollary 17.

5.3 Kato classes vs averages over spherical shells

For the Laplacian, it is known that V is a Kato potential provided that $V \in L^p_{loc,unif}$, i.e.

$$\sup_{y \in \mathbb{R}^d} \int_{\{|x| \leq 1\}} (V(y+x))^p dx < +\infty,$$

for some $p > \frac{d}{2}$ (see [1] Theorem 1.4 (iii)). By averaging in angles, it is easy to improve slightly this result (i.e. we gain something in "angles"). Actually, we state this for general rotationally invariant λ -potential kernels.

Theorem 19 Let $E_\lambda(\cdot)$ be rotationally invariant, i.e. $E_\lambda(x) = \widehat{E}_\lambda(|x|)$ and let

$$h(y, \rho) := \int_{S^{d-1}} V(y + \rho\omega) dS(\omega).$$

Let

$$s_\infty := \sup \left\{ s > 1; \int_{\{|x| \leq 1\}} (E_\lambda(x))^s dx < +\infty \right\}$$

If there exists $s > s_\infty^*$ (the conjugate exponent of s_∞) such that

$$C_s := \sup_{y \in \mathbb{R}^d} \left(\int_0^1 (h(y, \rho))^s \rho^{d-1} d\rho \right) < +\infty$$

then V is a Kato potential.

Proof. Note that V being T_1 -bounded in $L^1(\mathbb{R}^d)$ we have at least

$$\sup_{y \in \mathbb{R}^d} \int_0^1 h(y, \rho) \rho^{d-1} d\rho < +\infty.$$

Since

$$\int_{\{|x| \leq \varepsilon\}} V(y+x) E_\lambda(x) dx = \int_0^\varepsilon h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho$$

then

$$\begin{aligned} & \sup_{y \in \mathbb{R}^d} \int_{\{|x| \leq \varepsilon\}} V(y+x) E_\lambda(x) dx \\ & \leq \left[\sup_{y \in \mathbb{R}^d} \left(\int_0^1 (h(y, \rho))^s \rho^{d-1} d\rho \right)^{\frac{1}{s}} \right] \left(\int_0^\varepsilon \left(\widehat{E}_\lambda(\rho) \right)^{s^*} \rho^{d-1} d\rho \right)^{\frac{1}{s^*}} \end{aligned}$$

and we are done since $\int_0^\varepsilon \left(\widehat{E}_\lambda(\rho) \right)^{s^*} \rho^{d-1} d\rho \rightarrow 0$ ($\varepsilon \rightarrow 0$). ■

We give now membership criteria to the Kato class in terms of asymptotics of the averages of V_y over spherical shells

$$\Omega_j = \left\{ z \in \mathbb{R}^d; 2^{-(j+1)} \leq |z| < 2^{-j} \right\}, \quad (j \in \mathbb{N}).$$

Let

$$\eta_j(y) := \int_{\Omega_j} V_y(z) dz \quad \text{and} \quad \widehat{\eta}_j := \sup_{y \in \mathbb{R}^d} \eta_j(y).$$

Theorem 20 *Let $E_\lambda(\cdot)$ be spherically symmetric and radially decreasing, (i.e. $E_\lambda(z) = \widehat{E}_\lambda(|z|)$ and $\rho \rightarrow \widehat{E}_\lambda(\rho)$ nonincreasing) and let V be locally integrable. Then*

$$\sum_{j=0}^{\infty} \left(\widehat{E}_\lambda(2^{-j}) \right) \eta_j(y) \leq \int_{\{|x| \leq 1\}} V_y(x) E_\lambda(x) dx \leq \sum_{j=0}^{\infty} \widehat{E}_\lambda(2^{-(j+1)}) \eta_j(y).$$

Moreover, V is a Kato potential if and only if the series

$$\sum_{j=1}^{\infty} \left(\widehat{E}_\lambda(2^{-j}) \right) \eta_j(y) \quad \text{converges uniformly in } y \in \mathbb{R}^d. \quad (29)$$

This occurs if

$$\limsup_{j \rightarrow \infty} \left(\widehat{\eta}_j \right)^{\frac{1}{j}} < \left(\limsup_{j \rightarrow \infty} \left(\widehat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}} \right)^{-1}.$$

Proof. We know that

$$\int_{\{|x| \leq 1\}} V_y(x) E_\lambda(x) dx = \int_0^1 h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho$$

where

$$h(y, \rho) := \int_{S^{d-1}} V_y(\rho\omega) dS(\omega).$$

The fact that

$$\begin{aligned}
\int_0^1 h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho &= \sum_{j=0}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho \\
&\leq \sum_{j=0}^{\infty} \widehat{E}_\lambda(2^{-(j+1)}) \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \rho^{d-1} d\rho \\
&= \sum_{j=0}^{\infty} \widehat{E}_\lambda(2^{-(j+1)}) \eta_j(y)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho &= \sum_{j=0}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho \\
&\geq \sum_{j=0}^{\infty} \widehat{E}_\lambda(2^{-j}) \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \rho^{d-1} d\rho \\
&= \sum_{j=0}^{\infty} \widehat{E}_\lambda(2^{-j}) \eta_j(y)
\end{aligned}$$

show the first claim.

We have also

$$\int_{\{|x| \leq \varepsilon\}} W_y(x) E_\lambda(x) dx = \int_0^\varepsilon h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho.$$

Since $2^{-(j+1)} < \varepsilon$ amounts to $\frac{\ln \varepsilon^{-1}}{\ln 2} - 1 \leq j$ then

$$\begin{aligned}
\int_0^\varepsilon h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho &\leq \sum_{\{j; j \geq \frac{\ln \varepsilon^{-1}}{\ln 2} - 1\}} \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho \\
&\leq \sum_{\{j; j \geq \frac{\ln \varepsilon^{-1}}{\ln 2} - 1\}} \widehat{E}_\lambda(2^{-(j+1)}) \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \rho^{d-1} d\rho \\
&= \sum_{\{j; j \geq \frac{\ln \varepsilon^{-1}}{\ln 2} - 1\}} \widehat{E}_\lambda(2^{-(j+1)}) \eta_j(y).
\end{aligned}$$

Note that $\varepsilon \rightarrow 0$ implies that $\frac{\ln \varepsilon^{-1}}{\ln 2} - 1 \rightarrow +\infty$ so that

$$\sup_{y \in \mathbb{R}^d} \int_0^\varepsilon h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho \rightarrow 0 \quad (\varepsilon \rightarrow 0) \tag{30}$$

if the series $\sum_{j \geq 0} \widehat{E}_\lambda(2^{-(j+1)})\eta_j(y)$ converges uniformly in $y \in \mathbb{R}^d$.

Conversely, $2^{-j} < \varepsilon$ amounts to $\frac{\ln \varepsilon^{-1}}{\ln 2} \leq j$ so

$$\begin{aligned} \int_0^\varepsilon h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho &\geq \sum_{\{j; j \geq \frac{\ln \varepsilon^{-1}}{\ln 2}\}} \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \widehat{E}_\lambda(\rho) \rho^{d-1} d\rho \\ &\geq \sum_{\{j; j \geq \frac{\ln \varepsilon^{-1}}{\ln 2}\}} \widehat{E}_\lambda(2^{-j}) \int_{2^{-(j+1)}}^{2^{-j}} h(y, \rho) \rho^{d-1} d\rho \\ &= \sum_{\{j; j \geq \frac{\ln \varepsilon^{-1}}{\ln 2}\}} \widehat{E}_\lambda(2^{-j}) \eta_j(y) \end{aligned}$$

shows that $\sum_{j \geq 0} \widehat{E}_\lambda(2^{-j})\eta_j(y)$ converges uniformly in $y \in \mathbb{R}^d$ if (30) holds. Thus the second claim follows from the fact that $\widehat{E}_\lambda(2^{-j}) \geq \widehat{E}_\lambda(2^{-(j+1)})$. A sufficient condition for uniform convergence is

$$\limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-j}) \widehat{\eta}_j \right)^{\frac{1}{j}} < 1$$

and this ends the proof since

$$\limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-j}) \widehat{\eta}_j \right)^{\frac{1}{j}} \leq \limsup_{j \rightarrow +\infty} (\widehat{\eta}_j)^{\frac{1}{j}} \limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}}.$$

■

We complement Theorem 20 by:

Lemma 21 *Let*

$$s := \sup \left\{ p \geq 1; \int_{\{|x| \leq 1\}} (E_\lambda(x))^p dx < +\infty \right\}. \quad (31)$$

Then

$$\limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}} \leq 2^{\frac{d}{s}}.$$

Proof. We already know that $\int_{\{|x| \leq 1\}} E_\lambda(x) dx < +\infty$. Let $s > 1$ and let $1 < p < s$. We have

$$\int_{\{|x| \leq 1\}} (E_\lambda(x))^p dx = |S^{d-1}| \int_0^1 \left(\widehat{E}_\lambda(\rho) \right)^p \rho^{d-1} d\rho$$

and

$$\begin{aligned} \int_0^1 \left(\widehat{E}_\lambda(\rho)\right)^p \rho^{d-1} d\rho &= \sum_{j=0}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} \left(\widehat{E}_\lambda(\rho)\right)^p \rho^{d-1} d\rho \\ &\geq \sum_{j=0}^{\infty} \left(\widehat{E}_\lambda(2^{-j})\right)^p \int_{2^{-(j+1)}}^{2^{-j}} \rho^{d-1} d\rho. \end{aligned}$$

Since $\int_{2^{-(j+1)}}^{2^{-j}} \rho^{d-1} d\rho = c_d 2^{-(j+1)d}$ with $c_d = d^{-1} (2^d - 1)$ (see the proof of theorem 25) then

$$\sum_{j=0}^{\infty} 2^{-(j+1)d} \left(\widehat{E}_\lambda(2^{-j})\right)^p < +\infty$$

which implies

$$\limsup_{j \rightarrow +\infty} \left(2^{-(j+1)d} \left(\widehat{E}_\lambda(2^{-j})\right)^p\right)^{\frac{1}{j}} \leq 1$$

i.e.

$$\limsup_{j \rightarrow +\infty} \left(\left(\widehat{E}_\lambda(2^{-j})\right)^p\right)^{\frac{1}{j}} \leq 2^d.$$

Since $\limsup_{j \rightarrow +\infty} \left(\left(\widehat{E}_\lambda(2^{-j})\right)^p\right)^{\frac{1}{j}} = \left(\limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-j})\right)^{\frac{1}{j}}\right)^p$ then

$$\limsup_{j \rightarrow +\infty} \left(\widehat{E}_\lambda(2^{-j})\right)^{\frac{1}{j}} \leq 2^{\frac{d}{p}} \quad (p < s).$$

Finally, letting $p \rightarrow s$ ends the proof when $s > 1$. If $s = 1$, the above calculations with $p = 1$ end the proof. ■

Remark 22 *If we consider the class of convolution semigroups considered in [4], we have $\widehat{E}_\lambda(\rho) \sim \widehat{E}_\beta(\rho)$ ($\rho \rightarrow 0$) ($\lambda, \beta > 0$) (see [4] Lemma III. 3) so that the parameter (31) is λ -independent.*

By combining Theorem 20 and Lemma 21 we get:

Corollary 23 *Let $E_\lambda(\cdot)$ be spherically symmetric and radially decreasing, (i.e. $E_\lambda(z) = \widehat{E}_\lambda(|z|)$ and $\rho \rightarrow \widehat{E}_\lambda(\rho)$ nonincreasing). Let V be locally integrable and let s be given by (31). Then V is a Kato potential provided that*

$$\limsup_{j \rightarrow \infty} \left(\widehat{\eta}_j\right)^{\frac{1}{j}} < 2^{-\frac{d}{s}}.$$

Remark 24 Note that for $d \geq 3$

$$\widehat{E}_0(\rho) \sim \frac{1}{\rho^{d-\alpha}} \quad (\rho \rightarrow 0) \quad (0 < \alpha \leq 2) \quad (32)$$

for the usual examples ($\alpha = 2$ for the heat semigroup, $0 < \alpha < 2$ for the α -stable semigroup and $\alpha = 1$ for the relativistic semigroup; see [4]). Thus $\widehat{E}_0(2^{-(j+1)}) \sim 2^{(j+1)(d-\alpha)}$ and

$$\limsup_{j \rightarrow +\infty} \left(\widehat{E}_0(2^{-j}) \right)^{\frac{1}{j}} = 2^{(d-\alpha)}$$

shows that Lemma 21 is optimal. In these cases, V is Kato potential provided that

$$\limsup_{j \rightarrow \infty} \left(\widehat{\eta}_j \right)^{\frac{1}{j}} < 2^{-(d-\alpha)}.$$

Actually, for subordinate Brownian semigroups (relative to a Bernstein function f), the behaviour of $\widehat{E}_0^f(\rho)$ ($\rho \rightarrow 0$) is determined by the asymptotics of $f(\lambda)$ at infinity. In particular, $\widehat{E}_0^f(\rho) \sim \frac{1}{\rho^{d-\alpha}}$ ($\rho \rightarrow 0$) ($0 < \alpha \leq 2$) if $f(\lambda) \sim \lambda^{\frac{\alpha}{2}}$ ($\lambda \rightarrow +\infty$); (see [24] Theorem 3.1). This is the case of the Bernstein function $f(\lambda) = \left(\lambda + m \frac{2}{\alpha} \right)^{\frac{\alpha}{2}} - m$ ($0 < \alpha < 2$) which defines the relativistic α -stable semigroup with generator

$$T = - \left(-\Delta + m \frac{2}{\alpha} \right)^{\frac{\alpha}{2}} + m$$

and characteristic exponent $F(\zeta) = \left(|\zeta|^2 + m \frac{2}{\alpha} \right)^{\frac{\alpha}{2}} - m$.

We are going to derive Theorem 19 from Theorem 20.

Theorem 25 Let $E_\lambda(x)$ be rotationally invariant, i.e. $E_\lambda(x) = \widehat{E}_\lambda(|x|)$, and let (32) be satisfied. If $V \in L_{loc,unif}^s$, i.e.

$$\sup_{y \in \mathbb{R}^d} \int_{\{|x| \leq 1\}} (V(y+x))^s dx < +\infty,$$

with $s > \frac{d}{\alpha}$ then V is Kato potential.

Proof. We have

$$\begin{aligned}
& \int_{2^{-j} \leq |z| \leq 2^{-(j-1)}} V_y(z) dz \\
& \leq \left(\int_{2^{-j} \leq |z| \leq 2^{-(j-1)}} V_y(z)^s dz \right)^{\frac{1}{s}} \left(\int_{2^{-j} \leq |z| \leq 2^{-(j-1)}} dz \right)^{\frac{1}{s^*}} \\
& \leq C_j(s) \left(\int_{2^{-j} \leq |z| \leq 2^{-(j-1)}} dz \right)^{\frac{1}{s^*}} = C_j(s) \left(|S^{d-1}| \int_{2^{-j}}^{2^{-(j-1)}} \rho^{d-1} d\rho \right)^{\frac{1}{s^*}}
\end{aligned}$$

Since

$$\int_{2^{-j}}^{2^{-(j-1)}} \rho^{d-1} d\rho = c_d 2^{-jd}$$

(where $c_d = d^{-1} (2^d - 1)$) then

$$\hat{\eta}_j := \sup_{y \in \mathbb{R}^d} \int_{2^{-j} \leq |z| \leq 2^{-(j-1)}} V_y(z) dz \leq C_j(s) \left(|S^{d-1}| c_d 2^{-jd} \right)^{\frac{1}{s^*}}.$$

Since $\hat{E}_\lambda(2^{-(j+1)}) \sim 2^{(j+1)(d-\alpha)}$ then

$$\begin{aligned}
\limsup_{j \rightarrow \infty} \left(\hat{\eta}_j \hat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}} & \leq \limsup_{j \rightarrow \infty} (\hat{\eta}_j)^{\frac{1}{j}} \limsup_{j \rightarrow \infty} \left(\hat{E}_\lambda(2^{-j}) \right)^{\frac{1}{j}} \\
& \leq \left(\limsup_{j \rightarrow \infty} (C_j(s))^{\frac{1}{j}} \right) 2^{-\frac{d}{s^*}} 2^{(d-\alpha)} \\
& = \left(\limsup_{j \rightarrow \infty} (C_j(s))^{\frac{1}{j}} \right) 2^{\frac{d}{s} - \alpha}.
\end{aligned}$$

Note that $\limsup_{j \rightarrow \infty} (C_j(s))^{\frac{1}{j}} \leq 1$ since $V \in L_{loc,unif}^s$. Finally, Theorem 20 ends the proof. ■

Remark 26 One can also derive Theorem 19 (or Theorem 25) from the equi-integrability criterion given in Theorem 16. Indeed, let $s > \frac{d}{\alpha}$. We know that $E_\lambda(\cdot) \in L_{loc}^p$ for all $1 \leq p < \frac{d}{d-\alpha}$ and $p^* \in \left(\frac{d}{\alpha}, +\infty\right]$. Then for

any $1 < r < s$ and p^* such that $rp^* < s$

$$\begin{aligned}
\int_{\{|x|\leq 1\}} V_y(x)^r E_\lambda(x) dx &\leq \left(\int_{\{|x|\leq 1\}} V_y(x)^{rp^*} dx \right)^{\frac{1}{p^*}} \left(\int_{\{|x|\leq 1\}} E_\lambda(x)^p dx \right)^{\frac{1}{tp}} \\
&\leq \left(\int_{\{|x|\leq 1\}} V_y(x)^s dx \right)^{\frac{1}{p^*\tau}} |B(0,1)|^{\frac{1}{\tau^*}} \left(\int_{\{|x|\leq 1\}} E_\lambda(x)^p dx \right)^{\frac{1}{tp}} \\
&\leq \left(\sup_y \left(\int_{\{|x|\leq 1\}} V_y(x)^s dx \right)^{\frac{1}{p^*\tau}} \right) |B(0,1)|^{\frac{1}{\tau^*}} \left(\int_{\{|x|\leq 1\}} E_\lambda(x)^p dx \right)^{\frac{1}{tp}}
\end{aligned}$$

where $rp^*\tau = s$ (i.e. $\tau = \frac{s}{rp^*} > 1$). By choosing p^* close to $\frac{d}{\alpha}$ and r close to 1, we have indeed $rp^* < s$ since $s > \frac{d}{\alpha}$. Hence Corollary 17 ends the proof.

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